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Geometry Working Group

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Theoretical Frameworks for the Learning of Geometrical Reasoning

With the growth in interest in geometrical ideas it is important to be clear about the nature of geometrical reasoning and how it develops. This paper provides an overview of three theoretical frameworks for the learning of geometrical reasoning: the van Hiele model of thinking in geometry, Fischbein's theory of figural concepts, and Duval's cognitive model of geometrical reasoning. Each of these frameworks provides theoretical resources to support research into the development of geometrical reasoning in students and related aspects of visualisation and construction. This overview concludes that much research about the deep process of the development and the learning of visualisation and reasoning is still needed.

It seems that while for most of the twentieth century the mathematical literature has been predominantly algebraic, a growing interest in geometrical ideas has been stimulated by the development of powerful computer-based geometry and visualisation packages. The prediction is that such computer technology will have a significant positive influence on the progress of mathematics (National Research Council 1990, Science and Engineering Research Council 1991).

As geometry evolves to encompass the understanding of diverse visual phenomena, it is important to be clear about what is meant by the geometrical reasoning necessary to solve mathematical problems involving visual phenomena, and how such reasoning develops. The purpose of this paper is to provide an overview of several theoretical models which have been put forward as useful frameworks for describing and understanding the development of geometrical reasoning. After a brief outline of the van Hiele model of thinking in geometry, and of Fischbein's theory of figural concepts, a somewhat fuller description is provided of Duval's cognitive model of geometrical reasoning.

the van Hiele model of thinking in geometry

One framework describing the development of geometrical reasoning that has been the subject of considerable research is the van Hiele model of thinking in geometry (see, for instance, van Hiele 1986). This is a teaching approach based on levels of thinking commonly known as the "van Hiele levels", originally aimed at the teaching and learning of geometry but which may be applicable more widely (Pegg 1992). In the van Hiele model there are at least 5 levels, although some writers discern as many as 8. The structure of the van Hiele model bears some similarity to the framework proposed by the UK Mathematical Association in 1923 which recognised three stages in the teaching and learning of geometry. These three stages were, briefly: stage A, intuitive, experimental work; stage B, 'locally' deductive work (formal symbolism and deductive reasoning is introduced, but intuition and induction are used to bridge logically difficult gaps); Stage C, globally rigorous work (Mathematical Association 1923). In a similar way, the van Hiele approach fosters the idea that students' initial curricular encounters with geometry should be of the intuitive, explanatory kind (van Hiele 1986 p 117). The learner then progresses through a series of 'levels' characterised by increasing abstraction.

Fuys *et al* give the following description of the different levels, based on their translations of the work of van Hiele from the original Dutch:

level 0 the student identifies, names, compares and operates on geometric figures

level 1 the student analyses figures in terms of their components and relationships between components and discovers properties/rules empirically

level 2 the student logically inter-relates previously discovered properties/rules by giving or following informal arguments

level 3 the student proves theorems deductively and establishes inter-relationships between networks of theorems

level 4 the student establishes theorems in different postulational systems and analyses/compares these systems

(Fuys *et al* 1988 p5)

The van Hiele model has been subject to some critical discussion including querying, for example, the discreteness of the levels and the precise nature of levels 0 and 4 (or 1 and 5 as some writers denote them). For further details see Fuys *et al* 1988 and, for reviews, Hershkowitz 1990 and Pegg 1992.

the theory of figural concepts

Fischbein (1993) observes that while a geometrical figure such as a square can be described as having intrinsically conceptual properties (in that it is controlled by a theory), it is not solely a concept, it is an image too. As he says " it possesses a property which usual concepts do not possess, namely it includes the mental representation of space property" (*ibid* p141). So, Fischbein argues, all geometrical figures represent mental constructs which possess, simultaneously, conceptual and figural properties. According to this notion of figural concepts, geometrical reasoning is characterised *by the interaction between these two aspects, the figural and the conceptual*. Mariotti (1995 p94), in discussing Fischbein's notion of figural concept, stresses the dialectic relationship between a geometrical figure and a geometrical concept. She argues that geometry is a field in which it is necessary for images and concepts to interact, but that from the student's perspective there can be a tension between the two.

Duval's cognitive model of geometrical reasoning

The French psychologist Duval approaches geometry from a cognitive and perceptual viewpoint. For example, in Duval (1995 p145-147) he provides an analytic resource in the form of a detailed framework for analysing the semiotics of geometric drawings. In this framework he identifies four types of what he calls "cognitive apprehension". These are:

1. *perceptual apprehension*: this is what is recognised at first glance; perhaps, for instance, sub-figures which are not necessarily relevant to the construction of the geometrical figure.
2. *sequential apprehension*: this is used when constructing a figure or when describing its construction. In this case, the figural units depend not on perception but on mathematical and technical constraints (in the latter case this could be ruler and compasses, or perhaps the primitives in computer software).
3. *discursive apprehension*: perceptual recognition depends on discursive statements because mathematical properties represented in a drawing cannot be determined solely through perceptual apprehension, some must first be given through speech.
4. *operative apprehension*: this involves operating on the figure, either mentally or physically, which can give insight into the solution of a problem.

As Duval explains (*ibid* p 155), there is always a potential conflict between perceptual apprehension of a figure and mathematical perception: "difficulties in moving from perceived features of a figure can mislead students as to the mathematical properties and objects represented by a drawing, and can obstruct appreciation of the need for the discovery of proofs".

According to Duval, operative apprehension does not work independently of the others, indeed discursive and perceptual apprehension can very often obscure operative apprehension. From a teaching perspective Duval argues for "special and separate learning of operative as well as of discusive and sequential apprehension are required". Duval suggests that work with computers may support not only the development of sequential apprehension, but also the development of operative apprehension, if the software has been designed with this in mind. He concludes that "a mathematical way of looking at figures only results from co-ordination between separate processes of apprehension over a long time".

While the above refers to working with geometric drawings, Duval (1998 p38-39) has gone further in proposing that geometrical reasoning involves three kinds of cognitive processes which fulfil specific epistemological functions. These cognitive processes are:

- *visualisation processes*, for example the visual representation of a geometrical statement, the or heuristic exploration of a complex geometrical situation.
- *construction processes* (using tools)
- *reasoning processes* - particularly discursive processes for the extension of knowledge, for explanation, for proof

Duval points out that these different processes *can* be performed separately. For example, visualisation does not necessarily depend on construction. Similarly, even if construction leads to visualisation, construction processes actually depend only on the connections between relevant mathematical properties and the constraints of the tools being used. Similarly, even if visualisation can be an aid to reasoning through, for instance, aiding the finding of a proof, in some cases visualisation can be misleading.

However, Duval argues, “these three kinds of cognitive processes are closely connected and their synergy is cognitively necessary for proficiency in geometry” (*ibid* p38). Duval illustrates the connections between these three kinds of cognitive processes in the way represented in figure 1 below.

In Figure 1, each arrow represents the way one kind of cognitive process can support another kind in any geometrical activity. Duval makes arrow 2 dotted because, as argued above, visualisation does not always help reasoning. Arrows 5A and 5B illustrate that reasoning can develop in a way independent of construction or visualisation processes.

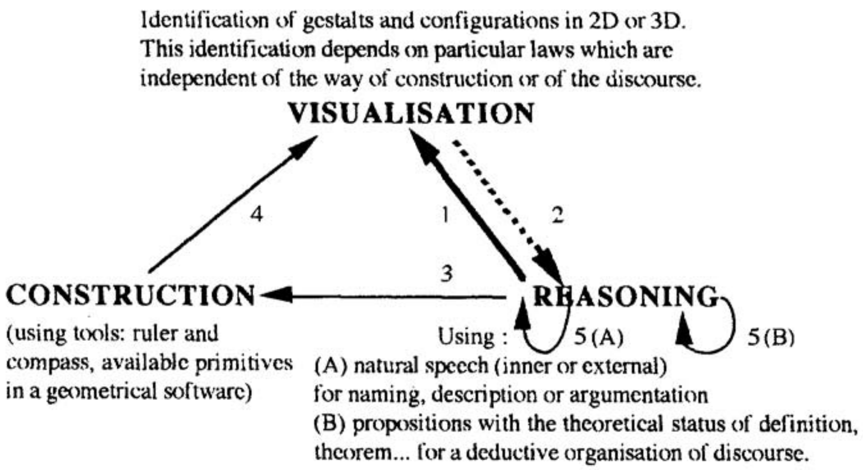

Figure 1
The underlying cognitive interactions involved in geometrical activity.

Given Duval’s argument that the synergy of these three cognitive processes is cognitively necessary for proficiency in geometry, the issue is, as Duval identifies, how to get pupils in school to see the communication between these three kinds of processes. Duval argues that in attempting to understand the development of geometrical reasoning, his research shows the following:

1. The three kinds of processes must be developed separately.
2. Work on differentiating visualisation processes and between different reasoning processes is needed in the curriculum.
3. The co-ordination of these three kinds of processes can really occur only after this work on differentiation.

Conclusions

The above overview of three fairly well-developed frameworks for describing and understanding the development of geometrical reasoning is intended to provide a brief idea of the theoretical resources available which may be useful in research in this area. It also underlines the cognitive complexity of geometry. As Duval concludes (*ibid* p51), "much research about the deep process of the development and the learning of visualisation and reasoning are still needed".

Acknowledgement

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BSRLM Geometry Working Group

The geometry working group focuses on the teaching and learning of geometrical ideas in its widest sense. The aim of the group is to share perspectives on a range of research questions which could become the basis for further collaborative work. Suggestions of topics for discussion are always welcome. The group is open to all.

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