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The dynamics of coupled structures possessing inhomogeneous attachments

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Abstract

Structures with hydraulic pipes and cable bundles attached to them are commonly found in engineering, therefore it is of interest to determine the effects of such attachments on the dynamics of the host structures and quantify them. Moreover, manufacturing processes often lead to deviations from the original design. One way to take these deviations into account is to consider small variations in the mechanical properties and/or geometry of the structures. This paper models the attachments as slowly varying inhomogeneous beams and uses the WKB (Wentzel, Kramers and Brillouin) approximation and wave propagation to derive the input and transfer mobilities of said beams. The properties of the beam are described as a random field and are expressed using the Karhunen-Loève expansion. Once the mobilities are known, a mobility approach can be used to couple the attachments to a host structure. Rigid connections are considered along with variations of the spacing between the attachments.

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Keywords: Mobility; Uncertainties; Inhomogeneous structures; WKB

1. Introduction

Structures comprising a one-dimensional vibration waveguide attached to a host-structure are commonly found in engineering. The one-dimensional waveguide can be flexible structures such as cable bundles, hydraulic pipes, reinforcement ribs etc., whilst the host structure can be an aircraft fuselage, a satellite body or a car body in white. To an extent, it is possible represent and model these assemblies as beams attached to plates to better understand how they behave when attached together through a set of point connections. It is also known that manufacturing processes produces variability in the nominal properties of the materials and that these deviations lead to different dynamic responses of nominally identical components [1].

Traditionally, design engineers frequently consider these attachments, the one-dimensional waveguide, as simply lumped masses at the attachment points, which is not accurate enough to predict all of the structural dynamic interaction of structures or determining when this interaction is relevant [2]. A wave propagation approach is used

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herein to describe the attachments and determine their mobility expressions. Once the input and the transfer mobilities are known, it is possible to couple the beam to the plate.

Another application is in the aerospace industry, which is always aiming to use lighter materials. Power and signal cable harnesses are often 10%, but they could be as much as 30% of the total mass of a spacecraft. Despite rarely being considered in the design phases, aerospace cables have a more appreciable effect on the dynamics of the structure, especially as the number of the electronic components onboard increase both in quantity and power demand. The current standard method of modelling cable bundles as lumped masses at attachment locations is not sufficiently accurate. A model able to predict the effects of cable harnesses has many benefits, especially in increasing the confidence in vibration control systems [2]. It was also highlighted that the effects on the structure are sensitive to relatively small changes in the cable properties. Therefore, it would be wiser to use the specific cable properties that will be coupled in the real structure when modelling [3], which are not always possible to know beforehand. This paper aims to get around this issue introducing spatially slowly varying properties and uncertainty for these attachments.

2. Modelling

In order to couple the structures, a mobility approach formulated in a matrix form [4] is used. Mobility is in general a complex function in the frequency domain and is defined as the ratio of velocity to force as a function of frequency. In a three-dimensional system, one can typically measure (or estimate) the translational velocity in the three coordinate directions, X, Y, and Z, and one can also measure the rotational response [5]. In this paper, only the coupled translational velocities are considered. The mobility method was chosen as it is a straightforward tool to analyse linear mechanical systems under periodic, transient or random loads. The mobility (or impedance) approach was developed so that the dynamic behaviour, in the frequency domain, of both a source and receiver of vibration are coupled in a manner analogous to what electrical engineers use for circuit analysis [5].

The method for coupling a beam to a plate using matrices when rigid links are considered can be found in [6] and written as:

$$\dot{\boldsymbol{w}}^p = \boldsymbol{Y}^p(\boldsymbol{f} - \boldsymbol{f}') \tag{1}$$

$$\boldsymbol{f}' = (\boldsymbol{Y}^p + \boldsymbol{Y}^b)^{-\mathbf{1}}\boldsymbol{Y}^p\boldsymbol{f} \tag{2}$$

where $\dot{\boldsymbol{w}}^p$ is the velocity vector of the plate with a beam attachment, $\boldsymbol{Y}^p$ is the mobility matrix of the plate, $\boldsymbol{f}$ is the vector of external forces applied to the plate, $\boldsymbol{f}'$ is the vector of transmitted or internal forces through the connection points to the beam and $\boldsymbol{Y}^b$ is the mobility matrix of the attached beam. The individual terms of the mobility matrices for homogeneous infinite beams and infinite plates are given in [4].

2.1. The application of the Wentzel–Kramers–Brillouin (WKB) approximation to evaluate mobilities

This method, initially developed in order to solve the Schrödinger equation, is named after Wentzel, Kramers and Brillouin. It is used for finding suitable modifications of plane-wave solutions for propagation in media varying slowly when compared to the wavelength [1] [7]. The fundamental assumption of this formulation is that the properties of the waveguide are varying slowly enough and do not lead to reflections due to local impedance changes or that they can be neglected, even if the net change is large [1] [8]. If the travelling wave reaches a local cut-off, or cut-on, region, the WKB approximation fails. These are known as turning points and they lead to internal reflection, when the main assumption of the method breaks down [1].

The WKB approximation is used to find the mobility of an infinite beam with slowly varying properties. Starting from the equation of motion for a beam in bending:

$$\frac{\partial^2}{\partial x}\left(EI(x)\frac{\partial^2 w(x,t)}{\partial x^2}\right)+\rho A(x)\frac{\partial^2 w(x,t)}{\partial t^2}=0 \tag{3}$$

where $EI(x)$ and $\rho A(x)$ are the bending stiffness and mass per unit length and $w(x,t)$ the flexural displacement

Assuming a harmonic response $w(x,t)=W(x)e^{i\omega t}$ and an eikonal solution $S(x)=\ln\widetilde{W}(x)-i\theta(x)$, it is possible to write $W(x)=e^{S(x)}=\widetilde{W}(x)e^{-i\theta(x)}$, which leads to:

$$\frac{\partial^2}{\partial x}\left(EI(x)\frac{\partial^2 \widetilde{W}(x)e^{-i\theta(x)}}{\partial x^2}\right)-\omega^2\rho A(x)\widetilde{W}(x)e^{-i\theta(x)}=0 \tag{4}$$

One can then calculate the derivatives and the expressions can be divided by $e^{-i\theta(x)}$. Considering the real and imaginary parts have to be equal to zero, neglecting the higher order terms of the real part and dividing them by $\omega^2\rho A(x)\widetilde{W}(x)$, it is possible to write the change of phase $\theta(x)$ between arbitrary points $x=x_1$ and $x=x_2$ as:

$$-1+\frac{EI(x)}{\omega^2\rho A(x)}\left[\frac{\partial\theta(x)}{\partial x}\right]^4=0\ \Rightarrow\ \theta(x)=\int\left[\frac{\rho A(x)}{EI(x)}\right]^{\frac{1}{4}}\sqrt{\omega}dx\ \equiv\int k_B(x)dx \tag{5}$$

where $k_B(x)$ is the local bending wavenumber along the beam at position x. $\theta(x)$ corresponds to the phase change.

The input and transfer mobilities of the slowly varying beam under a point harmonic load are then expressible using a wave propagation method.

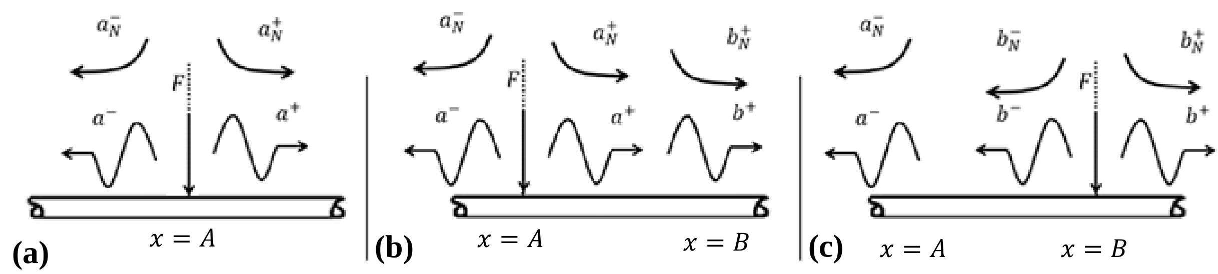

Fig. 1. Infinite beam under a point excitation and the propagating waves and near field waves. (a) is for the input mobility, (b) is for the transfer mobility from $x=A$ to $x=B$ and (c) is for the transfer mobility from $x=B$ to $x=A$.

Fig. 1 shows the propagating and near field waves that are necessary for finding the input and transfer mobilities applying the method derived in [1], but applying it for an infinite beam. From the force-displacement relationship for a beam excited by a harmonic load, one can find the corresponding forces related to these waves:

$$F=2EI(A)\left.\frac{\partial^3\boldsymbol{a}^+(x)}{\partial x^3}\right|_{x=A}=\ 2EI(A)\left.\frac{\partial^3\boldsymbol{a}^-(x)}{\partial x^3}\right|_{x=A} \tag{6}$$

where the wave vectors $\boldsymbol{a}^\pm=\ \{a^\pm\ a_N^\pm\}^T$ are the vectors to the right and left of the excitation point respectively.

The WKB assumption says [9] [10]:

$$\frac{\partial^n a^\pm(x)}{\partial x^n}=[\mp ik_B(x)]^n W^\pm(x);\ \frac{\partial^n a_N^\pm(x)}{\partial x^n}=[\mp k_B(x)]^n W_N^\pm(x) \tag{7}$$

After applying the slope boundary conditions and some algebraic manipulation, the amplitude of the waves are given by:

$$a^{+} = a^{-} = \frac{-iF}{4EI(A)k_B^3(A)};\ a_N^{+} = a_N^{-} = \frac{-F}{4EI(A)k_B^3(A)} \quad (8)$$

The input mobility can then be written as:

$$Y_{AA} = \frac{i\omega}{F}(a^{+} + a_N^{+}) = \frac{i\omega}{F}\frac{-F}{4EI(A)k_B^3(A)}(i+1) = \frac{\omega(1-i)}{4EI(A)k_B^3(A)} \quad (9)$$

For the transfer mobility from A to B, the amplitude of the waves at the point B can be found using $\boldsymbol{b}^{+} = \boldsymbol{\Lambda a}^{+}$, while the transfer mobility from B to A, the amplitude of the waves at the point A can be found using $\boldsymbol{a}^{-} = \boldsymbol{\Lambda b}^{-}$, where the propagation matrix $\boldsymbol{\Lambda}$ is given by:

$$\boldsymbol{\Lambda}_{\mathbf{mn}} = \begin{bmatrix} e^{-i\theta+\gamma} & 0 \\ 0 & e^{-\theta+\gamma} \end{bmatrix}; \theta = \int_{x_m}^{x_n} k_B(x)dx \text{ and } \gamma = \ln\frac{\widetilde{W}(x_n)}{\widetilde{W}(x_m)} \quad (10)$$

where γ is the change in the amplitude of the wave, $\widetilde{W}(x)$ can be found using the imaginary part of Eq. (4), which has a solution in the form of $EI(x)^{-1/8}\rho A(x)^{-3/8}$ and m and n correspond to the indexes for the start and end positions along the beam.

Applying the same procedure of Eq. (9), the transfer mobilities can be written as:

$$Y_{mn} = \frac{\widetilde{W}(x_n)}{\widetilde{W}(x_m)}\frac{\omega}{4EI(x_m)k_B^3(x_m)}\left[e^{-i\int_{x_m}^{x_n} k_B(x)dx} - ie^{-\int_{x_m}^{x_n} k_B(x)dx}\right] = Y_{nm} \quad (11)$$

2.2. Random fields using the Karhunen-Loève Expansion

One useful way to represent deviations that manufacturing process produces is to treat properties as random fields [11], which can be modelled using spatially correlated variability [1]. A usual technique for representing a sample function of a random field is the Karhunen-Loève (KL) expansion [12]. Moreover, the KL is said to be bi-orthogonal, which means that not only the deterministic basis functions are orthogonal, but also the corresponding random coefficients are orthogonal [11]. The KL results in a Gaussian distribution with zero mean and unity standard-deviation. This property allows for the optimal encapsulation of the random field into uncorrelated random variables [11] [13]. In addition, the KL is the optimal expansion in the sense that the mean-square error associated by approximating the infinite series with a finite number of terms is minimized and it is especially suited for strongly correlated random fields, i.e. slowly varying [1]. In this paper, the KL expansion is used to describe a random field for the Young's modulus as $E(x) = E_0[1 + \sigma H(x)]$, which, when small dispersion is considered, leads to:

$$\int k_B(x)dx \approx \sqrt{\omega}\left(\frac{\rho A}{E_0 I}\right)^{1/4}\int\left[1 - \frac{\sigma}{4}H(x)\right]dx \quad (12)$$

σ is the dispersion term that sets the limits of $H(x)$ and $H(x)$ is the random field.

For the domain $-L/2 \le x \le L/2$, it is possible to write, using the details of the KL given in [1]:

$$\int_{x_1}^{x_2} k_B(x)dx \approx \sqrt{\omega}\left(\frac{\rho A}{E_0 I}\right)^{\frac{1}{4}}\left[x + \sum_{j=1}^{N_{kl}}\frac{\alpha_j\xi_{1j}\sigma\cos(w_{1j}x)}{4w_{1j}} - \frac{\beta_j\xi_{2j}(q)\sigma\sin(w_{2j}x)}{4w_{2j}}\right]\Bigg|_{x=x_1}^{x=x_2} \quad (13)$$

Finally, it is possible to use Eq. (13) to rewrite Eq. (11) to find the transfer mobilities of the beam necessary for Eqs. (1) and (2).

3. Numerical results and discussion

Using the model developed in the previous section, numerical results for an infinite beam attached to an infinite plate through 5 rigid connection points are presented. Cases considered include different properties of the KL expansion and random spacing between the connection points. The homogeneous beam and plate are considered to be made of standard steel (0.2% of carbon) and have their nominal mechanical properties. The cross section of the beam is a square of side 20 mm and the thickness of the plate is 2 mm. The regular spacing between the five connection points is 37.5 mm.

Three different KL expansions were considered, with a different number of modes or correlation lengths, the results are shown in Fig. 3. In all three cases, the standard deviation divided by the mean of each case varies between 0.08 and 0.10. Increasing the number of modes in the expansion or setting a smaller correlation length causes the Young's modulus to vary more rapidly along the length of the domain. In this analysis, it is assumed that the variation outside of the analyzed domain is also slow enough such that there are no reflections.

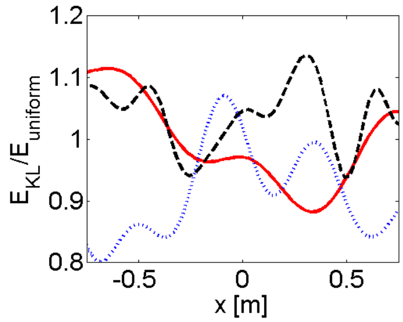

Fig. 3. Different Young's modulus along the length of the beam. Case 1 (—) has a correlation length of 1, 4 modes in the KL expansion and $\sigma = 0.2$. Case 2 (— —) has a correlation length of 1, 8 modes in the KL expansion and $\sigma = 0.2$. Case 3 (······) has a correlation length of 0.2, 4 modes in the KL expansion and $\sigma = 0.2$.

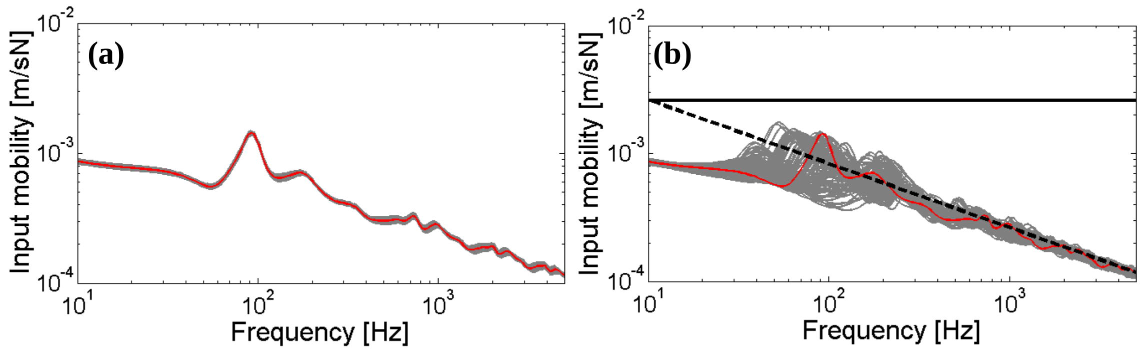

Fig. 2(a) shows the results for the input mobility of 100 slowly varying beams when their Young's moduli are

Fig. 2. Input mobility of the connected infinite beam-plate system. Excitation applied to a central attachment of the beam to the plate. Case 2 in (a) — Uniform beam, — 100 realisations with slowly varying E. In (b), Case 3 and with random spacing between the connections. — Uniform beam, — 100 realisations with slowly varying E, — is the input mobility of the host plate and — — the input mobility of the attached homogeneous beam.

described by the random field for Case 2. On a dB scale, referenced to the case when the attached beam is uniform with the nominal Young's modulus of 200 GPa, there is a difference of less than 1 dB. This behaviour repeats itself when the other two cases are considered. Fig. 2(b) shows the results for the input mobility of 100 slowly varying beams when their Young's moduli are described by the random field of Case 3 along with a random spacing between the connections. The connection points are allowed to vary up to 15% around their equally spaced original positions. A uniform distribution for sampling the random connection points was assumed. The combination of the random spacing along with the inhomogeneous beam, when plotted in a dB scale referenced to the case when the attached beam is a uniform one with the nominal Young's modulus, can differ by up to 10 dB. This is similar to results obtained when only the random spacing is considered [6]. The other two cases for the random field also produce similar results.

4. Conclusions

In this paper, expressions were derived for the input and transfer mobilities of an infinite beam with slowly varying properties. The Wentzel–Kramers–Brillouin approximation used assumes that the properties vary slowly enough along the length of the waveguide in a way that there are no reflections. The expressions were complemented using a Karhunen-Loève Expansion to describe random Young's moduli. Comparisons between when inhomogeneous beams are attached to a host uniform plate versus when a uniform beam is attached were made, considering firstly that such connections are rigid links and equally spaced and then when the spacing between the attachment points are random.

Numerical results for the input mobilities of the connected systems show that small differences occur due to the presence of the random field, as these mobilities are highly dependent only on the driving point properties which are not allowed to vary significantly to keep the WKB approximation valid. In principle, the expressions are straightforward to develop and apply to finite systems, when even small variations would lead to variations in the natural frequencies and resonances, which would then produce a more noticeable effect on the response.

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