BSRLM Geometry Working Group

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Providing the Motivation to Prove in a Dynamic Geometry Environment

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The use of dynamic geometry software may provide opportunities to improve the teaching and learning of mathematical proof within the context of plane geometry. Yet, it seems, if the approach to proving continues to emphasise a standardised linear deductive presentation, little improvement in student conceptions may result. This paper considers the design of geometrical tasks that could provide the motivation to prove.

Introduction

Traditionally, the main function of proof is that of verification (conviction or justification) of the correctness of a mathematical statement. Proof is needed in order to remove individual or social doubt about a proposition. As de Villiers (1998) expresses it, "the only purpose of proof is to give the final stamp of approval", the proof being the absolute guarantee of the truth of a proposition. When working in a dynamic geometry environments, however, students may be convinced of the validity of a statement by the use of dragging, through which they produce many instances of the same object. Therefore in using dynamic geometry software with pupils, "the challenge for mathematics educators is to find ways in which geometric proof has communicatory, exploratory, and explanatory functions alongside those of justification and verification" (Hoyles and Jones 1998 p122). The task is to find problem settings is which proof is a means of giving an insight-illumination into why a result that can be seen on the screen is true.

Open Mathematical Problems

The idea of open problems in mathematics was introduced by Arsac *et al* (1988), in trying to characterise a teaching and learning activity which allows students to 'do' mathematics. Our focus is on open problems in geometry, whose general structure is characterised as follows:

- the statement is short, and does not suggest any particular solution methods or the solution itself. It usually consists of a simple description of a configuration and a generic request of a statement about relationships between elements of the configuration or properties of the configuration.

- the questions are expressed in the form “which configuration does…assume when…?” “which relationship can you find between…?” “What kind of figure can…be transformed into?". These requests are different from traditional closed expressions such as "prove that…", which present students with an already established result.

A problem presented in an open form cannot be reduced to the implementation of a procedure or a routine that has to be remembered by heart. On the contrary, students have to make their own decisions in choosing a solution path. They are in a situation in which they really have to ‘discover’ a result, one that may even not be unique. An open problem allows freedom in producing conjectures. It requires students to pose questions rather than only answer predetermined ones. In order to solve an open problem students have to undergo the following process: exploring a situation, making conjectures, validating conjectures and proving them. The process of solution becomes as important as the solution itself; the attention is not only on producing ‘the correct result’, but on ‘how to produce a result’. In this sense open problems seem to be suitable to stimulate productive thinking: “problem formulating is a good companion to problem solving…Problem formulating should be viewed not only as a goal of instruction but also as a means of instruction" (Kilpatrick, 1987).

In geometrical problem solving, the suggestions is that proving a result that has been discovered and validated by students themselves is more meaningful than proving something they are given, but do not understand. Ongoing research (Boero *et al* 1996; Arzarello *et al* 1998) suggests that providing students with tasks which state “prove that…” might actually inhibit students’ capacity for proving. In contrast, open tasks which favour both a dynamic exploration of a statement and transformational reasoning (Simon 1998) might allow students to reconstruct, in terms of properties and relationships, all the elements needed in the proof.

Posing and Solving Problems in a Dynamic Geometry Environment

Dynamic software environments offer new tools and a new mediation system for the solution of geometric problems. Taking up the idea introduced by Pea (1987), computers are not only amplifiers of human cognitive capacities: they act as conceptual reorganisers. As a consequence, setting problem solving within these environments requires a careful design of activities, which need to take into account the interaction between three elements: the dynamic software, as an instance of the *milieu*, a problem, and a situation, through which the devolution of the problem takes place (Brousseau 1986).

The complexity of this interaction is such that we cannot account for all the aspects in this paper: our focus is on the design of particular tasks which exploit the features of the environment as well as providing a motivation to prove.

For this purpose, a problem would have to be presented within a situation which is defined under precise learning objectives and obeys the particular constraints set by the *milieu*. Problem posing becomes an important issue in this context and mathematics educator are faced with the challenge to design tasks which foster productive thinking and uncover aspects of the mathematical activity, which, in many situations remain unusual in school mathematics. Thinking in terms of the proving process, a dynamic geometry software, like *Cabri-Géomètre* (Baulac, Bellemain, Laborde, 1988) seems to highlight exploratory and explanatory functions of proof, which might be an important element in the introduction of pupils into the 'game' of producing theorems (in the sense of Garuti *et al* 1996), whereby a theorem is a cognitive unity of statement, proof and theory of reference.

Breaking down the process of solving problems into its main phases, we might identify construction, exploration, conjecture and justification as the key moments. When the geometric problem is tackled within the *milieu* provided by *Cabri*, the operations performed during these phases through the mediation of the software differ from those usually enacted in a paper and pencil environment.

As far as the construction is concerned the tools provided by Cabri require a deep reflection on the problems' givens, in terms of their conceptual nature, as well as on the way to represent these givens by means of the commands available.

The text needs to be interpreted and thought through in terms of hypotheses, which find a counterpart in the in-built *Cabri* primitives: the actual construction of the given objects and the relationships between them helps the pupil make explicit the starting points and therefore it might support the initial phase of the proving process.

Drawing a figure with *Cabri* and investigating its properties helps students to enact transformational reasoning processes, to grasp invariant elements while dragging elements of the figure, to see the components of the figure in a relation of functional dependence with each other and to find out under which hypotheses a certain configuration has certain properties. Construction tasks are particularly useful in terms of fostering the idea of justification if there is a shift from validating by dragging to explaining the 'proof by dragging' (Mariotti, 1997). In this part of the process the correspondence between *Cabri* primitives and axioms of Euclidean geometry may be a supporting tool, provided that the tasks are designed in order to highlight the procedure more than the outcome of the construction itself.

Exploration activities, favoured by a dynamic environment like Cabri, have two main aims, concerning:

– content, since they favour the emergence of properties in the form of theorems;

– method, since they 'force' the learner to pose questions and make conjectures.

In the phase of exploration of the configuration obtained through the construction process, conjectures might be formulated as suggested by the direct manipulation of the objects on the screen. These conjectures constitute the first germ of a theorem and their validation makes use of the invariance of the geometrical properties characterising the figure when the configuration is modified by dragging points and objects around the screen. The figures being manipulated are "general" in the sense that they represent multiple configurations in one “object” and therefore they are more likely to highlight properties which hold in every case under the given hypotheses.

We will not deal, in this paper, with all the elements involved in the exploration phase, but it seems important for our purposes, to mention that in this sort of problem solving activities, process of dynamic and transformational reasoning might be enacted, leading to the identification of invariants and to possible generalisations of the conjectures produced.

Across the different phases of the solution process *Cabri* provides feedback which has a visual as well as a conceptual nature (Laborde 1995): the tool is such that the figure produced needs to respect geometrical constraints in order not to be messed up under dragging. This feature provokes possible conflicts between the students' productions and the required (expected) figures: these conflicts lie at a conceptual as well as at a figural level. We suggest that at this point a motivation for proving can arise and be ‘cultivated’ through purposefully designed activities.

Developing Open Geometrical Problems

From the above discussion it seems clear that *milieu*, problems and situations are deeply intertwined: the challenge for mathematics educator being that of posing problems and devolving them in situations such that pupils are stimulated to explore and make conjectures, with a feeling of uncertainty underlying the whole process.

Geometrical problems, traditionally presented in textbooks in a closed form can appear to pupils leave little room for exploration and critical analysis of the task and often require explicitly a proof for a given property. The characteristics of dynamic software like *Cabri-Géomètre* call for a change in the nature of the tasks, which need to be more dynamic and open in order to exploit the features of the software and enhance productive thinking. A first step in this process of design of new tasks can be performed by taking a closed task and turning it into an open one, using the dynamic exploration and direct manipulation provided by *Cabri*. The process should possibly continue towards a design of tasks which are not inspired by traditional ones and relate directly to the work within the microworld environment.

During the working group, the participants have been asked to work on some closed problems and turn them into open ones. Problems have been chosen on the basis of their richness in terms of geometrical properties and configurations involved, and the possible links with other related properties which can be found through an open exploration. In the following we report the problems, as they have been proposed:

Problem 1. Let C and C' be two circles (with centres O and O' respectively) intersecting at two distinct points A and B. Let AD and AE be two diameters of C and C' respectively. Prove that D, B and E are collinear. Prove that DE and OO' are parallel segments.

Problem 2. Prove that the diagonals of the parallelogram obtained by joining the midpoints of the sides of a quadrilateral meet on the line joining the midpoints of the diagonals of the given quadrilateral.

Problem 3. Let ABCD be a parallelogram. Construct the angle bisectors of its four internal angles and their intersection points H, K, L, M. Prove that HKLM is a rectangle.

Problem 4. A chord AB of a given length slides on a given circle. Let P and Q be the orthogonal projections of A and B onto a fixed diameter. The midpoint M of the chord and the points P and Q are vertexes of a triangle. Prove that MPQ is always isosceles and keeps the same shape as the chord AB moves on the circle.

In order to give a flavour of the potential richness of these tasks we can analyse problem 2, in terms of the possibilities it offers: the fact that joining the midpoints of the sides of a quadrilateral you obtain a parallelogram is stated in the text as a given, while it might be a first task for an exploration, if not already known. The initial quadrilateral could be taken as a rectangle, or a trapezium and so forth and this could raise new questions and conjectures to be explored and validated. A second task then might be that of constructing the required diagonals and study the number and mutual position of their midpoints in the possible particular cases. Furthermore, the midpoints of the inscribed parallelogram might be joined with the midpoints of the diagonals of the initial parallelogram and the obtained quadrilaterals could be studied. These possibilities stem from a rough analysis of the text: a dynamic exploration with *Cabri* can possibly suggest other ideas and sub-tasks leading to generalisations about properties of quadrilaterals.

Concluding comments

The issue of designing tasks which can foster the idea of justification in geometry is a challenging one: the introduction of computers as cognitive tools in the picture raises some other questions related to the transformation of traditional tasks into open and dynamic ones. A follow-up paper will address the transforming of closed problem into open ones and discuss the use of *Cabri* in such process, as well as in the following exploration of the open task obtained.

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The BSRLM geometry working group focuses on the teaching and learning of geometrical ideas in its widest sense. The aim of the group is to share perspectives on a range of research questions which could become the basis for further collaborative work. Suggestions of topics for discussion are always welcome. The group is open to all.

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