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Geometry Working Group

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Visualisation, Imagery and the Development of Geometrical Reasoning

A report based on the meeting at the University of Birmingham, 20th June 1998
by

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This report focuses on some aspects of the nature and role of visualisation and imagery in the teaching and learning of mathematics, particularly as a component in the development of geometrical reasoning. Issues briefly addressed include the relationship between imagery and perception, imagery and memory, the nature of dynamic images, and the interaction between imagery and concept development. The report concludes with a series of questions that may provide a suitable programme for research and lays the foundation for further work of the BSRLM geometry working group.

The nature and role of visualisation and imagery in the teaching and learning of mathematics is complex. Much has been written about the value of visualisation and imagery in terms of the potential to enhance a global and intuitive view and understanding of various areas of mathematics (Bishop 1989, Fischbein 1987, Usiskin 1987, Zimmermann and Cunningham 1991). Fischbein (1987 p104), for example, comments that "a visual image not only organises the data at hand in meaningful structures, but is also an important factor guiding the analytical development of a solution". Bishop (1989) concludes his review by saying that "there is value in emphasising visual representations in all aspects of the mathematics classroom".

Yet it is also recognised that there are difficulties concerned with visualisation and imagery (Dreyfus 1991, Love 1995). If mathematical visualisation is taken to be "the process of forming images (mentally, or with pencil and paper, or with the aid of technology) and using such images effectively for mathematical discovery and understanding" (Zimmermann and Cunningham 1991 p3), then such difficulties can relate to the process of forming images as well as using them in solving problems. Similarly, if mental imagery is taken as involving: "constructing an image from pictures, words or thoughts; re-presenting the image as needed; and transforming that image" (Wheatley 1991), then difficulties can arise from the processes of constructing, re-presenting, and transforming. Love (1995 p125) suggests that in geometry the relationship between "mental objects and physical images is an especially difficult one". From a slightly different perspective, Dreyfus (1991) comments on the low status often accorded to visual aspects of mathematics in the classroom.

Recognising the complex nature of visualisation and imagery, especially its role in the development of geometrical reasoning, this paper presents a consideration of the visualisation process and the images formed. The French psychologist Raymond Duval (1998 p39) has suggested that: “ differentiating between different visualisation processes ... is needed in the curriculum”. So our central questions are:

What are the different visualisation processes?
What are the different types of mental images formed?

We begin with an outline of the role of visualisation in the model of the development of geometrical reasoning proposed by Duval (1998 p38-39), which is under consideration by the BSRLM Geometry working group (Jones 1998). This leads to a consideration of various aspects of visualisation and imagery in mathematics education including the relationship between imagery and perception, imagery and memory, the nature of dynamic images, and the interaction between imagery and concept development. While this discussion raises more questions than it can answer, the resulting questions provide a suitable programme for research and lays the foundation for further work of the BSRLM geometry working group.

The Role of Visualisation in the Development of Geometrical Reasoning

Duval suggests that geometrical reasoning involves three kinds of cognitive processes which fulfil specific epistemological functions. The three cognitive processes are :

- *visualisation processes*, for example the visual representation of a geometrical statement, or the heuristic exploration of a complex geometrical situation.
- *construction processes* (using tools)
- *reasoning processes* - particularly discursive processes for the extension of knowledge, for explanation, for proof

Duval points out that these different processes *can* be performed separately. For example, visualisation does not necessarily depend on construction. Even if a construction leads to a visualisation, construction processes, Duval contends, actually depend only on the connections between relevant mathematical properties and the constraints of the tools being used. Similarly, visualisation can be an aid to reasoning (for instance by aiding the finding of a proof) but visualisation can also be misleading (if our visualised image is a special case, for example).

Duval argues, however, that, “these three kinds of cognitive processes are closely connected and their synergy is cognitively necessary for proficiency in geometry” (*ibid* p38). Duval illustrates the connections between these three kinds of cognitive processes as represented in Figure 1.

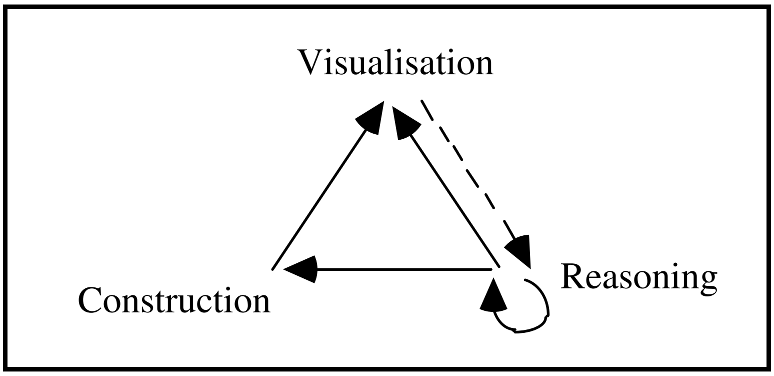

Figure 1
Underlying cognitive interactions involved in geometric activity
(from Duval 1998 p38)

In Figure 1 each arrow represents the way one kind of cognitive process can support another kind in any geometrical activity. Duval makes the arrow from visualisation to reasoning dotted because, as argued above, visualisation does not always help reasoning. The 'circular' arrow illustrates that reasoning can develop in a way that is independent of construction or visualisation processes.

Given that the synergy of these three processes is cognitively necessary for proficiency in geometry, the issue, as identified by Duval, is how to get pupils in school to see the communication between the three kinds of processes. Duval argues that, in attempting to understand the development of geometrical reasoning, his research has shown the following:

1. The three kinds of processes must be developed separately.
2. Work on differentiating visualisation processes and between different reasoning processes is needed in the curriculum.
3. The co-ordination of these three kinds of processes can really occur only after this work on differentiation.

Visualisation and Imagery Processes in Mathematics

This section attempts to summarise some of the issues concerned with visualisation, which may be mental or physical, and imagery, which may be pictorial. In particular we consider the relationship between imagery and perception, imagery and memory, the nature of dynamic images, and the interaction between imagery and concept development.

imagery and perception

While Duval may appear to prioritise visual perception (his area of expertise in psychology), 'seeing' is not the only source of mental imagery in mathematics. 'Feeling' physical objects without looking (in other words, touch perception) is another source of mental image creation. An exploratory study involving 3D geometrical objects undertaken by Triadafillidis (1995) shows some of the potential. It is worth noting, however, that there is no consensus about how perceptions are coded by the mind, nor how these codes are represented mentally. Love (1995 p125) suggests that there is disagreement “over whether such things as ‘pictures in the mind’ can exist independently of thought and language or even whether they exist at all”. A question that we do not have space to address here is whether visualisation even needs sight.

imagery and memory

Another open question is the relationship between memory and imagery, or perhaps better, the role of memory in imagery and visualisation. Are mental images formed from visual experiences necessarily pictures that can be viewed in the mind, or simply memories of that experience? Presmeg (1986), in a study of what she called ‘visualisers’ (those who prefer to use visual methods when attempting to solve mathematical problems that could be solved by both visual and non-visual methods), identified five kinds of visual imagery, which she referred to as:

- pictorial (picture-in-the-mind)
- pattern (relationships depicted spatially)
- memory (recreating images from experience)
- kinesthetic (involving muscular activity)
- dynamic (moving)

In Presmeg’s classification an image recreated from memory of a visual experience may or may not be pictorial.

dynamic images

As Presmeg identified, some visual thinkers are able to make use of dynamic mental images. The impact of the forms of dynamic diagrams available in computer-based mathematical learning environments on the development of such imagery is not known, although Gorgorió and Jones (1996) suggest that the use of a dynamic geometry package such as Cabri-géomètre can support the development of important visualisation skills necessary for the understanding of visual phenomena.

imagery and concept development

While many have suggested that the use of imagery aids conceptual development, there is still some way to go to understanding the precise relationship. Mariotti (1995 p104) suggests that geometrical reasoning can be interpreted in terms of “a dialectical process between the figural and conceptual aspects”. In other words, geometrical reasoning involves an inter-dependent relationship between images and concepts.

Simpson and Tall (1998) make the distinction between *passive*, *organisational*, and *generative* figures, we could use the same classification for mental images. A *passive* image could be merely associated with a concept whilst an *organisational* image allows information to be represented compactly. Alternatively a *generative* image is used by the learner to guide their learning and it may be *conceptually* or *formally generative*. In geometry, the *passive* image of a regular pentagon that many learners visualise can positively inhibit the development of the concept of a pentagon as any five sided shape. Images of objects being "dragged" as in a dynamic geometry package, on the other hand, can enhance that conceptual development and could thus be *conceptually generative*. *Formally generative* imagery would include: a "proof-without-words" of Pythagoras' Theorem, the visualisation we might employ to guide a formal proof or a visualised "sketch" used to generate a more formal construction.

Conclusions

While the above discussion probably raises more questions than it can answer, the resulting questions provide a suitable programme for research and the foundation for further work of the BSRLM geometry working group. To paraphrase Dreyfus (1995 p16-17), the overarching need is for theory building, with input both from classroom experiences and carefully-designed research. In particular we need:

- to understand the precise role of diagrams in problem solving and learning about specific mathematical concepts and processes
- to find out for what kinds of reasoning processes and in what kinds of learning situations, diagrams and/or visual imagery are particularly helpful
- to understand the impact on mathematical reasoning of dynamic diagrams available in computer-based mathematical learning environments
- to find out what are efficient means for communication about, and by means of, diagrams, and their associated interpretations.

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BSRLM Geometry Working Group

The geometry working group focuses on the teaching and learning of geometrical ideas in its widest sense. The aim of the group is to share perspectives on a range of research questions which could become the basis for further collaborative work. Suggestions of topics for discussion are always welcome. The group is open to all.

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