On Higher-order Linear Port-Hamiltonian Systems and Their Duals ⋆

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Abstract: We formulate a behavioral approach to higher-order linear port-Hamiltonian systems. We formalize constitutive laws such as power conservation, storage and (anti-)dissipative relations, and we study several properties of such systems. We also define the dual of a given port-Hamiltonian behavior.

Keywords: Bilinear- and quadratic differential forms; port-Hamiltonian systems; behavioral system theory; duality; switched linear differential systems

1. INTRODUCTION

The usual approach to port-Hamiltonian systems is representation-oriented, especially in state space terms (see van der Schaft and Jeltsema (2014)). Such approach captures the underlying principles and unveils an accurate mathematical representation of physical systems in terms of power and energy quantities, and consequently it has been largely successful in the analysis of dynamical systems in the linear and nonlinear setting (see e.g. Duindam et al. (2009); Ortega et al. (2002)). Motivated by the fact that, in many cases, state space models and variables are not necessarily a given, in this paper we develop a *trajectory-oriented* approach to port-Hamiltonian systems. For instance energy-balance equations obtained by modelling physical systems are often in higher-order terms, since they derive from accounting for energy exchange between subsystems, themselves modelled in terms of higher-order differential equations (see Willems (2007) and Willems (2010)). Moreover, in many real-life scenarios the state variables of the system are not necessarily known (see e.g. Mazloum et al. (2016) and Raju and Khaitan (2012)). Another common situation, e.g. in electrical systems involves the study of grids whose impedance specification (in higher-order terms) is directly identified from phasor-measurements (see e.g. Ardakanian et al. (2017)), but its state space structure is generally unknown, e.g. due to the fact that *node voltages* and *mesh currents* are not necessarily state variables. For such reasons, in this paper we develop a framework in which we can use the port-Hamiltonian system formalism, and simultaneously accommodate first principle models in the form of sets of higher-order differential equations.

Another inspiration to develop a higher-order, trajectory-based approach to linear port-Hamiltonian systems is stimulated by recent results in the *switched linear differential systems* (*SLDS*) framework developed by the authors (see Mayo-Maldonado and Rapisarda (2016a,b, 2013, 2014); Mayo-Maldonado et al. (2014); Rocha et al. (2011)). While such approach offers some advantages over the classical state-space based one, several important issues are still open. Among these is the automatic derivation of *gluing conditions* and related *reset rules* from physical principles (e.g. conservation of energy). In van der Schaft and Çamlibel (2009) a compelling mathematical formalization of *state transfer principles* for switched port-Hamiltonian systems has been given. We plan to explore similar ideas to solve the above mentioned open problems in the SLDS framework; this work is preparatory to such end.

We study port-Hamiltonian systems from the *behavioral* viewpoint, see Polderman and Willems (1997). We define the variables of such a system as *observables* induced by higher-order polynomial differential operators acting on an auxiliary variable. Each such observable has associated a *conjugate* one, just as in the classical framework each flow has associated a corresponding effort variable. The observables obey certain *constitutive relations* (power conservation, dissipativity, etc.) expressed in terms of bilinear- and quadratic differential forms (see Willems and Trentelman (1998)). We describe properties of such variables that can be derived from the constitutive relations, and we introduce the concept of a *dual port-Hamiltonian behaviour*, that satisfies an "anti-dissipative" constitutive relation.

2. BACKGROUND MATERIAL

A thorough treatment of the notions illustrated in this section can be found, respectively, in Polderman and Willems (1997), Rapisarda and Willems (1997); van der Schaft and Rapisarda (2011) and in Willems and Trentelman (1998).

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2.1 Notation

The space of $\mathtt{n}$ dimensional real vectors is denoted by $\mathbb{R}^{\mathtt{n}}$; that of $\mathtt{m}\times\mathtt{n}$ real matrices by $\mathbb{R}^{\mathtt{m}\times\mathtt{n}}$; and $\mathbb{R}^{\bullet\times\mathtt{m}}$ denotes the space of real matrices with $\mathtt{m}$ columns and an unspecified finite number of rows. Given matrices $A, B \in \mathbb{R}^{\bullet\times\mathtt{m}}$, $\operatorname{col}(A,B)$ denotes the matrix obtained by stacking A over B. The ring of polynomials with real coefficients in the indeterminate s is denoted by $\mathbb{R}[s]$; the ring of two-variable polynomials with real coefficients in the indeterminates ζ and η is denoted by $\mathbb{R}[\zeta,\eta]$. $\mathbb{R}^{\mathtt{r}\times\mathtt{w}}[s]$ denotes the set of all $\mathtt{r}\times\mathtt{w}$ matrices with entries in s, and $\mathbb{R}^{\mathtt{n}\times\mathtt{m}}[\zeta,\eta]$ that of $\mathtt{n}\times\mathtt{m}$ polynomial matrices in ζ and η. The set of rational $\mathtt{m}\times\mathtt{n}$ matrices is denoted by $\mathbb{R}^{\mathtt{m}\times\mathtt{n}}(s)$. Given $G = G^\top \in \mathbb{R}^{\mathtt{m}\times\mathtt{m}}$, $\sigma_+(G)$ denotes the number of positive eigenvalues of G. The set of infinitely differentiable functions from $\mathbb{R}$ to $\mathbb{R}^{\mathtt{w}}$ is denoted by $\mathfrak{C}^\infty(\mathbb{R},\mathbb{R}^{\mathtt{w}})$. $\mathfrak{D}(\mathbb{R},\mathbb{R}^{\mathtt{w}})$ is the subset of $\mathfrak{C}^\infty(\mathbb{R},\mathbb{R}^{\mathtt{w}})$ consisting of compact support functions.

2.2 Linear differential behaviors

A *linear differential behavior* is a linear subspace $\mathfrak{B} \subseteq \mathfrak{C}^\infty(\mathbb{R},\mathbb{R}^{\mathtt{w}})$ consisting of the solutions of a finite system of constant-coefficient linear differential equations. Such a set can be represented as

$$\mathfrak{B} = \left\{ w \in \mathfrak{C}^\infty(\mathbb{R},\mathbb{R}^{\mathtt{w}}) \;\mid\; R\left(\frac{d}{dt}\right) w = 0 \right\} = \ker\, R\left(\frac{d}{dt}\right) , \tag{1}$$

with $\mathbb{R} \in \mathbb{R}^{\mathtt{g}\times\mathtt{w}}[s]$. Equation (1) is called a *kernel representation* of $\mathfrak{B}$. We denote by $\mathcal{L}^{\mathtt{w}}$ the set of linear time-invariant differential behaviors with $\mathtt{w}$ variables.

The property of *controllability* is discussed in sect. 5.2 of Polderman and Willems (1997); if $\mathfrak{B}$ is controllable, it can be also represented in *image form* , i.e. there exist $M \in \mathbb{R}^{\mathtt{w}\times m}[s]$ and an *auxiliary variable* ℓ such that

$$\begin{aligned}\mathfrak{B} &= \left\{ w \mid \exists\, \ell \in \mathfrak{C}^\infty(\mathbb{R},\mathbb{R}^m) \text{ s.t. } w = M\left(\frac{d}{dt}\right)\ell \right\} \\ &= \operatorname{im}\, M\left(\frac{d}{dt}\right) .\end{aligned} \tag{2}$$

The number of *input variables* (see Def. 3.3.1 of Polderman and Willems (1997)) of a behavior $\mathfrak{B}$ is denoted by $\mathtt{m}(\mathfrak{B})$; the remaining $\mathtt{p}(\mathfrak{B}) := \mathtt{w} - \mathtt{m}(\mathfrak{B})$ variables are *outputs.*

2.3 State maps

An auxiliary variable x is a *state variable* for $\mathfrak{B}$ if $\mathfrak{B}$ has a representation of first order in x and zeroth order in w, i.e. there exist $E, F \in \mathbb{R}^{\bullet\times\bullet}$, $G \in \mathbb{R}^{\bullet\times\mathtt{w}}$ such that $\mathfrak{B} = \left\{ w \mid \exists\, x \text{ s.t. } E\frac{d}{dt}x + Fx + Gw = 0 \right\}$. The minimal number of state variables needed to represent $\mathfrak{B}$ in such way is called the *McMillan degree* of $\mathfrak{B}$, denoted by $\mathtt{n}(\mathfrak{B})$.

A state variable for $\mathfrak{B}$ can be computed as the image of a polynomial differential operator called a *state map* (see Rapisarda and Willems (1997),van der Schaft and Rapisarda (2011)). Algebraic characterizations of state maps and minimal state maps for systems in kernel and image form are given Rapisarda and Willems (1997); van der Schaft and Rapisarda (2011).

2.4 Bilinear and quadratic differential forms

Let $\Phi \in \mathbb{R}^{\mathtt{w}_1\times\mathtt{w}_2}[\zeta,\eta]$; then $\Phi(\zeta,\eta) = \sum_{h,k} \Phi_{h,k}\zeta^h\eta^k$, where $\Phi_{h,k} \in \mathbb{R}^{\mathtt{w}_1\times\mathtt{w}_2}$ and the sum is finite. $\Phi(\zeta,\eta)$ induces the *bilinear differential form* (BDF) L_Φ from $\mathfrak{C}^\infty(\mathbb{R},\mathbb{R}^{\mathtt{w}_1}) \times \mathfrak{C}^\infty(\mathbb{R},\mathbb{R}^{\mathtt{w}_2})$ to $\mathfrak{C}^\infty(\mathbb{R},\mathbb{R})$ defined by

$$L_\Phi(w_1,w_2) := \sum_{h,k} \left(\frac{d^h w_1}{dt^h}\right)^\top \Phi_{h,k} \frac{d^k w_2}{dt^k} .$$

$\Phi \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\zeta,\eta]$ also induces a *quadratic differential form* (QDF) Q_Φ from $\mathfrak{C}^\infty(\mathbb{R},\mathbb{R}^{\mathtt{w}})$ to $\mathfrak{C}^\infty(\mathbb{R},\mathbb{R})$ defined by $Q_\Phi(w) := L_\Phi(w,w)$. Associated to $\Phi \in \mathbb{R}^{\mathtt{w}_1\times\mathtt{w}_2}[\zeta,\eta]$ is its *coefficient matrix* defined by $\widetilde{\Phi} := [\Phi_{i,j}]_{i,j=0,1,2\ldots}$. $\widetilde{\Phi}$ is an infinite matrix with only a finite number of nonzero entries. $\Phi(\zeta,\eta)$ is called *symmetric* if $\Phi(\zeta,\eta) = \Phi(\eta,\zeta)^\top$, or equivalently if $\widetilde{\Phi} = \widetilde{\Phi}^\top$.

The *derivative* of Q_Ψ is the QDF Q_Φ defined by $Q_\Phi(w) := \frac{d}{dt}(Q_\Psi(w))$ for all $w \in \mathfrak{C}^\infty(\mathbb{R},\mathbb{R}^{\mathtt{w}})$; this holds if and only if $\Phi(\zeta,\eta) = (\zeta+\eta)\Psi(\zeta,\eta)$ (see Willems and Trentelman (1998), p. 1710).

Q_Φ is *nonnegative along* $\mathfrak{B} \in \mathcal{L}^{\mathtt{w}}$, denoted by $Q_\Phi \overset{\mathfrak{B}}{\geq} 0$ if $Q_\Phi(w) \geq 0$ for all $w \in \mathfrak{B}$; and *positive along* $\mathfrak{B}$, denoted by $Q_\Phi \overset{\mathfrak{B}}{>} 0$, if $Q_\Phi \overset{\mathfrak{B}}{\geq} 0$ and $[Q_\Phi(w) = 0] \implies [w = 0]$. If $\mathfrak{B} = \mathfrak{C}^\infty(\mathbb{R},\mathbb{R}^{\mathtt{w}})$, then we call Q_Φ simply *nonnegative*, respectively *positive.* Algebraic characterisations of such properties are on pp. 1712-1713 of Willems and Trentelman (1998).

2.5 Dissipative linear differential behaviors

Let $\mathfrak{B} \in \mathcal{L}^{\mathtt{w}}$ be controllable and let $\Phi \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\zeta,\eta]$. $\mathfrak{B}$ is called Φ-*dissipative* if for all $w \in \mathfrak{B} \cap \mathfrak{D}(\mathbb{R},\mathbb{R}^{\mathtt{w}})$ it holds that $\int_{-\infty}^{\infty} Q_\Phi(w)dt \geq 0$. The QDF Q_Φ is called a *supply rate.* A QDF Q_Ψ is a *storage function* for $\mathfrak{B}$ with respect to a supply rate Q_Φ if $\frac{d}{dt}Q_\Psi(w) \leq Q_\Phi(w)$ for all $w \in \mathfrak{B}$. A QDF Q_Δ is a *dissipation function* for $\mathfrak{B}$ with respect to Q_Φ if $Q_\Delta(w) \geq 0$ for all $w \in \mathfrak{B}$ and $\int_{-\infty}^{\infty} Q_\Phi(w)dt = \int_{-\infty}^{\infty} Q_\Delta(w)dt$ for all $w \in \mathfrak{B} \cap \mathfrak{D}(\mathbb{R},\mathbb{R}^{\mathtt{w}})$.

Q_Φ, Q_Ψ and Q_Δ are related to each other through the *dissipation equality* (see Trentelman and Willems (1997), Th. 4.3.): $\frac{d}{dt}Q_\Phi = Q_\Phi - Q_\Delta$. If $\mathfrak{B} = \mathfrak{C}^\infty(\mathbb{R},\mathbb{R}^{\mathtt{w}})$, such equality holds true if and only if

$$(\zeta+\eta)\Psi(\zeta,\eta) = \Phi(\zeta,\eta) - \Delta(\zeta,\eta) .$$

It follows from Prop. 5.2 of Willems and Trentelman (1998) that the inequality $\int_{-\infty}^{\infty} Q_\Phi(w)dt \geq 0$ is equivalent with the condition $\Phi(-j\omega, j\omega) \geq 0 \;\forall\; \omega \in \mathbb{R}$. Consequently a dissipation function can be computed by factorising $\Phi(-s,s) = D(-s)^\top D(s)$ with $D \in \mathbb{R}^{\bullet\times\mathtt{w}}[s]$ and defining $\Delta(\zeta,\eta) := D(\zeta)^\top D(\eta)$.

3. HIGHER-ORDER LINEAR PORT-HAMILTONIAN BEHAVIORS

Let $E_x, F_x \in \mathbb{R}^{n_x\times m}[s]$, $E_p, F_p \in \mathbb{R}^{n_p\times m}[s]$, $E_r, F_r \in \mathbb{R}^{n_r\times m}[s]$. Such polynomial matrices induce polynomial differential operators acting on free trajectories $\ell \in \mathfrak{C}^\infty(\mathbb{R},\mathbb{R}^m)$ that define the following *effort-* and *flow variables*:

$$e_x := E_x\left(\frac{d}{dt}\right)\ell\ ,\ e_p := E_p\left(\frac{d}{dt}\right)\ell\ ,\ e_r := E_r\left(\frac{d}{dt}\right)\ell$$
$$f_x := F_x\left(\frac{d}{dt}\right)\ell\ ,\ f_p := F_p\left(\frac{d}{dt}\right)\ell\ ,\ f_r := F_r\left(\frac{d}{dt}\right)\ell \tag{3}$$

We call e_x and f_x the *state* effort and flow variables, e_p and f_p the *port* effort and flows, and e_r and f_r the *resistive* effort and flow variables. Define

$$M_e(s) := \mathrm{col}\left(E_x(s), E_p(s), E_r(s), F_x(s), F_p(s), F_r(s)\right)\ ; \tag{4}$$

then we call $\mathfrak{B}_e = \mathrm{im}\left(M_e\left(\frac{d}{dt}\right)\right)$ the *efforts and flows behaviour*. The *projection* $\pi_r(\mathfrak{B}_e)$ of $\mathfrak{B}_e$ on the resistive variables is defined by

$$\begin{aligned}&\pi_r : \mathfrak{B}_e \to \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{2n_r})\\ &\pi_r\left(\mathrm{col}\left(e_x, e_p, e_r, f_x, f_p, f_r\right)\right) := \mathrm{col}\left(e_r, f_r\right)\ .\end{aligned} \tag{5}$$

The projections on the port variables are defined analogously. Such maps define the *resistive-* and *port behaviours* $\mathfrak{B}_r := \pi_r\left(\mathfrak{B}_e\right)$ and $\mathfrak{B}_p := \pi_p\left(\mathfrak{B}_e\right)$, respectively.

We assume that $\mathfrak{B}_e$ satisfies *constitutive relations* induced by bilinear- or quadratic functionals of the efforts and flows. Define

$$J_e := \begin{bmatrix} 0 & I_{n_x+n_p+n_r} \\ I_{n_x+n_p+n_r} & 0\end{bmatrix}$$

The *power relation* is

$$\mathrm{col}\left(e_x^1, e_p^1, e_r^1, f_x^1, f_p^1, f_r^1\right)^\top J_e\, \mathrm{col}\left(e_x^2, e_p^2, e_r^2, f_x^2, f_p^2, f_r^2\right) = 0\ , \tag{6}$$

for all $\mathrm{col}\left(e_x^i, e_p^i, e_r^i, f_x^i, f_p^i, f_r^i\right) \in \mathfrak{B}_e$, $i = 1, 2$. The *storage relation* is

$$e_x^{1\top} f_x^2 + f_x^{1\top} e_x^2 = -\frac{d}{dt}\left(e_x^{1\top} e_x^2\right)\ , \tag{7}$$

for all $\mathrm{col}\left(e_x^i, f_x^i\right) \in \pi_x\left(\mathfrak{B}_e\right)$, $i = 1, 2$. The *dissipative relation* is

$$e_r^{1\top} f_r^2 + f_r^{1\top} e_r^2 \leq 0\ , \tag{8}$$

for all $\mathrm{col}\left(e_r^i, f_r^i\right) \in \pi_r\left(\mathfrak{B}_e\right)$, $i = 1, 2$; and the *anti-dissipative relation* is

$$e_r^{1\top} f_r^2 + f_r^{1\top} e_r^2 \geq 0\ , \tag{9}$$

for all $\mathrm{col}\left(e_r^i, f_r^i\right) \in \pi_r\left(\mathfrak{B}_e\right)$, $i = 1, 2$. If $\mathfrak{B}_e$ satisfies (6)-(7) and one or both of (8) and (9), we call it a *dissipative-*, respectively *anti-dissipative port-Hamiltonian* behavior.

In the following example we illustrate a modelling procedure for flows an efforts as in (3) using first principles.

Example 1. Consider the following impedance:

$$Z_1(s) := \frac{v_1(s)}{i_1(s)} = \frac{2s^2 + 0.2s + 100}{10s + 1}\ .$$

To unveil the port-Hamiltonian structure of a circuit with such impedance, we use the Brune synthesis (see Wing (2008), Ch. 7). Such procedure enables modelling of flows and efforts directly in higher-order terms, using fundamental physical principles. We illustrate the steps of the *Brune synthesis* in Fig. 1; the procedure consists in removing poles/zeros at infinity, equivalently in removing series inductors and shunt capacitors. We thus obtain a circuit with impedance $Z_1(s)$, and we also obtain in intermediate stages the following impedances as remainders:

$$Z_2(s) = \frac{v_2(s)}{i_1(s)} = \frac{100}{10s+1}\ ;\ \ Z_3(s) = \frac{v_2(s)}{i_2(s)} = 100\ .$$

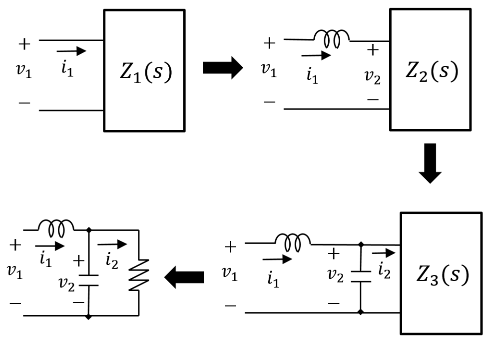

Fig. 1. Circuit synthesis of impedance $Z_1(s)$.

$Z_1(s)$, $Z_2(s)$ and $Z_3(s)$ can be represented in image form (see Willems and Trentelman (2002), Sec. VI) i.e.

$$\begin{bmatrix} i_1 \\ v_1\end{bmatrix} = \begin{bmatrix} 10\frac{d}{dt} + 1 \\ 2\frac{d^2}{dt^2} + 0.2\frac{d}{dt} + 100\end{bmatrix} i_2\ ;$$

$$\begin{bmatrix} i_1 \\ v_2\end{bmatrix} = \begin{bmatrix} 10\frac{d}{dt} + 1 \\ 100\end{bmatrix} i_2\ ;\ \ \begin{bmatrix} i_2 \\ v_2\end{bmatrix} = \begin{bmatrix} 1 \\ 100\end{bmatrix} i_2\ .$$

Using these equations and following the traditional physical definition of flows and efforts for electric and magnetic components (see App. B of van der Schaft and Jeltsema (2014)), we obtain the following set of variables as in (3)

$$\begin{aligned}
e_p &:= v_1 = \left(2\frac{d^2}{dt^2} + 0.2\frac{d}{dt} + 100\right) i_2\ ;\\
f_p &:= i_1 = \left(10\frac{d}{dt} + 1\right) i_2\ ;\\
e_x &:= \begin{bmatrix} i_1 \\ v_2\end{bmatrix} = \begin{bmatrix} 10\frac{d}{dt} + 1 \\ 100\end{bmatrix} i_2\ ;\\
f_x &:= \begin{bmatrix} v_1 - v_2 \\ i_1 - i_2\end{bmatrix} = \begin{bmatrix} 2\frac{d^2}{dt^2} + 0.2\frac{d}{dt} \\ 10\frac{d}{dt}\end{bmatrix} i_2\ ;\\
e_r &:= v_2 = 100 i_2\ ;\\
f_r &:= i_2\ .
\end{aligned}$$

It is a matter of straightforward verification that such effort- and flow- variables satisfy the relations (6)-(8).

4. FLOW- AND EFFORT VARIABLES

Our first result follows directly from (6)-(8).

Proposition 1. Assume that $\mathfrak{B}_e$ satisfies (6)-(8). Then its port behavior $\mathfrak{B}_p = \pi_p\left(\mathfrak{B}_e\right)$ is dissipative with respect to the supply rate induced by

$$J_p := \begin{bmatrix} 0 & I_{n_p} \\ I_{n_p} & 0\end{bmatrix}\ .$$

The functionals $e_x^\top e_x$ and $e_r^\top f_r + f_r^\top e_r$ are respectively a storage- and associated dissipation function for $\pi_p\left(\mathfrak{B}_e\right)$.

Proof. It follows from the constitutive relations that $e_p^\top f_p + f_p^\top e_p = \frac{d}{dt}\left(e_x^\top e_x\right) - \left(e_r^\top f_r + f_r^\top e_r\right)$ for all trajectories in $\mathfrak{B}_e$, and that (8) holds. Consequently $\mathfrak{B}_p$ is J_p-dissipative. The rest of the claim is straightforward.

Analogously, if $\mathfrak{B}_e$ satisfies (6)-(7) and (9), then $\mathfrak{B}_p$ is J_p-*anti-dissipative*, meaning that $e_p^\top f_p + f_p^\top e_p = \frac{d}{dt}\left(e_x^\top e_x\right) - e_r^\top f_r - f_r^\top e_r$ (equivalently, $\mathfrak{B}_p$ is $(-J_e)$-dissipative).

We now show that e_x is a linear function of the state of $\mathfrak{B}_p$ and that under suitable assumptions on the input cardinality of $\mathfrak{B}_p$, it is a state variable for $\mathfrak{B}_p$.

Proposition 2. Let $\mathfrak{B}_e$ be a port-Hamiltonian behavior, and let x be a minimal state variable for $\mathfrak{B}_p$. Then e_x is a linear function of x, i.e. there exists $L \in \mathbb{R}^{n_x \times \mathtt{n}(\mathfrak{B}_p)}$ such that $e_x = Lx$.

If $m(\mathfrak{B}_p) = n_p$, then $E_x\left(\frac{d}{dt}\right)$ is a state map for $\mathfrak{B}_p$, and L has full column rank.

Proof. Let $X\left(\frac{d}{dt}\right)$ be a minimal state map for $\mathfrak{B}_p$ acting on ℓ and producing x. Since $e_x^\top e_x$ is a storage function for $\mathfrak{B}_p$, there exists $K = K^\top \in \mathbb{R}^{\mathtt{n}(\mathfrak{B}_p)\times\mathtt{n}(\mathfrak{B}_p)}$, $K \geq 0$, such that $E_x(\zeta)^\top E_x(\eta) = X(\zeta)^\top K X(\eta)$ (see Th. 5.5 of Willems and Trentelman (1998)). Conclude that $\widetilde{E_x}^\top \widetilde{E_x} = \widetilde{X}^\top K \widetilde{X}$, with $\widetilde{X}$ of full row rank because x is minimal. Factorize $K = F^\top F$ in a rank-revealing way; it follows that $\left(\widetilde{X}^\top F^\top\right)\left(F\widetilde{X}\right)$ is a rank-revealing factorization of $\widetilde{E_x}^\top \widetilde{E_x}$. It follows that $\mathrm{rowspan}_{\mathbb{R}}\ \widetilde{E_x} = \mathrm{rowspan}_{\mathbb{R}}\ F\widetilde{X} \subseteq \mathrm{rowspan}_{\mathbb{R}}\ \widetilde{X}$ and the first claim (choose $L = F$).

We prove the second part of the claim. An argument analogous to that in the proof of (4) $\iff$ (7) of Th. 6.4 p. of Willems and Trentelman (1998) shows that since $X(\zeta)^\top K X(\eta)$ induces a nonnegative storage function and $m(\mathfrak{B}_p) = n_p = \sigma_+(J_p)$, K is not only semi-definite positive, but also positive-definite. It follows that $\mathrm{rowspan}_{\mathbb{R}}\ F\widetilde{X} = \mathrm{rowspan}_{\mathbb{R}}\ \widetilde{X} = \mathrm{rowspan}_{\mathbb{R}}\ \widetilde{E_x}$, which is equivalent with $\mathrm{rowspan}_{\mathbb{R}}\ X(s) = \mathrm{rowspan}_{\mathbb{R}}\ E_x(s)$. The claim on L having full column rank follows from such equality.

We prove an important consequence of (7).

Proposition 3. Let $\mathfrak{B}_e$ be a port-Hamiltonian behavior. Let x be a minimal state map for $\mathfrak{B}_p = \pi_p(\mathfrak{B}_e)$, and let $L \in \mathbb{R}^{n_x \times n(\mathfrak{B}_p)}$ be as in Prop. 2. Let $L^\perp \in \mathbb{R}^{n_x \times (n_x - \mathrm{rank}(L))}$ be a basis matrix for $\mathrm{im}(L)$.

There exists $G_x \in \mathbb{R}^{(n_x - \mathrm{rank}(L))\times m}[s]$ such that $F_x(s) + sE_x(s) = L^\perp G_x(s)$.

Proof. From the storage relation conclude that for every $\mathrm{col}(e_x, f_x) \in \pi_x(\mathfrak{B}_e)$ it holds that $\left(f_x + \frac{d}{dt}e_x\right)^\top e_x = 0$. From $e_x = Lx$ and the minimality of x conclude that for every $\mathrm{col}(e_x, f_x) \in \pi_x(\mathfrak{B}_e)$ it holds that $\left(f_x(0) + L\frac{d}{dt}x(0)\right)^\top L = 0$. Let $\mathrm{col}(V_1, V_2)$ be a basis matrix for the set of all $\mathrm{col}(v_1, v_2) \in \mathbb{R}^{2n_x}$ for which there exists $\mathrm{col}(e_x, f_x) \in \pi_x(\mathfrak{B}_e)$ such that $\mathrm{col}\left(\frac{d}{dt}e_x(0), f_x(0)\right) = \mathrm{col}(v_1, v_2)$. Then $L^\top V_1 + L^\top V_2 = 0$, from which it follows that there exists $H \in \mathbb{R}^{\bullet\times\bullet}$ such that $V_2 = -V_1 + L^\perp H$. Denote by $V_{j,k}$ the k-th column of V_j, $j = 1, 2$, and by ℓ_k an auxiliary variable trajectory such that $\frac{d}{dt}E_x\left(\frac{d}{dt}\right)\ell_k(0) = V_{1,k}$ and $F_k\left(\frac{d}{dt}\right)\ell_k(0) = V_{2,k}$. It is straightforward to see that there exists $G \in \mathbb{R}^{(n_x - \mathrm{rank}(L))\times m}[s]$ such that $G\left(\frac{d}{dt}\right)\ell_k(0) = H_k$, the k-th column of H. Conclude that for every $\ell \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^m)$ it holds that $\frac{d}{dt}E_x\left(\frac{d}{dt}\right)\ell(0) + F_x\left(\frac{d}{dt}\right)\ell(0) = L^\perp G\left(\frac{d}{dt}\right)\ell(0)$; this yields the claim.

In physical port-Hamiltonian systems the relation between e_x and f_x is often differential or integral in nature, i.e. $G(s) = 0$ in Prop. 3. In the rest of the paper we will assume that this is the case.

Now define $\mathcal{V}_r$ to be the set consisting of all $\mathrm{col}(v_1, v_2) \in \mathbb{R}^{2n_r}$ for which there exists $\mathrm{col}(e_r, f_r) \in \pi_r(\mathfrak{B}_e)$ such that $\mathrm{col}(e_r(0), f_r(0)) = \mathrm{col}(v_1, v_2)$. Let $\mathrm{col}(V_1, V_2) \in \mathbb{R}^{2n_r\times\bullet}$ be a basis matrix for $\mathcal{V}_r$. If the equality $V_1^\top V_2 = V_2^\top V_1$ holds true for any such basis matrix, we say that $\mathfrak{B}_e$ is *resistively symmetric*. The resistive symmetry condition is implied by the property of *reciprocity* satisfied by e.g. a large class of electrical circuits. We call the resistive effort- and flow variables *faithful* if $e_r(0)^\top f_r(0) + f_r(0)^\top e_r(0) = 0$ implies that $e_r(0) = 0$ and $f_r(0) = 0$.

Proposition 4. Let $\mathfrak{B}_e$ be a port-Hamiltonian behavior. Assume it is resistively symmetric and that the resistive variables are faithful. Then there exists $R = R^\top \in \mathbb{R}^{n_r \times n_r}$, $R \geq 0$ such that $F_r(s) = RE_r(s)$.

Proof. Let $\mathrm{col}(V_1, V_2) \in \mathbb{R}^{2n_r\times\bullet}$ be a basis matrix for $\mathcal{V}_r$. It follows from the properties of dissipativity and resistive symmetry that

$$-V_1^\top V_2 - V_2^\top V_1 = -2V_1^\top V_2 \geq 0\ .$$

We now prove that $V_1^\top V_2$ is nonsingular. Assume by contradiction that there exists $v \neq 0$ such that $V_1^\top V_2 v = 0$; then it also holds that $(V_1 v)^\top (V_2 v) = 0$. Faithfulness implies that $V_1 v = 0$ and $V_2 v = 0$; since $\mathrm{col}(V_1, V_2)$ is a basis matrix, this implies $v = 0$, a contradiction.

It follows that $V_1^\top V_2$ is nonsingular. Factor it as $V_1^\top V_2 = F^\top F$ with F nonsingular. Define $V_i' := V_i F^{-1}$, $i = 1, 2$; then $\mathrm{col}(V_1', V_2')$ is a basis for $\mathcal{V}_r$, and moreover $V_1'^\top V_2' = V_2'^\top V_1' = -I$. It follows that $-V_2' V_2'^\top V_1' = V_2'$.

Define $R := V_2' V_2'^\top$; an argument similar to that used at the end of the proof of Prop. 3 shows that $F_r(s) = -RE_r(s)$.

We call the resistive effort variables *independent* if they are not related to each other by algebraic relations. It is straightforward to prove that if a port-Hamiltonian behavior is resistively symmetric, then the resistive efforts are independent if and only if $\widetilde{E}_r$ is surjective.

From Prop. 1 and Th. 5.5 of Willems and Trentelman (1998) it follows that e_x is a function of a state of $\mathfrak{B}_p$, and e_r is a function of a state and the input of $\mathfrak{B}_p$. If the resistive variables are faithful, then it follows from Prop. 4 that also f_r is a function of the state and input of $\mathfrak{B}_p$. If $G_x(s)$ in Prop. 3 is zero, then also f_x is a function of a state of $\mathfrak{B}_p$. Under such assumptions it follows that the number m of auxiliary variables on which the polynomial differential operators (3) act equals $\mathtt{m}(\mathfrak{B}_p)$, the number of inputs of $\mathfrak{B}_p$. We will assume this to be the case in the rest of this paper.

5. DUAL PORT-HAMILTONIAN BEHAVIORS

We now define a port-Hamiltonian behavior constructed from the dual of $\mathfrak{B}_p$.

5.1 Dual linear differential systems

$\mathfrak{B}_p$ is J_p-*strictly dissipative* if for all $\mathrm{col}(e_p, f_p) \in \mathfrak{B}_p \cap \mathcal{D}\left(\mathbb{R}, \mathbb{R}^{2n_p}\right)$ it holds that

$$\int_{-\infty}^{+\infty} e_p^\top f_p + f_p^\top e_p \; dt = 0 \implies \mathrm{col}(e_p, f_p) = 0 \; .$$

The following is an algebraic characterization of strict dissipativity.

Proposition 5. Let $\mathfrak{B}_e$ be a port-Hamiltonian behavior. The following statements are equivalent:

(1) $\mathfrak{B}_p$ is strictly dissipative;
(2) For all $\mathrm{col}\left(e_x, e_p, e_r, f_x, f_p, f_r\right) \in \mathfrak{B}_e$ of compact support

$$\int_{-\infty}^{+\infty} e_r^\top f_r + f_r^\top e_r \; dt = 0 \Longrightarrow \mathrm{col}(e_p, f_p) = 0 \; ;$$

(3) (a) $E_p(-i\omega)^\top F_p(i\omega) + F_p(-i\omega)^\top E_p(i\omega) \geq 0 \; \forall\, \omega \in \mathbb{R}$;
(b) $\det\left(E_p(-s)^\top F_p(s) + F_p(-s)^\top E_p(s)\right) \neq 0$;
(4) (a) $E_r(-i\omega)^\top F_r(i\omega) + F_r(-i\omega)^\top E_r(i\omega) \geq 0 \; \forall\, \omega \in \mathbb{R}$;
(b) $\det\left(E_r(-s)^\top F_r(s) + F_r(-s)^\top E_r(s)\right) \neq 0$;
(5) There exists $G_p \in \mathbb{R}^{m \times m}[s]$ nonsingular such that

$$E_p(-s)^\top F_p(s) + F_p(-s)^\top E_p(s) = G_p(-s)^\top G_p(s) \; .$$

(6) There exists $G_r \in \mathbb{R}^{m \times m}[s]$ nonsingular such that

$$E_r(-s)^\top F_r(s) + F_r(-s)^\top E_r(s) = -G_r(-s)^\top G_r(s) \; .$$

Proof. To prove the equivalence of (1) and (2), observe that since $\mathfrak{B}_e$ is port-Hamiltonian, for all its trajectories $e_p^\top f_p + f_p^\top e_p = \frac{d}{dt}\left(e_x^\top e_x\right) - e_r^\top f_r - f_r^\top e_r$. For each such trajectory of compact support, integrate both sides of the equality, obtaining $\int_{-\infty}^{+\infty} e_p^\top f_p + f_p^\top e_p \; dt = -\int_{-\infty}^{+\infty} e_r^\top f_r + f_r^\top e_r \; dt$. The claim follows.

The equivalence of (1) and (3) follows from statement (*ii*) of Prop. 5.2 in Willems and Trentelman (1998).

The equivalence of (3) and (4) is straightforward using the calculus of BDFs. The equivalence of (3) and (5) follows from standard results on symmetric factorization of para-Hermitian polynomial matrices nonnegative on the imaginary axis (see e.g. Coppel (1972)). The equivalence of (5) and (6) follows from the two-variable version, $\Phi(\zeta, \eta) := M_e(\zeta)^\top J_e M(\eta) = 0$, of the power relation by substitution of $-s$ in place of ζ and s in place of η.

Let $\mathfrak{B}_e$ be port-Hamiltonian and assume that $\mathtt{m}(\mathfrak{B}_p) = \sigma_+(J_p) = n_p$; then $M_p(s) = \mathrm{col}\left(E_p(s), F_p(s)\right) \in \mathbb{R}^{2n_p \times n_p}[s]$. Let $R_p \in \mathbb{R}^{n_p \times 2n_p}[s]$ induce a minimal kernel representation of $\mathfrak{B}_p = \mathrm{im}\; M_p\left(\frac{d}{dt}\right)$. The behavior $\mathfrak{B}_p' := \mathrm{im}\; J_p R_p\left(-\frac{d}{dt}\right)^\top$ is called the J_p-*dual* of $\mathfrak{B}_p$ (see sect. 10 of Willems and Trentelman (1998)). The following result holds.

Proposition 6. Let $X\left(\frac{d}{dt}\right)$ be a minimal state map for $\mathfrak{B}_p$. There exists a minimal state map $Z\left(\frac{d}{dt}\right)$ for $\mathfrak{B}_p'$ such that

$$R_p(-\zeta)^\top M_p(\eta) = (\zeta + \eta) Z(\zeta)^\top X(\eta) \; . \tag{10}$$

Proof. Follows from Prop. 10.1 of Willems and Trentelman (1998).

A state map $Z\left(\frac{d}{dt}\right)$ for $\mathfrak{B}_p'$ satisfying (10) is called *matched* with $X\left(\frac{d}{dt}\right)$.

Proposition 7. Let $\mathfrak{B}_e$ be port-Hamiltonian. Assume that $\mathtt{m}(\mathfrak{B}_p) = \sigma_+(J_p) = n_p$ and that $\mathfrak{B}_p$ is strictly J_p-dissipative. Then $\mathfrak{B}_p'$ is strictly $(-J_p)$-dissipative.

Let $X\left(\frac{d}{dt}\right)$ be a minimal state map for $\mathfrak{B}_p$, and let $Z\left(\frac{d}{dt}\right)$ be a matched state map for $\mathfrak{B}_p'$. If $X(\zeta)^\top K X(\eta)$ is a storage function, then K is nonsingular, and the quadratic functional induced by $Z(\zeta)^\top \left(-K\right)^{-1} Z(\eta)$ is a storage function for $\mathfrak{B}_p'$.

Proof. The first claim follows from statement (ii) in Th. 10.2 of Willems and Trentelman (1998). The second claim follows from statement (iv) of Th. 10.2 *ibid.*

In the rest of this section, we assume that $\mathtt{m}(\mathfrak{B}_p) = \sigma_+(J_p) = n_p$, that the resistive efforts are flows are faithful, and that $\mathfrak{B}_p$ is strictly J_p-dissipative. We proceed to define a *dual* port-Hamiltonian system of $\mathfrak{B}_e$ by constructing effort- and flow-variables from the representation of $\mathfrak{B}_p'$ and the effort- and flow-variables of $\mathfrak{B}_e$.

5.2 Dual port efforts and flows

Partition

$$R_p(-s) =: \left[F_p'^\top(s) \;\; E_p'^\top(s)\right] \tag{11}$$

with $E_p', F_p' \in \mathbb{R}^{n_p \times n_p}[s]$. We define the *dual port efforts* by $e_p' := E_p'\left(\frac{d}{dt}\right) \ell'$ and the *dual port flows* by $f_p' := F_p'\left(\frac{d}{dt}\right) \ell'$, with $\ell' \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^m)$.

5.3 Dual state efforts and flows

Let $X\left(\frac{d}{dt}\right)$ be a minimal state map for $\mathfrak{B}_p$, and let $L \in \mathbb{R}^{n_x \times \mathtt{n}(\mathfrak{B}_p)}$ be a full column rank matrix such that $E_x(s) = LX(s)$ (see Prop. 2). Recall that $E_x(\zeta)^\top E_x(\eta) = X(\zeta)^\top L^\top L X(\eta)$ is a positive-definite storage function. Factorize $L^\top L = F^\top F$ with F square, and define $X'(s) := FX(s)$. $X'\left(\frac{d}{dt}\right)$ is also a minimal state map, and $E_x(s) = LF^{-1}X'(s)$. Moreover, $E_x(\zeta)^\top E_x(\eta) = X'(\zeta)^\top X'(\eta)$ is a storage function for $\mathfrak{B}_p$.

Let $Z'\left(\frac{d}{dt}\right)$ be a state map for $\mathfrak{B}_p'$ matched with $X'\left(\frac{d}{dt}\right)$; since $\mathtt{m}(\mathfrak{B}_p) = \sigma_+(J_p)$, it follows from Prop. 7 that $-Z'(\zeta)^\top Z'(\eta) = Z'(\zeta)^\top(-I)Z'(\eta)$ is a storage function for $\mathfrak{B}_p'$. Define the *dual state efforts* matrix by

$$E_x'(s) := LF^{-1}Z'(s) \; . \tag{12}$$

It follows from the series of equalities $-E_x'(\zeta)^\top E_x'(\eta) = -Z'(\zeta)^\top F^{-\top} L^\top L F^{-1} Z'(\eta) = -Z'(\zeta)^\top Z'(\eta)$ that the polynomial matrix $-E_x'(\zeta)^\top E_x'(\eta)$ induces a storage function for $\mathfrak{B}_p'$.

The *dual state flows* f_x' are defined by

$$F_x'(s) := -sE_x'(s) \; . \tag{13}$$

5.4 Dual resistive efforts and flows

From the fact that $-E_x'(\zeta)^\top E_x'(\eta)$ is a storage function for $\mathfrak{B}_p'$ and from Prop. 7 it follows that there exists a semi-definite negative $\Delta' \in \mathbb{R}^{n_p \times n_p}[\zeta, \eta]$ such that

$$R(-\zeta) J_p R(-\eta)^\top - (\zeta + \eta) E_x'(\zeta)^\top E_x'(\eta) = \Delta'(\zeta, \eta) \; . \tag{14}$$

The result of Prop. 5 implies that there exists $G'_r \in \mathbb{R}^{n_p\times n_p}[s]$ nonsingular such that $\Delta'(\zeta,\eta) = -G'_r(\zeta)^\top G'_r(\eta)$. Now denote the dissipation function corresponding to the storage function $E_x(\zeta)^\top E_x(\eta)$ of $\mathfrak{B}_p$ by

$$\Delta(\zeta,\eta) := E_p(\zeta)^\top F_p(\eta)+F_p(\zeta)^\top E_p(\eta)-(\zeta+\eta)E_x(\zeta)^\top E_x(\eta) .$$

Since $\mathfrak{B}_p$ is strictly J_p-dissipative, from Prop.5 it follows that $\Delta(\zeta,\eta)$ admits a factorization $G_r(\zeta)^\top G_r(\eta)$ with $G_r \in \mathbb{R}^{n_p\times n_p}[s]$ nonsingular. Use Prop. 4 to conclude that there exists $R \geq 0$ such that $\Delta(\zeta,\eta) = E_r(\zeta)^\top R E_r(\eta)$. Consequently, the coefficient-matrix equality $\widetilde{E}_r^\top R\widetilde{E}_r = \widetilde{G}_r^\top \widetilde{G}_r$ holds. Note that $\widetilde{G}_r$ has full row-rank, since $G_r(s)$ is nonsingular. Define $N := \widetilde{G}_r^\top \left(\widetilde{G}_r\widetilde{G}_r^\top\right)^{-1} \in \mathbb{R}^{\bullet\times m(\mathfrak{B}_p)}$; then $N^\top \widetilde{E}_r^\top R\widetilde{E}_r N = I_{m(\mathfrak{B}_p)}$.

The *dual resistive efforts* e'_r are defined by

$$E'_r(s) := \widetilde{E}_r N G'_r(s) . \tag{15}$$

The *dual resistive flows* f'_r are defined by

$$R' := R \text{ and } F'_r(s) := R'E'_r(s) . \tag{16}$$

The following is the main result of this paper.

Theorem 8. Assume that $\mathtt{m}(\mathfrak{B}_p) = \sigma_+(J_p) = n_p$, that $\mathfrak{B}_p$ is strictly J_p-dissipative, and that the resistive flows and efforts are faithful. Define

$$M'_e(s) := \operatorname{col}\left(E'_x(s), E'_p(s), E'_r(s), F'_x(s), F'_p(s), F'_r(s)\right) , \tag{17}$$

where E'_p, F'_p are defined in (11), F'_x is defined in (13), E'_r is defined in (15), and F'_r is defined in (16).

$\mathfrak{B}'_e = \operatorname{im}\ M'_e\left(\frac{d}{dt}\right)$ is an anti-dissipative port-Hamiltonian behaviour.

Proof. Use the definitions (15) and (16) of $E'_r(s)$ and of R' to conclude that the anti-dissipation function $\Delta'(\zeta,\eta)$ corresponding to the storage function $E'_x(\zeta)^\top E'_x(\eta)$ can be written as $\Delta'(\zeta,\eta) = -G'_r(\zeta)^\top G'_r(\eta) = -E'_r(\zeta)^\top R'E'_r(\eta) = F'_r(\zeta)^\top E'_r(\eta) + E'_r(\zeta)^\top F'_r(\eta)$.

The power relation for $\mathfrak{B}'_e$ follows from (14). The storage- and anti-dissipation relations are also satisfied. The claim is proved.

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