Cite as: Jones, K. (1997) Student teachers' conceptions of mathematical proof.
Mathematics Education Review, volume 9, pp21-32.

Student Teachers' Conceptions of Mathematical Proof

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The successful teaching of mathematical proof depends crucially on the subject knowledge of mathematics teachers. Yet the knowledge that teachers have of mathematics has become a matter of major concern in both pre-service and in-service teacher education. While this debate has largely focused on primary teachers, much less has been said about the subject knowledge of secondary mathematics teachers. This paper reports on an initial analysis of a small-scale investigation into trainee secondary mathematics teachers' conceptions of mathematical proof. Some tentative implications are drawn from this preliminary analysis. For example, it may be that while the least well-qualified trainee secondary teachers may have the poorest grasp of mathematical proof, the most highly qualified may not necessarily have the specific kind of subject matter knowledge needed for the most effective teaching. The methodology of concept-maps used in this study may provide a valuable approach to gathering insights into students' understanding of mathematical proof.

Concerns about the Teaching of 'Proof'

A number of concerns have recently emerged about the teaching of mathematical proof at school level. The London Mathematical Society, for instance, suggests that "most students entering higher education no longer understand that mathematics is a precise discipline in which …. logical exposition and proof play essential roles" (LMS 1995 p8). Amongst the possible causes they infer that the UK National Curriculum for mathematics may be distorting the notion of mathematical proof (*ibid* p25). In a similar vein, the Dearing review of UK qualifications for 16-19 year olds expresses concerns about the "limited perceptions of the role of …. proof" amongst A-level candidates and recommends that the mandatory core at A-level should be reviewed (Dearing1996 p96-98). Such a review has taken place and the new core for A-level mathematics does indeed contain a greater emphasis on mathematical proof.

The LMS report also notes that "to improve what is taught and how it is taught, we must raise the competence and confidence of those who choose to become mathematics teachers" (*ibid* p22). A crucial factor in this is the knowledge that new entrants into mathematics teaching have of mathematics. In this paper I focus on one pertinent aspect of the mathematical subject knowledge of pre-service secondary mathematics teachers, specifically their conceptions of mathematical proof. I present a preliminary analysis of data from a small-scale investigation which suggests that whilst the least well qualified trainee secondary teachers may have the poorest grasp of mathematical proof, the most highly qualified may not have the specific kind of subject matter knowledge needed for the most effective teaching. This accords with the current focus on the subject knowledge base of mathematics teachers.

The Subject Knowledge Base of Mathematics Teachers

The subject knowledge that teachers have as a basis for their teaching has become a matter of particular concern in both pre-service and in-service teacher education. This is especially so in the case of the teaching of mathematics in primary schools. Ofsted, for instance, claim that whilst teachers' command of mathematics is adequate in 90% of schools at key stage 1 and 75% of schools at key stage 2, in only 10% of primary schools is it good or very good (Ofsted 1995 p 9). Research by Aubrey, amongst others, has provided more detail on aspects of the influence of primary teachers' mathematics subject knowledge on how they teach (see, for example, Aubrey 1996). As a result of this concern about primary teachers' knowledge of mathematics there have been a number of initiatives to support teacher development. These include the provision of short courses, supported by publications such as Haylock (1995). More recently, the Teacher Training Agency has seen fit to impose an "Initial Teacher Training National Curriculum for Primary Mathematics" which specifies the essential mathematics subject knowledge that must be taught to all trainee primary teachers (TTA 1997a).

In contrast, the situation is often viewed as being somewhat different for secondary teachers of mathematics. After all, the argument goes, secondary mathematics teachers are, in general, mathematics specialists and so subject knowledge ought to be secure. Indeed, Ofsted suggest that 60% of mathematics teachers have a good or very good command of their subject and infer that anything less than adequate is due to schools using some non-specialist teachers at key stage 3 (Ofsted 1995 p 10). Yet it is worth investigating further whether all is well with the subject knowledge of secondary mathematics teachers. Certainly, evidence from a study of biology and geography teachers suggests that, for teachers of these subjects, both subject knowledge and pedagogical knowledge can be unsatisfactory (Hoz *et al* 1990). What is more, this particular study found that gaining experience does not necessarily improve this knowledge, although the teachers in the study did appear to master subject knowledge better than they did pedagogical knowledge. There is some similar evidence about secondary mathematics teachers. In a detailed study of prospective mathematics teachers' knowledge of functions, Even, for instance, found that these student teachers did not have a complete conception of this important part of mathematics (Even 1993). For example, appreciation of the arbitrary nature of functions was missing, and very few could explain the importance and origin of the univalence requirement. This limited conception of function appeared to influence the student teachers' pedagogical thinking.

Given that the majority of UK secondary mathematics teachers enter the profession by completing a one-year postgraduate course of initial teacher education (the PGCE), the basis for their subject knowledge is, in the main, developed during their specialist undergraduate course. Indeed, government requirements state that the content of a PGCE entrants' previous education must provide the necessary foundation for work as a mathematics teacher. During the one-year initial teacher education course, the emphasis has to be on transforming sound subject knowledge into secure pedagogical content knowledge (Ruthven 1993). Consequently, teacher educators need to be confident that student teachers on a PGCE course have sound content knowledge, particularly with respect to essential components of mathematics such as mathematical proof.

The Role of Proof in Mathematics and Mathematics Teaching

Mathematical proof is the essential component of mathematics and is arguably what distinguishes mathematics from other disciplines. As such it should be a key component in mathematics education. Yet providing a mathematics curriculum that makes proof accessible to school students appears to be difficult. Proving, it seems, either appears as an obscure ritual or it disappears in a series of innocuous classroom tasks in which students learn to 'spot patterns' but not much else (Hewitt 1994). For example, Schoenfeld (1989), reports that even when students can reproduce a formally taught Euclidean proof, a significant proportion conjecture a solution to the corresponding geometrical construction problem that "*flatly violates* the results they have just proven" (emphasis added). On the other hand, when the chosen proof contexts are data-driven, with students expected to form generalised conjectures and search for counter examples, Coe and Ruthven (1994) find that students' proof strategies are primarily empirical. It seems that the generation of numerical data becomes the object of the exercise and any notion of deductive argument is rejected. Balacheff (1988 p 222) similarly reports the occurrence of what he refers to as "naive empiricism".

A likely relevant issue is that proofs are often thought of solely as standardised linear deductive presentations. Indeed, this is how proofs are frequently presented and the form of two-column proofs taught in a number of countries entirely fits such a model. Proof can, however, take a number of forms. Balacheff (1988 p 216), for instance, contrasts what he calls pragmatic and conceptual proof. From a different perspective, Leron (1985) talks about "direct" and "indirect" proofs. A further distinction suggested by Hanna (1989; Hanna and Jahnke 1996) is that there are proofs that prove (and do no more) and proofs that *explain*. This latter form of proof, Hanna suggests, demonstrates not only that a statement is true, but also *why it is true*.

While underlining the central importance of mathematical proof, the above considerations say something about the difficulties pupils have in learning what constitutes a proof and indicate some ways in which proof might be taught in a more meaningful way. Of course there is a model of progression in mathematical reasoning embedded in the UK national curriculum for mathematics (DFE 1995). This indicates that, in the earliest years, pupils can be taught to recognize simple patterns and make predictions about them, ask questions such as "what would happen if?", and understand simple general statements such as "all even numbers divide by 2". Following this, pupils are expected to make conjectures, make and test generalizations, and appreciate the difference between mathematical explanation and experimental evidence. Only the older, more able, pupils in the 15-16 year age range are expected to extend their mathematical reasoning into understanding and using more rigorous argument, leading to notions of proof.

All this suggests that the teaching of mathematical proof places significant demands on both the subject knowledge and pedagogical knowledge of secondary mathematics teachers. Yet to be awarded Qualified Teacher Status in the UK, intending secondary mathematics teacher have to demonstrate, amongst other things, that they have a secure knowledge and understanding of the concepts and skills in mathematics at a standard equivalent to degree level to enable them to teach mathematics "confidently and accurately" (TTA 1997b). The above discussion of mathematical proof suggests that intending secondary mathematics teachers need to have an extremely secure subject knowledge base of mathematical proof if they are to teach it accurately and with confidence.

In the next section of this paper I describe some aspects of a small-scale study designed to illuminate the conceptions of mathematical proof held by trainee secondary teachers of mathematics. The aim of the study is to provide evidence of how secure mathematics student teachers are in their conception of mathematical proof. The methodological tool chosen to reveal the student conceptions is based on the idea of the *concept map* advocated by Novak and Gowin (1984) principally as an aid to meaningful learning. More recently concept mapping has been suggested both as a tool for assisting the teacher to teach and the learner to learn, and as a research and evaluation tool (Markham *et al* 1994). Because this study is very small-scale, clearly any conclusions must be very tentative. However, readers may find the methodology described an interesting approach for gaining insights into students' understanding of the nature of proof.

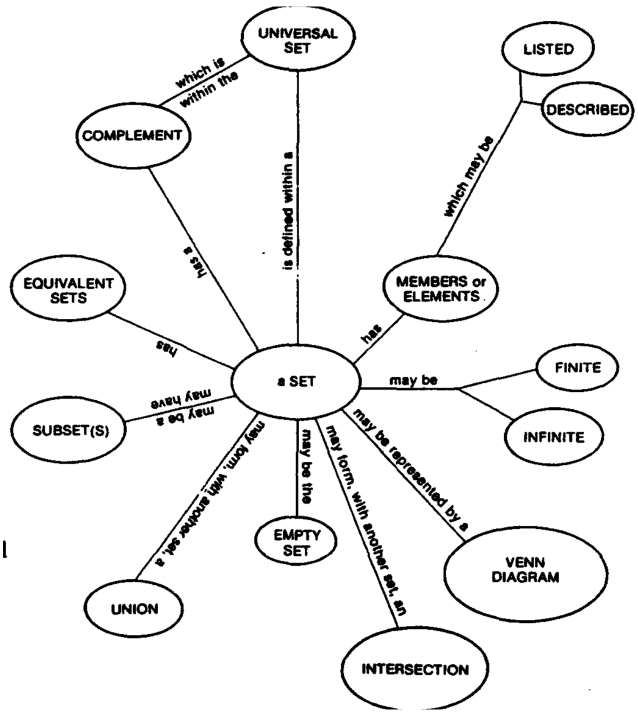

Figure1

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Concepts Maps of Student Teacher Knowledge of Proof

A concept map is, at its simplest, a graphical representation of domain material generated by the learner in which nodes are used to represent domain key concepts, and links between them denote the relationships between these concepts. In this way, a concept map provides an explicit representation of knowledge. An example of a concept map focusing on elementary set theory is provided by Orton (1992 p 169).

Characteristic elements to note are the nodes, which represent key ideas, and the linking lines which express the relationship between these ideas.

The theoretical foundation of concept mapping is Ausubel's theory of learning, which suggests that meaningful learning depends on integrating new information in a cognitive structure laid down during previous learning. The argument put forward by Novak and Gowin, amongst others, is that concept mapping resembles the cognitive structure developed during learning. It appears that neurologists tend to agree with this proposition.

To generate a map of their conception of mathematical proof, student teachers followed a version of the suggested method for producing concept maps. Step one is to produce a list of key terms that the students associate with mathematical proof through a group brainstorming session. The following is the list one group of 25 students generated (in no particular order):

Euclidean	*observation*
logic	*general case*
trial and improvement	*theorem*
graphical	*assumptions*
axioms	*irrefutable*
syllogism	*deduction*
definitive	*postulate*
lemma	*by contradiction*
explanation	*hypothesis*
examples	*implies*
precision	*proposition*
reasoning	*abstraction*

Step two is for each student teacher, individually, to produce their own representation of their conception of mathematical proof using any or all of the above key terms (or others that they might choose), arranging them on a blank piece of paper from their own perspective, and joining key terms in what they consider a meaningful way for them, using lines and words indicating relationships between the key terms they use.

Below are examples from three student teachers, all of whom have degrees in mathematics but with varying classifications. The maps were drawn during week 16 of a 36 week long course. These particular examples were selected to illustrate the range of responses from the trainee teachers to the task of drawing up a concept map and to test the method of analysis. Further analysis of data from a larger sample of trainee teachers is needed before any firm conclusions could be drawn either about the quality of the subject knowledge of trainee mathematics teachers or of the validity and reliability of the concept map method. Nevertheless, the data given below does raise some interesting questions. Figure 2 is of a student teacher A, whose degree classification is a pass.

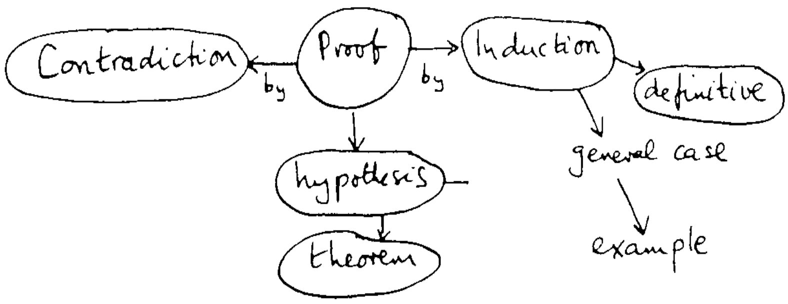

Figure 2

With figure 3, student teacher B has a third class honours.

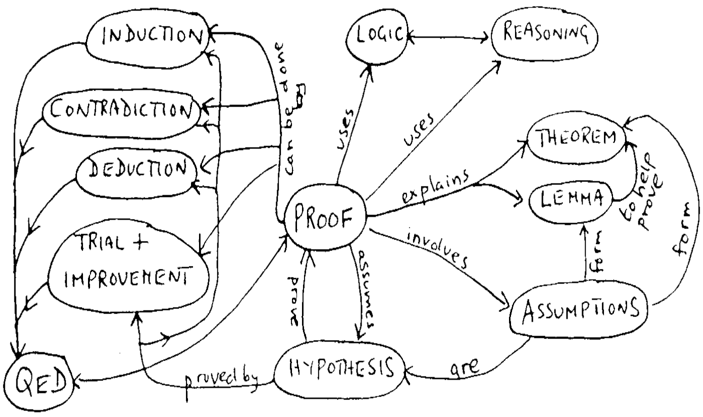

Figure 3

Figure 4 is of a student teacher C, with an upper second class honours degree.

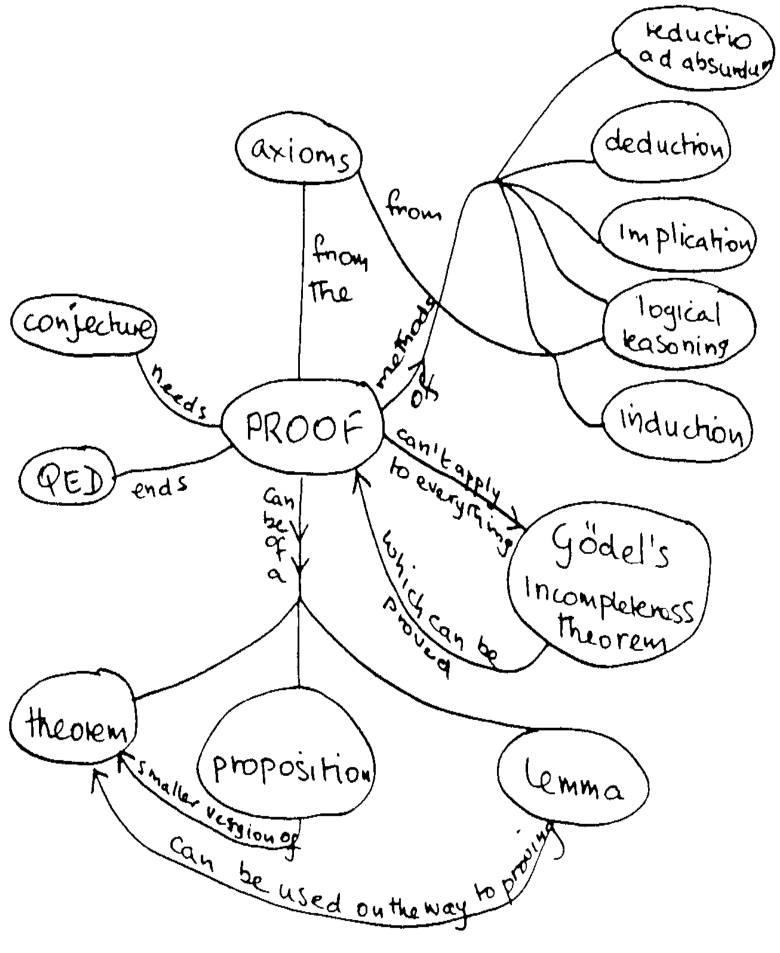

Figure 4

Before embarking on any analysis of these concept maps, it is important to recognise that cognitive structures and concept mappings are highly personal as each individual's knowledge is unique. Hence, concept maps are idiosyncratic. There is no one "correct" concept map. However, this does not mean that all concept maps are correct. It is possible, for instance, to examine the key terms used and the way in which relationships between these key terms are specified. It may also be possible to identify errors, such as the absence of essential concepts or inappropriate relationships between concepts.

Analysis and Discussion

In this section I present a preliminary analysis of the above three concept maps based on two main criteria:

- the use of key terms: how many key terms are used and which ones are included.
- the specified relationship between key terms: how many relationships are specified, how they are specified, and whether cross-links or multiple relationships are indicated.

Such an analysis of the three concept maps given above is shown in Table 1. The first column indicates that the better the qualification, the more key terms the student uses. There is also evidence from the maps themselves that the most highly qualified student teacher produces what could be considered a more sophisticated map by introducing additional terms not in the list given above.

	Number of key terms	Number of relationships	Number of relationships specified	Number of crosslinks specified
Student A (pass)	6	7	2	0
Student B (third)	12	25	18	12
Student C (2.1)	13	16	16	3

Table 1

Nevertheless, despite using more key terms, the most mathematically-able student teacher does not specify either the most number of relationships between key terms, nor the most number of cross-links between key terms. It is student B who does that.

Such a preliminary analysis only allows some very tentative conclusions to be drawn but these may fit in with some other findings. On the one hand, one conclusion might be that trainee mathematics teachers with the barest minimum qualification of a pass degree in mathematics need considerable support in developing a secure knowledge base of mathematics. Given the largely school-based nature of initial teacher education this may be rather difficult to provide.

On the other hand, the evidence here also suggests that having the best qualification does not necessarily mean that the student teacher will make the most effective mathematics teacher. For example, while student C arguably has the most sophisticated knowledge of mathematical proof, the concept map of student teacher B might be considered somewhat the richer as it has more relationships between the key terms present. This fits with the findings of a study by the US National Center for Research on Teacher Learning (NCRTL 1993) that “majoring in an academic subject in college does not guarantee that teachers have the specific kind of subject matter knowledge needed for teaching”.

Given that the concept maps provided above were produced during week 16 of a 36 week long course, it is possible, on the one hand, that the concept map of student teacher B has developed some of the linkages that reflect the transformation of subject knowledge into pedagogical content knowledge that is the aim of the initial teacher education course. Consequently, student teacher B may well be the most effective and successful of the three teachers chosen for this analysis.

On the other hand, some of the differences between the concept maps of students B and C may be due to differences in their undergraduate courses. This possibility is supported by some of the findings of the NCRTL report which is highly critical of undergraduate University courses, such as many in mathematics, that require students to memorize massive amounts of information, while paying little attention to the meaning or significance of the material covered. The suggestion is that once students graduate, they often think about mathematics as lengthy lists of facts with little or no consideration given to relationships among principles and concepts learned. This has the effect of making the transformation to effective pedagogical content knowledge all the more difficult. The NCRTL researchers did find a university-based course that seemed to make a difference. This course requires students to reason about the subject, to argue about alternative explanations for what they encounter, and to test their ideas and those of others. Such academic interaction, the study found, tended to improve students' understanding of important concepts in the subject matter and, along with that, their ability to explain concepts.

Given the central importance of proof in mathematics and in mathematics education, the development of successful and confident secondary mathematics teachers depends both on sound subject knowledge built up at undergraduate level, and secure pedagogical knowledge developed during postgraduate initial teacher education courses. This demands that attention is paid both to undergraduate courses in mathematics as well as to courses in initial teacher education.

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