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Children building on success: Arithmetical problem solving in the early years

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Abstract

This paper provides evidence of how children can build on their initial *successful* problem solving approaches in a way that promotes their creative thinking and triggers further learning. The findings come from a project that studied the process of change in 5-6 year old children's successful problem solving approaches when tacking a multiple-step arithmetic task. Micro-developmental changes in children's successful strategies were studied by closely observing children's problem solving behaviour, during a sequence of sessions during which they engaged in solving the same *form* of addition task more than once, and after they had already been successful. Changes in successful problem solving behaviour were analysed using Karmiloff-Smith's model of Representational Redescription (RR). The paper presents examples of different types of children's organisational strategies and shows that children do move beyond success, and do introduce qualitative changes to their successful problem solving strategies.

1. Introduction

Problem solving, creative thinking, and reasoning are emphasised as key aspects of children's learning in many countries, including, for example, in the UK National Primary Strategy (DfES, 2003). In the area of mathematical development, UK national guidelines for teachers address these aspects of learning within the development of numerical skills, including the teaching of a repertoire of computational skills and the ability to solve problems in a variety of contexts. Yet, as Ofsted observe, problem solving in mathematics remains a longstanding weakness in teaching across the primary years (Ofsted, 2005).

The argument that is developed in this paper is that deeper understanding and conceptualisation of a problem situation, and of the factors involved, develops in the course of children's engagement in a problem situation *not only before but also after* the achievement of an efficient solution. The aim of the project was to explore changes that children introduce to their successfully employed problem solving strategies, using, where appropriate, the theory of Representational Redescription (Karmiloff-Smith, 1992) which supports a success-based view on cognitive change. Studying these after-success changes has an important implication for teaching. If qualitative changes in the solution procedure do actually take place, even after a successful solution has been achieved, then a correct solution should not be taken as the end-point of thinking and reasoning, but also as the starting point of such a procedure.

1.1. Children's problem solving strategies: different definitions

In cognitive development, children's goal-directed behaviour has been explored by means of studying the strategies that children invent and apply in problem solving situations. Among researchers, different definitions have been employed to account for *strategies* and the term *strategy* has been used in various, different ways. There is a general agreement that strategies are goal oriented operations employed to facilitate task performance (Harnishfeger & Bjorklund, 1990; Fayol, 1994). However, two particular components of children's strategies - their deliberate implementation, and possible availability for conscious evaluation - constitute the object of long discussions and contradictions (Harnishfeger & Bjorklund, 1990).

According to the traditional view, the intention of achieving a goal is sufficient to define strategies, and aspects of behaviour such as planning and conscious awareness are not involved in the definition (Bjorklund & Harnishfeger, 1990). In contrast, Bisanz and Lefevre (1990), argue that strategies should be differentiated from any procedures or class of operations used to accomplish a task. Therefore, a definition of strategic behaviour should account for the ability to make decisions when more than one options are available, and also for the ability to adapt one's decisions and actions in a flexible way. In these terms, a *strategy* is defined as "a procedure that is invoked in a flexible, goal-oriented manner and that influences the selection and implementation of subsequent procedures" (p. 236). Thus, in Bisanz's and LeFevre's view, different procedures may be involved in the execution of a strategy but in their view these procedures constitute the products of a strategy, they are not equated with the strategy itself.

Although a consensus concerning what can, and what cannot, be considered as strategy seems difficult to establish, researchers agree generally that strategies are fundamentally characterised by their ability to develop. Research on children's strategies has focused particularly on how problem solving strategies, and their use, change and develop as children grow older. The study of developmental differences in strategy use has followed two directions; the different strategies that children of different ages use are investigated in terms of *inter-individual differences*, whereas the variety of strategies that a single child uses in similar tasks, in different contexts are investigated in terms of *intra-individual differences* (Harnishfeger & Bjorklund, 1990).

Strategies and their development are studied in consideration with the domain of knowledge and context in which they are applied. Arithmetic is a domain in which the term strategy has been used in a rather loose way. The terms *procedures*, *methods*, *strategies*, have been employed almost interchangeably by researchers who study the development of children's different approaches to arithmetical problem situations. Furthermore, these terms are employed in a broad sense and include the deliberate as well as unconscious and automatic procedures (Ashcraft 1990).

1.2. Children's strategies in additive tasks

The strategies that children of school age use to solve problems which involve addition have been studied mainly in the context of word problems. Even though Carpenter *et al*. (1981) suggest that comparisons between the strategies that children apply in word problems, and the strategies applied for the solution of number sentences, should be made cautiously, Gray (1991) argues that the basic categories of strategies that children apply for number calculations have proved consistent with those employed by children in solving basic computational problems. A comprehensive body of research has focused on the identification, classification and development of children's counting and non-counting strategies for solving addition problems (see for example, Houlihan and Ginsburg, 1981; Hiebert, et al., 1982; Fuson, 1982, 1992; Carpenter and Moser, 1983; Baroody and Gannon, 1984; Baroody and Ginsburg, 1986; Steffe and Cobb, 1988; MacLellan, 1995; Thompson, 1995).

The category of non-counting strategies that children employ to solve simple addition problems comprises the use of known and derived facts. Ashcraft (1982) argues that number fact efficiency develops following a shift from relying on procedural knowledge of counting to relying on declarative knowledge (a stored network of number facts). Baroody (1983, 1985) contradicts Ashcraft's model by arguing that it underestimates the role of procedural knowledge. In Baroody's view, number fact efficiency involves a developmental shift from slow counting procedures to a reliance on automatic "principled" procedural knowledge (p. 227). The notion of "principled" procedural knowledge implies that number facts are not stored individually in a retrieval network. Rather, the generation of basic number facts depends on stored rules and principles of arithmetic. "Principled" procedural knowledge develops from exploiting already internalised regularities and relationships and thus the need for one to learn and store numerous individual number combinations is eliminated.

During school years, or even before, children have the opportunity to observe rules and principles that underlie numerical relationships. For example, addition combinations arranged in a table produce various patterns, including N + 1 progressions (e.g. 1 + 0 = 1, 1 + 1 = 2, 1 + 2 = 3, e.t.c.). When such numerical relationships are observed, abstracted and stored in memory, different *families* of number combinations which are underlain by different principles can be very easily generated (Baroody et al., 1983; Baroody & Ginsburg, 1986). Baroody's model however, does not exclude the possibility for certain combinations in a family of number combinations to be stored in the form of a specific association *and* represented by a rule. Baroody (1985) gives the following example; 1 + 0 is a number combination which belongs in the (N + 0 and 0 + N) family of combinations ("zero addition family"). This specific combination (1 + 0) may be stored as the fact 1 + 0 = 1 and represented by the (N+0=N and 0+N=N) rule. It is assumed that less familiar members of the family (e.g. 0+9867) would probably be represented only by the rule (Baroody, 1985, p. 91).

1.3. Variability among addition strategies

Strategy *variability* in additive problems has been recognised, as an important aspect of young children's problem solving behaviour in additive tasks. Research in the field of mathematics education, has shown that children of a given age use a variety of strategies when dealing with arithmetic, additive problems (e.g. Carpenter & Moser, 1983; Fuson 1982; Baroody, 1987). These findings come in opposition to earlier research in mathematics education which tended to depict children of a particular age as consistently using a specific addition strategy (e.g. the research related to the *min model*, Groen and Parkman, 1972; Ashcraft, 1982).

Siegler and his colleagues, in a series of studies, have argued against the idea of 'stage' theories in mathematics thinking development, according to which, children of a given age are said to use a specific strategy when dealing with additive problems (Siegler,1987; Siegler and Robinson, 1982; Siegler and Shrager, 1984; Siegler, 1996; Siegler and Stern, 1998; Siegler, 2000). This series of microgenetic studies, which had as a result the development of the *overlapping waves theory,* showed that even very young children do not use invariably a single strategy for solving addition problems, rather, they use multiple, diverse strategies. What is more, children's strategies develop and change during childhood. Development does not only involve changes in the mix of existing strategies, but also construction of new ones and abandonment of old ones. These studies showed that *variability* is a crucial aspect of young children's strategies in additive tasks; it is even evident within a single trial, between children of a given age, as well as within individual children, i.e. within an individual solving the same problem on two occasions close in time (Siegler & Shrager, 1984). The issue of how children choose which addition strategy to use in order to solve a given task, and on which grounds they construct and discover new addition strategies has been also extensively studied by Siegler and colleagues and are discussed in Siegler and Robinson (1982), Siegler and Shrager (1984) and Siegler and Jenkins (1989). In a later discussion of the results of these studies, Siegler (1996) argued that children's adaptive decisions and choices can be produced through application of either implicit or explicit knowledge.

2. Aims and rationale of the study

This paper presents a study in which the focus is on methods and types of knowledge that children call upon in order to create strategies for solving a task which requires the production of all the additive number bonds which result in a given sum. In this sense, this study is not looking at the methods/strategies that children use to solve a single-step additive problem, as a comprehensive body of previous research on children's addition strategies has focused on (that is problems of the type: 6+5=?), rather, this study is looking at the methods/strategies that children use in a multiple-step task, in order to identify all the possible additive number combinations which result to a specific sum: for example: ? + ? = 9. Because the task in the context of which the micro-development of a strategy is studied involves multiple steps, the term *strategy* is used to refer to the unified technique that children will possibly apply for the whole of the task. When different steps are approached in a different way, the term *method* is employed. This differentiation does not imply though any particular stance regarding the definition of what constitutes a strategy. Rather, different terms are used for reasons of clarity and also because the development of a strategy for the whole of the task, after the initial application of different separate methods for each step, constitutes a significant shift in children's problem solving behaviour; a shift that this study particularly focuses on.

This research is focusing on studying the micro-development and changes that occur in individual children's problem solving approaches when children keep engaging with the task after their initial success in solving it. The hypothesis is that, initially, children will employ various methods calling upon different types of their mathematical knowledge (knowledge of procedures, facts, concepts and principles) in order to solve the task. As the children keep engaging with the same task after their initial success, the different pieces of knowledge underlying the application of various methods will be organised in a strategy applied consistently for every step, to the whole of the task. It is hypothesised that this organisational strategy will also develop as the children keep working on the task. The questions which are addressed in this paper are: What types of changes are observed in children's problem solving approaches; that is changes in what children do to solve the task and furthermore, are these strategy changes accompanied by changes at the level of problem solving efficiency and control that children have over the task? What types of inter-individual differences, among children of the same age working on the same task, can be observed in the development of after-success organisational strategies?

The post-success changes occurring in children's problem solving behaviour are analysed using Karmiloff-Smith's (1992) developmental model of Representational Redescription (RR), which postulates that after procedural success certain types of cognitive change may take place and that in this process of change, implicit information embedded in an efficient problem-solving procedure progressively becomes more explicit, manipulable and flexible. Karmiloff-Smith (1992) presents evidence of knowledge explicitation by exploring qualitative changes that take place both macro-developmentaly (i.e. over a period of years) and micro-developmentaly (i.e. within the boundaries of a specific number of experimental sessions). Her micro-developmental explorations cover a variety of domains such as language, science and psychology. Within the context of problem solving, however, the idea of knowledge explicitation has been studied in physics, spatial, linguistic and notational, but not in mathematical tasks (Karmiloff-Smith, 1984). The theory and model of representational redescription, involve many complex aspects and account for more than one complex process. In this paper only certain aspects of the model and theory which were used for the analysis of the data are presented.

3. Theoretical framework: The process of Representational Redescription

Karmiloff-Smith's (1992) theory of Representational Redescription (RR), situated midway between nativism and constructivism, suggests that development and learning are two complementary directions, both relevant to cognitive change. The hypothesised process of representational redescription is considered as a fundamental aspect of human development and it is defined as "a process by which implicit information in the mind subsequently becomes explicit knowledge to the mind, first within a domain and sometimes across domains" (Karmiloff-Smith, 1992, p. 18). It is presumed that this process of redescription occurs spontaneously and it is driven by an internal process of representational change after which elements of the same knowledge are re-represented at higher levels of abstraction (Karmiloff-Smith, 1990, 1993, 1994).

3.1 The RR model

According to the RR theory, it is the learners' continuous search for understanding and control over their external environment and internal representations that cognitive change emanates from. It is not failure nor reasons of economy that constitute the primary motivations for cognitive change to emerge (Karmiloff-Smith, 1984). A developmental model which is called the RR model was built to describe the process of change towards the acquisition of such understanding and control. The RR model is process-oriented and accounts for children's movement beyond their successful procedural behaviour to working on their internal representations for themselves as this being a problem itself (Karmiloff-Smith, 1981). The model describes phases of post-success problem solving behaviour as well as levels of explicitness at which the problem solvers' knowledge is re-described and re-represented in. This paper focuses on that part of the model which describes the phases of problem solving behaviour which follow procedural success, because the aim here is to present and analyse different paths of post success changes in children's strategies in arithmetic. For this reason, more details related to the theoretical description of the RR-phases rather than the RR-levels are presented below.

3.1.1. Phases of after success problem solving behaviour

The RR model is a recurrent 3-phase model. In contrast to stage models (e.g. Piaget's model of cognitive development) which describe the way that children of a particular age think, the RR model is not age-related and does not assume domain-general changes. The RR model focuses on the process rather than the structure. Hence, changes are considered to be recurrent and not as occurring simultaneously across the entire cognitive system. The assumption is that children, when involved in a problem solving situation, pass through the same three phases within the various micro-domains and also across different domains. After having specified the meaning of the terms that constitute the scaffold of the RR model, a detailed description of the recurrent phases will be presented as these apply to development and children's problem solving in particular.

Phase 1: 'Procedural phase'

At this phase, the external environment primarily controls the child's behaviour which is generated by adaptation to external stimuli. At the end point of this phase consistent successful performance is achieved and this is what Karmiloff-Smith calls "behavioural mastery" (ibid, p. 19). The notion of "behavioural mastery" implies that successful performance can be generated by a series of representations which are independently stored and are not yet consistently linked into a system. In a problem solving situation the behaviour of children who are in phase 1 is "success-oriented". The child reaches procedural success by adapting both to positive and negative external feedback. However, the child's adaptations constitute separate "behavioural units". These units of behaviour are not brought in relation one to another.

Phase 2: 'Metaprocedural phase'

At this phase, children's internal representations are brought into the focus at the expense of information coming from the external environment. In phase 2, overlooking environmental data at this phase, may cause errors and inflexibilities. However, the regression that may be observed at the behavioural level does not coincide with regression at the representational level. Ignorance and violation of environmental information is viewed as a consequence of an overall organisation of the internal representations which takes place at the same time (Karmiloff-Smith, 1985, 1991, 1992).

As a result of children's attempt to organise and connect the procedures involved in the problem, there is a deterioration (in comparison to phase 1) in their successful performance. This deterioration in successful behaviour is only superficial as it is compensated by the profit gained from children's movement beyond procedural success to theory building. It is this movement that results in the internal representational organisation into a single format leading to the generation of a unified, single approach for all the parts of the problem.

Phase 3: 'Conceptual phase'

During this phase the interaction between external data and internal representations is regulated and balanced as a result of the search for both internal and external control. Although children's behaviour in this phase is successful and seems identical to the behavioural output at phase 1, the similarity is only superficial. Representations that sustain children's behaviour in the third phase are richer and more coherent. "Phase 3 is the result of the reorganisational processes at work in phase 2 which, once consolidated, can take environmental feedback into account without the overall organisation being jeopardised" (Karmiloff-Smith, 1984, p. 44).

Accepting the hypothesis of a process of knowledge redescription entails that the aforementioned developmental phases are sustained by different formats of internal representations of knowledge. In the framework of the RR model, it is argued that there are at least four levels at which knowledge is represented and re-represented. Karmiloff-Smith (1992) names these levels Implicit (I), Explicit-1 (E1), Explicit-2 (E2), and Explicit-3 (E3). These levels of knowledge redescription, like phases, are considered as part of the cyclical process of knowledge explicitation that takes place repeatedly within each micro-domain and not as age-related stages of developmental change. It is postulated that different representational formats correspond to different levels (A publication related to the ascription of levels of knowledge explicitness to the conceptual and representational changes that where observed and analysed in this study is under preparation).

4. Method

4.1. Participants

The sample for the study consisted of ten 5-6 year old children, selected from a Year 1 class from an infant school in southern England. Given that the study focused on children's evolving strategies during a relatively limited number of sessions, and also on changes that children introduce to their problem solving behaviour after success, children most competent in addition were selected to participate. Thus, less time would be devoted to considering arithmetical misunderstandings and errors.

4.2. Design

The study follows the qualitative paradigm within which the methodological assumption is that research is an inductive process of framing a generalisation from particular cases. Detailed descriptions of individual cases have the advantage of communicating the sense of the quality of the cognitive activity under examination (Siegler and Jenkins, 1989). Thus, the choice to work with a certain number of cases rather than with an extended sample was grounded on the need to carry out dense observation and intensive analysis of qualitative changes in each individual's performance, as the micro-developmental exploration entails. The micro-developmental method (Siegler and Crowley, 1991) was used as the overall framework of research and approach to investigation. That is, changes in children's successful strategies were studied by observing children's problem solving behaviour very densely, in the course of a sequence of five sessions close in time, during which children were individually involved in solving a specific form of addition task more than once, and after they had already been successful in solving that form of task. The clinical interview was the research tool and was used in its 'revised' form (Ginsburg, 1983) involving a specific task to be solved, specific material and verbal questions posed by the researcher (For more on this methodological combination and the characteristics of the combined methods see Voutsina and Jones, 2005).

4.3. *The 'card' task*

$$\boxed{\square + \triangle = 7}$$

The main arithmetic task which was used and the related to which results are presented in this paper, required the children to find all the possible pairs of number bonds that add up to a specific number each time, i.e. the 'target' number. In one of the tasks, a pile of cards with incomplete number sentences, such as the one above, was at children's disposal. Children had to pick up one card at a time, and complete the number bond, by writing the first addend in the square and the second addend in the triangle, until there were no more possible ways to do so. The produced number bonds were put in a column. The task was repeated with different 'target' numbers which increased gradually. On each occasion the children were asked to describe how they completed each solution step and to explain the rationale and effectiveness of the strategy used. Children's overt behaviour (verbalisations, movements, gestures, hesitations) was video-recorded and analysed.

5. Findings

The tables that follow present the micro-development of organisational strategies that four cases of children developed after they had already been successful in solving the task. In tables 1 and 2 it is possible to observe the different itineraries of strategy development that different children followed and also, see different types of after-success organisational strategies that different children developed. In both tables, in certain cases, two different runs with the task from the same session are presented. This is because in the same session notable changes in two different runs with the task were observed. These are separated by a dotted line. Table 1 presents two cases of children whose problem solving approach had most of the characteristics of the 'meta-procedural' phase in the RR model by the end of the fifth session. Table 2 presents two cases of children whose problem solving approach had most of the characteristics of the 'conceptual' phase in the RR model, by the end of the fifth session and their participation in the study. The presentation of different organisational strategies is followed by a short discussion of each case.

Explanation of abbreviations used in tables

- ***n-bond***: number bond.
- ***n-line***: number line.
- ***D***: Declarative knowledge.
- ***F***: Factual knowledge.
- ***D/F***: For number bonds of the type n+1 and n+0, according the research which presented in section 1.2, children may call upon their declarative knowledge (knowledge of rules) or their factual knowledge (knowledge of number facts) or, in Baroody's view, their 'principled procedural' knowledge, as such number combinations may be stored in memory in different representational formats. The purpose of this study was not to differentiate the different formats in which such number combinations are represented. For this reason D/F stands for declarative/factual knowledge in cases where children's verbal report about the production of a number bond was of the type: "I just knew it".
- ***C***: Counting.
- ***2-step/C***: the number bond is produced by a 2-step process where the number to be used as 1st addend is chosen and written down first and the 2nd addend which completes the n-bond is identified by counting.
- ***S***: Swapping: children changed around the addends of a previously produced number bond to create a new one.
- ***DRV***: The 'deriving' method is the method where a new number bond is derived by an already produced number bond. For example: if the target is 7 and the number bond 5+2 has been already produced, the new number bond 4+3 is derived by taking away 1 from 5, the 1st addend of the number bond-reference and adding 1 on 2, the second addend of the number bond-reference.
- ***DRV report***: means that the child provided a verbal report of using a previous number bond to derive a new one, but did not provide explanation of how exactly the method works, which are the operations involved.
- ***Order***: the child identifies numbers to be used either as 1st or 2nd or both addends following a sequential (either ascending or descending) order.
- ***NR***: no report of the method used. Also not available other type of indication.
- ***Q***: When there is an extract from dialogue with a child, Q stands for the researcher's question. The child's reply is introduced by the child's initial.

When there is not indication in the table of the method used for the production of a number bond then the method used is the same method used for the production of the previous number bond.

<u>Table 1</u>:
After-success development of organisational strategies in two cases of **meta-procedural** problem solving behaviour.

	1^{st} session		2^{nd} session	3^{d} session	4^{th} session	5^{th} session
RAKHI	<u>Target 6</u> [4+2] 2-step/C [5+1] D/F [2+4] 2-step/C [3+3] F [1+5] 2-step/C [0+6] D/F [6+0] S Not quite certain for completion of the task. Not many verbal explanations.	<u>Target 8</u> [1+7] C [0+8] D/F [5+3] 2step / C [4+4] [2+6] [3+5] Order to check which 1^{st} addends missing – uses n-line. [6+2] S [7+1] D/F [8+0] D/F Certain of the completion right after the production of the last n-bond. Used number line to check if all numbers were used as 1^{st} addends.	<u>Target 12</u> [0+12] [1+11] [2+10] [3+9] [4+8] [5+7] [6+6] [7+5] [8+4] [9+3] [10+2] [11+1] [12+0] **2-step:** **Order** for 1^{st} addends Mixture of methods for 2^{d} addends. (F, C, S)	<u>Target 13</u> [0+13] [1+12] [2+11] [3+10] [4+9] [5+8] [6+7] [7+6] [8+5] [9+4] [10+3] [11+2] [12+1] [13+0] **2-step** as in session-2. Greater flexibility: started with 1+12 then 0+13 and put on top. **Described** strategy as 'doing it in order'. **Noticed** order of 2^{d} addends.	<u>Target 17</u> [0+17] [1+16] [2+15] [3+14] [4+13] [5+12] [6+11] [7+10] [8+9] [9+8] [10+7] [11+6] [12+5] [13+4] [14+3] [15+2] [16+1] [17+0] **'Ordering'** for 1^{st} and 2^{nd} addends. Report of strategy. No explanation for different order in two columns.	<u>Target 19</u> [0+19] [1+18] [2+17] [..+..] Continues **orally** up to the end. **Describes** strategy as going 'forwards' and 'backwards'. Does not explain why. Could not apply strategy when target number was 20 and [20+0] was given to her as first n-bond and she was asked to produce the rest. **No reversibility** of the strategy.
ISA	<u>Target 9</u> [0+9] D/F [1+8] [9+0] S [7+2] F **[6+3]** **Derived** from 7+2 [8+1] F [5+4] NR [4+5] S [3+6] S [2+7] S After completion, looks, for a while, at produced number bonds trying to find missing numbers. No further explanations.	<u>Target 7</u> [1+6] F [6+1] S [0+7] D/F [7+0] S [5+2] F [2+5] S [4+3] **DRV** from 5+2 [3+4] S **'Instant swapping' strategy:** n-bonds produced in pairs. No report of mechanisms involved in deriving method.	<u>Target 14</u> [7+7] F [13+1] D/F [12+2**]** **DRV** from 13+1 [14+0] D/F [0+14] S [2+12] S [11+3] **DRV** from 12+2 [10+4**]** **DRV** from 11+3 [4+10] S [1+13] S [3+11] S Put all the n-bonds in pairs as in previous session, with 7+7 on top. Said these were all because all in pairs. But task was not complete: **DETERIORATION**	<u>Target 10</u> [1+9] D/F [5+5] F [9+1] D/F [10+0] D/F [0+10] S [8+2] **DRV** from 9+1 [2+8] S [7+3] **DRV** from 8+2 [3+7] S [6+4] **DRV** from 7+3 [4+6] S **[3+…]** Did not complete. Realised she had done that before. **DRV report** Checked the use of all numbers with n-line.	<u>Target 11</u> [10+1] D/F [1+10] S [11+0] D/F [0+11] S **[9+2]** **DRV** from 10+1 [2+9] S **[8+3]** **DRV** from 9+2 [3+8] S **[7+4]** **DRV** from 8+3 [4+7] S **[6+5]** **DRV** from 7+4 [5+6] **DRV report** Immediately certain for the completion of the task but **no justification** of this certainty.	<u>Target 12</u> [6+6] [11+1] [1+11] [10+2] [2+10] [9+3] [3+9] [4+?] **Violation**: card introduced by researcher. Isa did not use DRV. She counted instead. Even after explicit prompt to use as reference the n-bond 3+9 in order to derive and complete the 4+? n-bond. Rigidity in application of strategy.

5.1 Cases of meta-procedural problem solving behaviour - Discussion

Rakhi and Isa

In the first session Rakhi and Isa approached each step of the solution process; that is, the production of each number bond, as a separate problem. They did not seem to refer to previously completed steps until the point that quite a lot of number bonds had been produced, and they needed to find the numbers that were missing. Their overt behaviour showed that they were not certain of the completion of the task and they gave very limited replies to the interviewer's relevant questions. As such, in this first run with the 'card' task, Rakhi and Isa gave signs of a problem solving approach which had the characteristics of the ***'procedural'*** phase.

In the second run with the task in the first session, Rakhi and Isa introduced interesting modifications in their approach to the task. Rakhi's first steps towards a systematic way in which she considered the numbers used in the course of the solution process, made her be certain for her success after the completion of the task. She justified her certainty on the basis of the same rationale which drove the choice of numbers (use of order and number line). The systematisation in the choice and use of numbers signalled the passage to a ***'meta-procedural'*** phase of problem solving. This systematisation entailed reference to number combinations which had been already produced.

In the second session, Rakhi extended the idea of 'ordering' for identifying the 1^{st} addends which drove this primary systematisation, and developed a unified strategy. The development of this strategy which involved the consistent combination of the aforementioned mechanisms could be considered as an indication of a 'conceptual' phase in problem solving. However, the following sessions showed that she was not aware of all the aspects of the task after the application of the 'ordering' strategy for the specification of the first addend. For example, Rakhi was not aware of the 'ordered' sequence of numbers in the column of second addends, which was the consequence of applying 'ordering' for the specification of the first addend of each new number bond. Rakhi was in control of only certain aspects of the task. In the third session, Rakhi discovered aspects which she had not noticed before (i.e. the order of 2^{nd} addends) and a new cycle of 'meta-procedural' work upon the 'ordering' strategy opened. Still, she could not provide answers for certain aspects of the task after the application of her strategy.

In the fourth session Rakhi introduced and reported the application of the 'ordering' strategy for the specification of the first and second addend of each new number bond. Moreover, in the fifth session she elaborated the verbal report of the procedures involved into her strategy. However, Rakhi did not seem to be in position to explain why her strategy worked the way it did. The development of the 'ordering' strategy seemed to be the result of Rakhi's observation and subsequent abstraction of a certain number regularity. Up to the final session, Rakhi did not seem to have conceptualised explicitly that the kind of order that she was following for the specification of the first addends was related to the order that she was following for the specification of the second addends, and the other way round. This was evident by the signs of *rigidity* in the way Rakhi was applying the 'ordering' strategy. Rakhi only applied the 'ordering' by following an ascending order in the column of first addends and a descending order in the column of second addends. She did not seem to have conceptualised the reversibility of the strategy.

The 'ordering' strategy allowed Rakhi to solve the task rapidly, avoid any type of calculation, and be certain of her success. However, she did not seem to have built an understanding of all the aspects of her strategy. There was *regression observed not at the level of performance, but on the level of the control* that Rakhi had over the strategy that she applied. It was not made possible to find in the framework of the RR model an explanation or prediction for this kind of regression: i.e. regression related to the degree of control that the problem solver had over the aspects of his/her strategy. This type of regression was not accompanied by unsuccessful efforts in solving the task.

Isa's first attempt at organising the solution process and her passage to the ***'meta-procedural'*** phase was also observed in the second run with the task in the first session. However this was a different type of organisation. Isa noticed that one important feature of the task was that at the end of the solution process all the number bonds had a corresponding 'other half': i.e. a number bond with the same addends in different order. She produced number bonds in pairs by applying 'instant swapping'. In this way she could be certain that for each number bond she had created the corresponding 'other half'. Isa's previous approach was successful regarding the completion of the task, but it did not allow her to be immediately certain that all the number bonds had been swapped. It must be emphasised that this organisational strategy integrated the use of a method (the 'deriving' method) that still did not appear to be explicit, that is verbalisable. The introduction of the 'instant swapping' strategy indicates a shift to an 'organisation-oriented' behaviour. 'Instant swapping' did allow Isa to know whether all the number bonds had been swapped. However, this was not enough for one to know that the task had been completed.

In the second session, and in the framework of Isa's 'meta-procedural' work on the task, the consideration of number bonds in pairs was used as a checking method instead of a strategy of producing number bonds. Isa seemed to be strongly focused on one particular aspect, and she based on that aspect the development of her strategy and, wrongly, the certainty for the completion of the task. In the third session this checking method was combined with the use of the number line as a tool for checking the use of all the possible numbers. The strategy of generating number bonds in pairs reappeared as a production strategy while the use of the number line was retained as a checking tool. The introduction of the number line indicated Isa's realisation that by considering the number bonds in pairs, she could not be certain of the completion of the task. This realisation was the result of a process of further explicitation that Isa's understanding regarding the requirements and conceptual aspects of the task had been subjected to.

At the same time of this back and forth regarding the use of the strategy of producing number bonds in pairs, for the first time Isa appeared to be in position to report the use of the 'deriving' method and refer to the relation between the first and second addends of the number bond-reference and the derived number bond. In subsequent sessions it was observed that the method was restrictedly employed when the two actions involved (i.e. choice of number 'before' and choice of number 'after') could be applied in this specific sequence: 'before/after'. Isa had not conceptualised the reversibility of the actions-components of the method. This had as an effect the *rigid and inflexible* use of the method (fifth session).

Because Isa was consistently following the pattern 'before/after' when she was applying the 'deriving' method, the number bonds produced appeared in a specific order. Isa strictly applied the strategy following a specific pattern: she was starting with the number bond that had the bigger number as first addend. This allowed her to go on with the production of number bonds following the 'before/after' sequence of actions that her 'deriving' method involved. It is considered that the limited conceptualisation and inflexible use of the 'deriving' method had as an effect the *inflexible* application of the overall strategy.

Table 2:
After-success development of organisational strategies in two cases of **conceptual** problem solving behaviour.

	1st session	**2nd session**	**3d session**	**4th session**		**5th session**
G R A C E	Target 9 [0+9] D/F **[1+8] D/F** [9+0] S [7+2] 2-step/C [6+3] [5+4] [4+5] S [3+6] [2+7] **[1+8] 2-step/C** **Repetition** of a n-bond - denial of the repetition. Not aware of the completion of the task: kept thinking for a while for new n-bonds.	Target 7 [0+7] D/F [3+4] 2-step/C [5+2] F [1+6] D/F [6+1] **S** [2+5] [4+3] [7+0] Introduction of the **2-part strategy** (production of 2 separate sets of n-bonds). No explanation of 1st addend choice in part-1. Report of shift to part-2 but no justification for the moment of the shift. Not certain for the task completion.	Target 9 [3+6] 2-step/C [8+1] D/F [7+2] 2-step/C [0+9] D/F [4+5] 2-step/C [6+3] **S** [1+8] [2+7] [9+0] [5+4] **Justification for completion and shift to part-2**: "I looked at all the numbers and *in order* to see if I have up to 9 (shows all number in part 1). I just needed to swap them around".	Target 12 [6+6] F [1+11] D/F [0+12] D/F [5+7] 2-step/C **[2+10] DRV** from 1+11 [3+9] 2-step/C [4+8] **DRV** from 3+9 [11+1] **S** [12+0] [7+5] [10+2] [9+3] [8+4] Indication of n-bonds used to derive new ones. No report or explanation of operations involved in the DRV method. Certain for completion.	Target 11 [0+11] D/F [1+10] **Order/DRV** [2+9] [3+8] [4+7] [5+6] [11+0] **S** [10+1] [9+2] [8+3] [7+4] [6+5] **Part-1**: choice of 1st addends in descending order. DRV for 2nd addends. Indication of n-bond reference. No report of operations involved in DRV.	Target 15 [0+15] D/F [1+14]**Order/DRV** [2+13] Explanation given for production of 2+13: "**G**: I thought of 1 less than 14 and it was 13 (she shows the 13 in the 2+13 card). **Q**: how did you know that you had to do 1 less than 14? **G**: (shows the 2 in the 2+13 card) Because 2 was 1 more than 1 (shows 1 in 1+14 card). Completed task in same way as previous session.
S E A N	Target 7 [6+1] D/F [5+2] **DRV** from previous [4+3] **DRV** from previous **[3+4] DRV** from previous [2+5] S [1+6] [7+0] D/F [0+7] S **Explains DRV** for production of [3+4] n-bond. "Take away 1 of that (shows 4 in [4+3] n-bond), makes 3, and then add 1 (shows 3 in [4+3] n-bond) is 4". W*rites*:**[3+4]**. Justifies completion checking all 1st addends in order.	Target 11 [11+0] D/F [10+1] **DRV/Order** [9+2] [8+3] [7+4] [6+5] [5+6] [4+7] [3+8] **[2+9]** [1+10] [0+11] **Explains** for [2+9] "Take away 1 (shows 3 in [3+8] n-bond) makes 2 add 9 because 9 is after 8 (shows 8 in [3+8]). And I changed it around (shows [9+2])". **Q**: Did you change the 9+2, or did you "take away 1"? (shows the [3+8]). S: Both. **Multiple represen/tions**	Target 12 **Same strategy**. **Justifies completion**: **S**: If you want to make 1, is only 1. If you want to make 2, is only 2. **Q**: So now that you want to make 12… **S**: Is 12 there. **Q**: 12 of what? Number bonds? **S**: (nods 'yes'. Counts the number bonds. Finds them to be 13). **S**: I don't know… Oh!Because that's a 0 (shows 0 as first addend at the last number bond, at the bottom).	Target 19 Writes **[19+0]** first. **Q**: Which is going to be the last number bond? **S**: 0 add 19. **Q**: What if you started with 0 add 19? Could you do that? **S** (nods 'yes') Writes: **[0+19]** [1+18] Completes in this way [2+17] [3+16] [4+15] **Explains** his strategy: **S**: It goes higher and lower (shows the columns of first and second addends correspondingly). Cause that's 0 add 19 and if you want to make 19 you have to go 1 and 1 and 1 (shows first addends) and then take away 1 makes 18, 17, 16, 15. Q: Why do you take away 1? S: Cause if it's 1 higher you need 1 less. I: Why is that? S: To make 19.		**Q**: Do you think that you can find all the number combinations for any number? **S**: Yeah. **Q**: Ok. Choose a really big number. **S**: 100. **Q**: How many ways you think there are? **S**: Uhm… 101. Writes these quickly. [100+0] [90+10] [80+20] [70+30] **Explains for 70+30:** **S**: I took away 10 (shows the 80) and I put 10 more on that (shows the 20). It's quicker to go up to 100. **Q**: Will you find all the ways counting in 10s? **S**: No, in 1s, but it's too many. I don't want to do it all.

5.2 Cases of conceptual problem solving behaviour - Discussion

Grace and Sean

Grace's initial problem solving approach was success-oriented. Multiple, different pieces of knowledge were activated for the goal to be attended. The fact that Grace repeated one number bond and then denied the repetition indicates the activation of different, unconnected knowledge representations at different moments during the solution process. Grace approached the production of each number bond as a separate problem in the context of her attempt to find all the possible number bonds. She did not give any evidence of an attempt to integrate each step of the solution process into a whole: a unified approach. This is why Grace's problem solving approach up to that point of her work with the task is classified as having the characteristics of the ***'procedural'*** phase.

The separation of the solution process in two parts, in the second session, is considered as Grace's first step towards an organisational strategy and signalled Grace's passage to a ***'meta-procedural'*** phase in her approach to the task. Still, the application of a mixture of methods could be observed. However, the various methods were consistently combined in the context of an organisational strategy. This combination seemed to be underlain by a specific rationale, which was not reported nor explained at the initial phase of the introduction of Grace's strategy. Also, a consistency regarding the timing in the use of each of these methods, was observed. In the second session, Grace did not provide an explanation of this 'timing': i.e. why she was shifting to the application of 'swapping' at that particular moment.

The explanations that Grace provided in the third session are a sign of further explicitation that Grace's organisational strategy and the rationale that sustained it had been subjected to. Grace seemed to have acquired a very good control over the task which indicated the passage to the ***'conceptual'*** phase. In the fourth session, the introduction of the 'deriving' method in the context of the same organisational strategy and without destabilising this strategy constituted a notable change. This change in the problem solving procedure took place when Grace seemed to control all the aspects of the task adequately. Little by little, the newly introduced method developed as the main method for the production of number bonds in the first part of the solution process. At the initial phase of the application of the method Grace did not report the procedures/operations involved in the 'deriving' method or explained the rationale that sustained the method. The consistent application of the 'deriving' method had as a consequence a new, ordered organisation of the number bonds in the first part of the solution process. Grace's solution times were remarkably shorter and her strategy was further elaborated and could be now described as a 'two-part' strategy each part of which was marked by the application of a unified, systematic method.

In the fifth session, Grace's explanations indicated that she had a good understanding of the fundamental concepts that underlay the 'deriving' method. Grace acknowledged that the operation that she was carrying out to specify the first addends was connected to the operation that she was carrying out to specify the second addend: She was adding 1 more because she had previously taken away 1. Grace's problem solving behaviour in the fifth session gives strong evidence of the high level of explicitness into which the 'deriving' method was represented. With the introduction of the 'deriving' method in the fourth session, and its subsequent elaboration and generalisation, Grace's approach to the task still had the characteristics of the 'conceptual phase'. The overall 'two-part' strategy was retained without being destabilised by the introduction of the 'deriving' method, as well as Grace's good control over all aspects of the task.

Sean was a remarkable case, very different from the other children because in this case a passage from all phases or at least the first two phases was not observed. Right from the first run with the task Sean gave signs of ‘organisation-oriented’ problem solving behaviour and very good control over all aspects of his approach and of the task. In the first session Sean reported the operations that he carried out to produce the new number bond using the ‘deriving’ method and proceeded to the application of ‘swapping’ after he realised that, after a certain point, a number bond that he produced by applying the ‘deriving’ method, had the same addends with a number bond produced at the initial phase of the solution process, just in a different order. This shows that, Sean, right from the first run with the ‘card’ task, had started constructing a complex representation of the features and requirements of the task in the context of which multiple pieces of knowledge coexisted, were harmoniously connected and Sean seemed to have a very good control on them. Also, there were no repetitions observed, or other types of error.

Very early, Sean’s approach to the problem gave signs of problem solving behaviour which was far beyond the ‘procedural’ and ‘meta-procedural’ phase. Even in the first run, Sean produced new number combinations by referring to previously produced number bonds. He seemed to consider and treat each solution step as a link of a chain of number bonds that he had to produce. Furthermore, he was in position to justify his success. In the second run with the ‘card’ task, Sean applied the ‘deriving’ method consistently. He clearly reported his intention to produce number bonds in “order” because it was “easy”. The ‘ordering’ strategy was developed by the consistent application of the ‘deriving’ method. Therefore, Sean’s organisation-oriented behaviour was supported by rich and well-balanced internal knowledge representations; which indicate the ***‘conceptual’*** phase.

In the second session, Sean elaborated the application of the ‘ordering’ strategy and gave a complete verbal report of the operations involved in the production of each of the two addends of a new number bond. His awareness that after a certain point, number bonds which were produced by the ‘deriving’ method were number bonds which consisted of numbers which had previously used in number combinations, but in a different order, was made evident. In the following sessions, Sean used different vocabulary (e.g. “take away”, “add”, “a number after”, “going in order”, “going higher and lower”) to report the relationship between the addends of the number bond-reference and the newly produced number bond. In this way, Sean indicated the rich, and explicit representational system which sustained his strategy. Sean used different vocabulary in the course of explaining and justifying the need to combine the two operations of adding and subtracting, or “going higher and lower” to produce number bonds “in order”.

Sean showed that he was in control of several aspects of the task (he was able to anticipate and overcome violations by reversing his strategy). Also, Sean abstracted the rule regarding the number of the possible combinations (target number +1), generalised it, and used it in the following runs, regardless of the target number. A final indication of the high level of explicitness of the knowledge representation which sustained his ‘ordering’ strategy, as well as of the representation of the task was given when Sean generalised the rationale of ‘1 more/1 less’, and he applied the ‘deriving/ordering’ strategy in steps of ‘10’. This was also an indication of Sean’s good conceptualisation of the regularities which underlay the system of decimal numeration.

Overall discussion: After success changes in problem solving behaviour

This study focused on the micro-developmental changes that occur in children’s problem solving behaviour during a sequence of sessions. Because the focus was on a sequence of sessions and not on a single run with the task or a single session, it was possible to observe and follow children's passage from more than one phase of problem solving as these are described in the RR model. Each of the phases that children passed from had the main characteristics that the RR model describes. However, certain aspects of children's problem solving behaviour while solving the particular task did not conform to certain characteristics of the phases as described in the RR model.

For example, within the 'procedural' phase, children approached each step of the solution process separately, as an isolated problem. They did not seem to have a sense of the end-point of the process of producing number bonds. Indicative of this was the fact that in the initial runs with the task children were not aware of the completion of the task. This approach conforms with Karmiloff-Smith's (1984) account on this phase. However, according to the RR model by the end of this phase, the problem solving behaviour is characterised by 'behavioural mastery'; that is the consolidated use of well-functioning, automatic procedures. This is something that was not identified in the context of this study. The methods that children used in the 'procedural' phase did not consist of automatic procedures, rather, children in this phase applied a mixture of different methods and showed that they had access to the components of the methods they used by giving a verbal report of the operations that they were carrying out while applying these methods. None of these methods was sustained by knowledge encoded merely in procedure-like representations. Procedural success in this case, did not entail 'behavioural mastery', that is the application of automatic, rigid procedures. Therefore, the data in this study show that 'behavioural mastery', does not necessarily precede the passage to a 'meta-procedural' phase of problem solving behaviour. This is in agreement with views of other researchers regarding this aspect of the model on the basis of their research findings (e.g. Goldin-Meadow & Alibali, 1994), and with the revised views of Karmiloff-Smith (1994) who tends to accept the idea that 'behavioural mastery' may not by essential for redescription to occur. It is certain though, that, as the RR model postulates, the movement to the 'meta-procedural' phase cannot occur unless the problem solver feels the need to search for, and acquire better control of the task at hand.

Children who participated in this study sought for better understanding of the task, and better control of their actions and strategies. Different children developed different strategies: different types of organisation of the solution process. According to the RR model, in this phase, children's focus on the organisational aspects of their approach, has as a result the deterioration of their performance. According to the model, children in this phase are less successful than in the previous, 'procedural' phase. In the cases which were discussed here, there was a deterioration observed but not at the level of success in solving the task, but at the level of *control* that children had over the aspects of their newly introduced organisational strategy. Children in this phase appeared to be partially and strongly focused on one aspect of the task. It was this particular aspect which drove the organisation of their behaviour. The loss of control in this case, had to do mainly with the concepts which supported the strategy not with the application of the procedures which were involved in the strategy. Therefore, even though children in that phase, applied a strategy of which they had limited control, they were still successful in solving the task. This finding does not necessarily confront the postulations of the RR model. Rather, it brings to light another aspect of this particular characteristic of the 'meta-procedural' phase. It shows that 'deterioration' is a characteristic of this phase of problem solving which may be detected in, and may concern, other aspects of problem solving behaviour, not only that of the performance.

Finally, certain cases of children exhibited behaviour which applies to the 'conceptual' phase. As the RR model predicts, in this phase, children were aware of, and in control of all the aspects, the procedural and conceptual, of their employed strategies, and were in position to report and explain them. In this phase, the problem solvers applied their strategy flexibly, and could adapt the rationale which underlay the application of the particular strategy in situations which diverged from the usual practice in the context of which the strategy had been developed. A notable difference between the behaviours observed in the meta-procedural and the conceptual phase was that changes observed in cases of meta-procedural behaviour were mainly changes occurring at the level of strategy development and strategy application, that is changes on what children were *doing* to solve the task, while changes observed in cases reaching the conceptual phase were changes at the level of what children were *saying* about their developed organisational strategies; that is they were changes at the degree of children's levels of more or less explicit understanding of their strategies. This difference can be grasped just by observing the two data tables.

The findings of this study revealed great strategy variability among different children and very interesting differences in the organisational strategies that different children of the same age develop while working in the same task. The study also revealed the activation of multiple representations of knowledge that children call upon for solving a task (something which conforms with previous research, see sections 1.2 and 1.3) and most importantly it revealed how these multiple representations connect to each other in the process of elaborating already successful approaches and developing new strategies.

The cases of children which were presented here followed very different itineraries of after-success change in problem solving but all moved towards the development of an organisational, unified, more efficient and less time consuming problem solving approach. All the cases of children sought better understanding and control over the problem situation in which they were already successful. The paper showed that children learn not only by employing their problem solving and reasoning skills in order to achieve initial success in problem solving, but also by employing these skills in order to elaborate their successful problem solving approaches and thus render knowledge which is already present in their cognitive system progressively more explicit and accessible. The implications of the study are that if the reported longstanding weakness in the teaching of problem solving across the primary years are to be remedied, then children need to be provided with structured activities and teaching that enables them to build on their initial successful problem solving in order to develop more powerful problem solving strategies.

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