ON CONJUGACY SEPARABILITY OF FIBRE PRODUCTS

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Abstract. In this paper we study conjugacy separability of subdirect products of two free (or hyperbolic) groups. We establish necessary and sufficient criteria and apply them to fibre products to produce a finitely presented group G_1 in which all finite index subgroups are conjugacy separable, but which has an index 2 overgroup that is not conjugacy separable. Conversely, we construct a finitely presented group G_2 which has a non-conjugacy separable subgroup of index 2 such that every finite index normal overgroup of G_2 is conjugacy separable. The normality of the overgroup is essential in the last example, as such a group G_2 will always posses an index 3 overgroup that is not conjugacy separable.

Finally, we characterize p-conjugacy separable subdirect products of two free groups, where p is a prime. We show that fibre products provide a natural correspondence between residually finite p-groups and p-conjugacy separable subdirect products of two non-abelian free groups. As a consequence, we deduce that the open question about the existence of an infinite finitely presented residually finite p-group is equivalent to the question about the existence of a finitely generated p-conjugacy separable full subdirect product of infinite index in the direct product of two free groups.

1. Introduction

Let $\mathcal{C}$ be a class of groups. A group G is said to be $\mathcal{C}$-conjugacy separable if one can distinguish its conjugacy classes by looking at the quotients of G in $\mathcal{C}$. More precisely, G is *$\mathcal{C}$-conjugacy separable* for any pair of non-conjugate elements $x, y \in G$ there must exist a group $M \in \mathcal{C}$ and a homomorphism $\varphi : G \to M$ such that $\varphi(x)$ and $\varphi(y)$ are not conjugate in M. In the case when $\mathcal{C}$ is the class of all *finite groups*, we omit the "$\mathcal{C}$-" and simply write that G is *conjugacy separable*; and if $\mathcal{C} = \mathcal{C}_p$ is the class of all *finite p-groups*, for some prime p, we will say that G is *p-conjugacy separable*.

Conjugacy separability is a basic and classical residual property of a group that is closely related to the solvability of the conjugacy problem. It is a natural strengthening of residual finiteness, which corresponds to the solvability of the word problem.

Many groups are known to be residually finite, including all finitely generated linear groups (Mal'cev [38]). Conjugacy separability, though, is a more delicate property. It is often difficult to check whether a given residually finite group is conjugacy separable. Classical examples of conjugacy separable groups include virtually polycyclic groups (Remeslennikov [53], Formanek [24]) and virtually free groups (Dyer [21]). The development of geometric methods in Group Theory prompted a lot of recent progress in this field, and the following classes of groups were found to be conjugacy separable:

- virtually surface groups and Seifert fibered 3-manifold groups (Martino [39]);
- limit groups and, more generally, finitely presented residually free groups (Chagas and Zalesskii [15, 14]);

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- right angled Artin groups (Minasyan [43]);
- most even Coxeter groups (Caprace and Minasyan [13]);
- 1-relator groups with torsion (Minasyan and Zalesskii [46]);
- compact orientable 3-manifold groups (Hamilton, Wilton and Zalesskii [32]);
- virtually compact special (Gromov) hyperbolic groups (Minasyan and Zalesskii [47]).

The most basic example of a residually finite group which is not conjugacy separable is $\mathrm{SL}(n,\mathbb{Z})$ (Remeslennikov [54], Stebe [58]), for $n \geq 3$. The proof of this fact uses the positive solution of the congruence subgroup problem for $\mathrm{SL}(n,\mathbb{Z})$, when $n \geq 3$, which was established by Bass, Lazard and Serre [3]. An example of a non-conjugacy separable finitely presented torsion-free metabelian group was given by Wehrfritz [61].

Presently it is unknown whether all (Gromov) hyperbolic groups are conjugacy separable. In [64] Wise noted that this question is closely linked to the well-known open problem asking if there exists a non-residually finite hyperbolic group.

Conjugacy separability has two main applications:

- a finitely presented conjugacy separable group has solvable conjugacy problem (Mostowski [48]);
- a finitely generated conjugacy separable group G, all of whose pointwise inner automorphisms are inner, has a residually finite outer automorphism group $Out(G)$ (Grossman [29]).

Much like the conjugacy problem, conjugacy separability may not behave well under passing to finite index subgroups and overgroups. Finitely presented conjugacy separable groups with non-conjugacy separable subgroups of any finite index were constructed by Martino and the author in [40]. A finitely generated conjugacy separable group possessing an overgroup of index 2 with unsolvable conjugacy problem was found by Goryaga in [25]. One can show that the overgroup from Goryaga's example is not conjugacy separable, even though this group is not finitely presented (and thus Mostowski's result [48] does not apply to it directly).

In this paper we investigate conjugacy separability of subdirect products of 'non-positively curved' (e.g., free, hyperbolic or acylindrically hyperbolic) groups. Recall that a subgroup $G \leqslant F_1 \times F_2$, of a direct product of two groups F_1, F_2, is called a *subdirect product*, if for each $i \in \{1,2\}$ the image of G under the natural projection $\rho_i : F_1 \times F_2 \to F_i$ is all of F_i: $\rho_i(G) = F_i$. If, in addition, $G \cap F_i \neq \{1\}$ for each $i = 1, 2$, then G is said to be a *full subdirect product* of F_1 and F_2. Note that if G is subdirect in $F_1 \times F_2$ then $G \cap F_i \lhd F_i$, $i = 1, 2$ (see Lemma 2.1).

A standard way for constructing subdirect products is to use *fibre products* (see Subsection 2.3). It provides a streamlined and powerful method for producing groups with exotic behaviour. The original idea belongs to Mihaĭlova [41], who applied the fibre product construction to a finitely presented group with unsolvable word problem to give an example of a finitely generated subgroup $G \leqslant F \times F$, where F is the free group of rank 2, such that the membership problem for G in $F \times F$ is undecidable. In [42] Miller showed that the same group G also has unsolvable conjugacy problem. Our first result shows, in the same spirit, that if one starts with a group which is not residually finite then the corresponding fibre product of free groups will not be conjugacy separable.

Theorem 1.1. *Suppose that $\mathcal{C}$ is a pseudovariety of groups and F_i is either a non-abelian free group (of arbitrary rank) or a non-elementary hyperbolic group without non-trivial finite normal subgroups, $i = 1, 2$. If $G \leqslant F_1 \times F_2$ is a full subdirect product such that F_1/N_1 is not residually-$\mathcal{C}$ (where $N_1 \coloneqq G \cap F_1$) then G is not $\mathcal{C}$-conjugacy separable.*

A special case of Theorem 1.1, when N_1 is finitely generated and $\mathcal{C}$ is the class of all finite groups, was proved by Martino and the author in [40, Prop. 7.6]. The proof of Theorem 1.1 is essentially done by a direct computation and does not require the groups F_1, F_2 or the subgroup $N_1 \lhd F_1$ to be finitely generated. This makes a difference, as Theorem 1.1 can be applied to fibre products of free groups.

The first application of Theorem 1.1 is in Example 4.11. It starts with Baumslag's non-residually finite 1-relator group [4], and shows that the corresponding symmetric fibre product $G \leqslant F \times F$, where F is the free group of rank 2, is not conjugacy separable. In fact we explicitly exhibit a pair of non-conjugate elements of G that are conjugate in every finite quotient. Thus we get a 3-generated residually free group that is not conjugacy separable. This can be contrasted with the result of Chagas and Zalesskii [14] mentioned above. Moreover, note that 3 is optimal, as any 2-generated residually free group is either free or abelian, and so it is conjugacy separable.

A group G is said to be *$\mathcal{C}$-hereditarily conjugacy separable* if for every subgroup $H \leqslant G$, open in the pro-$\mathcal{C}$ topology on G, H is $\mathcal{C}$-conjugacy separable. In Subsection 4.1 we give a sufficient criterion for $\mathcal{C}$-conjugacy separability of subdirect products (see Proposition 4.5), and in Subsection 4.3 we combine this criterion with Theorem 1.1 to obtain the following complete characterization of $\mathcal{C}$-conjugacy separability for subdirect products of finite index.

Corollary 1.2. *Let $\mathcal{C}$ be a non-trivial extension-closed pseudovariety of finite groups. Let F_i either be a non-abelian free group or a non-elementary $\mathcal{C}$-hereditarily conjugacy separable hyperbolic group without non-trivial finite normal subgroups, $i = 1, 2$. If $G \leqslant F_1 \times F_2$ is a subdirect product of finite index in $F_1 \times F_2$ then the following statements are equivalent:*

(1) *G is $\mathcal{C}$-conjugacy separable;*
(2) *$F_1/N_1 \in \mathcal{C}$, where $N_1 \coloneqq G \cap F_1$;*
(3) *G is open in the pro-$\mathcal{C}$ topology on $F_1 \times F_2$.*

If $\mathcal{C} = \mathcal{C}_p$ is the class of all finite p-groups, for some prime p, then the above corollary shows that a subdirect product of two free groups is p-conjugacy separable if and only if its index is a power of p (see Corollary 4.13). A result of Toinet [59] states that the subgroups of right angle Artin groups which are open in the pro-p topology are p-conjugacy separable. Since direct products of free groups are right angled Artin groups, Corollary 1.2 shows that the openness assumption in Toinet's theorem is indeed necessary and cannot be dropped (see Example 4.14).

In Subsection 4.4 we investigate a gap between the sufficient criterion for $\mathcal{C}$-conjugacy separability of subdirect products of free groups provided by Proposition 4.5 and the necessary criterion given by Theorem 1.1. Theorem 4.18 shows that this gap is quite small. For some pseudovarieties of groups the gap does not exist at all (e.g., when $\mathcal{C} = \mathcal{C}_p$), though the general case is unclear: see Question 4.20.

In Sections 5 and 6 we use fibre products to construct finitely presented groups demonstrating exotic behaviour with respect to conjugacy separability. In these two sections we are concerned with the case when $\mathcal{C}$ is the class of all finite groups. Thus a *hereditary conjugacy separable group* is a group where every subgroup of finite index is conjugacy separable. The following statement is a special case of Theorem 5.6:

Theorem 1.3. *There exists a finitely presented hereditarily conjugacy separable group G which has an overgroup K such that $|K : G| = 2$, K is not conjugacy separable and has unsolvable conjugacy problem.*

Note that every finite index subgroup of the group G from Theorem 1.3 has solvable conjugacy problem by Mostowski's result [48]. First examples of finitely presented groups with solvable conjugacy problem having index 2 overgroups with unsolvable conjugacy problem were constructed by Collins and Miller in [16] and, independently, by Goryaga and Kirkinskii in [26] (conjugacy problem for finite index subgroups in these examples was not investigated).

Theorem 1.3 shows that hereditary conjugacy separability is not stable under commensurability, which was previously unknown. It is now natural to ask about the converse of Theorem 1.3:

Question 1.4. Does there exist a group G such that every finite index overgroup K, of G, is conjugacy separable (resp. has solvable conjugacy problem), but G possesses a subgroup of finite index which is not conjugacy separable (resp. has unsolvable conjugacy problem)?

Surprisingly, we discovered that the answers to both versions of Question 1.4 are negative. More precisely, in Corollary 6.3 (resp. Corollary 6.2) we show that if a group G has an index 2 subgroup H such that H is not conjugacy separable (resp. H has unsolvable conjugacy problem), then there is an overgroup K, of G, with $|K : G| = 3$ such that K is not conjugacy separable (resp. K has unsolvable conjugacy problem). More generally, we use permutational wreath products to prove

Corollary 1.5. *If a group G is not hereditarily conjugacy separable subgroup, then G has an overgroup K, with $|K : G| < \infty$, such that K is not conjugacy separable.*

A similar statement about the conjugacy problem is given in Corollary 6.5. According to the construction in Corollary 1.5, G will not be normal in its overgroup K. This turns out to be the only obstruction. The next result is a special case of Theorem 6.10.

Theorem 1.6. *There exists a finitely presented group G, containing a subgroup $G' \lhd G$, of index 2, such that G' is not conjugacy separable, but for every group K, with $G \lhd K$ and $|K : G| < \infty$, K is conjugacy separable.*

The first example of a conjugacy separable group with a non-conjugacy separable subgroup of finite index was constructed by Chagas and Zalesskii in [14]; the first finitely generated and finitely presented such examples were given in [40]. However, these examples did not provide any information about conjugacy separability of finite index (normal) overgroups.

Our proofs of Theorems 1.3 and 1.6 rely on a combination of fibre products with Rips-type constructions. Such a combination was pioneered by Baumslag, Bridson, Miller and Short [5], who gave a sufficient criterion for finite presentability of symmetric fibre products. To prove Theorem 1.3 we modify the original construction of Rips [56], to ensure that the resulting small cancellation group admits an automorphism of finite order whose fixed subgroup projects onto any given finitely generated subgroup of the original finitely presented group (see Proposition 5.5). The proof of Theorem 1.6, on the other hand, uses the Rips-type construction introduced by Bumagina and Wise in [11]. It allows one to minimize the group of automorphisms of the finitely generated normal subgroup in the resulting small cancellation group, providing us with some control over the centralizers in any finite index normal overgroup of this group (see Lemma 6.9).

Conjugacy separability in both Theorems 1.3 and 1.6 is established using the sufficient criterion (Proposition 4.5) together with the fact that any group commensurable to a classical finitely presented $C'(1/6)$ small cancellation group is hereditarily conjugacy separable. This fact is a consequence of the results of Wise [62] and Agol [1], implying that such groups are virtually compact special (in the sense of Haglund and Wise [30]), and a theorem of the author and Zalesskii [47], claiming that virtually compact special hyperbolic groups are conjugacy separable.

The last Section 7 is devoted to the study of p-conjugacy separability of subdirect products. In fact, our methods work more generally, when $\mathcal{C}$ is a pseudovariety of Q'-groups, for some set of primes Q (see Definition 7.1). In this case we obtain a criterion which is both necessary and sufficient for $\mathcal{C}$-conjugacy separability of full subdirect products of non-abelian free groups (or of $\mathcal{C}$-hereditarily conjugacy separable torsion-free hyperbolic groups) – see Theorem 7.6. The following corollary is a special case of that criterion (recall that, for any prime p, a *p-group* is a periodic group where every element has a p-power order).

Corollary 1.7. *Suppose that p is a prime, $G \leqslant F_1 \times F_2$ is a full subdirect product of non-abelian free groups F_1, F_2, and $N_1 \coloneqq G \cap F_1$. Then the following are equivalent:*

- (i) *G is p-conjugacy separable;*
- (ii) *F_1/N_1 is a residually finite p-group.*

Note that by a *residually finite p-group* we mean a p-group which is residually finite. Corollary 1.7 reveals an interesting and unexpected connection between p-conjugacy separability of the subdirect product G and periodicity of the quotient F_1/N_1. There is no such connection in the case of standard conjugacy separability, when $\mathcal{C}$ is the class of all finite groups (cf. Example 7.8). Using the well-known fact that a subdirect product $G \leqslant F_1 \times F_2$, of finitely generated free groups F_1 and F_2, is finitely generated if and only if the quotient F_1/N_1 is finitely presented, where $N_1 \coloneqq G \cap F_1$, we prove

Corollary 1.8. *For each prime p the following statements are equivalent:*

1. *there exists an infinite finitely presented residually finite p-group;*
2. *there exists a finitely generated p-conjugacy separable subgroup $G \leqslant H \times H$, where H is the free group of rank 2, such that G is not virtually a direct product of two free groups;*
3. *there exists a finitely generated p-conjugacy separable full subdirect product $G \leqslant F_1 \times F_2$, where F_1, F_2 are free groups, such that $|(F_1 \times F_2) : G| = \infty$.*

The existence of an infinite finitely presented residually finite p-group is a long standing open problem, and Corollary 1.8 shows that this problem can be reformulated in terms of p-conjugacy separable subgroups in the direct product of two free groups. This can be considered as a further motivation for the study of p-conjugacy separability.

Corollary 1.7 also shows that p-conjugacy separability of a subdirect product is an extremely sensitive and rare condition. In particular, a proper full subdirect product of two non-abelian free groups can be p-conjugacy separable for at most one prime p (see Corollary 7.10). More generally, we obtain the following statement.

Corollary 1.9. *A subgroup G, of a direct product of two free groups, is p-conjugacy separable for at least two distinct primes p if and only if G is itself isomorphic to a direct product of two free groups (one or both of which may be trivial).*

2. Background

2.1. **Notation.** Given a group G, a subgroup $H \leqslant G$ and an element $g \in G$, we will write $g^H \coloneqq \{hgh^{-1} \mid h \in H\} \subseteq G$ to denote the *H-conjugacy class of g* and $\mathrm{C}_H(g) \coloneqq \{h \in H \mid hgh^{-1} = g\}$ to denote the *centralizer of g in H*. We will also let $\mathrm{C}_G(H) \coloneqq \bigcap_{h \in H} \mathrm{C}_G(h)$ denote the *centralizer of H in G*.

Throughout the paper we will be working with direct products $F_1 \times F_2$, so to simplify the notation, we will often identify F_1 with the subgroup $F_1 \times \{1\} = \{(f,1) \mid f \in F_1\} \leqslant F_1 \times F_2$; similarly, we will identify F_2 with $\{1\} \times F_2 = \{(1,g) \mid g \in F_2\} \leqslant F_1 \times F_2$.

2.2. **Subdirect products.** The following statement summarizes basic properties of subdirect products.

Lemma 2.1. *Let $G \leqslant F_1 \times F_2$ be a subdirect product of some groups F_1, F_2. Then*

(i) *for any normal subgroup $N \lhd G$ and any $i \in \{1,2\}$, the intersection $N \cap F_i$ is normal in F_i and in $F_1 \times F_2$. In particular, $N_i \coloneqq G \cap F_i$ is normal in F_i, for each $i = 1,2$, and in G.*

(ii) $F_1/N_1 \cong G/(N_1 \times N_2) \cong F_2/N_2$.

(iii) $|(F_1 \times F_2) : G| < \infty$ *if and only if* $|F_1/N_1| < \infty$, *in which case* $|(F_1 \times F_2) : G| = |F_1/N_1|$.

Proof. (i) Any element of the intersection $N \cap F_1$ has the form $(h,1)$, for some $h \in F_1$. Since $G \leqslant F_1 \times F_2$ is subdirect, for each $f_1 \in F_1$, there exists $f_2 \in F_2$ such that $(f_1, f_2) \in G$. Since $N \lhd G$, we have

$$(f_1,1)(h,1)(f_1,1)^{-1} = (f_1,f_2)(h,1)(f_1,f_2)^{-1} \in N.$$

On the other hand, obviously $(f_1,1)(h,1)(f_1,1)^{-1} = (f_1 h f_1^{-1}, 1) \in F_1$, hence $N \cap F_1$ is normal in F_1. Similarly one can show that $N \cap F_2$ is normal in F_2. Evidently any normal subgroup of F_i is also normal in the direct product $F_1 \times F_2$, for $i = 1,2$.

(ii) Let $\rho_i : F_1 \times F_2 \to F_i$ be the natural projection, $i = 1,2$. Then $\ker(\rho_2) = F_1$ and $\rho_2(G) = F_2$, as G is subdirect. Hence $F_2 = \rho_2(G) \cong G/(G \cap F_1) = G/N_1$, and since $\rho_2(N_2) = N_2$, we get $F_2/N_2 = \rho_2(G)/\rho_2(N_2) \cong G/(N_1 \times N_2)$. Similarly, $F_1/N_1 \cong G/(N_1 \times N_2)$, as required.

(iii) Clearly, if $|(F_1 \times F_2) : G| < \infty$, then $|F_1/N_1| = |F_1 : (F_1 \cap G)| \le |(F_1 \times F_2) : G| < \infty$. On the other hand, if F_1/N_1 is finite, then so is F_2/N_2 by claim (ii), hence the quotient $(F_1 \times F_2)/(N_1 \times N_2) \cong F_1/N_1 \times F_2/N_2$ is finite as well. Since $N_1 \times N_2 \subseteq G$, we can deduce that $|(F_1 \times F_2) : G| < \infty$.

Now, assuming that $|F_1/N_1| < \infty$, we have

$$|(F_1 \times F_2) : G| = \left| \frac{F_1 \times F_2}{N_1 \times N_2} : \frac{G}{N_1 \times N_2} \right| = \frac{|F_1/N_1|\,|F_2/N_2|}{|F_1/N_1|} = |F_1/N_1|,$$

as $|F_1/N_1| = |F_2/N_2| = |G/(N_1 \times N_2)|$ by claim (ii). □

2.3. **Constructing subdirect products.** In this subsection we will review two main methods for constructing subdirect products of two groups.

Let F_1, F_2, P be groups with epimorphisms $\psi_i : F_i \twoheadrightarrow P$, $i = 1,2$. The *fibre product of F_1 and F_2 corresponding to ψ_1 and ψ_2* is defined as the subgroup $G \leqslant F_1 \times F_2$ given by

$$G \coloneqq \{(f_1,f_2) \in F_1 \times F_2 \mid \psi_1(f_1) = \psi_2(f_2)\}. \tag{1}$$

If $F_1 = F_2$ and $\psi_1 = \psi_2$ then G is said to be the *symmetric fibre product of F_1 corresponding to ψ_1*.

The fibre product G, given by (1), is always subdirect in $F_1 \times F_2$ (because $\psi_1(F_1) = \psi_2(F_2) = P$). Moreover, $N_i \coloneqq G \cap F_i = \ker \psi_i$, so that $F_i/N_i \cong P$, $i = 1,2$.

Conversely, it's not hard to show that if $G \leqslant F_1 \times F_2$ is a subdirect product of groups F_1, F_2, then G is the fibre product of F_1 and F_2 with respect to some epimorphisms $\psi_i : F_i \to P$, where $P = G/(N_1 \times N_2) \cong F_1/N_1 \cong F_2/N_2$, and $N_i \coloneqq G \cap F_i = \ker \psi_i$, $i = 1,2$.

Another standard method for constructing subdirect products is to start with any group A which has two normal subgroups $L_i \lhd A$, and let $F_i \coloneqq A/L_i$, $i = 1, 2$, and $G \coloneqq A/(L_1 \cap L_2)$. Then the map $\eta : G \to F_1 \times F_2$, $a(L_1 \cap L_2) \overset{\eta}{\mapsto} (aL_1, aL_2)$, gives rise to a natural subdirect embedding of G into $F_1 \times F_2$. In particular, if $L_1 \cap L_2 = \{1\}$, then $A = G$ is itself a subdirect product of F_1 and F_2.

Fibre products have been used to construct numerous (counter-)examples in Group Theory. The first such construction is due to Mihaĭlova [41], who showed that the direct product of two non-abelian free groups contains a finitely generated subgroup with unsolvable membership problem. In fact, Mihaĭlova explicitly listed the finite generating set for her group, essentially proving the first claim of the following lemma.

Lemma 2.2 (cf. [17, Prop. 3.28.2)]). *Let $G \leqslant F_1 \times F_2$ be a subdirect product of groups F_1, F_2 and let $P \coloneqq F_1/N_1$, where $N_1 \coloneqq F_1 \cap G$.*

(a) *If the groups F_1, F_2 are finitely generated and N_1 is the normal closure of finitely many elements in F_1 (which holds when P is finitely presented) then G is finitely generated.*
(b) *If G is finitely generated and F_1, F_2 are finitely presented then P is finitely presented.*

Proof. (a) Suppose that $F_1 = \langle x_1, \dots, x_k\rangle$, $F_2 = \langle v_1, \dots, v_l\rangle$, and N_1 is the normal closure of finitely many elements $h_1, \dots h_m$ in F_1. Since $G \leqslant F_1 \times F_2$ is subdirect, there exist $y_1, \dots, y_k \in F_2$ and $u_1, \dots, u_l \in F_1$ such that $(x_1, y_1), \dots, (x_k, y_k) \in G$ and $(u_1, v_1), \dots, (u_l, v_l) \in G$. Since $G/N_1 \cong F_2$, it follows that

$$G = \langle (u_1, v_1), \dots, (u_l, v_l), N_1\rangle.$$

By the assumptions, every $h \in N_1$ can be expressed as a product of conjugates of $h_1, \dots, h_m$ by elements from $\{x_1, \dots, x_k\}$, hence $(h, 1)$ is the product of the conjugates of $(h_1, 1), \dots, (h_m, 1)$ by elements from $\{(x_1, y_1), \dots, (x_k, y_k)\}$. Consequently,

$$N_1 \subseteq \langle (x_1, y_1), \dots, (x_k, y_k), (h_1, 1) \dots, (h_m, 1)\rangle,$$

which implies that G is generated by the elements (u_1, v_1), … ,(u_l, v_l), (x_1, y_1), …, (x_k, y_k), $(h_1, 1)$, …, $(h_m, 1)$.

(b) Since $F_2 \cong G/N_1$ is finitely presented and G is finitely generated, N_1 is the normal closure of finitely many elements $(h_1, 1), \dots, (h_m, 1)$ in G. But the action of G on N_1 by conjugation coincides with the action of F_1 on N_1 by conjugation (because $(x, y)(h, 1)(x, y)^{-1} = (xhx^{-1}, 1)$), hence N_1 is the normal closure of $h_1, \dots, h_m$ in F_1. And since F_1 is finitely presented, we can conclude that F_1/N_1 is also finitely presented. □

Lemma 2.2 provides a criterion for the subdirect product $G \in F_1 \times F_2$ to be finitely generated. The next *asymmetric 1-2-3 theorem*, proved by Dison [20, Thm. 9.4] (see also [8, Thm. B]), allows one to produce finitely presented subdirect products (in the same spirit, Lemma 2.2 can be called the *0-1-2 lemma*). The original *symmetric 1-2-3 theorem* (when $F_1 = F_2$) was proved by Baumslag, Bridson, Miller and Short [5, Thm. B].

Lemma 2.3 ([20, Thm. 9.4]). *Let F_1, F_2 be finitely presented groups and let $G \leqslant F_1 \times F_2$ be a subdirect product. If $N_1 \coloneqq G \cap F_1$ is finitely generated and $P \coloneqq F_1/N_1$ is of type* F_3 *then G is finitely presented.*

The "input" for the 1-2-3 theorem can be very conveniently provided by the famous Rips's construction [56] or by its numerous enhancements/modifications (cf. [5, Thm. 3.1], [63, Thm. 3.1], [11, Thm. 15] or [30, Thm. 10.1]), claiming that for every finitely presented group P there is a hyperbolic group F (usually with many "nice" properties) and a normal subgroup

$N \lhd F$ such that N is finitely generated and $F/N \cong P$. Thus, if P is of type F_3, then, by Lemma 2.3, the corresponding symmetric fibre product $G \leqslant F \times F$ is finitely presented.

2.4. **Acylindrically hyperbolic groups.** In this paper we will mostly work with subdirect products of groups acting on δ-hyperbolic spaces in some controlled way. One of the most general classes of such groups is the class of acylindrically hyperbolic groups, introduced by Osin in [52]. It includes non-abelian free groups (of arbitrary rank) and non-elementary hyperbolic groups (in the sense of Gromov). The reader is referred to [7, Ch. III] for the basic theory of hyperbolic spaces and groups.

Recall that a group F is said to be *elementary* if it possesses a cyclic subgroup of finite index. Following Osin [52], we will say that a group F is *acylindrically hyperbolic* if it is non-elementary and admits an acylindrical cobounded isometric action on a hyperbolic metric space with unbounded orbits. We will not define the notion of acylindricity of the action here, as we will only use properties of such groups that have already been established elsewhere (the interested reader is referred to [52] for the background). If G is a hyperbolic group then it acts acylindrically and coboundedly on any Cayley graph corresponding to a finite generating set. If G splits as a free product of two non-trivial groups (e.g., if G is free and non-abelian), then it acts acylindrically and co-boundedly on the Bass-Serre tree corresponding to this splitting (cf. [45, Lemma 4.2]).

Given a group F acting by isometries on a δ-hyperbolic metric space $(\mathcal{S}, d)$, an element $f \in F$ is said to be *loxodromic* if for some point $s \in \mathcal{S}$ the function $\mathbb{Z} \to \mathcal{S}$, $n \mapsto f^n(s)$, is a quasi-isometric embedding; in particular, the order of f must be infinite. If the action of F on $\mathcal{S}$ is acylindrical then every loxodromic element $f \in F$ satisfies the *WPD condition* of Bestvina and Fujiwara [9] – see [52, Def. 2.5]. It follows from the work of Dahmani, Guirardel and Osin [18, Lemma 6.5, Cor. 6.6] that such f is contained in a *unique maximal elementary subgroup* $\mathrm{E}_F(f)$, and

$$\mathrm{E}_F(f) = \{g \in F \mid g f^n g^{-1} = f^{\pm n} \text{ for some } n \in \mathbb{N}\}. \tag{2}$$

Now, if F acts on a δ-hyperbolic metric space $\mathcal{S}$ coboundedly and $H \leqslant F$ is a non-elementary subgroup containing at least one loxodromic element then, by [2, Lemma 5.6], there is a largest finite subgroup $\mathrm{E}_F(H) \leqslant F$, normalized by H. In particular, F itself has a maximal finite normal subgroup $\mathrm{E}_F(F)$ (cf. [18, Thm. 2.24]).

Lemma 2.4. *Suppose that F is an acylindrically hyperbolic group without non-trivial finite normal subgroups. Then any non-trivial normal subgroup $N \lhd F$ is non-elementary, satisfies $\mathrm{E}_F(N) = \{1\}$, and there is an element $h \in N$ such that $\mathrm{C}_F(h) = \langle h \rangle \subseteq N$. Moreover, for any $x \in F$ there exists $n \in \mathbb{Z}$ such that the element $f \coloneqq h^n x \in F$ satisfies $\mathrm{C}_F(f) = \langle f \rangle$.*

Proof. Fix some non-elementary cobounded acylindrical action of F on a δ-hyperbolic metric space $\mathcal{S}$. Since $N \neq \{1\}$ is normal in F, it must be infinite by the assumptions, so, according to [52, Lemma 7.2], N is non-elementary and has at least one loxodromic element. Therefore we can consider $\mathrm{E}_F(N)$, the maximal finite subgroup of F, normalized by N.

Now, let us show that $\mathrm{E}_F(N) = \mathrm{E}_F(F)$. Indeed, $\mathrm{E}_F(F) \subseteq \mathrm{E}_F(N)$, as $\mathrm{E}_F(F)$ is finite and is normalized by N, and $\mathrm{E}_F(N)$ is the largest subgroup of F with this property. On the other hand, it is easy to check that since $N \lhd F$, $\mathrm{E}_F(N) \lhd F$ (because for each $f \in F$, $f\mathrm{E}_F(N)f^{-1}$ is also normalized by N), hence $\mathrm{E}_F(N) \subseteq \mathrm{E}_F(F)$. Thus we can conclude that $\mathrm{E}_F(N) = \mathrm{E}_F(F)$, but $\mathrm{E}_F(F) = \{1\}$ as F contains no non-trivial finite normal subgroups, therefore $\mathrm{E}_F(N) = \{1\}$.

We can now apply [2, Lemma 5.12] to find a loxodromic element $h \in N$ such that $\mathrm{E}_F(h) = \langle h \rangle \mathrm{E}_F(N) = \langle h \rangle$. Since $\mathrm{C}_F(h) \subseteq \mathrm{E}_F(h)$ by (2), we deduce that $\mathrm{C}_F(h) = \langle h \rangle$, as required.

If $x \notin \langle h \rangle = \mathrm{E}_F(h)$, then the final claim of the lemma follows from [2, Lemma 5.13]. Otherwise, if $x = h^m$ for some $m \in \mathbb{Z}$, then we choose $n = 1 - m \in \mathbb{Z}$, so that $f = h^n x = h$, and the required equality $\mathrm{C}_F(f) = \mathrm{E}_F(f) = \langle f \rangle$ holds for f because it holds for h. □

3. The pro-$\mathcal{C}$ topology on subdirect products

Our main tool for studying $\mathcal{C}$-conjugacy separability is the pro-$\mathcal{C}$ topology. In this section we will investigate some basic properties of this topology on subdirect products of two groups.

3.1. **Pseudovarieties and pro-$\mathcal{C}$ topology.** We will say that $\mathcal{C}$ is a *pseudovariety of groups* if $\mathcal{C}$ is a class of groups closed under isomorphisms, subgroups, direct products and quotients. In other words, to be a pseudovariety $\mathcal{C}$ must satisfy the following conditions:

- if $A \in \mathcal{C}$ and $B \cong A$ then $B \in \mathcal{C}$;
- if $A \in \mathcal{C}$ and $B \leqslant A$ then $B \in \mathcal{C}$;
- if $A, B \in \mathcal{C}$ then $A \times B \in \mathcal{C}$;
- if $A \in \mathcal{C}$ and $N \lhd A$ then $A/N \in \mathcal{C}$.

By a *pseudovariety of finite groups* we will mean a pseudovariety $\mathcal{C}$ such that each member of $\mathcal{C}$ is a finite group. For example, the class of all finite groups or the class of all finite p-groups, for some prime p, are pseudovarieties of finite groups.

Let $\mathcal{C}$ be a pseudovariety of groups. Given a group G and $N \lhd G$, N is said to be a *co-$\mathcal{C}$ subgroup* of G if $G/N \in \mathcal{C}$. One can define the *pro-$\mathcal{C}$ topology* on any group G by letting the cosets of co-$\mathcal{C}$ subgroups be the basic open sets. Since $\mathcal{C}$ is closed under taking subgroups, it is easy to see that any group homomorphism $G \to G_1$ is continuous with respect to the pro-$\mathcal{C}$ topologies on G and G_1 (see [23, pp. 8–11] for background on pro-$\mathcal{C}$ topologies). In particular, we can make the following observation.

Remark 3.1. If H is any subgroup of a group G and $Y \subset G$ is closed in the pro-$\mathcal{C}$ topology on G then $H \cap Y$ is closed in the pro-$\mathcal{C}$ topology on H.

We will say that a subset $X \subseteq G$ is *$\mathcal{C}$-closed* (respectively, *$\mathcal{C}$-open*) if X is closed (respectively, open) in the pro-$\mathcal{C}$ topology on G. Since G, equipped with its pro-$\mathcal{C}$ topology, is a topological group, for any $\mathcal{C}$-closed (respectively, $\mathcal{C}$-open) subset $X \subset G$, and any $g \in G$, gX and Xg are also $\mathcal{C}$-closed (respectively, $\mathcal{C}$-open) in G. Since the complement of a subgroup is a union of cosets modulo this subgroup, the complement of a $\mathcal{C}$-open subgroup is itself $\mathcal{C}$-open, so that the subgroup is $\mathcal{C}$-closed. More generally, the following holds:

Remark 3.2 (cf. [23, Lemma 2.6 on p. 10]). If G is a group and $H \leqslant G$ is a subgroup containing a $\mathcal{C}$-open subgroup then H is both $\mathcal{C}$-open and $\mathcal{C}$-closed in G, and H contains some co-$\mathcal{C}$ subgroup of G.

Lemma 3.3. *Suppose that G and P are groups and $\psi : G \to P$ is an epimorphism.*

(i) *For every $\mathcal{C}$-open subset X of G, $\psi(X)$ is $\mathcal{C}$-open in P (in other words, ψ is an open map when G and P are equipped with their pro-$\mathcal{C}$ topologies).*
(ii) *The pro-$\mathcal{C}$ topology on P coincides with the quotient topology induced by ψ from the pro-$\mathcal{C}$ topology on G.*
(iii) *Given any subset $Y \subseteq P$, Y is $\mathcal{C}$-closed in P if and only if $\psi^{-1}(Y)$ is $\mathcal{C}$-closed in G.*

Proof. (i) Let $N \coloneqq \ker \psi$. Clearly, to show that ψ is an open map, it is enough to prove that $\psi(H)$ is $\mathcal{C}$-open in P for every co-$\mathcal{C}$ subgroup $H \lhd G$. This is indeed the case because $P/\psi(H) \cong G/(HN)$ can be viewed as a quotient of $G/H \in \mathcal{C}$, and $\mathcal{C}$ is closed under taking quotients by the assumptions.

Claim (ii) is a direct consequence of claim (i), as any open continuous map is a quotient map (cf. [49, Sec. 2-11]). Claim (iii) is, of course, simply a restatement of claim (ii). □

The pro-$\mathcal{C}$ topology on G is Hausdorff if and only if G is *residually-$\mathcal{C}$*, i.e., for every $g \in G \setminus \{1\}$ there is a group $M \in \mathcal{C}$ and a homomorphism $\varphi : G \to M$ such that $\varphi(g) \neq 1$ in M. This is also equivalent to the statement that the singleton $\{1\}$ is $\mathcal{C}$-closed in G.

If H is a subgroup of a group G, we will say that the pro-$\mathcal{C}$ topology on H is a *restriction of the pro-$\mathcal{C}$ topology topology on G* if every subset $X \subseteq H$, closed in the pro-$\mathcal{C}$ topology on H, is also closed in the pro-$\mathcal{C}$ topology on G (in particular, H must be $\mathcal{C}$-closed in G); in other words, the converse of the claim of Remark 3.1 holds.

A pseudovariety of groups $\mathcal{C}$ is said to be *extension-closed* if for any group G, containing a normal subgroup $N \lhd G$ such that $N \in \mathcal{C}$ and $G/N \in \mathcal{C}$, one has $G \in \mathcal{C}$.

Lemma 3.4. *Let $\mathcal{C}$ be a pseudovariety of finite groups. Suppose that H is a subgroup of a group G.*

(a) *The pro-$\mathcal{C}$ topology on H is a restriction of the pro-$\mathcal{C}$ topology on G if and only if every co-$\mathcal{C}$ subgroup of H is $\mathcal{C}$-closed in G.*
(b) *If the pseudovariety $\mathcal{C}$ is extension-closed and H is $\mathcal{C}$-open in G then the pro-$\mathcal{C}$ topology on H is the restriction of the pro-$\mathcal{C}$ topology on G.*

Proof. (a) The necessity is clear by Remark 3.2. To prove the sufficiency, assume that every co-$\mathcal{C}$ subgroup of H is $\mathcal{C}$-closed in G. Evidently, to prove that every $\mathcal{C}$-closed subset of H is $\mathcal{C}$-closed in G it is enough to show this for each basic $\mathcal{C}$-closed subset X in H. Thus $X = \bigcup_{i \in I} N h_i$, for some co-$\mathcal{C}$ subgroup N of H, and some $h_i \in H$, $i \in I$. Since $H/N \in \mathcal{C}$ and $\mathcal{C}$ consists of finite groups by the assumption, we can deduce that $|H : N| < \infty$, so that X is a finite union of cosets modulo N. Consequently, X is $\mathcal{C}$-closed in G as a finite union of $\mathcal{C}$-closed sets, because each coset $N h_i$ is $\mathcal{C}$-closed in G.

(b) See the proof of [55, Lemma 3.1.4.(a)]. □

3.2. **Subdirect products.**

Remark 3.5. Suppose that $G \leqslant F_1 \times F_2$ is a subdirect product, $N_1 \coloneqq G \cap F_1$ and $Y \subseteq F_1$ is any subset. Then $F_1 \cap YG = YN_1$. (Indeed, since $Y \subseteq F_1$, $F_1 \cap YG = Y(F_1 \cap G) = YN_1$.)

The next lemma will be used throughout the paper.

Lemma 3.6. *Let F_1, F_2 be groups and let $G \leqslant F_1 \times F_2$ be a subdirect product. If $\mathcal{C}$ is a pseudovariety of groups and $X \subseteq F_1$ is any subset, then the product $XG = \{(x,1)g \mid x \in X,\ g \in G\} \subseteq F_1 \times F_2$ is closed in the pro-$\mathcal{C}$ topology on $F_1 \times F_2$ if and only if XN_1 is closed in the pro-$\mathcal{C}$ topology on F_1, where $N_1 \coloneqq F_1 \cap G$.*

Proof. If XG is $\mathcal{C}$-closed in $F_1 \times F_2$, then $XN_1 = XG \cap F_1$ (cf. Remark 3.5) is $\mathcal{C}$-closed in F_1 by Remark 3.1.

Now, suppose that XN_1 is $\mathcal{C}$-closed in F_1. Let $\mathcal{N}_{\mathcal{C}}(F_i)$ denote the set of all co-$\mathcal{C}$ subgroups of F_i, $i = 1, 2$. First, let us show that

$$XG = \bigcap_{A \in \mathcal{N}_{\mathcal{C}}(F_1)} AXG \quad \text{in } F_1 \times F_2. \tag{3}$$

Indeed, evidently, the right-hand side contains the left-hand side. For the opposite inclusion, assume that $(f_1, f_2) \in AXG$, for some $f_1 \in F_1$, $f_2 \in F_2$ and for every $A \in \mathcal{N}_{\mathcal{C}}(F_1)$. Then, since G is subdirect in $F_1 \times F_2$, there is $h_1 \in F_1$ such that $(h_1, f_2) \in G$, hence $(f_1 h_1^{-1}, 1) \in AXG$ for all $A \in \mathcal{N}_{\mathcal{C}}(F_1)$. The latter, combined with Remark 3.5, yields that

$$(f_1 h_1^{-1}, 1) \in AXG \cap F_1 = AXN_1, \text{ for all } A \in \mathcal{N}_{\mathcal{C}}(F_1). \tag{4}$$

However, since XN_1 is $\mathcal{C}$-closed in F_1, for every $(h, 1) \in F_1 \setminus XN_1$ there is $A_h \in \mathcal{N}_{\mathcal{C}}(F_1)$ such that $A_h(h, 1) \cap XN_1 = \emptyset$, which is equivalent to $(h, 1) \notin A_h XN_1$. Therefore, in view of (4), we can conclude that $(f_1 h_1^{-1}, 1) \in XN_1$, consequently $(f_1, f_2) = (f_1 h_1^{-1}, 1)(h_1, f_2) \in XN_1 G = XG$. Thus we have shown that $\bigcap_{A \in \mathcal{N}_{\mathcal{C}}(F_1)} AXG \subseteq XG$, so (3) holds.

Thus it remains to prove that for each $A \in \mathcal{N}_{\mathcal{C}}(F_1)$ the subset AXG is $\mathcal{C}$-closed in $F_1 \times F_2$. Take any $A \in \mathcal{N}_{\mathcal{C}}(F_1)$, and note that $P \coloneqq AG$ is a subgroup of $F_1 \times F_2$ (because $A \lhd F_1$, and so $A \lhd F_1 \times F_2$), and it is subdirect as it contains G. Combining Lemma 2.1.(ii) with Remark 3.5, we obtain

$$F_2/(F_2 \cap P) \cong F_1/(F_1 \cap P) = F_1/(AN_1).$$

It follows that $F_2/(F_2 \cap P) \in \mathcal{C}$, as it is a quotient of the group $F_1/A \in \mathcal{C}$ and $\mathcal{C}$ is closed under taking quotients. Thus $B \coloneqq F_2 \cap P \in \mathcal{N}_{\mathcal{C}}(F_2)$, and so $(F_1 \times F_2)/(A \times B) \cong F_1/A \times F_2/B \in \mathcal{C}$, i.e., $A \times B$ is a co-$\mathcal{C}$ subgroup of $F_1 \times F_2$.

We have shown that $P = AG$ contains the $\mathcal{C}$-open subgroup $A \times B$ of $F_1 \times F_2$, hence it is itself $\mathcal{C}$-open by Remark 3.2. Observe that $AXG = XAG = XP$ is a union of cosets modulo P, so its complement in $F_1 \times F_2$ is also a union of cosets modulo P, hence this complement is $\mathcal{C}$-open, so that AXG is $\mathcal{C}$-closed in $F_1 \times F_2$, as claimed.

Now, (3) implies that XG is $\mathcal{C}$-closed in $F_1 \times F_2$, and the lemma is proved. □

Corollary 3.7. *Suppose that $G \leqslant F_1 \times F_2$ is a subdirect product of groups F_1 and F_2, and $\mathcal{C}$ is a pseudovariety of groups. Then the following are equivalent:*

(i) *G is $\mathcal{C}$-closed in $F_1 \times F_2$;*
(ii) *$N_1 \coloneqq G \cap F_1$ is $\mathcal{C}$-closed in F_1;*
(iii) *F_1/N_1 is residually-$\mathcal{C}$.*

Proof. The equivalence of (i) and (ii) follows from Lemma 3.6 (set $X = \{1\}$), and the equivalence of (ii) and (iii) is given by the standard fact (left as an exercise for the reader) that a normal subgroup N of a group F is $\mathcal{C}$-closed if and only if the quotient F/N is residually-$\mathcal{C}$. □

We will now aim to give a sufficient criterion for the pro-$\mathcal{C}$ topology on a subdirect product to be a restriction of the pro-$\mathcal{C}$ topology on the direct product. To this end we will need the following definition.

Definition 3.8. Let M be a group and let $\mathcal{C}$ be a pseudovariety of groups. We will say that M is *highly residually-$\mathcal{C}$* if for every group L and every $K \in \mathcal{C}$ fitting in the short exact sequence

$$\{1\} \to K \to L \to M \to \{1\},$$

L is residually-$\mathcal{C}$.

In other words, M is highly residually-$\mathcal{C}$ if each extension of a $\mathcal{C}$-group by M is residually-$\mathcal{C}$. Our terminology is motivated by that of Lorensen [35, p. 1710], where he calls a group M *highly residually finite* if each (finite-by-M) group is residually finite.

Lemma 3.9. *Let F_1, F_2 be groups, let $G \leqslant F_1 \times F_2$ be a subdirect product, and let $\mathcal{C}$ be an extension-closed pseudovariety of finite groups. If F_1/N_1 is highly residually-$\mathcal{C}$, for $N_1 \coloneqq F_1 \cap G$, then the pro-$\mathcal{C}$ topology on G is a restriction of the pro-$\mathcal{C}$ topology on $F_1 \times F_2$.*

Proof. In view of Lemma 3.4.(a), it is enough to show that every co-$\mathcal{C}$ subgroup G' of G is $\mathcal{C}$-closed in $F_1 \times F_2$. Let $\rho_i : F_1 \times F_2 \to F_i$ denote the canonical projection, and set $F_i' = \rho_i(G')$, $i = 1, 2$. Then $F_i' \lhd F_i$, $i = 1, 2$, as $G' \lhd G$, and $F_1/F_1' = \rho_1(G)/\rho_1(G') \cong G/(G'N_2)$, where $N_2 \coloneqq G \cap \ker(\rho_1) = G \cap F_2$. Thus $F_1/F_1' \in \mathcal{C}$, as a quotient of the group $G/G' \in \mathcal{C}$; similarly, $F_2/F_2' \in \mathcal{C}$. Therefore, $(F_1 \times F_2)/(F_1' \times F_2') \cong F_1/F_1' \times F_2/F_2' \in \mathcal{C}$; in particular, $F_1' \times F_2'$ is a $\mathcal{C}$-open subgroup of $F_1 \times F_2$.

Obviously $G' \leqslant F_1' \times F_2'$ is a subdirect product, by construction. Now, observe that

$$N_1' \coloneqq G' \cap F_1' = G' \cap F_1 = G' \cap N_1$$

is a normal subgroup of F_1 by Lemma 2.1.(i), and $N_1/N_1' \cong G'N_1/G' \leqslant G/G' \in \mathcal{C}$, which yields that $N_1/N_1' \in \mathcal{C}$ as $\mathcal{C}$ is closed under taking subgroups. Recalling that F_1/N_1 is highly residually-$\mathcal{C}$, the short exact sequence

$$\{1\} \to N_1/N_1' \to F_1/N_1' \to F_1/N_1 \to \{1\}, \tag{5}$$

implies that F_1/N_1' is residually-$\mathcal{C}$, and hence its subgroup F_1'/N_1' is residually-$\mathcal{C}$ as well.

It remains to conclude that G' is $\mathcal{C}$-closed in $F_1' \times F_2'$ by Corollary 3.7, which, in view of Lemma 3.4.(b), implies that G' is $\mathcal{C}$-closed in $F_1 \times F_2$, as required. □

3.3. **Cyclic subgroup separability.** Given a pseudovariety of groups $\mathcal{C}$, a group M is said to be *cyclic subgroup $\mathcal{C}$-separable* if every cyclic subgroup is closed in the pro-$\mathcal{C}$ topology on M. As usual, if $\mathcal{C}$ is the class of all finite groups, then we will simply write that M is cyclic subgroup separable.

Since the trivial subgroup is cyclic, any cyclic subgroup $\mathcal{C}$-separable group is residually-$\mathcal{C}$. The converse is not true in general; for example, the metabelian Baumslag-Solitar group $BS(1,2) \coloneqq \langle a, t \parallel tat^{-1} = a^2 \rangle$ is residually finite ([31, Thm. 1]) but the cyclic subgroup $\langle a \rangle$ is not closed in the profinite topology on $BS(1,2)$, as it is conjugate to a proper subgroup of itself.

In Proposition 4.5 below we will see that cyclic subgroup $\mathcal{C}$-separability of the quotient F_1/N_1 is important in proving that a subdirect product $G \leqslant F_1 \times F_2$ is $\mathcal{C}$-conjugacy separable. In this subsection we will discuss some permanence properties related to cyclic subgroup $\mathcal{C}$-separability, that will be useful later on.

Lemma 3.10. *Let $\mathcal{C}$ be an extension-closed pseudovariety of finite groups and let F be any group.*

(i) *If F is cyclic subgroup $\mathcal{C}$-separable then so is every subgroup $H \leqslant F$.*
(ii) *If some $\mathcal{C}$-open subgroup $H \leqslant F$ is cyclic subgroup $\mathcal{C}$-separable then so is F.*
(iii) *If F is cyclic subgroup $\mathcal{C}$-separable and $K \lhd F$ is a finite normal subgroup then F/K is cyclic subgroup $\mathcal{C}$-separable.*
(iv) *Suppose that $K \lhd F$, $K \in \mathcal{C}$, F/K is highly residually-$\mathcal{C}$ and cyclic subgroup $\mathcal{C}$-separable. Then F is itself cyclic subgroup $\mathcal{C}$-separable.*

Proof. Claim (i) is a trivial consequence of the definition and Remark 3.1.

To prove claim (ii), suppose that H is cyclic subgroup $\mathcal{C}$-separable and $C \leqslant F$ is any cyclic subgroup. Then the cyclic subgroup $C' \coloneqq C \cap H$ is closed in the pro-$\mathcal{C}$ topology on H, and Lemma 3.4.(b) implies that C' is also $\mathcal{C}$-closed in F. Now, $|C : C'| \leq |F : H| < \infty$ as $\mathcal{C}$ consists of finite groups and H is $\mathcal{C}$-open in F (thus H contains some co-$\mathcal{C}$ subgroup of F by Remark 3.2). Therefore C is a finite union of cosets modulo C', hence it is also $\mathcal{C}$-closed in F.

To establish claim (iii) assume that F is cyclic subgroup $\mathcal{C}$-separable and $K \lhd F$, $|K| < \infty$. Then the cyclic subgroup $\{1\}$ is $\mathcal{C}$-closed in F, so, since K is finite, there exists a co-$\mathcal{C}$ subgroup $H \lhd F$ such that $K \cap H = \{1\}$. It follows that the image $HK/K \cong H/(H \cap K)$, of H in F/K, is naturally isomorphic to H and is a co-$\mathcal{C}$-subgroup of F/K because $\mathcal{C}$ is closed under taking quotients. Now, according to claim (i), H is cyclic subgroup $\mathcal{C}$-separable, hence so is F/K by claim (ii).

It remains to prove claim (iv). By the assumptions, F/K is highly residually-$\mathcal{C}$, so F is residually-$\mathcal{C}$, hence there exists a co-$\mathcal{C}$ subgroup $H \lhd F$ such that $H \cap K = \{1\}$ ($|K| < \infty$ as $K \in \mathcal{C}$). As before, H is isomorphic to its image HK/K in F/K, hence it is cyclic subgroup $\mathcal{C}$-separable by claim (i) as F/K is cyclic subgroup $\mathcal{C}$-separable. Therefore, in view of claim (ii), we can conclude that F is cyclic subgroup $\mathcal{C}$-separable. □

The next lemma will be useful for showing that every $\mathcal{C}$-open subgroup of a subdirect product is $\mathcal{C}$-conjugacy separable.

Lemma 3.11. *Let $\mathcal{C}$ be an extension-closed pseudovariety of finite groups, let $G \leqslant F_1 \times F_2$ be a subdirect product of some groups F_1, F_2 and let $N_1 \coloneqq G \cap F_1$. Suppose that F_1/N_1 is highly residually-$\mathcal{C}$ and cyclic subgroup $\mathcal{C}$-separable. If $H \leqslant G$ is any $\mathcal{C}$-open subgroup then H is a subdirect product in $J_1 \times J_2$, for some $\mathcal{C}$-open subgroups J_i of F_i, $i = 1, 2$, and $J_1/(H \cap J_1)$ is cyclic subgroup $\mathcal{C}$-separable.*

Proof. Naturally, we let $J_i \leqslant F_i$ be the image of H under the projection to the i-th coordinate group, $i = 1, 2$. Then $H \leqslant J_1 \times J_2$ is subdirect, by construction. Now, by Remark 3.2, H contains some co-$\mathcal{C}$ subgroup G' of G. Using the same notation as in Lemma 3.9, let $F_i' \leqslant F_i$ denote the projection of G' to the i-th coordinate group, $i = 1, 2$. Then $F_i' \subseteq J_i$ and F_i' is a co-$\mathcal{C}$ subgroup of F_i, hence J_i is $\mathcal{C}$-open in F_i, $i = 1, 2$.

Now, if we let $N_1' \coloneqq F_1 \cap G'$, then, by the argument from the proof of Lemma 3.9, $N_1' \lhd F_1$, $N_1/N_1' \in \mathcal{C}$ and the quotient F_1/N_1' fits into the short exact sequence (5). Therefore, in view of Lemma 3.10.(iv), our assumptions on F_1/N_1 imply that F_1/N_1' is cyclic subgroup $\mathcal{C}$-separable.

Clearly, $N_1' = G' \cap F_1 \subseteq H \cap F_1 = H \cap J_1 \subseteq G \cap F_1 = N_1$, so the group $J_1/(H \cap J_1)$ is isomorphic to the quotient of the group $J_1/N_1' \leqslant F_1/N_1'$ by the finite normal subgroup $(H \cap J_1)/N_1' \leqslant N_1/N_1'$. Thus we can apply claims (i) and (iii) of Lemma 3.10 to deduce that the group $J_1/(H \cap J_1)$ is cyclic subgroup $\mathcal{C}$-separable, as required. □

4. Conjugacy separability of subdirect products

In this section we will give necessary and sufficient criteria for $\mathcal{C}$-conjugacy separability of subdirect products of two groups.

Remark 4.1. Observe that a group G is $\mathcal{C}$-conjugacy separable if and only if the G-conjugacy class g^G is $\mathcal{C}$-closed in G, for each $g \in G$.

We will say that a pseudovariety of groups is *non-trivial* if it contains at least one non-trivial group. Basic examples of $\mathcal{C}$-conjugacy separable groups are free groups:

Lemma 4.2. *Suppose that $\mathcal{C}$ is a non-trivial extension-closed pseudovariety of groups and F is a free group of arbitrary rank. Then F is $\mathcal{C}$-hereditarily conjugacy separable.*

Proof. Since any subgroup of F is also free, it is enough to show that F is $\mathcal{C}$-conjugacy separable.

By the assumptions, the class $\mathcal{C}$ is closed under taking subgroups and contains at least one non-trivial group, so it must contain some non-trivial cyclic group, and since $\mathcal{C}$ is closed under quotients we deduce that $\mathbb{Z}/p\mathbb{Z} \in \mathcal{C}$, for some prime p. Now, every non-trivial finite p-group P has a normal series where the sections are cyclic groups of order p, hence $P \in \mathcal{C}$, as $\mathcal{C}$ is closed under taking extensions. Therefore $\mathcal{C}$ contains the class $\mathcal{C}_p$, of all finite p-groups. Since free groups are well-known to be $\mathcal{C}_p$-conjugacy separable (cf. [54, Prop. 5]), we can conclude that F is $\mathcal{C}$-conjugacy separable. □

In [22, Thm. 1.2] Ferov proved that for an extension-closed pseudovariety of finite groups, any graph product of $\mathcal{C}$-hereditarily conjugacy separable groups is also $\mathcal{C}$-hereditarily conjugacy separable. For our purposes we will need a much easier special case:

Lemma 4.3 ([23, Lemma 4.2 on p. 18])**.** *If $\mathcal{C}$ is an extension-closed pseudovariety of finite groups, then the direct product of two $\mathcal{C}$-hereditarily conjugacy separable groups is $\mathcal{C}$-hereditarily conjugacy separable.*

In the case when $\mathcal{C}$ is the class of all finite groups, Lemma 4.3 was originally proved by Martino and the first author in [40, Lemma 7.3].

4.1. **Criteria for conjugacy separability.** Throughout this subsection we will assume that $\mathcal{C}$ is an extension-closed pseudovariety of finite groups.

The following general criterion was proved by Ferov in [22, Cor. 4.7 and Thm. 4.2], it naturally extends the criterion found by the author in [43, Cor. 3.5 and Prop. 3.2].

Lemma 4.4. *Let F be a $\mathcal{C}$-hereditarily conjugacy separable group and let $G \leqslant F$ be a subgroup. Suppose that for each $g \in G$ the double coset $\mathrm{C}_F(g)G$ is $\mathcal{C}$-closed in F. Then g^G is $\mathcal{C}$-closed in F for each $g \in G$, in particular, G is $\mathcal{C}$-conjugacy separable.*

We will say that a group F *has cyclic centralizers* if the centralizer $\mathrm{C}_F(f)$ is cyclic for each $f \in F \setminus \{1\}$. Basic examples of groups with cyclic centralizers are free groups ([36, Prop. 2.19 in Sec. I.2]), torsion-free hyperbolic groups [7, Cor. 3.10 in Ch. III.Γ], 1-relator groups with torsion [50, Thm. 2] and $C'(1/6)$ small cancellation groups ([60]).

The following proposition generalizes [40, Prop. 7.5].

Proposition 4.5. *Suppose that $\mathcal{C}$ is an extension-closed pseudovariety of finite groups. Let F_1, F_2 be $\mathcal{C}$-hereditarily conjugacy separable groups with cyclic centralizers, let $G \leqslant F_1 \times F_2$ be a subdirect product and let $N_1 := G \cap F_1$. If F_1/N_1 is cyclic subgroup $\mathcal{C}$-separable then G is $\mathcal{C}$-conjugacy separable.*

Proof. We will aim to apply the criterion from Lemma 4.4. So, consider any element $(g_1, g_2) \in G$. If $g_1 = 1$ in F_1 (or $g_2 = 1$ in F_2), then $\mathrm{C}_{F_1 \times F_2}((g_1, g_2))$ contains all of F_1 (or all of F_2), and since $G \leqslant F_1 \times F_2$ is subdirect, we would have $\mathrm{C}_{F_1 \times F_2}((g_1, g_2))G = F_1 \times F_2$, which is evidently $\mathcal{C}$-closed in $F_1 \times F_2$.

Thus we can suppose that $g_1 \neq 1$ and $g_2 \neq 1$. Then $\mathrm{C}_{F_i}(g_i) = \langle f_i \rangle$, for some $f_i \in F_i$, by the assumptions, and $g_i = f_i^{m_i}$ for some $m_i \in \mathbb{Z} \setminus \{0\}$, $i = 1, 2$. Therefore $\mathrm{C}_{F_1 \times F_2}((g_1, g_2)) = \langle (f_1, 1), (1, f_2) \rangle \cong \langle f_1 \rangle \times \langle f_2 \rangle$, and, so the subgroup $H \coloneqq \langle (g_1, 1), (1, g_2) \rangle = \langle (g_1, 1), (g_1, g_2) \rangle$ has finite index in $\mathrm{C}_{F_1 \times F_2}((g_1, g_2))$. Note that $HG = \langle (g_1, 1) \rangle G$ because $(g_1, g_2) \in G$ commutes with $(g_1, 1)$. Thus, for any transversal $(a_1, b_1), \ldots, (a_k, b_k)$ for the left cosets in $\mathrm{C}_{F_1 \times F_2}((g_1, g_2))/H$, we have

$$\mathrm{C}_{F_1 \times F_2}((g_1, g_2))G = \bigcup_{j=1}^{k} (a_j, b_j)HG = \bigcup_{j=1}^{k} (a_j, b_j)\langle (g_1, 1) \rangle G. \tag{6}$$

Now, recall that, by Lemma 3.6, the double coset $\langle (g_1, 1) \rangle G$ is $\mathcal{C}$-closed in $F_1 \times F_2$, provided the double coset $\langle g_1 \rangle N_1$ is $\mathcal{C}$-closed in F_1. Since $N_1 \lhd F_1$, in view of Lemma 3.3.(iii) the latter is equivalent to saying that the cyclic subgroup $\psi(\langle g_1 \rangle)$ is $\mathcal{C}$-closed in F_1/N_1, which is true by our assumptions, where $\psi : F_1 \to F_1/N_1$ is the natural homomorphism. Thus we can deduce that $\langle (g_1, 1) \rangle G$ is $\mathcal{C}$-closed in $F_1 \times F_2$, and so (6) implies that $\mathrm{C}_{F_1 \times F_2}((g_1, g_2))G$ is $\mathcal{C}$-closed in $F_1 \times F_2$.

We have shown that $\mathrm{C}_{F_1 \times F_2}((g_1, g_2))G$ is $\mathcal{C}$-closed in $F_1 \times F_2$ for every $(g_1, g_2) \in G$, and since $F_1 \times F_2$ is $\mathcal{C}$-hereditarily conjugacy separable (by Lemma 4.3), we can use Lemma 4.4 to conclude that G is $\mathcal{C}$-conjugacy separable. □

Proposition 4.5 can be combined with Lemma 3.11 to establish $\mathcal{C}$-hereditary conjugacy separability of subdirect products.

Corollary 4.6. *Let $\mathcal{C}$ be an extension-closed pseudovariety of finite groups and let F_1, F_2 be $\mathcal{C}$-hereditarily conjugacy separable groups with cyclic centralizers. If $G \leqslant F_1 \times F_2$ is a subdirect product such that F_1/N_1 is highly residually-$\mathcal{C}$ and cyclic subgroup $\mathcal{C}$-separable, where $N_1 \coloneqq G \cap F_1$, then G is $\mathcal{C}$-hereditarily conjugacy separable.*

Proof. Consider any $\mathcal{C}$-open subgroup H of G. Then, according to Lemma 3.11, there is a $\mathcal{C}$-open subgroup $J_i \leqslant F_i$, $i = 1, 2$, such that $H \leqslant J_1 \times J_2$ is a subdirect product and $J_1/(H \cap J_1)$ is cyclic subgroup $\mathcal{C}$-separable.

Note that for each $i = 1, 2$, J_i has cyclic centralizers, as a subgroup of F_i, and every $\mathcal{C}$-open subgroup K of J_i is also $\mathcal{C}$-open in F_i. Indeed, since the class $\mathcal{C}$ consists of finite groups, we have $|F_i : J_i| < \infty$ and $|J_i : K| < \infty$, hence $|F_i : K| < \infty$. Now, K is $\mathcal{C}$-closed in F_i by Lemma 3.4.(b) and Remark 3.2, so its complement $F_i \setminus K$ is also $\mathcal{C}$-closed in F_i, being a finite union of cosets modulo K. Therefore K must be $\mathcal{C}$-open in F_i, as the complement of a $\mathcal{C}$-closed set.

Recalling that F_i is $\mathcal{C}$-hereditarily conjugacy separable, we can conclude that so is J_i, $i = 1, 2$. It remains to apply Proposition 4.5 to deduce that H is $\mathcal{C}$-conjugacy separable. Since the latter is true for any $\mathcal{C}$-open subgroup $H \leqslant G$, we have shown that G is $\mathcal{C}$-hereditarily conjugacy separable. □

We will later see why the assumptions that F_i have cyclic centralizers and F_1/N_1 is cyclic subgroup $\mathcal{C}$-separable are essential in Proposition 4.5 (see Remark 5.3 and Subsection 4.4). It is also worth mentioning that some criteria for solvability of the conjugacy problem in subdirect products were studied by Kulikova in [34].

4.2. **Criteria for non-conjugacy separability.** In this subsection $\mathcal{C}$ will denote a pseudovariety of groups, unless specified otherwise.

We will start with the following general statement.

Lemma 4.7. *Let F_1, F_2 be groups and let $\rho_1 : F_1 \times F_2 \to F_1$ denote the natural projection. Assume that $G \leqslant F_1 \times F_2$ is a subgroup such that $\rho_1(G) = F_1$ and $N_i \coloneqq G \cap F_i$, for $i = 1, 2$. If $h_i \in N_i$, $i = 1, 2$, are arbitrary elements and x_1 belongs to the closure of N_1 in the pro-$\mathcal{C}$ topology on F_1 then the element $(x_1h_1x_1^{-1}, h_2)$ belongs to the closure of the N_1-conjugacy class $(h_1, h_2)^{N_1} \subseteq (h_1, h_2)^G$ in the pro-$\mathcal{C}$ topology on G.*

Proof. First, note that N_1 is normal in G, and hence it is normal in F_1 as $\rho_1(G) = F_1$ (see the proof of Lemma 2.1.(i)). Therefore $x_1h_1x_1^{-1} \in N_1$ and, thus, $(x_1h_1x_1^{-1}, h_2) \in N_1 \times N_2 \subseteq G$.

Since $\rho_1(G) = F_1$, we can find some element $x_2 \in F_2$ such that $(x_1, x_2) \in G$. If $K \lhd G$ is any co-$\mathcal{C}$ subgroup then $\rho_1(K)$ is a co-$\mathcal{C}$ subgroup of F_1, as $\mathcal{C}$ is closed under taking quotients, hence $x_1\rho_1(K) \cap N_1 \neq \emptyset$ in F_1, because x_1 belongs to the closure of N_1 in the pro-$\mathcal{C}$ topology on F_1. Since $(x_1, x_2) \in \rho_1^{-1}(x_1)$ and $\rho_1^{-1}(N_1) = N_1 \times N_2$, it follows that $(x_1, x_2)K \cap (N_1 \times N_2) \neq \emptyset$. The latter holds for every co-$\mathcal{C}$ subgroup K of G, showing that (x_1, x_2) belongs to the closure of $N_1 \times N_2$ in the pro-$\mathcal{C}$ topology on G.

Now, consider any homomorphism $\varphi : G \to M$, where $M \in \mathcal{C}$. Then $\varphi((x_1, x_2)) \in \varphi(N_1 \times N_2)$, i.e., there exist $a_i \in N_i$, $i = 1, 2$, such that $\varphi((x_1, x_2)) = \varphi((a_1, a_2))$ in M. Since the elements $(h_1, 1)$, $(x_1h_1x_1^{-1}, 1)$, $(1, h_2)$, (x_1, x_2), (a_1, a_2) and $(a_1, 1)$ all belong to G, their φ-images are defined, and we have

$$\begin{aligned}
&\varphi\left((x_1h_1x_1^{-1}, h_2)\right) = \varphi\left((x_1h_1x_1^{-1}, 1)\right)\varphi((1, h_2)) \\
&= \varphi((x_1, x_2))\varphi((h_1, 1))\varphi((x_1, x_2))^{-1}\varphi((1, h_2)) = \varphi((a_1, a_2))\varphi((h_1, 1))\varphi((a_1, a_2))^{-1}\varphi((1, h_2)) \\
&\qquad = \varphi\left((a_1h_1a_1^{-1}, h_2)\right) = \varphi\left((a_1, 1)(h_1, h_2)(a_1, 1)^{-1}\right) \in \varphi\left((h_1, h_2)^{N_1}\right).
\end{aligned}$$

Thus we have shown that $\varphi\left((x_1h_1x_1^{-1}, h_2)\right) \in \varphi\left((h_1, h_2)^{N_1}\right)$ for every homomorphism φ from G to a group $M \in \mathcal{C}$. This proves that $(x_1h_1x_1^{-1}, h_2)$ belongs to the closure of $(h_1, h_2)^{N_1}$ in the pro-$\mathcal{C}$ topology on G. □

The following elementary fact will be useful:

Remark 4.8. Suppose that F is any group, $G \leqslant F$ is any subgroup and $f \in F$ is any element. Then, for an arbitrary $h \in F$, $hfh^{-1} \in f^G$ if and only if $h \in G\mathrm{C}_F(f)$.

We can now formulate the first basic criterion of non-conjugacy separability of subdirect products.

Theorem 4.9. *Let $\mathcal{C}$ be a pseudovariety of groups, let $G \leqslant F_1 \times F_2$ be a subdirect product of groups F_1 and F_2, and let $N_i \coloneqq G \cap F_i$, $i = 1, 2$. Suppose that F_1/N_1 is not residually-$\mathcal{C}$ and there are elements $h_i \in N_i$ such that $\mathrm{C}_{F_i}(h_i) \subseteq N_i$, for $i = 1, 2$. Then G is not $\mathcal{C}$-conjugacy separable.*

Proof. The assumption that F_1/N_1 is not residually-$\mathcal{C}$ is equivalent to the statement that N_1 is not $\mathcal{C}$-closed in F_1, i.e., there is $x_1 \in F_1 \setminus N_1$ such that x_1 belongs to the closure of N_1 in the pro-$\mathcal{C}$ topology on F_1. Also, note that

$$\mathrm{C}_{F_1 \times F_2}((h_1, h_2)) = \mathrm{C}_{F_1}(h_1) \times \mathrm{C}_{F_2}(h_2) \subseteq N_1 \times N_2 \subseteq G,$$

therefore $G\mathrm{C}_{F_1 \times F_2}((h_1, h_2)) = G$.

In view of Lemma 4.7, to prove the theorem it is enough to check that $(x_1h_1x_1^{-1}, h_2) \notin (h_1, h_2)^G$ in G. Indeed, since $(x_1h_1x_1^{-1}, h_2) = (x_1, 1)(h_1, h_2)(x_1, 1)^{-1}$ in $F_1 \times F_2$, Remark 4.8

tells us that this element belongs to $(h_1, h_2)^G$ if and only if $(x_1, 1) \in G\mathrm{C}_{F_1\times F_2}((h_1, h_2)) = G$. But the latter is equivalent to $(x_1, 1) \in G \cap F_1 = N_1$, contradicting the choice of x_1.

We can now conclude that $(x_1h_1x_1^{-1}, h_2) \notin (h_1, h_2)^G$, but this element belongs to the closure of $(h_1, h_2)^G$ in the pro-$\mathcal{C}$ topology on G by Lemma 4.7. It follows that $(h_1, h_2)^G$ is not $\mathcal{C}$-closed in G, thus G is not $\mathcal{C}$-conjugacy separable by Remark 4.1. □

Recall that, according to Lemma 2.4, the existence of elements $h_i \in N_i$ such that $\mathrm{C}_{F_i}(h_i) \subseteq N_i$, for $i = 1, 2$, holds as long as F_i are acylindrically hyperbolic groups without non-trivial finite normal subgroups, and $N_i \neq \{1\}$, $i = 1, 2$. The latter condition simply means that $G \leqslant F_1 \times F_2$ is a *full* subdirect product. Theorem 4.9 together with Lemma 2.4 immediately yield the following:

Corollary 4.10. *Suppose that $\mathcal{C}$ is a pseudovariety of groups and F_i is an acylindrically hyperbolic group without non-trivial finite normal subgroups, $i = 1, 2$. If $G \leqslant F_1 \times F_2$ is a full subdirect product such that F_1/N_1 is not residually-$\mathcal{C}$ (where $N_1 \coloneqq G \cap F_1$) then G is not $\mathcal{C}$-conjugacy separable.*

Theorem 1.1 from the Introduction is a special case of Corollary 4.10, since every non-abelian free group or a non-elementary hyperbolic group is acylindrically hyperbolic (see Subsection 2.4).

We end this subsection by giving an explicit application of Theorem 4.9 and Lemma 4.7.

Example 4.11. Let $P \coloneqq \langle a, b \,\|\, bab^{-1}aba^{-1}b^{-1} = a^2\rangle$ be the 1-relator group introduced by Baumslag in [4]. Baumslag proved that the element a is contained in every subgroup of finite index in P, that is, it belongs to the closure of the identity element in the profinite topology on P.

Now, let F be the free group with the free generating set $\{x, y\}$, and let $\psi : F \to P$ be the epimorphism given by $\psi(x) \coloneqq a$, $\psi(y) \coloneqq b$. We can construct the symmetric fibre product $G \leqslant F \times F$ corresponding to ψ as in Subsection 2.3. It can be deduced from the proof of Lemma 2.2.(a) that $G = \langle (x, x), (y, y), (h, 1)\rangle$, where $h \coloneqq yxy^{-1}xyx^{-1}y^{-1}x^{-2}$, because $N \coloneqq \ker \psi$ is the normal closure of h in F, by construction.

Note that x belongs to the closure of N in the profinite topology of F (by Lemma 3.3.(iii)), but $x \notin N$ as $a \neq 1$ in P. Moreover, $\mathrm{C}_F(h) = \langle h\rangle \subseteq N$, as h is not a proper power in the free group F. Therefore the elements $(h, h), (xhx^{-1}, h) \in N \times N \leqslant G$ are not conjugate in G (by Remark 4.8), but are conjugate in every finite quotient of G by Lemma 4.7. In particular, G is not conjugacy separable.

4.3. **Characterizing conjugacy separable subdirect products of finite index.** Corollary 4.10 can be combined with Lemma 4.3 to give a complete characterization of $\mathcal{C}$-conjugacy separable subdirect products that have finite index

Corollary 4.12. *Let $\mathcal{C}$ be a non-trivial extension-closed pseudovariety of finite groups, and let F_i be a $\mathcal{C}$-hereditarily conjugacy separable acylindrically hyperbolic group without non-trivial finite normal subgroups, $i = 1, 2$. If $G \leqslant F_1 \times F_2$ is a subdirect product of finite index in $F_1 \times F_2$ then the following statements are equivalent:*

(1) *G is $\mathcal{C}$-conjugacy separable;*
(2) *$F_1/N_1 \in \mathcal{C}$, where $N_1 \coloneqq G \cap F_1$;*
(3) *G is $\mathcal{C}$-open in $F_1 \times F_2$.*

Proof. Note that $G \cap F_i$ is non-trivial as it has finite index in F_i, $i = 1, 2$. Thus $G \leqslant F_1 \times F_2$ is a full subdirect product.

First let us assume that G is $\mathcal{C}$-conjugacy separable. Then F_1/N_1 is residually-$\mathcal{C}$ by Corollary 4.10. But $|F_1/N_1| = |(F_1 \times F_2) : G| < \infty$ (by Lemma 2.1.(iii)) and a finite group is residually-$\mathcal{C}$ if and only if it belongs to $\mathcal{C}$, thus $F_1/N_1 \in \mathcal{C}$, and we have shown that (1) implies (2).

If $F_1/N_1 \in \mathcal{C}$ then $F_2/N_2 \in \mathcal{C}$ by Lemma 2.1.(ii) (where $N_2 \coloneqq G \cap F_2$), so $(F_1 \times F_2)/(N_1 \times N_2) \cong F_1/N_1 \times F_2/N_2 \in \mathcal{C}$. Thus $N_1 \times N_2$ is a co-$\mathcal{C}$ subgroup of $F_1 \times F_2$ contained in G, so G is $\mathcal{C}$-open in $F_1 \times F_2$ by Remark 3.2. Hence (2) implies (3).

Finally, let us assume (3) and deduce (1). Note that $F_1 \times F_2$ is $\mathcal{C}$-hereditarily conjugacy separable by Lemma 4.3. Therefore G is $\mathcal{C}$-conjugacy separable, as it is $\mathcal{C}$-open in $F_1 \times F_2$ by the assumption. Thus (3) implies (1). □

Corollary 1.2 from the Introduction is a special case of Corollary 4.12 because of Lemma 4.2.

Corollary 4.13. *Suppose that p is a prime, F_1, F_2 are non-abelian free groups and $G \leqslant F_1 \times F_2$ is a subdirect product of finite index. Then the following are equivalent:*

1. *G is p-conjugacy separable;*
2. *F_1/N_1 is a finite p-group, where $N_1 \coloneqq G \cap F_1$;*
3. *the index $|(F_1 \times F_2) : G|$ is a power of p.*

Proof. Evidently in this statement $\mathcal{C} = \mathcal{C}_p$ is the class of all finite p-groups, so the equivalence of (1) and (2) has already been proved in Corollary 1.2.

The equivalence of (2) and (3) follows from the fact that $|(F_1 \times F_2) : G| = |F_1/N_1|$ (see Lemma 2.1.(iii)). □

Example 4.14. Let F be the free group of rank 2. By Lemmas 4.2 and 4.3 every finite index subgroup of $F \times F$ is conjugacy separable (with respect to the class of all finite groups), but, in view of Corollary 4.13, it is easy to construct finite index subgroups that are not p-conjugacy separable for any prime p.

- Let p be any prime and let $G \leqslant F \times F$ be the symmetric fibre product corresponding to any epimorphism from F to $\mathbb{Z}/p\mathbb{Z}$. Then $G \cap F \times \{1\} = N \times \{1\}$, where $F/N \cong \mathbb{Z}/p\mathbb{Z}$, and Corollary 4.13 tells us that G is p-conjugacy separable but not q-conjugacy separable for any prime $q \neq p$.
- We can also take $G \leqslant F \times F$ to be the symmetric fibre product corresponding to any epimorphism from F to $\mathbb{Z}/6\mathbb{Z}$. In this case, since $\mathbb{Z}/6\mathbb{Z}$ is not a p-group, G is not p-conjugacy separable for any prime p, by Corollary 4.13.

More generally, we obtain the following statement.

Corollary 4.15. *Let $\mathcal{C}$ and $\mathcal{D}$ be two non-trivial extension-closed pseudovarieties of finite groups such that $\mathcal{C} \nsubseteq \mathcal{D}$, and let H be the free group of rank 2. Then there exists a finite index subgroup $G \leqslant H \times H$ such that G is $\mathcal{D}$-conjugacy separable but not $\mathcal{C}$-conjugacy separable.*

Proof. By the assumptions, there is some finite group $P \in \mathcal{D} \setminus \mathcal{C}$. Let $F \leqslant H$ be a finite index subgroup admitting an epimorphism $\psi : F \to P$. Then we can construct the symmetric fibre product $G \leqslant F \times F$ corresponding to ψ, so that $G \cap (F \times \{1\}) = N \times \{1\}$, where $N \coloneqq \ker \psi$.

Thus $F/N \cong P \in \mathcal{D} \setminus \mathcal{C}$, hence G is $\mathcal{D}$-conjugacy separable but not $\mathcal{C}$ -conjugacy separable by Corollary 1.2. Recalling that $|(F \times F) : G| = |F/N| < \infty$, by Lemma 2.1.(iii), and $|(H \times H) : (F \times F)| < \infty$ by construction, we deduce that $|(H \times H) : G| < \infty$, as claimed. $\square$

4.4. **Necessity of cyclic subgroup separability of the quotient.** If the reader compares the assumptions on the quotient F_1/N_1 in Proposition 4.5 and Corollary 4.10, they will immediately notice that to establish $\mathcal{C}$-conjugacy separability of a subdirect product G, Proposition 4.5 requires F_1/N_1 to be cyclic subgroup $\mathcal{C}$-separable, while Corollary 4.10 only shows that F_1/N_1 must be residually-$\mathcal{C}$ if G is $\mathcal{C}$-conjugacy separable. The goal of this subsection is to address this "gap": we will give an example showing that it is indeed necessary to require cyclic subgroup $\mathcal{C}$-separability of F_1/N_1, and just residual-$\mathcal{C}$-ness of F_1/N_1 is insufficient. We will also prove that, in general, if the former condition fails then the corresponding subdirect product possesses a finite index subgroup which is not $\mathcal{C}$-conjugacy separable.

The following observation can help in showing that a subgroup of a group is not conjugacy separable (cf. [43, Remark 3.6]); it is a converse of Lemma 4.4.

Lemma 4.16. *Let $\mathcal{C}$ be a pseudovariety of groups, let F be a group, $G \leqslant F$ a subgroup and $f \in F$ an element. If the G-conjugacy class f^G is $\mathcal{C}$-closed in F then the double coset $\mathrm{C}_F(f)G$ is $\mathcal{C}$-closed in F.*

Proof. Suppose that $h \in F \setminus G\mathrm{C}_F(f)$. Then $hfh^{-1} \notin f^G$ by Remark 4.8, so, since f^G is $\mathcal{C}$-closed in F, there must exist a co-$\mathcal{C}$ subgroup $N \lhd F$ such that $\varphi(hfh^{-1}) \notin \varphi(f)^{\varphi(G)}$ in F/N, where $\varphi : F \to F/N$ is the natural homomorphism. It follows that $\varphi(h) \notin \varphi(G)\mathrm{C}_{F/N}(\varphi(f))$ in F/N (see Remark 4.8), which obviously yields that $\varphi(h) \notin \varphi(G\mathrm{C}_F(f))$ in F/N.

Therefore for each $h \in F \setminus G\mathrm{C}_F(f)$ we found a co-$\mathcal{C}$ subgroup $N \lhd F$ such that $h \notin G\mathrm{C}_F(f)N$ (equivalently, $hN \cap G\mathrm{C}_F(f) = \emptyset$), which shows that the double coset $G\mathrm{C}_F(f)$ is $\mathcal{C}$-closed in F. Since the map $a \mapsto a^{-1}$ is a homeomorphism of F, with respect to its pro-$\mathcal{C}$ topology, we can conclude that $\mathrm{C}_F(f)G = (G\mathrm{C}_F(f))^{-1}$ is also $\mathcal{C}$-closed in F. $\square$

A priori, it may happen that for a subgroup G of a group F, G is $\mathcal{C}$-conjugacy separable but g^G is not $\mathcal{C}$-closed in F, for some $g \in G$ (even though this conjugacy class is $\mathcal{C}$-closed in G). However, this is certainly impossible if the pro-$\mathcal{C}$ topology on G is a restriction of the pro-$\mathcal{C}$ topology on F. After combining this observation with Lemma 4.16 we obtain the following corollary.

Corollary 4.17. *Suppose that $\mathcal{C}$ is a pseudovariety of groups and G is a $\mathcal{C}$-conjugacy separable subgroup of a group F. If the pro-$\mathcal{C}$ topology on G is a restriction of the pro-$\mathcal{C}$ topology on F then $\mathrm{C}_F(g)G$ is $\mathcal{C}$-closed in F for each $g \in G$.*

The next statement can be regarded as nearly a converse to Proposition 4.5.

Theorem 4.18. *Let $\mathcal{C}$ be an extension-closed pseudovariety of finite groups and let F_1, F_2 be acylindrically hyperbolic groups without non-trivial finite normal subgroups. Suppose that $G \leqslant F_1 \times F_2$ is a full subdirect product such that G is $\mathcal{C}$-conjugacy separable but F_1/N_1 is not cyclic subgroup $\mathcal{C}$-separable, where $N_1 \coloneqq G \cap F_1$. Then all of the following must hold:*

(a) *G is $\mathcal{C}$-closed in $F_1 \times F_2$ but some co-$\mathcal{C}$ subgroup G' of G is not $\mathcal{C}$-closed in $F_1 \times F_2$;*
(b) *the quotient F_1/N_1 is residually-$\mathcal{C}$ but not highly residually-$\mathcal{C}$;*
(c) *G' is not $\mathcal{C}$-conjugacy separable, thus G is not $\mathcal{C}$-hereditarily conjugacy separable.*

Proof. First, note that for each $i = 1, 2$, $N_i := G \cap F_i$ is an infinite normal subgroup of the acylindrically hyperbolic group F_i. Let $h_i \in N_i$ be the loxodromic element provided by Lemma 2.4, such that $C_{F_i}(h_i) = \langle h_i \rangle \subseteq N_i$, $i = 1, 2$.

Since G is $\mathcal{C}$-conjugacy separable, we can apply Theorem 4.9 to deduce that F_1/N_1 must be residually-$\mathcal{C}$. Therefore G is $\mathcal{C}$-closed in $F_1 \times F_2$ by Corollary 3.7. Let us now show that the pro-$\mathcal{C}$ topology on G is not a restriction of the pro-$\mathcal{C}$ topology on $F_1 \times F_2$.

By the assumptions, F_1/N_1 is not cyclic subgroup $\mathcal{C}$-separable, so there exists an element $\bar{x} \in F_1/N_1$ such that $\langle \bar{x} \rangle$ is not $\mathcal{C}$-closed in F_1/N_1. Let $\psi : F_1 \to F_1/N_1$ denote the natural epimorphism and let $u \in \psi^{-1}(\bar{x})$ be any preimage of $\bar{x}$ in F_1. Then, by Lemma 2.4, there exists $m \in \mathbb{Z}$ such that the element $x_1 := h_1^m u \in F_1$ satisfies $\mathrm{C}_{F_1}(x_1) = \langle x_1 \rangle$. Observe that $\psi(x_1) = \psi(u) = \bar{x}$ in F_1/N_1.

Now, since $G \leqslant F_1 \times F_2$ is subdirect, there exists $v \in F_2$ such that $(x_1, v) \in G$. After applying Lemma 2.4 once again, we can find some $n \in \mathbb{Z}$ such that the element $x_2 := h_2^n v \in F_2$ satisfies $\mathrm{C}_{F_2}(x_2) = \langle x_2 \rangle$. Moreover, since $(1, h_2^n) \in N_2 \subseteq G$, we have that $(x_1, x_2) = (1, h_2^n)(x_1, v) \in G$.

Observe that in $F_1 \times F_2$ we have

$$\mathrm{C}_{F_1 \times F_2}((x_1, x_2))G = \langle (x_1, 1), (1, x_2) \rangle G = \langle (x_1, 1) \rangle \langle (x_1, x_2) \rangle G = \langle (x_1, 1) \rangle G.$$

So, according to Lemma 3.6, to show that the double coset $\mathrm{C}_{F_1 \times F_2}((x_1, x_2))G$ is not $\mathcal{C}$-closed in $F_1 \times F_2$, it is enough to prove that $\langle x_1 \rangle N_1$ is not $\mathcal{C}$-closed in F_1. However, $\langle x_1 \rangle N_1 = \psi^{-1}(\langle \bar{x} \rangle) \leqslant F_1$, and since $\langle \bar{x} \rangle$ is not $\mathcal{C}$-closed in P, it follows that $\langle x_1 \rangle N_1$ is not closed in the pro-$\mathcal{C}$ topology on F_1 (see Lemma 3.3.(iii)). Thus $\mathrm{C}_{F_1 \times F_2}((x_1, x_2))G$ is not $\mathcal{C}$-closed in $F_1 \times F_2$, even though G is $\mathcal{C}$-conjugacy separable by the assumption. Consequently, we can use Corollary 4.17 to conclude that the pro-$\mathcal{C}$ topology on G is not a restriction of the pro-$\mathcal{C}$ topology on $F_1 \times F_2$. In view of Lemma 3.4.(a), this means that there is a co-$\mathcal{C}$ subgroup $G' \lhd G$ such that G' is not closed in the pro-$\mathcal{C}$ topology on $F_1 \times F_2$. Hence F_1/N_1 cannot be highly residually-$\mathcal{C}$ by Lemma 3.9. Thus we have proved claims (a) and (b), and it remains to prove claim (c).

Let F_i' denote the projection of G' to the i-th coordinate group, and let $N_i' := G' \cap F_i' = G' \cap F_i$, $i = 1, 2$. Since $G/G' \in \mathcal{C}$, $|G : G'| < \infty$ and F_i' is a co-$\mathcal{C}$ subgroup of F_i, $i = 1, 2$, by our assumptions on $\mathcal{C}$. Hence $F_1' \times F_2'$ is a co-$\mathcal{C}$ subgroup of $F_1 \times F_2$. It also follows that $|N_i : N_i'| < \infty$, so N_i' must be infinite as $|N_i| = \infty$, $i = 1, 2$. Therefore $G' \leqslant F_1' \times F_2'$ is a full subdirect product, and G' is not $\mathcal{C}$-closed in $F_1' \times F_2'$ by Lemma 3.4.(b), because G' is not $\mathcal{C}$-closed in $F_1 \times F_2$. Therefore F_1'/N_1' is not residually-$\mathcal{C}$ by Corollary 3.7.

Finally, for each $i = 1, 2$, F_i' is an infinite normal subgroup of the acylindrically hyperbolic group F_i by construction, so F_i' is itself acylindrically hyperbolic by [52, Cor. 1.5] and $\mathrm{E}_{F_i'}(F_i') \subseteq \mathrm{E}_{F_i}(F_i') = \{1\}$ by Lemma 2.4. Thus G' satisfies all the assumptions of Corollary 4.10, which implies that G' is not $\mathcal{C}$-conjugacy separable, and so claim (c) holds. □

We are now ready to construct an example showing that it is necessary to assume cyclic subgroup $\mathcal{C}$-separability of F_1/N_1 in Proposition 4.5.

Example 4.19. Let $P = \langle a, t \,\|\, tat^{-1} = a^2 \rangle$ be a metabelian Baumslag-Solitar group. Then P is highly residually finite (e.g., by [35, Cor. 4.17]). It follows that P is highly residually-$\mathcal{C}$, where $\mathcal{C}$ is either the class of all finite groups or the class of solvable finite groups.

However, P is not cyclic subgroup $\mathcal{C}$-separable, because $\langle a \rangle$ is conjugate in P to its proper subgroup $\langle a^2 \rangle$, which implies that in any finite quotient of P these two cyclic subgroups have the same image. Hence a belongs to the closure of $\langle a^2 \rangle$ in the pro-$\mathcal{C}$ topology on P, i.e., the cyclic subgroup $\langle a^2 \rangle$ is not $\mathcal{C}$-closed in P.

Now, consider any epimorphism $\psi : F \to P$, where F is the free group of rank 2, and let $G \leqslant F \times F$ be the resulting symmetric fibre product (see Subsection 2.3). Then G is a full subdirect product in $F \times F$ and $F/(G \cap F) \cong P$ is highly residually-$\mathcal{C}$. However, G cannot be $\mathcal{C}$-conjugacy separable by claim (b) of Theorem 4.18.

Theorem 4.18 does not quite close the gap between cyclic subgroup $\mathcal{C}$-separability and residual-$\mathcal{C}$-ness of F_1/N_1. Thus the answer to the following natural question is still unknown:

Question 4.20. Does there exist an extension-closed pseudovariety of finite groups $\mathcal{C}$ and a full subdirect product $G \leqslant F_1 \times F_2$, of two non-abelian free groups F_1, F_2, such that G is $\mathcal{C}$-conjugacy separable but F_1/N_1 is not cyclic subgroup $\mathcal{C}$-separable, where $N_1 \coloneqq G \cap F_1$?

In Section 7 we will show that the answer to Question 4.20 is negative when $\mathcal{C} = \mathcal{C}_p$ is the class of finite p-groups; we do not have an answer in case when $\mathcal{C}$ is the class of all finite groups.

5. An exotic hereditarily conjugacy separable group

In this section we will see how the criteria from Section 4 can be combined with the constructions of subdirect products from Subsection 2.3 to produce hereditarily conjugacy separable groups that possess finite index overgroups with unsolvable conjugacy problem. We will first give finitely generated (but not finitely presented) examples, as these are easier to construct, before giving finitely presented examples in Subsection 5.2 (see Theorem 5.6).

Throughout this section $\mathcal{C}$ will always be the class of all finite groups, so we will simply talk about conjugacy separability, cyclic subgroup separability, etc. (suppressing $\mathcal{C}$).

5.1. A finitely generated example.

The following lemma will be used to construct finite index overgroups with unsolvable conjugacy problem.

Lemma 5.1. *Let P be a finitely generated group with a finitely generated subgroup $Q \leqslant P$. Then for every integer $k \geq 2$ there exists a finitely generated free group F, an automorphism $\sigma \in Aut(F)$ and an epimorphism $\psi : F \to P$ such that*

- *the order of σ is k;*
- *$\psi(\mathrm{Fix}(\sigma)) = Q$, where $\mathrm{Fix}(\sigma) \coloneqq \{h \in F \mid \sigma(h) = h\}$ is the subgroup of fixed points of σ;*
- *σ induces the identity automorphism of P, i.e., $\sigma(\ker\psi) = \ker\psi$ and $\psi(\sigma(f)) = \psi(f)$ for all $f \in F$.*

Proof. Clearly we can suppose that P is generated by some elements $a_1, \dots, a_m, b_1, \dots, b_n \in P$, such that $a_1, \dots, a_m$ generate Q. Take $F = F(Z)$ to be the free group on a set

$$Z \coloneqq \{x_1, \dots, x_m, y_{11}, \dots, y_{1k}, \dots, y_{n1}, \dots, y_{nk}\}$$

of cardinality $m + kn$. Let $\psi : F \to P$ be the epimorphism defined by $\psi(x_l) \coloneqq a_l$ and $\psi(y_{ij}) \coloneqq b_i$, for $l = 1, \dots, m$, $i = 1, \dots, n$ and $j = 1, \dots, k$.

Now let $\sigma : F \to F$ be the automorphism given by the following permutation of X:

$$\sigma(x_l) \coloneqq x_l, \text{ for all } l = 1, \dots, m, \ \sigma(y_{ij}) \coloneqq y_{i,j+1}, \text{ for all } i = 1, \dots, n, j = 1, \dots, k,$$

where the addition of indices is done modulo k. Evidently σ has order k in $Aut(F)$ and induces the identity automorphism of P.

Clearly, $\langle x_1, \dots, x_m \rangle \subseteq \mathrm{Fix}(\sigma)$. On the other hand, if $w \in F$ is a reduced word fixed by σ, then $\sigma(w)$ is also a reduced word of the same length. So $\sigma(w) = w$ can only occur if

w consists entirely of letters from $\{x_1, \dots, x_m\}^{\pm 1}$, which shows that $\mathrm{Fix}(\sigma) \subseteq \langle x_1, \dots, x_m \rangle$. Hence $\mathrm{Fix}(\sigma) = \langle x_1, \dots, x_m \rangle$, and so $\psi(\mathrm{Fix}(\sigma)) = \langle \psi(x_1), \dots \psi(x_m) \rangle = Q$, as required. □

Let F be a group generated by a finite set Z and let $Y \subseteq F$ be a subset. We will say that the *membership problem for Y in F is solvable* if there is an algorithm which, given any word W over $Z^{\pm 1}$, decides whether or not W represents an element of Y in F. The *conjugacy problem in F is solvable* if there exists an algorithm taking on input two words over $Z^{\pm 1}$, and deciding whether or not the elements of F represented by these words are conjugate in F.

By a *normal overgroup* of a group G we mean a group K, which contains a normal subgroup isomorphic to G (to simplify the notation we will identify G with this normal subgroup).

Theorem 5.2. *For every integer $k \geq 2$ there exists a finitely generated subdirect product $G \leqslant F \times F$, where F is a finitely generated non-abelian free group, satisfying the following. The group G is hereditarily conjugacy separable but there is a normal overgroup K, of G, such that $|K : G| = k$, K is not conjugacy separable and has unsolvable conjugacy problem.*

Proof. Let P be a finitely presented group satisfying the following three conditions:

1. P is highly residually finite;
2. P is cyclic subgroup separable;
3. there is a finitely generated subgroup $Q \leqslant P$ such that the membership problem for Q in P is unsolvable.

For example, we can take P to be the direct product of two free groups of rank 2. Indeed, such a group P is highly residually finite because for finitely generated groups this property is stable under direct products (cf. [35, Cor. 2.11]) and free groups obviously have it (as any finite-by-free group is virtually free). That direct products of free groups are cyclic subgroup separable can be extracted from [12, Thm. 4.4]. Finally, the existence of a finitely generated subgroup Q with unsolvable membership problem in the direct product of two free groups of rank 2 was proved by Mihaĭlova in [41, Thm. 1].

Let F be the finitely generated free group, $\sigma \in Aut(F)$ be the automorphism and $\psi : F \to P$ be the epimorphism given by Lemma 5.1. We also let $G \leqslant F \times F$ be the symmetric fibre product corresponding to ψ. Then G is finitely generated by Lemma 2.2.(a), and G is hereditarily conjugacy separable by Corollary 4.6 and Lemma 4.2.

Now, let $\tilde{F} := F \rtimes_\sigma \langle t \rangle_k$ be the semidirect product of the free group F with the cyclic group $\langle t \rangle_k$ of order k, where $tft^{-1} := \sigma(f)$ for all $f \in F$. Since $\sigma(N) = N$, where $N := \ker \psi$, and σ induces the identity automorphism of P, ψ can be naturally extended to an epimorphism $\tilde{\psi} : \tilde{F} \to P \times \langle t \rangle_k$, by defining $\tilde{\psi}(f) = \psi(f)$, for all $f \in F$, and $\tilde{\psi}(t) = t$. Then $\ker \tilde{\psi} = N$, so if $K \leqslant \tilde{F} \times \tilde{F}$ is the symmetric fibre product corresponding to $\tilde{\psi}$ then $K \cap (\tilde{F} \times \{1\}) = N \times \{1\}$. Evidently we can identify G with the subgroup of K defined by

$$\{(g_1, g_2) \in F \times F \leqslant \tilde{F} \times \tilde{F} \mid \tilde{\psi}(g_1) = \tilde{\psi}(g_2)\} \leqslant K.$$

In other words, $G = K \cap (F \times F)$ in $\tilde{F} \times \tilde{F}$. By construction, F is normal in $\tilde{F}$, yielding that G is normal in K. Recall that $K/(N \times \{1\}) \cong \tilde{F}$ and $G/(N \times \{1\}) \cong F$, so

$$K/G \cong \frac{K/(N \times \{1\})}{G/(N \times \{1\})} \cong \tilde{F}/F \cong \langle t \rangle_k, \text{ thus } |K : G| = k.$$

It remains to show that K is not conjugacy separable and has unsolvable conjugacy problem. Observe that for every $x \in F$, $\tilde{\psi}(x^{-1}tx) = \tilde{\psi}(t)$, because $t = \tilde{\psi}(t)$ is central in $P \times \langle t \rangle_k$. Therefore $(x^{-1}tx, t) \in K$ for each $x \in F$.

Now, for any $x \in F$, since $(x^{-1}tx, t) = (x^{-1}, 1)(t, t)(x^{-1}, 1)^{-1}$, the element $(x^{-1}tx, t)$ is conjugate to (t, t) in K if and only if $(x^{-1}, 1) \in K\mathrm{C}_{\tilde{F}\times\tilde{F}}((t, t))$ (see Remark 4.8) if and only if $(x, 1) \in \mathrm{C}_{\tilde{F}\times\tilde{F}}((t, t))K$ if and only if $(x, 1) \in (\mathrm{C}_{\tilde{F}}(t) \times \mathrm{C}_{\tilde{F}}(t))K = (\mathrm{C}_{\tilde{F}}(t) \times \{1\})K$ (because K contains the diagonal subgroup of $\tilde{F} \times \tilde{F}$, by definition) if and only if $x \in \mathrm{C}_{\tilde{F}}(t)N$ (cf. Remark 3.5) if and only if $x \in \mathrm{C}_F(t)N$ (as $x \in F$ and $N \subseteq F$, so $F \cap \mathrm{C}_{\tilde{F}}(t)N = \mathrm{C}_F(t)N$).

Thus, if we were able to solve the conjugacy problem in K, then we would be able to solve the membership problem for $\mathrm{C}_F(t)N$ in F. But $\mathrm{C}_F(t) = \mathrm{Fix}(\sigma)$ and $\psi(\mathrm{Fix}(\sigma)) = Q$, hence $\psi^{-1}(Q) = \mathrm{C}_F(t)N$. And so the membership problem for $\mathrm{C}_F(t)N$ in F is equivalent to the membership problem for Q in P, which is undecidable by construction. Therefore the conjugacy problem in K is unsolvable.

Finally, let us show that K is not conjugacy separable. Note that we cannot use Mostowski's result [48, Thm. 3] for this, as K is not finitely presented by a theorem of Baumslag and Roseblade [6, Thm. B]. Since P is a finitely presented group and the membership problem for the finitely generated subgroup Q in P is unsolvable, Q cannot be closed in the profinite topology of P by a standard Mal'cev-type argument (see [37, §7] or [48, Thm. 2]). Therefore $\mathrm{C}_F(t)N = \psi^{-1}(Q)$ is not closed in the profinite topology of F by Lemma 3.3.(iii). Observe that $F\times\{1\}\cap\mathrm{C}_{\tilde{F}\times\tilde{F}}((t, t))K = \mathrm{C}_F(t)N\times\{1\}$ (we have essentially shown this earlier in the proof), so the double coset $\mathrm{C}_{\tilde{F}\times\tilde{F}}((t, t))K$ is not closed in the profinite topology of $\tilde{F} \times \tilde{F}$ by Remark 3.1. Therefore $(t, t)^K$ is not closed in the profinite topology of $\tilde{F} \times \tilde{F}$ by Lemma 4.16. But the profinite topology on K is the restriction of the profinite topology on $\tilde{F} \times \tilde{F}$ by Lemma 3.9, because $\tilde{F}/N \cong P \times \mathbb{Z}/k\mathbb{Z}$ is highly residually finite (e.g., by [35, Cor. 2.11]). Consequently, we can conclude that the conjugacy class $(t, t)^K$ is not closed in the profinite topology of K, thus K is not conjugacy separable and the proof is complete. □

Remark 5.3. Theorem 5.2 shows that it is necessary to assume in Proposition 4.5 that the groups F_i have cyclic centralizers, $i = 1, 2$. Indeed, the group $\tilde{F}$, the subdirect product $K \leqslant \tilde{F}\times\tilde{F}$ and the normal subgroup $N \lhd \tilde{F}$ which were constructed in the proof of Theorem 5.2 clearly satisfy all the remaining assumptions of Proposition 4.5, nevertheless, K is not conjugacy separable.

5.2. **Finitely presented examples.** To prove Theorem 5.6 we will need an enhancement of Lemma 5.1. It will use classical small cancellation theory, and we refer the reader to [36, Sec. V.2] for the background and definitions. Some of the basic properties of small cancellation groups are summarized in the next statement.

Lemma 5.4. *Suppose that $\lambda \in (0, 1/6]$ and F is a group given by a finite presentation $\langle Z \,\|\, \mathcal{R}\rangle$ satisfying the small cancellation condition $C'(\lambda)$. Then*

(i) *F is hyperbolic;*
(ii) *F has cyclic centralizers;*
(iii) *if $f \in F$ is an element of finite order then there is a word R over $Z^{\pm 1}$ and an integer $n \geq 2$ such that $R^n \in \mathcal{R}$ is a defining relator and f is conjugate in F to an element represented by a power of R.*

Proof. (i) The group F is hyperbolic because, by Greendlinger's lemma [36, Thm. 4.5 in Sec. V.4], any $C'(1/6)$ presentation is necessarily a *Dehn presentation* and any group with a finite Dehn presentation is hyperbolic (cf. [7, Thm. 2.6 in Ch. III.Γ]).

(ii) That centralizers of non-trivial elements in a $C'(1/6)$-group are cyclic was proved by Truffault [60].

The statement (iii), in this generality, is due to Greendlinger [27, Thm. VIII]. □

The main technical tool in the proof of Theorem 5.6 is the following proposition, which may also have other applications in the future.

Proposition 5.5. *Let P be a finitely presented group and let $Q \leqslant P$ be a finitely generated subgroup. Then for each real number $\lambda > 0$ and every integer $k \geq 2$ there exists a torsion-free group F, with a finite $C'(\lambda)$ small cancellation presentation, an automorphism $\sigma \in Aut(F)$ and an epimorphism $\psi : F \to P$ such that*

- $\ker\psi$ *is generated by k elements;*
- *the order of σ is k;*
- $\psi(\mathrm{Fix}(\sigma)) = Q$, *where* $\mathrm{Fix}(\sigma) \coloneqq \{h \in F \mid \sigma(h) = h\}$ *is the subgroup of fixed points of σ;*
- *σ induces the identity automorphism of P, i.e., $\sigma(\ker\psi) = \ker\psi$ and $\psi(\sigma(f)) = \psi(f)$ for all $f \in F$.*

Proof. The argument is essentially a combination of Rips's original idea from [56, Thm.] with the proof of Lemma 5.1.

As before, we can suppose that P is generated by some elements $a_1, \dots, a_m, b_1, \dots, b_n$ and $Q = \langle a_1, \dots, a_m\rangle$. Let $\langle a_1, \dots, a_m, b_1, \dots, b_n \,\|\, R_1, \dots, R_s\rangle$ be a finite presentation of P.

Let F be the group given by a finite presentation $\langle Z \,\|\, \mathcal{R}\rangle$, where

$$Z \coloneqq \{x_1, \dots, x_m, y_{11}, \dots, y_{1k}, \dots, y_{n1}, \dots, y_{nk}, z_1, \dots, z_k\}$$

is a generating set of cardinality $m+nk+k$. To specify the set of defining relators $\mathcal{R}$, for all $t \in \{1, \dots, s\}$ and $j \in \{1, \dots, k\}$, let R_{tj} be the word over the alphabet $\{x_1, \dots, x_m, y_{1j} \dots, y_{nj}\}^{\pm 1}$ obtained from the word R_t by replacing each letter a_l with x_l and each b_i with y_{ij}, $l = 1, \dots, m$, $i = 1, \dots, n$.

Let

$$\alpha_t < \beta_t,\ \gamma_i < \delta_i,\ \varepsilon_l < \zeta_l,\ \varepsilon'_l < \zeta'_l,\ \eta_{ip} < \theta_{ip} \text{ and } \eta'_{ip} < \theta'_{ip}, \tag{7}$$

$t = 1, \dots, s$, $i = 1, \dots, n$, $l = 1, \dots, m$, $p = 0, \dots, k-1$, be some large positive integers that we will specify later. The set of defining relators $\mathcal{R}$, of F, will consist of the following words (where the addition of lower indices at z_* and $y_{i,*}$ is done modulo k):

$$R_{tj} z_j z_{j+1}^{\alpha_t} z_j z_{j+1}^{\alpha_t+1} z_j z_{j+1}^{\alpha_t+2} \dots z_j z_{j+1}^{\beta_t},\ t = 1, \dots, s,\ j = 1, \dots, k, \tag{8}$$

$$y_{ij}^{-1} y_{i,j+1} z_j z_{j+1}^{\gamma_i} z_j z_{j+1}^{\gamma_i+1} z_j z_{j+1}^{\gamma_i+2} \dots z_j z_{j+1}^{\delta_i},\ i = 1, \dots, n,\ j = 1, \dots, k, \tag{9}$$

$$x_l^{-1} z_j x_l z_j z_{j+1}^{\varepsilon_l} z_j z_{j+1}^{\varepsilon_l+1} z_j z_{j+1}^{\varepsilon_l+2} \dots z_j z_{j+1}^{\zeta_l},\ l = 1, \dots, m,\ j = 1, \dots, k, \tag{10}$$

$$x_l z_j x_l^{-1} z_j z_{j+1}^{\varepsilon'_l} z_j z_{j+1}^{\varepsilon'_l+1} z_j z_{j+1}^{\varepsilon'_l+2} \dots z_j z_{j+1}^{\zeta'_l},\ l = 1, \dots, m,\ j = 1, \dots, k, \tag{11}$$

$$y_{i,p+j}^{-1} z_j y_{i,p+j} z_j z_{j+1}^{\eta_{ip}} z_j z_{j+1}^{\eta_{ip}+1} \dots z_j z_{j+1}^{\theta_{ip}},\ i = 1, \dots, n,\ p = 0, \dots, k-1,\ j = 1, \dots, k, \tag{12}$$

$$y_{i,p+j} z_j y_{i,p+j}^{-1} z_j z_{j+1}^{\eta'_{ip}} z_j z_{j+1}^{\eta'_{ip}+1} \dots z_j z_{j+1}^{\theta'_{ip}},\ i = 1, \dots, n,\ p = 0, \dots, k-1,\ j = 1, \dots, k. \tag{13}$$

Clearly for $\mu := \min\{\lambda, 1/8\}$ we can choose the positive integers (7) in such a way that all the intervals $[\alpha_t, \beta_t]$, $[\gamma_i, \delta_i]$, $[\varepsilon_l, \zeta_l]$, $[\varepsilon'_l, \zeta'_l]$, $[\eta_{ip}, \theta_{ip}]$ and $[\eta'_{ip}, \theta'_{ip}]$, $t = 1, \dots, s$, $i = 1, \dots, n$, $l = 1, \dots, m$, $p = 0, \dots, k-1$, are very long (compared to the maximum of the lengths of the words R_{tj}) and pairwise disjoint, so that the set $\mathcal{R}$ satisfies the small cancellation condition $C'(\mu)$. It follows that $\mathcal{R}$ satisfies both $C'(\lambda)$ and $C'(1/8)$. Note that no defining relator from $\mathcal{R}$ is a proper power, hence F is torsion-free by Lemma 5.4.(iii).

Since the indices l, i and p are independent of j in (10)–(13), these relators ensure that $N := \langle z_1, \dots, z_k \rangle$ is normal in F, and the relators (8)–(9) ensure that the quotient F/N is naturally isomorphic to P. More precisely, we can define the epimorphism $\psi : F \to P$, with $\ker \psi = N$, by setting $\psi(x_l) := a_l$, $\psi(y_{ij}) := b_i$ and $\psi(z_j) = 1$, for $l = 1, \dots, m$, $i = 1, \dots, n$ and $j = 1, \dots, k$.

Now, define a permutation σ of the generating set Z of F by

$$\sigma(x_l) := x_l,\ \sigma(y_{ij}) := y_{i,j+1},\ \sigma(z_j) := z_{j+1},$$

for all $l = 1, \dots, m$, $i = 1, \dots, n$, $j = 1, \dots, k$. Naturally, σ extends to a permutation of the set of words over $Z^{\pm 1}$, so that each group of the defining relators (8)–(13) is invariant under σ by construction (one can see that σ acts by adding 1 to the index j modulo k). Therefore $\sigma(\mathcal{R}) = \mathcal{R}$, hence σ defines an automorphism of F, which, by abusing the notation, we will again denote by $\sigma \in Aut(F)$. Clearly σ^k is the identity automorphism of F, but $\sigma^\varkappa$ is non-trivial in $Aut(F)$, if $1 \le \varkappa < k$, because $z_1 \neq z_{1+\varkappa} = \sigma^\varkappa(z_1)$ in F (otherwise, by Greendlinger's lemma [36, Thm. 4.5 in Sec. V.4], the cyclically reduced word $z_1^{-1} z_{1+\varkappa}$, of length 2, would contain more than a half of a cyclic permutation of a word from $\mathcal{R}^{\pm 1}$, but the length of any word from $\mathcal{R}$ is greater than 8, by construction). Therefore the order of σ in $Aut(F)$ is k. Obviously σ induces the identity automorphism of P.

It remains to show that $\mathrm{Fix}(\sigma) = \langle x_1, \dots, x_m \rangle$ in F. Evidently, $\langle x_1, \dots, x_m \rangle \subseteq \mathrm{Fix}(\sigma)$ by the definition of σ. Arguing by contradiction, assume that the converse inclusion does not hold, and take an element $w \in \mathrm{Fix}(\sigma) \setminus \langle x_1, \dots, x_m \rangle$ such that w has the shortest possible length when expressed as a word over the alphabet $Z^{\pm 1}$. Choose a geodesic (i.e., of minimal length) word W over this alphabet representing w in F.

Since $\sigma(w) = w$, the word $W^{-1}\sigma(W)$ represents the identity element of F. Obviously, the words W^{-1} and $\sigma(W)$ are both freely reduced and geodesic in F, as W is geodesic. Also, note that W cannot start with x_l^ξ, where $\xi = \pm 1$ and $l \in \{1, \dots, m\}$, because otherwise $x_l^{-\xi} w$ would be an element of $\mathrm{Fix}(\sigma) \setminus \langle x_1, \dots, x_m \rangle$ which is strictly shorter than w. Similarly, W cannot end with a letter from $\{x_1, \dots, x_m\}^{\pm 1}$. It follows that the first letter of $\sigma(W)$ is not the inverse of the last letter of W^{-1}, and the last letter of $\sigma(W)$ is not the inverse of the first letter of W^{-1}. Therefore the word $W^{-1}\sigma(W)$ is freely cyclically reduced, and since $\mathcal{R}$ satisfies $C'(\mu)$, we can apply [36, Thm. 4.4 in Sec. V.4]. This theorem claims that there is a cyclic permutation S, of a word from $\mathcal{R}^{\pm 1}$, and a prefix U, of S, such that U is a subword of $W^{-1}\sigma(W)$ and $\|U\| > (1 - 3\mu)\|S\|$, where $\|U\|$ denotes the length of the word U.

Now, observe that $1 - 3\mu > 1/2$ as $\mu \le 1/8$, so the word U contains more than a half of the relator S, of F, and hence it is not geodesic in F. Therefore U cannot be solely a subword of W^{-1} or of $\sigma(W)$, each which is geodesic by construction. Consequently, we can write $U \equiv U_1 U_2$, where U_1 is the suffix of W^{-1} and U_2 is a prefix of $\sigma(W)$, such that $\max\{\|U_1\|, \|U_2\|\} \le \frac{1}{2}\|S\|$. Since $\|U_1\| + \|U_2\| = \|U\| > (1 - 3\mu)\|S\|$, we deduce that $\min\{\|U_1\|, \|U_2\|\} > (1/2 - 3\mu)\|S\|$.

We will now assume that $\|U_1\| \le \|U_2\|$, as the case when $\|U_2\| < \|U_1\|$ can be treated similarly. Let us replace U_2 with its prefix that has the same length as U_1. Then U_1, U_2 are disjoint subwords of S satisfying $\|U_1\| = \|U_2\| > (1/2 - 3\mu)\|S\|$. Moreover, since U_1 is the suffix of W^{-1} and U_2 is the prefix of $\sigma(W)$, the word $\sigma(U_1)^{-1}$ coincides with the word U_2. But $\sigma(U_1)^{-1}$ is a suffix of the word $\sigma(S)^{-1}$, which is a cyclic permutation of some defining relator from $\mathcal{R}^{\pm 1}$ because $\sigma(\mathcal{R}) = \mathcal{R}$. Thus $U_2 \equiv \sigma(U_1)^{-1}$ is a common subword of two cyclic permutations S and $\sigma(S)^{-1}$, of words from $\mathcal{R}^{\pm 1}$. Note that by the construction of the defining relators (8)–(13), no relator T is equal to a cyclic permutation of $\sigma(T)^{-1}$. Therefore the word U_2 is a piece of S (in the terminology of [36, Sec. V.2]), whose length is greater than $(1/2 - 3\mu)\|S\| \geq \frac{1}{8}\|S\|$, as $\mu \leq 1/8$, contradicting the fact that the set $\mathcal{R}$ satisfies $C'(1/8)$.

Therefore there can be no elements $w \in \mathrm{Fix}(\sigma) \setminus \langle x_1, \dots, x_m \rangle$. Thus $\mathrm{Fix}(\sigma) = \langle x_1, \dots, x_m \rangle$, so

$$\psi(\mathrm{Fix}(\sigma)) = \langle \psi(x_1) \dots, \psi(x_m) \rangle = \langle a_1, \dots, a_m \rangle = Q.$$

This finishes the proof of the proposition. □

Theorem 5.6. *For each real number $\lambda > 0$ and every integer $k \geq 2$ there exists a finitely presented subdirect product $G \leqslant F \times F$, where F is a finitely presented torsion-free $C'(\lambda)$-group, satisfying the following. The group G is hereditarily conjugacy separable but there is a normal overgroup K, of G, such that $|K : G| = k$, K is not conjugacy separable and has unsolvable conjugacy problem.*

Proof. Take any finitely presented group P satisfying the following four conditions:

1. P is highly residually finite;
2. P is cyclic subgroup separable;
3. there is a finitely generated subgroup $Q \leqslant P$ such that the membership problem for Q in P is unsolvable;
4. P is of type F_3.

As before, we can take P to be the direct product of two free groups of rank 2 (in the proof of Theorem 5.2 we have already explained that it would satisfy conditions 1–3, and, obviously, it would also satisfy condition 4).

Take $\mu := \min\{\lambda, 1/6\}$ and apply Proposition 5.5 to find a torsion-free $C'(\mu)$ group F, an automorphism $\sigma \in Aut(F)$ and an epimorphism $\psi : F \to P$ from its claim. Then F has cyclic centralizers by Lemma 5.4.(ii).

Let $G \leqslant F \times F$ be the symmetric fibre product corresponding to ψ. Then G is finitely presented by Lemma 2.3, because $N := \ker \psi$ is finitely generated, F is finitely presented and P is of type F_3. Observe that F is hereditarily conjugacy separable by [47, Cor. 1.3], and, hence, so is G by Corollary 4.6.

As before, we let $\tilde{F} := F \rtimes_\sigma \langle t \rangle_k$ be the semidirect product of the group F with the cyclic group $\langle t \rangle_k$ of order k, where $tft^{-1} := \sigma(f)$ for all $f \in F$. By construction, ψ extends to an epimorphism $\tilde{\psi} : \tilde{F} \to P \times \langle t \rangle_k$, where $\tilde{\psi}(f) = \psi(f)$, for all $f \in F$, $\tilde{\psi}(t) = t$ and $\ker \tilde{\psi} = N$. Finally, we let $K \leqslant \tilde{F} \times \tilde{F}$ be the symmetric fibre product corresponding to $\tilde{\psi}$.

It remains to repeat the arguments from the proof of Theorem 5.2 to show that G can be identified with a normal subgroup of index k in K, K has unsolvable conjugacy problem and K is not conjugacy separable. (Observe that in this case the fact that K is not conjugacy separable can be deduced from Mostowski's result [48, Thm. 3], because K is finitely presented and has unsolvable conjugacy problem.) □

6. Conjugacy separability of finite index overgroups

In this section we will first show that a group which is not hereditarily conjugacy separable always has a finite index overgroup which is not conjugacy separable. Our second goal will be to produce an example of a finitely presented group G possessing an non-conjugacy separable subgroup of index 2, such that every finite index normal overgroup of G is conjugacy separable. This section is still concerned with the case when $\mathcal{C}$ is the class of all finite groups.

6.1. **Constructing non-conjugacy separable overgroups from subgroups.** Given a group G, a subgroup $H \leqslant G$ and elements $x, y \in G$, we will write $x \sim_H y$ if there exists $h \in H$ such that $x = hyh^{-1}$. If no such $h \in H$ exists, then we will write $x \not\sim_H y$.

Let us start with the following easy observation.

Lemma 6.1. *Let G be a group with a subgroup $H \leqslant G$ of index 2. Then there are an overgroup K, of G, with $|K : G| = 3$, and an element $a \in K$, centralizing H, such that for any $x \in H$ and $y \in G$, $x \sim_H y$ if and only if $ax \sim_K ay$ in K.*

Proof. Let $K \coloneqq \langle a \rangle_3 \rtimes G$, where G acts on the cyclic group $\langle a \rangle_3$, of order 3, as follows:

$$gag^{-1} = \begin{cases} a & \text{if } g \in H \\ a^2 & \text{if } g \notin H \end{cases}.$$

This action is well-defined since $|G/H| = 2$ and $a \mapsto a^2$ is an automorphism of $\langle a \rangle_3$ of order 2.

Clearly $|K : G| = 3$ and a centralizes H, so it remains to check the last claim. Let $x \in H$ and $y \in G$ be arbitrary elements. If there is $h \in H$ such that $x = hyh^{-1}$ then $h(ay)h^{-1} = ahyh^{-1} = ax$, because h commutes with a, hence ax is conjugate to ay in K.

Conversely, suppose that $ax \sim_K ay$. Then there exist $g \in G$ and $\varepsilon \in \{0, 1, 2\}$ such that $a^\varepsilon g(ay)g^{-1}a^{-\varepsilon} = ax$ in K. This is equivalent to $g(ay)g^{-1} = ax$, as a^ε commutes with $ax \in aH$, thus we have $(gag^{-1})(gyg^{-1}) = ax$. Since $gag^{-1}, a \in \langle a \rangle$, $gyg^{-1}, x \in G$ and K is the semidirect product of $\langle a \rangle$ with G, we can deduce that $gag^{-1} = a$ and $gyg^{-1} = x$. But the former equality implies that $g \in H$, so the latter equality yields $x \sim_H y$.

Thus we have proved that $x \sim_H y$ is equivalent to $ax \sim_K ay$, as required. □

Lemma 6.1 immediately gives the following corollary.

Corollary 6.2. *If G is a finitely generated group possessing a subgroup H, of index 2, which has unsolvable conjugacy problem, then G has an overgroup K, with $|K : G| = 3$, such that K has unsolvable conjugacy problem.*

We can also deduce the analogous fact for conjugacy separability.

Corollary 6.3. *If G is a group possessing a subgroup H, of index 2, which is not conjugacy separable, then G has an overgroup K, with $|K : G| = 3$, such that K is not conjugacy separable.*

Proof. Let K be the overgroup of G and let $a \in K$ be the element centralizing H, given by Lemma 6.1. Since H is not conjugacy separable by the assumptions, there are two elements $x, y \in H$ such that $x \not\sim_H y$ but x is conjugate to y in every finite quotient of H.

Then $ax \not\sim_K ay$, but for every homomorphism $\varphi : K \to M$, where M is a finite group, we have $\varphi(x) \sim_{\varphi(H)} \varphi(y)$, which implies that $\varphi(ax) \sim_M \varphi(ay)$, as $\varphi(a)$ commutes with every element of $\varphi(H)$ in M. Therefore K is not conjugacy separable. □

Theorem 6.10 below shows that the index $|K : G| = 3$ is optimal in Corollary 6.3, as an index 2 overgroup would necessarily be normal. The next proposition deals with the general case. It gives an exponential bound on the index $|K : G|$ in terms of $|G : H|$, which is not always optimal.

Proposition 6.4. *Let G be a group with a subgroup $H \leqslant G$ of index $k \in \mathbb{N}$. Then there are an overgroup K, of G, with $|K : G| = 2^k$, and an element $a \in K$, centralizing H, such that for any $x \in H$ and $y \in G$, $x \sim_H y$ if and only if $ax \sim_K ay$ in K.*

Proof. Let $A = \mathbb{Z}/2\mathbb{Z}$ be the group of residues modulo 2. The natural action of G on the left cosets modulo H gives rise to the action of G on the group $L \coloneqq A^{G/H}$, which can be thought of as the set of all functions from the set of left cosets G/H to A, under addition. The resulting semidirect product $K \coloneqq L \rtimes G$ is the so-called *permutational wreath product* of A with G. More explicitly, for every $f \in A^{G/H}$, thought of as a function $f : G/H \to A$, and any $g \in G$, we define $gfg^{-1} \in A^{G/H}$ by the formula $(gfg^{-1})(uH) \coloneqq f(g^{-1}uH)$, for all $uH \in G/H$.

Let $a \in L$ be the characteristic function of $H \in G/H$, that is $a(H) = \bar{1} \in A$ and $a(uH) = \bar{0}$ if $uH \neq H$, where $A = \mathbb{Z}/2\mathbb{Z} = \{\bar{0}, \bar{1}\}$. Clearly $|K : G| = |L| = 2^{|G:H|} = 2^k$ and $hah^{-1} = a$ for every $h \in H$, i.e., a centralizes H.

Consider any $x \in H$ and $y \in G$. Evidently, if $x \sim_H y$ then $ax \sim_K ay$, because a centralizes H. Conversely, assume that $ax \sim_K ay$. Then there are $b \in L$ and $g \in G$ such that $bg(ay)g^{-1}b^{-1} = ax$ in K. Since $a, b \in L$ and L is abelian, we get $(gag^{-1})(gyg^{-1}) = ab^{-1}xb = (ab^{-1}xbx^{-1})x$. As before, since $gag^{-1}, ab^{-1}xbx^{-1} \in L$, $gyg^{-1}, x \in G$ and K is the semidirect product of L and G, we must have

$$gag^{-1} = ab^{-1}xbx^{-1} \quad \text{and} \quad gyg^{-1} = x \quad \text{in } K. \tag{14}$$

Suppose that $g \notin H$. Then $g^{-1}H \neq H$, so $(gag^{-1})(H) = a(g^{-1}H) = \bar{0}$ by the definition of a. On the other hand, since $x^{-1}H = H$, we have the following equality in A:

$$(ab^{-1}xbx^{-1})(H) = a(H) + b^{-1}(H) + (xbx^{-1})(H) = \bar{1} - b(H) + b(x^{-1}H) = \bar{1} - b(H) + b(H) = \bar{1},$$

contradicting the first equation in (14). Therefore $g \in H$, and the second equation in (14) yields that $x \sim_H y$. This completes the proof of the proposition. □

Corollary 1.5 from the Introduction can be deduced from Proposition 6.4 in the same way as Corollary 6.3 is deduced from Lemma 6.1. Evidently one can draw a similar conclusion for the conjugacy problem:

Corollary 6.5. *Let G be a finitely generated group possessing a subgroup of finite index with unsolvable conjugacy problem. Then G has a finite index overgroup with unsolvable conjugacy problem.*

Remark 6.6. A theorem of Remeslennikov [54, Thm. 1] states that the restricted wreath product of two conjugacy separable groups is conjugacy separable provided the base group is abelian and the acting group is cyclic subgroup separable. The argument from Proposition 6.4 shows that these conditions are no longer sufficient for conjugacy separability of a permutational wreath product (with finite orbits), because there exist conjugacy separable and cyclic subgroup separable groups possessing non-conjugacy separable subgroups of finite index (see Theorem 6.10 or [40, Thm. 1.1]).

6.2. **A non-hereditarily conjugacy separable group with conjugacy separable normal overgroups.** In this subsection we will prove Theorem 6.10 which is a stronger version of Theorem 1.6 from the Introduction. The proof will require several auxiliary statements.

Lemma 6.7. *Suppose that F_1, F_2 are acylindrically hyperbolic groups with cyclic centralizers and $G \leqslant F_1 \times F_2$ is a full subdirect product. Let $N_i \coloneqq G \cap F_i$, $i = 1, 2$. Then for every automorphism $\sigma \in Aut(G)$, either $\sigma(N_1) = N_1$ and $\sigma(N_2) = N_2$ or $\sigma(N_1) = N_2$ and $\sigma(N_2) = N_1$. Moreover, the latter is only possible if $F_1 \cong F_2$.*

Proof. Observe that for each $i = 1, 2$, F_i cannot have non-trivial finite normal subgroups: the centralizer of such a normal subgroup must have finite index in F_i and it also must be cyclic, but F_i is not virtually cyclic by definition. Therefore $N_i \lhd F_i$ is non-elementary by Lemma 2.4, and hence it is non-abelian (as F_i has cyclic centralizers).

Now, note that $N_2 \subseteq \mathrm{C}_G(N_1)$ and $N_1 \subseteq \mathrm{C}_G(N_2)$, so $\mathrm{C}_G(N_i)$ is non-abelian, $i = 1, 2$. However, for any element $(g_1, g_2) \in G$, if $g_1 \neq 1$ and $g_2 \neq 1$ then $\mathrm{C}_G((g_1, g_2)) \leqslant \mathrm{C}_{F_1}(g_1) \times \mathrm{C}_{F_2}(g_2)$ is abelian, because $\mathrm{C}_{F_i}(g_i)$ are cyclic for $i = 1, 2$. It follows that N_1 and N_2 are the only maximal subgroups of G with the property that $\mathrm{C}_G(N_i)$ is non-abelian. Hence any automorphism of G either fixes both N_1 and N_2 or it interchanges them.

For the final claim, assume that $\sigma \in Aut(G)$ is an automorphism satisfying $\sigma(N_1) = N_2$. Then σ naturally induces an isomorphism between the quotients $G/N_1 \cong F_2$ and $G/N_2 \cong F_1$, sending fN_1 to $\sigma(f)N_2$, for all $fN_1 \in F/N_1$. Hence $F_1 \cong F_2$. □

The next statement was proved by Bumagina and Wise [11] and is, in some sense, an amplification of Rips's original construction [56].

Lemma 6.8. *For any finitely presented group P and each integer $p > 92$ there exist a group F, given by a finite presentation $\langle Z \,\|\, \mathcal{R} \rangle$ satisfying the small cancellation condition $C'(1/11)$, and an epimorphism $\psi : F \to P$ such that all of the following hold.*

- (i) *There are $U, V \in Z$ such that $U^p, V^p \in \mathcal{R}$, and no other words in $\mathcal{R}$ are proper powers;*
- (ii) *$N \coloneqq \ker \psi$ is generated by two elements $u, v \in F$ of order p, represented by the words U, V respectively;*
- (iii) *N is non-cyclic, infinite and characteristic in F;*
- (iv) *the natural action of F on N by conjugation gives rise to a surjective homomorphism $P \to Out(N)$.*

Proof. Claims (i),(ii) and (iv) were proved in [11, Lemma 9] (that the orders of u and v are exactly p is an easy consequence of Greendlinger's lemma [36, Thm. 4.5 in Sec. V.4]).

The fact that N is non-cyclic was noted in [11, Lemma 10]. Now, suppose that N is finite. Then $\mathrm{C}_F(N)$ has finite index in F, but $\mathrm{C}_F(N) = \{1\}$, as F has cyclic centralizers (by Lemma 5.4.(ii)) and N is not cyclic. This implies that F must also be finite. However, $|Z| \geq 2$ and no defining relator from $\mathcal{R}$ has length 1 (by construction in [11]), so the small cancellation group F must be non-torsion by [27, Thm. VII]. This contradiction shows that $|N| = \infty$.

Finally, the fact that N is characteristic is an easy consequence of claims (i) and (ii). Indeed, (i), (ii) together with Lemma 5.4.(iii) show that every element of finite order is conjugate in F to an element of N. Since $N = \langle u, v \rangle$ and u, v have order p, we can deduce that N is the normal closure of the torsion elements in F. The latter clearly implies that N is characteristic in F. □

In the next lemma we observe a key property of centralizers in finite index normal overgroups of the group F, produced by the Bumagina-Wise construction from [11], which will be important in the proof of Theorem 6.10.

Lemma 6.9. *Let F be the group given by Lemma 6.8 (for some P and p) and let $\tilde{F}$ be a normal overgroup of F, with $|\tilde{F} : F| < \infty$. Then $\tilde{F}$ is hyperbolic and for every element $f \in \tilde{F}$, either $|\tilde{F} : \mathrm{C}_{\tilde{F}}(f)| < \infty$ or $|\mathrm{C}_{\tilde{F}}(f) : \langle f \rangle| < \infty$.*

Proof. Recall that F that is hyperbolic by Lemma 5.4.(i), hence so is $\tilde{F}$: since $|\tilde{F} : F| < \infty$, the natural inclusion of F in $\tilde{F}$ induces a quasi-isometry between the Cayley graphs of these groups (with respect to some finite generating sets), and hyperbolicity is preserved by quasi-isometries (see [7, Thm. 1.9 in Ch. III.H]).

Consider any $f \in \tilde{F}$. If f has infinite order then $|\mathrm{C}_{\tilde{F}}(f) : \langle f \rangle| < \infty$ by [7, Cor. 3.10 in Ch. III.Γ]. Thus we can now suppose that f has finite order $n \in \mathbb{N}$ in $\tilde{F}$.

Let $N \lhd F$ be the normal subgroup from Lemma 6.8. Since N is characteristic in F and $F \lhd \tilde{F}$, we deduce that $N \lhd \tilde{F}$. Therefore conjugation by f induces an automorphism of N. But then, according to Lemma 6.8.(iv), there is an element $g \in F$ such that

$$fhf^{-1} = ghg^{-1} \text{ for all } h \in N. \tag{15}$$

It follows that $g^n h g^{-n} = f^n h f^{-n} = h$ for all $h \in H$, thus $g^n \in \mathrm{C}_F(N)$. Note that $\mathrm{C}_F(N) = \{1\}$ as N is not cyclic (by Lemma 6.8.(iii)) and F has cyclic centralizers (by Lemma 5.4.(ii)). Hence g must have finite order in F.

Let $L \coloneqq \mathrm{C}_{\tilde{F}}(g^{-1}f)$. Then, evidently,

$$fhf^{-1} = ghg^{-1} \text{ for all } h \in L, \tag{16}$$

and $N \subseteq L$ by (15). It is well known that centralizers of elements are quasiconvex in any hyperbolic group (cf. [7, Prop. 3.9 in Ch. III.Γ]), therefore L is quasiconvex in $\tilde{F}$. However, L contains N, which is an infinite normal subgroup of $\tilde{F}$ by Lemma 6.8.(iii), hence $|\tilde{F} : L| < \infty$ by [44, Cor. 2].

Now we need to consider two cases. If $g = 1$ in F, then $L = \mathrm{C}_{\tilde{F}}(f)$ has finite index in $\tilde{F}$, as required. Otherwise, $g \in F$ is a non-trivial element of finite order, so $\mathrm{C}_F(g)$ is a finite cyclic group (as F has cyclic centralizers), hence $|\mathrm{C}_{\tilde{F}}(g)| \le |\mathrm{C}_F(g)|\,|\tilde{F} : F| < \infty$. Recalling (16), we deduce that $\mathrm{C}_{\tilde{F}}(f) \cap L = \mathrm{C}_{\tilde{F}}(g) \cap L$ is finite, and so $|\mathrm{C}_{\tilde{F}}(f)| < \infty$ as $|\tilde{F} : L| < \infty$. Consequently, $|\mathrm{C}_{\tilde{F}}(f) : \langle f \rangle| < \infty$, and the lemma is proved. □

We are now ready to prove the main result of this section.

Theorem 6.10. *For each integer $k \ge 2$ there exists a finitely presented subdirect product $G \leqslant F_1 \times F_2$, where F_1, F_2 are finitely presented $C'(1/11)$-groups, satisfying the following. There is a subgroup $G' \lhd G$, of index k, such that G' is not conjugacy separable, but for every group K, with $G \lhd K$ and $|K : G| < \infty$, K is conjugacy separable.*

Proof. It was shown in [40, Example 6.1], using a result of Deligne [19], that there is a finite index subgroup $Q \leqslant \mathrm{Sp}(4, \mathbb{Z})$ and a short exact sequence of groups

$$\{1\} \to O \to P \xrightarrow{\theta} Q \to \{1\}$$

such that $O \cong \mathbb{Z}/k\mathbb{Z}$ is central in P and P is not residually finite. Note that Q has type F_3 by the work of Borel and Serre [10] and Q is cyclic subgroup separable as any subgroup of $\mathrm{GL}(4, \mathbb{Z})$ (see [57, Thm. 5 in Sec. 4.C]).

Now, denote $p_1 \coloneqq 93$ and $p_2 \coloneqq 94$. For each $i = 1, 2$, we can use Lemma 6.8 to find a $C'(1/11)$-small cancellation group F_i, an epimorphism $\psi'_i : F_i \to P$ and the normal subgroup $N'_i \coloneqq \ker \psi'_i$, generated by two elements of order p_i, from its claim. Note that F_1 has an element of order $p_1 = 93$, but the order of any torsion element in F_2 divides 94 by Lemma 5.4.(iii), hence $F_1 \ncong F_2$.

The group F_i is non-elementary (as it maps onto the non-elementary group Q) and hyperbolic (by Lemma 5.4.(i)), $i = 1, 2$. Moreover, F_i has cyclic centralizers by Lemma 5.4.(ii), and so it cannot have any non-trivial finite normal subgroups (as the centralizer of such a subgroup would be cyclic and of finite index).

Set $\psi_i \coloneqq \theta \circ \psi'_i : F_i \to Q$, and let $G' \leqslant F_1 \times F_2$ be the fibre product corresponding to ψ'_1, ψ'_2 and $G \leqslant F_1 \times F_2$ be the fibre product corresponding to ψ_1, ψ_2. Clearly, $G' \leqslant G$. Denote $N_i \coloneqq G \cap F_i = \ker \psi_i \lhd F_i$, $i = 1, 2$, and observe that $N_1/N'_1 \cong \ker \theta = O$ has order k, thus $N_1 = \bigsqcup_{j=1}^{k} s_j N'_1$, for some $s_1, \dots, s_k \in N_1$. It is easy to see that $G = N_1 G'$ (as $G' \leqslant F_1 \times F_2$ is subdirect and $N_1 = G \cap F_1$), which implies that $G = \bigsqcup_{j=1}^{k} s_j G'$, i.e., $|G : G'| = k$. The fact that $G' \lhd G$ easily follows from the fact that O is central in P. We can also deduce that N_1 is finitely generated, as this is true for N'_1, hence G is finitely presented by Lemma 2.3 (because $F_1/N_1 \cong Q$ is of type F_3).

By Theorem 1.1, the group G' is not conjugacy separable since $F_1/N'_1 \cong P$ is not residually finite.

Now, suppose that K is a normal overgroup of G, with $|K : G| < \infty$. Since $F_1 \ncong F_2$, Lemma 6.7 tells us that $N_1, N_2 \lhd K$. Then, for every $i = 1, 2$, $\tilde{F}_i \coloneqq K/N_i$ can be naturally considered as a normal overgroup of $F_i \cong G/N_i$, with $|\tilde{F}_i/F_i| = |K/G| < \infty$. Note that since N_1 has trivial intersection with N_2, we can think of K as a subdirect product in $\tilde{F}_1 \times \tilde{F}_2$, with $K \cap \tilde{F}_i = N_i$, $i = 1, 2$ (see Subsection 2.3).

We will now aim to apply Lemma 4.4 to show that K is conjugacy separable. First we need to check that all the assumptions of this lemma are satisfied. According to Lemma 6.9, the groups $\tilde{F}_1$ and $\tilde{F}_2$ are hyperbolic. Moreover, since F_i is a group possessing a finite presentation satisfying $C'(1/11)$, it is virtually compact special (in the terminology of Haglund and Wise [30]) by a combination of the results of Wise [62, Thm. 1.2] and Agol [1, Thm. 1.1], $i = 1, 2$. Therefore $\tilde{F}_1$ and $\tilde{F}_2$ are also virtually compact special, hence they must be hereditarily conjugacy separable by a theorem of the author and Zalesskii [47, Thm. 1.1]. Thus $\tilde{F}_1 \times \tilde{F}_2$ is hereditarily conjugacy separable by Lemma 4.3.

Consider any element $(f_1, f_2) \in K$, where $f_i \in \tilde{F}_i$, $i = 1, 2$, and denote $C \coloneqq \mathrm{C}_{\tilde{F}_1 \times \tilde{F}_2}((f_1, f_2))$. We need to check that the double coset CK is closed in the profinite topology on $\tilde{F}_1 \times \tilde{F}_2$.

First, assume that $|\tilde{F}_1 : \mathrm{C}_{\tilde{F}_1}(f_1)| < \infty$. Then there is a finite index normal subgroup $L_1 \lhd F_1$, which is contained in $\mathrm{C}_{\tilde{F}_1}(f_1)$. Since $K \leqslant \tilde{F}_1 \times \tilde{F}_2$ is subdirect, $T \coloneqq L_1 K$ will be a finite index subgroup of $\tilde{F}_1 \times \tilde{F}_2$. But $L_1 \leqslant C$, hence $CK = CL_1K = CT$ is equal to a union of left cosets modulo T. There are only finitely many of such cosets, so CK is closed in the profinite topology on $\tilde{F}_1 \times \tilde{F}_2$, as a finite union of translates of T.

Obviously, if $|\tilde{F}_2 : \mathrm{C}_{\tilde{F}_2}(f_2)| < \infty$, we can show that CK is closed in the profinite topology on $\tilde{F}_1 \times \tilde{F}_2$ using a similar argument. Thus, we can further suppose that $|\tilde{F}_i : \mathrm{C}_{\tilde{F}_i}(f_i)| = \infty$ for $i = 1, 2$. Therefore $|\mathrm{C}_{\tilde{F}_i}(f_i) : \langle f_i \rangle| < \infty$ for $i = 1, 2$, by Lemma 6.9. This implies that the subgroup $H \coloneqq \langle (f_1, 1), (1, f_2) \rangle$ has finite index in C, and we can argue as in the proof of Proposition 4.5. Indeed, suppose that $C = \bigcup_{j=1}^{k} (a_i, b_j) H$. Then, as $(f_1, f_2) \in K$ and

$H = \langle (f_1, 1)\rangle \langle (f_1, f_2)\rangle$, we have $HK = \langle (f_1, 1)\rangle K$. Consequently,

$$CK = \bigcup_{j=1}^{k} (a_i, b_j)HK = \bigcup_{j=1}^{k} (a_i, b_j)\langle (f_1, 1)\rangle K \ \text{ in } \tilde{F}_1 \times \tilde{F}_2. \tag{17}$$

Now, the double coset $\langle (f_1, 1)\rangle K$ is closed in the profinite topology on $\tilde{F}_1 \times \tilde{F}_2$ if and only if $\langle f_1\rangle N_1$ is closed in the profinite topology on $\tilde{F}_1$, by Lemma 3.6, which, in its own turn, happens if and only if the cyclic subgroup $\langle \psi(f_1)\rangle$ is closed in the profinite topology on $\tilde{F}_1/N_1$ (see Lemma 3.3.(iii)). However, recall that $F_1/N_1 \cong Q$ has finite index in $\tilde{F}_1/N_1$, and Q is cyclic subgroup separable. Therefore $\tilde{F}_1/N_1$ is also cyclic subgroup separable by Lemma 3.10.(ii). Thus we conclude that $\langle (f_1, 1)\rangle K$ is closed in the profinite topology on $\tilde{F}_1 \times \tilde{F}_2$, which, in view of (17), implies that CK is closed as well.

We have checked that $\tilde{F}_1 \times \tilde{F}_2$ and $K \leqslant \tilde{F}_1 \times \tilde{F}_2$ satisfy all the assumptions of Lemma 4.4. Therefore we can use this lemma to deduce that K is conjugacy separable. Thus the proof of the theorem is complete. □

7. Conjugacy separability with respect to Q'-groups

This section investigates $\mathcal{C}$-conjugacy separability of subdirect products when $\mathcal{C}$ is a class of p-groups, or, more generally, a class of Q'-groups.

Definition 7.1. Let $Q \subset \mathbb{N}$ be a set of prime numbers and let F be a group. We will say that F is a *Q'-group* if every element of F has finite order which is coprime to each $q \in Q$.

If p is a prime then the class of p-groups is precisely the class of all Q'-groups, where $Q \coloneqq \mathbb{P} \setminus \{p\}$ and $\mathbb{P}$ denotes the set of all prime numbers.

It is easy to see that for any $Q \subseteq \mathbb{P}$ the class of *all* Q'-groups is an extension-closed pseudovariety (which is trivial if and only if $Q = \mathbb{P}$), and every group in this class is periodic. By a *pseudovariety of Q'-groups* we will mean a pseudovariety which consists only of Q'-groups (but it does not have to contain all Q'-groups).

The following theorem is an improvement of Corollary 4.10 in the case when $\mathcal{C}$ is a class of Q'-groups.

Theorem 7.2. *Let $Q \subseteq \mathbb{P}$ be a non-empty set of primes and let $\mathcal{C}$ be a pseudovariety of Q'-groups. Suppose that F_1, F_2 are acylindrically hyperbolic groups without non-trivial finite normal subgroups, $G \leqslant F_1 \times F_2$ is a full subdirect product and $N_1 \coloneqq G \cap F_1$. If G is $\mathcal{C}$-conjugacy separable then F_1/N_1 is a residually-$\mathcal{C}$ Q'-group.*

The proof of Theorem 7.2 will employ the following lemma.

Lemma 7.3. *Suppose that $Q \subseteq \mathbb{P}$ is a set of primes and $\mathcal{C}$ is a pseudovariety of Q'-groups. Let F_1, F_2 be groups, let $G \leqslant F_1 \times F_2$ be a subgroup such that $\rho_1(G) = F_1$, where $\rho_1 : F_1 \times F_2 \to F_1$ is the natural projection, and let $N_1 \coloneqq G \cap F_1$. If $x_1 \in F_1$ and $(y_1, y_2) \in G$ are elements such that $y_1 \in N_1 x_1^q$, for some $q \in Q$, then $(x_1 y_1 x_1^{-1}, y_2) \in G$ belongs to the closure of the N_1-conjugacy class $(y_1, y_2)^{N_1} \subseteq (y_1, y_2)^G$ in the pro-$\mathcal{C}$ topology on G.*

Proof. First, observe that $N_1 \lhd F_1$, as $\rho_1(G) = F_1$, and, since there is $h \in N_1$ such that $y_1 = h x_1^q$, for each $k \in \mathbb{Z}$ we have

$$(x_1^k y_1 x_1^{-k}, y_2) = (x_1^k h x_1^{-k} h^{-1} y_1, y_2) = (x_1^k h x_1^{-k} h^{-1}, 1)(y_1, y_2) \in N_1(y_1, y_2) \subseteq G.$$

Thus $(x_1^k y_1 x_1^{-k}, y_2) \in G$ for all $k \in \mathbb{Z}$. We also see that for each $n \in \mathbb{Z}$ there is $h_1 \in N_1$ such that $x_1^{nq} = h_1 y_1^n$, hence

$$(x_1^{nq} y_1 x_1^{-nq}, y_2) = (h_1 y_1 h_1^{-1}, y_2) = (h_1, 1)(y_1, y_2)(h_1, 1)^{-1} \sim_{N_1} (y_1, y_2) \text{ in } G. \tag{18}$$

By the assumptions, there is $x_2 \in F_2$ such that $(x_1, x_2) \in G$. Let $M \in \mathcal{C}$ be any group and let $\varphi : G \to M$ be a homomorphism. Then the order of $\varphi((x_1, x_2))$ in M is some $l \in \mathbb{N}$ which is coprime to q, because M is a Q'-group and $q \in Q$.

Set $y_1' \coloneqq x_1 y_1 x_1^{-1} \in F_1$. Let us now show that for every integer $m \in \mathbb{Z}$ we have

$$\varphi\left((x_1^{ml} y_1' x_1^{-ml}, y_2)\right) = \varphi\left((y_1', y_2)\right) \text{ in } M. \tag{19}$$

Indeed, clearly $y_1' = g x_1^q$, where $g \coloneqq x_1 h x_1^{-1} \in N_1$. Since the elements (y_1', y_2), (x_1, x_2) and $(g, 1)$ all belong to G and $\varphi\left((x_1, x_2)\right)^{ml} = 1$ in M, we obtain

$$\begin{aligned}\varphi\left((x_1^{ml} y_1' x_1^{-ml}, y_2)\right) &= \varphi\left((x_1^{ml} g x_1^{-ml} g^{-1} y_1', y_2)\right) = \varphi\left((x_1^{ml} g x_1^{-ml} g^{-1}, 1)(y_1', y_2)\right) \\ &= \varphi\left((x_1, x_2)^{ml}(g, 1)(x_1, x_2)^{-ml}(g, 1)^{-1}(y_1', y_2)\right) \\ &= \varphi\left((x_1, x_2)\right)^{ml} \varphi\left((g, 1)\right) \varphi\left((x_1, x_2)\right)^{-ml} \varphi\left((g, 1)\right)^{-1} \varphi\left((y_1', y_2)\right) = \varphi\left((y_1', y_2)\right).\end{aligned}$$

Thus we have established the validity of equation (19).

Finally, since q and l are coprime, there exist $m, n \in \mathbb{Z}$ such that $nq = ml + 1$. Therefore we can combine (18) with (19) to achieve

$$\varphi\left((y_1, y_2)\right) \sim_{\varphi(N_1)} \varphi\left((x_1^{nq} y_1 x_1^{-nq}, y_2)\right) = \varphi\left((x_1^{ml} y_1' x_1^{-ml}, y_2)\right) = \varphi\left((y_1', y_2)\right) \text{ in } M.$$

Thus $\varphi\left((y_1, y_2)\right) \sim_{\varphi(N_1)} \varphi\left((y_1', y_2)\right)$ in M. Since $M \in \mathcal{C}$ is arbitrary, we can conclude that $(y_1', y_2) = (x_1 y_1 x_1^{-1}, y_2)$ belongs to the closure of $(y_1, y_2)^{N_1}$ in the pro-$\mathcal{C}$ topology on G, as claimed. □

Remark 7.4. If $\mathcal{C}$ is a pseudovariety of groups and P is a residually-$\mathcal{C}$ group then every finite subgroup of P belongs to $\mathcal{C}$. In particular, if P is periodic and $\mathcal{C}$ consists of Q'-groups, for some $Q \subseteq \mathbb{P}$, then P is itself a Q'-group.

Indeed, if P is residually-$\mathcal{C}$, then every finite subgroup $A \leqslant P$ is $\mathcal{C}$-closed in P, hence it injects into some quotient $M \in \mathcal{C}$, of G. Therefore $A \in \mathcal{C}$ as $\mathcal{C}$ is closed under taking subgroups.

Proof of Theorem 7.2. By Corollary 4.10, F_1/N_1 must be residually-$\mathcal{C}$, and so, in view of Remark 7.4, it remains to prove that this group is periodic. Arguing by contradiction, suppose that there is an element $\bar{x} \in F_1/N_1$ of infinite order, and let $x_1 \in F_1$ be any preimage of $\bar{x}$ in F_1.

Choose some $q \in Q$. By the assumptions, the normal subgroups $N_1 \lhd F_1$ and $N_2 \coloneqq G \cap F_2 \lhd F_2$ must be infinite. Therefore, according to Lemma 2.4, there is $h_1 \in N_1$ and $m \in \mathbb{Z}$ such that $y_1 \coloneqq h_1^m x_1^q \in N_1 x_1^q \subseteq F_1$ satisfies $\mathrm{C}_{F_1}(y_1) = \langle y_1 \rangle$. Take $v \in F_2$ so that $(y_1, v) \in G$, and apply Lemma 2.4 again, to find $h_2 \in N_2$ and $n \in \mathbb{Z}$ such that the element $y_2 \coloneqq h_2^n v \in F_2$ satisfies $\mathrm{C}_{F_2}(y_2) = \langle y_2 \rangle$. Observe that $(y_1, y_2) = (1, h_2^n)(y_1, v) \in N_2 G = G$.

By Lemma 7.3, the element $(x_1 y_1 x_1^{-1}, y_2) \in G$ belongs to the closure of $(y_1, y_2)^G$ in the pro-$\mathcal{C}$ topology on G. Let us now check that $(x_1 y_1 x_1^{-1}, y_2) \notin (y_1, y_2)^G$. Indeed, since $(x_1 y_1 x_1^{-1}, y_2) =$

$(x_1,1)(y_1,y_2)(x_1,1)^{-1}$, this element is conjugate to (y_1,y_2) in G if and only if $(x_1^{-1},1) \in \mathrm{C}_{F_1\times F_2}((y_1,y_2))G$ by Remark 4.8. But

$$\mathrm{C}_{F_1\times F_2}((y_1,y_2)) = \mathrm{C}_{F_1}(y_1) \times \mathrm{C}_{F_2}(y_2) = \langle y_1\rangle \times \langle y_2\rangle = \langle (y_1,1)\rangle\langle (y_1,y_2)\rangle,$$

hence $\mathrm{C}_{F_1\times F_2}((y_1,y_2))G = \langle (y_1,1)\rangle G$, as $(y_1,y_2) \in G$. It follows from Remark 3.5, that $(x_1^{-1},1) \in \mathrm{C}_{F_1\times F_2}((y_1,y_2))G$ if and only if $x_1^{-1} \in \langle y_1\rangle N_1$ in F_1. Since $y_1N_1 = x_1^q N_1$, the latter is equivalent to $\bar{x}^{-1} \in \langle \bar{x}^q\rangle$ in F_1/N_1, which is impossible as $q \geq 2$ and $\bar{x}$ has infinite order in F_1/N_1. Thus $(x_1^{-1},1) \notin \mathrm{C}_{F_1\times F_2}((y_1,y_2))G$, implying that $(x_1y_1x_1^{-1},y_2) \notin (y_1,y_2)^G$. This means that G is not $\mathcal{C}$-conjugacy separable, contradicting our assumption. So we can conclude that every element in F_1/N_1 must have finite order.

Finally, since F_1/N_1 is residually-$\mathcal{C}$ and periodic, we can finish the proof of the theorem by using Remark 7.4 to deduce that F_1/N_1 is a Q'-group. □

Corollary 7.5. *Let $\mathcal{C}$ be a pseudovariety of Q'-groups, for some non-empty subset $Q \subseteq \mathbb{P}$. Suppose that there exists a finitely generated subdirect product $G \leqslant F_1\times F_2$ such that F_1, F_2 are finitely presented acylindrically hyperbolic groups without non-trivial finite normal subgroups, G is $\mathcal{C}$-conjugacy separable and $|(F_1 \times F_2) : G| = \infty$. Then there exists a finitely presented infinite residually-$\mathcal{C}$ Q'-group.*

Proof. This statement is an immediate consequence of Lemma 2.2.(b), Lemma 2.1.(iii) and Theorem 7.2. □

For example, let p be a prime and $\mathcal{C}$ be the class of all p-groups (including the infinite ones). If there is a finitely generated subdirect product $G \leqslant F_1\times F_2$, of non-abelian free groups F_1, F_2, such that G is $\mathcal{C}$-conjugacy separable and $|(F_1\times F_2) : G| = \infty$ then, according to Corollary 7.5, there exists a finitely presented infinite p-group.

In Subsection 4.4 we discussed the gap between the sufficient criterion for $\mathcal{C}$-conjugacy separability of subdirect products given by Proposition 4.5 and the necessary criterion provided by Theorem 4.9. Theorem 7.2 allows us to close this gap in the case when $\mathcal{C}$ consists of Q'-groups, for some non-empty $Q \subseteq \mathbb{P}$, because a periodic group is cyclic subgroup $\mathcal{C}$-separable if and only if it is residually-$\mathcal{C}$.

Theorem 7.6. *Let $Q \subseteq \mathbb{P}$ be a non-empty set of primes and let $\mathcal{C}$ be an extension-closed pseudovariety of finite Q'-groups. Suppose that F_1, F_2 are $\mathcal{C}$-hereditarily conjugacy separable acylindrically hyperbolic groups with cyclic centralizers and without non-trivial finite normal subgroups, and $G \leqslant F_1 \times F_2$ is a full subdirect product. Then the following are equivalent:*

(a) *G is $\mathcal{C}$-conjugacy separable;*
(b) *F_1/N_1 is a residually-$\mathcal{C}$ Q'-group, where $N_1 \coloneqq G \cap F_1$;*
(c) *F_1/N_1 is a residually-$\mathcal{C}$ periodic group.*

Proof. The fact that (a) implies (b) is given by Theorem 7.2, and (b) implies (c) by Definition 7.1. If (c) holds, then F_1/N_1 is a residually-$\mathcal{C}$ periodic group, so every cyclic subgroup is finite and, hence, $\mathcal{C}$-closed in F_1/N_1. Therefore we can deduce (a) from Proposition 4.5. □

We will now focus on the applications of the above theorem in the case when $\mathcal{C} = \mathcal{C}_p$ is the class of finite p-groups. In this case instead of writing residually-$\mathcal{C}_p$ we will write *residually-p*. Let us first prove Corollary 1.7 from the Introduction.

Corollary 7.7. *Suppose that p is a prime, $G \leqslant F_1\times F_2$ is a full subdirect product of non-abelian free groups and $N_1 \coloneqq G \cap F_1$. Then the following are equivalent:*

(1) *G is p-conjugacy separable;*
(2) *F_1/N_1 is a residually-p periodic group;*
(3) *F_1/N_1 is a residually finite p-group.*

Proof. Let $Q \coloneqq \mathbb{P} \setminus \{p\}$, then the class of all finite p-groups coincides with the class of all finite Q'-groups. Therefore (1) is equivalent to (2) by Theorem 7.6 (recall that F_1 and F_2 are p-hereditarily conjugacy separable by Lemma 4.2). And (2) is equivalent to (3) because a periodic group is residually-p is and only if it is a residually finite p-group. □

Example 7.8. Let F be the free group of rank 2, let $\psi : F \to \mathbb{Z}$ be any epimorphism and let $G \leqslant F \times F$ be the corresponding symmetric fibre product. Then G is finitely generated (Lemma 2.2.(a)) and hereditarily conjugacy separable (Corollary 4.6), but it is not p-conjugacy separable for any prime p by Corollary 7.7, as $\mathbb{Z}$ is not a p-group.

Moreover, arguing as in Lemma 3.11, we see that if $H \leqslant G$ is any finite index subgroup then $H \leqslant J_1 \times J_2$ is a subdirect product of some finite index subgroups $J_1, J_2 \leqslant F$, so that $J_1/(H \cap J_1)$ maps onto $J_1/(J_1 \cap \ker\psi)$, which has finite index in $F/\ker\psi \cong \mathbb{Z}$. It follows that $J_1/(H \cap J_1)$ cannot be a periodic group, hence H is not p-conjugacy separable for any prime p by Corollary 7.7. Thus G is not even virtually p-conjugacy separable, for any $p \in \mathbb{P}$.

Example 7.9. Let P be the first Grigorchuk's group [28]. This is an infinite residually finite 2-group generated by 3 elements (cf. [28, Thm.] and [33, Prop. 6 and Remark 11 in Ch. VIII]). Therefore there is an epimorphism $\psi : F \to P$, where F is the free group of rank 3, and we can construct the corresponding symmetric fibre product $G \leqslant F \times F$.

By Corollary 7.7, G is 2-conjugacy separable. Note that G has infinite index in $F \times F$ because P is infinite (cf. Lemma 2.1.(iii)). Moreover, G is not finitely generated by Lemma 2.2.(b) because P is not finitely presented (see [28, Thm.] or [33, Thm. 55 in Sec. VIII.E]).

Let us now prove Corollary 1.8 mentioned in the Introduction.

Proof of Corollary 1.8. First, suppose that (1) holds and let P be an infinite finitely presented residually finite p-group. Let F be a finitely generated free group possessing an epimorphism $\psi : F \to P$, and let $G \leqslant F \times F$ be the corresponding symmetric fibre product. Since F is finitely generated, it can be embedded into the free group H of rank 2, so $G \leqslant H \times H$. Since P is finitely presented and infinite, G is finitely generated by Lemma 2.2.(a) and has infinite index in $F \times F$ by Lemma 2.1.(iii). The latter implies that G cannot be finitely presented by a result of Baumslag and Roseblade [6, Thm. B], as P is not free. Therefore G is not virtually a direct product of free groups (the free groups would have to be finitely generated as G is finitely generated). Finally, G is p-conjugacy separable by Corollary 7.7, because P is a residually finite p-group. Hence (2) holds.

Now let us show that (2) implies (3). Let F_i be the projection of G to the i-th coordinate group, $i = 1, 2$. Then F_i is a finitely generated free group, $i = 1, 2$, and G can be considered as a subdirect product in $F_1 \times F_2$. If $G \cap F_i = \{1\}$ for some $i \in \{1, 2\}$, then G is free, contradicting our assumption. Therefore G is a full subdirect product in $F_1 \times F_2$. Now, if $|(F_1 \times F_2) : G| < \infty$ then $F_1/N_1 \cong G/(N_1 \times N_2)$ would be finite (see claims (ii) and (iii) of Lemma 2.1), where $N_i \coloneqq G \cap F_i$, $i = 1, 2$. This would yield that the direct product of the free groups N_1 and N_2 has finite index in G, which is again impossible by the assumptions of (2). Thus G has infinite index in $F_1 \times F_2$, so (3) holds.

Finally, let $G \leqslant F_1 \times F_2$ be the full subdirect product satisfying claim (3) and let $N_i \coloneqq G \cap F_i$, $i = 1, 2$. If F_i was cyclic for some $i \in \{1, 2\}$ then $|F_i : N_i| < \infty$, as $N_i \neq \{1\}$ by the assumption, which would imply that $|(F_1 \times F_2) : G| < \infty$ by Lemma 2.1. The latter would contradict another assumption of (3), hence F_1 and F_2 must both be non-abelian.

We can now apply Lemma 2.1.(iii), Lemma 2.2.(b) and Corollary 7.7 to conclude that F_1/N_1 is an infinite finitely presented residually finite p-group. Thus (3) implies (1). □

Our last two corollaries show that p-conjugacy separable subgroups in direct products of two free groups are very rare. The first one is an immediate consequence of Theorem 7.2 and the fact that no non-trivial group can be a p-group for two distinct primes p.

Corollary 7.10. *Suppose that $G \leqslant F_1 \times F_2$ is a full subdirect product of acylindrically hyperbolic groups F_1 and F_2, which do not have non-trivial finite normal subgroups. If G is p-conjugacy separable for at least two distinct primes p, then $G = F_1 \times F_2$.*

Corollary 1.9 from the Introduction treats the general case of arbitrary subgroups in direct products of two free groups.

Proof of Corollary 1.9. The sufficiency is clear, as the direct product of two free groups is p-conjugacy separable for any $p \in \mathbb{P}$ by Lemmas 4.2 and 4.3.

Thus it remains to establish the necessity. So, suppose that G is a subgroup in $F_1 \times F_2$, where F_1, F_2 are free groups, and G is p-conjugacy separable for at least two distinct primes p. As before, without loss of generality, we can assume that G is subdirect in $F_1 \times F_2$. If $G \cap F_i = \{1\}$, for some $i \in \{1, 2\}$, then G is free. Otherwise, $G \leqslant F_1 \times F_2$ is a full subdirect product.

If $F_1 \cong \mathbb{Z}$, then $N_1 \coloneqq G \cap F_1$ is infinite cyclic and central in $F_1 \times F_2$. Moreover, $G/N_1 \cong F_2$ is free, therefore G is a split extension of N_1 by a free subgroup, isomorphic to F_2. It follows that $G \cong N_1 \times F_2 \cong \mathbb{Z} \times F_2$, as N_1 is central.

Similarly, if $F_2 \cong \mathbb{Z}$, we can show that $G \cong F_1 \times \mathbb{Z}$. Thus we can further suppose that F_1 and F_2 are non-abelian. In this case all the assumptions of Corollary 7.10 are satisfied, which yields that $G = F_1 \times F_2$. □

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