THE HOMOTOPY TYPES OF G_2-GAUGE GROUPS

DAISUKE KISHIMOTO, STEPHEN THERIAULT, AND MITSUNOBU TSUTAYA

Abstract. The equivalence class of a principal G_2-bundle over S^4 is classified by the value $k \in \mathbb{Z}$ of the second chern class. In this paper we consider the homotopy types of the corresponding gauge groups $\mathcal{G}_k$, and determine the number of homotopy types up to one factor of 2.

1. Introduction

Let G be a simple, simply-connected, compact Lie group and let BG be its classifying space. Since $[S^4, BG] \cong \mathbb{Z}$, a principal G-bundle $P \longrightarrow S^4$ is classified by the integer value k of its second chern class. The *gauge group* $\mathcal{G}_k(G)$ of this bundle is the group of G-equivariant automorphisms of P which fix S^4. As there are countably many equivalence classes of principal G-bundles, potentially there are countably many inequivalent gauge groups. However, in [CS] it was shown that the gauge groups $\{\mathcal{G}_k(G) \mid k \in \mathbb{Z}\}$ have only finitely many distinct homotopy types.

There has been considerable interest recently in determining precisely when $\mathcal{G}_k(G) \simeq \mathcal{G}_{k'}(G)$ in special cases. If a, b are two integers, let (a, b) be the greatest common divisor of $|a|$ and $|b|$. It is known that $\mathcal{G}_k(SU(2)) \simeq \mathcal{G}_{k'}(SU(2))$ if and only if $(12, k) = (12, k')$ [K]; $\mathcal{G}_k(SU(3)) \simeq \mathcal{G}_{k'}(SU(3))$ if and only if $(24, k) = (24, k')$ [HK]; and in a non-simply connected case, $\mathcal{G}_k(SO(3)) \simeq \mathcal{G}_{k'}(SO(3))$ if and only if $(12, k) = (12, k')$ [KKKT]. Slightly weaker results hold for $Sp(2)$ and $SU(5)$: a homotopy equivalence $\mathcal{G}_k(Sp(2)) \simeq \mathcal{G}_{k'}(Sp(2))$ implies $(40, k) = (40, k')$ and if $(40, k) = (40, k')$ then $\mathcal{G}_k(Sp(2)) \simeq \mathcal{G}_{k'}(Sp(2))$ when localized rationally or at any prime [Th2]; while a homotopy equivalence $\mathcal{G}_k(SU(5)) \simeq \mathcal{G}_{k'}(SU(5))$ implies $(120, k) = (120, k')$ and if $(120, k) = (120, k')$ then $\mathcal{G}_k(SU(5)) \simeq \mathcal{G}_{k'}(SU(5))$ when localized rationally or at any prime [Th3]. The localized statements for $Sp(2)$ and $SU(5)$ stem from the fact that there are integral homotopy classes in dimensions larger than 5, which leads to potential obstructions when attempting to produce integral statements from local statements via a Sullivan arithmetic square.

The goal of this paper is to investigate the homotopy types of the gauge groups $\mathcal{G}_k(G_2)$. A pivotal step in determining the homotopy types of gauge groups for any simply-connected simple compact Lie group G is determining the order of the Samelson product $S^3 \wedge G \xrightarrow{\langle i, 1 \rangle} G$, where i is the canonical group homomorphism $S^3 = SU(2) \longrightarrow G$ and 1 is the identity map on G. Remarkably, despite decades of intense research in the topology of Lie groups, very little is known about such

2010 *Mathematics Subject Classification.* Primary 55P15, Secondary 54C35.

Key words and phrases. gauge group, Lie group, homotopy type.

Samelson products. Most of the effort in the $SU(3)$, $Sp(2)$ and $SU(5)$ cases mentioned above was devoted to finding the precise order of $\langle i, 1\rangle$. As the Lie groups increase in complexity it becomes less likely that the precise order of $\langle i, 1\rangle$ can be determined. Instead, it is more reasonable to look for good upper and lower bounds on the order. We do this for G_2 and are able to determine the order of $\langle i, 1\rangle$ up to a single factor of 2.

Theorem 1.1. *The Samelson product* $S^3 \wedge G_2 \xrightarrow{\langle i,1\rangle} G_2$ *has order* $4 \cdot 3 \cdot 7 = 84$ *or* $8 \cdot 3 \cdot 7 = 168$.

The 2 and 3-primary information in Theorem 1.1 demands a delicate analysis of the very subtle homotopy theory of G_2: the 3-primary problem is resolved using methods from unstable K-theory, while the 2-primary problem involves extracting as much commutativity information as possible out of the non-commutative group G_2. This theorem greatly improves a result of Oshima [O], which states that the Samelson product $\langle \pi_3(G_2), \pi_{11}(G_2)\rangle$ has order $2 \cdot 3 \cdot 7 = 42$; note that Oshima's result is about certain homotopy groups of G_2 while Theorem 1.1 is more fundamentally about the space G_2 itself.

Theorem 1.1 is applied to examine the homotopy types of the gauge groups $\mathcal{G}_k(G_2)$.

Theorem 1.2. *The following hold:*

(a) *if there is a homotopy equivalence* $\mathcal{G}_k(G_2) \simeq \mathcal{G}_{k'}(G_2)$ *then* $(84, k) = (84, k')$*;*
(b) *if* $(168, k) = (168, k')$ *then* $\mathcal{G}_k(G_2)$ *and* $\mathcal{G}_{k'}(G_2)$ *are homotopy equivalent when localized rationally or at any prime.*

In particular, there are either 12 or 24 homotopy types for $\mathcal{G}_k(G_2)$, and all odd primary homotopy types are completely determined.

The authors would like to thank the London Mathematical Society and the Great Britain Sasakawa Foundation for supporting mutual research visits which made this project possible. We would also like to thank the referee for carefully reading the paper and making many helpful comments.

2. A method for determining the homotopy types of gauge groups

In this section we describe the basic method for determining the homotopy types of gauge groups that will be used to prove Theorem 1.2. We first establish a context in which gauge groups are more easily studied, from a homotopy theoretic point of view. As we will have to consider $SU(3)$-gauge groups as well as G_2-gauge groups at a certain point, the context will be written for the general case of any simple, simply-connected compact Lie group G. We also suppress the group G by writing $\mathcal{G}_k = \mathcal{G}_k(G)$.

By [AB], there is a homotopy equivalence $B\mathcal{G}_k \simeq \mathrm{Map}_k(S^4, BG)$ between the classifying space $B\mathcal{G}_k$ of $\mathcal{G}_k$ and the component of the space of continuous maps from S^4 to BG which contains the map inducing P. Further, there is a fibration $\mathrm{Map}^*_k(S^4, BG) \longrightarrow \mathrm{Map}_k(S^4, BG) \xrightarrow{ev} BG$, where ev evaluates a map at the basepoint of S^4 and $\mathrm{Map}^*_k(S^4, BG)$ is the k^{th}-component of the space of pointed continuous maps from S^4 to BG. It is well known that there is a homotopy equivalence $\mathrm{Map}^*_k(S^4, BG) \simeq \mathrm{Map}^*_0(S^4, BG)$ for every $k \in \mathbb{Z}$; the latter space is usually written as $\Omega^3_0 G$. Putting all this together, for each k the evaluation fibration induces a homotopy fibration sequence

$$G \xrightarrow{\partial_k} \Omega^3_0 G \longrightarrow B\mathcal{G}_k \xrightarrow{ev} BG$$

where ∂_k is the fibration connecting map.

The following lemma describes the triple adjoint of ∂_k and was proved in [L]. Recall that $S^3 \xrightarrow{i} G$ is the inclusion of the bottom cell and $G \xrightarrow{1} G$ is the identity map.

Lemma 2.1. *The adjoint of the map $G \xrightarrow{\partial_k} \Omega^3_0 G$ is homotopic to the Samelson product $S^3 \wedge G \xrightarrow{\langle ki, 1\rangle} G$. Consequently, the linearity of the Samelson product implies that $\partial_k \simeq k \circ \partial_1$.* □

The order of ∂_1 plays a crucial role in determining whether two gauge groups are homotopy equivalent. The following general lemma was proved in [Th2].

Lemma 2.2. *Let X be a space and Y be an H-space with a homotopy inverse. Suppose that $X \xrightarrow{f} Y$ is a map of order m, where m is finite. Let F_k be the homotopy fiber of the composite $X \xrightarrow{f} Y \xrightarrow{k} Y$. If $(m, k) = (m, k')$ then F_k and $F_{k'}$ are homotopy equivalent when localized rationally or at any prime.* □

In our case, suppose that the order of $G \xrightarrow{\partial_1} \Omega^3_0 G$ is m. Observe that, by Lemma 2.1, the homotopy fiber of $k \circ \partial_1$ is $\mathcal{G}_k$. So Lemma 2.2 implies that if $(m, k) = (m, k')$ then $\mathcal{G}_k$ and $\mathcal{G}_{k'}$ are homotopy equivalent when localized rationally or at any prime. Therefore Theorem 1.2 (b) will be proved once we show that the order of $G_2 \xrightarrow{\partial_1} \Omega^3_0 G_2$ is 84 or 168. This will occupy the bulk of the paper.

The order of ∂_1 also plays a role in proving Theorem 1.2 (a). Given a homotopy equivalence $\mathcal{G}_k \simeq \mathcal{G}_{k'}$, there is an isomophism $[X, \mathcal{G}_k] \cong [X, \mathcal{G}_{k'}]$ for any space X. The idea is to find an appropriate space X for which the isomorphism $[X, \mathcal{G}_k] \cong [X, \mathcal{G}_{k'}]$ directly implies that $(84, k) = (84, k')$ when the underlying Lie group is G_2. This is a more *ad hoc* approach than for Theorem 1.2 (b), but in determining the order of ∂_1 it will become clear which choice of X we should choose.

The following lemma proved by Kono and Oshima in [KO, O] gives some preliminary information on the order of ∂_1. By [M], $\pi_{11}(G_2) \cong \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$. There is a choice of the integral generator $c\colon S^{11} \longrightarrow G_2$ such that the following holds.

Lemma 2.3. *The composite $S^{11} \xrightarrow{c} G_2 \xrightarrow{\partial_1} \Omega^3_0 G_2$ has order $3 \cdot 7$.* □

In particular, Lemma 2.3 implies that ∂_1 has order at least 21. To get the exact value of the order, we will localize and work one prime at a time. At primes $p \geq 3$, it will turn out that the lower bounds given by Lemma 2.3 are exact bounds. However, at $p = 2$ it will be shown that the order of ∂_1 is 4 or 8, which is not accurately seen by Lemma 2.3. The 2-primary case is the most difficult.

3. The case when $p \geq 5$

For the remainder of the paper we work in the p-local setting. Fix a prime p and localize all spaces and maps at p. When referring to homotopy group calculations we restrict to the rational or p-component of the homotopy group.

In this section we determine the p-component of the order of $G_2 \xrightarrow{\partial_1} \Omega_0^3 G_2$ for primes $p \geq 5$, and use it to prove the $p \geq 5$-primary information in Theorem 1.2.

Lemma 3.1. *The map $G_2 \xrightarrow{\partial_1} \Omega_0^3 G_2$ has order 1 if $p = 5$ or $p \geq 11$, and it has order 7 if $p = 7$.*

Proof. By Lemma 2.1, the adjoint of ∂_1 is the Samelson product $S^3 \wedge G \xrightarrow{\langle i,1 \rangle} G$.

When $p = 5$, McGibbon [Mc] showed that G_2 is homotopy commutative. Thus $\langle i, 1 \rangle$ is null homotopic, implying that ∂_1 has order 1.

When $p \geq 7$, Serre [S] showed that $G_2 \simeq S^3 \times S^{11}$. Thus $S^3 \wedge G_2 \simeq S^6 \vee S^{14} \vee S^{17}$, implying that ∂_1 may be regarded as a map $S^6 \vee S^{14} \vee S^{17} \longrightarrow G_2$. The calculations for $\pi_*(G_2)$ in [M] show that $\pi_6(G_2)$, $\pi_{14}(G_2)$ and $\pi_{17}(G_2)$ are all zero for $p \geq 7$, except for $\pi_{14}(G_2) \cong \mathbb{Z}/7\mathbb{Z}$. Thus if $p \geq 11$ then $\langle i, 1 \rangle$ is null homotopic, implying that ∂_1 has order 1. If $p = 7$ then $\langle i, 1 \rangle$ may have order 7. In fact, by Lemma 2.3, the composite $S^3 \wedge S^{11} \xrightarrow{1 \wedge c} S^3 \wedge G_2 \xrightarrow{\langle i,1 \rangle} G_2$ is nontrivial. Thus it represents the generator of $\pi_{14}(G_2) \cong \mathbb{Z}/7\mathbb{Z}$ and so ∂_1 has order 7. □

Proposition 3.2. *If $p = 5$ or $p \geq 11$ then $\mathcal{G}_k \simeq G_2 \times \Omega_0^4 G_2$ for all $k \in \mathbb{Z}$. If $p = 7$ then there is a 7-local homotopy equivalence $\mathcal{G}_k \simeq \mathcal{G}_{k'}$ if and only if $(7, k) = (7, k')$.*

Proof. By Lemma 3.1, if $p = 5$ or $p \geq 11$ then ∂_1 is null homotopic. By Lemma 2.1, $\partial_k \simeq k \circ \partial_1$, and therefore $G_2 \xrightarrow{\partial_k} \Omega_0^3 G_2$ is null homotopic for every $k \in \mathbb{Z}$. Hence the fiber $\mathcal{G}_k$ of ∂_k splits as $G_2 \times \Omega_0^4 G_2$.

If $p = 7$, since ∂_1 has order 7, the p-local analogue of Lemma 2.2 implies that $\mathcal{G}_k \simeq \mathcal{G}_{k'}$ if $(7, k) = (7, k')$. On the other hand, if $\mathcal{G}_k \simeq \mathcal{G}_{k'}$ then $\pi_{10}(\mathcal{G}_k) \cong \pi_{10}(\mathcal{G}_{k'})$, implying that $\pi_{11}(B\mathcal{G}_k) \cong \pi_{11}(B\mathcal{G}_{k'})$. In what follows, the information on the 7-component of $\pi_*(G_2)$ is from [M]. Consider the homotopy fibration sequence $G_2 \xrightarrow{\partial_k} \Omega_0^3 G_2 \longrightarrow B\mathcal{G}_k \longrightarrow BG_2$. When $k = 1$, adjointing in Lemma 2.3, the composite $S^{11} \xrightarrow{c} G_2 \xrightarrow{\partial_1} \Omega_0^3 G_2$ is nontrivial. Since $\pi_{11}(\Omega_0^3 G_2) \cong \mathbb{Z}/7\mathbb{Z}$, $\partial_1 \circ c$ therefore represents a generator of $\pi_{11}(\Omega_0^3 G_2)$. So for a general value of k, since $\partial_k \simeq k \circ \partial_1$ by Lemma 2.1, $\partial_k \circ c$ is null homotopic if $(7, k) = 7$ and represents a generator of $\mathbb{Z}/7\mathbb{Z}$ if $(7, k) = 1$. That is, the cokernel of $\pi_{11}(\partial_k)$ is $\mathbb{Z}/t_k\mathbb{Z}$ where $t_k = 7/(7, k)$. On the other hand, since $\pi_{11}(BG_2) = 0$, we obtain that $\pi_{11}(B\mathcal{G}_k)$ is isomorpic to the cokernel of $\pi_{11}(\partial_k)$. Thus $\pi_{11}(B\mathcal{G}_k) \cong \mathbb{Z}/t_k\mathbb{Z}$. Therefore if $\mathcal{G}_k \simeq \mathcal{G}_{k'}$ then $t_k = t_{k'}$, implying that $(7, k) = (7, k')$. □

4. The case when $p = 3$

In this section we determine the 3-component of the order of $G_2 \xrightarrow{\partial_1} \Omega^3_0 G_2$ and use it to help prove the 3-primary information in Theorem 1.2. By Lemma 2.1, to find the order of ∂_1 it is equivalent to adjoint and find the order of the Samelson product $S^3 \wedge G_2 \xrightarrow{\langle i,1\rangle} G_2$.

Localize all spaces and maps at 3. Let $\mathbb{Z}_{(3)}$ be the integers localized at 3. It is well known that $H^*(G_2;\mathbb{Z}_{(3)}) \cong \Lambda(y_3, y_{11})$ but we wish to make a precise choice of generators. There is a canonical group homomorphism $G_2 \longrightarrow Spin(7)$. Let $Spin(\infty)$ be the infinite Spinor group, and consider the composite

$$g\colon G_2 \longrightarrow Spin(7) \longrightarrow Spin(\infty).$$

Recall that $H^*(BSpin(\infty);\mathbb{Z}_{(3)}) \cong \mathbb{Z}_{(3)}[p_1, p_2, \ldots]$ where p_i is the universal Pontrjagin class. Let $x_{4i-1} \in H^*(Spin(\infty);\mathbb{Z}_{(3)})$ be the cohomology suspension of p_i. For $i \in \{1, 3\}$, define $\overline{x}_{4i-1} \in H^*(G_2;\mathbb{Z}_{(3)})$ by $\overline{x}_{4i-1} = g^*(x_{4i-1})$. Then

$$H^*(Spin(\infty);\mathbb{Z}_{(3)}) \cong \Lambda(x_3, x_7, x_{11}, \ldots) \qquad \text{and} \qquad H^*(G_2;\mathbb{Z}_{(3)}) \cong \Lambda(\overline{x}_3, \overline{x}_{11}).$$

Let A be the 11-skeleton of G_2 and let $j\colon A \longrightarrow G_2$ be the skeletal inclusion. Consider the composite

$$\langle i,j\rangle\colon S^3 \wedge A \xrightarrow{1\wedge j} S^3 \wedge G_2 \xrightarrow{\langle i,1\rangle} G_2.$$

As a step in finding the order of $\langle i,1\rangle$ we first find the order of $\langle i,j\rangle$. Notice that as $Spin(\infty)$ is an infinite loop space, it is homotopy commutative, so $g \circ \langle i,j\rangle$ is null homotopic. Therefore, there is a lift

(1)
$$\begin{array}{ccc} & & \Omega(Spin(\infty)/G_2) \\ & \overset{\ell}{\nearrow} & \downarrow \\ S^3 \wedge A & \xrightarrow{\langle i,j\rangle} & G_2 \end{array}$$

for some map ℓ. We wish to choose a lift ℓ with good properties. To do so we use methods from unstable K-theory.

Let $Z = Spin(\infty)/G_2$. We identify ΩZ up to dimension 18. Consider the Serre spectral sequence of the homotopy fibration $\Omega Z \longrightarrow G_2 \longrightarrow Spin(\infty)$. Then we easily see that

$$H^*(\Omega Z;\mathbb{Z}_{(3)}) \cong \mathbb{Z}_{(3)}[y_6, y_{14}]$$

for $* < 18$ where y_i transgresses to x_{i+1} for $i = 6, 14$. Let E be the homotopy fibre of the map $\mathcal{P}^1\colon K(\mathbb{Z}_{(3)}, 6) \longrightarrow K(\mathbb{Z}/3\mathbb{Z}, 10)$. Then we have

$$H^*(E;\mathbb{Z}_{(3)}) \cong \mathbb{Z}_{(3)}[u_6]$$

for $* < 18$. Since $\mathcal{P}^1 y_6 = 0$, the map $(y_6, y_{14})\colon \Omega Z \longrightarrow K(\mathbb{Z}_{(3)}, 6) \times K(\mathbb{Z}_{(3)}, 14)$ lifts to a map $\Omega Z \longrightarrow E \times K(\mathbb{Z}_{(3)}, 14)$ which is an isomorphism in cohomology with $\mathbb{Z}_{(3)}$-coefficients up to dimension 18. Thus ΩZ is homotopy equivalent to $E \times K(\mathbb{Z}_{(3)}, 14)$ up to dimension 18.

Lemma 4.1. *Let Y be a CW-complex of dimension < 17 such that $H^9(Y;\mathbb{Z}/3\mathbb{Z}) = 0$. Then $[Y,\Omega Z]$ is a free $\mathbb{Z}_{(3)}$-module and the map*

$$\Phi\colon [Y,\Omega Z] \longrightarrow H^6(Y;\mathbb{Z}_{(3)}) \oplus H^{14}(Y;\mathbb{Z}_{(3)})$$

defined by $\Phi(\alpha) = (\alpha^*(y_6), \alpha^*(y_{14}))$ is injective.

Proof. Since ΩZ is homotopy equivalent to $E \times K(\mathbb{Z}_{(3)}, 14)$ in dimensions < 18 and Y has dimension < 17, there is an isomorphism

$$[Y,\Omega Z] \cong [Y, E \times K(\mathbb{Z}_{(3)}, 14)].$$

So to prove that Φ is injective it suffices to show that the map $[Y,E] \longrightarrow [Y, K(\mathbb{Z}_{(3)}, 6)]$ is injective. By considering the homotopy fibration $K(\mathbb{Z}/3\mathbb{Z}, 9) \longrightarrow E \longrightarrow K(\mathbb{Z}_{(3)}, 6)$, the injectivity follows from the assumption that $H^9(Y;\mathbb{Z}/3\mathbb{Z}) = 0$. □

Arguing as in [H], we obtain the following proposition.

Proposition 4.2. *Let Y be a CW-complex of dimension ≤ 14 satisfying the hypotheses of Lemma 4.1. Then there is an exact sequence*

$$\widetilde{KO}^{-2}(Y) \xrightarrow{\Theta} \mathrm{Im}\,\Phi \longrightarrow [Y, G_2] \longrightarrow \widetilde{KO}^{-1}(Y)$$

where $\Theta(\xi) = \Sigma^{-2}(3!\mathrm{ch}_3(\xi\otimes\mathbb{C}), 7!\mathrm{ch}_7(\xi\otimes\mathbb{C}))$ for $\xi \in \widetilde{KO}^{-2}(Y) = \widetilde{KO}(\Sigma^2 Y)$. □

Notice that $Y = S^3 \wedge A$ is of dimension 14, so it satisfies the hypotheses of Lemma 4.1 and Proposition 4.2. Thus, arguing as in [H], we obtain the following.

Proposition 4.3. *The lift ℓ in (1) may be chosen so that*

$$\ell^*(u_6, u_{14}) = (i^*(\overline{x}_3) \times j^*(\overline{x}_3), i^*(\overline{x}_3) \times j^*(\overline{x}_{11})) \in \mathrm{Im}\,\Phi.$$ □

As spaces are localized at 3, the complexification $c\colon \widetilde{KO}^{-2}(S^3\wedge A) \longrightarrow \widetilde{K}^{-2}(S^3\wedge A) \cong \widetilde{K}(S^3\wedge A)$ is an isomorphism since, if $r\colon \widetilde{K}^{-1}(S^3\wedge A) \longrightarrow \widetilde{KO}^{-2}(S^3\wedge A)$ is the realification map, we have $r\circ c = 2$. By [Wa] there exists an element $\zeta \in \widetilde{K}(S^3\wedge A)$ such that

$$\mathrm{ch}(\zeta) = \Sigma^3\overline{x}_3 + \frac{1}{5!}\Sigma^3\overline{x}_{11}.$$

So $\widetilde{K}(S^3\wedge A)$ is a free $\mathbb{Z}_{(3)}$-module generated by ζ and $\eta\colon S^3\wedge A \longrightarrow S^{14} \longrightarrow BU(\infty)$ where

$$\mathrm{ch}(\eta) = \Sigma^3\overline{x}_{11}.$$

Proposition 4.4. *The Samelson product $S^3\wedge A \xrightarrow{\langle i,j\rangle} G_2$ has order 3.*

Proof. First note that the restriction of $\langle i, j\rangle$ to $S^3 \wedge S^3$ is the Samelson product $\langle i, i\rangle$, which has order 3 by [O]. So the order of $\langle i, j\rangle$ is at least 3.

By Proposition 4.2 there is an exact sequence

$$\widetilde{K}(S^3 \wedge A) \cong \widetilde{KO}^{-2}(S^3 \wedge A) \xrightarrow{\Theta} \operatorname{Im}\Phi \longrightarrow [S^3 \wedge A, G_2] \longrightarrow \widetilde{KO}^{-1}(S^3 \wedge A).$$

Consider the map $S^3 \wedge A \xrightarrow{\langle i,j\rangle} G_2$. By Proposition 4.3, $\langle i, j\rangle$ is in $\operatorname{Im}\Phi$, and within this image, it is represented by $\Sigma^3\overline{x}_3 + \Sigma^3\overline{x}_{11}$. Observe that $\Sigma^3\overline{x}_3 + \Sigma^3\overline{x}_{11} \notin \operatorname{Im}\Theta$, but

$$3(\Sigma^3\overline{x}_3 + \Sigma^3\overline{x}_{11}) = \Theta(\frac{1}{2}\zeta - \frac{18}{7!}\eta).$$

Therefore, by exactness, $3\langle i, j\rangle$ is null homotopic. Thus the order of $\langle i, j\rangle$ is at most 3. □

Proposition 4.5. *Localized at* 3*, the map* $G_2 \xrightarrow{\partial_1} \Omega^3_0 G_2$ *has order* 3*.*

Proof. It is equivalent to adjoint and show that the map $S^3 \wedge G_2 \xrightarrow{\langle i,1\rangle} G_2$ has order 3. Since G_2 is an H-space with the property that $\tilde{H}_*(G;\mathbb{Z}/3\mathbb{Z}) \cong \Lambda(\tilde{H}_*(A;\mathbb{Z}/3\mathbb{Z}))$ and $\tilde{H}_*(A;\mathbb{Z}/3\mathbb{Z})$ has m generators with $m < p$ ($m = 2$ and $p = 3$ here), by [CN] there is a homotopy equivalence $\Sigma G_2 \simeq \Sigma A \vee S^{15}$. Thus $\langle i, 1\rangle$ can be regarded as a map $(S^3 \wedge A) \vee S^{17} \longrightarrow G_2$. By Proposition 4.4, the composite $S^3 \wedge A \longrightarrow S^3 \wedge G_2 \xrightarrow{\langle i,1\rangle} G_2$ has order 3. By [M], the 3-component of $\pi_{17}(G_2)$ is zero. Thus $\langle i, 1\rangle$ has order 3. □

Proposition 4.6. *There exists a* 3*-local homotopy equivalence* $\mathcal{G}_k \simeq \mathcal{G}_{k'}$ *if and only if* $(3, k) = (3, k')$*.*

Proof. Argue exactly as in the $p = 7$ case of Proposition 3.2. □

5. Preliminary homotopy theory for the $p = 2$ case

In Sections 5 to 8 we determine the 2-component of the order of $G_2 \xrightarrow{\partial_1} \Omega^3_0 G_2$ and use it to prove the 2-primary information in Theorem 1.2. This case is by far the most delicate, so the work has been spread out over several sections. In this section we state some background facts about the 2-primary homotopy theory of G_2 and mod-2 Moore spaces which will be used later on. Throughout Sections 5 to 8 we assume all spaces and maps have been localized at 2 and homology is taken with mod-2 coefficients.

Lemma 5.1 contains information on the homotopy groups of G_2; all statements were proved in [M].

Lemma 5.1. *The following hold:*

(a) $\pi_6(G_2) = 0$*;*

(b) $\pi_7(G_2) = 0$*;*

(c) $\pi_8(G_2) \cong \mathbb{Z}/2\mathbb{Z}$*;*

(d) $\pi_9(G_2) \cong \mathbb{Z}/2\mathbb{Z}$*;*

(e) *if* $S^8 \longrightarrow G_2$ *represents a generator of* $\pi_8(G_2)$*, then the composite* $S^9 \xrightarrow{\eta} S^8 \longrightarrow G_2$ *represents a generator of* $\pi_9(G_2) \cong \mathbb{Z}/2\mathbb{Z}$*;*

(f) $\pi_{14}(G_2) \cong \mathbb{Z}/8\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$.

□

For $r \geq 1$ and $m \geq 3$, the mod-2^r *Moore space* $P^m(2^r)$ is the cofiber of the degree 2^r map on S^{m-1}. Lemma 5.2 describes several properties concerning the 2-primary homotopy theory of Moore spaces; a modern reference is [N, Proposition 6.1.6]. For $m \geq 4$, let $\eta\colon S^m \longrightarrow S^{m-1}$ represent the generator of the stable group $\pi_m(S^{m-1}) \cong \mathbb{Z}/2\mathbb{Z}$.

Lemma 5.2. *Let $m \geq 4$. The following hold:*

(a) *the degree 2 map on $P^m(2)$ is nontrivial, and is homotopic to the composite $P^m(2) \xrightarrow{q} S^m \xrightarrow{\eta} S^{m-1} \xrightarrow{j} P^m(2)$ where q is the pinch map to the top cell and j is the inclusion of the bottom cell;*

(b) *the degree 4 map on $P^m(2)$ is null homotopic;*

(c) *the degree 4 map on $P^m(4)$ is null homotopic.*

□

6. A lower bound on the order of ∂_1 at 2

Let $i'\colon S^3 \longrightarrow SU(3)$ be the inclusion of the bottom cell. Notice that the composite $S^3 \xrightarrow{i'} SU(3) \longrightarrow G_2$ is homotopic to the map $S^3 \xrightarrow{i} G_2$. Since the inclusion $SU(3) \longrightarrow G_2$ is a homomorphism, we obtain a homotopy commutative diagram

$$\begin{array}{ccc} S^3 \wedge SU(3) & \xrightarrow{\langle i',1\rangle} & SU(3) \\ \downarrow & & \downarrow \\ S^3 \wedge G_2 & \xrightarrow{\langle i,1\rangle} & G_2. \end{array} \tag{2}$$

There is a canonical map $\Sigma\mathbb{C}P^2 \longrightarrow SU(3)$ which induces the inclusion of the generating set in homology (see [Wh, Chapter IV, Section 10] for example). Let ϵ be the composite

$$\epsilon\colon S^3 \wedge \Sigma\mathbb{C}P^2 \longrightarrow S^3 \wedge SU(3) \xrightarrow{\langle i',1\rangle} SU(3).$$

Lemma 6.1. *The composite $S^3 \wedge \Sigma\mathbb{C}P^2 \xrightarrow{\epsilon} SU(3) \longrightarrow G_2$ is nontrivial.*

Proof. Consider the long exact sequence in homotopy groups induced by the fibration sequence $\Omega G_2 \longrightarrow \Omega S^6 \longrightarrow SU(3) \longrightarrow G_2 \longrightarrow S^6$. By Lemma 5.1, $\pi_6(\Omega G_2) = 0$, so $\pi_6(\Omega S^6)$ injects into $\pi_6(SU(3))$. By [To1] and [M], both $\pi_6(\Omega S^6)$ and $\pi_6(SU(3))$ are isomorphic to $\mathbb{Z}/2\mathbb{Z}$. Thus the injection $\pi_6(\Omega S^6) \longrightarrow \pi_6(SU(3))$ is an isomorphism. A representative of the generator of $\pi_6(\Omega S^6)$ is the map $\overline{\eta}\colon S^6 \longrightarrow \Omega S^6$ which is adjoint to $S^7 \xrightarrow{\eta} S^6$. Thus a representative of the generator of $\pi_6(SU(3))$ is the composite $S^6 \xrightarrow{\overline{\eta}} \Omega S^6 \longrightarrow SU(3)$. On the other hand, by [B] the Samelson product $\langle i', i'\rangle\colon S^6 \longrightarrow SU(3)$ is non-trivial, so it represents the generator of $\pi_6(SU(3)) \cong \mathbb{Z}/2$. Since the composite $S^6 \longrightarrow S^3 \wedge \Sigma\mathbb{C}P^2 \xrightarrow{\epsilon} SU(3)$ is the Samelson product $\langle i', i' \rangle$, there is a homotopy commutative diagram

$$\begin{array}{ccc} S^6 & \longrightarrow & S^3 \wedge \Sigma\mathbb{C}P^2 \\ \downarrow{\scriptstyle \overline{\eta}} & & \downarrow{\scriptstyle \epsilon} \\ \Omega S^6 & \longrightarrow & SU(3) \end{array}$$

where the top horizontal map is the inclusion of the bottom cell. Now consider the diagram

$$\begin{array}{ccccccc} S^6 & \longrightarrow & S^3 \wedge \Sigma\mathbb{C}P^2 & \longrightarrow & S^8 & \xrightarrow{\eta} & S^7 \\ \downarrow{\scriptstyle \overline{\eta}} & & \downarrow{\scriptstyle \epsilon} & & \downarrow{\scriptstyle \gamma} & & \downarrow{\scriptstyle \eta} \\ \Omega S^6 & \longrightarrow & SU(3) & \longrightarrow & G_2 & \longrightarrow & S^6. \end{array} \tag{3}$$

The bottom row is a fibration sequence while the top row is a cofibration sequence. So the homotopy commutativity of the left square implies that there is a map γ which makes the middle and right squares homotopy commute. Since the composite $S^8 \xrightarrow{\eta} S^7 \xrightarrow{\eta} S^6$ represents a generator of $\pi_8(S^6) \cong \mathbb{Z}/2\mathbb{Z}$, the map γ must be nontrivial. In particular, by Lemma 5.1, $\pi_8(G_2) \cong \mathbb{Z}/2\mathbb{Z}$, so γ represents a generator of this homotopy group.

Focus on the middle square in (3). The assertion of the lemma is that the composite $S^3 \wedge \Sigma\mathbb{C}P^2 \xrightarrow{\epsilon} SU(3) \longrightarrow G_2$ is nontrivial. If it were trivial, then the homotopy commutativity of this middle square would imply that the composite $S^3 \wedge \mathbb{C}P^2 \longrightarrow S^8 \xrightarrow{\gamma} G_2$ is null homotopic. This implies that γ extends across $S^8 \xrightarrow{\eta} S^7$ to a map $\gamma' \colon S^7 \longrightarrow G_2$. But by Lemma 5.1, $\pi_7(G_2) \cong 0$, so γ' is null homotopic, which implies that γ is null homotopic, a contradiction. Thus the composite $S^3 \wedge \Sigma\mathbb{C}P^2 \xrightarrow{\epsilon} SU(3) \longrightarrow G_2$ is nontrivial. □

Let X be the 6-skeleton of G_2 and let $X \longrightarrow G_2$ be the skeletal inclusion. Observe that X is a three-cell complex satisfying $H^*(X) \cong \{x_3, x_5, x_6\}$ with $Sq^2(x_3) = x_5$ and $Sq^1(x_5) = x_6$, and there is a homotopy cofibration $S^3 \longrightarrow X \longrightarrow P^6(2)$. As well, for dimensional reasons, there is a homotopy commutative diagram

$$\begin{array}{ccc} \Sigma\mathbb{C}P^2 & \longrightarrow & SU(3) \\ \downarrow & & \downarrow \\ X & \longrightarrow & G_2. \end{array}$$

This diagram together with (2) implies that there is a homotopy commutative diagram

$$\begin{array}{ccccc} S^3 \wedge \Sigma\mathbb{C}P^2 & \longrightarrow & S^3 \wedge SU(3) & \xrightarrow{\langle i,1 \rangle} & SU(3) \\ \downarrow & & \downarrow & & \downarrow \\ S^3 \wedge X & \longrightarrow & S^3 \wedge G_2 & \xrightarrow{\langle i,1 \rangle} & G_2. \end{array} \tag{4}$$

Notice that the top row is the definition of ϵ so by Lemma 6.1 the upper direction around (4) is nontrivial. Thus the lower direction around (4) is nontrivial. Therefore, if we let f be the composite

$$f\colon S^3\wedge X\longrightarrow S^3\wedge G_2\xrightarrow{\langle i,1\rangle}G_2$$

then we immediately obtain the following.

Corollary 6.2. *The composite* $S^3\wedge\Sigma\mathbb{C}P^2\longrightarrow S^3\wedge X\xrightarrow{f}G_2$ *is nontrivial.* □

Observe that there is a homotopy commutative diagram of homotopy cofibrations

$$\begin{array}{ccccc}
S^3\wedge S^3 & \longrightarrow & S^3\wedge\Sigma\mathbb{C}P^2 & \longrightarrow & S^3\wedge S^5\\
\| & & \downarrow & & \downarrow{\scriptstyle 1\wedge j}\\
S^3\wedge S^3 & \longrightarrow & S^3\wedge X & \longrightarrow & S^3\wedge P^6(2)
\end{array}$$

where j is the inclusion of the bottom cell. By Lemma 5.1, $\pi_6(G_2)\cong 0$, so the composite $S^3\wedge S^3\longrightarrow S^3\wedge X\xrightarrow{f}G_2$ is null homotopic. Thus f extends through $S^3\wedge P^6(2)$ and we obtain a homotopy commutative diagram

$$\begin{array}{ccc}
S^3\wedge\Sigma\mathbb{C}P^2 & \longrightarrow & S^3\wedge S^5\\
\downarrow & & \downarrow{\scriptstyle 1\wedge j}\\
S^3\wedge X & \longrightarrow & S^3\wedge P^6(2)\\
\downarrow{\scriptstyle f} & \swarrow{\scriptstyle g} & \\
G_2 & &
\end{array}\tag{5}$$

for some map g. By Corollary 6.2, the left column in this diagram is nontrivial. Therefore the upper direction around the diagram is nontrivial. This implies that the composite $g\circ(1\wedge j)$ is nontrivial. By Lemma 5.1, $\pi_8(G_2)\cong\mathbb{Z}/2\mathbb{Z}$, so we obtain the following.

Lemma 6.3. *The composite* $S^3\wedge S^5\xrightarrow{1\wedge j}S^3\wedge P^6(2)\xrightarrow{g}G_2$ *represents a generator of* $\pi_8(G_2)\cong\mathbb{Z}/2\mathbb{Z}$. □

We now calculate the order of f. In general, if Y is an H-space let $2^r\colon Y\longrightarrow Y$ be the 2^r-power map, and if A is a co-H-space let $\underline{2}^r\colon A\longrightarrow A$ be the map of degree 2^r.

Lemma 6.4. *The map* $S^3\wedge X\xrightarrow{f}G_2$ *has order* 4.

Proof. Focus on (5). By Lemma 5.2, the identity map on $S^3\wedge P^6(2)\simeq P^9(2)$ has order 4. Thus the order of g is at most 4, which implies that the order of f is at most 4.

We next show that the order of f is at least 4. Consider the composite $S^3\wedge X\xrightarrow{f}G_2\xrightarrow{2}G_2$. The factorization of f through g in (5) implies that $2\circ f$ factors as $S^3\wedge X\longrightarrow S^3\wedge P^6(2)\xrightarrow{g}G_2\xrightarrow{2}G_2$. Observe that the homotopy cofiber of $S^3\wedge X\longrightarrow S^3\wedge P^6(2)$ is $S^3\wedge S^4\simeq S^7$ and by Lemma 5.1,

$\pi_7(G_2) = 0$. Therefore $[S^3 \wedge P^6(2), G_2]$ injects into $[S^3 \wedge X, G_2]$. Thus to show that $2 \circ f$ is nontrivial it suffices to show that $2 \circ g$ is nontrivial.

It remains to show that $2 \circ g$ is nontrivial. Equivalently, we show that the composite $S^3 \wedge P^6(2) \xrightarrow{\underline{2}} S^3 \wedge P^6(2) \xrightarrow{g} G_2$ is nontrivial. By Lemma 5.2, the degree 2 map on $S^3 \wedge P^6(2)$ factors as the composite $S^3 \wedge P^6(2) \xrightarrow{1\wedge q} S^3 \wedge S^6 \xrightarrow{1\wedge\eta} S^3 \wedge S^5 \xrightarrow{1\wedge j} S^3 \wedge P^6(2)$, where q is the pinch map to the top cell. By Lemma 6.3, the composite $g \circ (1 \wedge j)$ represents a generator of $\pi_8(G_2) \cong \mathbb{Z}/2\mathbb{Z}$. Lemma 5.1 (e) therefore implies that the composite $g \circ (1 \wedge j) \circ (1 \wedge \eta)$ represents a generator of $\pi_9(G_2) \cong \mathbb{Z}/2\mathbb{Z}$. To compress notation, let $g' = g \circ (1 \wedge j) \circ (1 \wedge \eta)$. If the composite $S^3 \wedge P^6(2) \xrightarrow{1\wedge q} S^3 \wedge S^6 \xrightarrow{g'} G_2$ was null homotopic, then g' would extend across the cofiber of $1 \wedge q$, implying that g' factors as a composite $S^3 \wedge S^6 \xrightarrow{1\wedge\underline{2}} S^3 \wedge S^6 \xrightarrow{g''} G_2$ for some map g''. But $\pi_9(G_2)$ has order 2 and $1 \wedge \underline{2}$ is homotopic to the degree 2 map on S^9, so $g'' \circ (1 \wedge \underline{2})$ is null homotopic. This implies that g' is null homotopic, contradicting the fact that it represents a generator of the nontrivial group $\pi_9(G_2)$. Hence $g' \circ (1 \wedge q)$ is nontrivial. That is, $g \circ (1 \wedge j) \circ (1 \wedge \eta) \circ (1 \wedge q) \simeq g \circ \underline{2}$ is nontrivial, as required. $\square$

Since f is the composite $S^3 \wedge X \longrightarrow S^3 \wedge G_2 \xrightarrow{\langle i,1\rangle} G_2$, taking adjoints we obtain the following.

Corollary 6.5. *The composite* $X \longrightarrow G_2 \xrightarrow{\partial_1} \Omega^3_0 G_2$ *has order* 4. $\square$

Consequently, we obtain a lower bound on the order of the map $G_2 \xrightarrow{\partial_1} \Omega^3_0 G_2$.

Corollary 6.6. *The map* $G_2 \xrightarrow{\partial_1} \Omega^3_0 G_2$ *has order at least* 4. $\square$

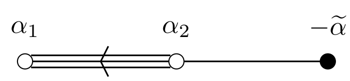
7. An upper bound on the order of ∂_1 at 2

By Corollary 6.6 the map $G_2 \xrightarrow{\partial_1} \Omega^3_0 G_2$ has order at least 4. In this section we will show that ∂_1 has order at most 8. It is equivalent to show that the triple adjoint of ∂_1 has order at most 8. Recall by Lemma 2.1 that the triple adjoint of ∂_1 is the Samelson product

$$\Sigma^3 G_2 \xrightarrow{\langle i,1\rangle} G_2.$$

We will show that $\langle i, 1\rangle$ factors through a space of smaller dimension.

It is well known that the extended Dynkin diagram of G_2 is given by the following:



Therefore, there is a maximal rank subgroup $SO(4)$ of G_2 corresponding to the roots α_1 and $-\widetilde{\alpha}$, implying that there are two distinct copies of $SU(2)$ in $SO(4) \subseteq G_2$ corresponding to α_1 and $-\widetilde{\alpha}$, which we denote by $SU(2)_-$ and $SU(2)_+$ respectively. Since $-\widetilde{\alpha}$ is the longest root, the inclusion $SU(2)_+ \longrightarrow G_2$ is a generator $i\colon S^3 \longrightarrow G_2$, and for $\frac{|\widetilde{\alpha}|^2}{|\alpha_1|^2} = 3$, the inclusion $j\colon SU(2)_- \longrightarrow G_2$ is three times a generator of $\pi_3(G_2)$. Therefore we obtain the following.

Lemma 7.1. *There is an equality* $\langle j, 1\rangle = 3\langle i, 1\rangle$, *where* 1 *is the identity map on* G_2. $\square$

The Samelson product $\langle j,1\rangle$ factors conveniently. Let $V=G_2/SU(2)_+$ and let $q\colon G_2\longrightarrow V$ be the projection.

Lemma 7.2. *The Samelson product* $\Sigma^3 G_2\xrightarrow{\langle j,1\rangle}G_2$ *factors as a composite* $\Sigma^3G_2\xrightarrow{\Sigma^3 q}\Sigma^3V\xrightarrow{\psi}G_2$ *for some map* ψ*.*

Proof. By construction, $SU(2)_+$ and $SU(2)_-$ commute in $SO(4)$, and hence in G_2, where $SO(4)=(SU(2)_+\times SU(2)_-)/(\mathbb{Z}/2\mathbb{Z})$. This implies the asserted factorization. □

By Lemma 7.1, $\langle j,1\rangle=3\cdot\langle i,1\rangle$, so as 3 is a unit when localized at 2, if we define $\psi'=(1/3)\circ\psi$ then Lemma 7.2 immediately implies the following.

Corollary 7.3. *The Samelson product* $\Sigma^3G_2\xrightarrow{\langle i,1\rangle}G_2$ *factors as* $\Sigma^3G_2\xrightarrow{\Sigma^3q}\Sigma^3V\xrightarrow{\psi'}G_2$*.* □

Consider the space Σ^3V more closely. Observe that $V=G_2/SU(2)_+$ is the Stiefel manifold $SO(7)/SO(5)$. Thus $H^*(V)\cong\Lambda(x_5,x_6)$, where $Sq^1(x_5)=x_6$. We show that there is a homotopy decomposition of Σ^2V.

Lemma 7.4. *There is a homotopy equivalence* $\Sigma^2V\simeq P^8(2)\vee S^{13}$*.*

Proof. The cohomology description for V implies that there is a homotopy cofibration

$$S^{10}\xrightarrow{f}P^6(2)\longrightarrow V$$

where f attaches the top cell to V. In particular, f represents an element in $\pi_{10}(P^6(2))$. Observe that Σ^2f is in the stable range, and by [Wu, Lemma 5.2(3)], the stable homotopy group $\pi^S_{n+4}(P^n(2))\cong 0$. Therefore Σ^2f is null homotopic, implying that there is a homotopy equivalence $\Sigma^2V\simeq P^8(2)\vee S^{13}$. □

Lemma 7.4 is used to obtain an upper bound on the order of ∂_1.

Proposition 7.5. *The map* $G_2\xrightarrow{\partial_1}\Omega^3_0G_2$ *has order at most* 8*.*

Proof. It is equivalent to show that the triple adjoint $\Sigma^3G_2\xrightarrow{\langle i,1\rangle}G_2$ of ∂_1 has order at most 8. By Corollary 7.3, $\langle i,1\rangle$ factors as the composite

$$\Sigma^3G_2\xrightarrow{\Sigma^3q}\Sigma^3V\xrightarrow{\psi'}G_2.$$

By Lemma 7.4, the map $\Sigma^3V\xrightarrow{\psi'}G_2$ is the same, up to a homotopy equivalence, as $P^9(2)\vee S^{14}\xrightarrow{a+b}G_2$ for some maps a and b. By Lemma 5.2 (b), the order of the identity map on $P^m(2)$ is 4 if $m\geq 4$, so the order of a is at most 4. By Lemma 5.1 (f), $\pi_{14}(G_2)\cong\mathbb{Z}/8\mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z}$, so the order of b is at most 8. Thus the order of $a+b$ is at most 8, implying that the order of $\psi'\circ\Sigma^3q$ is at most 8. □

Remark 7.6. In the proof of Proposition 7.5, it is not clear what the order of the map b is, and determining this order seems to be a very delicate problem. Observe that if b has order at most 4 then ∂_1 has order at most 4. We suspect that this is actually the case.

Combining Corollary 6.6 with Proposition 7.5 we obtain the following.

Proposition 7.7. *The map* $G_2 \xrightarrow{\partial_1} \Omega^3_0 G_2$ *has order* 4 *or* 8. □

8. Towards the 2-types of G_2-gauge groups

In this section we prove the 2-primary information in Theorem 1.2 (a): if there is a homotopy equivalence $\mathcal{G}_k \simeq \mathcal{G}_{k'}$ then $(4, k) = (4, k')$. This requires several preliminary lemmas. Again, througout this section, assume that spaces and maps are localized at 2.

For a space X, let $X\langle 3\rangle$ be the three-connected cover of X. In the case of G_2, observe that $\Omega^3_0 G_2 \simeq \Omega^3 G_2\langle 3\rangle$. Adjointing ∂_1 twice, we obtain a map

$$\phi\colon \Sigma^2 G_2 \longrightarrow \Omega G_2\langle 3\rangle.$$

It is easier to work with $\Omega G_2\langle 3\rangle$ than $G_2\langle 3\rangle$ because of the following lemma. Consider the canonical fibration $SU(3) \xrightarrow{j} G_2 \xrightarrow{p} S^6$. Take 3-connected covers and loop. Composing with the second James-Hopf invariant $H\colon \Omega S^6 \longrightarrow \Omega S^{11}$, we obtain a homotopy commutative diagram of homotopy fibrations

$$\begin{array}{ccccc}
Y & \longrightarrow & S^5 & \longrightarrow & SU(3)\langle 3\rangle \\
\downarrow & & \downarrow{\scriptstyle E} & & \| \\
\Omega G_2\langle 3\rangle & \xrightarrow{\Omega p} & \Omega S^6 & \longrightarrow & SU(3)\langle 3\rangle \\
\downarrow & & \downarrow{\scriptstyle H} & & \downarrow \\
\Omega S^{11} & = & \Omega S^{11} & \longrightarrow & *
\end{array}$$

which defines the space Y. Let $\tilde{c}\colon S^{10} \longrightarrow \Omega G_2\langle 3\rangle$ be the adjoint of three-connected cover of the map $S^{11} \xrightarrow{c} G_2$ appearing in Lemma 2.3.

Lemma 8.1. *The homotopy fibration* $Y \longrightarrow \Omega G_2\langle 3\rangle \longrightarrow \Omega S^{11}$ *has the following properties:*

(a) *the* 14*-skeleton of* Y *is* $P^8(2)$ [To2]*;*

(b) *the composite* $S^{10} \xrightarrow{\tilde{c}} \Omega G_2\langle 3\rangle \longrightarrow \Omega S^{11}$ *is of degree* 4 [M, Theorem 6.1]. □

Recall from Section 6 that X is the 6-skeleton of G_2 and that there is a cofibration $S^3 \longrightarrow X \xrightarrow{q} P^6(2)$, where q is the pinch map to the top two cells. By Lemma 8.1, the 8-skeleton of $\Omega G_2\langle 3\rangle$ is $P^8(2)$, implying that the 6-skeleton of $\Omega^3_0 G_2 \simeq \Omega^3 G_2\langle 3\rangle$ is $P^6(2)$. Let $j\colon P^6(2) \longrightarrow \Omega^3_0 G_2$ be the skeletal inclusion.

Lemma 8.2. *There is a homotopy commutative diagram*

$$\begin{array}{ccc}
X & \xrightarrow{u\cdot q} & P^6(2) \\
\downarrow & & \downarrow{\scriptstyle j} \\
G_2 & \xrightarrow{\partial_1} & \Omega^3_0 G_2
\end{array}$$

where u *is a unit in* $\mathbb{Z}_{(2)}$.

Proof. Recall from (5) that there is a homotopy commutative square

$$\begin{array}{ccc} S^3\wedge X & \xrightarrow{1\wedge q} & S^3\wedge P^6(2) \\ \Big\downarrow f & & \Big\downarrow g \\ G_2 & = & G_2 \end{array}$$

where f is the composite $S^3 \wedge X \longrightarrow S^3 \wedge G_2 \xrightarrow{\langle i,1\rangle} G_2$ and, by Lemma 6.3, the restriction of g to S^8 is a generator of $\pi_8(G_2) \cong \mathbb{Z}/2\mathbb{Z}$. Adjointing, we obtain a homotopy commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{q} & P^6(2) \\ \Big\downarrow & & \Big\downarrow \gamma \\ G_2 & \xrightarrow{\partial_1} & \Omega^3_0 G_2 \end{array}$$

where the restriction of γ to S^5 represents a generator of $\pi_5(\Omega^3_0 G_2) \cong \mathbb{Z}/2\mathbb{Z}$. That is, up to multiplication by a unit in $\mathbb{Z}_{(2)}$, the restriction of γ to S^5 is the inclusion of the bottom cell into $\Omega^3_0 G_2$. The Bockstein therefore implies that, up to multiplication by a unit u in $\mathbb{Z}_{(2)}$, γ is the skeletal inclusion j. □

The homotopy cofibration $S^3 \longrightarrow X \xrightarrow{q} P^6(2)$ is induced by a map $P^5(2) \xrightarrow{\overline{\eta}} S^3$. This is because, as a CW-complex, X has three cells in dimensions 3, 5 and 6 and the top two cells are connected by a Bockstein. Further, as the bottom two cells are connected by the Steenrod operation Sq^2, the map $\overline{\eta}$ is an extension of the map $S^4 \xrightarrow{\eta} S^3$ representing a generator of $\pi_4(S^3) \cong \mathbb{Z}/2\mathbb{Z}$. For $k \in \mathbb{Z}$, define the space C_k and the maps c_k and d_k by the homotopy pushout

$$\begin{array}{ccc} P^5(2) & \xrightarrow{\overline{\eta}} & S^3 \\ \Big\downarrow \underline{k} & & \Big\downarrow d_k \\ P^5(2) & \xrightarrow{c_k} & C_k. \end{array} \tag{6}$$

The following lemma gives some information about the cohomology of the spaces C_k.

Lemma 8.3. *The following hold:*

(a) *if* $(4, k) = 1$ *then* $C_k \simeq S^3$*, so* $H^*(C_k) \cong H^*(S^3)$*;*
(b) *if* $(4, k) = 2$ *or* $(4, k) = 4$ *then, as a module,* $H^*(C_k) \cong H^*(S^3) \oplus H^*(P^5(2)) \oplus H^*(P^6(2))$*;*
(c) *if* $(4, k) = 2$ *then* Sq^2 *is nontrivial on the degree* 4 *generator in cohomology;*
(d) *if* $(4, k) = 4$ *then* Sq^2 *is trivial on the degree* 4 *generator in cohomology.*

Proof. If $(4, k) = 1$ then k is a unit in $\mathbb{Z}_{(2)}$ so the map $\underline{k}$ is a homotopy equivalence. Thus the pushout in (6) implies that d_k is also a homotopy equivalence and the cohomology assertion follows.

If $(4,k)=2$ or $(4,k)=4$ then 2 divides k so $(\underline{k})^*=0$. Also for dimensional reasons, $(\overline{\eta})^*=0$. So the Mayer-Vietoris sequence determined by the pushout defining C_k implies that for each $m\geq 1$ there is a short exact sequence

$$0\longrightarrow H^m(P^6(2))\longrightarrow H^m(C_k)\longrightarrow H^m(S^3)\oplus H^m(P^5(2))\longrightarrow 0$$

of $\mathbb{Z}/2\mathbb{Z}$-modules. By inspection, this splits for each $m\geq 1$ and part (b) follows.

Observe that there is a single generator $x\in H^4(C_k)$, and $(c_k)^*$ is an isomorphism in degree 4. So as Sq^2 detects η, $Sq^2(x)$ is nontrivial or trivial depending on whether $\underline{k}$ factors through η. If $(4,k)=2$ then $k=2\cdot v$ for some unit $v\in\mathbb{Z}_{(2)}$, so $\underline{k}$ is homotopic, up to an equivalence, to the degree 2 map. By Lemma 5.2, $\underline{2}$ factors as the composite $P^5(2)\xrightarrow{q}S^5\xrightarrow{\eta}S^4\xrightarrow{j}P^5(2)$. Thus $Sq^2(x)$ is nontrivial in $H^6(C_k)$. On the other hand, if $(4,k)=4$ then k is divisible by 4 so Lemma 5.2 implies that $\underline{k}$ is null homotopic. Therefore $Sq^2(x)$ is zero in $H^6(C_k)$. □

Since X is the cofiber of $\overline{\eta}$, the pushout in (6) implies that for each $k\in\mathbb{Z}$ there is a homotopy cofibration sequence

$$P^5(2)\xrightarrow{c_k}C_k\longrightarrow X\xrightarrow{k\cdot q}P^6(2)$$

where $k\cdot q=\underline{k}\circ q$. Now consider the diagram

(7)
$$\begin{array}{ccccccc}
P^5(2) & \xrightarrow{c_k} & C_{ku} & \longrightarrow & X & \xrightarrow{ku\cdot q} & P^6(2)\\
\downarrow{\scriptstyle\widetilde{j}} & & \downarrow{\scriptstyle\theta_k} & & \downarrow & & \downarrow{\scriptstyle j}\\
\Omega^4_0G_2 & \longrightarrow & \mathcal{G}_k & \longrightarrow & G_2 & \xrightarrow{\partial_k} & \Omega^3_0G_2
\end{array}$$

where $\widetilde{j}$ is the adjoint of j and the map θ_k will be defined momentarily. By Lemma 2.1, $\partial_k\simeq k\circ\partial_1$. So Lemma 8.2 implies that the right square homotopy commutes. Since the top row is a homotopy cofibration sequence and the bottom row is a homotopy fibration sequence, the commutativity of the right square implies that the composite $C_{ku}\longrightarrow X\longrightarrow G_2$ lifts to a map $\theta_k\colon C_{ku}\longrightarrow\mathcal{G}_k$ which can be chosen to make both the middle and left squares homotopy commute. Since spaces are localized at 2 and u is a unit in $\mathbb{Z}_{(2)}$, there is a homotopy equivalence $C_{ku}\simeq C_k$.

Lemma 8.4. *The composite $C_k\xrightarrow{\simeq}C_{ku}\xrightarrow{\theta_k}\mathcal{G}_k$ induces an isomorphism in homology in dimensions ≤ 6.*

Proof. We first need some information about $H_*(\Omega^4_0G_2)$. By Lemma 8.1, there is a homotopy fibration $\Omega^3Y\longrightarrow\Omega^4_0G_2\longrightarrow\Omega^4S^{11}$ where the 7-skeleton of Ω^3Y is $P^5(2)$. Notice that the 13-skeleton of Ω^4S^{11} is S^7. So for dimensional reasons, the homology Serre spectral sequence converging to $H_*(\Omega^4_0G_2)$ collapses in degrees ≤ 7. This implies that the composite $P^5(2)\longrightarrow\Omega^3Y\longrightarrow\Omega^4_0G_2$ is an isomorphism in degrees ≤ 6. Notice that this map is precisely $\widetilde{j}$. In particular, this isomorphism implies that $\Omega^4_0G_2$ is 3-connected and $H_6(\Omega^4_0G_2)\cong 0$.

Now consider the homotopy fibration $\Omega_0^4 G_2 \longrightarrow \mathcal{G}_k \longrightarrow G_2$. Since G_2 is 2-connected and $\Omega_0^4 G_2$ is 3-connected, the Serre exact sequence gives a long exact sequence

$$H_6(\Omega_0^4 G_2) \longrightarrow H_6(\mathcal{G}_k) \longrightarrow H_6(G_2) \longrightarrow H_5(\Omega_0^4 G_2) \longrightarrow \cdots .$$

Thus (7) induces a commutative diagram of long exact sequences

$$\begin{array}{ccccccccc} H_6(P^5(2)) & \longrightarrow & H_6(C_{ku}) & \longrightarrow & H_6(X) & \longrightarrow & H_5(P^5(2)) & \longrightarrow & \cdots \\ \downarrow{\scriptstyle \tilde{j}_*} & & \downarrow{\scriptstyle (\theta_k)_*} & & \downarrow & & \downarrow{\scriptstyle \tilde{j}_*} & & \\ H_6(\Omega_0^4 G_2) & \longrightarrow & H_6(\mathcal{G}_k) & \longrightarrow & H_6(G_2) & \longrightarrow & H_5(\Omega_0^4 G_2) & \longrightarrow & \cdots \end{array}$$

Since $H_6(P^5(2)) \cong H_6(\Omega_0^4 G_2) \cong 0$, we in fact have a commutative diagram of long exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H_6(C_{ku}) & \longrightarrow & H_6(X) & \longrightarrow & H_5(P^5(2)) & \longrightarrow & \cdots \\ \| & & \downarrow{\scriptstyle (\theta_k)_*} & & \downarrow & & \downarrow{\scriptstyle \tilde{j}_*} & & \\ 0 & \longrightarrow & H_6(\mathcal{G}_k) & \longrightarrow & H_6(G_2) & \longrightarrow & H_5(\Omega_0^4 G_2) & \longrightarrow & \cdots \end{array} \tag{8}$$

By definition of X, the map $X \longrightarrow G_2$ is the inclusion of the 6-skeleton. Notice that as G_2 has no 7-cell, this map induces an isomorphism in homology in degrees ≤ 6. We have just seen that the map $\tilde{j}$ induces an isomorphism in degrees ≤ 6. The five-lemma applied to the long exact sequences in (8) therefore implies that $(\theta_k)_*$ is an isomorphism in degrees ≤ 5. But observe as well in (8) that because of the two lefthand zeroes, a diagram chase implies that $(\theta_k)_*$ is also an isomorphism in degree 6. □

Proposition 8.5. *Suppose there is a homotopy equivalence $\mathcal{G}_k \simeq \mathcal{G}_{k'}$. Then $(4, k) = (4, k')$.*

Proof. A homotopy equivalence $\mathcal{G}_k \simeq \mathcal{G}_{k'}$ induces an isomorphism in cohomology as algebras over the Steenrod algebra. By Lemma 8.4, this implies that $H^*(C_k)$ is isomorphic to $H^*(C_{k'})$ as an algebra over the Steenrod algebra in dimensions ≤ 6 (and therefore in all dimensions as Lemma 8.3 implies that, for any k, C_k has dimension at most 6). By Lemma 8.3, such an isomorphism can exist if and only if $(4, k) = (4, k')$. □

9. The proofs of Theorems 1.2 and 1.1

In this section we prove our main results by putting together the partial results for individual primes. We work integrally.

Proof of Theorem 1.1. To determine the order of $S^3 \wedge G \xrightarrow{\langle i,1 \rangle} G$ it is equivalent to calculate the order of its adjoint $G \longrightarrow \Omega_0^3 G$. By Lemma 2.1, this adjoint is ∂_1. For a prime p, the p-components of the order of ∂_1 have been determined in Lemma 3.1 and Propositions 4.5 and 7.7 as: 1 if $p = 5$ or $p \geq 11$; 7 if $p = 7$; 3 if $p = 3$; and 4 or 8 if $p = 2$. Thus the order of ∂_1 is either $4 \cdot 3 \cdot 7 = 84$ or $8 \cdot 3 \cdot 7 = 168$. □

Proof of Theorem 1.2. For part (a), if $\mathcal{G}_k \simeq \mathcal{G}_{k'}$ then Propositions 3.2, 4.6 and 8.5 imply that $(84, k) = (84, k')$. For part (b), by Theorem 1.1, the order of $G_2 \xrightarrow{\partial_1} \Omega_0^3 G_2$ is at most 84. By Lemma 2.1, $\partial_k \simeq k \circ \partial_1$, and the homotopy fiber of ∂_k is $\mathcal{G}_k$. Lemma 2.2 therefore implies that if $(168, k) = (168, k')$ then $\mathcal{G}_k \simeq \mathcal{G}_{k'}$ when localized rationally or at any prime. □

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Department of Mathematics, Kyoto University, Kyoto 606-8502 Japan

E-mail address: kishi@math.kyoto-u.ac.jp

Mathematical Sciences, University of Southampton, Southampton SO17 1BJ, United Kingdom

E-mail address: S.D.Theriault@soton.ac.uk

Faculty of Mathematics, Kyushu University, Fukuoka 819-0395, Japan

E-mail address: tsutaya@math.kyushu-u.ac.jp