

Journal of Engineering Mathematics

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--Manuscript Draft--

Manuscript Number:	ENGI-D-16-00319R2	
Full Title:	Free vibration analysis of functionally graded beams with non-uniform cross-section using the differential transform method	
Article Type:	Manuscript	
Keywords:	free vibrations; non-uniform cross-section beam; Functionally graded material; differential transform Method	
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Funding Information:	Erasmus Mundus program ACTIVE (2013-2523/001-001)	Not applicable
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Response to Reviewers:	<p>Dear Editors,</p> <p>We want once more to extend our appreciation to editors and reviewers for their time and efforts spent to provide us with insightful guidance. We have carefully considered all the comments and answered them as outlined below.</p> <p>Kind regards, The authors</p>	

Free vibration analysis of functionally graded beams with non-uniform cross-section using the differential transform method

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Abstract

Free vibrations of non-uniform cross-section and axially functionally graded (AFG) Euler-Bernoulli beams with various boundary conditions were studied using the differential transform method. The method was applied to a variety of beam configurations that are either axially non-homogeneous or geometrically non-uniform along the beam length or both. The governing equation of an Euler-Bernoulli beam with variable coefficients was reduced to a set of simpler algebraic recurrent equations by means of the differential transformations. Then, transverse natural frequencies were determined by requiring the non-trivial solution of the eigenvalue problem stated for a transformed function of the transverse displacement with appropriately transformed its high derivatives and boundary conditions. To show the generality and effectiveness of this approach, natural frequencies of various beams with variable cross-section and functionally graded non-homogeneity profiles were calculated and compared with analytical and numerical results available in the literature. The benefit of the differential transform method to solve eigenvalue problems for beams with

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arbitrary axial geometrical non-uniformities and axial material gradient profiles is clearly demonstrated.

Keywords: free vibrations, non-uniform cross-section beam, functionally graded material, differential transform method

1. Introduction

Natural frequencies are fundamental to the understanding of the dynamics of mechanical systems. Their knowledge is required for analysing and designing a structure in dynamic environment. Contemporary in-service conditions impose additional requirements on the properties of materials used in structures. This is particularly true for structures that have to perform under extremely severe thermal loading and adverse environment. In this regard, functionally graded materials (FGM), a type of composite materials, are able to withstand high temperatures, thermal fracture and corrosion [1–4]. Therefore, the vibration analysis of FGM structures is of critical importance from the standpoint of their safety and effective exploitation.

Because beams are the basic structural elements in engineering constructions on the one hand and the simplest models for theoretical research on the other hand, free vibration analysis of FGM beams has been extensively studied and continues to receive much attention in the literature. Many studies have been carried out on vibrations of beams containing a material gradient in the thickness direction. Various assumptions on deformation of beams and different approaches used for obtaining exact analytical and approximate numerical solutions have been reported. The comparison between different beam theories being exploited for natural frequency analysis of FGM beams is given in [5].

Analytical solutions for FGM beams with axially graded material properties are difficult to obtain. The complexity of the analysis lies in the presence of variable coefficients in the governing beam equation caused by the dependence of material parameters on the axial coordinate. From the mathematical point of view this issue is similar to the problem, in which the terms of governing beam

equation are functions of the axial coordinate due to variable cross-sectional area and flexural rigidity along the beam length. In the past this problem received a wide consideration among many researchers; see e.g. [6]. It is well-known that exact solutions of this problem can be obtained only for a few special cross-sectional profiles and boundary conditions. Free vibration analysis solved for beams with continuously and stepped varying cross-sectional parameters has been reviewed in recent publications, e.g. [7–9]. Both analytical solutions in terms of special functions including Bessel functions, hypergeometric series, power series, Bernstein polynomials and approximate solutions obtained by using Rayleigh-Ritz approach, finite element method, dynamic stiffness method, differential quadrature method, differential transform method have been reported there. A series of analytical solutions for prismatic beams with inhomogeneous material parameters prescribed in the form of certain polynomials has been obtained in [10, 11]. Li [12] was able to derive an analytical solution in terms of Bessel functions for the free vibration problem of beams containing determined both an axial geometrical non-uniformity and axial mass and shear stiffness distributions. Another analytical solution for natural frequencies of FGM beams is shown in [13], where the authors have considered exponential material gradient in the beam thickness direction and arbitrary variable cross-section along the beam length. Exact solutions of the free vibration problem for FGM beams with material profiles and cross-sectional parameters varying exponentially in the axial direction have been deduced in [14, 15], where assumptions of Euler-Bernoulli and Timoshenko beam theories have been applied respectively.

Apart from analytical solutions for analysing limited classes of beams satisfying defined assumptions of inhomogeneity and non-uniformity, many numerical approaches have been developed. Some of researchers have used the finite element method (FEM) for AFGM beams with arbitrary varying cross-section using appropriate homogeneous finite elements, e.g. in [16, 17] or developing special graded finite elements, e.g. in [18]. In [19] the authors studied the free vibration of beams with axial material gradation and non-uniform cross-section

by transforming the governing differential equation with variable coefficients into Fredholm integral equations. The natural frequencies were determined by requiring that the resulting Fredholm equation has non-trivial solutions. The natural frequencies of non-uniform and AFGM beams with various boundary conditions and cross-sectional parameters have been calculated by using Haar wavelets in [20]. The differential quadrature method has been employed for studying free vibrations of tapered homogeneous and AFGM rotating cantilever beams in [21, 22], respectively. In the latter paper, for the sake of comparison the author has also used the differential transform method (DTM). However, only beams with polynomial forms of non-uniform cross-sectional parameters and material inhomogeneities were considered there.

Differential transformations as an approach for solving variety types of differential equations were firstly proposed by G.E. Pukhov [23] in 1976. In the beginning, the method invented by him was applied for investigation of electrical circuits [24–26] and thereafter he published a series of books [27–30], where foundations of differential transformations and their application to various pure mathematical and engineering problems were laid out and developed in detail. Recently, Bervillier [31] presented the state-of-the-art of the differential transform method in modern science, but there appear to be notable exclusions. In particular, the book of Simonyan and Avetisyan [32], where problems such as finding invariants (eigenvalue, determinant, inverse and pseudo-inverse) of non-autonomous matrices, solving linear and non-linear non-autonomous systems of finite equations have been considered and solved. Also, optimal control problems and non-autonomous matrix equations have been studied and methods of their solutions based on DTM have been proposed in [33, 34].

The differential transform method has already been applied to the vibration analysis of non-uniform cross-section and inhomogeneous beams. In [35] and [36] the authors have utilized the differential transformations to analyse free lateral vibrations of rotating tapered Euler-Bernoulli beams. Abdelghany et al. in [37] have used the DTM to compute natural frequencies and appropriate mode shapes of homogeneous beams with smooth and continuous variations of non-

uniform cross-sections. Free vibrations of stepped FGM beams having abrupt changes of geometrical characteristics have been studied by means the DTM as well. However, it should be noticed that since cross-section and moment of inertia of the stepped beam are no longer continuous functions along the beam length in the governing differential equations, the stepped beam is analysed as an assemblage of several piecewise continuous segments and the differential transformations are applied to each of the segments accounting additionally for continuous conditions between the segments. Thus, the computational cost of the DTM increases with increasing the number of the beam sections, because more recurrent equations have to be solved. In [38] the DTM has been exploited to the free vibration analysis of FGM two-section-beams with thickwise material gradients. Natural frequencies of similar stepped FGM two-section-beams with elastically constrained ends have been calculated with the DTM in [39]. In [40] and [41], the authors have proposed a differential transform element method (DTEM) improving the convergence of the DTM to examine free vibrations and stability of axially FGM tapered Euler-Bernoulli and Timoshenko beams, respectively. A vibration analysis of spinning exponentially functionally graded Timoshenko beams has been carried out with the DTM in [42].

The literature search has reveals that studies on free vibrations of axially FGM beams with varying cross-sectional parameters has been a subject of active recent research. To the best of our knowledge, DTM has not been applied to the study of free vibrations of FGM beams with arbitrary axial material gradation and cross-sectional non-uniformities varying along the beam length. Existing works are limited in the consideration to the choice of material gradients, cross-sectional parameters and boundary conditions. Having identified the gap in the literature, the objective of this paper is to present a novel approach based on the DTM for analysing free vibrations of axially functionally graded and non-uniform cross-section beams. The differential transformations will be used as a primary mathematical tool for finding natural frequencies of those beams. We transform the governing differential beam equation with arbitrary variable coefficients in connection with appropriate end supports to a set

of recurrent algebraic equations with respect to unknown coefficients. Natural frequencies are determined from the existence condition of a non-trivial solution in the resulting system of algebraic equations. The results obtained are compared with those solutions available in the literature. In this regard, the convergence of DTM is also evaluated in the paper. A high accuracy, efficiency and versatility of the DTM is demonstrated by various examples. The examples considered are supplemented by results concerning the convergence of the DTM solutions. Finally, we analyse natural frequencies of axially FGM beams made of Aluminium Zirconia alloy (AlZrO_2) subjected to different end supports, material gradation profiles and cross-sectional parameters. The effects of the gradation parameter and the degree of non-uniformity on the natural frequencies are shown. Some computational issues being arisen in those calculations with the DTM are discussed in detail.

This paper is organized as follows. Differential transforms are introduced and their best known properties are discussed in the next section. A mathematical solution of the differential beam equation with variable coefficients arising from the free vibration of axially functionally graded beams with non-uniform cross-section along the beam length is formulated in Section 3. Numerical results are presented and compared with existing results, where possible, in Section 4. Finally, concluding remarks are presented in Section 5.

2. A brief description of differential transformations

Basic mathematical aspects of the DTM were presented in the original publications of the inventor of differential transformations [27–30], to which we refer for more details of this method. Throughout this paper, we use definitions and denotations introduced in those books.

Let us consider a function $A(x)$ and suppose that $A(x)$ and its all derivatives are smooth functions in a given interval (x_0, x_1) . The K th derivative of $A(x)$ in a certain x_v point can be considered as an $A(K)$ image (or discrete) of the original function $A(x)$. The totality of $A(K)$ images when the index goes $K = [0, \infty)$

will be a differential spectrum of the $A(x)$ function. Then, the basic expression to get images for a given original is the following:

$$A(K) = \frac{H^K}{K!} \left. \frac{d^K A(x)}{dx^K} \right|_{x=x_v}, \quad (1)$$

where $A(K)$ are images of $A(x)$, H is a given constant (or so-called the scale factor), x_v is a centre of approximation.

If a spectrum of $A(K)$ images of the original $A(x)$ is known, the original $A(x)$ may be reconstructed. This can be achieved by various ways, depending on what type of series could be used to recover the original function. In particular, using the Taylor series, the Taylor differential transformations or the T-transformations, called so by G.E. Pukhov, take the following form:

$$A(x) = \sum_{K=0}^{\infty} \left(\frac{x - x_v}{H} \right)^K A(K). \quad (2)$$

The basic operations in the domain of differential transformations (DT) and some their properties as those in [27–30] are listed in Table 1. It is assumed that the originals $u(x)$ and $v(x)$ have appropriate $U(K)$ and $V(K)$ differential spectra. Then, for given $U(K)$ and $V(K)$ discretised we calculate the discrete $Z(K)$, which is a differential spectrum of an $z(x)$ original. The latter is a result of algebraic manipulations between the originals $u(x)$ and $v(x)$ and differentiation of $u(x)$. It should be noted that the scale factor H is taken equal to one there.

3. DTM applied to free vibrations of inhomogeneous and non-uniform beams

The partial differential equation that governs free vibrations of beams of length L , which are axially inhomogeneous and/or have a continuously variable cross-section area along the beam axis is given by [43]:

$$\frac{\partial^2}{\partial x^2} \left[D(x) \frac{\partial^2 w}{\partial x^2} \right] + m(x) \frac{\partial^2 w}{\partial t^2} = 0, 0 \leq x \leq L, \quad (3)$$

where x is the axial coordinate, w is the transverse beam displacement, $D(x) = E(x)I(x)$ is the flexural rigidity presented as a function of the axial coordinate

Table 1: Main operations of differential transformations

Original domain	DT domain
$z(x) = u(x) + v(x)$	$Z(K) = U(K) + V(K)$
$z(x) = cu(x)$	$Z(K) = cU(K)$
$z(x) = u(x)v(x)$	$Z(K) = \sum_{p=0}^K U(p)V(K-p)$
$z(x) = \frac{u(x)}{v(x)}$	$Z(K) = \frac{U(K) - \sum_{p=0}^K U(p)V(K-p)}{V(0)}$
$z(x) = \frac{du(x)}{dx}$	$Z(K) = (K+1)U(K+1)$
$z(x) = \frac{d^2u(x)}{dx^2}$	$Z(K) = (K+1)(K+2)U(K+2)$
$z(x) = \frac{d^m u(x)}{dx^m}$	$Z(K) = \frac{(K+m)!}{m!}U(K+m)$
$z(x) = x^n$	$Z(K) = \delta(K-n) = \begin{cases} 1 & K = n \\ 0 & K \neq n \end{cases}$

x and depends on both Youngs modulus $E(x)$ and second moment of the cross-section area $I(x)$, $m(x) = \rho(x)A(x)$ is the mass of the beam per unit length depending upon variable cross-sectional area $A(x)$ and mass density $\rho(x)$.

It is well-known that to solve (3) it is necessary to seek a temporally harmonic solution for the deflection in the form:

$$w(x, t) = \bar{w}(x)e^{i\omega t}, \quad (4)$$

where $\bar{w}(x)$ is the amplitude of the beam deflection, ω is the angular frequency and i is the imaginary unit. Substituting (4) into (3) reduces the initially partial differential equation (3) into an ordinary differential equation of the beam:

$$\frac{d^2}{dx^2} \left[D(x) \frac{d^2 \bar{w}(x)}{dx^2} \right] - m(x)\omega^2 \bar{w}(x) = 0, \quad 0 \leq x \leq L. \quad (5)$$

Thus, in the present study, a beam with inhomogeneous material properties and non-uniform cross-sectional parameters varying continuously along the axial direction is completely determined by the ordinary differential equation (5).

In order to find a solution of 5 using the differential transformations, it is

convenient to rewrite it as:

$$\frac{d^4 \bar{w}}{dx^4} + \bar{D}_1(x) \frac{d^3 \bar{w}}{dx^3} + \bar{D}_2(x) \frac{d^2 \bar{w}}{dx^2} - \bar{M}(x) \omega^2 \bar{w} = 0, \quad (6)$$

where the variable coefficients will be denoted as $\bar{D}_1(x) = 2D'(x)/D(x)$, $\bar{D}_2(x) = D''(x)/D(x)$ and $\bar{M}(x) = m(x)/D(x)$. Here, and in what follows, the prime denotes a derivative with respect to the x -coordinate.

Applying the differential transformations defined in Table 1 to Eq. (6), the latter in the differential transformation domain takes a form of a set of recurrent algebraic equations:

$$W(K+4) = \frac{1}{(K+1)(K+2)(K+3)(K+4)} \cdot \left[\omega^2 \sum_{p=0}^K W(p)M(K-p) - \sum_{p=0}^K (p+1)(p+2)(p+3)W(p+3)D_1(K-p) - \sum_{p=0}^K (p+1)(p+2)W(p+2)D_2(K-p) \right], \quad (7)$$

where $W(K)$ and $D_1(K)$, $D_2(K)$ and $M(K)$ are the discretized originals of the unknown function $\bar{w}(x)$ and the given expressions $\bar{D}_1(x)$, $\bar{D}_2(x)$ and $\bar{M}(x)$, respectively. The last discretized values are calculated using appropriate transformation rules presented in Table 1. Note that each recurrent equation of the system (7) is obtained by sequentially changing the K index.

The system (7) can be reduced to a matrix equation as follows:

$$W(K+4) = B_K W(0) + C_K W(1) + G_K W(2) + H_K W(3), \quad (8)$$

where the recurrent expressions for the coefficients $B_K(\omega)$, $C_K(\omega)$, $G_K(\omega)$ and $H_K(\omega)$ are presented in Appendix A.

In (8), the discretized values $W(0)$, $W(1)$, $W(2)$ and $W(3)$ are taken as unknown variables because their actual values in the differential spectrum cannot be calculated from the system of recurrent equations (7).

If all $W(K)$ images are computed, the original function $\bar{w}(x)$ can be reconstructed according to Eq. (2) as

$$\begin{aligned} \bar{w}(x, \omega) = & W(0) + W(1)(x - x_v) + W(2)(x - x_v)^2 + W(3)(x - x_v)^3 + \\ & W(4)(x - x_v)^4 + \cdots + W(K)(x - x_v)^K, \end{aligned} \quad (9)$$

where the discretized $W(K)$ at $K \geq 4$ can be expressed via the unknown discretized $W(0)$, $W(1)$, $W(2)$ and $W(3)$. Hence, one can get the original function in the form:

$$\begin{aligned} \bar{w}(x, \omega) = & W(0) + W(1)(x - x_v) + W(2)(x - x_v)^2 + W(3)(x - x_v)^3 + \\ & [B_0W(0) + C_0W(1) + G_0W(2) + H_0W(3)](x - x_v)^4 + \\ & [B_1W(0) + C_1W(1) + G_1W(2) + H_1W(3)](x - x_v)^5 + \cdots + [B_{K-4} + \\ & W(0) + C_{K-4}W(1) + G_{K-4}W(2) + H_{K-4}W(3)](x - x_v)^K. \end{aligned} \quad (10)$$

Substituting the expressions for the coefficients (A.1) to (A.4) into (10) and collecting common terms with respect to the unknown discretized, the original function becomes as

$$\begin{aligned} \bar{w}(x, \omega) = & (1 + B_0(x - x_v)^4 + B_1(x - x_v)^5 + \cdots + B_{K-4}(x - x_v)^K)W(0) + \\ & ((x - x_v) + C_0(x - x_v)^4 + C_1(x - x_v)^5 + \cdots + C_{K-4}(x - x_v)^K)W(1) + \\ & ((x - x_v)^2 + G_0(x - x_v)^4 + G_1(x - x_v)^5 + \cdots + G_{K-4}(x - x_v)^K)W(2) + \\ & ((x - x_v)^3 + H_0(x - x_v)^4 + H_1(x - x_v)^5 + \cdots + H_{K-4}(x - x_v)^K)W(3). \end{aligned} \quad (11)$$

Boundary conditions imposed on the beam should be taken into account to calculate its natural frequencies. Hence, physical quantities such as rotation angle θ , bending moment M and shear force Q involved in Eq. (3) should be defined in their explicit forms, i.e.

$$\theta = \frac{d\bar{w}}{dx}, \quad M = -D(x)\frac{d^2\bar{w}}{dx^2} \quad \text{and} \quad Q = -\frac{d}{dx} \left[D(x)\frac{d^2\bar{w}}{dx^2} \right]. \quad (12)$$

In the present paper, the following end supports of the beam are considered:

- Cantilever (clamped-free) beam (C-F):

$$\bar{w}(0) = 0, \quad \theta(0) = 0, \quad M(L) = 0 \quad \text{and} \quad Q(L) = 0;$$

- Simply supported beam (S-S):
 $\bar{w}(0) = 0, M(0) = 0, \bar{w}(L) = 0$ and $M(L) = 0$;
- Clamped-pinned beam (C-P):
 $\bar{w}(0) = 0, \theta(0) = 0, \bar{w}(L) = 0$ and $M(L) = 0$;
- Clamped-clamped beam (C-C):
 $\bar{w}(0) = 0, \theta(0) = 0, \bar{w}(L) = 0$ and $\theta(L) = 0$;
- Clamped-guided beam (C-G):
 $\bar{w}(0) = 0, \theta(0) = 0, \theta(L) = 0$ and $Q(L) = 0$.

Using the direct differentiation of Eq. (11) with respect to the x coordinate, we are able to compute any of the first three derivatives of the reconstructed original to provide the boundary conditions mentioned above. Therefore, using Eq. (8) for \bar{w} in the DT domain together with appropriately transformed into the DT domain boundary conditions, which are expressed via \bar{w} and its derivatives, we arrive at an eigenvalue problem for each case in the following form:

$$\begin{bmatrix} f_{11}(\omega) & f_{12}(\omega) & f_{13}(\omega) & f_{14}(\omega) \\ f_{21}(\omega) & f_{22}(\omega) & f_{23}(\omega) & f_{24}(\omega) \\ f_{31}(\omega) & f_{32}(\omega) & f_{33}(\omega) & f_{34}(\omega) \\ f_{41}(\omega) & f_{42}(\omega) & f_{43}(\omega) & f_{44}(\omega) \end{bmatrix} \begin{pmatrix} W(0) \\ W(1) \\ W(2) \\ W(3) \end{pmatrix} = 0, \quad (13)$$

where the functions $f_{ij}(\omega)$ are polynomials of ω .

It should be noticed that the system (13) is composed of the original and its derivatives regardless boundary conditions subjected, while a discrete of the original substituted into equations of defined boundary conditions is used to formulate a similar matrix equation within a traditional application of the DTM to the eigenvalue problem, e.g. [36]. This proposed novelty will further allow us more efficiently to handle different types of constraints, and geometrical and mechanical parameters of beams without implementation of a new code for each problem.

For the non-trivial solutions of the system (13), the determinant of the matrix $[f_{ij}(\omega)]$ is set to zero, i.e.

$$\det[f_{ij}(\omega)] = 0. \quad (14)$$

The eigenvalue problem in (14) is a polynomial root finding task, where the i th estimated eigenvalue is determined using an iterative scheme with a total number of iterations related to the accuracy of calculations. In the present work, a computational program implementing DTM for solving the free vibration problem of inhomogeneous and non-uniform beams has been developed within the Matlab environment. The eigenvalue problem is solved by the standard algorithm provided by the Matlab code with a computer using IntelTMCore[®]i7-4770 Quad-Core Desktop Processor 3.4 GHz with 16Gb RAM. In the calculations, the tolerance parameter between the $(n - 1)$ th and n th iterations corresponding to four-digit precision in the calculated eigenvalues has been utilized.

It is well-known that the accuracy of solutions, obtained using the DTM, essentially depends on a number of discretises used in the Taylor series for restoring the original. The bigger number of discretises would be involved, the more accurate the approximate solution would be achieved [23]. However, in practice, the maximum numbers of discretises are always restricted by computational power and memory capacity due to quickly growing calculations needed for recurrent type equations. This fact results in increasing an error in simulations. On the other hand, the value at which the T-transformations is to be evaluated affect the accuracy of the solution as well. Therefore, to accomplish higher accuracy of the approximate solution, we will reconstruct the original (2) through the discrete spectrum found at a non-zero centre of approximation in a given solution interval. This is in contrast to the traditional approach, where a zero centre of approximation is usually used for restoring the original, e.g. [36–42].

4. Numerical results and discussions

To demonstrate the general applicability and effectiveness of DTM for solving the free vibration problem for axially functionally graded and non-uniform

Table 2: *Non-dimensional natural frequencies Ω_n of a tapered cantilever beam with $A/A_0 = I/I_0 = 1 - 0.5x$*

Mode	Present	Abrate[6]	Huang&Li[19] (N=10)
1	4.315170	4.315170	4.315170
2	23.51926	23.51926	23.51926
3	63.19919	63.19919	63.19919
4	122.4377	122.4396	

cross-section beams, we consider a variety of specific problems and compare the results obtained using the DTM with those available in the literature when possible. A general case, where there is a full arbitrariness in choosing both an axial material non-homogeneity and a cross-sectional non-uniformity simultaneously is also considered here.

4.1. Homogeneous beams with non-uniform cross-sections

First, a homogeneous tapered cantilever beam with cross-sectional parameters varying linearly along the beam length as functions $A/A_0 = I/I_0 = 1 - 0.5x$ is considered. The first four non-dimensional natural frequencies $\Omega_n = \omega_n \sqrt{\frac{\rho A_0 L^4}{E J_0}}$ calculated with the DTM are compared with those obtained by the Rayleigh-Ritz method in [6] and using the Fredholm integral equation in [19] are presented in Table 2. As seen from Table the present results are practically coincide with those given in the literature including higher vibration modes also.

The rate of convergence of the DTM used for calculating first four natural frequencies of the tapered cantilever beam and its dependence on the centre of approximation chosen are presented in Fig. 1. One can see that the higher frequencies require more number of discretises than lower ones to reach the exact solution. Moreover, the fastest convergence of the solution is at the central point of the given interval, while as one calculates at the zero point, the rate of convergence slows down and dramatically decelerate for the high fourth frequency.

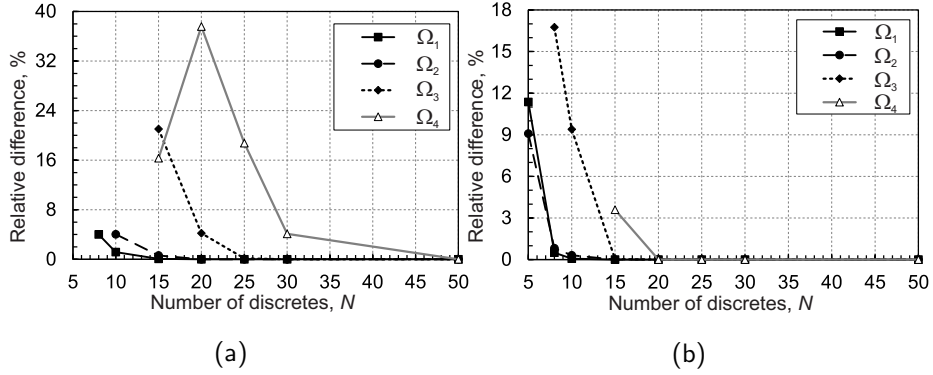


Figure 1: Convergence of the first four normalized natural frequencies of the tapered cantilever beam with $A/A_0 = I/I_0 = 1 - 0.5x$ found: (a) at $x_v = 0$; and (b) at $x_v = 0.5$.

This choice among the other points of the solution interval appeared the best from the standpoint of reducing the solution error. Thereby, in what follows the calculations have been carried out in the middle of given interval $x_v = 0.5$, until another is said.

A homogeneous non-uniform beam with linearly variable cross-section area $A(x) = A_0(1 + \alpha x)$, but with a cubic variation of the moment of inertia $I(x) = I_0(1 + \alpha x)^3$ is studied next. The natural frequencies of such homogeneous beam subjected to different boundary conditions have been found depending on the non-uniformity parameter α in [6] and [19], where the Rayleigh-Ritz method, and the Fredholm integral method and the FEM have been used, respectively. In Table 3, we present comparisons for clamped-pinned and clamped-clamped beams between the non-dimensional frequencies Ω_n obtained with the DTM and those results that are available in the mentioned papers.

The natural frequencies computed with the DTM coincide to those found by the other methods for the beams with the both types of boundary conditions and a variety of the non-uniformity parameter α used in the calculations. The dependence of the rate of convergence of the approximate fundamental frequency of the C-P beam on the non-uniformity parameter is shown in Fig. 2. It is obvious that the convergence rate of the solution practically is not sensitive

Table 3: *Non-dimensional natural frequencies Ω_n of beams with $A(x) = A_0(1 + \alpha x)$ and $I(x) = I_0(1 + \alpha x)^3$*

BC	α	Mode	Present	Huang&Li [19]	Abrate[6]	FEM [19]
C - P	-0.1	1	14.848896	14.848896	14.848896	14.92
		2	47.637037	47.637037	47.637037	
		3	99.171635	99.171653	99.171635	
	0	1	15.418206	15.418206		
		2	49.964862			
		3	104.247696			
	0.1	1	15.9687099	15.9687099	15.9687099	15.997
		2	52.237227	52.237227	52.237227	
		3	109.20235	109.20235	109.20235	
	0.2	1	16.502899	16.502899	16.502899	16.561
		2	54.4614625	54.4614625	54.4614625	
		3	114.051623	114.051631	114.051623	
	1.0	1	20.3666		20.3666	
		2	71.04797		71.04797	
		3	150.20086		150.20086	
	2.0	1	24.5826		24.5826	
		2	89.98368		89.98368	
		3	191.44814		191.44814	
C - C	-0.1	1	21.240978	21.240978		
		2	58.550055	58.550055		
		3	114.780242	114.780278		
	0	1	22.373285	22.373285	22.3732854	22.373
		2	61.672823	61.672823	61.672823	
		3	120.903392	120.90340	120.90339	
	0.1	1	23.479607	23.479607	23.479607	23.521
		2	64.721068	64.721068	64.721086	
		3	126.87802	126.87805	126.87804	
	0.2	1	24.563418	24.563418	24.563418	24.647
		2	67.704755	67.704755	67.704755	
		3	132.72398	15132.72407	132.72398	

to small variations of the non-uniformity parameter (Fig. 2a), while it changes more clearly with increasing the parameter value (Fig. 2b). Also, it should be noticed that this tendency holds for the higher frequencies as well.

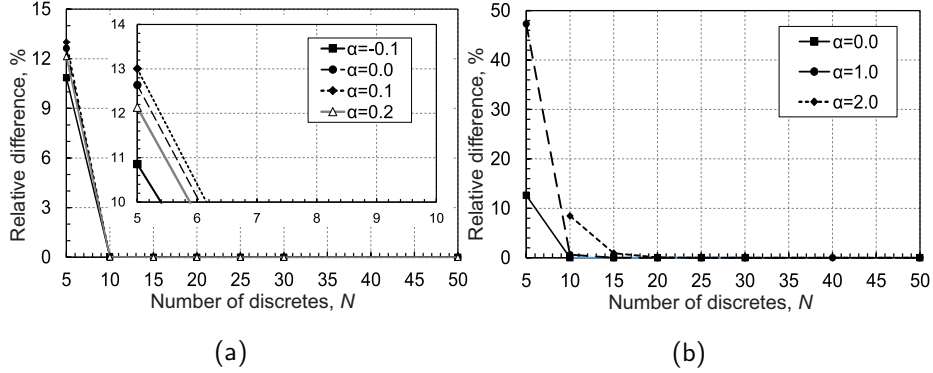


Figure 2: Convergence of the normalized fundamental frequency depending on the parameter α for the C - P beam with $A(x) = A_0(1 + \alpha x)$ and $I(x) = I_0(1 + \alpha x)^3$ if: (a) α varies from -0.1 to 0.2; (b) α varies from 0.0 to 2.0.

4.2. Inhomogeneous uniform cross-section beams

Further, we use the proposed approach implementing the DTM to compute natural frequencies of a series of uniform inhomogeneous beams, containing Young modulus $E(x)$ and mass density $\rho(x)$ that are arbitrary variable along the beam length. In the calculations we adopt there parameters in the following forms:

$$\rho(x) = \rho_0 \sum_{j=0}^J a_j x^j, \quad E(x) = E_0 \sum_{r=0}^R b_r x^r, \quad (15)$$

where J and R are any positive integers, and a_j with $(0 \leq j \leq J)$ and b_r with $(0 \leq r \leq R)$ are constants satisfying to requirements $\rho(x) > 0$, $E(x) > 0$ for all $x \in [0, L]$. Closed-form and approximate solutions for such inhomogeneous cantilever beam are known in [10] and [19], respectively.

Three types of beam's mass density such as: (1) constant $\rho(x)/\rho_0 = 1$, (2) linearly changing $\rho(x)/\rho_0 = 1 + x$, and (3) varying as a quadratic function $\rho(x)/\rho_0 = 1.5954 + 0.04x + x^2$ are considered in the calculations. Each of Tables 4 to 6 represents the first five non-dimensional frequencies $k_n = \omega_n \left(\frac{\rho A_0 L^4}{E J_0} \right)$

Table 4: *Non-dimensional natural frequencies k_n of the cantilever beam with $\rho(x)/\rho_0 = 1$ and $D(x)/E_0I = \sum_{r=0}^R b_{0r}x^r$*

R	Source	1st	2nd	3rd	4th	5th
0	Present	17.928230	112.354302	314.595300	616.481229	1019.087644
	Huang&Li [19]	17.928232	112.355863			
	Weaver[43]	17.928232	112.354350			
1	Present	18.899661	124.961973	355.014816	698.216086	1155.920215
	Huang&Li [19]	18.899664	124.961794			
2	Present	18.994321	127.178034	363.037901	714.808453	1183.929860
	Huang&Li [19]	18.994320	127.177553			
3	Present	18.971053	126.344324	359.531108	707.281060	1171.009309
	Huang&Li [19]	18.971054	126.343737			
4	Present	18.973666	126.471840	360.156611	708.689040	1173.480918
	Huang&Li [19]	18.973666	126.471301			
	Elishakoff[10]	18.973666				

for each of the mass density and flexural rigidity relations. For the latter, appropriate coefficients can be found in Appendix B. In Tables the first two of the frequencies are compared with those available for the same cases of inhomogeneity in [10], [19] and [43]. The compared results are in very good agreement with each other for all cases of inhomogeneity adopted in the calculations.

The convergence analysis showed us that the higher is the degree of the polynomial of Young's modulus used in (15), the lower is the difference between the known results and the results obtained with DTM, and the faster the DTM is able to approximate the exact solution as seen in Fig. 3. This conclusion is valid for all calculated frequencies regardless the degree of a polynomial of the density function. However, the higher frequencies require more number of discretises in the approximate solution than in the case of lower ones.

To demonstrate the generality of the DTM approach for modelling completely arbitrary forms of the material inhomogeneity in the vibration analysis of FGM beams, we consider the flexural rigidity and mass density as a trigono-

Table 5: *Non-dimensional natural frequencies k_n of the cantilever beam with $\rho(x)/\rho_0 = 1 + x$ and $D(x)/E_0I = \sum_{r=0}^R b_{1r}x^r$*

R	Source	1st	2nd	3rd	4th	5th
0	Present	21.055898	140.928477	403.479493	795.279422	1317.894448
	Huang&Li [19]	21.055897	140.930444			
1	Present	22.298672	158.371332	460.631754	911.241917	1512.256542
	Huang&Li [19]	22.298670	158.372567			
2	Present	22.455308	162.368406	475.380847	941.746354	1563.727521
	Huang&Li [19]	22.455309	162.369548			
3	Present	22.452019	162.239233	474.830118	940.570801	1561.716527
	Huang&Li [19]	22.452020	162.240346			
4	Present	22.449333	162.094285	474.110529	938.967473	1558.918394
	Huang&Li [19]	22.449332	162.095281			
5	Present	22.449944	162.136198	474.350328	939.528605	1559.920652
	Huang&Li [19]	22.449944	162.137195			
	Elishakoff[10]	22.449944				

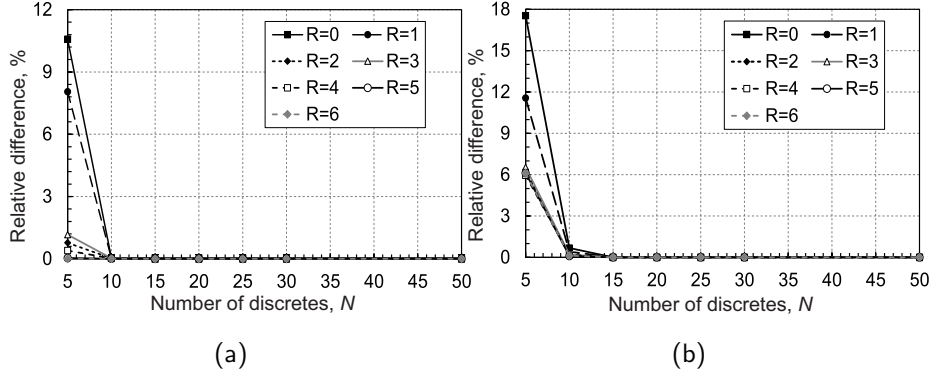


Figure 3: *Convergence of normalized natural frequencies of the cantilever beam with $\rho(x)/\rho_0 = 1.5954 + 0.04x + x^2$ depending on the degree R of the polynomial $D(x)/E_0I = \sum_{r=0}^R b_{2r}x^r$ for: (a) the fundamental frequency; (b) the 2nd natural frequency.*

metric functions of the axial coordinate in the calculations as follows:

$$D(x) = D_0[1 + \alpha \cos(\pi x)], \rho(x) = \rho_0[1 + \beta \cos(\pi x)], \quad (16)$$

Table 6: *Non-dimensional natural frequencies k_n of the cantilever beam with $\rho(x)/\rho_0 = 1.5954 + 0.04x + x^2$ and $D(x)/E_0I = \sum_{r=0}^R b_{2r}x^r$*

R	Source	1st	2d	3rd	4th	5th
0	Present	24.294479	162.797108	462.716163	909.677354	1505.583556
	Huang&Li [19]	24.294444	162.797574			
1	Present	25.737129	182.906725	527.889460	1041.632735	1726.572091
	Huang&Li [19]	25.737133	182.906151			
2	Present	25.921234	187.554244	544.822636	1076.596904	1785.535421
	Huang&Li [19]	25.921235	187.553379			
3	Present	25.918839	187.461469	544.432450	1075.764990	1784.112810
	Huang&Li [19]	25.918839	187.460556			
4	Present	25.924098	187.739992	545.793135	1078.791133	1789.389455
	Huang&Li [19]	25.924097	187.739261			
5	Present	25.922736	187.648647	545.279363	1077.591252	1787.248519
	Huang&Li [19]	25.922735	187.647888			
6	Present	25.922963	187.666960	545.396356	1077.879518	1787.776220
	Huang&Li [19]	25.922963	187.666166			
	Elishakoff[10]	25.922963				

where $|\alpha| < 1$, $|\beta| < 1$ are some parameters.

Tables 7 and 8 show the first six non-dimensional natural frequencies $\Omega_n = \omega_n \sqrt{\frac{\rho A_0 L^4}{D_0}}$ for the two cases of parameters $\beta = 4\alpha$ and $\beta = \alpha$, respectively, depending on the inhomogeneity parameter α and boundary conditions imposed.

The first three of the frequencies computed with DTM are compared with those presented in [19, 20] for the same cases of material inhomogeneity and boundary conditions. The results match extremely well for the first two frequencies for all cases of the boundary conditions and the inhomogeneity parameter. For the third frequency the differences are moderate with the maximum distinction up to 7% in the comparisons, as seen in Fig. 4. This discrepancy results from the difference between the techniques employed for finding the approximate solutions. It is worth to notice that the DTM did not experience convergence difficulties. Though the rate of convergence depended on the boundary conditions imposed and the parameter α used. Moreover, the rate of convergence was slightly quicker and the computational time was somewhat lesser in the case of $\beta = \alpha$ than those in the case of $\beta = 4\alpha$. It means that increasing in-

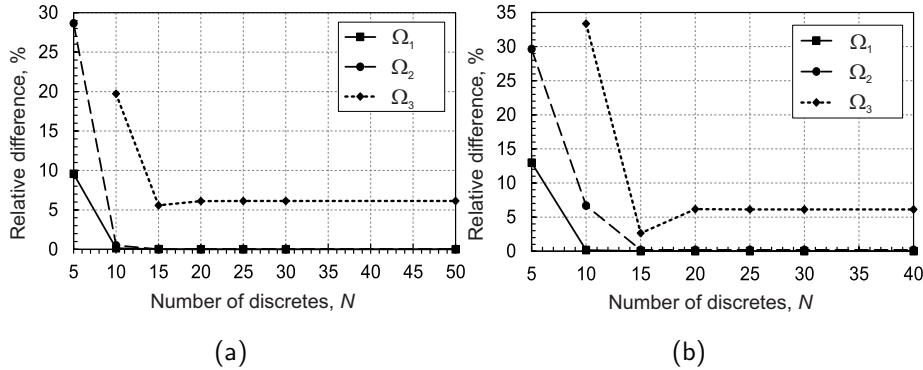


Figure 4: Convergence of approximate solutions of normalized natural frequencies of the C - C beam with $D(x) = D_0[1 + \alpha \cos(\pi x)]$ and $\rho(x) = \rho_0[1 + \beta \cos(\pi x)]$, and $\alpha = 0.2$ if: (a) $\beta = \alpha$; (b) $\beta = 4\alpha$.

homogeneity of the mechanical parameter along the beam length increases the computational cost of the DTM needed for restoring an exact solution. Also we revealed that unlike the method used in [19] all natural frequencies obtained

Table 7: *Non-dimensional natural frequencies Ω_n of beams with $D(x) = D_0[1 + \alpha \cos(\pi x)]$ and $\rho(x) = \rho_0[1 + \beta \cos(\pi x)]$, if $\beta = 4\alpha$*

BC	n	Source	α							
			-0.2	-0.15	-0.1	0	0.1	0.15	0.2	
C - F	1	Present	2.56899	2.75721	2.97143	3.51602	4.34283	4.96987	5.89084	
		Huang&Li [19]	2.5690	2.7572	2.9714	3.5160	4.3428	4.9699	5.8908	
	2	Present	20.54575	20.84940	21.18732	22.03449	23.40750	24.64429	27.05785	
		Huang&Li [19]	20.5462	20.8498	21.1877	22.0345	23.4080	24.6452	27.0597	
	3	Present	64.03571	62.80216	61.99739	61.69721	63.47331	65.61956	69.88894	
		Huang&Li [19]	64.1287	62.8517	62.0246	61.7151	63.5303	65.7357	70.1756	
	4	Present	128.86866	124.88568	122.43363	120.90192	123.54569	127.15568	134.15760	
	5	Present	215.28680	207.44420	202.86979	199.85950	203.75458	209.32144	220.05414	
	6	Present	324.69770	311.43252	303.79480	298.55314	303.94195	312.28903	333.53432	
	S - S	1	Present	9.86960	9.86960	9.86960	9.86960	9.86960	9.86960	9.86960
			Huang&Li [19]	9.8696	9.8696	9.8696	9.8696	9.8696	9.8696	9.8696
		2	Present	42.53832	41.13895	40.19695	39.47842	40.19695	41.13895	42.53832
Huang&Li [19]			42.5405	41.1404	40.1979	39.4791	40.1979	41.1404	42.5522	
3		Present	96.74878	92.75051	90.42512	88.82644	90.42512	92.7505	96.74878	
		Huang&Li [19]	98.9439	92.4350	90.4469	88.8481	90.3370	92.4350	98.6659	
4		Present	172.31388	164.80929	160.69538	157.91367	160.69538	164.80929	172.31388	
5		Present	269.24338	257.39772	251.03594	246.74011	251.03594	257.39772	269.24338	
6		Present	387.46288	370.51003	361.43981	355.30580	361.43981	370.51003	387.46288	
C - P		1	Present	14.21170	14.49167	14.78504	15.41821	16.12346	16.50640	16.91055
			Huang&Li [19]	14.2117	14.4917	14.7850	15.4182	16.1235	16.5065	16.9107
		2	Present	51.51352	50.72796	50.19312	49.96486	51.14342	52.46453	54.46396
	Huang&Li [19]		51.5819	50.7722	50.2210	49.9742	51.1459	52.4695	54.4812	
	3	Present	110.84949	107.62986	105.59270	104.24770	106.29373	109.07530	113.90842	
		Huang&Li [19]	112.9319	110.0300	108.2707	107.4485	110.4157	113.7307	119.2732	
	4	Present	191.82757	185.02266	180.99163	178.26973	181.53790	186.20726	194.74699	
	5	Present	294.30289	282.91292	276.42480	272.03097	276.87027	283.89975	296.98341	
	6	Present	418.41898	401.33012	391.90015	385.53145	392.27584	402.09670	420.24635	
	C - C	1	Present	22.36955	22.37119	22.37235	22.37329	22.37235	22.37119	22.36955
			Huang&Li [19]	22.3700	22.3715	22.3726	22.3735	22.3726	22.3715	22.3700
		2	Present	64.64605	63.32288	62.39804	61.67282	62.39804	63.32288	64.64605
Huang&Li [19]			64.7658	63.3937	62.4327	61.6883	62.4330	63.3897	64.7668	
3		Present	129.32930	125.27614	122.73650	120.90339	122.73650	125.27614	129.32930	
		Huang&Li [19]	138.6441	132.6284	131.3240	129.2174	131.2343	132.3560	137.759	
4		Present	215.63076	207.74116	203.10268	199.85945	203.10268	207.74116	215.63076	
5		Present	323.43250	310.72825	303.52343	298.55554	303.52343	310.72825	323.43250	
6		Present	452.43908	434.13230	423.96477	416.99101	423.96477	434.13230	452.43908	

Table 8: *Non-dimensional natural frequencies Ω_n of beams with $D(x) = D_0[1 + \alpha \cos(\pi x)]$ and $\rho(x) = \rho_0[1 + \beta \cos(\pi x)]$, if $\beta = \alpha$*

BC	n	Source	α							
			-0.2	-0.15	-0.1	0	0.1	0.15	0.2	
C - F	1	Present	3.02240	3.14158	3.26323	3.51602	3.78526	3.92769	4.07635	
		Huang&Li [19]	3.0024	3.1416	3.2632	3.5160	3.7853	3.9277	4.0763	
	2	Present	21.20726	21.41809	21.62542	22.03449	22.44457	22.65349	22.86721	
		Huang&Li [19]	21.2069	21.4179	21.6255	22.0345	22.4447	22.6534	22.8668	
	3	Present	61.22461	61.33987	61.45613	61.69721	61.95882	62.10089	62.25261	
		Huang&Li [19]	61.2666	61.3741	61.4838	61.7151	61.9758	62.1192	62.2737	
	4	Present	120.57744	120.65392	120.73270	120.90192	121.09441	121.20274	121.32110	
	5	Present	199.61965	199.67285	199.73064	199.85953	200.01244	200.10102	200.19772	
	6	Present	298.35875	298.40832	298.45270	298.55552	298.68294	298.76147	298.83443	
	S - S	1	Present	9.83953	9.85280	9.86217	9.86960	9.86217	9.85280	9.83953
			Huang&Li [19]	9.8395	9.8528	9.8622	9.8696	9.8622	9.8528	9.8395
		2	Present	39.52073	39.50227	39.48903	39.47842	39.48903	39.50227	39.52073
Huang&Li [19]			39.5239	39.4045	39.4905	39.4791	39.4905	39.4045	39.5239	
3		Present	88.87424	88.85299	88.83814	88.82644	88.83814	88.85299	88.87424	
		Huang&Li [19]	90.2491	90.2874	90.3149	88.8481	90.3149	90.2874	90.2491	
4		Present	157.96291	157.94097	157.92570	157.91367	157.92570	157.94097	157.96291	
5		Present	246.79126	246.76778	246.75224	246.74011	246.75224	246.76778	246.79126	
6		Present	355.36156	355.33150	355.31812	355.30580	355.31812	355.33150	355.36156	
C - P		1	Present	14.91970	15.05273	15.17990	15.41821	15.63677	15.73901	15.83660
			Huang&Li [19]	14.9196	15.0527	15.1799	15.4182	15.6367	15.7389	15.8365
		2	Present	49.67240	49.74997	49.82387	49.96486	50.10256	50.17204	50.24281
	Huang&Li [19]		49.6719	49.7506	49.8265	49.9742	50.1206	50.1944	50.2691	
	3	Present	104.05672	104.10517	104.15266	104.24770	104.34732	104.40054	104.45702	
		Huang&Li [19]	107.5159	107.5407	107.5357	107.4485	107.2753	107.1613	107.0311	
	4	Present	178.13492	178.16704	178.19982	178.26973	178.34898	178.39352	178.44241	
	5	Present	271.93307	271.95304	271.97670	272.03097	272.09750	272.13663	272.18249	
	6	Present	385.44723	385.47001	385.48867	385.53145	385.58880	385.62228	385.68087	
	C - C	1	Present	22.29882	22.33172	22.35491	22.37329	22.35491	22.33172	22.29882
			Huang&Li [19]	22.2984	22.3316	22.3549	22.3735	22.3549	22.3316	22.2984
		2	Present	61.62357	61.64563	61.66089	61.67282	61.66089	61.64563	61.62357
Huang&Li [19]			61.6542	61.6699	61.6804	61.6883	61.6804	61.6699	61.6542	
3		Present	120.87930	120.89014	120.89759	120.90339	120.89759	120.89014	120.87930	
		Huang&Li [19]	128.7765	128.9739	129.1098	129.2174	129.1098	128.9739	128.7780	
4		Present	199.85145	199.85512	199.85759	199.85945	199.85759	199.85512	199.85145	
5		Present	298.55941	298.55688	298.55625	298.55554	298.55625	298.55688	298.55941	
6		Present	417.00069	416.99397	416.99399	416.99101	416.99399	416.99397	417.00069	

using the DTM are exactly even functions of the parameter α in the case of the symmetric boundary conditions S-S and C-C, while in the compared method, the results are symmetrical with respect to the sign of α only if $\beta = \alpha$.

4.3. Axially functionally graded beams with uniform and non-uniform cross-sections

We illustrate now an application of DTM to the free vibration analysis of beams made of functionally graded materials with any sophisticated axial gradation profiles. At the beginning, let us consider a beam with uniform cross section area A_0 and second moment of area I_0 , but with variable Young modulus $E(x)$ and mass density $\rho(x)$ as the following functions of the x -coordinate:

$$\begin{cases} E(x) = E_1 \left(1 - \frac{e^{\alpha x} - 1}{e^\alpha - 1} \right) + E_2 \frac{e^{\alpha x} - 1}{e^\alpha - 1}, & \alpha \neq 0 \\ E(x) = E_1 (1 - x) + E_2 x, & \alpha = 0 \end{cases} \quad (17)$$

and

$$\begin{cases} \rho(x) = \rho_1 \left(1 - \frac{e^{\alpha x} - 1}{e^\alpha - 1} \right) + \rho_2 \frac{e^{\alpha x} - 1}{e^\alpha - 1}, & \alpha \neq 0 \\ \rho(x) = \rho_1 (1 - x) + \rho_2 x, & \alpha = 0, \end{cases} \quad (18)$$

where E_1 , ρ_1 and E_2 , ρ_2 are corresponding Young modulus and mass density of material at the ends $x = 0$ and $x = L$, respectively, and α is the gradation parameter describing the volume fraction change along the beam length.

For the sake of comparison, in the calculations we accept functionally graded metal/ceramic Aluminium Zirconia alloy (AlZrO_2), the material properties of which are presented in [19] and are given as follows:

$$\begin{aligned} \text{Al} : \quad E_a &= 70 \text{ GPa}, & \rho_a &= 2702 \text{ kg/m}^3 \\ \text{ZrO}_2 : \quad E_z &= 200 \text{ GPa}, & \rho_z &= 5700 \text{ kg/m}^3 \end{aligned}$$

The results of the free vibration analysis of axially functionally graded beams with two types of gradation profiles, when the metal phase Al is rich near the end $x = 0$ and the ceramic phase ZrO_2 is rich near the end $x = L$ (case 1), and the metal phase Al is rich near the end $x = L$ and the ceramic phase ZrO_2 is

Table 9: *Non-dimensional fundamental frequencies Ω_n of axially graded AlZrO₂ beams*

α	Source	C - F		S - S		C - P		C - C	
		Case 1	Case 2	Case 1	Case 2	Case 1	Case 2	Case 1	Case 2
-10	Present	3.47486	4.05056	11.46179	9.93313	16.40307	17.31412	24.06385	24.79734
	Huang&Li [19]	3.5656	4.1800	11.4532	9.9358	16.4775	17.2993	24.0576	24.7949
	Hein&Feklistova [20]			11.4481		16.3837		24.0269	
	Error, %	2.545	3.097	0.095	0.027	0.452	0.086	0.0260	0.056
-3	Present	3.14088	4.83168	11.24428	10.36686	16.02546	17.86998	23.94328	24.93636
	Huang&Li [19]	3.1421	4.8317	11.2443	10.3669	16.0307	17.8701	23.9456	24.9375
	Hein&Feklistova [20]			11.2422		16.0307		23.9384	
	Error, %	0.0401	0.0004	0.0002	0.0004	0.0326	0.0007	0.0097	0.0046
0	Present	2.92561	5.01564	10.86634	10.86634	15.87338	17.91475	24.37535	24.37535
	Huang&Li [19]	2.9256	5.0156	10.8663	10.8663	15.8734	17.9147	24.3752	24.3752
	Hein&Feklistova [20]			10.8660		15.8729		24.3749	
	Error, %	0.0003	0.0008	0.0004	0.0004	0.0001	0.0003	0.0006	0.0138
3	Present	2.85446	4.84663	10.36686	11.24428	15.71686	17.88728	24.93636	23.94329
	Huang&Li [19]	2.8544	4.8466	10.3669	11.2443	15.7171	17.8867	24.9375	23.9456
	Hein&Feklistova [20]			10.3670		15.7171		24.9371	
	Error, %	0.0021	0.0006	0.0286	0.0002	0.0015	0.0032	0.0046	0.0097
10	Present	3.09642	4.46502	9.93277	11.4582	15.47935	17.91852	24.81462	24.05657
	Huang&Li [19]	3.0431	4.4629	9.9358	11.4532	15.4930	17.9050	24.7949	24.0576
	Hein&Feklistova [20]			9.9366		15.4930		24.8080	
	Error, %	1.752	0.048	0.031	0.044	0.088	0.076	0.080	0.004
HB ¹		3.5160		9.8696		15.4182		22.3733	

¹ HB is a homogeneous beam

rich near the end $x = 0$ (case 2), subjected to different boundary conditions are collected in Tables 9 and 10. Table 9 demonstrates non-dimensional fundamental frequencies $\Omega_n = \omega_n \sqrt{\frac{\rho_a A_0 L^4}{E_a I_0}}$ of the beams compared the present results with those available in the literature. Table 10 shows the other first five non-dimensional frequencies Ω_n depending on the gradation parameter α and the boundary conditions.

One can see in Table 9 that the frequencies calculated with DTM are mainly in a good agreement with those obtained by the other methods in [19, 20]. The maximum relative differences between the results vary from 1.7% to 3% in the case of the cantilever beam with the highest material parameter $\alpha = \pm 10$, but they are hundredths and thousandths of a percent for the smaller values of this parameter and more restrained boundary conditions.

The convergence analysis of these solutions with respect to the gradation parameter α and boundary conditions has been performed and revealed some

computational issues. It was found out that in the case of $\alpha = -10$, the approximate solution of the FGM beam with the loosest C-F constraints for the both gradation patterns case 1 and case 2 exhibits oscillations and grows rapidly with slow increasing N , when the number of discretises surpasses a certain threshold as shown in Fig. 5a. This limit value was equal to $N = 53$ in the calculations. The DTM solutions for that beam under more restrained boundary conditions than the C-F constraints do not show a strong oscillatory behaviour, but they feature a divergence phenomenon after $N = 53$ as seen in Fig. 6a. In the presence of

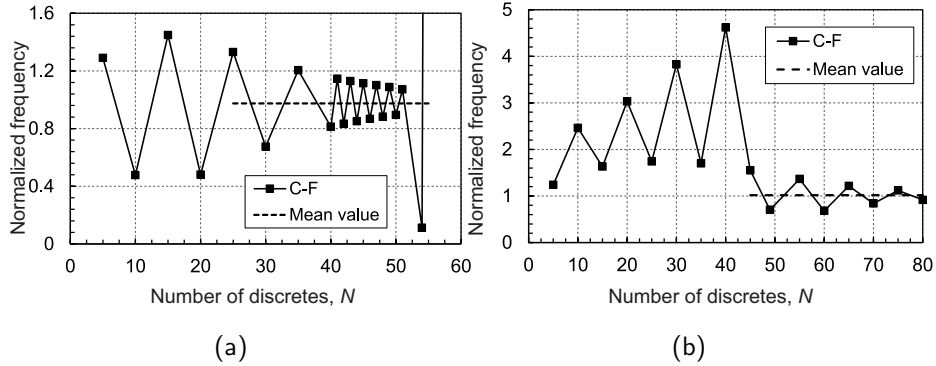


Figure 5: Convergence of the fundamental frequency normalized with respect to the solution in [19], $\frac{\Omega_1}{\Omega_1^*}$ for the C-F beam in the case 1: (a) $\alpha = -10$; and (b) $\alpha = 10$.

such computational issues the accuracy of the DTM deteriorates. This is due to the well known phenomenon for the higher order polynomial terms which could lead to round-off errors and ill-conditioning for finding the roots of a polynomial in the eigenvalue problem [44]. This computational instability cannot be merely overcome by increasing a number of discretises. Perhaps, a multistep approach within the DTM [22, 27, 40] is more suitable for solving problems involving high material gradients in FGM structures. Nevertheless, one can see in Table 10 that the frequencies averaged over a range, where spurious oscillations occur, are of a good compliance with the results in the literature, and the differential transform of 50 discretises is an reasonable quantity to achieve a demand for high accuracy of the solutions as well.

In the case of $\alpha = 10$, the DTM solutions are less sensitive to rounding errors. The approximate solution of the C-F beam oscillates near an averaged value and tends to it with increasing N , Fig. 5b. The approximate solutions for beams subjected to the other boundary conditions are non-oscillatory and show the convergence, but the rate of convergence slows down remarkably in compared with the examples previously mentioned in the paper as illustrated in Fig. 6b. In the cases of $\alpha = 0$ and ± 3 , the convergence of DTM solutions

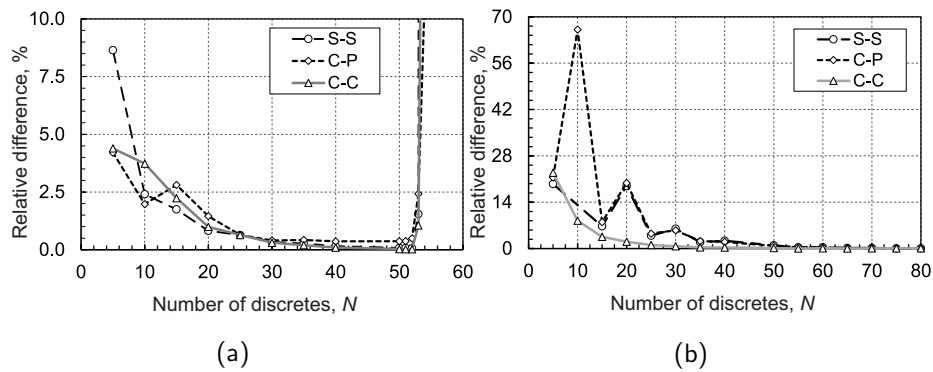


Figure 6: Convergence of the fundamental frequencies normalized with respect to the solutions in [19] for the beams under S-S, C-P and C-C constraints in the case 1: (a) $\alpha = -10$; and (b) $\alpha = 10$.

does not demonstrate any computational issues and possesses a fast rate of convergence regardless boundary conditions applied to the FGM beam.

The results listed in Table 10 present higher mode frequencies of the considered FGM beam. It is worth to notice that the DTM solutions of the higher frequencies have the same convergence behaviour as those described above for the fundamental frequency. Therefore, in the cases of C-F beam with $\alpha = \pm 10$ the averaging and truncation techniques mentioned above were applied for recovering the results. As seen in Table, the higher frequencies of the beams under symmetric boundary conditions in the case 1 are identical to ones in the case 2 for the equal, but opposite sign inhomogeneity parameter α . Moreover, the calculated frequencies indicate a complex variation tendency depending on that parameter. This fact proves a strong dependence of free vibrations on FGM

Table 10: *Non-dimensional natural frequencies Ω_n of axially graded AlZrO₂ beams*

α	Mode	C - F		S - S		C - P		C - C	
		Case 1	Case 2	Case 1	Case 2	Case 1	Case 2	Case 1	Case 2
-10	2	23.75440	24.29463	45.70803	40.01590	54.92971	54.01091	68.11671	66.33548
	3	68.13491	66.30623	102.72185	90.70289	116.42524	111.04684	135.35756	128.25552
	4	135.43341	128.78960	182.41317	160.67124	200.98782	186.87381	225.52083	210.67467
	5	225.49512	210.07984	284.85677	254.31263	308.11954	290.92369	338.63110	311.36029
	6	338.79814	334.51364	410.13045	354.47104	438.49036	409.84991	474.60218	448.14160
-3	2	23.05185	25.49215	44.85368	41.96811	54.46761	54.92984	67.75525	67.10406
	3	67.47564	67.78661	100.81154	94.50370	115.51745	112.79363	134.43877	130.22820
	4	134.15354	130.83415	179.10915	167.99936	199.05597	191.57750	223.57897	214.26212
	5	223.31328	214.83999	279.74462	262.46802	305.02380	291.31830	335.14367	319.25327
	6	334.88852	319.81271	402.71983	377.91596	433.39133	412.03012	469.10471	445.21482
0	2	22.35382	26.46480	43.66381	43.66381	53.95855	55.97100	67.58746	67.58746
	3	66.16920	70.26689	98.11772	98.11772	113.77010	115.83950	132.83907	132.83907
	4	131.42546	135.61105	174.30881	174.30881	195.35569	197.45609	219.85463	219.85463
	5	218.46231	222.69830	272.24998	272.24998	298.70768	300.82891	328.64103	328.64103
	6	327.26046	331.53089	391.94625	391.94625	423.82371	425.95976	459.19322	459.19322
3	2	21.49489	27.14143	41.96811	44.85372	52.80201	56.89965	67.10406	67.75527
	3	63.67157	72.77856	94.50370	100.81159	110.60535	118.43481	130.22820	134.43881
	4	126.57717	140.33777	167.99936	179.10929	189.36052	202.30712	214.26212	223.57908
	5	210.50632	230.10699	262.46802	279.74473	289.08361	308.52179	319.25327	335.14381
	6	315.42943	342.13279	377.91596	402.72007	409.78356	437.07867	445.21482	469.10516
10	2	21.06942	27.07263	39.98589	45.69440	50.75565	57.79663	66.26926	68.09948
	3	60.59987	74.40182	90.72143	102.66569	106.34852	120.46246	127.89884	135.31837
	4	123.75115	144.11101	160.71032	182.38506	182.12175	205.87632	209.61761	225.46878
	5	208.28604	236.40866	254.25563	284.88178	278.30040	314.05743	311.40310	338.61609
	6	332.59071	351.30339	354.46540	410.12223	386.07054	444.99713	447.88063	474.45889

material parameters and the importance of their predictions at the design stage.

Finally, we consider functionally graded beams with axially graded material properties defined by the relations (17) and (18) for the two gradation patterns (case 1) and (case 2) mentioned in the previous simulations with non-uniform cross-sectional parameters. The beams are assumed to have a variable cross-section given by functions of x as follows: $A(x) = A_0(1 + \beta x)^2$ and $I(x) = I_0(1 + \beta x)^4$. These geometry relations may correspond to, for instance, a beam with a rectangular cross-section whose height and width vary linearly with the same taper ratio or a beam with a circular cross-section and linearly tapering diameter. Tables 11 to 14 contain non-dimensional natural frequencies $\Omega_n = \omega_n \sqrt{\frac{\rho_0 AL^4}{E_0 I}}$ computed depending on values of inhomogeneity α and non-uniformity β parameters at different boundary conditions.

The convergence analysis has been performed for each case study. Analogously to the previous calculations, the oscillations and divergence behaviour of the DTM solutions in the regions of high material gradients $\alpha = \pm 10$ occur. Herewith, these phenomena become stronger with extending the cross-sectional taper towards the free end of C-F beam as seen in Fig. 7. To recover the natural frequencies from the results suffered from the computational instabilities, the averaging and truncation techniques mentioned above have been used in the post-processing stage. In the other examples, where computational issues did not arise, we have observed only a slight slowing down the convergence rate with increasing the taper ratio towards unity, for instance, in the case of C-C beam as shown in Fig. 8. The DTM solutions of frequencies of beams with low material gradients $\alpha = 0$ and $\alpha = \pm 3$ have had no any computational difficulties and have featured fast enough convergence rate and relatively low computational cost at the variety of taper ratios and boundary conditions.

The results in Tables 11 to 14 indicate that for axially functionally graded beams, the natural frequencies show a complex variation behaviour with respect to the both material inhomogeneity and geometrical non-uniformity and they differ significantly from beams having only material inhomogeneity or only non-uniform cross-section at the same boundary conditions. In doing so, the higher

Table 11: *Non-dimensional natural frequencies Ω_n of an axially functionally graded and non-uniform cross-section cantilever beam with $A(x) = (1 + \beta x)^2$ and $I(x) = (1 + \beta x)^4$, and $E(x)$ and $\rho(x)$ changing in accordance with (17) and (18)*

β	Mode	α										
		-10		-3		0		3		10		
		Case 1	Case 2	Case 1	Case 2	Case 1	Case 2	Case 1	Case 2	Case 1	Case 2	
C - F	-0.6	1	5.24863	5.66395	4.71243	6.50997	4.37675	6.86693	4.22066	6.72950	4.42416	6.26775
		2	20.6782	21.0110	20.0040	21.9602	19.4586	22.7719	22.7719	23.4807	17.8584	23.5657
		3	50.3367	49.2191	49.8027	50.4430	48.8317	52.1948	47.1362	54.0286	44.5712	55.3852
		4	94.8355	90.4947	94.4329	92.4857	92.6090	95.7134	89.4361	98.9904	85.1080	101.883
		5	156.055	145.153	154.034	148.298	150.930	153.537	145.763	158.611	140.205	163.268
	-0.4	1	4.47769	4.99796	3.98421	5.73764	3.69879	6.02308	3.57797	5.87318	3.86578	5.4441
		2	21.8076	22.0237	21.0035	23.1441	20.4178	23.9995	19.6468	24.6895	19.9372	24.7168
		3	56.5137	55.3678	55.9637	56.5601	54.8844	58.5515	52.9266	60.6072	57.3247	62.0384
		4	108.434	104.582	108.535	106.177	106.403	109.940	102.654	113.718	105.458	116.900
		5	180.932	168.481	178.831	172.181	175.119	178.346	168.971	184.244	166.759	189.462
	-0.2	1	3.96618	4.56252	3.49561	5.21665	3.24952	5.44536	3.15675	5.28434	3.51417	4.88086
		2	22.8699	23.3752	22.0423	24.3356	21.4026	25.2486	20.5870	25.9309	21.6115	25.9061
		3	62.8528	61.0208	61.8333	62.3096	60.6416	64.5452	58.4149	66.8288	62.9876	68.3468
		4	121.556	116.251	121.676	118.852	119.243	123.129	114.938	127.389	130.355	130.857
		5	202.072	189.284	201.711	194.147	197.421	201.178	190.349	207.850	225.875	213.596
	0.0	1	3.63209	4.23374	3.14051	4.83168	2.92561	5.01564	2.85446	4.84663	3.10906	4.46439
		2	23.6312	24.7248	23.0523	25.4922	22.3538	26.4648	21.4949	27.1414	22.2552	27.0680
		3	68.2915	66.1484	67.4751	67.7866	66.1692	70.2669	63.6716	72.7786	66.7292	74.387
		4	135.212	128.603	134.154	130.834	131.425	135.611	126.577	140.338	137.669	144.080
		5	225.468	208.583	223.312	214.840	218.462	222.698	210.506	230.107	238.091	236.355
0.2	1	3.36348	3.97340	2.86822	4.53043	2.67867	4.67861	2.62432	4.50426	2.79864	4.14012	
	2	24.4510	25.6833	24.0188	26.6073	23.2602	27.6376	22.3619	28.3074	22.8564	28.1876	
	3	75.2230	70.6983	72.9399	73.0562	71.5191	75.7805	68.7499	78.5189	70.6037	80.2199	
	4	149.469	146.281	146.146	142.308	143.123	147.574	137.742	152.755	145.524	156.764	
	5	239.044	302.382	243.985	234.611	238.588	243.269	229.776	251.387	249.979	258.115	
0.4	1	3.11747	3.54799	2.65117	4.28528	2.48280	4.40441	2.44172	4.22666	2.55922	3.87825	
	2	25.9216	27.5856	24.9404	27.6825	24.1220	28.7669	23.1893	29.4278	23.4731	29.2636	
	3	80.4953	74.6903	78.2643	78.1623	76.7284	81.1298	73.6872	84.0933	74.5624	85.8872	
	4	156.165	260.079	157.761	153.390	154.444	159.137	148.540	164.761	153.999	169.029	
	5	263.324	370.126	263.946	253.675	258.011	263.111	248.365	271.917	263.455	279.107	

Table 12: *Non-dimensional natural frequencies Ω_n of an axially functionally graded and non-uniform cross-section simply supported beam with $A(x) = (1 + \beta x)^2$ and $I(x) = (1 + \beta x)^4$, and $E(x)$ and $\rho(x)$ changing in accordance with (17) and (18)*

β	Mode	α										
		-10		-3		0		3		10		
		Case 1	Case 2	Case 1	Case 2	Case 1	Case 2	Case 1	Case 2	Case 1	Case 2	
S - S	-0.6	1	7.25496	6.27605	7.28940	6.21801	7.19290	6.45971	6.89874	6.77262	6.41542	7.10246
		2	31.1844	27.5181	30.5010	28.6164	29.5438	29.8912	28.4334	30.5705	27.3279	30.9338
		3	69.5049	61.8984	68.0843	63.8432	66.1358	66.4256	63.7265	68.1800	60.9766	69.1615
		4	122.863	110.685	120.496	112.767	117.221	117.146	113.027	120.401	107.935	122.462
		5	191.313	177.037	187.806	175.496	182.845	182.186	176.367	187.340	161.089	190.740
	-0.4	1	8.89776	7.72741	8.85331	7.79734	8.65017	8.14671	8.26256	8.50296	7.78256	8.80140
		2	36.3025	32.0805	35.5434	33.3036	34.5075	34.7140	33.2038	35.5687	31.7405	36.1189
		3	81.3263	72.5706	79.7036	74.7443	77.4950	77.6798	74.6752	79.7541	71.5480	81.0578
		4	144.172	129.806	141.463	132.538	137.649	137.598	132.713	141.392	125.297	143.879
		5	224.884	209.699	220.842	206.745	214.979	214.543	207.324	220.528	194.146	224.549
	-0.2	1	10.2669	8.95252	10.1383	9.15491	9.84220	9.58773	9.38966	9.96201	8.93653	10.2188
		2	41.1055	36.2726	40.2920	37.7238	39.1778	39.2763	37.6780	40.3008	35.9936	41.0045
		3	92.3162	82.2952	90.5167	84.8736	88.0613	88.1522	84.8407	90.5396	81.2987	92.1251
		4	163.910	145.777	160.808	150.766	156.490	156.465	150.854	160.771	143.342	163.661
		5	255.791	229.961	251.166	235.431	244.466	244.249	235.720	251.009	223.400	255.592
	0.0	1	11.4625	10.0301	11.2443	10.3669	10.8663	10.8663	10.3669	11.2443	10.0322	11.4623
		2	45.7174	40.2024	44.8537	41.9681	43.6638	43.6638	41.9681	44.8537	40.2049	45.7191
		3	102.701	91.1871	100.811	94.5037	98.1177	98.1177	94.5037	100.812	91.1415	102.704
		4	182.505	158.809	179.109	167.999	174.309	174.309	167.999	179.109	158.752	182.481
		5	284.975	240.013	279.745	262.468	272.250	272.250	262.468	279.745	240.272	284.985
0.2	1	12.5243	11.0063	12.2236	11.4737	11.7727	12.0274	11.2388	12.3999	10.8700	12.5686	
	2	50.1747	43.9139	49.2845	46.0847	48.0217	47.9236	46.1301	49.2760	44.0152	50.2408	
	3	112.850	99.5920	110.739	103.777	107.812	107.721	103.810	110.716	99.4961	112.8049	
	4	200.535	170.229	196.668	184.525	191.399	191.425	184.437	196.705	176.742	200.338	
	5	312.533	252.314	307.082	288.332	298.821	299.038	288.043	307.240	276.450	312.856	
0.4	1	13.4962	11.9115	13.1073	12.5001	12.5911	13.0984	12.0319	13.4592	11.7095	13.5845	
	2	54.5467	47.4764	53.6179	50.1028	52.2846	52.0848	50.1968	53.5980	47.8578	54.6819	
	3	122.512	107.629	120.389	112.780	117.233	117.050	112.847	120.341	108.144	122.664	
	4	217.773	181.455	213.664	200.514	207.939	207.990	200.338	213.738	192.316	217.721	
	5	339.974	268.653	333.480	313.308	324.473	324.908	312.729	333.794	300.969	339.895	

Table 13: *Non-dimensional natural frequencies Ω_n of an axially functionally graded and non-uniform cross-section clamped-pinned beam with $A(x) = (1 + \beta x)^2$ and $I(x) = (1 + \beta x)^4$, and $E(x)$ and $\rho(x)$ changing in accordance with (17) and (18)*

β	Mode	α										
		-10		-3		0		3		10		
		Case 1	Case 2	Case 1	Case 2	Case 1	Case 2	Case 1	Case 2	Case 1	Case 2	
C - P	-0.6	1	12.9193	13.7797	12.4695	14.4721	12.4513	14.3477	12.5458	14.1075	12.4197	14.1012
		2	38.9033	38.6426	38.3811	39.5267	37.9677	40.2060	37.3309	40.6393	36.0462	40.9910
		3	80.0231	76.5424	79.2618	78.1420	78.0815	80.1744	76.1066	81.8057	72.3270	82.9396
		4	136.370	127.832	135.110	130.501	132.743	134.420	128.905	137.656	120.597	139.957
		5	207.966	192.950	205.936	196.686	201.968	203.020	195.752	208.260	177.064	212.038
	-0.4	1	14.2144	15.1079	13.8179	15.7182	13.7694	15.6379	13.7661	15.4944	13.5657	15.5339
		2	44.6125	44.1992	44.1116	45.0560	43.6753	45.8430	42.879	46.4377	41.2507	47.0013
		3	93.0592	88.8644	92.2442	90.6095	90.8761	92.9839	88.5003	94.9406	84.1295	96.4028
		4	159.556	149.551	158.106	152.519	155.278	157.126	150.687	160.925	140.008	163.682
		5	243.998	227.280	241.665	230.845	236.872	238.315	229.445	244.426	204.932	248.841
	-0.2	1	15.3578	16.2641	14.9846	16.8374	14.8876	16.8181	14.8008	16.7429	14.6032	16.7884
		2	49.9003	49.2813	49.4237	50.1377	48.9538	51.0479	47.9790	51.8091	46.2008	52.5517
		3	105.089	100.233	104.210	102.038	102.653	104.750	99.8803	107.029	95.5192	108.789
		4	180.910	169.133	179.197	172.657	175.927	177.912	170.621	182.247	160.805	185.432
		5	276.925	259.249	274.333	262.039	268.763	270.557	260.210	277.480	237.082	282.484
	0.0	1	16.4002	17.3141	16.0255	17.8670	15.8734	17.9148	15.7169	17.8873	15.5435	17.9201
		2	54.9317	54.0109	54.4677	54.9298	53.9586	55.9710	52.8020	56.8997	50.8730	57.8023
		3	116.374	111.047	115.517	112.794	113.770	115.840	110.605	118.435	106.244	120.473
		4	200.920	186.874	199.056	191.578	195.356	197.456	189.361	202.307	180.689	205.897
		5	308.074	290.924	305.024	291.318	298.708	300.829	289.084	308.522	270.827	314.084
0.2	1	17.3490	18.3014	16.9720	18.8380	16.7634	18.9450	16.5486	18.9503	16.4064	18.9608	
	2	59.7676	58.4471	59.3224	59.5157	58.7708	60.6922	57.4282	61.7884	55.3210	62.8390	
	3	127.349	121.578	126.356	123.068	124.417	126.444	120.863	129.352	116.383	131.654	
	4	220.271	202.588	218.036	209.629	213.912	216.112	207.247	221.464	199.242	225.441	
	5	337.244	323.997	334.304	319.232	327.262	329.695	316.608	338.128	302.651	344.223	
0.4	1	18.2423	19.2524	17.8444	19.7553	17.5798	19.9208	17.3166	19.9475	17.2087	19.9302	
	2	64.4810	62.6023	64.0353	63.9453	63.4388	65.2602	61.9070	66.5238	59.5994	67.7141	
	3	137.723	131.998	136.839	132.976	134.707	136.681	130.766	139.897	126.073	142.451	
	4	238.763	216.519	236.348	227.021	231.805	234.095	224.484	239.935	216.717	244.284	
	5	366.410	362.219	362.513	346.108	354.760	357.494	343.107	366.643	331.697	373.251	

Table 14: *Non-dimensional natural frequencies Ω_n of an axially functionally graded and non-uniform cross-section fully clamped beam with $A(x) = (1 + \beta x)^2$ and $I(x) = (1 + \beta x)^4$, and $E(x)$ and $\rho(x)$ changing in accordance with (17) and (18)*

β	Mode	α										
		-10		-3		0		3		10		
		Case 1	Case 2	Case 1	Case 2	Case 1	Case 2	Case 1	Case 2	Case 1	Case 2	
C - C	-0.6	1	16.0941	17.0724	15.7308	17.6020	15.9232	17.2096	16.3018	16.7718	16.4644	16.6712
		2	45.5124	44.9199	45.0176	45.7723	44.7470	46.2067	44.3980	46.2577	43.8359	46.2752
		3	90.3178	86.0362	89.5264	87.7129	88.3928	89.5854	86.6294	90.7248	84.8628	91.1810
		4	150.404	140.630	149.050	143.462	146.630	147.324	142.906	150.019	135.297	151.435
		5	225.756	209.349	223.574	213.069	219.455	219.458	213.244	224.146	195.689	226.694
	-0.4	1	18.8783	19.8346	18.6189	20.1879	18.8965	19.7273	19.3310	19.3019	19.4258	19.2732
		2	53.5683	52.6230	53.1176	53.4274	52.8930	53.8611	52.5145	53.9365	51.9323	54.0689
		3	106.495	101.225	105.653	103.040	104.366	105.161	102.318	106.445	100.635	107.060
		4	177.470	165.970	175.916	169.034	173.044	173.508	168.662	176.557	162.530	178.130
		5	266.478	248.095	263.881	251.458	258.922	258.927	251.574	264.259	244.036	267.131
	-0.2	1	21.5180	22.3963	21.3364	22.6145	21.6912	22.1015	22.1877	21.6759	22.1918	21.7179
		2	61.0333	59.6901	60.6269	60.4651	60.4349	60.9180	60.0093	61.0350	59.3664	61.2832
		3	121.353	115.127	120.468	117.056	119.025	119.423	116.695	120.863	114.858	121.628
		4	202.253	189.014	200.485	192.367	197.180	197.413	192.181	200.804	187.796	202.566
		5	303.693	283.659	300.647	286.446	294.900	294.903	286.504	300.836	284.301	304.007
	0.0	1	24.0629	24.8279	23.9433	24.9364	24.3754	24.3754	24.9364	23.9433	24.8283	24.0629
		2	68.1190	66.3359	67.7553	67.1041	67.5875	67.5875	67.1041	67.7553	66.3363	68.1192
		3	135.351	128.158	134.439	130.228	132.839	132.839	130.228	134.439	128.152	135.352
		4	225.540	210.309	223.579	214.262	219.855	219.855	214.262	223.579	210.300	225.536
		5	338.605	317.091	335.144	319.253	328.641	328.641	319.253	335.144	317.100	338.604
0.2	1	26.5366	27.1668	26.4723	27.1827	26.9839	26.5740	27.6093	26.1329	27.3717	26.3381	
	2	74.9315	72.6668	74.6131	73.4587	74.4633	73.9802	73.9145	74.2051	72.9893	74.6847	
	3	148.777	140.559	147.811	142.801	146.051	145.654	143.162	147.416	140.813	148.475	
	4	247.821	230.039	245.625	235.136	241.489	241.256	235.322	245.306	231.192	247.457	
	5	371.844	347.994	368.027	350.510	360.791	360.788	350.453	367.838	347.282	371.580	
0.4	1	28.9584	29.4358	28.9432	29.3711	29.5375	28.7138	30.2256	28.2631	29.8445	28.5613	
	2	81.5429	78.7483	81.2668	79.5978	81.1300	80.1633	80.5099	80.4498	79.4116	81.0453	
	3	161.708	152.446	160.729	154.919	158.807	158.013	155.641	159.938	153.010	161.143	
	4	269.260	248.332	266.878	255.235	262.334	261.869	255.607	266.239	251.098	268.583	
	5	404.054	375.904	399.688	380.593	391.736	391.730	380.477	399.310	376.170	403.327	

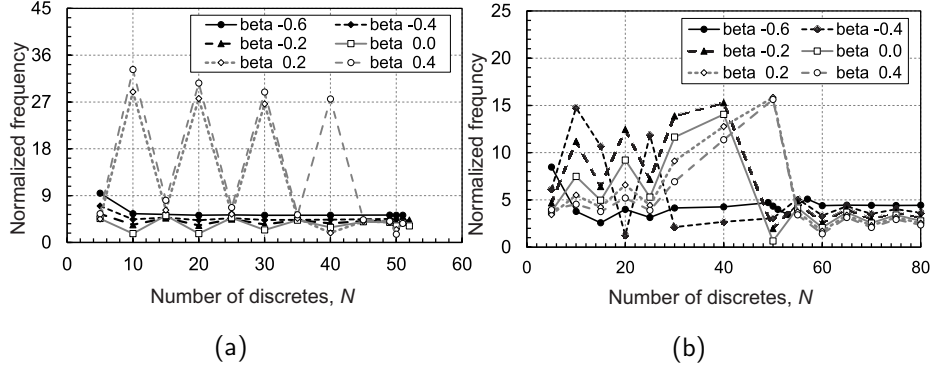


Figure 7: Convergence of the fundamental normalized frequencies for the C-F beam in the case 1: (a) $\alpha = -10$; and (b) $\alpha = 10$.

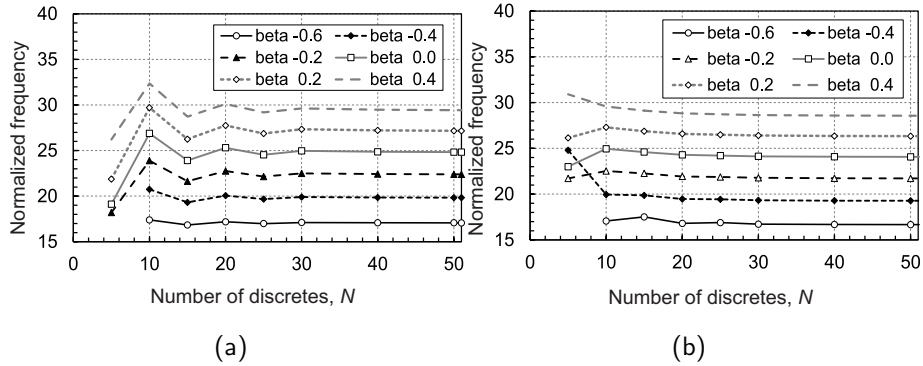


Figure 8: Convergence of the fundamental normalized frequencies for the C-C beam in the case 2: (a) $\alpha = -10$; and (b) $\alpha = 10$.

mode frequencies of such AFGM non-uniform beams often do not follow the variation tendency which is established for their fundamental frequencies and each mode may have its own variation law. Moreover, the variations of frequencies with changing α and β are different for the material gradation patterns called as case 1 and case 2, and for symmetric boundary conditions the natural frequencies, in general, show opposite trends in these two cases.

The results obtained by the proposed approach implementing the DTM demonstrate its good versatility and efficiency for solving the free vibration problem of beams for a variety of cross-sectional non-uniformity, axial mate-

rial inhomogeneity and boundary conditions. Moreover, using the results of the convergence analyses of the obtained DTM solutions, we have proven a high enough accuracy of the developed algorithm as well as have established some issues that may happen in the computations of the natural frequencies of axially functionally graded beams with non-uniform cross-section subjected to different end supports. Therefore, these results can be considered as benchmarks for other researchers.

5. Conclusions

The differential transform method is effectively applied to the free vibration problem of axially functionally graded non-uniform cross-section Euler-Bernoulli beams with arbitrary gradation profiles and non-uniform cross-sectional parameters. The method provides an effective way for solving the governing differential beam equation with arbitrarily varying coefficients. For various beam configurations and different boundary conditions, we have used the differential transformations to reduce the differential beam equation to a set of recurrent algebraic equations. This set of equations together with appropriately transformed boundary conditions results in to a polynomial eigenvalue problem that allows us to find required natural frequencies. By comparing the results of calculations carried out with the DTM and those available in the literature for various axially graded and non-uniform cross-section beams subjected to different end supports, effectiveness and versatility of the DTM has been demonstrated. The superiority of the DTM to other numerical methods has been also shown by its capability to treat any arbitrary gradation profiles and cross-sectional non-uniformities. This has allowed us to demonstrate the influence of the inhomogeneity and non-uniformity parameters on the natural frequencies of the metal/ceramic AlZrO_2 beams.

The accuracy of the method is proven by supplement studies concerning the convergence of solutions. We have concluded that the fastest convergence of the DTM is at the centre of approximation interval and as one moves away from the

centre, the rate of convergence slows down dramatically. Furthermore, we have observed that though the DTM provides very accurate results for most of cases studied in the paper, some computational issues arise when the DTM is applied to the problems which feature high material gradients. Hence, a convergence analysis is always needed to ensure the reliability of the solution and to avoid inaccuracy of results. Based on the revealed advantages and drawbacks of the DTM solutions we would like to believe that the results presented in this paper can be useful for other researchers and can be used by them as references to validate their results.

Acknowledgements

This research has been done under financial support of the Erasmus Mundus post-doc exchange program ACTIVE, Grant Agreement No. 2013-2523/001-001 at the University of Southampton.

Appendix A

The coefficients $B_K(\omega)$, $C_K(\omega)$, $G_K(\omega)$ and $H_K(\omega)$ denoted in (8) are presented by the following recurrent expressions:

$$\begin{aligned}
 B_K(\omega) = & \frac{1}{(K+1)(K+2)(K+3)(K+4)} \cdot \\
 & \left[\omega^2 \left(\sum_{p=0}^{K-4} B_p M(K-4-p) + M(K) \right) - \right. \\
 & \sum_{p=0}^{K-1} (p+2)(p+3)(p+4) B_p D_1(K-1-p) - \\
 & \left. \sum_{p=0}^{K-2} (p+3)(p+4) B_p D_2(K-2-p) \right], \tag{A.1}
 \end{aligned}$$

$$\begin{aligned}
C_K(\omega) &= \frac{1}{(K+1)(K+2)(K+3)(K+4)} \\
&\left[\omega^2 \left(\sum_{p=0}^{K-4} C_p M(K-4-p) + M(K-1) \right) - \right. \\
&\sum_{p=0}^{K-1} (p+2)(p+3)(p+4) C_p D_1(K-1-p) - \\
&\left. \sum_{p=0}^{K-2} (p+3)(p+4) C_p D_2(K-2-p) \right], \tag{A.2}
\end{aligned}$$

$$\begin{aligned}
G_K(\omega) &= \frac{1}{(K+1)(K+2)(K+3)(K+4)} \\
&\left[\omega^2 \left(\sum_{p=0}^{K-4} G_p M(K-4-p) + M(K-2) \right) - \right. \\
&\sum_{p=0}^{K-1} (p+2)(p+3)(p+4) G_p D_1(K-1-p) - \\
&\left. \left(\sum_{p=0}^{K-2} (p+3)(p+4) G_p D_2(K-2-p) + 2D_2(K) \right) \right], \tag{A.3}
\end{aligned}$$

$$\begin{aligned}
H_K(\omega) &= \frac{1}{(K+1)(K+2)(K+3)(K+4)} \\
&\left[\omega^2 \left(\sum_{p=0}^{K-4} H_p M(K-4-p) + M(K-3) \right) - \right. \\
&\left(\sum_{p=0}^{K-1} (p+2)(p+3)(p+4) H_p D_1(K-1-p) + 6D_1(K) \right) - \\
&\left. \left(\sum_{p=0}^{K-2} (p+3)(p+4) H_p D_2(K-2-p) + 6D_2(K-1) \right) \right]. \tag{A.4}
\end{aligned}$$

Appendix B

The constants taken from [19] for calculations of the natural frequencies presented in Tables 4-6 have been used in the following forms:

$$b_{00} = 26a_0, b_{01} = 16a_0, b_{02} = 6a_0, b_{03} = -4a_0, b_{04} = a_0, \text{ where } a_0 = 1 \quad (\text{B.1})$$

$$b_{10} = \frac{2(71a_1 + 91a_0)}{5}, b_{11} = \frac{2(51a_1 + 56a_0)}{5}, b_{12} = \frac{2(31a_1 + 21a_0)}{5},$$

$$b_{13} = \frac{2(11a_1 - 14a_0)}{5}, b_{14} = \frac{-18a_1 + 7a_0}{5}, b_{15} = a_1,$$

$$\text{where } a_0 = 1, a_1 = 1 \quad (\text{B.2})$$

$$b_{20} = \frac{465a_2 + 568a_1 + 728a_0}{15}, b_{21} = \frac{2(181a_2 + 204a_1 + 224a_0)}{15},$$

$$b_{22} = \frac{259a_2 + 248a_1 + 168a_0}{15}, b_{23} = \frac{4(39a_2 + 22a_1 - 28a_0)}{15},$$

$$b_{24} = \frac{53a_2 - 72a_1 + 28a_0}{5}, b_{25} = \frac{2(-5a_2 + 2a_1)}{3}, b_{26} = a_2,$$

$$\text{where } a_0 = 1.5954, a_1 = 0.04, a_2 = 1 \quad (\text{B.3})$$

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