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On the Nature and Role of Mathematical Intuition

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Introduction

Both the recent US National Council of Teachers of Mathematics publication 'Curriculum and Evaluation Standards for School Mathematics' (NCTM 1989) and the latest proposals for the UK National Curriculum from the UK School Curriculum and Assessment Authority in the form of the draft proposals for the new UK mathematics curriculum (SCAA 1994) allude to mathematical intuition. In the NCTM Curriculum Standards for Grades K-4, standard 6 is 'number sense and numeration'which is explained as follows (ibid p 39): "Number sense is an intuition about numbers that is drawn from all the varied meanings of number". Similarly, standard 9 is 'Geometry and spatial sense', given as (ibid p 49) "Spatial sense is an intuitive feel for one's surroundings and the objects in them" For the latest UK curriculum, the programme of study for key stage 2 (for pupils aged 7 to 11) declares that one of the "main emphases at this key stage" is on "developing an intuitive understanding of probability" (SCAA 1994 p 7).

These examples point to the importance attached to the notion of mathematical intuition within the field of mathematics education. Indeed this emphasis is not new and can be traced back over the centuries through the work of many mathematicians, educators and philosophers. The purpose of this paper is to discuss some of the issues surrounding the current notion of mathematical intuition and how these might be researched. Details of a study I recently completed have already been published (Jones 1993a and b) and these will not be considered in detail here. Rather, the mechanism that I will use will be to present a series of mathematical problems with which I hope you, as reader, will engage. In this way I hope to raise various theoretical and methodological issues.

Intuitive Responses to Mathematical Problems

The intention in presenting these problems is to provide a starting point for discussion and as such it would be helpful if you could approach the problems in a particular way. What I am interested in is what comes to your mind in the first few moments of tackling each problem. As this is the case, I suggest that before tackling each problem you attempt to clear your mind and then, as you read the problem, see if you can focus on the following: does an answer or a problem-solving strategy come to mind immediately? if so, what is that answer or strategy? and, thirdly, do you know where this answer or strategy comes from? You might well find that thinking about this as you tackle the problem interferes with your tackling of the problem I will return to this point later in this paper. We begin with problem 1 (source: Schoenfeld 1985 p 16).

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Problem 1
Three points are chosen on the circumference of a circle radius R, and the triangle containing them is drawn. What choice of points results in the triangle with the largest possible area? Justify your answer as well as you can.

What came to mind? A visual image? A visual image with movement? Did you think that the answer must be an equilateral triangle? If so, why did you think this? Or did you immediately think of calculus? Was symmetry an important consideration? Or did you recollect other similar problems that you have encountered in the past?

Certainly, Schoenfeld, who studied students tackling this problem, records one of his subjects saying almost immediately "Right, I think that the largest triangle should probably be equilateral" (Schoenfeld 1985 p 320). Similarly one of a pair of subjects that Schoenfeld studied comments at the outset". my basis hunch would be that it would be .. an equilateral.." (*ibid* p 324). In a study I have made (Jones 1993a) I recorded pairs of subjects tackling this problem using Cabri-Geometre and then asked them the basis of their decisions. One pair I studied said "It had to be regular and the only way to get it regular was .. to make all the sides equal .. (*ibid* p 120). Another pair said" . it's just an instinctive reaction that you have to have equal dimensions to maximise the area" (*ibid* p 121).

My intention in using this problem is to provide a potential example of mathematical intuition. If you had a strong initial reaction, and many people that I have used this problem with have had a strong initial reaction (that the solution is an equilateral triangle), then, I venture, this is an example of your mathematical intuition at work. Below is Fischbein's outline (Fischbein 1987) that an intuition is a cognition characterised by the following properties (ibid p 43-56)]

self-evidence and immediacy - extrinsic justification is not needed

intrinsic certainty - self-evidence and certainty are not the same

perseverance - intuitions are stable

coerciveness

theory status

extrapolativeness

globality - intuitions offer a unitary global view

implicitness - although the result of selection, globalisation and inference, intuitions will appear to be implicit.

How do some of the terms that Fischbein uses correspond to your experience with problem 1? If you did immediately think of an equilateral triangle, how certain did you feel? Both the students Schoenfeld studied and the students that I worked with all felt certain that the solution was an equilateral triangle. Schoenfeld was not interested in mathematical intuition in his study, although he does mention it on several occasions. The notion that I was researching in my study was the behavioural

task of intuition which Fischbein suggests, and which I am also arguing, is to "prepare and guide our mental or practical activity".

With this in mind let us try problem 2 (source: Avita and Barbeau 1991). Again, try to clear your mind and see if you can monitor your reaction to this problem.

Problem 2
An isosceles triangle has equal sides of fixed length and a base of variable length. What is the shape of the triangle for which the area is maximum?

Did you have a strong reaction this time? Was there a strong visual image? Did it involve movement? Did you immediately think of an equilateral triangle? Was this in terms of an image or did the word 'equilateral' come to mind?

Of course it happens that this time the answer is not an equilateral triangle. While I will leave the actual solution for you to take up at some other time should you care to, I would suggest again that, if you did have a strong feeling that the solution should be an equilateral triangle, then again this is an example of your mathematical intuition at work. However this time your mathematical intuition led you astray! Of course, this is what can (and does) happen. Mathematical intuition, as Avital and Barbeau point out, is not necessarily a "basis for working out a solid solution" However, the case I would wish to argue is that intuition not only provides conjectures that may be worthy of following up by analytical means but more than that there is actually a symbiotic relationship between analytical and intuitive thinking in mathematics.

Before we consider this symbiotic relationship in more detail let us turn to a third problem (source: Avita and Barbeau 1991). Again, try to clear your mind and see if you can monitor your reaction to the problem.

Problem 3
A bag contains a certain number of black and white balls. Two balls are drawn at random. It is known that the probability that one is black and the other white is 1/2. What can be said about the number of balls of the two colours in the bag?

What was your reaction to this problem? Was your reaction to be very cautious. If so, why was this? Or was your immediate thought that there are the same number of black and white balls?

Whatever your reaction was to this problem there may be several factors at work here. As Fischbein points out (*op cit* p 85) "Experience is a fundamental factor in shaping intuitions" and in this instance two sources of experience may be operating. Firstly, if you did have a strong intuition that the solution to problem 2 was an equilateral triangle only to find that this is incorrect, then you may now suspect any intuitive reaction to problem 3. You then risk only feeling very cautious about how to proceed which may actually prevent you from solving the problem. Alternatively, your past experience of being taught probability may have been such that you were already so suspicious of probability questions that the effect is the same and you are at least momentarily unable to proceed. As Fischbein found (Fischbein 1971), the development of what he refers to as secondary intuition, that is "those which are systematically constructed during the teaching process", are not well developed in the area of probability and that there is an "almost total lack of understanding of the compound nature of some events".

Discussion

One of the intentions of this paper is to provide a way of identifying the difference between intuition and a wild guess. I hope that your experience with the problems in this paper has illustrated that. In addition to this, in my research I have found that Fischbein's definition of intuition given above also provides this distinction. In the discussion at the BSRLM conference these are some of the meanings that participants attached to the term mathematical intuition:

- it is trying to make sense of a problem;
- it is located in experience and success;
- it can be misleading;
- it is coming up with a solution and a method without being aware of all the steps;
- it is suspending conscious use of formal methods in favour of allowing the problem to speak to you;
- it is linked to the underlying mathematical structure.

A second intention of this paper is to illustrate that, although mathematical intuition may play different roles in different parts of the mathematics curriculum, it is not as simple as saying that in geometry intuition plays a strong positive role linked to the visual image and that in probability, for instance, mathematical intuition tends to perhaps inhibit problem-solving. Certainly the underlying mathematical structures are different. Furthermore our primary (experiential) and secondary (taught) experiences of geometry and probability are certainly different too. However, as with problem 2 above, mathematical intuition can lead one astray even with geometrical problems.

It may be that if your experience of your primary mathematical intuitions is positive in particular areas of the mathematics curriculum, as it may be in terms of your experience of geometry, then you tend to trust your intuitive reaction when you are tackling mathematical problems in these areas. In some respects then it does not matter if your intuition can occasionally lead you astray because at least you are tackling the problem. The argument that I am putting forward is that if your experience of your mathematical intuition is not positive then this might prevent you from tackling the problem at all.

In terms of the relationship between analytic and intuitive thinking there seems to be a number of ways of looking at this. Piaget, for instance, seems to suggest a hierarchy when he writes:

> *Although effective at all stages and remaining fundamental from the point of view of invention, the cognitive role of intuition diminishes (in a relative sense) during development. .... there then results an internal tendency towards formalisation which, without ever being able to cut itself off entirely from its intuitive roots, progressively limits the field of intuition (in the sense of non-formalised operational thought).*
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> *(Piaget 1966, p 225)*

Fischbein, however, amongst others, suggests either a plurality or a dialectic. In his most recent article, Fischbein suggests that:

> *The interactions and conflicts between the formal, the algorithmic, and the intuitive components of a mathematical activity are very complex and usually not easily identified or understood.*
>
> *(Fischbein 1994, p 244)*

Finally, Papert, for instance, tends to argue that the intuitive mode is the natural one; analytical thinking is then merely a useful tool. Recently he has written that "The basic kind of thought is intuitive: formal logical thinking is an artificial, though certainly enormously useful construct: logic is on tap, not on top" (Papert 1993, p 167). All this suggests that there is a rich vein of research to tackled. As has been pointed out, however, this are of research is very complex.

Areas for Research

It may not surprise you to find out that there has not been a great deal of research in this area. There are a number of reasons for this. Firstly there is the problem of defining intuition and providing a suitably robust theoretical framework which will inform the design of empirical studies and the analysis of their results. Secondly there are significant methodological problems. For instance, as I mentioned above, as you began tackling problem 1 the very request to focus on your thought process may well have interfered with that thought process. In addition relying on subjects' own verbal reports as data brings its own problems. Thirdly, techniques for analysing verbal data, such as protocol analysis, are predicated on an information-processing model of human problem-solving. This virtually rules out a role for intuition.

However some progress has been made. I have found, for instance, that Fischbein's work provides a reasonably good theoretical starting point. On the other hand, the significant methodological problems do remain. It is possible, however, to establish some lines of enquiry such as researching the following:

- the relationship between experience and mathematical intuition
- the nature and role of visualisation
- the relationship between intuition and proof
- the relationship between what Fischbein calls primary and secondary intuitions

The intention is that a better understanding of the nature of mathematical intuition and its role in solving mathematical problems may be able to inform both the design of the mathematics curriculum and the way in which it is taught.

Note

It may be that for some readers these problems are actually no such thing and that for them the problems are merely exercises. When this is the case then the reader's reaction to the problems is not necessarily appropriate evidence relating to the nature of mathematical intuition.

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