Dynamic Geometry Contexts for Proof as Explanation[1]

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Abstract
Providing a mathematics curriculum that makes proof accessible to school students appears to be difficult. This paper describes work carried out in a secondary school mathematics class in which students worked on tasks designed to enable them to experience the necessity *of certain geometrical facts that are true in Euclidean geometry. In these tasks, the students were asked to construct figures using the dynamic geometry package* Cabri-Géomètre *such that each figure was invariant when any basic object used in the construction was dragged. It is argued that working on these tasks provided the students with suitable experiences to enable them to explain* why *these geometrical facts are* necessarily *true. The changing quality of the students' mathematical analysis suggests that working on suitable tasks with a dynamic geometry package may allow some students to develop an appreciation of proof as explanation.*

Introduction

Mathematical proof is the essential component of mathematics and is arguably what distinguishes mathematics from other disciplines. Yet providing a mathematics curriculum that makes proof accessible to school students appears to be difficult. Proving, it seems, either appears as an obscure ritual or it disappears in a series of innocuous classroom tasks in which students learn to `spot patterns'. Schoenfeld (1989), for instance, reports that even when students can reproduce a formally taught Euclidean proof, a significant proportion conjecture a solution to the corresponding geometrical construction problem that "*flatly violates* the results they have just proven" (emphasis added). On the other hand, when the chosen proof contexts are data-driven, with students expected to form generalised conjectures and search for counter examples, Coe and Ruthven (1994) find that "students' proof strategies were primarily empirical". In this case, the generation of numerical data becomes the object of the exercise and any notion of deductive argument appears abandoned. Balacheff (1988, p 222) similarly reports the occurrence of what he refers to as "naive empiricism".

It seems that to begin creating a more meaningful experience of proof for school students we firstly need contexts for proof with which students can engage. Secondly, we need ways of working in the classroom that provide opportunities for students to explain *why* they

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obtain a particular outcome. Some initial findings from my work in a secondary mathematics class suggests that the use of a dynamic geometry package such as *Cabri-Géomètre* may provide an opportunity for some students to consider the "why" in addition to the "what if.." and the "what if not...". Such an approach demands the provision of appropriate tasks and in designing them there have been two (connected) issues. The first issue involves ensuring the students experience the *necessity* of geometrical facts that are true in Euclidean geometry. The second concerns providing the students with suitable experiences to allow them to explain *why* these geometrical facts are *necessarily true*.

In this paper I describe the approach that I have been developing with a lower secondary school mathematics class (students aged 12). My approach has been to seek to provide the students with opportunities to experience what I will call `proof as explanation'. In the next section I will explain what I mean by `proof as explanation' and in a following section describe the development of some dynamic geometry tasks that might usefully support such an approach to proof. The quality of the students' mathematical analysis, I will argue, suggests that the use of a dynamic geometry package such as *Cabri-Géomètre*, coupled with suitable tasks, may provide an opportunity for some students to develop the basis for a fuller appreciation of the nature and purpose of mathematical proof.

Proof as Explanation

Proofs are often thought of solely as standardised linear deductive presentations. The form of two-column proofs taught in a number of countries precisely fits such a model. Proof can, however, take a number of forms. Balacheff (1988, p 216), for instance, contrasts what he calls pragmatic and conceptual proof. From a different perspective, Leron (1985) talks about "direct" and "indirect" proofs. In this paper I will employ the distinction suggested by Hanna (1989), that there are proofs that prove (and do no more) and proofs that *explain*. This latter form of proof demonstrates not only that a statement is true, but also *why it is true*. As an example, Hanna contrasts two proofs for the following:

Prove that the sum of the first n positive integers, $S(n)$, is equal to $n(n+1)/2$

She argues that, while a proof by induction is mathematically satisfactory, the format suggested by Gauss (employing a geometrical representation) not only provides a satisfactory proof but also explains why the proof is true. It is this form of proof - *proof as explanation* - that is the focus of this paper. When proof is viewed as *explanation*, such a proof may be more successful as a means of communication and a way of convincing others. Given such an approach, proof as experienced by school students might be more meaningful for all concerned.

In developing a more meaningful approach to proof for school students, I want to emphasise the importance of the choice of context. The traditional context for proof in the school curriculum has been geometry, while in University it has been analysis. Neither context has so far proved wholly successful. Indeed, within school mathematics, geometry as a context for proof has not, it seems, been successful either when the dominant emphasis is on the abstract and technical aspects of proof or when it is on gathering data. It appears that proof activities do not work because either the property is so obvious that it is hard for the students to see why a proof is needed, or that while a generalisation can be ascertained and no counter-examples found, a mathematical proof is beyond almost all the students.

Schoenfeld (1985), for example, argues that, when proof is taught formally, then, for most students "mathematical proof is irrelevant to both the discovery and (personal, rather than formal) verification process". He goes on to say that if a student finds that a proof is "absolutely necessary - that is, the teacher demands it - [the student] can probably verify a result using proof techniques. But this is simply playing the rules of the game, verifying under duress those things that one already knows to be correct" (p 160- 161). From the empirically-based, data-driven, perspective, an analysis of student responses to such tasks by Coe and Ruthven (1994) found that only two pieces of student work out of 60 "could be said to contain a strong deductive proof". The remainder contained "at best empirical proof, and in one or two cases, not even that".

Thus, as Hoyles, Healy and Noss (1995 p 101) summarise, many students:

- fail to appreciate the crucial distinction between empirical and deductive arguments
- show a preference for empirical arguments
- behave as if deductive proof provides no more than contributory evidence
- regard proof as an irrelevant activity

This suggests that we need to continue to look for ways of laying the foundation for a deeper appreciation of the role of proof. There are a number of reasons why using a dynamic geometry package such as *Cabri-Géomètre* may present us with an opportunity to look afresh at our way of approaching proof with secondary school students. Firstly, a dynamic geometry package allows *direct* manipulation of geometrical objects (or, at least, the *appearance* of such direct manipulation). The drawing on the screen can be manipulated by means of the mouse. Objects can be 'dragged' while, all the time, all the geometric properties used to construct the drawing are preserved.

Underpinning a package such as *Cabri* is the idea of geometric relationships which require clear definition. As Laborde (1993 p 57) observes "explicit description leads to an emphasis on the functional and analytic aspects of geometry".

Hoyles, Healy and Noss (1995 p 102) confirm that "the computer may offer just such a new context" for proof as long as, they stress, tasks are carefully designed to provide students with what they call "the bones of a proof" (*ibid* p 104). The remainder of this paper describes the design and use of a series of dynamic geometry tasks created in an attempt to take advantage of the potential offered by a dynamic geometry package.

Developing Dynamic Geometry Contexts

Given the above discussion, the aim of developing such a series of tasks was based on the following factors all of which were considered critical:

- encouraging the students to make conjectures
- focusing on the *relationships* between geometrical objects
- providing the means for the students to *explain* their actions and results

It has been observed before that starting out in an English mathematics classroom with a dynamic geometry package such as *Cabri* is far from straightforward. Constructions in *Cabri* are based in the Euclidean tradition, something that has not been a significant part of the UK mathematics curriculum since the early 1970s (Fielker 1986). Consequently, several authors (including Ainley and Pratt (1995), Healy *et al* (1994), and Winbourne and Wrigley (1993),) have suggested a logo-like start with students being introduced to some of the menu items in *Cabri*. The students are then free to choose their own goals.

However, as Healy *et al* (1994) observe, there are risks in such an approach in that it can allow students "to avoid interacting with mathematics at all" (*op cit*). To counteract this possibility, Healy *et al* introduced to their students the idea of "messing up" (or, more accurately, *not* messing up). They defined "messing up" in the following way: "After a figure was drawn it could be dragged to see if it became unrecognisable, that is whether the different objects within the design moved together in a sensible way or not" (*op cit*). They conclude (in Holzl *et al* 1994) that, while they commend the idea of "messing up", the idea of functional dependency may be difficult for students to grasp.

The students in the present study were unfamiliar with *Cabri* and so a similar approach to that suggested by Healy *et al* was adopted. Pairs of students were introduced to some of the menu items in *Cabri* and then allowed to choose their own goal. The notion of "messing up" was introduced with the students being encouraged to formulate *mathematically* challenging goals. These introductory sessions did mean that the students gained some familiarity with *Cabri* but it became clear that the students would need more explicit mathematical goals if they were not going to use *Cabri* solely as a drawing tool rather than a tool for constructing geometrical figures.

It is important, here, to stress the importance of distinguishing between *drawing* and *figure*. Laborde (1993 p49) makes clear the distinction in the following way: "*drawing* refers to the material entity while *figure* refers to a theoretical object". In terms of a dynamic geometry package, a *drawing* can be a juxtaposition of geometrical objects resembling closely the intended construction. In contrast, a *figure* additionally captures the relationships between the objects in such a way that the figure is invariant when any basic object used in the construction is dragged. As I shall subsequently argue, the ability to *check* a construction *by dragging* appears to be particularly important part of experiencing the *necessity* of relevant geometrical facts.

The search for a suitable context for work on "proof as explanation" led to a focus on the *analysis of static figures* and their construction in the dynamic *Cabri* environment. In other words, tasks would be designed which asked the students to construct, given a geometrical figure drawn on paper, the corresponding geometric figure in *Cabri* such that the *Cabri* figure can not be "messed up".

Another important consideration was to utilise some of the students existing mathematical knowledge and document how this developed as the students used *Cabri*. A long-standing component of the primary geometry curriculum in the UK is the recognising and sorting of various simple plane shapes. This entails such matters as identifying circles and various specific triangles and quadrilaterals. In the secondary curriculum, students are expected to know and use the properties of quadrilaterals and be able to classify them on the basis of their properties. Given these factors, and the fact that there has been both discussion about the classification of quadrilaterals (for example, de Villiers 1994) and research involving students' ability to classify them (for instance, Fuys et al 1988), the 'family of quadrilaterals' evolved as the choice of context for the *Cabri* tasks.

In constructing quadrilaterals, a key idea is the use of the circle as a length-carrier. This, in turn, depends on the locus definition of the circle. Just as the idea of functional dependency may not be easy for lower secondary school students to grasp, neither might the locus definition of the circle (or, at least, it contrasts with what is claimed by Bishop (1983) to be the notion of the circle commonly-held by upper primary and lower secondary pupils: that is, that a circle is a disc). Some way was required to introduce the students to using the circle in the construction of other geometrical objects.

What might be a promising solution to this problem, illustrated by the work of one pair of pupils on a selection of the resulting *Cabri* tasks, is described in the next section.

Children working on Dynamic Geometry Contexts

The students reported on here are 12 year olds with no previous experience of using a dynamic geometry package although they had all used various drawing packages and other IT resources. The class is an above-average mathematics class in a city comprehensive school whose results in mathematics at age 16 are at the national average. The mathematics teachers use a resource-based approach to teaching mathematics and the students usually work in pairs or small groups. The class has three 50-minute mathematics lessons per week. At the beginning of the study, computer use for *Cabri* was restricted to one computer in the classroom (the students have access to computer laboratories for other computer applications). This meant that, as student pairs took it in turn to use the computer, it was often several weeks between sessions for particular pairs. Towards the end of the study, up to four machines were able to be used at the same time in the classroom. The version of *Cabri* used was *Cabri I* for the PC.

Each of the classroom tasks requires the students, in pairs, to analyse a figure presented on paper. The students are asked to construct the figure using *Cabri* such that the figure is invariant when any basic point used in the construction is dragged. This means that the students have to focus on the *relationship* between the basic objects (points, lines and circles) necessary to construct the figure.

This section presents the work of one pair of pupils, H and R, as they progress from the initial exploratory session through two of a series of ‘starter’ tasks to one task from a series on quadrilaterals. Four sessions will be described in total. They took place over a period of five months. During that time all students in the class used *Cabri* and the work they did was sometimes shared. Certainly, students discussed between themselves the work they did. Also during that time, of course, the students continued with their regular mathematics programme. This included some work on certain aspects of `shape and space’ although none of it could be said to be directly relevant to the area of the mathematics curriculum that was the focus of the *Cabri* work.

As described above, the initial session introduced the students to some of the menu items of *Cabri* and the students were then free to choose their own goal.

The Exploratory Session

After being introduced to the creation menu and told that the idea was to use these geometric objects to construct other geometric objects, H and R agreed to start with *basic point* and work their way down the menu trying out each item. In this way they created a juxtaposition of shapes. After clearing this design they embarked on creating a picture that they called `the crooked house' (see figure 1).

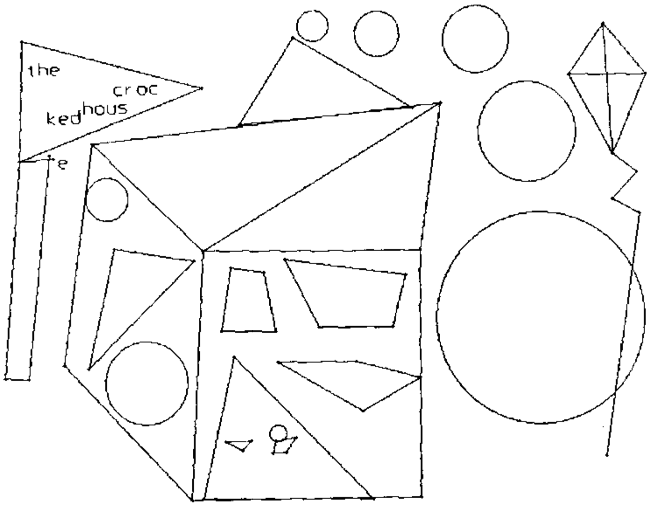

Figure 1

While H and R enjoyed themselves and became competent at using the basic components of *Cabri*, their project involved little mathematics and certainly did not seem to challenge them in any serious mathematical way. None of the critical factors itemised at the start of the previous section were present. To include these factors, a series of suitable tasks were designed and H and R tackled the first of these some four weeks after their introductory session.

The `Starter' Tasks

The `starter' tasks consisted of a series of simple patterns made up of lines and circles. The students were asked to construct the patterns so that they could not be "messed up". In this section, I describe first how H and R tackle the pattern shown in figure 2.

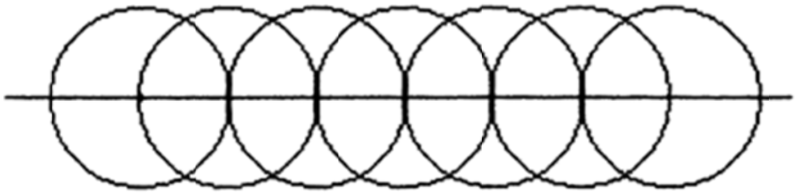

Figure 2

H and R discuss the pattern and begin constructing it using basic circles starting from the left. They form three basic circles of approximately the correct size and drag them into approximately the correct orientation. With three circles in place, I point to the instruction on the sheet (in the transcripts that follow, *I* stands for interviewer/researcher):

I: Before you do any more, it says here "so they can not be messed up". What does that mean?

H; They can't overlap if they are not supposed to.

I: What else could it mean?

R: It shouldn't get muddled up.

I invite them to drag one of the circles away. One of the students immediately exclaims

R: So, you can mess it up!

I: You have to find a way so that you can't mess it up.

Having used *basic circle* and failed, they try using *circle by centre and radius point*. They begin by constructing two circles such that the radius point of the first circle is the centre of the second circle. This illustrates that H and R have some understanding of the relationship between the objects they need to construct. Then, importantly, they *check by dragging* one of the points and find that their figure can not be `messed up'. At this point they are not sure how to proceed. After being referred back to the figure on paper, they construct the line and thence confidently complete the task.

I: Why can't it be messed up?

H: Because they [the circles] are all linked with points.

R: The line links them all up.

The pair is pleased with their result. Through this task, they were beginning to appreciate the need to focus on the *relationship* between the objects they were creating. This focus developed in the next task they tackled, figure 3, some four weeks later.

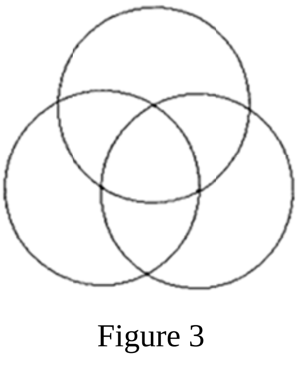

Figure 3

H and R confidently create two circles that could not be "messed up" but, rather than using a *point of intersection* (in *Cabri I*) they use *point on object* to create the centre of the third circle. As a result, their final construction could still be "messed up" albeit not drastically. Nevertheless, the pair is aware that something about their construction is unsatisfactory. After being reminded of their previous work, they looked again at the construction menu and this time correctly chose *point of intersection*. In this way, they show that they are still becoming accustomed to the special nature of relationships in *Cabri I* and of the notion of not "messing up".

Following work on these `starter' tasks, H and R embarked on a series of tasks that focused on the `family of quadrilaterals'. The next section describes them tackling one of the tasks. The lesson occurred after a long school holiday.

A Quadrilateral Task
The students are asked to construct the figure given in figure 4.

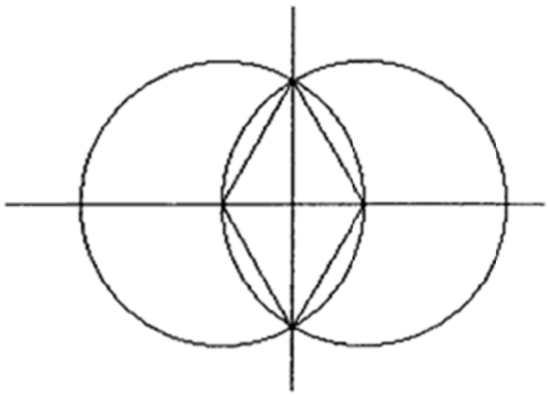

Figure 4

and thereby obtain figure 5

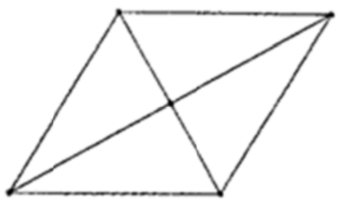

Figure 5

H and R tackle this with consummate ease, checking after each individual action that the figure does not mess up. They are expected not only to construct the figure but also to explain *why* the shape is a rhombus. Although the task refers to the shape as a rhombus, in part of their explanation, they refer to it as a diamond. This is what they say:

> The sides are all the same because, if the centre is in the right place, the sides are bound to be the same.
>
> The diagonals of the diamond cross in the middle, though they are different sizes [lengths]. The diagonals bisect each other. The angles [where the diagonals cross] are the same. They are right angles.
>
> The opposite angles [of the rhombus] are the same. Two are more than 90 degrees but less than 180 degrees, and the others are less than 90 degrees but more than zero degrees.
>
> This shape is a rhombus because the sides are the same, the diagonals bisect at right angles and the opposites have the same angles.

In order to write down their explanation, H and R needed appropriate support. The task on paper contained certain prompts. For instance, the task suggested that attention be paid to the sides, the angles, and the diagonals. I also prompted the pair by focusing their attention on these attributes and providing technical language (such as bisect) when appropriate in order to assist the precision of their explanation.

Discussion

The above provides some illustration of how the quality of these particular students' mathematical analysis of such geometrical figures improved significantly and observably within three sessions using *Cabri*. Nevertheless, it has to be stressed that this experience is founded on the following factors: a well-established and carefully-nurtured classroom culture that values mathematical thinking, a sequence of carefully selected tasks, and a range of appropriate prompts from the researcher and teacher (and possibly other classmates). It might well be that, without this combination of factors, the outcome would have been very different.

It is also worth noting that H and R are but one study-pair in the class and not the most able mathematically. In addition, their experience was spread over several months. Again, the influence of factors such as these on the outcome is unknown. It may be that the time sequence was an important factor; or it might be that the same outcome would have been achieved if the students had completed the same tasks within a much shorter period. Only further investigation can address that particular issue.

Of course, a further, and perhaps, critical factor is the mediating effect of the computer. Papert has claimed that, in the right circumstances, the computer can "help bridge the gap between formal knowledge and intuitive understanding" (Papert 1980 p 145). It does this, Turkle and Papert (1991 p 162) suggest, by standing "betwixt and between the world of formal systems and physical things: it has the ability to make the abstract concrete". From this perspective, a dynamic geometry package such as *Cabri* provides the semblance of direct manipulation of abstract geometrical objects. Yet in *Cabri* (or, more accurately, in *Cabri I*), there is a need to be both precise *and* explicit about the relationships between geometrical objects in the figure being created. This means that, in the design of *Cabri I*, it became necessary to have different types of point: *basic point, point on object,* and *point of intersection*. Consequently, it could be argued, the nature of the specification of relationships in *Cabri I* is overly particular to *Cabri I*. This is another factor that demands further investigation (given the release of *Cabri II* where the demands for precision are different).

All the above factors need to be borne in mind in any evaluation of the results of the empirical work described in this paper. After all, only the experience of one pair of pupils is documented here. Nevertheless, I would suggest that through working on tasks such as the ones described above, the students gained some insight into the structure of plane geometry. For H and R this meant that they were able to explain the properties of a rhombus. For other pairs it meant, for example, that when they came to constructing a square they not only knew that the shape *had to be a square* but they also knew *why*.

In this process, both of experiencing the necessity of a result (for H and R, that the figure *has* to be a rhombus) and of coming to know *why* that particular result is obtained, the ability to *check by dragging* appears to be particularly important. This facility allows conjectures to be tested by focusing attention on the relationships between the geometrical objects that have been constructed. In my work, the commands *Delete object* (and the object's dependants), together with *undo*, have also shown themselves to be useful in a similar way. My argument is that experiencing the *necessity* of such geometrical facts involves perceiving (at some level) the axioms of the mathematical structure of plane geometry. Explaining *why* these geometrical facts are necessarily true involves constructing chains of reasoning. These are two of the essential components in a meaningful experience of proof.

I am carrying out further classroom work with these able 12 year olds and also doing some parallel work with a class of 14 year olds. From the evidence so far gathered in this study, it may be that carefully chosen tasks using a dynamic geometry package provide a suitable context for some students to develop an appreciation of mathematical proof. In order to see how this view of proof as explanation develops further in the classroom, the process of acceptance within such a setting is a vital aspect. In due course, attention will need to be paid to that process and to how this contributes to developing in students the basis for a fuller appreciation of the nature and purpose of mathematical proof.

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