Optimising Social Welfare in Multi-Resource Threshold Task Games

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A Proofs

This section provides the omitted proofs in the text.

Theorem 1. *The CSG problem on MR-TTGs is $\mathcal{NP}$-hard.*

Proof. We reduce from the bounded multidimensional knapsack problem, which is formally defined as:

$$max \quad \sum_{k=1}^{q} p_k \cdot x_k \qquad x_k = 0, \ldots, b_k \tag{1}$$

$$s.t. \sum_{k=1}^{q} w_{jk} \cdot x_k \leq c_j \; j = 1, \ldots, m. \tag{2}$$

The proof proceeds as follows: (1) The BMKP is mapped to an instance of an MR-TTG; (2) a solution for the BMKP is constructed from the optimal solution to the CSG on MR-TTG; (3) we show that the optimality of the solution to the CSG guarantees the optimality of the BMKP from step (2).

First, given an instance of a BMKP, an MR-TTG of 2 players is constructed as follows: the knapsack dimensions are mapped directly to the resource types $R = \{1, \ldots, m\}$ and item types are mapped to task types $T = \{1, \ldots, q\}$. Consequently, the number of copies of every item type b_k is mapped to the task demand d_k, $\forall k \in q$. Moreover, each task type $k \in T$ is described by a value $v_k = p_k$ and a vector of thresholds $\tau_k = w_k$. The knapsack capacity can be divided between the players to represent the resources they posses. We divide the capacity such as player 1 gets the first $\lfloor \frac{m}{2} \rfloor$ resources and player 2 gets the rest. Thus, the resource vectors for the first and second players are $r^1 = (c_1, \ldots, c_{\lfloor \frac{m}{2} \rfloor}, 0, \ldots, 0)$ and $r^2 = (0, \ldots, 0, c_{\lfloor \frac{m}{2} \rfloor + 1}, \ldots, c_m)$ consecutively.

Secondly, let CS^* be an optimal coalition structure for the constructed MR-TTG. A solution, $\mathbf{x}$, to the BMKP can be derived from CS^* as: the number of items to be packed in the knapsack is equal to the size of the optimal coalition structure $|CS^*|$, more specifically, there are $|CS^*_k|$ copies of each item type k, i.e., $x_k = |CS^*_k|$. This solution respects the BMKP constraints eq (**??**) since $|CS^*_k|$, the number of accomplished tasks of type k, is bounded by $d_k = b_k$.

Finally, we prove by contradiction that $\mathbf{x}$ is optimal. Let us assume that the derived solution, $\mathbf{x}$, is not optimal. This assumption implies the existence of another solution, $\mathbf{x}'$, to the BMKP with a greater payoff than $\mathbf{x}$, i.e., $\sum_{k=1}^{q} p_k \cdot x'_k > \sum_{k=1}^{q} p_k \cdot x_k$ and $\sum_{k=1}^{q} w_k \cdot x'_k \leq c$. Since $c = r^1 + r^2$ and $w_k = \tau_k$ we can write the last inequality as $\sum_{k=1}^{q} \tau_k \cdot x'_k \leq r^1 + r^2$. This means that there exist a coalition structure CS such that players 1 and 2 can jointly accomplish x'_k tasks of type k, i.e., $|CS_k| = x'_k$, $k = 1, \ldots, q$. The value of this coalition structure can be calculated as $\sum_{k=1}^{q} v_k \cdot |CS_k|$ which is equal to is $\sum_{k=1}^{q} p_k \cdot x'_k$ since $p_k = v_k$. This leads to a contradiction since $v(CS) = \sum_{k=1}^{q} p_k \cdot x'_k > \sum_{k=1}^{q} p_k \cdot x_k = v(CS^*)$.

Theorem 2. *The Coalition Structure Generation problem for a MR-TTG can be reduced in a polynomial time to a BMKP.*

Proof. The proof proceeds as follows: (1) The CSG problem is mapped to an instance of the BMKP; (2) a coalition structure is constructed from the optimal solution to the BMKP; (3) we show that the optimality of the solution to the BMKP guarantees the optimality of the coalition structure from step (2).

First, a BMKP is constructed from a CSG problem for an MR-TTG. Given the input to the CSG problem, each task type k can be mapped to an item type k in the corresponding BMKP. Item type k is described by a profit $p_k = v_k$ and the weights vector $w_k = \tau_k$. In addition, the demand d_k corresponds to the bound b_k (number of copies available of each item). Finally, the capacity of the knapsack is calculated as $c_j = \sum_{i=1}^{n} r^i_j, \forall\, j = 1, \ldots, m$.

Secondly, we give a polynomial time algorithm of complexity $O(\sum_{k=1}^{q} b_k \cdot m \cdot n)$ to translate an outcome of a BMKP to a coalition structure for an MR-TTG (see algorithm **??**). The outcome has x_k copies of item k and a profit of $\sum_{k=1}^{q} p_k \cdot x_k$. The input of the algorithm is the set of agents A, agents' possessions r^i, the set of task types T, and the outcome of the BMKP. The algorithm distributes agents' resources among partial coalitions so that each partial coalition is satisfied; it assigns a value to the variable w^i_{jkl} (line 7). From the loops in lines 3 and 4 we can see that the number of partial coalitions formed is $\sum_{k=1}^{q} x_k$, and specifically the number of partial coalitions formed of type k is x_k. Therefore, $|CS_k| = x_k \leq b_k = d_k$, from the definition of the BMKP. In addition, the condition in line 9 ensures that an agent i does not contribute more than a total of $r^i_j, \forall\, j \in R$ in all the partial coalitions i has joined. Hence, the outcome of a BMKP satisfies the properties of a coalition structure after algorithm **??** has run.

Finally, we prove by contradiction that the generated coalition structure CS is optimal. Let us assume that CS is not optimal, this assumption leads to the existence of another coalition structure CS' with a value greater than the value of CS; $\exists\, CS' s.t. \sum_{C_{kl} \in CS'} v(C_{kl}) > \sum_{C_{kl} \in CS} v(C_{kl})$. Suppose that the number of partial coalitions working on a task of type k in CS' is $x'_k (|CS'_k| = x'_k)$ and for CS we know from algorithm **??** that $|CS_k| = x_k$ then the previous inequality can be written as $\sum_{k=1}^{q} v_k \cdot x'_k > \sum_{k=1}^{q} v_k \cdot x_k$. Also, from the feasibility of CS', the resource constraint $\sum_{C_{kl} \in CS'} w^i_{jkl} \leq r^i_j$ can be generalised to $\sum_{C_{kl} \in CS'} \sum_{i=1}^{n} w_{jkl}^{i} \leq \sum_{i=1}^{n} r_j^i$. Since it is optimal to allocate exactly τ_k to each partial coalition C_{kl} due to monotonicity ($\sum_{i=1}^{n} w_{jkl}^{i} = \tau_{jk}, \forall\, j \in R$), the resource constraint can be re-written as $\sum_{k=1}^{q} \tau_{jk} \cdot x'_k \leq \sum_{i=1}^{n} r_j^i, \forall\, j \in R$. Since $\tau_k = \bar{w_k}, d_k = b_k$ and $v_k = p_k \,\forall\, k \in T$ and $c_j = \sum_{i=1}^{n} r_j^i \,\forall\, j \in R$ then there is a feasible solution to the BMKP with x'_k copies of item k, resource requirement $w_{jk} \cdot x'_k \leq c_j, \forall\, j = 1, \ldots, m$ and value $\sum_{k=1}^{q} p_k \cdot x'_k > \sum_{k=1}^{q} p_k \cdot x_k$ - contradiction since it outvalues the optimal solution.

Algorithm 1: Redistribute agents' resources

$w_{jkl}^{i} = 0, \forall\, i \in A, j \in R, k \in T, l \leq x_k$
$i = 1$
for all tasks types $k \in T$ **do**
  for all copies $l \leq x_k$ of k **do**
    for all $j \in R$ **do**
      repeat
        $w_{jkl}^{i} = min(r_j^i, \tau_{jk} - \sum_{i=1}^{n} w_{jkl}^{i})$
        $r_j^i = r_j^i - w_{jkl}^{i}$
        if $r_j^i = 0$ **then**
          $i = (i+1)\, mod\, n$
      until $\sum_{i=1}^{n} w_{jkl}^{i} = \tau_{jk}$

Theorem 3. *A bounded multi-dimensional knapsack problem can be reduced to a multiple-choice multi-dimensional knapsack problem.*

Proof. The MMKP is formally defined as:

$$max \qquad \sum_{y=1}^{\upsilon} \sum_{g=1}^{h^y} p_g^y \cdot x_g^y \qquad y = 1, \ldots, \upsilon \tag{3}$$

$$s.t. \sum_{y=1}^{\upsilon} \sum_{g=1}^{h^y} w_{jg}^y \cdot x_g^y \leq C_j \;\; j = 1, \ldots, m \tag{4}$$

$$\sum_{g=1}^{h^y} x_g^y = 1 \qquad g = 1, \ldots, h^y \tag{5}$$

$$x_g^y \in \{0, 1\} \tag{6}$$

Let C be a multiset such that $k \in C$, $\forall\, k = 1, \ldots, q$ and the recurrence of item $k \in C$ (multiplicity of k) is denoted by $m_{k \in C} = b_k$, $\forall\, k \in C$. Thus, $|C| = \sum_{k=1}^{q} b_k$.

Suppose that C is partitioned into an arbitrary number υ of multisets C^y, $y = 1, \ldots, \upsilon$ and let $Y = \cup_{y=1}^{\upsilon} \{\mathcal{P}(C^y)\}$, where $\mathcal{P}(C^y)$ denotes the power set of C^y. For an arbitrary multiset $S \in \mathcal{P}(C^y)$, assuming all the items it holds $k \in S$ are required to be packed at a time, the profit and vector of weights for S are $p^S = \sum_{k \in S} p_k$ and $w^S = \sum_{k \in S} w_k$ respectively.

The set Y represents a multiple-choice multidimensional knapsack problem with υ classes. Furthermore, the power set $\mathcal{P}(C^y)$ constitutes the classes $y = 1, \ldots, \upsilon$. The objective function, as shown below, restricts the number of sets of items to be packed in the knapsack out of each class to exactly one. Formally, the transformed MMKP problem is defined as:

$$\begin{aligned} &max \quad \textstyle\sum_{y=1}^{\upsilon} \sum_{S \in \mathcal{P}(C^y)} p^S \cdot x^S &(7)\\ &s.t. \textstyle\sum_{y=1}^{\upsilon} \sum_{S \in \mathcal{P}(C^y)} w_j^S \cdot x^S \leq c_j \ \forall j = 1, \ldots, m &(8)\\ &\textstyle\sum_{S \in \mathcal{P}(C^y)} x^S = 1 \qquad \forall y = 1, \ldots, \upsilon &(9) \end{aligned}$$

Lemma 1. *The union of the selected set of each class $y = 1, \ldots, \upsilon$ of the outcome of the MMKP $\cup_{y=1}^{\upsilon} S$ s.t. $x^S = 1$ is the optimal solution for the BMKP.*

Proof. The union set is a valid outcome for the BMKP since it is a subset of the multiset C holding the maximum number of items that could be packed in the knapsack. In addition, the union set could be any subset of the multiset C, since it is formed of the union of the power sets of all partitions of C. Hence, the mapping does not affect the possible outcomes of the original problem. Finally, the maximisation in the objective function guarantees that optimality of the solution.