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Researching Geometrical Intuition

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Introduction

The term 'intuition' occurs frequently in discourse about mathematics. Rarely, however, is the term analysed or investigated empirically. In this paper I will describe a recent exploratory study of the nature and role of geometrical intuition in the solving of geometrical problems.

What is intuition?

The most substantial work on intuition in mathematics learning is that carried out by Fischbein (1987). In my work I have used Fischbein's definition which is that intuition is a cognition characterised by the following properties (*ibid* p 43-56): *self-evidence and immediacy* (in that extrinsic justification is not needed), *intrinsic certainty* (note that self-evidence and certainty are not the same), *perseverance* (so that intuitions are stable), *coerciveness, theory status, extrapolativeness, globality* (in that intuitions offer a unitary global view), *implicitness* (so that although the result of selection, globalisation and inference, intuitions will appear to be implicit). The behavioural task of intuition then is to prepare and guide our mental or practical activity. This links with Fischbein's further suggestion that there are "problem solving intuitions", for which, however, he provides no empirical evidence. Thus, in attempting to study intuition, the context of problem solving appears to be an appropriate place to begin as the following quotation shows:

> "If we seriously want to recognise the role of intuition in problem-solving - and virtually all mathematicians recognise its contribution - then we need to fill out this stage of representation with concrete accounts rather than detailed and prespecific theoretical accounts."
>
> Noddings (1985 p 348)

Why study geometrical intuition?

The role of intuition in geometrical understanding has been emphasised for some time (for instance see Hilbert 1932 and Van Heile 1986). Indeed Fischbein's work stresses the role of visualisation in the generation of intuitions. Furthermore, for instance, the final chapter of Piaget's *The Child's Conception of Space* (Piaget and Inhelder 1956) is called 'The 'Intuition' of Space'). Since then, however, although there has been work on the intuition of number (see, for instance, Resnick 1986) there appears to be little which directly considers the intuition of geometry.

In addition, the advent of computer packages such as *Cabri-Geometre* provides new opportunities for studying the approaches used in solving geometrical problems. Indeed, such use of the computer may make intuition more accessible for study. In the context of Logo, Papert says:

> "I see the computer as helping in two ways. First the computer allows, or

obliges, the child to externalise intuitive expectations. When the intuition is translated into a program it becomes more obtrusive and more accessible to reflection. Second, computational ideas can be taken up as materials for remodelling intuitive knowledge. ... a turtle model can help bridge the gap between formal knowledge and intuitive understanding"

Papert (1980 p 145)

One aim of my current work is to discern whether *Cabri* has a similar effect.

A Framework for Discerning Geometrical Intuition

One of the most productive researchers in the area of mathematical problem solving has been Schoenfeld (Schoenfeld 1985, for example). This particular work offers not only a theoretical framework within which to base the research questions in the present study, but also a rich source of guidance on issues of methodology and on the design of the empirical work. In his theoretical framework Schoenfeld makes only passing reference to the role of intuition.

In his work Schoenfeld has used a number of geometrical construction problems in which he found that "insight and intuition come from drawing" and that two factors dominate in generating and rank ordering hypotheses for solution, the first being what he refers to as the "intuitive apprehensibility" of a solution. In other words, Schoenfeld's study shows that geometric construction tasks may be ones in which it may be possible to discern the role of geometrical intuition. However, Schoenfeld makes little other reference to the nature and role of intuition in problem solving.

My approach was then as follows:

step 1: analyse Schoenfeld's transcripts of students working on the geometrical construction problems to help discern *critical decisions in the solution of these problems*. The conjecture being proposed here is that geometrical intuition plays a part in the critical decisions that problem-solvers make when tackling geometrical problems.

step 2: use the *categories of intuition* proposed by Fischbein to discern examples of the use of geometrical intuition by Schoenfeld's subjects at moments of critical decision.

step 3: by interpreting the definitions of the *mechanisms of intuition* given by Fischbein, the factors which participate in the generation of these geometrical intuitions will be isolated. Thus the second conjecture is that certain mechanisms of intuition are influential in certain episodes in the problem solving process.

In this way I devised a framework for discerning geometrical intuition.

Experimental work

The following methodology was developed:

1) The categories and mechanisms of intuition proposed by Fischbein are integrated into the framework for characterising mathematical problem solving developed by Schoenfeld.

2) Protocol analysis is adopted as the method for generating and analysing empirical evidence with the techniques of Schoenfeld being used to design the experimental work.

3) The focus for the experimental work is the solving of three geometric construction problems (as used by Schoenfeld) by pairs of subjects working with the computer-based

geometry package *Cabri-Geometre* and their solution process will be analysed by reference to the techniques developed in 1) and 2).

The experimental work produced three case studies of pairs of subjects engaged in solving geometric problems using the geometry package *Cabri-Geometre*.

Method of Analysis

The approach to the analysis of the empirical data generated by the experimental work carried out for this exploratory study can be summarised as follows:

1) analyse the problem-solving sessions for critical decisions.
2) analyse the problem-solving protocols for evidence of anticipatory and conclusive intuitions at moments of critical decision
3) analyse the protocols using the episodes proposed by Schoenfeld and to discern the participation of the mechanisms of intuition.

What did I find?

In this paper I only have space to describe problem 1 which is given below:

Problem 1 You are given two intersecting straight lines and a point *P* marked on one of them, as in Figure 1 below. Show how to construct, using straightedge and compass, a circle that is tangent to both lines and that has the point *P* as its point of tangency to one of the lines.

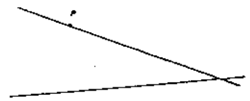

Figure 1

From Schoenfeld (1985)

Critical decisions in the solution of Problem 1 are as follows:

1. Constructing a perpendicular line through P
2. Constructing the angle bisector of the angle between the two intersecting lines

However, while each of the pairs made critical decision 1 almost straight away, only one member of pair 2 immediately saw that decision 2 was also critical. The other pairs constructed a second perpendicular, perpendicular to the lower of the intersecting lines, and proceeded to move this into approximately the correct position. For pairs 1 and 3 this prompted them into drawing the angle bisector of the intersecting lines and the problem was solved.

For pair 1 the suggestion to draw the angle bisector was made quite tentatively:

TC: Yes ... Ah! Now would the centre of the circle lie .. I'm just thinking something slightly different now, because I'm just trying to think, there must be a way of securing the centre accurately .. and I'm thinking .. does the centre of the circle ..sit on the bisector of the angle that's made by those two lines ..

In contrast, for pair 3 one of the students was more certain

KH: I tell you the other thing we could do and that's to bisect that angle to find out where they should cross.

My claim is that these are examples of the influence of geometrical intuition. This is how the students accounted for their actions. In the case of pair 1:

TC: .. [long pause] .. well, partly previous knowledge. I wasn't .. completely sure. I wasn't saying 'Oh, yes. This is what does happen'. I just had a sneaky feeling that we were missing something and I couldn't work out what it was, but I thought, well I'm sure the angle .. there must be some connection between the angle between the two lines and the centre [of the circle]. So, let's put the line in and see what happens.

It turned out to be right, but it was just a sort of stab .. well, it wasn't a stab in the dark completely ...

I can't think why, but I was sure we should be bisecting the angle.

In the case of pair 3:

KH: Ohhh! .. [laughs] .. That's quite interesting because, maybe, .. the fact that there's a cross there [where the two perpendicular lines intersect 'opposite' where the original two lines intersect] actually encouraged me to think well, we need to know where the cross is going to be. Perhaps if we hadn't have drawn the other perpendicular it would not have come so quickly.

Looking at that picture now I think .. it's ..er .. er .. I mean just having that sort of cross there on the screen opposite the angle there, I mean, that just spells it out. I think perhaps that's why it just came so quickly.

In both cases the students had some difficulty explaining their actions although both previous experience and the visual image played a part in determining the course of action they were suggesting. In this context, Fischbein says, "Experience is a fundamental factor in shaping intuitions" (Fischbein 1987 p 85). However, Fischbein then goes on to say that "There is little systematic evidence available supporting that view, ie evidence demonstrating that new intuitions can be shaped by practice" (*ibid* p 85).

In terms of the visual image, Fischbein claims that visualisation "is the main factor contributing to the production of the effect of immediacy" (*ibid* p 103). Fischbein then goes on to relate visualisation to the domain of mental models. The evidence available from this study seems to support Fischbein's views in the domain of solving geometrical problems. However, further work is needed in order to say much more than that.

Conclusions

I began with two conjectures. Firstly that geometrical intuition plays a part in the critical decisions that problem-solvers make when tackling geometrical problems and secondly thatcertain mechanisms of intuition are influential in certain episodes in the problem solving process. It seems that the first of these conjectures is well-founded: geometrical intuition does indeed play an important part in the critical decisions that problem-solvers make when

tackling geometrical problems.

The second conjecture, however, remains tentative. One reason for this is that the analysis was examining points of critical decision for the *successful* solution of the problem, instances of geometrical intuition may, inevitably, tend to form points of transition in the problem-solving process or occur during planning and implementing episodes.

Further work currently being considered could examine, for instance,:
- the relationship between experience and intuition
- the nature and role of visualisation in the generation of intuitions
- developing the methodology in order to encourage the subjects to explain more
- the use of different geometrical problems
- the relationship between intuition and proof
- more on the effects of using *Cabri*, in particular, focusing on the potential of *Cabri* to provide a 'window' on geometrical intuition

On this last point, further work is necessary before any firm conclusions can be made. Correspondence on this point, or any other raised in this paper, would be welcomed (my e-mail address is DKJ@uk.ac.soton.mail

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