INSPECTION POLICIES FOR THE DETECTION OF SYSTEM FAILURE

BY

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ABSTRACT

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This thesis is concerned with the problem of deciding when to inspect a system in order to detect a failure that would not otherwise be apparent.

The first chapter includes a review of some of the inspection models that have appeared in the literature. A new linear cost model is then proposed, and the chapter ends with a discussion of optimality criteria.

The second chapter is devoted entirely to 'optimal' inspection policies, i.e. those which minimise a certain cost function. The necessary theory is developed to deal with the new cost model proposed in Chapter 1.

Some computational problems that arise in connection with the optimal policies motivate a study of suboptimal policies, and Chapter 3 is concerned with an investigation into the properties of periodic (regular) inspection policies. Chapters 4 and 5 introduce two heuristic inspection policies which are constructed so that the times between inspections are influenced by the mean residual life function in one case, and the hazard rate function in the other. This latter policy (designated $\underline{x}_p$) has some attractive properties, and tables are given for computing the best $\underline{x}_p$ policy in the Weibull case. A sensitivity analysis of $\underline{x}_p$ policies is carried out in Chapter 6.

In the last chapter some tables of expected cost in the Weibull case are presented for the 3 suboptimal policies. A comparison with the optimal policy reveals that the suboptimal policies are highly efficient in many cases.

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To Roberta

Chapter 1

1.1 Introduction

A system may be regarded as a collection of items working together as a unit to perform some defined function. A feature of many systems is that their ability to perform this function varies with time; some systems improve with age, but many do not and their performance worsens as they get older. This deterioration in performance may be caused by external factors such as accidental damage, or it may be due to an intrinsic ageing process.

The performance of a system may deteriorate gradually, as in the case of a continuous production process running in time, where some output parameter such as mean length may shift gradually from its target value. On the other hand, the change in performance may be abrupt, and sometimes the system can suddenly cease to perform at all. We will be concerned with systems of the latter type and assume that originally the system is working, but at some later time it may suddenly fail. It will be convenient to label the working and failed states as E_0 and E_1 respectively. In some cases (an electric light bulb, for example) it will be obvious when the failure has occurred. In other cases, however, the failure can only be detected by an inspection (e.g. a safety valve) and it is this type of system that we shall consider in this thesis.

There is usually some loss incurred each time a system is inspected. This can be a cost, or it may be that the system cannot operate while it is being inspected, as in the case of a computer when a test run is made. In general there will be a cost of some kind associated with each inspection, so we would not wish to inspect too often. On the other hand infrequent inspections could lead to the system remaining in the failed state E_1 for an undesirably long period of time. Thus we wish to achieve a balance between the cost of inspection and the possible consequences of an undetected failure.

1.2 Some models for the inspection problem

Throughout this thesis we consider a system which is originally known to be in a working state, but which may suddenly fail. In other words at $t = 0$ we assume that the system is in state E_0, but at some later time $t = T$ the state of the system will change from E_0 to E_1. We will also assume that:

(i) The state of the system is revealed only by inspection.

(ii) Inspection has no harmful effect on the system.

(iii) Inspection always reveals the true state of the system.

(iv) The duration of each inspection is negligible so that the system cannot fail while it is being inspected

(v) Once in state E_1 the system remains there until it is repaired or replaced.

A time to failure model

If T, the time to failure is known in advance, no inspection problem exists for in this case we could merely leave the system running until time T, and then take the necessary corrective action of repairing or replacing it. We take the uncertainty about T into account by supposing that T is a (continuous) random variable with probability density function (p.d.f.) $f(t),\ t \geq 0$, cumulative ~~density~~ distribution function (c.d.f.)

$$F(t) = P(T \leq t) = \int_0^t f(u)du,$$ and reliability function

$$\bar{F}(t) = P(T > t) = \int_t^\infty f(u)du \ .$$

The probability that a working system, aged t, will fail in the interval $(t, t + dt)$ is

$$\frac{F(t + dt) - F(t)}{1 - F(t)} \simeq \frac{f(t)dt}{\bar{F}(t)} \ .$$

The function

$$h(t) = \frac{f(t)}{\bar{F}(t)} = -\frac{d}{dt} \ln \bar{F}(t)$$

is called the failure rate or hazard rate function, and its integral

$$H(t) = \int_0^t h(u)du = -\ln \bar{F}(t) \qquad (1.2.1)$$

is called the hazard function (Saunders, 1968). Note that (1.2.1) implies

$$\bar{F}(t) = \exp[-H(t)] \quad , \qquad (1.2.2)$$

differentiation gives

$$f(t) = h(t) \exp[-H(t)] \quad . \qquad (1.2.3)$$

Typically, inspection problems arise in connection with systems which are ageing in some statistical sense. Bryson and Siddiqui (1969) propose seven interrelated criteria for ageing systems, one of which is that $h(t)$ is increasing. A somewhat weaker, but intuitively appealing condition is that the mean residual life of a working system $\mu(t) = E(T - t|T > t)$ is decreasing with t. By considering the conditional p.d.f. of T given $T > t$ we see that

$$\mu(t) = \frac{\int_t^\infty uf(u)du}{\bar{F}(t)} - t = \frac{\int_t^\infty \bar{F}(u)du}{\bar{F}(t)} \quad . \qquad (1.2.4)$$

$\mu(t)$ is particularly useful in empirical studies since it can be estimated from sample data by

$$\hat{\mu}(t) = \frac{1}{n(t)} \sum_{j=1}^{n(t)} (t_j - t)$$

where t_j is the time to failure of the jth element in the surviving population of $n(t)$ survivors at time t.

The hazard rate function $h(t)$ is difficult to study empirically, since it involves all the problems of estimating $f(t)$ (Watson and Leadbetter, 1964a, 1964b), but the hazard rate average

$$\frac{H(t)}{t} = -\frac{\ell n\ \bar{F}(t)}{t}$$

can easily be estimated by

$$-\frac{1}{t}\ \ell n\ \frac{n(t)}{n(0)}\quad .$$

A density for which

$$\frac{f(t)}{F(t + a) - F(t)}$$

is increasing in t for all $a > 0$ is said to be a Polya frequency function of order $2(PF_2)$ (Karlin, Proschan and Barlow, 1961). Letting a tend to infinity we see that PF_2 densities have increasing hazard rate. An equivalent definition of a PF_2 density is that $\frac{f(t - a)}{f(t)}$ is increasing in t for all $a > 0$.

Typical failure laws

The exponential distribution $\bar{F}(t) = \exp(- t/\alpha);\ \ t \geq 0,\ \alpha > 0$ has the property that both the hazard rate and mean residual life are constant. In fact no other distribution has either of these properties. Further, for all $t_0 \geq 0$

$$P(T > t + t_0 | T > t_0) = P(T > t)\quad ,$$

so that the exponential distribution is invariant to left truncation, and systems which have this failure law therefore do not 'age'. Epstein (1958) has been largely responsible for most of the work done on the exponential distribution as a failure law, and has published a stream of papers on the problems of estimating α and the associated problems of testing hypotheses. If system failure is due to accidental damage which is likely to occur at random in time, then the exponential distribution would seem to be a reasonable model, but this would not be the case if the system 'aged' in any sense. A result due to Drenick (1960) explains why the exponential law may be used to describe the time between failures of a certain class of systems. If

(i) a system has n statistically independent components

(ii) component failure causes system failure

(iii) each component is repaired or replaced immediately upon failure

then after the system has been running for a long time, for large n the times between system failures are exponentially distributed (subject to some mild conditions on the components' failure distributions). This result explains why the stationary distribution of time between failures of a piece of complex equipment such as a computer can be approximately described by the exponential failure law. Zelen and Dannemiller (1961) have shown that life testing procedures derived from the exponential distribution are not, in general, robust.

Assuming a simple power law for the hazard rate function $h(t)$ leads to the Weibull distribution (Weibull, 1951)

$$\bar{F}(t) = \exp\left[-\left(\frac{t}{\alpha}\right)^{\beta}\right] ; \quad t \geq 0, \quad \alpha, \beta > 0 ,$$

for which

$$E(T) = \alpha\Gamma(1 + \frac{1}{\beta}) \quad \text{and} \quad h(t) = \frac{\beta}{\alpha}\left(\frac{t}{\alpha}\right)^{\beta-1} .$$

Thus the Weibull distribution has an increasing (decreasing) hazard rate if $\beta > 1$ ($\beta < 1$). The Weibull distribution has found applications in a variety of fields, for example Kao (1956, 1958) has found that the time to failure of a certain type of electron tube is best described by a Weibull distribution with $\beta = 1.7$. With $\beta = 3.4$ the Weibull distribution is very nearly normal. A considerable interest has been shown in the Weibull distribution in recent years, and the literature is reviewed in Johnson and Kotz (1970) and Mann, Schafer and Singpurwalla

(1974). Particular attention is given to the Weibull distribution in this thesis because of its wide range of application and the nature of the failure rate function.

The gamma family of distributions has p.d.f.

$$f(t) = \frac{1}{\alpha}\left(\frac{t}{\alpha}\right)^{\beta-1} \frac{\exp(-\,t/\alpha)}{\Gamma(\beta)} \qquad t \geqslant 0; \quad \alpha,\ \beta > 0$$

with increasing (decreasing) hazard rate tending to $\frac{1}{\alpha}$ for $\beta > 1$ $(\beta < 1)$. For general β, $\bar{F}(t)$ is given by

$$\bar{F}(t) = \frac{\Gamma(\beta,\ t/\alpha)}{\Gamma(\beta)} \quad \text{where} \quad \Gamma(\nu,\ x) = \int_x^\infty y^{\nu-1}\, e^{-y}\, dy$$

is the incomplete gamma function (see, for example Abramovitz and Stegun, 1965). However, for integral β, the distribution is of the special Erlangian form, and it can be shown that

$$\bar{F}(t) = \sum_{j=0}^{\beta-1} \left(\frac{t}{\alpha}\right)^j \frac{\exp(-\,t/\alpha)}{j!} .$$

The mean and variance are $\alpha\beta$ and $\alpha^2\beta$ respectively, and the standardised variate

$$\frac{T - \alpha\beta}{\alpha\sqrt{\beta}}$$

is asymptotically $N(0,\ 1)$ as $\beta \to \infty$. However for practical purposes we would use the result due to Fisher (1922) that

$$2\sqrt{\frac{T}{\alpha}} - \sqrt{4\beta - 1}$$

is asymptotically $N(0,\ 1)$. When $\beta = 1$, both the Weibull and Gamma distributions reduce to the exponential distribution.

Davis (1952) suggested that under certain conditions a normal distribut of times to ~~theory of~~ failure seems to fit sample data fairly well. However histograms of failure data are often highly skewed, indicating that this is not always the case. With p.d.f.

$$f(t) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left\{-\frac{1}{2\sigma^2}(t - \mu)^2\right\} \qquad -\infty < \mu < \infty, \quad \sigma > 0$$

we have

$$\frac{f(t - a)}{f(t)} = \exp\left\{\frac{a}{\sigma^2}\,(t - \mu - \frac{a}{2})\right\}$$

which is increasing in t for $a > 0$ and so normal densities are PF_2 and hence have increasing failure rate. The normal distribution is only meaningful as a failure law if the coefficient of variation $\frac{\sigma}{\mu}$ is sufficiently small to ensure near zero probability of negative life.

If $\ell n\ T$ is normally distributed with mean μ and variance σ^2 then T is said to have a lognormal distribution (Finney, 1941) and

$$f(t) = \frac{1}{t\sigma\sqrt{2\pi}} \exp\left\{- \frac{(\ell n\ t - \mu)^2}{2\sigma^2}\right\} \qquad t \geqslant 0 \quad .$$

The distribution function $F(t)$ is given by

$$F(t) = \Phi\left\{\frac{\ell n\ t - \mu}{\sigma}\right\}$$

where

$$\Phi(x) = \int_{-\infty}^{x} \frac{e^{-\frac{u^2}{2}}}{\sqrt{2\pi}}\, du \quad .$$

The lognormal distribution finds applications in the biological sciences and manpower studies, but recent opinion is that it is not a good candidate for the time to failure model (Freudenthal, 1960 and Saunders, 1968). The reason for this is that for all $(\mu, \sigma^2), h(t)$ increases from zero to a maximum, and then decreases as t tends to infinity. Watson and Wells (1961) have exploited this fact and shown that the expected life of items governed by such a law can be increased by using them under normal conditions to eliminate the early failures; the surviving fraction will then have a greater mean residual life than the original batch. A detailed account of the lognormal distribution is given by Aitchison and Brown (1957).

Inspection cost models

Having proposed a model for T, the time to failure, a reasonable approach is to define a cost or loss function which depends on the time to system failure, and the times at which inspections are made; for a given cost function we can then propose a criterion of optimality to determine when the inspections should best be scheduled. This criterion may or may not assume full knowledge about the random variable T.

Definition. An inspection policy $\underline{x} = \{x_1, x_2, \dots\}$ is a sequence of times at which the system is to be inspected. Inspection ceases as soon as the failure (state E_1) is detected. Associated with any given policy $\underline{x}$ is the sequence of inter inspection times $\{\delta_i = x_i - x_{i-1} : i = 1, 2, \dots\}$ where x_0 is defined to be zero. Clearly $\delta_i > 0$.

An early approach was due to Savage (1956) who proposed the so-called preparedness model

$$c = c(t; \underline{x}) = c_1 n + \sum_{i=1}^{n} G(\delta_i) + H(t - x_n)$$

where t is the time to failure, n is such that $x_n \leqslant t < x_{n+1}$, c_1 is the cost of each inspection, and G and H are increasing functions satisfying $G(0) = H(0) = 0$. The three components of this model are therefore the cost of performing each inspection, a cost depending on the times between inspections before the failure occurs, and a cost depending on the time elapsed between the last inspection and the failure. Savage proposed this model for the problem of inspecting standby equipment such as fire fighting apparatus, which is somewhat different from the problem considered in this thesis in that the 'failure' at time t represents some emergency (e.g. a fire) rather than the failure of the equipment itself. Hence in this model the 'failure' is apparent as soon as it happens, and the penalty cost $H(t - x_n)$ is a function of the time between the 'failure' and the previous inspection, which was the time when the equipment was last known to be working properly. Savage assumed that the time to failure had an exponential distribution and considered the two cases $G(x) = 0$, $H(x) = \lambda x$ and $G(x) = H(x) = 1 - \exp(-\lambda x)$. He then found the policy which minimised the expected total cost per failure.

Barlow and Hunter (1960) proposed a model in which each inspection costs c_1, and the penalty due to leaving the system in a failed state is c_2 per unit time. This gives

$$c(t; \underline{x}) = c_1 n + c_2(x_n - t) \tag{1.2.5}$$

where now n is such that $x_{n-1} < t \leqslant x_n$, so that the penalty cost is a function of the elapsed time between the failure and the <u>next</u> inspection (c.f. Savage). This model might be appropriate for the inspection of systems such as early warning systems, since the penalty cost $c_2(x_n - t)$ takes account of the vulnerable period of time $x_n - t$ during which the system would be inoperative.

If the failure of a piece of emergency standby equipment (such as fire fighting apparatus) occurred at time t, and was detected at time x_n, then a real cost would only be incurred if an emergency actually happened in the interval (t, x_n). If the cost incurred when an emergency happens and no equipment is available is c_0, then the expected penalty cost upon failure of equipment would be c_0 P {emergency during (t, x_n)}. Assuming that emergencies occur at random every Δ time units on average, then this becomes

$$c_0\left\{1 - \exp\left(-\frac{(x_n - t)}{\Delta}\right)\right\} .$$

However, for effective inspection, the intervals $\left\{x_n - x_{n-1}\right\}$ would usually be small compared with Δ so that

$$\frac{x_n - t}{\Delta} << 1$$

and

$$c_0\left\{1 - \exp\left(-\frac{(x_n - t)}{\Delta}\right)\right\} \simeq c_0 \frac{(x_n - t)}{\Delta} \quad ,$$

which gives us cost model (1.2.5) with $c_2 = \frac{c_0}{\Delta}$.

In the case of a continuous production process producing items at a constant rate, the penalty cost incurred would largely be due to the cost of scrapping or reworking the defective items produced in the interval (t, x_n). However, in general the actual time of failure would not necessarily be known, so that all items produced since the process was last known to be working properly, that is those produced during the interval (x_{n-1}, x_n) would have to be reworked or scrapped. We therefore propose a new cost model

$$c(t; \underline{x}) = c_1 n + c_2(x_n - x_{n-1})^{\dagger} \quad . \tag{1.2.6}$$

In the subsequent chapters of this thesis we will consider only the cost models given by (1.2.5) and (1.2.6) which we shall refer to as model I and model II respectively.

† In which $c_1 n$ is the inspection cost and $c_2(x_n - x_{n-1})$ is the penalty cost.

1.3 Some criteria for choosing an inspection policy

If the probability distribution of the time to system failure is known, then in principle we can compute such quantities as $E\{c(T; \underline{x})\}$ and $P(c(T; \underline{x}) \leq \text{constant})$ for particular cost models. These quantities can then be used to define a policy which is 'best' in some sense, for example we may wish to choose that policy $\underline{x}^*$ for which $E\{c(T; \underline{x})\}$ is as small as possible, or for which $P(c(T; \underline{x}) \leq c)$ is as large as possible. These 'best' policies may or may not exist.

We first consider the case when complete knowledge about the distribution of the time to system failure is not available.

Derman's minimax policy

Derman (1961) considered the case when the life of the system cannot exceed some finite time τ say, and assuming no other knowledge about the p.d.f. of T, derived a minimax inspection policy for Barlow and Hunter's (1960) cost model I. He showed that the inspection policy which minimises the maximum possible expected cost over all possible densities $f(t)$, is given by

$$x_i = ip\left\{\frac{\tau}{np + 1} + \frac{c_1}{2c_2}\left(\frac{n\left[(n + 1)p + 2\right]}{np + 1} - (i + 1)\right)\right\} \quad i = 1, \ldots, n$$

where n is the largest integer such that

$$c_1\, p^2 n^2 + c_1\, p(2 - p)n + 2(c_1 - p\, c_2\, \tau) \leq 0$$

and p is the probability that an inspection reveals state E_1 when E_1 is the true state of the system. By assumption (iii) of section 1.2 we consider only the case $p = 1$.

Minimising the expected cost

When the p.d.f. of T is known, the minimax criterion ceases to be meaningful, and Barlow, Hunter and Proschan (1963) defined an optimal policy to be one which minimises the expected value of $c(T; \underline{x})$. In many cases a renewal takes place as soon as the failure is detected. For example when a continuous production process is found to be out of control, necessary adjustments or repairs are made, production resumes, and inspection continues. In such cases a more suitable definition of an optimal policy is that which minimises the expected cost per unit time over an infinite time span. We now assume that upon detection of failure, a repair taking time r is made, at a cost c_3, and that the system is then taken to be as good as now and inspection resumes. Although the model is now slightly more complicated and a new objective is being defined, the following result due to Brender (1963) shows that in principle, the problem can be formulated as one of minimising the expected cost per failure.

Let

$$C(\underline{x}) = c_1 E(N) + c_2\left\{E(x_N - T)\right\} + c_3$$

be the expected cost per cycle under policy $\underline{x}$, and let

$$T(\underline{x}) = E(x_N) + r$$

be the expected length of each cycle under policy $\underline{x}$. Then we wish to minimise the expected cost per unit time, which is

$$R(\underline{x}) = \frac{C(\underline{x})}{T(\underline{x})} .$$

Define

$$D(\alpha, \underline{x}) = C(\underline{x}) - \alpha T(\underline{x}) = c_1 E(N) + (c_2 - \alpha)E(x_N - T) + \{c_3 - \alpha(r + \mu)\}$$

where $\mu = E(T)$.

For fixed α find $\underline{x}(\alpha)$ which minimises $D(\alpha, \underline{x})$, and then find that value of α, α^* say, for which $D(\alpha^*, \underline{x}(\alpha^*)) = 0$. (It has been shown that for nontrivial values of c_1, c_2, c_3 and r, such an α^* must exist.) Then $\underline{x}(\alpha^*)$ minimises $R(\underline{x})$ and $\alpha^* = \min_{\underline{x}} R(\underline{x})$.

The result relies on the fact that if

$$\min_{\underline{x}} D(\alpha^*, \underline{x}) = 0$$

then

$$\min_{\underline{x}} \{C(\underline{x}) - \alpha^* T(\underline{x})\} = 0$$

or

$$\min_{\underline{x}} T(\underline{x})\{R(\underline{x}) - \alpha^*\} = 0$$

so that

$$\min_{\underline{x}} R(\underline{x}) = \alpha^* \quad \text{since} \quad T(\underline{x}) > 0 \quad .$$

Thus in principle the problem of minimising expected cost per unit time, and minimising expected cost per cycle reduces to the same computational problem.

Chapter 2

Optimal Inspection Policies

In this chapter we compute optimal inspection policies in the exponential case, and when f is a PF_2 density, for both cost model I and the new cost model II introduced in section 1.2. We conclude that these optimal policies pose a real computational problem, especially in the case of model II. This motivates the search in later chapters for a class of computationally simpler, near-optimal policies.

Preliminary

A component of the expected total cost for both models I and II is the average number of inspections needed to detect the failure, $E(N)$. Since $E(N)$ may or may not exist, we give a necessary and sufficient condition for existence, namely that

$$\sum_{j=0}^{\infty} \bar{F}(x_j)$$

converges, and in this case

$$E(N) = \sum_{j=0}^{\infty} \bar{F}(x_j) \quad .$$

Proof

$$E(N) = \lim_{n\to\infty} \sum_{j=1}^{n} j\ P(x_{j-1} < T \leqslant x_j)$$

$$= \lim_{n\to\infty} \sum_{j=1}^{n} j\{\bar{F}(x_{j-1}) - \bar{F}(x_j)\} \ .$$

Writing $\bar{F}_k$ for $\bar{F}(x_k)$ we have

$$\sum_{j=1}^{n} j(\bar{F}_{j-1} - \bar{F}_j) = \sum_{j=1}^{n} (j - 1)\bar{F}_{j-1} + \sum_{j=1}^{n} \bar{F}_{j-1} - \sum_{j=1}^{n} \bar{F}_j \ .$$

The first and third terms on the right hand side differ by $n\bar{F}_n$ and so

$$\sum_{j=1}^{n} j(\bar{F}_{j-1} - \bar{F}_j) = \sum_{j=1}^{n} \bar{F}_{j-1} - n\bar{F}_n \quad .$$

The convergence of $\sum_{j=0}^{\infty} \bar{F}_j$ is sufficient since the convergence of a series of positive decreasing terms $\sum_{n=0}^{\infty} a_n$ implies $na_n \to 0$ as $n \to \infty$ (see, for example Flett, 1966 page 242). To prove that the convergence of $\sum_{j=0}^{\infty} \bar{F}_j$ is necessary we must show that if

$$\sum_{j=1}^{\infty} j(\bar{F}_{j-1} - \bar{F}_j)$$

exists then $n\bar{F}_n \to 0$ as $n \to \infty$. If

$$\sum_{j=1}^{\infty} j(\bar{F}_{j-1} - \bar{F}_j)$$

converges then

$$\sum_{j=n+1}^{\infty} j(\bar{F}_{j-1} - \bar{F}_j) \to 0 \quad \text{as} \quad n \to \infty \quad .$$

But

$$\sum_{j=n+1}^{\infty} j(\bar{F}_{j-1} - \bar{F}_j) > (n + 1) \sum_{j=n+1}^{\infty} (\bar{F}_{j-1} - \bar{F}_j) = (n + 1)\bar{F}_n > 0 \quad .$$

Hence $(n + 1)\bar{F}_n \to 0$ so that $n\bar{F}_n \to 0$. This completes the proof.

It is interesting to note that the existence of $E(N)$ depends on $\{x_n\}$ as much as $\bar{F}$; that $x_j \to \infty$ as $j \to \infty$ is not sufficient for existence. As an example consider the inspection policy given by $x_n = \alpha \, \ell n(n + 1)$. Note that $x_0 = 0$, $x_\infty = \infty$, and suppose that $\bar{F}(t) = \exp(- t/\alpha)$. Then

$$\bar{F}(x_n) = \exp\left[- \alpha \, \ell n(n + 1)/\alpha\right] = \frac{1}{n + 1} \quad .$$

So that

$$\sum_{j=0}^{n} \bar{F}(x_j) = \sum_{j=0}^{n} \frac{1}{j+1} \to \infty \quad \text{as} \quad n \to \infty \quad ,$$

which means that the expected number of inspections is infinite in this case.

2.1 Optimal inspection policy in the exponential case; models I and II

A consequence of the fact that systems with an exponential failure distribution do not age is that the optimal policy must be periodic, i.e. of the form $x_k = kx$, $k = 0, 1, \ldots,$ for some $x > 0$. Barlow and Hunter (1960) showed that for model I when

$$f(t) = \frac{1}{\alpha} e^{-x/\alpha}$$

this gives an expected total cost per cycle of

$$\frac{c_1 + c_2 x}{1 - e^{-x/\alpha}} - c_2\alpha \tag{2.1.1}$$

which is minimised when

$$e^{x/\alpha} - \frac{x}{\alpha} - 1 = \frac{c_1}{\alpha c_2} \ . \tag{2.1.2}$$

If α is large so that $e^{x/\alpha} \simeq 1 + \frac{x}{\alpha} + \frac{1}{2}\left(\frac{x}{\alpha}\right)^2$ then

$$x \simeq \sqrt{\frac{2\alpha c_1}{c_2}} \ . \tag{2.1.3}$$

It is interesting to extend their analysis a little further, for if x^* is the optimal inspection interval, then from (2.1.2)

$$1 - e^{-\frac{x^*}{\alpha}} = \frac{\frac{x^*}{\alpha} + \frac{c_1}{\alpha c_2}}{1 + \frac{x^*}{\alpha} + \frac{c_1}{\alpha c_2}} \ .$$

Substituting in (2.1.1) we find that after a little algebra

$$\min_{\underline{x}} E\left[c(T;\ \underline{x})\right] = c_1 + c_2 x^* \ .$$

So that the minimum expected cost is equal to the sum of the cost of a single inspection and the penalty cost incurred over an optimal inspection interval.

For model II note that $x_k - x_{k-1} = x$ for all k so that

$$E(x_N - x_{N-1}) = x$$

and

$$E[c(T)] = c_1\ E(N) + c_2\ E(x_N - x_{N-1})$$

$$= c_1 \sum_{j=0}^{\infty} e^{-\frac{jx}{\alpha}} + c_2 x$$

$$= \frac{c_1}{1 - \exp\left(-\frac{x}{\alpha}\right)} + c_2 x \quad .$$

This is minimised when

$$\frac{-\ c_1 \exp\left(-\frac{x}{\alpha}\right)}{\left[1 - \exp\left(-\frac{x}{\alpha}\right)\right]^2} + \alpha c_2 = 0$$

$$x^* = -\ \alpha\ \ell n\left(1 + \frac{K}{2} - \sqrt{\frac{K^2}{4} + K}\right) \quad \text{where} \quad K = \frac{c_1}{\alpha c_2} \quad . \qquad (2\cdot1\cdot4)$$

It is interesting that the optimal inspection interval for model II in the exponential case can be determined explicitly, unlike model I. However the subsequent sections of this chapter reveal that this computational simplification is not evident in the general case.

2.2 Optimal inspection policies for PF_2 densities; model I

In this section we present the results of Barlow, Hunter and Proschan (1963) who have given conditions for the existence of an inspection policy which minimises $E\{c(T;\underline{x})\}$ for model I, and in the special case when $f(t)$ is PF_2 have proposed an algorithm for the calculation of such a policy.

Optimal inspection policies

For model I

$$E\{c(T;\underline{x})\} = c_1 \sum_{j=0}^{\infty} \bar{F}(x_j) + c_2 \sum_{n=1}^{\infty} \int_{x_{n-1}}^{x_n} (x_n - t)\, f(t)dt \tag{2.2.1}$$

and Barlow, Hunter and Proschan (1963) showed that if $F(t)$ is continuous with finite mean, then a policy $\underline{x}^*$ which minimises $E\{c(T;\underline{x})\}$ exists. In the case when it is known that the system will fail in some given interval $[0,\tau]$ say, they showed that a necessary and sufficient condition for the optimal policy to consist of a single inspection at time τ is that

$$F(t) \leqslant \frac{c_1}{c_1 + c_2(\tau - t)} \quad \text{for all} \quad 0 \leqslant t \leqslant \tau \ .$$

In the general case $E\{c(T;\underline{x})\}$ is given by (2.2.1) and for a minimum

$$\frac{\partial}{\partial x_k} E\{c(T;\underline{x})\} = 0 \qquad k = 1, 2, \ldots$$

This leads to

$$x_{n+1} - x_n = \frac{F(x_n) - F(x_{n-1})}{f(x_n)} - \frac{c_1}{c_2} \tag{2.2.2}$$

so that $\underline{x}^* = \{x_k^*\}$ can be calculated once x_1^* is known.

Barlow, Hunter and Proschan (1963) showed that $\left\{\delta_k^* = x_k^* - x_{k-1}^*\right\}$ is a decreasing sequence if f is PF_2, and in this case proposed the following algorithm for computing x_1^* .

Algorithm 2.2.1

As an approximation to x_1^* choose x_1 so that

$$\int_0^{x_1} F(t)dt = \frac{c_1}{c_2} \ .$$

This is that value of x_1 for which the cost of undetected failure during $[0, x_1]$ is balanced against the cost of the first inspection.

(i) Compute $\{x_k\}$ recursively from (2.2.2),

(ii) if $\delta_k > \delta_{k-1}$ for some k, reduce x_1 and repeat; if $\delta_k < 0$ for some k, increase x_1 and repeat.

The motivation of algorithm 2.2.1 is the following theorem due to Barlow, Hunter and Proschan (1963):

Theorem 2.2.1

Let f be PF_2 with $F(x)/f(x)$ strictly increasing and with $f(t) > 0$. Then if $\{x_k^*\}$ is the optimal policy

(i) if $x_1 > x_1^*$, $\delta_k > \delta_{k-1}$ for some k

(ii) if $x_1 < x_1^*$, $\delta_k < 0$ for some k .

Optimal policy in the Weibull case

The Weibull distribution

$$f(t) = \frac{\beta}{\alpha}\left(\frac{t}{\alpha}\right)^{\beta-1} \exp\left\{-\left(\frac{t}{\alpha}\right)^{\beta}\right\}$$

is PF_2 for $\beta \geq 1$, and

$$\frac{F(t)}{f(t)} = \frac{1 - \exp\left\{-\left(\frac{t}{\alpha}\right)^{\beta}\right\}}{\frac{\beta}{\alpha}\left(\frac{t}{\alpha}\right)^{\beta-1} \exp\left\{-\left(\frac{t}{\alpha}\right)^{\beta}\right\}} = \frac{t}{\beta}\left\{\frac{\exp\left(\frac{t}{\alpha}\right)^{\beta} - 1}{\left(\frac{t}{\alpha}\right)^{\beta}}\right\} .$$

Using the fact that x, $(e^x - 1)/x$, and x^β are all strictly increasing functions, and that the composition and product of strictly increasing positive functions is strictly increasing, we see that $F(t)/f(t)$ is strictly increasing in the Weibull case. Hence, theorem 2.2.1 holds and algorithm 2.2.1 can be used to find the optimal policy when $\beta \geq 1$.

Now

$$\frac{F(x_n) - F(x_{n-1})}{f(x_n)} = \frac{\exp\left\{-\left(\frac{x_{n-1}}{\alpha}\right)^{\beta}\right\} - \exp\left\{-\left(\frac{x_n}{\alpha}\right)^{\beta}\right\}}{\frac{\beta}{\alpha}\left(\frac{x_n}{\alpha}\right)^{\beta-1} \exp\left\{-\left(\frac{x_n}{\alpha}\right)^{\beta}\right\}} .$$

So that (2.2.2) gives

$$x_{n+1} - x_n = \frac{\exp\left\{\left(\frac{x_n}{\alpha}\right)^{\beta} - \left(\frac{x_{n-1}}{\alpha}\right)^{\beta}\right\} - 1}{\frac{\beta}{\alpha}\left(\frac{x_n}{\alpha}\right)^{\beta-1}} - \frac{c_1}{c_2} .$$

Putting $z_n = \frac{x_n}{\alpha}$, $K = \frac{c_1}{\alpha c_2}$ this simplifies to

$$z_{n+1} - z_n = \frac{\exp\left\{z_n^{\beta} - z_{n-1}^{\beta}\right\} - 1}{\beta\, z_n^{\beta-1}} - K . \tag{2.2.3}$$

This standardisation has the advantage that instead of having to find the inspection times as a function of the four parameters α, β, c_1 and c_2, we need only two parameters, namely K and β to compute $\{z_i\}$. Using algorithm 2.2.1 and (2.2.3) to compute $\{z_i\}$ the optimal policy is then given by $x_i = \alpha z_i$.

2.3 Optimal inspection policies for PF_2 densities; model II

Calculation of a policy which minimises the total expected cost for model I leads to an equation giving x_{n+1} explicitly in terms of x_n and x_{n-1}. For the new cost model II proposed in section 1.2 we now show that minimising the expected total cost leads to an implicit equation for x_{n+1} in terms of x_n and x_{n-1}, which adds to the already difficult computational problem. For this new model we give results which justify the use of a modified version of the model I algorithm, when $f(t)$ is PF_2 .

Optimal model II policies

For the new cost model II proposed in section 1.2

$$c(t;\underline{x}) = c_1 n + c_2(x_n - x_{n-1}) \quad \text{where} \quad x_{n-1} < t \leqslant x_n \ .$$

So that

$$E\{c(T;\underline{x})\} = c_1\, E(N) + c_2\, E(x_N - x_{N-1})$$

$$= c_1\, E(N) + c_2 \sum_{n=1}^{\infty} (x_n - x_{n-1}) P(x_{n-1} < T \leqslant x_n)$$

$$E\{c(T;\underline{x})\} = c_1 \sum_{j=0}^{\infty} \bar{F}(x_j) + c_2 \sum_{n=1}^{\infty} (x_n - x_{n-1})\{F(x_n) - F(x_{n-1})\} \ . \tag{2.3.1}$$

By a modification of theorem 1 of Barlow, Hunter and Proschan (1963) we can show that an inspection policy $\underline{x}^*$ which minimises (2.3.1) exists.

For such a policy, we must have

$$\frac{\partial}{\partial x_k} E\{c(T;\underline{x})\} = 0, \qquad k = 1, 2, \ldots$$

Now from (2.3.1), the terms involving x_k in $E\{c(T;\underline{x})\}$ are

$$c_1\, \bar{F}(x_k) + c_2(x_k - x_{k-1})\{F(x_k) - F(x_{k-1})\}$$
$$+ c_2(x_{k+1} - x_k)\{F(x_{k+1}) - F(x_k)\}$$

so that

$$\frac{\partial}{\partial x_k} E\{c(T;\, \underline{x})\} = - c_1\ f(x_k) + c_2\{F(x_k) - F(x_{k-1})\}$$
$$+ c_2\ f(x_k)(x_k - x_{k-1}) - c_2\{F(x_{k+1}) - F(x_k)\}$$
$$- c_2\ f(x_k)(x_{k+1} - x_k) \quad .$$

Equating the right hand side to zero gives

$$(x_{k+1} - x_k) - (x_k - x_{k-1}) = \frac{\{F(x_k) - F(x_{k-1})\} - \{F(x_{k+1}) - F(x_k)\}}{f(x_k)} - \frac{c_1}{c_2} \ ,$$

$$k = 1,\ 2,\ \ldots \qquad (2.3.2)$$

Like model I, $\left\{x_k^*\right\}$ for model II is determined by x_1^* . However, we now have the added computational difficulty that x_{k+1} cannot be written explicitly in terms of x_k and x_{k-1} as before, because of the presence of the term $F(x_{k+1})$ in (2.3.2). For this reason the results of Barlow, Hunter and Proschan (1963) for model I do not carry over to model II. However, we now proceed to show that if $\underline{x}^*$ is the optimal policy, and $\delta_k^* = x_k^* - x_{k-1}^*$, then once again $\left\{\delta_k^*\right\}$ is decreasing. Moreover if $x_1 > x_1^*$ then $\delta_k > \delta_{k-1}$ for some k; if $x_1 < x_1^*$, $\delta_k < 0$ for some k. We first need the following lemmas.

<u>Lemma 1</u> Let $\{x_k\}$ be defined by (2.3.2) with $x_0 = 0$, and $\delta_k = x_k - x_{k-1} \geq 0$ for $k = 1, 2, \ldots$ Then $x_k \to \infty$ as $k \to \infty$.

<u>Proof</u> Suppose to the contrary that $x_k \to c < \infty$ as $k \to \infty$.
From (2.3.2)

$$\delta_{k+1} = \delta_k - \frac{F(x_{k+1}) - F(x_k)}{f(x_k)} + \frac{F(x_k) - F(x_{k-1})}{f(x_k)} - \frac{c_1}{c_2}$$

so that

$$\delta_{k+1} \leq \delta_k + \frac{F(x_k) - F(x_{k-1})}{f(x_k)} - \frac{c_1}{c_2} \quad .$$

But, if $x_k \to c$ as $k \to \infty$, then $\delta_k \to 0$, $F(x_k)$, $F(x_{k-1}) \to F(c)$, and $f(x_k) \to f(c) > 0$. So that eventually

$$\delta_{k+1} \leqslant -\frac{c_1}{c_2} \ ,$$

contradicting the fact that $\delta_k \geqslant 0$ for all k.

In lemmas 2, 3, 4 and 5, f is taken to be PF_2, $y > x$, $\Delta > 0$ and m is the mode of f.

<u>Lemma 2</u>

$$\frac{F(x+\Delta) - F(x)}{f(x)} \quad \text{is decreasing in } x \ ,$$

<u>Lemma 3</u>

$$\frac{F(x) - F(x-\Delta)}{f(x)} \quad \text{is increasing in } x \ ,$$

<u>Lemma 4</u> for $r \geqslant 1$ and $x - r\Delta \geqslant m$

$$\frac{F(y) - F(y - r\Delta)}{f(y)} \geqslant r \, \frac{F(x) - F(x-\Delta)}{f(x)} \ ,$$

<u>Lemma 5</u> for $r \geqslant 1$ and $x \geqslant m$

$$\frac{F(y + r\Delta) - F(y)}{f(y)} \leqslant r \, \frac{F(x+\Delta) - F(x)}{f(x)} \ ,$$

<u>Proof</u> The proofs of Lemmas 2, 3 and 4 are given in Barlow, Hunter and Proschan (1963); for Lemma 5 note that since f is PF_2

$$\frac{f(x+t)}{f(x)} \geqslant \frac{f(y+t)}{f(y)} \quad \text{for } t \geqslant 0, \text{ and since } x \geqslant m$$

$$\frac{f(y+t)}{f(y)} \quad \text{is decreasing in } t.$$

$$\therefore \int_0^{r\Delta} \frac{f(y+t)}{f(y)}\,dt \leqslant r \int_0^{\Delta} \frac{f(y+t)}{f(y)}\,dt \leqslant r \int_0^{\Delta} \frac{f(x+t)}{f(x)}\,dt$$

$$\frac{F(y + r\Delta) - F(y)}{f(y)} \leqslant r \, \frac{F(x + \Delta) - F(x)}{f(x)}$$

Lemma 6 Let $\delta_k = x_k - x_{k-1}$, $\{x_k\}$ satisfy (2.3.2) with $\delta_k > 0$ for all k. If f is PF_2 and for some k $\delta_k > \delta_{k-1}$ then $\delta_{k+1} > \delta_k$.

Proof Since

$$(x_{n+1} - x_n) - (x_n - x_{n-1}) = \frac{\left[F(x_n) - F(x_{n-1})\right] - \left[F(x_{n+1}) - F(x_n)\right]}{f(x_n)} - \frac{c_1}{c_2}$$

$$n = 1, 2, \ldots$$

$$\begin{aligned}
(\delta_{k+1} - \delta_k) - (\delta_k - \delta_{k-1}) &= \frac{\left[F(x_k) - F(x_{k-1})\right] - \left[F(x_{k+1}) - F(x_k)\right]}{f(x_k)} \\
&\quad - \frac{\left[F(x_{k-1}) - F(x_{k-2})\right] - \left[F(x_k) - F(x_{k-1})\right]}{f(x_{k-1})} \\
&= \frac{F(x_k) - F(x_{k-1})}{f(x_k)} - \frac{F(x_{k-1}) - F(x_{k-2})}{f(x_{k-1})} \\
&\quad + \frac{F(x_k) - F(x_{k-1})}{f(x_{k-1})} - \frac{F(x_k + \delta_k) - F(x_k)}{f(x_k)} \\
&\quad + \frac{F(x_k + \delta_k) - F(x_{k+1})}{f(x_k)} \\
&> \frac{F(x_k) - F(x_k - \delta_k)}{f(x_k)} - \frac{F(x_{k-1}) - F(x_{k-1} - \delta_k)}{f(x_{k-1})} \\
&\quad + \frac{F(x_{k-1} + \delta_k) - F(x_{k-1})}{f(x_{k-1})} - \frac{F(x_k + \delta_k) - F(x_k)}{f(x_k)} \\
&\quad + \frac{F(x_k + \delta_k) - F(x_{k+1})}{f(x_k)} ,
\end{aligned}$$

since $x_{k-2} = x_{k-1} - \delta_{k-1} > x_{k-1} - \delta_k$. Therefore

$$(\delta_{k+1} - \delta_k) - (\delta_k - \delta_{k-1}) > \frac{F(x_k + \delta_k) - F(x_{k+1})}{f(x_k)}$$

using lemma 3 and lemma 2 respectively.

Since $\delta_k - \delta_{k-1} > 0$ by assumption,

$$\delta_{k+1} - \delta_k > \frac{F(x_k + \delta_k) - F(x_k + \delta_{k+1})}{f(x_k)}$$

$$\left[\delta_{k+1} - \delta_k\right] + \left[\frac{F(x_k + \delta_{k+1}) - F(x_k + \delta_k)}{f(x_k)}\right] > 0 \quad .$$

Since F is increasing, both expressions contained in square brackets have the same sign; since their sum is positive, each must be positive and hence $\delta_{k+1} > \delta_k$.

<u>Lemma 7</u> Let $\delta_k = x_k - x_{k-1}$, $\{x_k\}$ satisfy (2.3.2) with $\delta_k > 0$ for all k. If f is PF_2 and for some k $\delta_k = r\,\delta_{k-1}$ where $r > 1$ and $x_{k-2} \geq m$ where m is the mode of f, then

$$\delta_{k+1} > r\,\delta_k \quad .$$

<u>Proof</u>

$$(\delta_{k+1} - \delta_k) - r(\delta_k - \delta_{k-1}) = \frac{\left[F(x_k) - F(x_{k-1})\right] - \left[F(x_{k+1}) - F(x_k)\right]}{f(x_k)}$$

$$- r\,\frac{\left[F(x_{k-1}) - F(x_{k-2})\right] - \left[F(x_k) - F(x_{k-1})\right]}{f(x_{k-1})}$$

$$+ (r - 1)\,\frac{c_1}{c_2} \quad .$$

$$\delta_{k+1} - r\,\delta_k = \frac{F(x_k) - F(x_k - r\,\delta_{k-1})}{f(x_k)} - r\,\frac{F(x_{k-1}) - F(x_{k-1} - \delta_{k-1})}{f(x_{k-1})}$$

$$+ r\,\frac{F(x_{k-1} + \delta_k) - F(x_{k-1})}{f(x_{k-1})} - \frac{F(x_k + r\,\delta_k) - F(x_k)}{f(x_k)}$$

$$+ \frac{F(x_k + r\,\delta_k) - F(x_{k+1})}{f(x_k)} + (r - 1)\,\frac{c_1}{c_2}$$

$$\delta_{k+1} - r\,\delta_k > \frac{F(x_k + r\,\delta_k) - F(x_{k+1})}{f(x_k)}$$

using lemma 4 and lemma 5 respectively.

$$\therefore \quad \left[\delta_{k+1} - r\,\delta_k\right] + \left[\frac{F(x_k + \delta_{k+1}) - F(x_k + r\,\delta_k)}{f(x_k)}\right] > 0 \quad .$$

Since F is increasing, both expressions contained in square brackets have the same sign; since their sum is positive, each must be positive, and hence $\delta_{k+1} > r\,\delta_k$.

<u>Lemma 8</u> Let $\delta_k = x_k - x_{k-1}$, $\{x_k\}$ satisfy (2.3.2) with $\delta_k \geqslant 0$ for all k. Then if f is PF_2

$$\frac{d\delta_{k+1}}{dx_1} \geqslant \frac{f(x_1) + f(x_2)}{f(x_k) + f(x_{k+1})}\,\frac{d\delta_2}{dx_1} > 0 \qquad k = 1, 2, \ldots$$

<u>Proof</u>

$$\delta_2 = x_1 + \frac{F(x_1)}{f(x_1)} - \frac{F(x_1 + \delta_2) - F(x_1)}{f(x_1)} - \frac{c_1}{c_2} \quad \text{by} \quad (2.3.2)$$

so that by the chain rule for partial differentiation

$$\frac{d\delta_2}{dx_1} = 1 + \frac{d}{dx_1}\left[\frac{F(x_1)}{f(x_1)}\right] - \frac{\partial}{\partial x_1}\left[\frac{F(x_1 + \delta_2) - F(x_1)}{f(x_1)}\right] - \frac{f(x_1 + \delta_2)}{f(x_1)}\,\frac{d\delta_2}{dx_1}$$

hence

$$\frac{d\delta_2}{dx_1}\left[1 + \frac{f(x_2)}{f(x_1)}\right] \geqslant 1$$

since $\frac{F(x)}{f(x)}$ is increasing, and $\frac{F(x + \Delta) - F(x)}{f(x)}$ is decreasing for $\Delta > 0$. So that the lemma is certainly true for $k = 1$.

Now suppose that the lemma is true for $k = 1, 2, \ldots, n-1$; we will show that this implies it is true for $k = n$, and hence for all k by induction. Note that

$$\delta_{n+1} = \Phi(x_n, \delta_n, \delta_{n+1})$$

where

$$\Phi(x_n, \delta_n, \delta_{n+1}) = \delta_n + \frac{F(x_n) - F(x_n - \delta_n)}{f(x_n)}$$

$$- \frac{F(x_n + \delta_{n+1}) - F(x_n)}{f(x_n)} - \frac{c_1}{c_2}$$

using (2.3.2). So that by the chain rule

$$\frac{d\delta_{n+1}}{dx_1}\left[1 - \frac{\partial\Phi}{\partial\delta_{n+1}}\right] = \frac{\partial\Phi}{\partial x_n}\frac{dx_n}{dx_1} + \frac{\partial\Phi}{\partial\delta_n}\frac{d\delta_n}{dx_1} .$$

Now $x_n = x_1 + \delta_2 + \dots + \delta_n$, so that

$$\frac{dx_n}{dx_1} = 1 + \sum_{k=2}^{n} \frac{d\delta_k}{dx_1}$$

and since $\frac{d\delta_k}{dx_1} > 0$ for $k = 2, \dots, n$ by hypothesis, $\frac{dx_n}{dx_1} > 0$.

Moreover, using lemma 3 and lemma 2, $\frac{\partial\Phi}{\partial x_n} > 0$.

$$\therefore \quad \frac{d\delta_{n+1}}{dx_1}\left\{1 - \frac{\partial\Phi}{\partial\delta_{n+1}}\right\} > \frac{\partial\Phi}{\partial\delta_n}\frac{d\delta_n}{dx_1}$$

and by the definition of $\Phi(x_n, \delta_n, \delta_{n+1})$,

$$\frac{d\delta_{n+1}}{dx_1}\left[1 + \frac{f(x_n + \delta_{n+1})}{f(x_n)}\right] > \left[1 + \frac{f(x_n - \delta_n)}{f(x_n)}\right]\frac{d\delta_n}{dx_1}$$

which implies

$$\frac{d\delta_{n+1}}{dx_1} > \frac{f(x_n) + f(x_{n-1})}{f(x_n) + f(x_{n+1})}\frac{d\delta_n}{dx_1}$$

but

$$\frac{d\delta_n}{dx_1} \geqslant \frac{f(x_1) + f(x_2)}{f(x_{n-1}) + f(x_n)}\frac{d\delta_2}{dx_1}$$

by hypothesis so that

$$\frac{d\delta_{n+1}}{dx_1} > \frac{f(x_n) + f(x_{n-1})}{f(x_n) + f(x_{n+1})} \frac{f(x_1) + f(x_2)}{f(x_{n-1}) + f(x_n)} \frac{d\delta_2}{dx_1}$$

$$\frac{d\delta_{n+1}}{dx_1} \geqslant \frac{f(x_1) + f(x_2)}{f(x_n) + f(x_{n+1})} \frac{d\delta_2}{dx_1} .$$

This completes the proof, since we have already seen that $\frac{d\delta_2}{dx_1} > 0$.

<u>Lemma 9</u> If $f_k(x)$ is continuous and $\lim_{k\to\infty} f_k(x) = \infty$ for $x \in [a, b]$ then for all λ there exists an m such that

$$f_n(x) > \lambda \quad \text{for all} \quad x \in [a, b] \quad \text{and} \quad n > m .$$

<u>Proof</u> Let $\varepsilon > 0$ and define $k(x)$ such that $f_n(x) > \lambda + \varepsilon$ for $n \geqslant k(x)$. Since $f_k(x)$ is continuous, then there exists some $\delta > 0$ such that

$$\left| f_k(y) - f_k(x) \right| < \varepsilon \quad \text{whenever} \quad |y - x| < \delta .$$

Therefore $- \varepsilon < f_k(y) - f_k(x) < \varepsilon$

$$f_k(y) > f_k(x) - \varepsilon > \lambda$$

for all y in some neighbourhood $(x - \delta, x + \delta)$ of x, U_x say.

Thus we can find a covering neighbourhood for every $x \in [a, b]$. The collection of these open intervals covers $[a, b]$, and hence by the Heine-Borel theorem we can select a finite subcover

$$U_{x_1}, \ldots U_{x_m} \quad \text{of} \quad [a, b] .$$

Define

$$K = \max_{i=1,\ldots m} \{k(x_i)\}$$

then $f_k(x) > \lambda$ for all $x \in [a, b]$, and all $k > K$.

Theorem 2.3.1 Let $\{x_k^*\}$, $\{\tilde{x}_k\}$, $\{x_k\}$ satisfy (2.3.2), where $\{x_k^*\}$ is the optimal inspection policy and $\tilde{x}_1 > x_1^*$, and let

$$\tilde{\delta}_k = \tilde{x}_k - \tilde{x}_{k-1} , \quad \delta_k^* = x_k^* - x_{k-1}^* , \quad \delta_k = x_k - x_{k-1} .$$

Then if f is PF_2, $\tilde{\delta}_n > \tilde{\delta}_{n-1}$ for some n.

Proof For all n $\delta_n^* > 0$, and by lemma 8 $\frac{d\delta_n}{dx_1} > 0$ if $\delta_n > 0$ so that $x_n = x_1 + \delta_2 + \ldots + \delta_n$ increases with x_1 for $x_1 > x_1^*$. Hence $x_n > x_n^*$, and by lemma 1 $\lim_{n\to\infty} x_n^* = \infty$, so that $\lim_{n\to\infty} x_n = \infty$ whenever $x_1 \geq x_1^*$.

Now by lemma 8,

$$\frac{d\delta_{n+1}}{dx_1} \geq \frac{f(x_1) + f(x_2)}{f(x_n) + f(x_{n+1})} \frac{d\delta_2}{dx_1} \quad \text{and} \quad \frac{d\delta_2}{dx_1} > 0 .$$

So that

$$\lim_{n\to\infty} \frac{d\delta_n}{dx_1} = \infty \quad \text{for} \quad x_1 \geq x_1^* ,$$

and in particular for all $x_1 \in [x_1^*, \tilde{x}_1]$. Using lemma 9, we can find an m such that whenever $x_1 \in [x_1^*, \tilde{x}_1]$

$$\frac{d\delta_n}{dx_1} > \frac{\tilde{x}_1}{\tilde{x}_1 - x_1^*} \quad \text{for} \quad n \geq m .$$

But, δ_m can be regarded as a function of x_1, so that

$$\tilde{\delta}_m - \delta_m^* = \int_{x_1^*}^{\tilde{x}_1} \frac{d\delta_m}{dx_1} dx_1 > \int_{x_1^*}^{\tilde{x}_1} \frac{\tilde{x}_1}{\tilde{x}_1 - x_1^*} dx_1 = \tilde{x}_1$$

$\therefore$ $\tilde{\delta}_m > \tilde{x}_1 = \tilde{\delta}_1$, so that $\tilde{\delta}_n > \tilde{\delta}_{n-1}$ for some $n \leq m$.

Theorem 2.3.2 Let $\{x_k^*\}$, $\{\tilde{x}_k\}$, $\{x_k\}$ satisfy (2.3.2) where $\{x_k^*\}$ is the optimal inspection policy and $\tilde{x}_1 < x_1^*$, and let

$$\tilde{\delta}_k = \tilde{x}_k - \tilde{x}_{k-1} \; , \quad \delta_k^* = x_k^* - x_{k-1}^* \; , \quad \delta_k = x_k - x_{k-1} \; .$$

Then if f is PF_2 $\tilde{\delta}_n < 0$ for some n.

Proof If $\delta_k^* > \delta_{k-1}^*$ for some k then using lemma 6 and lemma 7 $\delta_n^* \to \infty$ geometrically fast from some point on as $n \to \infty$ and hence, by a modification of Theorem 5 in Barlow, Hunter and Proschan (1963), we can show that this contradicts the fact that $\{x_k^*\}$ is an optimal policy. Hence $\{\delta_k^*\}$ is a decreasing sequence.

We will show that $\tilde{\delta}_k \geq 0$ for all k leads to a contradiction. By lemma 8

$$\frac{d\delta_k}{dx_1} > 0 \quad \text{if} \quad \delta_k \geq 0 \; ,$$

so that $x_k = x_1 + \delta_2 + \ldots + \delta_k$ increases with x_1 for $x_1 > \tilde{x}_1$.

Hence $x_k > \tilde{x}_k$, and by lemma 1 $\lim_{k\to\infty} \tilde{x}_k = \infty$, so that $\lim_{k\to\infty} x_k = \infty$ whenever $x_1 \geq \tilde{x}_1$. Now, by lemma 8

$$\frac{d\delta_{k+1}}{dx_1} \geq \frac{f(x_1) + f(x_2)}{f(x_k) + f(x_{k+1})} \frac{d\delta_2}{dx_1} \quad \text{and} \quad \frac{d\delta_2}{dx_1} > 0 \; .$$

So that

$$\lim_{k\to\infty} \frac{d\delta_k}{dx_1} = \infty \quad \text{for} \quad x_1 \geq \tilde{x}_1 \; ,$$

and in particular for all $x_1 \in [\tilde{x}_1, x_1^*]$. Using lemma 9, we can find an m such that whenever $x_1 \in [\tilde{x}_1, x_1^*]$

$$\frac{d\delta_k}{dx_1} > \frac{x_1^*}{x_1^* - \tilde{x}_1} \quad \text{for all} \quad k \geq m \; .$$

But, δ_m can be regarded as a function of x_1, so that

$$\delta_m^* - \tilde{\delta}_m = \int_{\tilde{x}_1}^{x_1^*} \frac{d\delta_m}{dx_1} dx_1 > \int_{\tilde{x}_1}^{x_1^*} \frac{x_1^*}{x_1^* - \tilde{x}_1} dx_1 = x_1^* = \delta_1^*$$

$$\therefore \quad \tilde{\delta}_m < \delta_m^* - \delta_1^* < 0 \quad \text{since} \quad \left\{\delta_k^*\right\} \quad \text{is decreasing.}$$

i.e. $\tilde{\delta}_m < 0$, contradicting the fact that $\tilde{\delta}_k \geq 0$ for all k.

This completes the proof.

Computing the optimal inspection policy for PF_2 densities

By virtue of theorems 2.3.1 and 2.3.2 we now propose the following algorithm for computing $\underline{x}^*$ when f is PF_2 .

As an approximation to x_1^* choose x_1 so that

$$x_1 F(x_1) = \frac{c_1}{c_2} \ .$$

This is that value of x_1 for which the cost of undetected failure during $[0, x_1]$ is balanced against the cost of the first inspection.

(i) Compute $\{x_k\}$ recursively from (2.3.2).

(ii) If $\delta_k > \delta_{k-1}$ for some k, reduce x_1 and repeat; if $\delta_k < 0$ for some k, increase x_1 and repeat.

Optimal policies in the Weibull case

For the Weibull distribution

$$f(t) = \frac{\beta}{\alpha}\left(\frac{t}{\alpha}\right)^{\beta-1} \exp\left\{-\left(\frac{t}{\alpha}\right)^{\beta}\right\}$$

and (2.3.2) becomes

$$(x_{k+1}-x_k)-(x_k-x_{k-1}) = \frac{\left[\exp\left\{-\left(\frac{x_{k-1}}{\alpha}\right)^{\beta}\right\}-\exp\left\{-\left(\frac{x_k}{\alpha}\right)^{\beta}\right\}\right]-\left[\exp\left\{-\left(\frac{x_k}{\alpha}\right)^{\beta}\right\}-\exp\left\{-\left(\frac{x_{k+1}}{\alpha}\right)^{\beta}\right\}\right]}{\frac{\beta}{\alpha}\left(\frac{x_k}{\alpha}\right)^{\beta-1}\exp\left\{-\left(\frac{x_k}{\alpha}\right)^{\beta}\right\}}$$

$$-\frac{c_1}{c_2}$$

with $z_k = \frac{x_k}{\alpha}$ and $K = \frac{c_1}{\alpha c_2}$ as before, after some algebra this can be written as

$$A_k(z_k)e^{-z_{k+1}^{\beta}} + B_k(z_k, z_{k-1}) = z_{k+1} \tag{2.3.3}$$

where

$$A_k(z_k) = \frac{z_k^{1-\beta}\exp\left(z_k^{\beta}\right)}{\beta}$$

and

$$B_k(z_k, z_{k-1}) = \left\{\exp\left(-z_{k-1}^{\beta}\right) - 2\exp\left(-z_k^{\beta}\right)\right\} A_k(z_k) + 2z_k - z_{k-1} - K .$$

Equation (2.3.3) is of the form $g(x) = x$, and can be solved by any of the standard numerical techniques, such as Wegsteins method. Thus, given z_1 the sequence $\{z_k\}$ can be calculated from (2.3.3), and the algorithm used to find z_1^* in terms of $K = \frac{c_1}{\alpha c_2}$ and β.

2.4 An approximation to the optimal policy when the inspection cost is low

If the cost of a single inspection is very small so that $c_1 << c_2E(T)$ then it is reasonable to assume that the optimal policy will consist of many inspections, i.e. $E(N)$ will be large. In a recent paper Keller (1974) considers the case when inspections are "... so frequent that they can be described by a smooth density ... which denotes the number of checks per unit time ...".

If the rate of inspections at time t is $\phi(t)$ then the time between inspections is approximately $\frac{1}{\phi(t)}$ so that Keller proposes that $\{x_i\}$ should be given by

$$x_1 = \frac{1}{\phi(0)}, \quad x_{n+1} = x_n + \frac{1}{\phi(x_n)}, \quad n = 1, 2, \ldots \tag{2.4.1}$$

Keller approximates $x_n - t$ in the interval $x_{n-1} < t < x_n$ by an 'average' value $(x_n - x_{n-1})/2$, so that $x_n - t \approx \frac{1}{2\phi(t)}$. Note that this is a time average and $f(t)$ is not taken into account at this stage.

Since the number of inspections made up to time t is approximately

$$\int_0^t \phi(u)du = \Phi(t) \quad \text{say,} \tag{2.4.2}$$

then if the failure occurs at time t the total cost for model I is

$$c(t; \phi) = c_1 \Phi(t) + \frac{c_2}{2\phi(t)}, \tag{2.4.3}$$

and the expected cost is

$$E\{c(T; \phi)\} = \int_0^\infty \left\{c_1 \Phi(t) + \frac{c_2}{2\phi(t)}\right\} f(t)dt \quad . \tag{2.4.4}$$

Using the calculus of variations Keller showed that $E\{c(T;\ \phi)\}$ is minimised when

$$\phi(t) = \left\{\frac{c_2\ h(t)}{2c_1}\right\}^{\frac{1}{2}} \tag{2.4.5}$$

where $h(t)$ is the hazard rate function, and the minimum cost is then

$$c_{min} = \sqrt{2c_1c_2}\int_0^\infty \sqrt{f(t)\ \bar{F}(t)}\ dt \quad . \tag{2.4.6}$$

If this method is applied to the new cost model II, the equation corresponding to (2.4.3) is

$$c(t;\ \phi) = c_1\ \Phi(t) + \frac{c_2}{\phi(t)} \quad ,$$

so that any results for model II are found by replacing c_2 by $2c_2$ in the corresponding results for model I.

Keller applied his method to the uniform distribution, and the exponential distribution with $c_1/\alpha c_2 << 1$ and showed that in the latter case this leads to a periodic policy with $x = \sqrt{2\alpha c_1/c_2}$. This is the same as (2.1.3), the approximate solution that Barlow and Hunter (1960) gave for large α. Using (2.4.6) we have $c_{min} = \sqrt{2\alpha c_1 c_2}$, which differs from the exact solution by only c_1 (which is small, by assumption).

The two distributions that Keller considered both have the property that $h(0) > 0$, so that from (2.4.1) x_1 is always finite. However if $h(0)\ (= f(0)) = 0$, then (2.4.1) gives $x_1 = \infty$; Keller did not consider such cases, but this difficulty can be avoided if $\{x_i\}$ is defined by

$$\int_0^{x_n} \phi(t)dt = n, \quad \text{i.e.} \quad \Phi(x_n) = n \quad \text{or} \quad x_n = \Phi^{-1}(n) \quad . \tag{2.4.7}$$

Unfortunately this can introduce a further problem, namely that

$$\phi(t) = \sqrt{\frac{c_2\ h(t)}{2c_1}}$$

is often difficult to integrate analytically. For example the gamma distribution with increasing hazard rate has

$$h(t) = \frac{\frac{1}{\alpha}\left(\frac{t}{\alpha}\right)^{\beta-1} e^{-t/\alpha}}{\Gamma(\beta,\ t/\alpha)}$$

so that $h(0) = 0$, and neither (2.4.1) nor (2.4.7) are suitable for evaluating $\{x_i\}$. In such cases Keller's method has no real advantage over the exact solution considered in the earlier sections of this chapter.

Keller's method applied to the Weibull distribution

The Weibull distribution $\bar{F}(t) = \exp\{-(t/\alpha)^\beta\}$ has the property that the hazard rate function takes on a particularly simple form.

With

$$h(t) = \frac{\beta}{\alpha}\left(\frac{t}{\alpha}\right)^{\beta-1},$$

(2.4.5) gives

$$\phi(t) = \sqrt{\frac{\beta c_2}{2c_1\ \alpha^\beta}}\ t^{\frac{\beta-1}{2}} .$$

Note that Keller's method (2.4.1) for evaluating $\{x_i\}$ cannot be used, since $\phi(0) = 0$. However (2.4.7) can be used, and this gives (for model I)

$$\sqrt{\frac{\beta c_2}{2c_1\ \alpha^\beta}}\ \frac{2}{\beta+1}\ x_n^{\frac{\beta+1}{2}} = n$$

$$x_n = \left\{\frac{(\beta+1)^2\ c_1\ \alpha^\beta\ n^2}{2\beta\ c_2}\right\}^{\frac{1}{\beta+1}} . \qquad (2.4.8)$$

Replacing c_2 by $2c_2$ in (2.4.8) gives

$$x_n = \left\{\frac{(\beta+1)^2\ c_1\ \alpha^\beta\ n^2}{4\beta\ c_2}\right\}^{\frac{1}{\beta+1}} \quad \text{for model II} . \qquad (2.4.9)$$

The minimum cost for model I is given by (2.4.6):

$$c_{min} = \sqrt{2c_1c_2} \int_0^\infty \sqrt{\frac{\beta}{\alpha}} \left(\frac{t}{\alpha}\right)^{\frac{\beta-1}{2}} \exp\left\{-\left(\frac{t}{\alpha}\right)^\beta\right\} dt$$

$$= \sqrt{2c_1c_2}\ \sqrt{\frac{\alpha}{\beta}} \int_0^\infty \left(\frac{t}{\alpha}\right)^{\frac{1-\beta}{2}} \exp\left\{-\left(\frac{t}{\alpha}\right)^\beta\right\} \frac{\beta}{\alpha} \left(\frac{t}{\alpha}\right)^{\beta-1} dt$$

$$= \sqrt{\frac{2c_1c_2\ \alpha}{\beta}} \int_0^\infty \left(\frac{t}{\alpha}\right)^{\beta\left(\frac{1}{2\beta} - \frac{1}{2}\right)} \exp\left\{-\left(\frac{t}{\alpha}\right)^\beta\right\} d\left(\frac{t}{\alpha}\right)^\beta$$

$$c_{min} = \sqrt{\frac{2c_1c_2\ \alpha}{\beta}}\ \Gamma\left(\frac{1}{2\beta} + \frac{1}{2}\right) \qquad (2.4.10)$$

2.5 General comments on optimal policies

With the exception of the exponential case, calculating optimal policies in general poses a severe computational problem. Moreover, no theory exists for calculating optimal policies when $f(t)$ is not PF_2. For failure distributions for which $\bar{F}(t)$ cannot be expressed in terms of elementary functions (such as the gamma and truncated normal), calculating the optimal policy even for model I requires the use of a computer and is slow. Since model II requires the numerical solution of an equation many times during each iteration, solving the problem becomes expensive in terms of computer time.

It would seem that the computational difficulties which arise from using the inspection times $\{x_i\}$ as control variables would be avoided if we considered only policies which depended on a single variable. Clearly, by considering policies which depend on only one variable, we are restricted to a subset of all possible policies, which in general, will not include the optimal policy. However the subsequent chapters of this thesis reveal that this restriction can sometimes be compensated by a considerable reduction in computational effort without paying too great an extra cost due to the use of a restricted subset of policies.

Chapter 3

Periodic Inspection Policies

In this chapter we consider the most commonly used inspection policy, the periodic policy given by $x_n = nx$ for some $x > 0$. Barlow, Hunter and Proschan (1963) have shown that the optimal inspection policy has decreasing inspection intervals when the failure density is PF_2, so that periodic policies cannot be optimal in this case. Although periodic policies are simple to use, the best periodic policy can sometimes given an expected cost in considerable excess of the optimal non periodic policy. We first derive an expression for the expected total costs for models I and II.

3.1 Periodic inspection policies

If $m(x)$ is the mean number of inspections needed to detect the failure when the inspection interval is x, then

$$m(x) = E(N) = \sum_{j=0}^{\infty} \bar{F}(jx) \quad . \tag{3.1.1}$$

The expected time at which the failure is detected is

$$E(x_N) = E(Nx) = xE(N) = xm(x)$$

and so the expected total cost for model I is

$$E\{c(T;\ x)\} = c_1\ m(x) + c_2\ xm(x) - c_2\ E(T)$$

$$= (c_1 + c_2\ x)\ m(x) - c_2\ E(T) \quad . \tag{3.1.2}$$

Since $x_n - x_{n-1} = x$ for $n \geqslant 1$, $E(x_N - x_{N-1}) = x$ and so the expected total cost for model II is

$$E\{c(T;\ x)\} = c_1\ m(x) + c_2\ x \quad . \tag{3.1.3}$$

The best periodic policy in either case is found by minimising (3.1.2) or (3.1.3) with respect to x.

We can obtain an approximation to m(x) (and hence the expected costs) in terms of E(T) and f(0) using the Euler-MacLaurin summation formula

$$\sum_{j=1}^{\infty} g(j) = \int_0^{\infty} g(u)du - \frac{1}{2} g(0) - \frac{1}{12} g'(0) - \ldots - \frac{B_{2n}}{(2n)!} g^{(2n-1)}(0) + \ldots$$

where B_n is the nth Bernoulli number (see, for example Knopp, 1947).

Putting $g(j) = \bar{F}(jx)$ we have, using (3.1.1)

$$m(x) = \bar{F}(0) + \int_0^{\infty} \bar{F}(ux)du - \frac{1}{2} \bar{F}(0) - \frac{x\bar{F}'(0)}{12}$$

approximately, or

$$m(x) \simeq \frac{1}{2} + \frac{\mu}{x} + \frac{xf(0)}{12} \qquad (3.1.4)$$

where $\mu = E(T)$. Knopp (p. 532) points out that usually the error in truncating the Euler-MacLaurin series is of the same sign as, but smaller in absolute value than, the first term neglected. The approximation (3.1.4) will be at its best when x is small since the first term neglected is

$$- \frac{x^3}{720} f''(0) \quad .$$

Note that the expected time to detection of failure within an interval is given by

$$E(T - x_{N-1}) = \mu - x[m(x) - 1]$$

$$\simeq \frac{x}{2} - \frac{x^2 f(0)}{12}$$

which is a generalisation of the result obtained by Duncan (1956) in the exponential case for small x.

3.2 Periodic policy for model I

Using (3.1.2) and (3.1.4) the expected cost for model I is approximately

$$(c_1 + c_2\, x)\left\{\frac{1}{2} + \frac{\mu}{x} + \frac{xf(0)}{12}\right\} - c_2\, \mu$$

which is minimised when

$$\frac{x^3 f(0)}{6} + \left\{\frac{1}{2} + \frac{c_1\, f(0)}{12\, c_2}\right\} x^2 - \frac{\mu\, c_1}{c_2} = 0 \quad . \qquad (3.2.1)$$

For most life distributions $f(0) = 0$ (The exponential is a notable exception), and in this case,

$$x = \sqrt{\frac{2\mu\, c_1}{c_2}} \quad . \qquad (3.2.2)$$

Exponential case

In this case we have $f(0) = 1/\alpha$, so that from (3.2.1) we get

$$\frac{1}{6}\left(\frac{x}{\alpha}\right)^3 + \left\{\frac{1}{2} + \frac{k}{12}\right\}\left(\frac{x}{\alpha}\right)^2 - k = 0 \qquad (3.2.3)$$

where $k = c_1/\alpha c_2$, and so for given k we can find x/α .

Now for the exponential case the periodic solution is in fact the optimal solution, and it is interesting to compare the exact solution and the approximate solution given by (3.2.3). We have seen from (2.1.2) that x/α is given exactly by

$$e^{x/\alpha} - x/\alpha - 1 = k \quad ,$$

and we need numerical procedures for determining x/α. Now let us approximate $e^{x/\alpha}$. For x/α small, we have

$$e^{x/\alpha} \simeq 1 + \frac{x}{\alpha} + \frac{1}{2}\left(\frac{x}{\alpha}\right)^2 \quad ,$$

which gives

$$\frac{x}{\alpha} \simeq \sqrt{2k} \quad .$$

Note that this approximation agrees with (3.2.2), the approximate solution with $f(0) = 0$. The next term in the series for $e^{x/\alpha}$ is $\frac{1}{6}\left(\frac{x}{\alpha}\right)^3$ and if we include this we have

$$\frac{1}{6}\left(\frac{x}{\alpha}\right)^3 + \frac{1}{2}\left(\frac{x}{\alpha}\right)^2 - k = 0 .$$

This equation differs from (3.2.3) by the term $\frac{k}{12}\left(\frac{x}{\alpha}\right)^2$.

Table 1 gives the exact inspection interval and the approximations given by (3.2.3) and $x/\alpha = \sqrt{2k}$. We see that, as expected (3.2.3) is the better of the two, holding over a greater range of k.

Weibull case

In the Weibull case with unit scale parameter,

$$\bar{F}(t) = \exp(-t^{\beta})$$

and the mean number of inspections, $m(x)$, is given by

$$m(x) = \sum_{j=0}^{\infty} \exp\{-(jx)^{\beta}\} \quad . \tag{3.2.4}$$

For increasing hazard rate distributions $\beta \geq 1$, so that from some point on each term in the series for $m(x)$ in (3.2.4) is not greater than the corresponding term in the series

$$\sum_{j=0}^{\infty} \exp\{-jx\} \quad .$$

The error in summing (3.2.4) to n terms will therefore be less than

$$\sum_{j=n}^{\infty} \exp\{-jx\} = \frac{\exp(-nx)}{1 - \exp(-x)}$$

so we can find $m(x)$ to an absolute error of less than ε by summing to n terms where

Table 1

Exact and approximate inspection intervals in the exponential case - model I (α=1). $\hat{x}$ and $\tilde{x}$ are given by

$$\frac{\hat{x}^3}{6} + \left(\frac{1}{2} + \frac{k}{12}\right)\hat{x}^2 - k = 0 \quad \text{and} \quad \tilde{x} = \sqrt{2k} \ .$$

k	Exact	$\hat{x}$	$\tilde{x}$
0.01	0.1382	0.1382	0.1414
0.03	0.2354	0.2353	0.2449
0.05	0.3004	0.3004	0.3162
0.07	0.3522	0.3521	0.3742
0.09	0.3963	0.3961	0.4243
0.1	0.4162	0.4161	0.4472
0.3	0.6863	0.6851	0.7746
0.5	0.8577	0.8549	1.0000
0.7	0.9893	0.9844	-
0.9	1.0979	1.0905	-
1.0	1.1463	1.1375	-
3.0	1.7490	1.7034	-
5.0	2.0908	2.0000	-
7.0	2.3357	2.1975	-
9.0	2.5282	2.3423	-

$$\frac{e^{-nx}}{1 - e^{-x}} < \varepsilon , \quad \text{i.e.} \quad n > - \frac{\ln\{\varepsilon(1 - e^{-x})\}}{x} \quad .$$

For given x, the expected total cost can be found by combining (3.1.2) and (3.2.4), and using a numerical search procedure the optimal value of x can be found as a function of $k = c_1/\alpha c_2$ and β .

When $\beta > 1$ $f(0) = 0$ and approximation (3.2.2) applies, and with $E(T) = \alpha\Gamma(1 + 1/\beta)$ this gives

$$\frac{x}{\alpha} \simeq \{2k\Gamma(1 + 1/\beta)\}^{\frac{1}{2}} \quad . \tag{3.2.5}$$

A table of values of this approximation is given in Table 2, along with the exact values obtained by search by golden section.

Table 2

Exact and approximate inspection intervals in the Weibull case - model I ($\alpha = 1$). Upper entries are square root approximations (3.2.5), lower entries are exact values.

k \ β	1.25	1.5	1.75	2.0	3.0	4.0
0.01	0.1365 0.1352	0.1344 0.1339	0.1335 0.1333	0.1331 0.1331	0.1336 0.1336	0.1346 0.1346
0.05	0.3052 0.2972	0.3005 0.2969	0.2984 0.2973	0.2977 0.2977	0.2988 0.2991	0.3011 0.3011
0.09	0.4094 0.3939	0.4031 0.3956	0.4004 0.3977	0.3994 0.3994	0.4009 0.4020	0.4039 0.4044
0.1	0.4316 0.4141	0.4249 0.4162	0.4220 0.4189	0.4210 0.4210	0.4226 0.4239	0.4258 0.4226
0.5	0.9651 0.8580	0.9501 0.8760	0.9437 0.9036	0.9414 0.9405	0.9450 1.2595	0.9521 1.2703
0.9	1.2948 1.0927	1.2747 1.1142	1.2661 1.1542	1.2630 1.2144	1.2678 1.3394	1.2773 1.3088
1.0	1.3648 1.1389	1.3437 1.1600	1.3346 1.2003	1.3313 1.2584	1.3364 1.3435	1.3464 1.3164
5.0	3.0519 1.9694	3.0046 1.8946	2.9843 1.8319	2.9770 1.7706	2.9883 1.5688	3.0107 1.4431

3.3 Periodic policy for model II

Using (3.1.3) and (3.1.4) the expected cost under model II is approximately

$$c_1\left\{\frac{1}{2} + \frac{\mu}{x} + \frac{x\ f(0)}{12}\right\} + c_2\ x$$

which is minimised when

$$x = \left\{\mu\ c_1 \Big/ \left(c_2 + \frac{c_1\ f(0)}{12}\right)\right\}^{\frac{1}{2}} \tag{3.3.1}$$

or $$x = \sqrt{\frac{\mu\ c_1}{c_2}} \quad \text{if} \quad f(0) = 0\ . \tag{3.3.2}$$

By comparing (3.2.2) with (3.3.2) we see that the approximate optimal intervals for models I and II differ by a factor of $\sqrt{2}$ in the case when $f(0) = 0$.

Exponential case

In the exponential case $f(0) = 1/\alpha$ and (3.3.1) gives

$$\frac{x}{\alpha} = \sqrt{\frac{k}{1 + k/12}} \tag{3.3.3}$$

which is an approximation to the exact solution

$$\frac{x}{\alpha} = -\ \ell n\left\{1 + \frac{k}{2} - \sqrt{\frac{k^2}{4} + k}\right\}\ .$$

From Table 3 we see that for all tabulated values of k the square root approximation (3.3.3) is good, and particularly accurate when $k < 1$.

Weibull case

In the Weibull case the expected total cost is found by combining (3.1.3) and (3.2.4), and using a numerical search procedure the optimmal inspection interval can be found as a function of $k = c_1/\alpha c_2$ and β. An approximation to this interval is given by (3.3.2) which

Table 3

Exact and approximate inspection intervals in the exponential case - model II ($\alpha = 1$).

The approximation is given by (3.3.3).

k	Exact	Approx.
0.01	0.1000	0.1000
0.03	0.1730	0.1730
0.05	0.2231	0.2231
0.07	0.2638	0.2638
0.09	0.2989	0.2989
0.1	0.3149	0.3149
0.3	0.5411	0.5410
0.5	0.6932	0.6928
0.7	0.8140	0.8133
0.9	0.9163	0.9150
1.0	0.9625	0.9608
3.0	1.5668	1.5492
5.0	1.9249	1.8787
7.0	2.1847	2.1026
9.0	2.3896	2.2678

gives

$$\frac{x}{\alpha} \simeq \sqrt{k}\,\Gamma(1 + 1/\beta) \quad . \qquad (3.3.4)$$

A comparison of this approximation with the exact values found by search by golden section is given in Table 4.

Table 4

Exact and approximate inspection intervals in the Weibull case - model II ($\alpha = 1$). Upper entries are square root approximation (3.3.4), lower entries are exact value.

k \ β	1.25	1.5	1.75	2.0	3.0	4.0
0.01	0.0965 0.0965	0.0950 0.0950	0.0944 0.0944	0.0941 0.0941	0.0945 0.0945	0.0952 0.0952
0.05	0.2158 0.2156	0.2125 0.2124	0.2110 0.2110	0.2105 0.2105	0.2113 0.2113	0.2129 0.2129
0.09	0.2895 0.2889	0.2850 0.2848	0.2831 0.2830	0.2824 0.2824	0.2835 0.2835	0.2856 0.2856
0.1	0.3052 0.3045	0.3005 0.3001	0.2984 0.2983	0.2977 0.2977	0.2988 0.2989	0.3011 0.3011
0.5	0.6824 0.6723	0.6718 0.6657	0.6673 0.6646	0.6657 0.6657	0.6682 0.6667	0.6732 0.6431
0.9	0.9156 0.8892	0.9014 0.8829	0.8953 0.8857	0.8931 0.8930	0.8965 1.0005	0.9032 1.1354
1.0	0.9651 0.9339	0.9501 0.9276	0.9437 0.9315	0.9414 0.9411	0.9450 1.0765	0.9521 1.1657
5.0	2.1580 1.8181	2.1246 1.7609	2.1102 1.7200	2.1050 1.6800	2.1130 1.5260	2.1289 1.4181

3.4 Comments on approximations to optimal periodic policies

The approximations (3.2.1) and (3.3.1), depending only on c_1, c_2, $E(T)$ and $f(0)$ have the advantage that they require only a minimum of information about the time to failure distribution, and since $f(0)$ is generally zero, we require only $E(T)$ to find an approximation to the optimal inspection interval.

Tables 2 and 4 give an indication of the accuracy of these approximations in the Weibull case, and we see that for both models I and II the approximations are fair for small and moderate values of k, but particularly good for model II with $k < 1$. A natural measure of the loss in using these approximations in the general case is the percentage increase in the expected total cost due to using the approximate interval in place of the exact interval. Tables 5 and 6 give this increase in the Weibull case.

In the exponential case this loss is less than 1% for k in the range $0.1 < k < 10$, and in the Weibull case it is less than 10% for $0.1 \leqslant k \leqslant 2$, $1 \leqslant \beta \leqslant 3$ with model I, and less than 5% for $0.1 \leqslant k \leqslant 2$, $1 \leqslant \beta \leqslant 4$ with model II. A feature of both tables is that the percentage increase in cost has turning points in the direction of k increasing.

Comparing Table 5 with Table 6 we see that for all tabulated values of k and β, the loss under model II is less than that for model I.

Table 5

Percentage increase in the mean total cost of the optimal periodic policy due to using the approximate inspection interval. Model I, Weibull case ($\alpha = 1$) .

k \ β	1.0	1.25	1.50	1.75	2.0	2.5	3.0	3.5	4.0
0.1	0	0	0	0	0	0	0	0	0
0.2	0	0	0	0	0	0	0	0	0
0.3	0	0	0	0	0	0	1	8	17
0.4	0	0	0	0	0	0	5	13	22
0.5	0	0	0	0	0	1	6	13	19
0.6	0	1	0	0	0	1	5	9	13
0.7	0	1	1	0	0	1	3	5	6
0.8	0	1	0	0	0	0	2	2	2
0.9	0	1	1	0	0	0	1	0	0
1.0	0	2	1	1	0	0	0	0	0
1.1	0	2	1	0	0	0	0	1	1
1.2	0	2	1	1	1	0	1	2	3
1.3	0	2	2	1	1	1	2	3	6
1.4	0	2	2	2	1	1	3	5	8
1.5	0	3	2	2	2	2	4	7	10
1.6	0	3	2	2	2	3	5	8	12
1.7	0	3	3	3	3	4	7	10	13
1.8	0	3	3	3	3	5	8	11	15
1.9	0	3	3	3	4	6	9	13	16
2.0	0	3	3	4	4	7	10	14	17
3.0	0	5	6	7	9	14	18	21	23
4.0	0	7	8	10	13	17	21	23	26
5.0	0	8	10	12	15	19	22	24	26
6.0	0	9	11	14	16	20	23	25	27
7.0	0	9	12	15	17	20	23	25	26
8.0	0	10	13	15	17	20	23	25	26
9.0	0	10	13	15	17	20	23	24	26
10.0	0	11	13	16	18	20	22	24	25

Table 6

Percentage increase in the mean total cost of the optimal periodic policy due to using the approximate inspection interval. Model II, Weibull case ($\alpha = 1$) .

k \ β	1.0	1.25	1.50	1.75	2.0	2.5	3.0	3.5	4.0
0.1	0	0	0	0	0	0	0	0	0
0.2	0	0	0	0	0	0	0	0	0
0.3	0	0	0	0	0	0	0	0	0
0.4	0	0	0	0	0	0	0	0	0
0.5	0	0	0	0	0	0	0	0	0
0.6	0	0	0	0	0	0	0	0	0
0.7	0	0	0	0	0	0	0	0	0
0.8	0	0	0	0	0	0	0	1	2
0.9	0	0	0	0	0	0	0	1	3
1.0	0	0	0	0	0	0	1	2	3
1.1	0	0	0	0	0	0	1	2	3
1.2	0	0	0	0	0	0	1	2	3
1.3	0	0	0	0	0	0	1	1	2
1.4	0	0	0	0	0	0	1	1	1
1.5	0	0	0	0	0	0	0	1	1
1.6	0	0	0	0	0	0	0	0	0
1.7	0	0	0	0	0	0	0	0	0
1.8	0	0	0	0	0	0	0	0	0
1.9	0	0	0	0	0	0	0	0	0
2.0	0	0	0	0	0	0	0	0	0
3.0	0	0	0	0	1	1	2	3	4
4.0	0	1	1	1	2	3	5	6	8
5.0	0	1	1	2	3	5	7	8	10
6.0	0	1	2	3	4	6	8	10	11
7.0	0	2	2	4	5	7	9	10	12
8.0	0	2	3	4	6	8	10	11	12
9.0	0	2	3	5	6	8	10	11	12
10.0	0	3	4	5	7	9	10	12	13

Chapter 4

Mean Residual Life Inspection Policies

4.1 Mean residual life inspection policies

We have mentioned in section 1.2 that Bryson and Siddiqui (1969) proposed decreasing mean residual life as one of seven criteria for classifying ageing systems. They showed that decreasing mean residual life is a consequence of an increasing hazard rate, and gave a counter example to show that the converse is not necessarily true.

The mean residual life function is defined by

$$\mu(t) = \frac{\int_t^\infty \bar{F}(x)\, dx}{\bar{F}(t)} .$$

That is, $\mu(t)$ is the expected life measured from time t given survival until at least t. Clearly $\mu(0) = E(T)$. We will consider densities for which $\mu(t)$ is decreasing in t, so that $\mu(t) < \mu(0)$, $(t > 0)$.

Suppose that the system was found to be working at time x_n, then $\mu(x_n)$ is the expected time that the system will remain in a working state, and this can be used as a guide to when the next inspection should be scheduled. It seems reasonable that the time to the next inspection should be an increasing function of $\mu(x_n)$, say $G\{\mu(x_n)\}$ where $G(0) = 0$. A natural choice for the function G is $G(x) = \lambda x$ where $\lambda > 0$. This leads us to:

Definition

A mean residual life (MRL) policy for a density f is a sequence of inspection times $\{x_1, x_2, \ldots\}$ satisfying

$$x_{n+1} - x_n = \lambda\mu(x_n) \qquad n = 0, 1, \ldots \quad \text{for some } \lambda > 0. \tag{4.1.1}$$

That is, each inspection is scheduled in such a way that the intervals are equal to a constant proportion of the mean residual life after the previous inspection. With $x_0 = 0$ we have $x_1 = \lambda\mu(0) = \lambda E(T)$.

When $f(t)$ has an increasing hazard rate $\mu(t)$ is decreasing and in particular

$$\mu(x_n) < \mu(x_{n-1}) \qquad n = 1, 2, \ldots$$

$$\frac{x_{n+1} - x_n}{\lambda} < \frac{x_n - x_{n-1}}{\lambda} \qquad \text{using (4.1.1)}$$

$$x_{n+1} - x_n < x_n - x_{n-1} \quad ,$$

i.e. the inspection intervals are decreasing if $h(t)$ is increasing. Similarly $\{x_n - x_{n-1}\}$ is an increasing sequence if $h(t)$ is decreasing.

Mean residual life policy for the exponential case

We have seen in section 1.2 that constant mean residual life is a property of only the exponential family of densities, and with $\mu(t) = \alpha$ we have from (4.1.1)

$$x_{n+1} - x_n = \lambda\alpha$$

so that the inspection intervals are constant, i.e. a MRL policy is a periodic policy in this case.

4.2 Mean residual life policy illustrated in the Weibull case

Although MRL policies are simple in conception, the amount of computation involved in finding the best MRL policy is comparable with that of finding the optimal policy. We will solve the problem in the Weibull case as an illustration of this computational difficulty.

With

$$\bar{F}(t) = \exp\left\{-\left(\frac{t}{\alpha}\right)^{\beta}\right\} ,$$

$$\int_t^\infty \bar{F}(u)du = \int_t^\infty \exp\left\{-\left(\frac{u}{\alpha}\right)^{\beta}\right\} du$$

$$= \frac{\alpha}{\beta} \int_{\left(\frac{t}{\alpha}\right)^{\beta}}^{\infty} e^{-u} u^{1/\beta - 1} du = \frac{\alpha}{\beta} \Gamma\{1/\beta, (t/\alpha)^{\beta}\} ,$$

where

$$\Gamma(a, x) = \int_x^\infty e^{-u} u^{a-1} du$$

is the incomplete gamma function. Therefore,

$$\mu(t) = \frac{\int_t^\infty \bar{F}(u)\, du}{\bar{F}(t)} = \frac{\alpha}{\beta} \exp\left\{\left(\frac{t}{\alpha}\right)^{\beta}\right\}\Gamma\left\{1/\beta, \left(\frac{t}{\alpha}\right)^{\beta}\right\} .$$

So that from (4.1.1) a MRL policy in the Weibull case is given by

$$x_{n+1} - x_n = \frac{\lambda\alpha}{\beta} \exp\left\{\left(\frac{x_n}{\alpha}\right)^{\beta}\right\}\Gamma\left\{1/\beta, \left(\frac{x_n}{\alpha}\right)^{\beta}\right\} \tag{4.2.1}$$

and since

$$x_n = \sum_{j=1}^{n} (x_j - x_{j-1})$$

this gives

$$x_n = \frac{\lambda\alpha}{\beta} \sum_{j=1}^{n} \exp\left\{\left(\frac{x_{j-1}}{\alpha}\right)^{\beta}\right\}\Gamma\left\{1/\beta, \left(\frac{x_{j-1}}{\alpha}\right)^{\beta}\right\} ,$$

$$n = 1, 2, \ldots \tag{4.2.2}$$

When $\beta = 1$, the failure distribution is exponential and (4.2.2) reduces to $x_n = n\lambda\alpha$. For $\beta > 1$, (4.2.2) cannot be simplified so the expected cost for a MRL policy must be found numerically.

Evaluation of the expected cost

Under cost model I, the expected penalty cost is proportional to $E(x_N - T)$ where

$$E(x_N - T) = \sum_{j=1}^{\infty} \int_{x_{j-1}}^{x_j} (x_j - t)\, f(t)\, dt \quad . \qquad (4.2.3)$$

Now let $\sum_n$ be the sum to n terms of the above series and let $\bar{F}_j = \bar{F}(x_j)$ so that

$$\textstyle\sum_n = \sum_{j=1}^{n} x_j\left(\bar{F}_{j-1} - \bar{F}_j\right) - \int_0^{x_n} t\, f(t)\, dt \quad .$$

Rearranging the terms of the finite sum on the right hand side of the above equation, and integrating by parts gives

$$\textstyle\sum_n = \sum_{j=1}^{n} \bar{F}_{j-1}\,(x_j - x_{j-1}) - x_n\,\bar{F}_n - \left\{- x_n\,\bar{F}_n + \int_0^{x_n} \bar{F}(t)\, dt\right\} \quad ,$$

$$\textstyle\sum_n = \frac{\lambda\alpha}{\beta} \sum_{j=1}^{n} \Gamma\left\{1/\beta,\ \left(\frac{x_{j-1}}{\alpha}\right)^{\beta}\right\} - \frac{\alpha}{\beta}\,\gamma\left\{1/\beta,\ \left(\frac{x_n}{\alpha}\right)^{\beta}\right\} \qquad (4.2.4)$$

using (4.2.1) where

$$\gamma(a, x) = \int_0^{x} e^{-u}\, u^{a-1}\, du \quad .$$

The error E_n in approximating $E(x_N - T)$ by $\sum_n$ is

$$\sum_{j=n+1}^{\infty} \int_{x_{j-1}}^{x_j} (x_j - t)\, f(t)\, dt \quad ,$$

now $x_j - t < x_j - x_{j-1}$ for $x_{j-1} < t < x_j$ so that

$$E_n < \sum_{j=n+1}^{\infty} (x_j - x_{j-1}) \int_{x_{j-1}}^{x_j} f(t)\, dt$$

and since $\{x_j - x_{j-1}\}$ is decreasing for $\beta > 1$

$$E_n < (x_{n+1} - x_n) \sum_{j=n+1}^{\infty} \int_{x_{j-1}}^{x_j} f(t)\, dt$$

$$E_n < (x_{n+1} - x_n)\, \bar{F}(x_n) \quad ,$$

so we may use (4.2.4) to calculate $E(x_N - T)$ as accurately as we please, since $(x_{n+1} - x_n)\, \bar{F}(x_n) \to 0$ as $n \to \infty$.

We have considered $E(x_N - T)$ rather than $E(x_N) - E(T)$ since it is difficult to obtain bounds on the error of the truncated series of $E(x_N)$.

For cost model II the expected penalty cost is proportional to $E(x_N - x_{N-1})$. Now

$$E(x_N - x_{N-1}) = \sum_{j=1}^{\infty} (x_j - x_{j-1})(\bar{F}_{j-1} - \bar{F}_j) \quad , \qquad (4.2.5)$$

and by summing the series in (4.2.5) to n terms the error is once again bounded by $(x_{n+1} - x_n)\, \bar{F}(x_n)$ provided $\beta > 1$.

For either cost model the expected number of inspections is

$$\sum_{j=0}^{\infty} \bar{F}(x_j)$$

and so using (4.2.2) to generate the $\{x_i\}$ and (4.2.4) to calculate the penalty cost for model I, or (4.2.5) for model II, we can find the expected total cost as a function of λ, and use the method of search by golden section to minimise this cost with respect to λ for values of $k = c_1/\alpha c_2$ and β .

The values of λ which minimise the expected cost are given in Tables 7 and 8 for models I and II respectively, for a selection of values of k and β. When $\beta = 1$, the policy is periodic and the tabulated value of λ is the same as the value of x in Tables 1 and 3. This provides a useful check on the computation.

k \ β	1.0	1.25	1.5	1.75	2.0	2.5	3.0	3.5	4.0
0.01	0.1382	0.1778	0.2131	0.2451	0.2745	0.3272	0.3742	0.4170	0.4564
0.05	0.3004	0.3827	0.4548	0.5189	0.5766	0.6754	0.7547	0.8164	0.8631
0.09	0.3963	0.5002	0.5884	0.6635	0.7273	0.8266	0.8956	0.9433	0.9767
0.1	0.4162	0.5243	0.6152	0.6917	0.7561	0.8539	0.9204	0.9654	0.9967
0.5	0.8577	1.0221	1.1302	1.1992	1.2425	1.2851	1.2980	1.2975	1.2906
0.9	1.0979	1.2687	1.3641	1.4140	1.4376	1.4454	1.4306	1.4082	1.3840
1.0	1.1462	1.3166	1.4082	1.4537	1.4731	1.4737	1.4534	1.4269	1.3996
5.0	2.0908	2.1650	2.1383	2.0766	2.0059	1.8713	1.7594	1.6694	1.5968
9.0	2.5280	2.5161	2.4167	2.2994	2.1870	1.9970	1.8519	1.7406	1.6539

Table 7

Value of λ giving the best Mean Residual Life policy in the Weibull case, model I ($\alpha = 1$).

k \ β	1.0	1.25	1.5	1.75	2.0	2.5	3.0	3.5	4.0
0.01	0.1000	0.1268	0.1502	0.1711	0.1899	0.2230	0.2518	0.2776	0.3009
0.05	0.2231	0.2786	0.3246	0.3634	0.3969	0.4521	0.4963	0.5328	0.5637
0.09	0.2989	0.3694	0.4256	0.4711	0.5085	0.5663	0.6089	0.6416	0.6678
0.1	0.3149	0.3884	0.4464	0.4929	0.5308	0.5884	0.6300	0.6616	0.6866
0.5	0.6931	0.8153	0.8926	0.9412	0.9721	1.0050	1.0192	1.0253	1.0276
0.9	0.9163	1.0519	1.1275	1.1691	1.1917	1.2093	1.2113	1.2075	1.2018
1.0	0.9624	1.0996	1.1741	1.2139	1.2345	1.2487	1.2479	1.2418	1.2340
5.0	1.9249	2.0127	2.0079	1.9682	1.9165	1.8099	1.7157	1.6370	1.5720
9.0	2.3896	2.4026	2.3282	2.2310	2.1337	1.9631	1.8288	1.7241	1.6413

Table 8

Value of λ giving the best Mean Residual Life policy in the Weibull case, model II ($\alpha = 1$).

Chapter 5

Constant Hazard Inspection Policies

In Chapter 4 we defined a family of inspection policies in such a way that the inspection intervals $\{x_i - x_{i-1}\}$ are decreasing if the system is ageing in the sense that the mean residual life function $\mu(t)$ is decreasing. We now consider a stronger criterion of ageing, namely that the hazard rate function $h(t) = f(t)/\bar{F}(t)$ is increasing, and define the constant hazard $(\underline{x}_p)$ family of inspection policies which have the property that $\{x_i - x_{i-1}\}$ is decreasing if $h(t)$ is increasing.

Constant hazard policies are a one parameter family, i.e. for a given survivor function $\bar{F}(t)$, the time at which the nth inspection is scheduled (x_n) can be written as a function of just one variable, p, which turns out to be the conditional probability of failure within any inspection interval. The mean number of inspections needed to detect the failure also depends on p and is quite simply $1/p$. This means that the expected total cost depends on p, so that p is chosen to be that value which minimises the expected total cost.

The best $\underline{x}_p$ policy is easier to compute than the optimal policy, and is highly efficient in that it achieves an expected cost which exceeds that of the optimal policy by only a few percent in most cases.

5.1 Definition and general properties

A constant hazard inspection policy for a density $f(t)$ is a sequence of inspection times $\{x_i\}$ satisfying

$$H(x_n) = n\theta , \qquad n = 1, 2, \dots \tag{5.1.1}$$

for some $\theta > 0$ where

$$H(t) = \int_0^t h(u)du$$

is called the hazard function (Saunders, 1968). From (5.1.1) we have

$$H(x_n) - H(x_{n-1}) = \int_{x_{n-1}}^{x_n} h(t)dt = \theta , \qquad n = 1, 2, \ldots$$

and we see immediately that if $h(t)$ is increasing (decreasing), then the inspection intervals $\{x_i - x_{i-1}\}$ are decreasing (increasing). For the exponential distribution with constant hazard rate, the inspection intervals are constant, and the policy is periodic.

The survivor function $\bar{F}$ is related to the hazard rate function by the equation

$$\bar{F}(t) = \exp\left\{- \int_0^t h(u)du\right\}$$

and so, using (5.1.1)

$$\bar{F}(x_n) = e^{-n\theta}$$

or

$$\bar{F}(x_n) = q^n$$

where

$$q = e^{-\theta} \quad . \tag{5.1.2}$$

The conditional probability of failure during $\left[x_{n-1}, x_n\right]$ given that the system was working at time x_{n-1} is

$$\frac{F(x_n) - F(x_{n-1})}{1 - F(x_{n-1})} = \frac{\bar{F}(x_{n-1}) - \bar{F}(x_n)}{\bar{F}(x_{n-1})}$$

$$= \frac{q^{n-1} - q^n}{q^{n-1}}$$

$$= 1 - q = p \quad \text{say}$$

i.e. $$P(x_{n-1} < T \leqslant x_n \mid T > x_{n-1}) = p, \quad n = 1, 2, \ldots \tag{5.1.3}$$

Thus for a given policy the conditional probability of failure during any inspection interval is constant, and equal to p, where $p = 1 - e^{-\theta}$.

Each inspection time can be written in terms of p using (5.1.2) since

$$\bar{F}(x_n) = 1 - F(x_n) = (1 - p)^n$$

so that if F^{-1} is the inverse function of F

$$x_n = F^{-1}\{1 - (1-p)^n\} \qquad n = 1, 2, \ldots \tag{5.1.4}$$

It is more convenient to use p rather than $\theta = -\ell n(1-p)$ to define constant hazard policies for a given distribution, and we will denote the sequence

$$\{F^{-1}\left[1 - (1-p)^n\right]\}$$

by $\underline{x}_p(F)$, or simply $\underline{x}_p$ if there is no risk of ambiguity.

From (5.1.4) we see that x_n increases with p so that the number of inspections performed up to time t decreases as p increases from 0 to 1. Also $\lim_{p \to 1} x_1 = \infty$, and $\lim_{p \to 0} x_1 = 0$.

Inspection cost

Let N be the number of inspections needed to detect the failure. Then

$$P(N = n) = P(x_{n-1} < T \leqslant x_n) = \bar{F}(x_{n-1}) - \bar{F}(x_n) = q^{n-1} - q^n$$

$$P(N = n) = q^{n-1}\, p, \qquad n = 1, 2, \ldots \tag{5.1.5}$$

Thus, N has a geometric distribution, and in particular the mean number of insepctions is

$$E(N) = \sum_{n=1}^{\infty} nq^{n-1}\, p = 1/p \quad . \tag{5.1.6}$$

The mean inspection cost is therefore $\frac{c_1}{p}$ for both cost model I and cost model II.

Penalty cost

We now derive expressions for the penalty costs for both models in terms of p, and show that a simple relationship exists between them. Now

$$E(x_N) = \sum_{n=1}^{\infty} x_n \, P(N = n) \quad ,$$

so that from (5.1.4) and (5.1.5) we get

$$E(x_N) = \sum_{n=1}^{\infty} q^{n-1} \, p \, \bar{F}^{-1}(q^n) \quad . \qquad (5.1.7)$$

The expected time between the detection of failure and the previous inspection $E(x_N - x_{N-1})$ is

$$\sum_{n=1}^{\infty} (x_n - x_{n-1}) P(N = n) = \sum_{n=1}^{\infty} x_n \, q^{n-1} \, p - q \sum_{n=1}^{\infty} x_{n-1} \, q^{n-2} \, p \quad .$$

Remembering that $x_0 = 0$, the second term on the right hand side of the above equation is just q times the first, so that

$$E(x_N - x_{N-1}) = (1-q) \sum_{n=1}^{\infty} x_n \, q^{n-1} \, p$$

$$E(x_N - x_{N-1}) = p \, E(x_N) \quad . \qquad (5.1.8)$$

Thus provided $E(x_N)$ can be calculated we can find the penalty costs for both cost models:

$$c_2 \, E(x_N) - c_2 \, E(T) \quad \text{for model I,}$$

and

$$c_2 p \, E(x_N) \qquad \text{for model II} \quad .$$

Example: The Exponential Distribution

We have already seen that like the optimal policy, $\underline{x}_p$ policies are periodic in the exponential case, and so the best $\underline{x}_p$ policy coincides with the optimal policy.

Since the policy is periodic, $x_n = nx_1$, and

$$E(x_N) = E(Nx_1) = x_1 E(N) = \frac{x_1}{p}$$

from (5.1.6). But from (5.1.1)

$$\int_0^{x_1} h(t)dt = \theta$$

and since $h(t) = \frac{1}{\alpha}$

$$\frac{x_1}{\alpha} = \theta \ , \qquad x_1 = -\alpha \ \ell n(1-p) \ .$$

So that

$$E(x_N) = -\frac{\alpha \ \ell n(1-p)}{p} \ .$$

For model I the expected total cost is

$$E(c) = c_1 E(N) + c_2 E(x_N) - c_2 E(T)$$

$$= \frac{c_1}{p} - \frac{\alpha c_2 \ \ell n(1-p)}{p} - \alpha c_2$$

which is minimised when

$$\frac{p}{1-p} + \ell n(1-p) = \frac{c_1}{\alpha c_2} \ . \qquad (5.1.9)$$

Equation (5.1.9) can be solved numerically to find p. For model II the expected total cost is

$$E(c) = \frac{c_1}{p} - \alpha c_2 \ \ell n(1-p)$$

which is minimised when

$$-\frac{c_1}{p^2} + \frac{\alpha c_2}{1-p} = 0$$

$$p = -\frac{K}{2} + \sqrt{\frac{K^2}{4} + K} \quad \text{where} \quad K = \frac{c_1}{\alpha c_2} \quad . \tag{5.1.10}$$

Table 9 gives the values of p which minimise the expected total cost, and the minimum cost for models I and II.

Note that (5.1.9) and (5.1.10) are the same as (2.1.2) and (2.1.4) respectively, with $p = 1 - e^{-\lambda/\alpha}$.

	Model I		Model II	
$\frac{c_1}{\alpha c_2}$	Optimal p	$E(c)$	Optimal p	$E(c)$
0.01	0.1290	0.1482	0.0951	0.2051
0.05	0.2595	0.3504	0.2000	0.4731
0.10	0.3405	0.5162	0.2701	0.6851
0.50	0.5759	1.3577	0.5000	1.6931
1.0	0.6821	2.1462	0.6180	2.5805
5.0	0.8764	7.0907	0.8541	7.7789
10.0	0.9265	12.6109	0.9161	13.3940

Table 9

Optimal p and minimum expected total cost in the exponential case (costs measured in units of αc_2).

5.2 $\underline{x}_p$ policy in the Weibull case

We have already seen that for any $\underline{x}_p$ policy the mean inspection cost is c_1/p and that the penalty cost is $c_2\{E(x_N) - E(T)\}$ for model I and $c_2 p\, E(x_N)$ for model II. We now derive an accurate approximation to $E(x_N)$ in the Weibull case using the incomplete gamma function. This enables us to compute the $\underline{x}_p$ policy for any

$$K = \frac{c_1}{\alpha c_2}$$

and $\beta \geq 1$. Tables of optimal p and the corresponding expected cost are given for both cost models. Given p, the computation of the inspection times raises no computational problems.

The expected time to detection of failure, $E(x_N)$

From (5.1.1)

$$\int_0^{x_n} h(t)dt = - n \ln(1-p) \quad ,$$

and the hazard rate function for the Weibull distribution is

$$h(t) = \frac{\beta}{\alpha}\left(\frac{t}{\alpha}\right)^{\beta-1} \quad ,$$

so that

$$\left(\frac{x_n}{\alpha}\right)^{\beta} = - n \ln(1-p)$$

$$x_n = \alpha\{- n \ln(1-p)\}^{1/\beta} \quad , \qquad n = 1, 2, \ldots \qquad (5.2.1)$$

(Alternatively we could have found x_n from the equation $\bar{F}(x_n) = q^n$).

$$\therefore \quad E(x_N) = \sum_{n=1}^{\infty} x_n \, q^{n-1} \, p = \alpha\theta^{1/\beta} \sum_{n=1}^{\infty} n^{1/\beta} \, q^{n-1} \, p$$

$$= \alpha \, \tau(p, \beta) \tag{5.2.2}$$

where

$$\tau(p, \beta) = \theta^{1/\beta} \sum_{n=1}^{\infty} n^{1/\beta} \, q^{n-1} \, p \tag{5.2.3}$$

and

$$\theta = - \ell n(1-p) \quad .$$

Let

$$\phi(p, \ell) = \sum_{n=1}^{\infty} n^{\ell} \, q^{n-1} \, p \quad ; \tag{5.2.4}$$

for increasing hazard rate distributions $\beta \geq 1$ and so we are interested in values of ℓ in the range $0 < \ell \leq 1$. From (5.2.4) we see that $\phi(p, \ell)$ is the ℓth moment of the geometric distribution with parameter p (ℓ fractional).

Note that ϕ is an increasing function of ℓ and so

$$\phi(p, 0) = 1 \leq \phi(p, \ell) \leq \phi(p, 1) = 1/p \ , \qquad 0 \leq p \leq 1 \quad .$$

We can bound the error in approximating $\phi(p, \ell)$ by a finite number of terms of the sum (5.2.4) as follows:

Define E_{n_0} by the equation

$$\phi(p, \ell) = \sum_{n=1}^{n_o-1} n^{\ell} \, q^{n-1} \, p + E_{n_0} \quad ,$$

so that E_{n_0} is the absolute error in approximating $\phi(p, \ell)$ by the first n_0-1 terms of (5.2.4). Since $\phi > 1$ for $0 < \ell \leq 1$, the relative error will be less than E_{n_0} .

Now

$$E_{n_0} = \sum_{n=n_0}^{\infty} n^{\ell} q^{n-1} p \leqslant \sum_{n=n_0}^{\infty} nq^{n-1} p = p \frac{d}{dq} \sum_{n=n_0}^{\infty} q^n$$

$$E_{n_0} \leqslant p \frac{d}{dq} \left\{\frac{q^{n_0}}{1-q}\right\}$$

$$E_{n_0} \leqslant q^{n_0-1} \left\{n_0 + \frac{q}{p}\right\} \quad . \tag{5.2.5}$$

The right hand side of (5.2.5) tends to zero as n_0 tends to infinity. Hence for any $\varepsilon > 0$ we can calculate $\phi(p, \ell)$ to an absolute error (and hence relative error) of at most ε by summing to n_0-1 terms where $n_0(p, \varepsilon)$ is the smallest integer satisfying

$$q^{n_0(p, \varepsilon)-1} \left\{n_0(p, \varepsilon) + \frac{q}{p}\right\} \leqslant \varepsilon \quad .$$

Some values of $n_0(p, \varepsilon)$ are:

$$n_0(0.98, 10^{-5}) = 5$$

$$n_0(0.70, 10^{-5}) = 13$$

$$n_0(0.02, 10^{-5}) = 911 \quad .$$

For values of p much less than $p = 0.7$, convergence of $\sum n^{\ell} q^{n-1} p$ is slow, and since the optimal p is to be found by search techniques the speed of computation is an important factor. However, an approximation to $\sum n^{\ell} q^{n-1} p$ can be found using the Euler-MacLaurin summation formula (Abramovitz and Stegun, 1965):

$$\sum_{j=1}^{\infty} g(j) = \int_0^{\infty} g(u)du - \frac{1}{2} g(0) - \sum_{n=1}^{\infty} \frac{B_{2n}}{(2n)!} g^{(2n-1)}(0) \quad .$$

As the derivatives of $n^{\ell} q^{n-1} p$ with respect to n do not exist at $n = 0$ for $\ell < 1$, we write $\phi(p, \ell)$ as

$$\phi(p, \ell) = p + p \sum_{j=1}^{\infty} (1+j)^{\ell} q^j$$

$$= p + p \sum_{j=1}^{\infty} (1+j)^{\ell} e^{-j\theta} \qquad (5.2.6)$$

where

$$\theta = - \ell n(1-p) \quad .$$

Using Leibnitz's rule for the derivatives of products, and the Euler-MacLaurin formula we have

$$\sum_{j=1}^{\infty} (1+j)^{\ell} e^{-j\theta} = \int_0^{\infty} (1+u)^{\ell} e^{-u\theta} du - \frac{1}{2}$$

$$- \sum_{n=1}^{\infty} \frac{B_{2n}}{(2n)!} \sum_{j=0}^{2n-1} \binom{2n-1}{j} (-\theta)^j [\ell]_{2n-1-j} \qquad (5.2.7)$$

where

$$[x]_k = x(x-1) \ldots (x-k+1) \quad .$$

On substituting $y = (1+u)\theta$ the integral in (5.2.7) becomes

$$\frac{e^{\theta}}{\theta^{1+\ell}} \int_{\theta}^{\infty} y^{\ell} e^{-y} dy = \frac{\Gamma(1+\ell, \theta)}{q \theta^{1+\ell}} \quad . \qquad (5.2.8)$$

Combining (5.2.6), (5.2.7) and (5.2.8) gives

$$\phi(p, \ell) = \frac{p}{2} + \frac{p \Gamma(1+\ell, \theta)}{q \theta^{1+\ell}} - p \sum_{n=1}^{\infty} \frac{B_{2n}}{(2n)!} \sum_{j=0}^{2n-1} (-\theta)^j [\ell]_{2n-1-j} \; . \qquad (5.2.9)$$

The contribution from the infinite series in (5.2.9) is small when $\theta < 1$ i.e. $- \ell n(1-p) < 1$

$$p < 1 - \frac{1}{e}$$

$p < 0.63$ approximately.

By summing n from 1 to 2 in (5.2.9) we can calculate $\phi(p, \ell)$ to 5 significant figures for p in the range $0 < p < 0.63$, and for $p \geq 0.63$, $\phi(p, \ell)$ can be calculated accurately by direct summation. Table 10 gives $\phi(p, \ell)$ for $p = 0.01\ (0.01)\ 1.00$, $\ell = 0.0\ (0.1)\ 1.0$.

Using Table 10, $\tau(p, \beta)$ can be calculated from (5.2.3) and (5.2.4), i.e.

$$\tau(p, \beta) = \{-\ \ell n(1-p)\}^{1/\beta}\ \phi(p, 1/\beta) \quad . \tag{5.2.10}$$

<u>Optimal p for models I and II</u>

The expected total cost is

$$E(c) = \frac{c_1}{p} + c_2\alpha\ \tau(p, \beta) - c_2\alpha\ \Gamma(1 + 1/\beta) \tag{5.2.11}$$

for model I, and

$$E(c) = \frac{c_1}{p} + p\ c_2\alpha\ \tau(p, \beta) \tag{5.2.12}$$

for model II.

By measuring the costs in units of αc_2 in both cases they become

$$\frac{K}{p} + \tau(p, \beta) - \Gamma(1 + 1/\beta) \quad ,$$

and

$$\frac{K}{p} + p\ \tau(p, \beta)$$

respectively, where

$$K = \frac{c_1}{\alpha c_2} \quad .$$

Using the method of search by golden section we can find that value of p which minimises the expected cost in either case for values of K and β. For both cost models the inspection times are given by (5.2.1):

Table 10 ℓth moment of the geometric distribution

p	ℓ=0.0	ℓ=0.1	ℓ=0.2	ℓ=0.3	ℓ=0.4	ℓ=0.5	ℓ=0.6	ℓ=0.7	ℓ=0.8	ℓ=0.9	ℓ=1.0
0.01	1.0000	1.5104	2.3121	3.5825	5.6127	8.8825	14.1881	22.8571	37.1154	60.7133	100.0000
0.02	1.0000	1.4112	2.0170	2.9168	4.2635	6.2942	9.3775	14.0899	21.3376	32.5512	50.0000
0.03	1.0000	1.3568	1.8634	2.5883	3.6332	5.1496	7.3652	10.6227	15.4411	22.6096	33.3333
0.04	1.0000	1.3198	1.7623	2.3792	3.2450	4.4684	6.2080	8.6967	12.2780	17.4604	25.0000
0.05	1.0000	1.2920	1.6882	2.2295	2.9739	4.0043	5.4390	7.4488	10.2801	14.2902	20.0000
0.06	1.0000	1.2699	1.6303	2.1148	2.7702	3.6621	4.8833	6.5648	8.8928	12.1333	16.6667
0.07	1.0000	1.2516	1.5832	2.0229	2.6094	3.3966	4.4589	5.9008	7.8581	10.5664	14.2857
0.08	1.0000	1.2361	1.5438	1.9469	2.4783	3.1828	4.1220	5.3809	7.0771	9.3741	12.5000
0.09	1.0000	1.2227	1.5099	1.8825	2.3685	3.0059	3.8466	4.9612	6.4463	8.4351	11.1111
0.10	1.0000	1.2109	1.4805	1.8269	2.2747	2.8565	3.6164	4.6141	5.9304	7.6755	10.0000
0.11	1.0000	1.2004	1.4544	1.7783	2.1934	2.7281	3.4205	4.3215	5.4997	7.0477	9.0909
0.12	1.0000	1.1909	1.4312	1.7352	2.1219	2.6162	3.2512	4.0710	5.1342	6.5197	8.3333
0.13	1.0000	1.1823	1.4102	1.6967	2.0584	2.5176	3.1032	3.8537	4.8198	6.0693	7.6923
0.14	1.0000	1.1744	1.3912	1.6618	2.0015	2.4298	2.9725	3.6631	4.5461	5.6802	7.1429
0.15	1.0000	1.1672	1.3737	1.6302	1.9501	2.3511	2.8559	3.4944	4.3056	5.3405	6.6667
0.16	1.0000	1.1605	1.3577	1.6013	1.9034	2.2799	2.7513	3.3439	4.0923	5.0413	6.2500
0.17	1.0000	1.1542	1.3429	1.5746	1.8607	2.2152	2.6566	3.2086	3.9017	4.7757	5.8824
0.18	1.0000	1.1484	1.3291	1.5500	1.8214	2.1560	2.5706	3.0862	3.7303	4.5382	5.5556
0.19	1.0000	1.1430	1.3162	1.5272	1.7850	2.1016	2.4918	2.9749	3.5753	4.3245	5.2632
0.20	1.0000	1.1378	1.3042	1.5059	1.7514	2.0514	2.4195	2.8731	3.4343	4.1312	5.0000
0.21	1.0000	1.1330	1.2929	1.4860	1.7200	2.0048	2.3528	2.7797	3.3054	3.9554	4.7619
0.22	1.0000	1.1284	1.2823	1.4673	1.6907	1.9615	2.2910	2.6936	3.1872	3.7948	4.5455
0.23	1.0000	1.1241	1.2722	1.4497	1.6632	1.9211	2.2336	2.6139	3.0783	3.6475	4.3478
0.24	1.0000	1.1200	1.2627	1.4331	1.6374	1.8833	2.1801	2.5399	2.9775	3.5119	4.1667
0.25	1.0000	1.1160	1.2537	1.4175	1.6131	1.8477	2.1300	2.4709	2.8841	3.3866	4.0000
0.26	1.0000	1.1123	1.2451	1.4026	1.5902	1.8143	2.0831	2.4065	2.7972	3.2705	3.8462
0.27	1.0000	1.1087	1.2369	1.3885	1.5685	1.7828	2.0390	2.3463	2.7160	3.1626	3.7037
0.28	1.0000	1.1053	1.2291	1.3751	1.5479	1.7530	1.9975	2.2897	2.6402	3.0620	3.5714
0.29	1.0000	1.1020	1.2216	1.3623	1.5283	1.7248	1.9582	2.2364	2.5690	2.9680	3.4483
0.30	1.0000	1.0989	1.2145	1.3501	1.5097	1.6980	1.9211	2.1862	2.5022	2.8800	3.3333
0.31	1.0000	1.0958	1.2076	1.3384	1.4919	1.6726	1.8860	2.1388	2.4392	2.7974	3.2258
0.32	1.0000	1.0929	1.2011	1.3272	1.4749	1.6484	1.8526	2.0939	2.3798	2.7197	3.1250
0.33	1.0000	1.0901	1.1947	1.3165	1.4587	1.6253	1.8209	2.0513	2.3237	2.6465	3.0303

Table 10 (continued)

p	ℓ=0.0	ℓ=0.1	ℓ=0.2	ℓ=0.3	ℓ=0.4	ℓ=0.5	ℓ=0.6	ℓ=0.7	ℓ=0.8	ℓ=0.9	ℓ=1.0
0.34	1.0000	1.0874	1.1887	1.3063	1.4432	1.6032	1.7906	2.0109	2.2705	2.5774	2.9412
0.35	1.0000	1.0848	1.1828	1.2964	1.4283	1.5821	1.7618	1.9725	2.2201	2.5120	2.8571
0.36	1.0000	1.0823	1.1772	1.2869	1.4140	1.5619	1.7343	1.9359	2.1722	2.4501	2.7778
0.37	1.0000	1.0798	1.1717	1.2777	1.4003	1.5426	1.7080	1.9010	2.1266	2.3914	2.7027
0.38	1.0000	1.0775	1.1664	1.2689	1.3871	1.5240	1.6828	1.8676	2.0833	2.3356	2.6316
0.39	1.0000	1.0752	1.1613	1.2604	1.3744	1.5061	1.6587	1.8357	2.0419	2.2825	2.5641
0.40	1.0000	1.0730	1.1564	1.2521	1.3622	1.4890	1.6355	1.8052	2.0024	2.2320	2.5000
0.41	1.0000	1.0708	1.1516	1.2442	1.3503	1.4725	1.6133	1.7760	1.9546	2.1838	2.4390
0.42	1.0000	1.0687	1.1470	1.2365	1.3389	1.4565	1.5919	1.7480	1.9285	2.1377	2.3810
0.43	1.0000	1.0667	1.1425	1.2290	1.3279	1.4412	1.5713	1.7210	1.8939	2.0938	2.3256
0.44	1.0000	1.0647	1.1382	1.2218	1.3172	1.4264	1.5515	1.6952	1.8607	2.0517	2.2727
0.45	1.0000	1.0628	1.1339	1.2148	1.3069	1.4121	1.5324	1.6703	1.8288	2.0114	2.2222
0.46	1.0000	1.0609	1.1298	1.2080	1.2969	1.3982	1.5139	1.6463	1.7982	1.9727	2.1739
0.47	1.0000	1.0591	1.1258	1.2014	1.2873	1.3849	1.4961	1.6232	1.7687	1.9357	2.1277
0.48	1.0000	1.0573	1.1219	1.1950	1.2779	1.3719	1.4789	1.6010	1.7404	1.9001	2.0833
0.49	1.0000	1.0555	1.1181	1.1888	1.2687	1.3594	1.4623	1.5795	1.7131	1.8658	2.0408
0.50	1.0000	1.0538	1.1144	1.1827	1.2599	1.3472	1.4462	1.5587	1.6868	1.8329	2.0000
0.51	1.0000	1.0522	1.1108	1.1768	1.2513	1.3354	1.4306	1.5387	1.6614	1.8012	1.9608
0.52	1.0000	1.0506	1.1073	1.1711	1.2429	1.3240	1.4156	1.5193	1.6369	1.7707	1.9231
0.53	1.0000	1.0490	1.1039	1.1655	1.2348	1.3129	1.4009	1.5005	1.6133	1.7413	1.8868
0.54	1.0000	1.0474	1.1005	1.1601	1.2269	1.3021	1.3867	1.4823	1.5904	1.7129	1.8519
0.55	1.0000	1.0459	1.0973	1.1547	1.2192	1.2916	1.3730	1.4647	1.5683	1.6854	1.8182
0.56	1.0000	1.0445	1.0941	1.1496	1.2117	1.2813	1.3596	1.4476	1.5469	1.6589	1.7857
0.57	1.0000	1.0430	1.0910	1.1445	1.2044	1.2714	1.3466	1.4311	1.5262	1.6333	1.7544
0.58	1.0000	1.0416	1.0879	1.1396	1.1972	1.2617	1.3340	1.4150	1.5061	1.6086	1.7241
0.59	1.0000	1.0402	1.0849	1.1347	1.1903	1.2523	1.3217	1.3994	1.4866	1.5846	1.6949
0.60	1.0000	1.0389	1.0820	1.1300	1.1835	1.2431	1.3097	1.3842	1.4677	1.5614	1.6667
0.61	1.0000	1.0375	1.0792	1.1254	1.1769	1.2342	1.2981	1.3695	1.4494	1.5389	1.6393
0.62	1.0000	1.0362	1.0764	1.1209	1.1704	1.2254	1.2868	1.3552	1.4316	1.5171	1.6129
0.63	1.0000	1.0350	1.0737	1.1166	1.1642	1.2170	1.2758	1.3413	1.4143	1.4960	1.5873
0.64	1.0000	1.0337	1.0710	1.1123	1.1580	1.2087	1.2650	1.3277	1.3976	1.4754	1.5625
0.65	1.0000	1.0325	1.0684	1.1081	1.1519	1.2006	1.2546	1.3145	1.3812	1.4555	1.5384
0.66	1.0000	1.0313	1.0658	1.1039	1.1450	1.1927	1.2443	1.3017	1.3653	1.4362	1.5151

Table 10 (continued)

p	ℓ=0.0	ℓ=0.1	ℓ=0.2	ℓ=0.3	ℓ=0.4	ℓ=0.5	ℓ=0.6	ℓ=0.7	ℓ=0.8	ℓ=0.9	ℓ=1.0
0.67	1.0000	1.0301	1.0633	1.0999	1.1403	1.1849	1.2344	1.2891	1.3499	1.4174	1.4925
0.68	1.0000	1.0289	1.0608	1.0959	1.1346	1.1774	1.2246	1.2769	1.3349	1.3992	1.4706
0.69	1.0000	1.0278	1.0584	1.0920	1.1291	1.1700	1.2151	1.2650	1.3202	1.3814	1.4492
0.70	1.0000	1.0267	1.0560	1.0882	1.1237	1.1627	1.2058	1.2534	1.3059	1.3641	1.4285
0.71	1.0000	1.0256	1.0536	1.0845	1.1184	1.1557	1.1967	1.2420	1.2920	1.3473	1.4084
0.72	1.0000	1.0245	1.0513	1.0808	1.1132	1.1487	1.1879	1.2310	1.2785	1.3309	1.3889
0.73	1.0000	1.0234	1.0491	1.0772	1.1081	1.1419	1.1792	1.2201	1.2652	1.3150	1.3698
0.74	1.0000	1.0224	1.0469	1.0737	1.1031	1.1353	1.1707	1.2096	1.2523	1.2994	1.3513
0.75	1.0000	1.0214	1.0447	1.0702	1.0982	1.1288	1.1624	1.1992	1.2397	1.2843	1.3333
0.76	1.0000	1.0203	1.0425	1.0668	1.0934	1.1224	1.1543	1.1892	1.2275	1.2695	1.3158
0.77	1.0000	1.0193	1.0404	1.0635	1.0886	1.1162	1.1463	1.1793	1.2154	1.2551	1.2987
0.78	1.0000	1.0184	1.0384	1.0602	1.0840	1.1101	1.1385	1.1696	1.2037	1.2411	1.2820
0.79	1.0000	1.0174	1.0363	1.0570	1.0795	1.1040	1.1309	1.1602	1.1923	1.2274	1.2658
0.80	1.0000	1.0164	1.0343	1.0538	1.0750	1.0982	1.1234	1.1510	1.1811	1.2140	1.2500
0.81	1.0000	1.0155	1.0324	1.0507	1.0706	1.0924	1.1161	1.1419	1.1701	1.2009	1.2346
0.82	1.0000	1.0146	1.0304	1.0476	1.0663	1.0867	1.1089	1.1331	1.1594	1.1881	1.2195
0.83	1.0000	1.0137	1.0285	1.0446	1.0621	1.0811	1.1018	1.1244	1.1489	1.1757	1.2048
0.84	1.0000	1.0128	1.0266	1.0417	1.0580	1.0757	1.0949	1.1159	1.1387	1.1635	1.1905
0.85	1.0000	1.0119	1.0248	1.0387	1.0539	1.0703	1.0882	1.1076	1.1286	1.1515	1.1765
0.86	1.0000	1.0110	1.0229	1.0359	1.0499	1.0651	1.0815	1.0994	1.1188	1.1399	1.1628
0.87	1.0000	1.0102	1.0212	1.0330	1.0459	1.0599	1.0750	1.0914	1.1092	1.1285	1.1494
0.88	1.0000	1.0093	1.0194	1.0303	1.0421	1.0548	1.0686	1.0836	1.0998	1.1173	1.1364
0.89	1.0000	1.0085	1.0176	1.0275	1.0382	1.0498	1.0623	1.0759	1.0905	1.1064	1.1236
0.90	1.0000	1.0077	1.0159	1.0248	1.0345	1.0449	1.0561	1.0683	1.0815	1.0957	1.1111
0.91	1.0000	1.0068	1.0142	1.0222	1.0308	1.0401	1.0501	1.0609	1.0726	1.0852	1.0989
0.92	1.0000	1.0060	1.0126	1.0196	1.0272	1.0353	1.0441	1.0536	1.0639	1.0750	1.0870
0.93	1.0000	1.0053	1.0109	1.0170	1.0236	1.0306	1.0383	1.0465	1.0554	1.0649	1.0753
0.94	1.0000	1.0045	1.0093	1.0145	1.0200	1.0261	1.0325	1.0395	1.0470	1.0551	1.0638
0.95	1.0000	1.0037	1.0077	1.0120	1.0166	1.0215	1.0269	1.0326	1.0388	1.0455	1.0526
0.96	1.0000	1.0029	1.0061	1.0095	1.0132	1.0171	1.0213	1.0259	1.0308	1.0360	1.0417
0.97	1.0000	1.0022	1.0046	1.0071	1.0098	1.0127	1.0159	1.0192	1.0229	1.0267	1.0309
0.98	1.0000	1.0015	1.0030	1.0047	1.0065	1.0084	1.0105	1.0127	1.0151	1.0177	1.0204
0.99	1.0000	1.0007	1.0015	1.0023	1.0032	1.0042	1.0052	1.0063	1.0075	1.0087	1.0101
1.00	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000

$$x_n = \alpha n^{1/\beta} \{- \ell n(1-p)\}^{1/\beta} , \qquad n = 1, 2, \ldots$$

Tables 11 and 12 give the optimal p for values of K and β. The corresponding expected costs (measured in units of αc_2) are given in Tables 13 and 14.

$\underline{x}_p$ policies in the Weibull case when c_1 and c_2 are not known

We now consider the case when the values of c_1 and c_2 are not known, and an inspection policy has to be chosen by the intuitive balancing of the number of inspections performed, and the consequences of undetected failure. If the time to failure distribution is known then the mean number of inspections and the mean time between the failure and its detection (or the mean time between the detection of failure and the previous inspection in the case of model II) provide good quantitative aids. Both of these functions depend on p. The mean number of inspections is quite simply $1/p$, and Figure 1 gives a graph of

$$\frac{1}{\alpha} E(x_N - T) = \tau(p, \beta) - \Gamma(1 + 1/\beta) ,$$

which can be used for model I; for model II we would use Figure 2 which gives a graph of

$$E(x_N - x_{N-1}) = p\, \tau(p, \beta) .$$

The behaviour of $\tau(p, \beta)$ as p tends to zero

From (5.2.9) and (5.2.10) we have

$$\tau(p, \beta) = \frac{p\theta^{1/\beta}}{2} + \frac{p}{q\theta} \Gamma(1 + 1/\beta, \theta)$$

$$- p\theta^{1/\beta} \sum_{n=1}^{\infty} \frac{B_{2n}}{(2n)!} \sum_{j=0}^{2n-1} (-\theta)^j \left[1/\beta\right]_{2n-1-j}$$

Table 11 MODEL 1 OPTIMAL P FOR WEIBULL DISTRIBUTION

K \ β	1.00	1.25	1.50	1.75	2.00	2.50	3.00	3.50	4.00	4.50
0.01	0.1290	0.1367	0.1431	0.1490	0.1546	0.1653	0.1757	0.1856	0.1953	0.2047
0.02	0.1759	0.1872	0.1968	0.2057	0.2141	0.2301	0.2453	0.2598	0.2737	0.2869
0.03	0.2097	0.2237	0.2357	0.2468	0.2573	0.2773	0.2960	0.3137	0.3304	0.3464
0.04	0.2367	0.2529	0.2669	0.2798	0.2921	0.3151	0.3366	0.3567	0.3757	0.3935
0.05	0.2595	0.2776	0.2933	0.3078	0.3214	0.3471	0.3708	0.3928	0.4134	0.4328
0.06	0.2793	0.2990	0.3162	0.3321	0.3470	0.3748	0.4003	0.4240	0.4459	0.4664
0.07	0.2969	0.3181	0.3366	0.3536	0.3696	0.3993	0.4264	0.4513	0.4743	0.4957
0.08	0.3127	0.3353	0.3550	0.3731	0.3900	0.4213	0.4497	0.4757	0.4996	0.5216
0.09	0.3272	0.3509	0.3717	0.3907	0.4085	0.4412	0.4708	0.4977	0.5222	0.5448
0.10	0.3405	0.3653	0.3871	0.4069	0.4255	0.4595	0.4900	0.5176	0.5427	0.5658
0.12	0.3643	0.3911	0.4145	0.4359	0.4557	0.4918	0.5239	0.5527	0.5786	0.6022
0.14	0.3851	0.4136	0.4385	0.4611	0.4820	0.5197	0.5530	0.5826	0.6091	0.6300
0.16	0.4037	0.4337	0.4598	0.4834	0.5052	0.5443	0.5784	0.6086	0.6300	0.6589
0.18	0.4204	0.4517	0.4789	0.5035	0.5260	0.5661	0.6010	0.6300	0.6580	0.6818
0.20	0.4358	0.4682	0.4963	0.5216	0.5448	0.5858	0.6211	0.6514	0.6783	0.7019
0.22	0.4498	0.4832	0.5122	0.5382	0.5618	0.6035	0.6389	0.6696	0.6963	0.7197
0.24	0.4627	0.4971	0.5268	0.5533	0.5774	0.6197	0.6560	0.6860	0.7125	0.7356
0.26	0.4748	0.5100	0.5404	0.5674	0.5918	0.6300	0.6702	0.7008	0.7271	0.7498
0.28	0.4860	0.5220	0.5529	0.5804	0.6052	0.6479	0.6839	0.7143	0.7403	0.7627
0.30	0.4965	0.5332	0.5646	0.5925	0.6175	0.6605	0.6965	0.7267	0.7524	0.7744
0.32	0.5065	0.5437	0.5756	0.6038	0.6291	0.6722	0.7081	0.7381	0.7634	0.7852
0.34	0.5158	0.5536	0.5860	0.6145	0.6397	0.6831	0.7189	0.7486	0.7737	0.7949
0.36	0.5247	0.5630	0.5957	0.6245	0.6498	0.6933	0.7288	0.7584	0.7830	0.8039
0.38	0.5331	0.5718	0.6049	0.6337	0.6593	0.7029	0.7383	0.7674	0.7917	0.8122
0.40	0.5410	0.5803	0.6137	0.6426	0.6683	0.7118	0.7470	0.7758	0.7998	0.8198
0.42	0.5486	0.5883	0.6219	0.6511	0.6769	0.7202	0.7551	0.7837	0.8072	0.8269
0.44	0.5559	0.5959	0.6298	0.6590	0.6849	0.7281	0.7628	0.7910	0.8142	0.8336
0.46	0.5628	0.6032	0.6371	0.6666	0.6925	0.7357	0.7701	0.7979	0.8207	0.8397
0.48	0.5695	0.6102	0.6443	0.6739	0.6997	0.7428	0.7769	0.8044	0.8269	0.8454
0.50	0.5758	0.6168	0.6511	0.6808	0.7066	0.7495	0.7833	0.8105	0.8326	0.8508
0.52	0.5820	0.6232	0.6576	0.6880	0.7132	0.7559	0.7894	0.8163	0.8380	0.8559
0.54	0.5879	0.6294	0.6639	0.6936	0.7195	0.7619	0.7952	0.8217	0.8431	0.8607
0.56	0.5936	0.6351	0.6699	0.6997	0.7254	0.7677	0.8007	0.8268	0.8479	0.8651
0.58	0.5990	0.6408	0.6757	0.7054	0.7312	0.7733	0.8059	0.8317	0.8524	0.8693
0.60	0.6043	0.6463	0.6813	0.7110	0.7367	0.7785	0.8108	0.8363	0.8567	0.8733

Table 11 MODEL 1 OPTIMAL P FOR WEIBULL DISTRIBUTION (CONTINUED)

K \ β	1.00	1.25	1.50	1.75	2.00	2.50	3.00	3.50	4.00	4.50
0.62	0.6094	0.6516	0.6866	0.7164	0.7420	0.7836	0.8156	0.8407	0.8608	0.8771
0.64	0.6143	0.6566	0.6917	0.7215	0.7470	0.7884	0.8201	0.8449	0.8647	0.8806
0.66	0.6191	0.6615	0.6967	0.7264	0.7519	0.7930	0.8244	0.8488	0.8683	0.8840
0.68	0.6237	0.6663	0.7015	0.7311	0.7565	0.7974	0.8285	0.8527	0.8718	0.8872
0.70	0.6300	0.6709	0.7061	0.7358	0.7610	0.8016	0.8324	0.8563	0.8751	0.8902
0.72	0.6324	0.6753	0.7106	0.7402	0.7654	0.8057	0.8362	0.8597	0.8783	0.8932
0.74	0.6366	0.6797	0.7149	0.7444	0.7695	0.8096	0.8398	0.8630	0.8813	0.8959
0.76	0.6407	0.6838	0.7191	0.7485	0.7735	0.8134	0.8432	0.8662	0.8842	0.8986
0.78	0.6447	0.6880	0.7231	0.7526	0.7774	0.8170	0.8465	0.8692	0.8870	0.9011
0.80	0.6486	0.6918	0.7270	0.7564	0.7812	0.8204	0.8497	0.8722	0.8897	0.9035
0.82	0.6523	0.6956	0.7308	0.7607	0.7848	0.8238	0.8528	0.8749	0.8922	0.9058
0.84	0.6560	0.6993	0.7345	0.7637	0.7883	0.8270	0.8558	0.8776	0.8945	0.9080
0.86	0.6595	0.7029	0.7381	0.7672	0.7917	0.8302	0.8586	0.8802	0.8969	0.9101
0.88	0.6630	0.7064	0.7416	0.7706	0.7950	0.8332	0.8614	0.8827	0.8991	0.9121
0.90	0.6664	0.7099	0.7450	0.7739	0.7982	0.8361	0.8640	0.8850	0.9013	0.9141
0.92	0.6697	0.7132	0.7483	0.7772	0.8015	0.8390	0.8666	0.8873	0.9033	0.9160
0.94	0.6729	0.7165	0.7515	0.7803	0.8043	0.8417	0.8690	0.8897	0.9053	0.9177
0.96	0.6760	0.7196	0.7546	0.7833	0.8072	0.8449	0.8714	0.8917	0.9072	0.9194
0.98	0.6791	0.7229	0.7576	0.7862	0.8100	0.8469	0.8737	0.8937	0.9091	0.9211
1.00	0.6821	0.7257	0.7607	0.7891	0.8128	0.8494	0.8759	0.8957	0.9109	0.9227
1.50	0.7401	0.7828	0.8158	0.8418	0.8626	0.8934	0.9145	0.9297	0.9408	0.9494
2.00	0.7780	0.8192	0.8499	0.8735	0.8918	0.9181	0.9355	0.9476	0.9564	0.9629
2.50	0.8053	0.8449	0.8732	0.8947	0.9110	0.9338	0.9485	0.9586	0.9657	0.9710
3.00	0.8260	0.8636	0.8903	0.9099	0.9246	0.9447	0.9574	0.9659	0.9719	0.9763
3.50	0.8425	0.8783	0.9033	0.9213	0.9347	0.9526	0.9638	0.9712	0.9763	0.9801
4.00	0.8558	0.8900	0.9135	0.9302	0.9424	0.9587	0.9686	0.9751	0.9796	0.9829
4.50	0.8670	0.8997	0.9219	0.9374	0.9486	0.9634	0.9723	0.9781	0.9821	0.9850
5.00	0.8764	0.9078	0.9287	0.9432	0.9537	0.9672	0.9753	0.9805	0.9841	0.9867
6.00	0.8915	0.9205	0.9394	0.9523	0.9613	0.9729	0.9797	0.9841	0.9871	0.9892
7.00	0.9032	0.9302	0.9474	0.9589	0.9669	0.9770	0.9829	0.9866	0.9891	0.9909
8.00	0.9126	0.9377	0.9535	0.9639	0.9711	0.9800	0.9852	0.9885	0.9907	0.9922
9.00	0.9202	0.9438	0.9584	0.9679	0.9744	0.9824	0.9870	0.9899	0.9918	0.9932
10.00	0.9265	0.9488	0.9623	0.9711	0.9770	0.9843	0.9884	0.9910	0.9927	0.9940

Table 12 MODEL 2 OPTIMAL P FOR WEIBULL DISTRIBUTION

K \ β	1.00	1.25	1.50	1.75	2.00	2.50	3.00	3.50	4.00	4.50
0.01	0.0951	0.0986	0.1002	0.1011	0.1016	0.1019	0.1020	0.1019	0.1019	0.1018
0.02	0.1318	0.1366	0.1390	0.1404	0.1412	0.1420	0.1424	0.1425	0.1425	0.1425
0.03	0.1589	0.1647	0.1678	0.1696	0.1708	0.1720	0.1726	0.1730	0.1731	0.1733
0.04	0.1810	0.1878	0.1915	0.1937	0.1951	0.1968	0.1977	0.1982	0.1986	0.1989
0.05	0.2000	0.2076	0.2118	0.2144	0.2161	0.2182	0.2195	0.2202	0.2208	0.2212
0.06	0.2168	0.2251	0.2298	0.2328	0.2348	0.2373	0.2389	0.2399	0.2406	0.2411
0.07	0.2319	0.2408	0.2460	0.2494	0.2517	0.2546	0.2565	0.2577	0.2586	0.2593
0.08	0.2457	0.2553	0.2609	0.2646	0.2671	0.2705	0.2727	0.2742	0.2753	0.2761
0.09	0.2584	0.2685	0.2746	0.2786	0.2815	0.2853	0.2878	0.2895	0.2908	0.2918
0.10	0.2702	0.2809	0.2874	0.2917	0.2949	0.2991	0.3019	0.3038	0.3053	0.3064
0.12	0.2916	0.3034	0.3106	0.3156	0.3193	0.3243	0.3277	0.3301	0.3320	0.3334
0.14	0.3107	0.3234	0.3314	0.3370	0.3412	0.3470	0.3510	0.3539	0.3561	0.3579
0.16	0.3279	0.3416	0.3503	0.3565	0.3611	0.3677	0.3723	0.3757	0.3783	0.3804
0.18	0.3437	0.3582	0.3676	0.3743	0.3794	0.3869	0.3920	0.3958	0.3989	0.4013
0.20	0.3583	0.3736	0.3836	0.3909	0.3964	0.4046	0.4103	0.4146	0.4180	0.4207
0.22	0.3718	0.3878	0.3985	0.4063	0.4123	0.4212	0.4275	0.4322	0.4360	0.4390
0.24	0.3844	0.4011	0.4124	0.4207	0.4271	0.4367	0.4436	0.4488	0.4529	0.4562
0.26	0.3962	0.4137	0.4255	0.4343	0.4411	0.4514	0.4588	0.4645	0.4690	0.4726
0.28	0.4074	0.4255	0.4378	0.4471	0.4544	0.4653	0.4732	0.4793	0.4842	0.4881
0.30	0.4179	0.4366	0.4495	0.4592	0.4669	0.4785	0.4870	0.4935	0.4987	0.5029
0.32	0.4279	0.4472	0.4606	0.4708	0.4788	0.4911	0.5000	0.5070	0.5125	0.5171
0.34	0.4374	0.4573	0.4712	0.4817	0.4902	0.5031	0.5125	0.5199	0.5258	0.5307
0.36	0.4464	0.4669	0.4813	0.4922	0.5010	0.5145	0.5245	0.5323	0.5385	0.5437
0.38	0.4551	0.4760	0.4909	0.5023	0.5114	0.5255	0.5360	0.5441	0.5507	0.5561
0.40	0.4633	0.4848	0.5001	0.5119	0.5214	0.5360	0.5470	0.5555	0.5624	0.5682
0.42	0.4713	0.4932	0.5089	0.5211	0.5309	0.5462	0.5576	0.5665	0.5737	0.5797
0.44	0.4789	0.5013	0.5174	0.5300	0.5401	0.5559	0.5677	0.5771	0.5846	0.5909
0.46	0.4862	0.5091	0.5256	0.5385	0.5490	0.5653	0.5776	0.5872	0.5951	0.6017
0.48	0.4932	0.5166	0.5335	0.5467	0.5575	0.5743	0.5870	0.5971	0.6052	0.6121
0.50	0.5000	0.5238	0.5411	0.5546	0.5657	0.5830	0.5962	0.6066	0.6151	0.6221
0.52	0.5066	0.5307	0.5484	0.5623	0.5737	0.5915	0.6050	0.6158	0.6245	0.6300
0.54	0.5129	0.5375	0.5555	0.5697	0.5813	0.5996	0.6136	0.6246	0.6336	0.6412
0.56	0.5190	0.5440	0.5623	0.5768	0.5887	0.6075	0.6218	0.6331	0.6425	0.6503
0.58	0.5249	0.5503	0.5690	0.5837	0.5959	0.6152	0.6298	0.6415	0.6511	0.6591
0.60	0.5307	0.5564	0.5754	0.5905	0.6029	0.6226	0.6375	0.6495	0.6594	0.6677

Table 12 MODEL 2 OPTIMAL P FOR WEIBULL DISTRIBUTION (CONTINUED)

K \ β	1.00	1.25	1.50	1.75	2.00	2.50	3.00	3.50	4.00	4.50
0.62	0.5362	0.5623	0.5816	0.5970	0.6096	0.6297	0.6450	0.6574	0.6675	0.6760
0.64	0.5416	0.5680	0.5877	0.6033	0.6162	0.6366	0.6523	0.6650	0.6753	0.6841
0.66	0.5469	0.5736	0.5935	0.6094	0.6226	0.6434	0.6594	0.6723	0.6829	0.6919
0.68	0.5520	0.5790	0.5992	0.6154	0.6287	0.6499	0.6663	0.6795	0.6903	0.6995
0.70	0.5569	0.5843	0.6047	0.6211	0.6346	0.6563	0.6730	0.6864	0.6975	0.7069
0.72	0.5617	0.5894	0.6101	0.6267	0.6405	0.6625	0.6795	0.6931	0.7044	0.7140
0.74	0.5664	0.5944	0.6154	0.6321	0.6461	0.6685	0.6858	0.6997	0.7112	0.7210
0.76	0.5710	0.5992	0.6205	0.6375	0.6517	0.6743	0.6919	0.7061	0.7178	0.7277
0.78	0.5755	0.6039	0.6254	0.6426	0.6570	0.6800	0.6979	0.7122	0.7242	0.7343
0.80	0.5798	0.6085	0.6300	0.6477	0.6623	0.6856	0.7037	0.7182	0.7304	0.7407
0.82	0.5840	0.6130	0.6349	0.6526	0.6674	0.6910	0.7093	0.7241	0.7364	0.7469
0.84	0.5882	0.6174	0.6395	0.6574	0.6723	0.6962	0.7148	0.7298	0.7423	0.7529
0.86	0.5922	0.6217	0.6440	0.6621	0.6772	0.7013	0.7202	0.7354	0.7480	0.7587
0.88	0.5962	0.6259	0.6484	0.6666	0.6819	0.7063	0.7254	0.7408	0.7536	0.7645
0.90	0.6000	0.6299	0.6527	0.6711	0.6865	0.7112	0.7304	0.7460	0.7590	0.7700
0.92	0.6038	0.6339	0.6568	0.6754	0.6910	0.7159	0.7354	0.7511	0.7642	0.7753
0.94	0.6075	0.6378	0.6609	0.6797	0.6954	0.7206	0.7402	0.7561	0.7693	0.7806
0.96	0.6111	0.6416	0.6649	0.6838	0.6997	0.7251	0.7449	0.7609	0.7743	0.7857
0.98	0.6146	0.6453	0.6688	0.6880	0.7038	0.7295	0.7495	0.7657	0.7791	0.7906
1.00	0.6180	0.6490	0.6726	0.6918	0.7079	0.7338	0.7539	0.7703	0.7839	0.7954
1.50	0.6861	0.7203	0.7468	0.7682	0.7862	0.8148	0.8368	0.8543	0.8686	0.8806
2.00	0.7320	0.7676	0.7949	0.8169	0.8350	0.8633	0.8843	0.9005	0.9133	0.9236
2.50	0.7655	0.8015	0.8286	0.8502	0.8677	0.8943	0.9133	0.9274	0.9382	0.9466
3.00	0.7913	0.8269	0.8535	0.8742	0.8907	0.9152	0.9321	0.9442	0.9532	0.9601
3.50	0.8117	0.8467	0.8725	0.8922	0.9076	0.9299	0.9449	0.9553	0.9629	0.9687
4.00	0.8284	0.8626	0.8874	0.9061	0.9204	0.9407	0.9540	0.9631	0.9696	0.9744
4.50	0.8423	0.8756	0.8994	0.9170	0.9304	0.9489	0.9608	0.9688	0.9744	0.9785
5.00	0.8541	0.8864	0.9092	0.9258	0.9383	0.9553	0.9660	0.9730	0.9780	0.9816
6.00	0.8730	0.9034	0.9243	0.9392	0.9501	0.9645	0.9733	0.9790	0.9830	0.9858
7.00	0.8875	0.9161	0.9353	0.9487	0.9583	0.9708	0.9782	0.9830	0.9862	0.9886
8.00	0.8990	0.9259	0.9437	0.9558	0.9643	0.9753	0.9817	0.9857	0.9885	0.9904
9.00	0.9083	0.9338	0.9502	0.9612	0.9689	0.9786	0.9842	0.9878	0.9902	0.9918
10.00	0.9161	0.9401	0.9554	0.9656	0.9726	0.9813	0.9862	0.9893	0.9914	0.9929

Table 13 MODEL 1 EXPECTED COST FOR WEIBULL DISTRIBUTION

K \ β	1.00	1.25	1.50	1.75	2.00	2.50	3.00	3.50	4.00	4.50
0.01	0.1482	0.1410	0.1360	0.1319	0.1282	0.1216	0.1158	0.1106	0.1059	0.1017
0.02	0.2135	0.2025	0.1947	0.1881	0.1823	0.1721	0.1632	0.1554	0.1485	0.1423
0.03	0.2653	0.2511	0.2408	0.2323	0.2247	0.2115	0.2001	0.1902	0.1815	0.1738
0.04	0.3101	0.2931	0.2806	0.2702	0.2611	0.2452	0.2317	0.2200	0.2098	0.2008
0.05	0.3504	0.3308	0.3163	0.3043	0.2937	0.2754	0.2600	0.2467	0.2352	0.2250
0.06	0.3875	0.3654	0.3491	0.3355	0.3236	0.3031	0.2859	0.2712	0.2584	0.2472
0.07	0.4222	0.3978	0.3797	0.3647	0.3515	0.3289	0.3101	0.2940	0.2801	0.2680
0.08	0.4550	0.4284	0.4086	0.3922	0.3778	0.3533	0.3329	0.3156	0.3007	0.2877
0.09	0.4863	0.4576	0.4362	0.4184	0.4029	0.3765	0.3546	0.3361	0.3202	0.3064
0.10	0.5162	0.4855	0.4625	0.4434	0.4268	0.3987	0.3755	0.3558	0.3390	0.3244
0.12	0.5730	0.5383	0.5124	0.4909	0.4722	0.4407	0.4149	0.3932	0.3747	0.3586
0.14	0.6263	0.5880	0.5593	0.5355	0.5148	0.4803	0.4520	0.4284	0.4083	0.3910
0.16	0.6770	0.6352	0.6038	0.5778	0.5553	0.5178	0.4874	0.4620	0.4405	0.4221
0.18	0.7255	0.6804	0.6464	0.6183	0.5941	0.5539	0.5213	0.4942	0.4715	0.4519
0.20	0.7722	0.7239	0.6874	0.6573	0.6315	0.5886	0.5540	0.5255	0.5014	0.4808
0.22	0.8174	0.7659	0.7271	0.6951	0.6676	0.6222	0.5858	0.5558	0.5305	0.5089
0.24	0.8612	0.8067	0.7656	0.7317	0.7027	0.6549	0.6167	0.5853	0.5589	0.5364
0.26	0.9039	0.8464	0.8030	0.7674	0.7369	0.6868	0.6469	0.6141	0.5867	0.5633
0.28	0.9455	0.8852	0.8396	0.8022	0.7703	0.7181	0.6764	0.6424	0.6139	0.5898
0.30	0.9862	0.9231	0.8754	0.8363	0.8031	0.7486	0.7054	0.6701	0.6407	0.6158
0.32	1.0261	0.9602	0.9105	0.8698	0.8351	0.7786	0.7339	0.6974	0.6671	0.6414
0.34	1.0653	0.9967	0.9449	0.9026	0.8667	0.8081	0.7619	0.7243	0.6931	0.6667
0.36	1.1037	1.0325	0.9788	0.9349	0.8978	0.8372	0.7895	0.7509	0.7188	0.6918
0.38	1.1415	1.0677	1.0121	0.9667	0.9283	0.8658	0.8168	0.7771	0.7442	0.7165
0.40	1.1788	1.1025	1.0449	0.9981	0.9584	0.8941	0.8437	0.8030	0.7693	0.7410
0.42	1.2155	1.1367	1.0773	1.0290	0.9882	0.9221	0.8704	0.8286	0.7942	0.7653
0.44	1.2517	1.1705	1.1092	1.0595	1.0175	0.9497	0.8967	0.8540	0.8189	0.7894
0.46	1.2874	1.2038	1.1408	1.0897	1.0466	0.9770	0.9228	0.8792	0.8434	0.8133
0.48	1.3228	1.2368	1.1721	1.1195	1.0753	1.0040	0.9487	0.9042	0.8676	0.8370
0.50	1.3577	1.2694	1.2029	1.1491	1.1037	1.0308	0.9743	0.9289	0.8917	0.8606
0.52	1.3922	1.3016	1.2335	1.1783	1.1319	1.0574	0.9997	0.9535	0.9157	0.8840
0.54	1.4264	1.3336	1.2638	1.2072	1.1598	1.0838	1.0250	0.9780	0.9395	0.9073
0.56	1.4603	1.3652	1.2937	1.2359	1.1875	1.1099	1.0500	1.0022	0.9631	0.9305
0.58	1.4938	1.3966	1.3235	1.2644	1.2150	1.1359	1.0749	1.0263	0.9866	0.9536
0.60	1.5271	1.4276	1.3529	1.2927	1.2422	1.1616	1.0997	1.0503	1.0100	0.9765

Table 13 MODEL 1 EXPECTED COST FOR WEIBULL DISTRIBUTION (CONTINUED)

K \ β	1.00	1.25	1.50	1.75	2.00	2.50	3.00	3.50	4.00	4.50
0.62	1.5600	1.4585	1.3822	1.3207	1.2693	1.1873	1.1243	1.0742	1.0333	0.9994
0.64	1.5927	1.4890	1.4112	1.3485	1.2961	1.2127	1.1487	1.0979	1.0565	1.0221
0.66	1.6251	1.5194	1.4400	1.3761	1.3228	1.2380	1.1730	1.1215	1.0796	1.0448
0.68	1.6573	1.5495	1.4686	1.4036	1.3493	1.2631	1.1972	1.1450	1.1026	1.0674
0.70	1.6893	1.5794	1.4970	1.4308	1.3757	1.2882	1.2213	1.1684	1.1255	1.0899
0.72	1.7210	1.6091	1.5253	1.4579	1.4019	1.3130	1.2453	1.1917	1.1483	1.1123
0.74	1.7525	1.6386	1.5533	1.4849	1.4279	1.3378	1.2692	1.2150	1.1710	1.1347
0.76	1.7838	1.6680	1.5812	1.5117	1.4539	1.3625	1.2929	1.2381	1.1937	1.1570
0.78	1.8149	1.6971	1.6089	1.5383	1.4797	1.3870	1.3166	1.2611	1.2163	1.1792
0.80	1.8458	1.7261	1.6365	1.5648	1.5053	1.4114	1.3402	1.2841	1.2388	1.2014
0.82	1.8766	1.7549	1.6640	1.5912	1.5309	1.4357	1.3637	1.3070	1.2612	1.2235
0.84	1.9072	1.7836	1.6913	1.6174	1.5563	1.4600	1.3871	1.3298	1.2836	1.2455
0.86	1.9376	1.8121	1.7184	1.6436	1.5816	1.4841	1.4104	1.3526	1.3059	1.2675
0.88	1.9678	1.8405	1.7455	1.6696	1.6068	1.5082	1.4337	1.3753	1.3282	1.2895
0.90	1.9979	1.8688	1.7724	1.6955	1.6319	1.5321	1.4568	1.3979	1.3504	1.3114
0.92	2.0278	1.8969	1.7992	1.7213	1.6569	1.5560	1.4800	1.4205	1.3726	1.3332
0.94	2.0576	1.9249	1.8258	1.7469	1.6818	1.5798	1.5030	1.4430	1.3947	1.3550
0.96	2.0873	1.9527	1.8524	1.7725	1.7067	1.6035	1.5260	1.4654	1.4168	1.3768
0.98	2.1168	1.9804	1.8788	1.7980	1.7314	1.6272	1.5489	1.4878	1.4388	1.3985
1.00	2.1462	2.0080	1.9052	1.8234	1.7560	1.6507	1.5718	1.5102	1.4608	1.4202
1.50	2.8474	2.6691	2.5378	2.4349	2.3513	2.2231	2.1290	2.0568	1.9998	1.9535
2.00	3.5052	3.2925	3.1373	3.0171	2.9206	2.7745	2.6690	2.5891	2.5265	2.4761
2.50	4.1363	3.8931	3.7172	3.5823	3.4749	3.3142	3.1995	3.1135	3.0466	2.9930
3.00	4.7490	4.4782	4.2840	4.1363	4.0195	3.8464	3.7241	3.6330	3.5625	3.5064
3.50	5.3482	5.0522	4.8414	4.6822	4.5572	4.3734	4.2445	4.1492	4.0758	4.0175
4.00	5.9368	5.6175	5.3917	5.2222	5.0899	4.8965	4.7619	4.6629	4.5870	4.5269
4.50	6.5172	6.1762	5.9365	5.7576	5.6186	5.4167	5.2771	5.1749	5.0967	5.0350
5.00	7.0907	6.7294	6.4768	6.2893	6.1443	5.9347	5.7906	5.6854	5.6053	5.5422
6.00	8.2215	7.8229	7.5471	7.3441	7.1884	6.9654	6.8134	6.7033	6.6198	6.5543
7.00	9.3356	8.9034	8.6069	8.3905	8.2255	7.9910	7.8324	7.7181	7.6318	7.5643
8.00	10.4368	9.9740	9.6590	9.4305	9.2574	9.0129	8.8486	8.7307	8.6419	8.5727
9.00	11.5280	11.0369	10.7050	10.4658	10.2854	10.0320	9.8626	9.7416	9.6508	9.5801
10.00	12.6109	12.0936	11.7462	11.4972	11.3103	11.0488	10.8750	10.7512	10.6585	10.5865

Table 14 MODEL 2 EXPECTED COST FOR WEIBULL DISTRIBUTION

K \ β	1.00	1.25	1.50	1.75	2.00	2.50	3.00	3.50	4.00	4.50
0.01	0.2051	0.1980	0.1948	0.1934	0.1927	0.1924	0.1927	0.1931	0.1936	0.1940
0.02	0.2931	0.2829	0.2783	0.2760	0.2750	0.2743	0.2745	0.2749	0.2753	0.2758
0.03	0.3618	0.3492	0.3434	0.3405	0.3390	0.3380	0.3379	0.3382	0.3387	0.3391
0.04	0.4207	0.4059	0.3990	0.3955	0.3937	0.3922	0.3919	0.3921	0.3925	0.3928
0.05	0.4731	0.4565	0.4486	0.4445	0.4423	0.4404	0.4399	0.4399	0.4401	0.4405
0.06	0.5211	0.5027	0.4939	0.4893	0.4866	0.4843	0.4835	0.4834	0.4835	0.4837
0.07	0.5657	0.5457	0.5359	0.5307	0.5278	0.5249	0.5239	0.5235	0.5235	0.5237
0.08	0.6076	0.5860	0.5754	0.5696	0.5663	0.5630	0.5617	0.5611	0.5610	0.5610
0.09	0.6472	0.6241	0.6127	0.6065	0.6027	0.5990	0.5973	0.5966	0.5963	0.5962
0.10	0.6851	0.6605	0.6483	0.6415	0.6374	0.6332	0.6313	0.6303	0.6299	0.6297
0.12	0.7563	0.7290	0.7152	0.7074	0.7026	0.6973	0.6948	0.6934	0.6926	0.6922
0.14	0.8227	0.7928	0.7775	0.7687	0.7631	0.7569	0.7537	0.7519	0.7508	0.7500
0.16	0.8853	0.8529	0.8361	0.8263	0.8201	0.8129	0.8090	0.8067	0.8052	0.8042
0.18	0.9448	0.9101	0.8919	0.8811	0.8741	0.8659	0.8613	0.8585	0.8567	0.8554
0.20	1.0018	0.9647	0.9451	0.9333	0.9256	0.9164	0.9112	0.9079	0.9056	0.9040
0.22	1.0566	1.0173	0.9962	0.9835	0.9751	0.9648	0.9589	0.9551	0.9525	0.9505
0.24	1.1095	1.0680	1.0456	1.0319	1.0227	1.0115	1.0049	1.0005	0.9975	0.9952
0.26	1.1607	1.1171	1.0933	1.0786	1.0688	1.0565	1.0492	1.0443	1.0409	1.0383
0.28	1.2105	1.1647	1.1396	1.1240	1.1135	1.1001	1.0921	1.0867	1.0828	1.0799
0.30	1.2590	1.2111	1.1847	1.1682	1.1569	1.1425	1.1338	1.1278	1.1235	1.1203
0.32	1.3063	1.2564	1.2287	1.2112	1.1992	1.1838	1.1743	1.1678	1.1631	1.1595
0.34	1.3525	1.3006	1.2716	1.2532	1.2405	1.2240	1.2138	1.2068	1.2016	1.1977
0.36	1.3978	1.3439	1.3136	1.2942	1.2808	1.2633	1.2523	1.2448	1.2392	1.2349
0.38	1.4421	1.3863	1.3547	1.3345	1.3203	1.3018	1.2901	1.2819	1.2759	1.2713
0.40	1.4857	1.4279	1.3951	1.3739	1.3590	1.3395	1.3270	1.3183	1.3118	1.3068
0.42	1.5285	1.4688	1.4347	1.4126	1.3971	1.3764	1.3632	1.3540	1.3471	1.3417
0.44	1.5706	1.5091	1.4737	1.4507	1.4344	1.4127	1.3988	1.3889	1.3816	1.3759
0.46	1.6120	1.5486	1.5120	1.4881	1.4711	1.4484	1.4337	1.4233	1.4155	1.4094
0.48	1.6529	1.5876	1.5498	1.5250	1.5073	1.4835	1.4680	1.4571	1.4488	1.4424
0.50	1.6931	1.6261	1.5870	1.5613	1.5429	1.5181	1.5018	1.4903	1.4816	1.4748
0.52	1.7329	1.6640	1.6238	1.5971	1.5780	1.5521	1.5351	1.5230	1.5139	1.5067
0.54	1.7721	1.7015	1.6600	1.6324	1.6126	1.5857	1.5680	1.5553	1.5457	1.5381
0.56	1.8109	1.7385	1.6958	1.6673	1.6468	1.6188	1.6003	1.5871	1.5770	1.5691
0.58	1.8492	1.7750	1.7311	1.7018	1.6806	1.6515	1.6323	1.6185	1.6080	1.5996
0.60	1.8871	1.8112	1.7661	1.7359	1.7139	1.6839	1.6639	1.6495	1.6385	1.6298

Table 14 MODEL 2 EXPECTED COST FOR WEIBULL DISTRIBUTION (CONTINUED)

K \ β	1.00	1.25	1.50	1.75	2.00	2.50	3.00	3.50	4.00	4.50
0.62	1.9246	1.8469	1.8007	1.7695	1.7469	1.7158	1.6951	1.6801	1.6686	1.6596
0.64	1.9617	1.8823	1.8349	1.8029	1.7796	1.7474	1.7259	1.7103	1.6984	1.6890
0.66	1.9984	1.9173	1.8687	1.8359	1.8119	1.7787	1.7564	1.7403	1.7279	1.7180
0.68	2.0348	1.9520	1.9023	1.8685	1.8438	1.8096	1.7866	1.7698	1.7570	1.7468
0.70	2.0709	1.9864	1.9355	1.9009	1.8755	1.8402	1.8165	1.7991	1.7858	1.7752
0.72	2.1067	2.0205	1.9684	1.9329	1.9069	1.8706	1.8460	1.8281	1.8143	1.8034
0.74	2.1421	2.0543	2.0011	1.9647	1.9380	1.9006	1.8753	1.8568	1.8426	1.8312
0.76	2.1773	2.0878	2.0334	1.9962	1.9688	1.9304	1.9044	1.8853	1.8706	1.8589
0.78	2.2122	2.1211	2.0655	2.0275	1.9994	1.9600	1.9332	1.9135	1.8983	1.8862
0.80	2.2468	2.1540	2.0974	2.0585	2.0297	1.9892	1.9617	1.9415	1.9258	1.9133
0.82	2.2812	2.1868	2.1290	2.0893	2.0598	2.0183	1.9900	1.9692	1.9531	1.9402
0.84	2.3153	2.2193	2.1604	2.1198	2.0896	2.0471	2.0181	1.9967	1.9802	1.9669
0.86	2.3492	2.2516	2.1916	2.1501	2.1193	2.0758	2.0460	2.0240	2.0070	1.9934
0.88	2.3828	2.2836	2.2225	2.1802	2.1487	2.1042	2.0736	2.0511	2.0336	2.0196
0.90	2.4163	2.3155	2.2533	2.2101	2.1779	2.1324	2.1011	2.0780	2.0601	2.0457
0.92	2.4495	2.3472	2.2838	2.2398	2.2070	2.1604	2.1284	2.1047	2.0863	2.0716
0.94	2.4825	2.3786	2.3142	2.2693	2.2358	2.1883	2.1555	2.1313	2.1124	2.0973
0.96	2.5154	2.4099	2.3443	2.2987	2.2645	2.2159	2.1824	2.1576	2.1383	2.1228
0.98	2.5480	2.4410	2.3743	2.3278	2.2930	2.2434	2.2092	2.1838	2.1641	2.1482
1.00	2.5805	2.4719	2.4041	2.3568	2.3213	2.2708	2.2358	2.2099	2.1897	2.1734
1.50	3.3449	3.1999	3.1064	3.0394	2.9882	2.9140	2.8618	2.8227	2.7920	2.7671
2.00	4.0490	3.8709	3.7539	3.6691	3.6039	3.5087	3.4416	3.3913	3.3518	3.3200
2.50	4.7161	4.5076	4.3692	4.2683	4.1906	4.0771	3.9973	3.9377	3.8913	3.8541
3.00	5.3581	5.1214	4.9633	4.8479	4.7589	4.6294	4.5388	4.4717	4.4197	4.3783
3.50	5.9816	5.7186	5.5424	5.4137	5.3147	5.1711	5.0714	4.9979	4.9414	4.8966
4.00	6.5912	6.3035	6.1105	5.9697	5.8616	5.7055	5.5979	5.5191	5.4588	5.4112
4.50	7.1896	6.8787	6.6700	6.5181	6.4018	6.2346	6.1201	6.0366	5.9731	5.9232
5.00	7.7789	7.4461	7.2228	7.0606	6.9368	6.7597	6.6390	6.5515	6.4852	6.4333
6.00	8.9364	8.5630	8.3131	8.1325	7.9955	7.8011	7.6700	7.5758	7.5049	7.4497
7.00	10.0721	9.6619	9.3883	9.1916	9.0432	8.8344	8.6947	8.5950	8.5204	8.4625
8.00	11.1914	10.7474	10.4525	10.2416	10.0833	9.8620	9.7151	9.6109	9.5332	9.4731
9.00	12.2978	11.8226	11.5084	11.2848	11.1177	10.8855	10.7324	10.6242	10.5439	10.4820
10.00	13.3940	12.8898	12.5578	12.3227	12.1478	11.9059	11.7473	11.6358	11.5532	11.4897

Figure 1

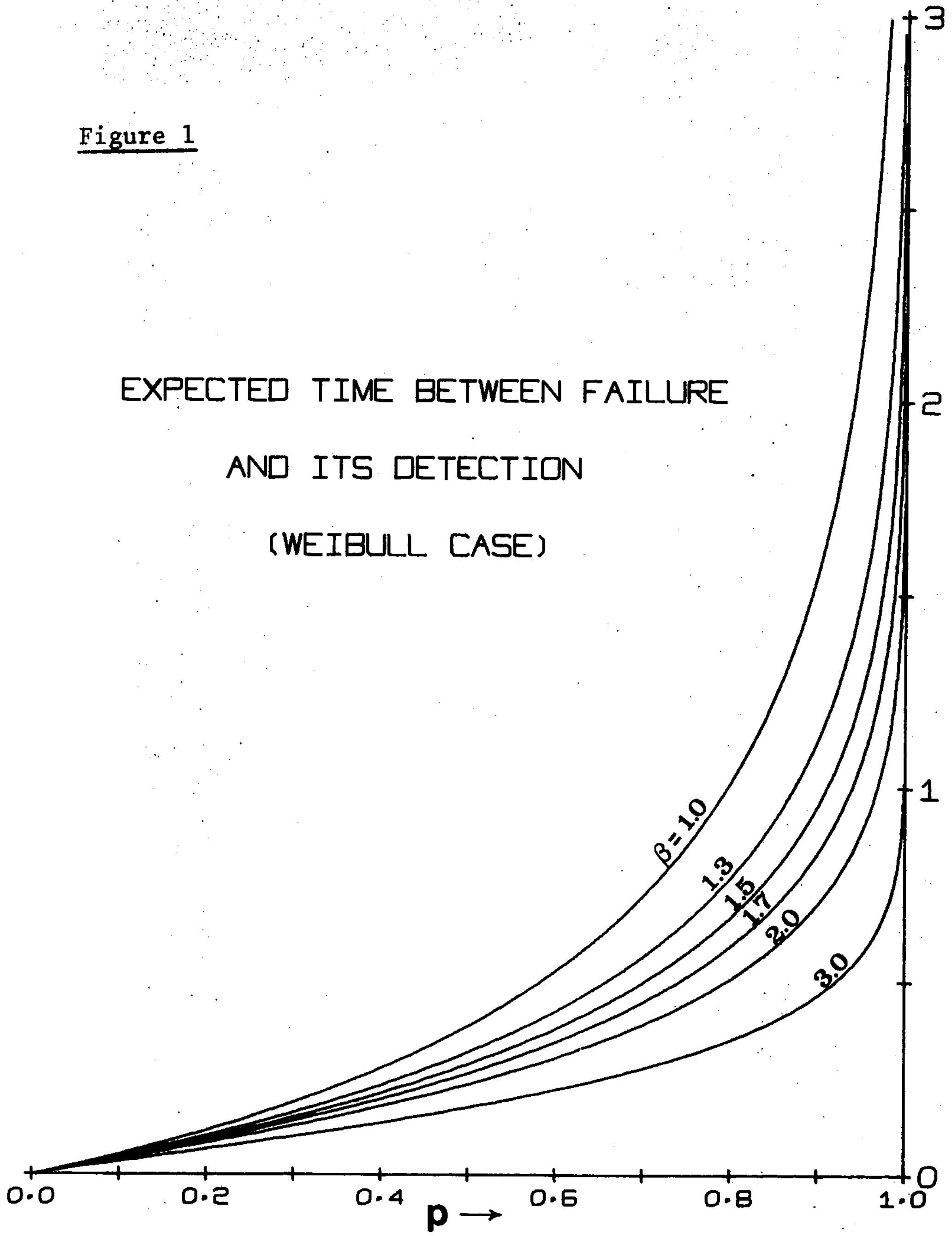
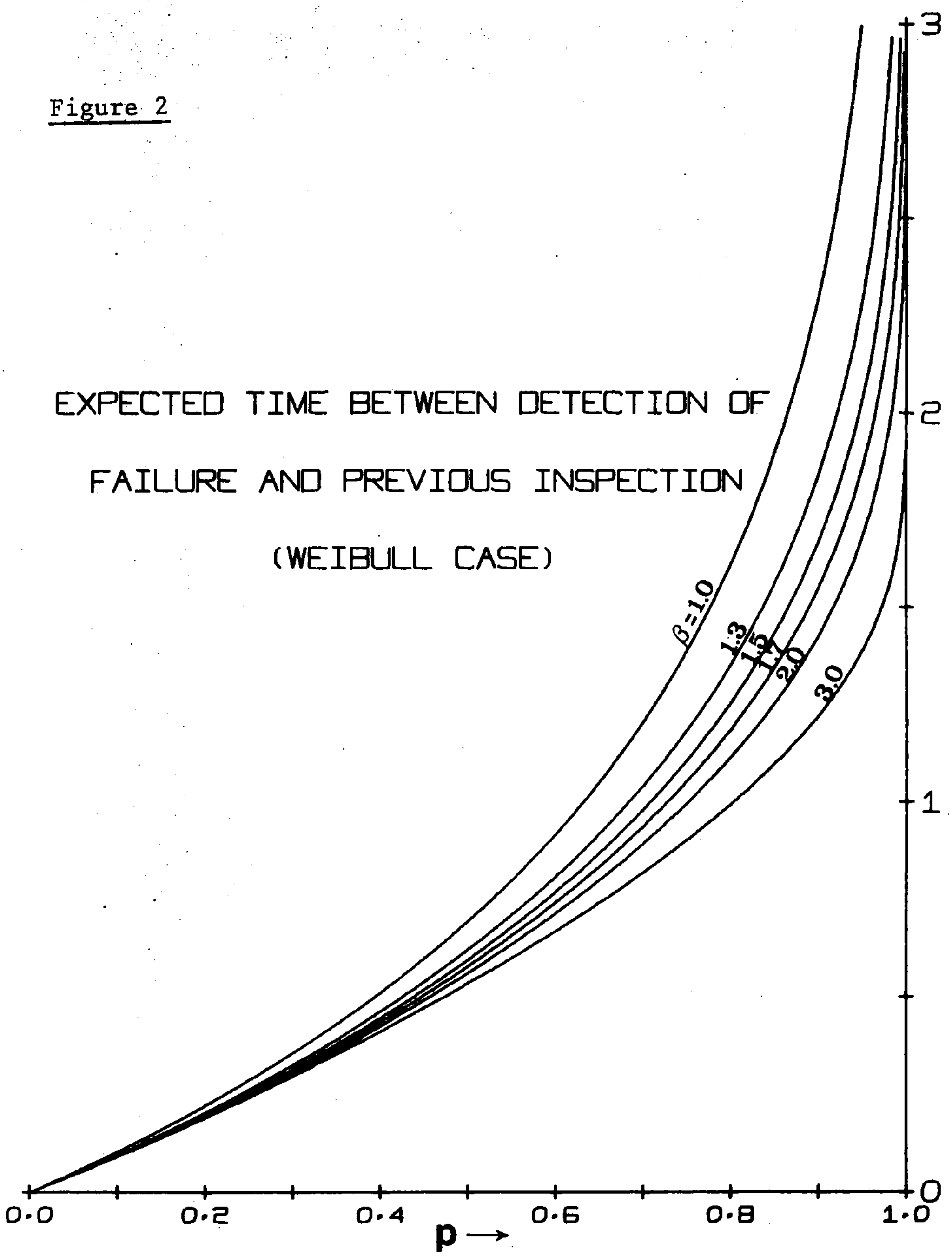

EXPECTED TIME BETWEEN FAILURE
AND ITS DETECTION
(WEIBULL CASE)
β=1.0
1.3
1.5
1.7
2.0
3.0
0
1
2
3
0·0
0·2
0·4
0·6
0·8
1·0
p →

Figure 2


EXPECTED TIME BETWEEN DETECTION OF
FAILURE AND PREVIOUS INSPECTION
(WEIBULL CASE)
β=1.0
1.3
1.5
1.7
2.0
3.0
3
2
1
0
0·0
0·2
0·4
0·6
0·8
1·0
p →

and since $\theta = -\ln(1-p) = p + \frac{p^2}{2} + \dots$,

$\theta \to 0$ and $p/\theta \to 1$ as $p \to 0$. Therefore

$$\lim_{p\to 0^+} \tau(p, \beta) = \Gamma(1 + 1/\beta, 0) = \Gamma(1 + 1/\beta) \quad ,$$

and since

$$\tau(p, \beta) = \theta^{1/\beta} \sum_{n=1}^{\infty} n^{1/\beta} q^{n-1} p \quad ,$$

this gives us the interesting limit:

$$\lim_{p\to 0^+} \{-\ln(1-p)\}^{\ell} \sum_{n=1}^{\infty} n^{\ell}(1-p)^{n-1} p = \Gamma(1+\ell) \qquad (5.2.13)$$

$$0 \leqslant \ell \leqslant 1 \quad .$$

Alternatively, note that

$$x_n - t \leqslant x_n - x_{n-1} = \delta_n$$

for

$$x_{n-1} < t \leqslant x_n \quad ,$$

and for $\beta \geqslant 1$ $h(t)$ is nondecreasing, which means that $\delta_n \leqslant \delta_1 = x_1$ for all n.

$$\therefore \quad x_n - t \leqslant x_1 \quad \text{for} \quad x_{n-1} < t \leqslant x_n \quad ,$$

so that

$$0 \leqslant E(x_N - T) \leqslant x_1$$

$$\Gamma(1 + 1/\beta) \leqslant E(x_N) \leqslant x_1 + \Gamma(1 + 1/\beta) \quad .$$

Now from section 5.1 we saw that

$$\lim_{p\to 0^+} x_1 = 0 \quad ,$$

so that

$$\lim_{p\to 0^+} E(x_N) = \Gamma(1 + 1/\beta) \quad ,$$

which is the same as (5.2.13).

5.3 An inspection rate function for $\underline{x}_p$ policies

In section 2.4 we saw that when the inspection intervals are small it is convenient to describe them by a smooth density $\phi(t)$ satisfying

$$\Phi(x_n) = \int_0^{x_n} \phi(t)dt = n \quad . \tag{5.3.1}$$

Even if the inspection intervals are not small this equation is still meaningful, and when the inspection sequence $\{x_i\}$ constitutes an $\underline{x}_p$ policy we can find a function $\phi(t)$ satisfying (5.3.1).

From (5.1.1), $\underline{x}_p$ policies are given by

$$\int_0^{x_n} h(t)dt = n\theta \quad \text{where} \quad \theta = -\ln(1-p) \quad ,$$

so that

$$\int_0^{x_n} \frac{h(t)}{\theta} dt = n$$

and the function

$$\phi_p(t) = \frac{h(t)}{\theta} \tag{5.3.2}$$

can be regarded as an inspection rate function. Note that the inspection rate function is not unique, for example

$$\hat{\phi}(t) = \frac{1}{x_n - x_{n-1}} \quad , \quad x_{n-1} < t \leqslant x_n$$

also satisfies (5.3.1), but

$$\phi_p(t) = \frac{h(t)}{\theta}$$

has the attraction that $\phi(t)$ is a smooth (differentiable) function of t provided that $f(t)$ is.

$\underline{x}_p$ policies when the inspection cost is low

In section 2.4 we gave Keller's (1974) approximation to the expected total cost for model I when $c_1 << c_2\, E(T)$, i.e.

$$E\{c(T;\ \phi)\} = \int_0^\infty \left\{c_1\ \Phi(t) + \frac{c_2}{2\phi(t)}\right\} f(t)dt \qquad (5.3.3)$$

which is minimised when

$$\phi(t) = \sqrt{\frac{c_2}{2c_1}\, h(t)} \quad . \qquad (5.3.4)$$

Comparing (5.3.4) with (5.3.2) we see that $\phi(t)$ is proportional to $\sqrt{h(t)}$ in the optimal case, and to $h(t)$ in the $\underline{x}_p$ case. However if $h(t)$ is not a simple function of t, it can be difficult to compute $\{x_i\}$ in the optimal case using (5.3.1).

We now consider the loss in using an $\underline{x}_p$ policy in the case $c_1 << c_2\, E(T)$ in place of Keller's optimal policy.

With

$$\phi_p(t) = \frac{h(t)}{\theta}\ , \quad \Phi_p(t) = \frac{1}{\theta}\int_0^t \frac{f(u)}{1 - F(u)}\, du \quad ,$$

i.e.

$$\Phi_p(t) = -\ \frac{\ln\ \bar{F}(t)}{\theta} \quad ,$$

and (5.3.3) becomes

$$E\{c(T)\} = \int_0^\infty \left\{-\ \frac{c_1\ \ln\ \bar{F}(t)}{\theta} + \frac{c_2\theta\ \bar{F}(t)}{2f(t)}\right\} f(t)dt \quad ,$$

where $\theta = -\ \ln(1-p)$ is to be determined.

$$E\{c(T)\} = \frac{c_1}{\theta} \int_0^\infty \ln \bar{F}(t)\, d\bar{F}(t) + \frac{c_2\theta}{2} \int_0^\infty \bar{F}(t)dt$$

$$= \frac{c_1}{\theta} + \frac{c_2\theta}{2} \quad , \tag{5.3.5}$$

where $\mu = E(T)$.

Differentiating (5.3.5) with respect to θ and equating to zero gives

$$-\frac{c_1}{\theta^2} + \frac{c_2\mu}{2} = 0$$

$$\theta^* = \sqrt{\frac{2c_1}{c_2\mu}} \quad . \tag{5.3.6}$$

That means that if an $\underline{x}_p$ policy is used, the minimum expected cost for model I is, from (5.3.5) and (5.3.6)

$$C_p^* = \sqrt{2c_1 \; c_2\mu} \quad . \tag{5.3.7}$$

In this case there is no difficulty in computing $\{x_i\}$ since from (5.3.6)

$$\theta^* = -\ln(1-p^*) = \sqrt{\frac{2c_1}{c_2\mu}} \quad ,$$

$$p^* = 1 - e^{-\sqrt{2c_1/c_2\mu}} \quad , \tag{5.3.8}$$

and using (5.1.2)

$$F(x_n) = 1 - (1-p^*)^n \; .$$

By assumption

$$\frac{c_1}{c_2\mu} << 1 \quad ,$$

so that (5.3.8) gives

$$p^* \simeq 1 - \left\{1 - \sqrt{\frac{2c_1}{c_2\mu}}\right\}$$

$$p^* \simeq \sqrt{\frac{2c_1}{c_2\mu}} \ .$$

These results are for model I; for model II simply replace c_2 by $2c_2$.

Table 15 gives some values of p^* calculated from (5.3.8) in the Weibull case when $c_1/\alpha c_2 = 0.01$. The corresponding exact values (which were calculated in section 5.2) are given for comparison.

Table 15

Values of the approximation to p^* given by (5.3.8) in the Weibull case for $c_1/\alpha c_2 = 0.01$.

The exact values are taken from Tables 11 and 12.

	Model I		Model II	
β	Approximate	Exact	Approximate	Exact
1.0	0.1319	0.1290	0.0952	0.0951
1.5	0.1383	0.1431	0.0999	0.1002
2.0	0.1395	0.1546	0.1008	0.1016
2.5	0.1394	0.1653	0.1007	0.1019
3.0	0.1390	0.1757	0.1004	0.1020
3.5	0.1385	0.1856	0.1001	0.1019
4.0	0.1380	0.1953	0.0997	0.1019
4.5	0.1376	0.2047	0.0994	0.1018

The efficiency of $\underline{x}_p$ policies when $c_1 \ll c_2\mu$

When T has a Weibull distribution and $c_1 \ll c_2\mu$ the minimum expected cost can be calculated quite easily, and from (2.4.10) this is

$$C_{min} = \sqrt{\frac{2c_1 c_2\alpha}{\beta}}\,\Gamma\left(\frac{1}{2\beta} + \frac{1}{2}\right) \quad .$$

Now $\mu = \alpha\Gamma(1 + 1/\beta)$ in this case so that if an $\underline{x}_p$ policy is used the minimum expected cost is, from (5.3.7)

$$C_p^* = \sqrt{2c_1 c_2 \alpha\Gamma(1 + 1/\beta)} \quad .$$

A measure of the efficiency of $\underline{x}_p$ policies is

$$E = \frac{C_{min}}{C_p^*} \leqslant 1 \quad .$$

This gives

$$E = \sqrt{\frac{2c_1 c_2\alpha}{\beta}}\,\Gamma\left(\frac{1}{2\beta} + \frac{1}{2}\right) \Big/ \sqrt{2c_1 c_2 \alpha\Gamma(1 + 1/\beta)}$$

$$E = \frac{\Gamma\left(\frac{1}{2\beta} + \frac{1}{2}\right)}{\sqrt{\beta\Gamma(1 + 1/\beta)}} \quad . \tag{5.3.9}$$

But $\Gamma(1 + 1/\beta) = 1/\beta\ \Gamma(1/\beta)$ so that

$$E(\beta) = \frac{\Gamma\left(\frac{1}{2\beta} + \frac{1}{2}\right)}{\sqrt{\Gamma\left(\frac{1}{\beta}\right)}} \tag{5.3.10}$$

for model I.

Since $E(\beta)$ is independent of c_2, replacing c_2 by $2c_2$ leaves E unchanged, thus (5.3.10) holds for both model I and model II.

Table 16

Limiting efficiency of $\underline{x}_p$ policies for

models I and II in the Weibull case

β	$E(\beta)$
1.0	1.0000
1.25	0.9904
1.5	0.9700
1.75	0.9457
2.0	0.9204
2.5	0.8716
3.0	0.8273
3.5	0.7881
4.0	0.7534

Table 16 gives the limiting efficiency of $\underline{x}_p$ policies for some values of β in the range $1 \leq \beta \leq 4$. From the table we see that they are fairly efficient, but particularly so when β is near 1. The efficiency of $\underline{x}_p$ policies for general values of $c_1/\alpha c_2$ is discussed in Chapter 7.

Chapter 6

Robustness of $\underline{x}_p$ policies in the Weibull case

6.1 Robustness of $\underline{x}_p$ policies: the approach

Using the results of section 5.2 it is possible to find the optimal $\underline{x}_p$ policy in the Weibull case when the values of the parameters α, β, c_1 and c_2 are known. However, in practical situations some, or all of these values will be estimated and therefore subject to error.

To investigate the robustness of $\underline{x}_p$ policies we will compare the expected total cost achieved by using an 'optimal' $\underline{x}_p$ policy calculated from estimated parameter values $\hat{\alpha}$, $\hat{\beta}$, $\hat{c}_1$ and $\hat{c}_2$ with the minimum expected cost due to using the optimal $\underline{x}_p$ policy calculated from the exact parameter values, where one or more of the estimated values is in error.

Model I

Suppose that $\hat{\alpha}$, $\hat{\beta}$, $\hat{c}_1$ and $\hat{c}_2$ are estimated values of α, β, c_1 and c_2 respectively. In section 5.2 we found that the value of p giving the optimal $\underline{x}_p$ policy depends on $K = \dfrac{c_1}{\alpha c_2}$ and β . Let this value be $p^* = p^*(K, \beta)$. From (5.2.1) the optimal $\underline{x}_p$ policy is $\{\alpha\, z_n^* (K, \beta)\}$ where

$$z_n^* (K, \beta) = \{- n\, \ell n(1-p^*)\}^{1/\beta} \quad ,$$

and the corresponding expected cost from (5.2.11) is

$$C^* = C^*(\alpha, \beta, c_1, c_2) = \frac{c_1}{p^*} + \alpha c_2\, \tau(p^*, \beta) - \alpha c_2\, \Gamma(1 + 1/\beta).$$

Using the values $\hat{\alpha}$, $\hat{\beta}$, $\hat{c}_1$ and $\hat{c}_2$, with $\hat{K} = \hat{c}_1/\hat{\alpha}\hat{c}_2$ the estimated optimal $\underline{x}_p$ policy is $\{\hat{x}_n\}$ where

$$\hat{x}_n = \hat{\alpha}\ z_n^*(\hat{K},\ \hat{\beta}) \quad ,$$

which from (2.2.1) gives an expected cost of

$$\hat{C} = \hat{C}(\alpha,\beta,c_1,c_2,\hat{\alpha},\hat{\beta},\hat{c}_1,\hat{c}_2) = c_1 \sum_{j=0}^{\infty} \bar{F}(\hat{x}_j) + c_2 \sum_{n=1}^{\infty} \int_{\hat{x}_{n-1}}^{\hat{x}_n} (\hat{x}_n - t)\ f(t)dt \tag{6.1.1}$$

$$= c_1 \sum_{j=0}^{\infty} \bar{F}(\hat{x}_j) + c_2 \sum_{n=1}^{\infty} \hat{x}_n \left\{F(\hat{x}_n) - F(\hat{x}_{n-1})\right\}$$

$$- \alpha c_2\ \Gamma(1 + 1/\beta)$$

where

$$\bar{F}(t) = \exp\left\{- \left(\frac{t}{\alpha}\right)^{\beta}\right\} \quad .$$

The difference in these costs as a proportion of C^* is

$$\Delta_I = \frac{\hat{C} - C^*}{C^*} \quad , \tag{6.1.2}$$

so that

$$\Delta_I = \frac{K \sum_{j=0}^{\infty} \bar{F}(\hat{x}_j) + \frac{\hat{\alpha}}{\alpha} \sum_{n=1}^{\infty} z_n^*\ (\hat{K},\ \hat{\beta}) \left\{F(\hat{x}_n) - F(\hat{x}_{n-1})\right\} - \frac{K}{p^*} - \tau(p^*,\ \beta)}{\frac{K}{p^*} + \tau(p^*,\ \beta) - \Gamma(1 + 1/\beta)}$$

on dividing numerator and denominator by αc_2 .

Now

$$F(\hat{x}_n) = 1 - \exp\left\{- \left[\frac{\hat{\alpha}}{\alpha}\ z_n^*\ (\hat{K},\ \hat{\beta})\right]^{\beta}\right\}$$

so that Δ_I depends on $\frac{\hat{\alpha}}{\alpha}$, β, $\hat{\beta}$, K, $\hat{K}$. Since Δ_I depends on α only through the ratio $\frac{\hat{\alpha}}{\alpha}$, Δ_I is independent of α when $\frac{\hat{\alpha}}{\alpha}$ is given, so that in this case we can perform all calculations with $\alpha = 1$, and tabulate Δ_I as a function of K and β for values of

$$\delta_\alpha = \frac{\hat{\alpha}}{\alpha}, \quad \delta_\beta = \frac{\hat{\beta}}{\beta}, \quad \delta_K = \frac{\hat{K}}{K} \quad .$$

Model II

The analysis for model II is similar to that of model I and is not given here to avoid repetition. It turns out that in this case with

$$\hat{C} = c_1 \sum_{j=0}^{\infty} \bar{F}(\hat{x}_j) + c_2 \sum_{n=1}^{\infty} \int_{\hat{x}_{n-1}}^{\hat{x}_n} (\hat{x}_n - \hat{x}_{n-1}) \, f(t)dt \tag{6.1.3}$$

and

$$C^* = \frac{c_1}{p^*} + \alpha c_2 \, p^* \, \tau(p^*, \beta)$$

$$\Delta_{II} = \frac{K \sum_{j=0}^{\infty} \bar{F}(\hat{x}_j) + \frac{\hat{\alpha}}{\alpha} \sum_{n=1}^{\infty} \left\{ z_n^*(\hat{K},\hat{\beta}) - z_{n-1}^*(\hat{K},\hat{\beta}) \right\} \left\{ F(\hat{x}_n) - F(\hat{x}_{n-1}) \right\} - \frac{K}{p^*} - p^* \tau(p^*,\beta)}{\frac{K}{p^*} + p^* \, \tau(p^*, \beta)}$$

so that again, Δ_{II} can be regarded as a function of $K, \beta, \delta_\alpha, \delta_\beta, \delta_K$.

6.2 Robustness of $\underline{x}_p$ policies: theory

The computation of Δ for model I and model II is tedious since $\hat{C}$ must be calculated by summing the series in (6.1.1) and (6.1.3), and to do this accurately using a digital computer we need first to bound the errors incurred by summing only a finite number of terms of these series. However, in the special case when only K and α are in error, a simpler method for calculating $\hat{C}$ is available.

For the remainder of this section, let

$$p^* = p^*(K, \beta) \quad ,$$
$$q^* = 1 - p^* \quad ,$$
$$\hat{p} = p^*(\hat{K}, \hat{\beta}) \quad ,$$
$$\hat{q} = 1 - \hat{p} \quad .$$

Special case $\hat{\beta} = \beta$

Using (5.2.1), $\hat{x}_n = \hat{\alpha}\{-n\ell n\hat{q}\}^{1/\hat{\beta}}$ and with $\bar{F}(t) = \exp\left\{-\left(\frac{t}{\alpha}\right)^{\beta}\right\}$

we have

$$\bar{F}(\hat{x}_n) = \exp\left\{-\left(\frac{\hat{\alpha}\left[-n\ \ell n\ \hat{q}\right]^{1/\beta}}{\alpha}\right)^{\hat{\beta}}\right\}$$

$$= \exp\left\{n\left(\frac{\hat{\alpha}}{\alpha}\right)^{\beta}\ell n\ \hat{q}\right\} \quad \text{when} \quad \hat{\beta} = \beta$$

$$\bar{F}(\hat{x}_n) = \left\{\hat{q}^{(\hat{\alpha}/\alpha)^{\beta}}\right\}^n \quad .$$

Writing $\tilde{q}$ for $\hat{q}^{(\hat{\alpha}/\alpha)^{\beta}}$ and letting $\tilde{p} = 1 - \tilde{q}$ we have

$$\bar{F}(\hat{x}_n) = (1 - \tilde{p})^n \quad ,$$

so that from (5.1.2) we see that $\{\hat{x}_n\}$ is an $\underline{x}_p$ policy for F(t) with $p = \tilde{p}$.

For model I, we have from (5.2.11) that

$$\hat{C} = \frac{c_1}{\tilde{p}} + \alpha c_2\ \tau(\tilde{p}, \beta) - \alpha c_2\ \Gamma(1 + 1/\beta) \quad ,$$

and with

$$C^* = \frac{c_1}{p^*} + \alpha c_2\ \tau(p^*, \beta) - \alpha c_2\ \Gamma(1 + 1/\beta)$$

$$\Delta_I = \frac{\hat{C} - C^*}{C^*} = \frac{\frac{K}{\tilde{p}} - \frac{K}{p^*} + \tau(\tilde{p}, \beta) - \tau(p^*, \beta)}{\frac{K}{p^*} + \tau(p^*, \beta) - \Gamma(1 + 1/\beta)} \quad .$$

Similarly for model II

$$\Delta_{II} = \frac{\frac{K}{\tilde{p}} - \frac{K}{p^*} + \tilde{p}\tau(\tilde{p}, \beta) - p^*\tau(p^*, \beta)}{\frac{K}{p^*} + p^*\tau(p^*, \beta)} \quad .$$

Since $\tilde{p}$ depends on $\frac{\hat{\alpha}}{\alpha}$, $\hat{K}$ and $\hat{\beta}$ (= β in this case), both Δ_I and Δ_{II} depend only on K, $\hat{K}$, β and $\frac{\hat{\alpha}}{\alpha}$, in accordance with the argument given at the end of the previous section.

<u>General case $\beta \neq \hat{\beta}$</u>

For model I we will consider the terms in (6.1.1) separately. First the expected number of inspections, which is

$$\sum_{j=0}^{\infty} \bar{F}(\hat{x}_j) = \sum_{n=1}^{\infty} \int_{\hat{x}_{n-1}}^{\hat{x}_n} n\ f(t)dt \quad . \tag{6.2.1}$$

Let E_k be the error in approximating the infinite sum on the right hand side of (6.2.1) by the first k terms. Then

$$E_k = \sum_{n=k+1}^{\infty} \int_{\hat{x}_{n-1}}^{\hat{x}_n} n\, f(t)dt \quad .$$

Now $\{\hat{x}_n\}$ is an $\underline{x}_p$ policy for the distribution

$$\hat{F}(t) = 1 - \exp\left\{- \left(\frac{t}{\hat{\alpha}}\right)^{\hat{\beta}}\right\} \quad ,$$

with $p = \hat{p}$. Therefore, using (5.1.1) and (5.1.2) and the fact that $H(t)$ is increasing, this gives

$$n - 1 < \frac{(t/\hat{\alpha})^{\hat{\beta}}}{-\ \ell n\ \hat{q}} < n \quad \text{for} \quad \hat{x}_{n-1} < t < \hat{x}_n \quad ,$$

so that

$$E_k < \sum_{n=k+1}^{\infty} \int_{\hat{x}_{n-1}}^{\hat{x}_n} \left\{1 + \frac{(t/\hat{\alpha})^{\hat{\beta}}}{-\ \ell n\ \hat{q}}\right\} f(t)dt$$

$$E_k < \frac{1}{-\ \ell n\ \hat{q}} \int_{\hat{x}_k}^{\infty} \left(\frac{t}{\hat{\alpha}}\right)^{\hat{\beta}} \left(\frac{\beta}{\alpha}\right) \left(\frac{t}{\alpha}\right)^{\beta-1} \exp\left\{- \left(\frac{t}{\alpha}\right)^{\beta}\right\} dt + \int_{\hat{x}_k}^{\infty} f(t)dt$$

$$E_k < \frac{1}{-\ \ell n\ \hat{q}} \left(\frac{\alpha}{\hat{\alpha}}\right)^{\hat{\beta}} \int_{\hat{x}_k}^{\infty} \left(\frac{t}{\alpha}\right)^{\hat{\beta}} \exp\left\{- \left(\frac{t}{\alpha}\right)^{\beta}\right\} d\left(\frac{t}{\alpha}\right)^{\beta} + \bar{F}(\hat{x}_k)$$

$$E_k < \frac{1}{-\ \ell n\ \hat{q}} \left(\frac{\alpha}{\hat{\alpha}}\right)^{\hat{\beta}} \int_{(\hat{x}_k/\alpha)^{\beta}}^{\infty} z^{\hat{\beta}/\beta}\, e^{-z}\, dz + \bar{F}(\hat{x}_k)$$

$$E_k < \frac{1}{-\ \ell n\ \hat{q}} \left(\frac{\alpha}{\hat{\alpha}}\right)^{\hat{\beta}} \Gamma(1 + \hat{\beta}/\beta,\ (\hat{x}_k/\alpha)^{\beta}) + \exp\left\{- (\hat{x}_k/\alpha)^{\beta}\right\} \quad . \qquad (6.2.2)$$

Where $\Gamma(\nu,\ x)$ is the incomplete gamma function.

Since the right hand side of (6.2.2) tends to zero as k tends to infinity, for any $\varepsilon > 0$ we can calculate

$$\sum_{j=0}^{\infty} \bar{F}(\hat{x}_j)$$

to an error of at most ε by summing to k terms, where $k(\varepsilon)$ is such that the right hand side of (6.2.2) is less than or equal to ε .

The second term in (6.1.1) can be dealt with in a similar manner as follows:

$$\sum_{n=1}^{\infty} \int_{\hat{x}_{n-1}}^{\hat{x}_n} (\hat{x}_n - t)\ f(t)dt = \sum_{n=1}^{\infty} \int_{\hat{x}_{n-1}}^{\hat{x}_n} \hat{x}_n\ f(t)dt - \Gamma(1 + 1/\beta) \quad , \tag{6.2.3}$$

so that if E_k' is the error in summing the series on the right hand side of (6.2.4) to k terms,

$$E_k' = \sum_{n=k+1}^{\infty} \int_{\hat{x}_{n-1}}^{\hat{x}_n} \hat{x}_n\ f(t)dt \quad .$$

For increasing failure rate distributions $\{x_n - x_{n-1}\}$ is decreasing, so that if $\hat{\beta} > 1$, $\hat{x}_n \leqslant n\hat{x}_1$ and

$$E_k' \leqslant \hat{x}_1 \sum_{n=k+1}^{\infty} \int_{\hat{x}_{n-1}}^{\hat{x}_n} n\ f(t)dt \quad .$$

This implies that if

$$\sum_{j=0}^{\infty} \bar{F}(\hat{x}_j)$$

is determined to within ε by summing the right hand side of (6.2.1) to $k(\varepsilon)$ terms, then

$$\sum_{n=1}^{\infty} \int_{\hat{x}_{n-1}}^{\hat{x}_n} (\hat{x}_n - t)\, f(t)dt$$

will be determined to within $\hat{x}_1\, \varepsilon$. Using this fact we see from (6.1.1), (6.2.1) and (6.2.3) that by summing both series to $k(\varepsilon)$ terms, we incur an absolute error of at most $c_1\, \varepsilon + c_2\, \hat{x}_1\, \varepsilon = (c_1 + c_2\, \hat{x}_1)\varepsilon$ in $\hat{C}$. Now since

$$\sum_{j=0}^{\infty} \bar{F}(\hat{x}_j) > 1 ,$$

$\hat{C} > c_1$, so that the relative error $\hat{\varepsilon}$ in $\hat{C}$ is such that

$$\hat{\varepsilon} < \frac{(c_1 + c_2\, \hat{x}_1)\varepsilon}{\hat{C}} < \frac{(c_1 + c_2\, \hat{x}_1)\varepsilon}{c_1} \tag{6.2.4}$$

Therefore if we wish to calculate $\hat{C}$ to within ε', say, we can take

$$\varepsilon = \frac{c_1\, \varepsilon'}{c_1 + c_2\, \hat{x}_1} ,$$

since from (6.2.4) this gives $\hat{\varepsilon} < \varepsilon'$. For cost model II, comparing (6.1.1) with (6.1.3) we see that an identical analysis follows, since the first term in both expressions for $\hat{C}$ is the same, and in (6.1.3) $\hat{x}_n - \hat{x}_{n-1} \leq \hat{x}_n$ for $n = 1, 2, \ldots$.

6.3 Robustness of $\underline{x}_p$ policies: results

In section 6.2 it was shown that for both cost models Δ, the percentage change in the expected cost, can be written as a function of the five variables K, β, δ_α, δ_β, δ_K, where $\delta_\alpha = \hat{\alpha}/\alpha$, $\delta_\beta = \hat{\beta}/\beta$, $\delta_K = \hat{K}/K$. Ideally, Δ should be small whenever the δ's are in the neighbourhood of unity i.e. small errors in the parameter values should cause small increases in the expected cost.

In the sensitivity analysis presented here we will consider errors of $\pm$ 10 per cent in the parameters α, β and K, i.e. δ_α, δ_β and δ_K will be given the values 0.9, 1.0 and 1.1. The tables of Δ for the 27 combinations of these δ values are presented in Tables 17 to 43 for model I, and in Tables 44 to 70 for model II.

Model I

A surprising feature of the tables is that some Δ values are negative, indicating that an erroneous parameter value can sometimes reduce the expected cost by a small amount. This can only happen when $\hat{\beta} \neq \beta$ since we have seen in section 6.2 that the inspection policy calculated from the values $\hat{\alpha}$, β and $\hat{K}$ is itself an $\underline{x}_p$ policy, and can therefore offer no improvement over the 'best' $\underline{x}_p$ policy. Perturbing β however, takes us outside the class of $\underline{x}_p$ policies, so that a reduction in the expected cost is then possible.

Tables 17 to 43 may briefly be summarized as follows:

(i) none of the tabulated values exceed 20 per cent, indicating a fair degree of robustness.

(ii) the smaller values of Δ occur when $\hat{\alpha} = \alpha$, i.e. when there is no error in α.

(iii) in nearly every case a positive error in β yields a larger Δ value than a negative error in β (all other variables fixed).

SENSITIVITY OF XP POLICIES IN THE WEIBULL CASE: MODEL I

TABLE 17
DA = -0.1, DB = -0.1, DK = -0.1

K	BETA 2.0	3.0	4.0	5.0	6.0
0.01	0.68	2.56	5.54	9.33	13.89
0.10	1.57	3.42	5.91	8.91	12.43
1.00	1.49	2.49	3.76	5.30	7.16
5.00	0.62	0.98	1.51	2.23	3.15
10.00	0.33	0.56	0.90	1.39	2.03

TABLE 18
DA = -0.1, DB = -0.1, DK = 0.0

K	BETA 2.0	3.0	4.0	5.0	6.0
0.01	-0.41	0.98	3.48	6.77	10.80
0.10	0.60	2.07	4.19	6.82	9.96
1.00	0.87	1.74	2.89	4.31	6.06
5.00	0.37	0.71	1.21	1.88	2.75
10.00	0.20	0.41	0.73	1.18	1.79

TABLE 19
DA = -0.1, DB = -0.1, DK = 0.1

K	BETA 2.0	3.0	4.0	5.0	6.0
0.01	-1.16	-0.20	1.86	4.72	8.31
0.10	-0.07	1.06	2.86	5.17	7.98
1.00	0.45	1.19	2.22	3.54	5.16
5.00	0.21	0.51	0.98	1.61	2.42
10.00	0.11	0.30	0.60	1.02	1.59

TABLE 20
DA = -0.1, DB = 0.0, DK = -0.1

K	BETA 1.0	2.0	3.0	4.0	5.0
0.01	1.23	3.18	5.91	9.45	13.84
0.10	1.17	2.86	5.13	8.00	11.50
1.00	0.98	2.12	3.54	5.30	7.43
5.00	0.70	1.28	2.02	2.95	4.12
10.00	0.56	0.94	1.44	2.11	2.97

TABLE 21
DA = -0.1, DB = 0.0, DK = 0.0

K	BETA 1.0	2.0	3.0	4.0	5.0
0.01	0.55	2.00	4.21	7.23	11.07
0.10	0.54	1.81	3.67	6.13	9.22
1.00	0.49	1.42	2.69	4.31	6.32
5.00	0.39	0.94	1.64	2.54	3.66
10.00	0.33	0.72	1.21	1.85	2.67

TABLE 22
DA = -0.1, DB = 0.0, DK = 0.1

K	BETA 1.0	2.0	3.0	4.0	5.0
0.01	0.17	1.16	2.94	5.49	8.85
0.10	0.17	1.07	2.57	4.68	7.41
1.00	0.20	0.92	2.04	3.54	5.42
5.00	0.19	0.69	1.36	2.21	3.29
10.00	0.17	0.55	1.02	1.64	2.43

TABLE 23
DA = -0.1, DB = 0.1, DK = -0.1

K	BETA 1.0	2.0	3.0	4.0	5.0
0.01	2.32	6.90	10.37	14.47	19.56
0.10	1.99	4.97	7.60	10.88	14.97
1.00	1.51	3.18	5.03	7.36	10.17
5.00	1.21	2.25	3.42	4.83	6.54
10.00	1.08	1.86	2.72	3.77	5.09

TABLE 24
DA = -0.1, DB = 0.1, DK = 0.0

K	BETA 1.0	2.0	3.0	4.0	5.0
0.01	1.52	5.60	8.53	12.07	16.52
0.10	1.27	3.81	6.00	8.84	12.47
1.00	0.96	2.38	4.07	6.25	8.92
5.00	0.82	1.82	2.96	4.33	5.99
10.00	0.77	1.56	2.40	3.43	4.70

TABLE 25
DA = -0.1, DB = 0.1, DK = 0.1

K	BETA 1.0	2.0	3.0	4.0	5.0
0.01	1.03	4.67	7.14	10.16	14.08
0.10	0.83	2.97	4.77	7.22	10.46
1.00	0.59	1.78	3.31	5.35	7.90
5.00	0.54	1.48	2.58	3.91	5.52
10.00	0.54	1.31	2.15	3.15	4.39

SENSITIVITY OF XP POLICIES IN THE WEIBULL CASE: MODEL I

TABLE 26

DA = 0.0, DB = -0.1, DK = -0.1

K	BETA 2.0	3.0	4.0	5.0	6.0
0.01	-1.94	-2.55	-2.69	-2.68	-2.63
0.10	-0.80	-0.91	-0.84	-0.75	-0.65
1.00	-0.10	-0.09	-0.06	-0.03	-0.01
5.00	0.01	0.05	0.07	0.08	0.08
10.00	0.05	0.08	0.08	0.08	0.08

TABLE 27

DA = 0.0, DB = -0.1, DK = 0.0

K	BETA 2.0	3.0	4.0	5.0	6.0
0.01	-2.09	-2.74	-2.90	-2.90	-2.84
0.10	-0.92	-1.03	-0.95	-0.84	-0.74
1.00	-0.11	-0.05	0.00	0.04	0.07
5.00	0.09	0.13	0.14	0.14	0.13
10.00	0.12	0.13	0.13	0.12	0.11

TABLE 28

DA = 0.0, DB = -0.1, DK = 0.1

K	BETA 2.0	3.0	4.0	5.0	6.0
0.01	-2.00	-2.68	-2.85	-2.86	-2.81
0.10	-0.83	-0.94	-0.86	-0.74	-0.63
1.00	0.01	0.08	0.13	0.17	0.19
5.00	0.20	0.22	0.21	0.20	0.18
10.00	0.21	0.19	0.17	0.15	0.14

TABLE 29

DA = 0.0, DB = 0.0, DK = -0.1

K	BETA 1.0	2.0	3.0	4.0	5.0
0.01	0.13	0.13	0.14	0.14	0.14
0.10	0.11	0.12	0.12	0.12	0.12
1.00	0.08	0.07	0.05	0.04	0.03
5.00	0.05	0.02	0.01	0.01	0.01
10.00	0.03	0.01	0.01	0.00	0.00

TABLE 30

DA = 0.0, DB = 0.0, DK = 0.0

K	BETA 1.0	2.0	3.0	4.0	5.0
0.01	-0.00	0.00	0.00	0.01	0.01
0.10	-0.01	0.00	0.01	0.02	0.02
1.00	0.00	0.00	0.00	0.00	0.00
5.00	0.00	0.00	0.00	0.00	0.00
10.00	0.00	0.00	0.00	0.00	0.00

TABLE 31

DA = 0.0, DB = 0.0, DK = 0.1

K	BETA 1.0	2.0	3.0	4.0	5.0
0.01	0.10	0.11	0.11	0.11	0.12
0.10	0.09	0.10	0.10	0.11	0.11
1.00	0.07	0.05	0.04	0.03	0.03
5.00	0.04	0.02	0.01	0.01	0.01
10.00	0.02	0.01	0.01	0.00	0.00

TABLE 32

DA = 0.0, DB = 0.1, DK = -0.1

K	BETA 1.0	2.0	3.0	4.0	5.0
0.01	0.91	3.37	3.78	3.80	3.72
0.10	0.69	1.72	1.64	1.48	1.32
1.00	0.37	0.49	0.39	0.33	0.30
5.00	0.24	0.25	0.23	0.21	0.20
10.00	0.22	0.22	0.19	0.17	0.16

TABLE 33

DA = 0.0, DB = 0.1, DK = 0.0

K	BETA 1.0	2.0	3.0	4.0	5.0
0.01	0.71	3.25	3.69	3.72	3.63
0.10	0.53	1.58	1.50	1.33	1.17
1.00	0.24	0.34	0.24	0.19	0.17
5.00	0.12	0.14	0.14	0.14	0.13
10.00	0.11	0.13	0.13	0.12	0.11

TABLE 34

DA = 0.0, DB = 0.1, DK = 0.1

K	BETA 1.0	2.0	3.0	4.0	5.0
0.01	0.76	3.38	3.84	3.87	3.78
0.10	0.59	1.65	1.57	1.38	1.21
1.00	0.26	0.30	0.19	0.13	0.10
5.00	0.07	0.08	0.08	0.08	0.09
10.00	0.05	0.08	0.08	0.09	0.08

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TABLE 35
DA = 0.1, DB = -0.1, DK = -0.1

K	BETA 2.0	3.0	4.0	5.0	6.0
0.01	-1.53	-1.29	-0.10	1.70	3.98
0.10	-0.35	0.44	1.84	3.63	5.74
1.00	0.63	1.64	2.78	3.89	4.89
5.00	0.89	1.41	1.73	1.93	2.04
10.00	0.81	1.05	1.15	1.20	1.21

TABLE 36
DA = 0.1, DB = -0.1, DK = 0.0

K	BETA 2.0	3.0	4.0	5.0	6.0
0.01	-0.85	-0.29	1.23	3.36	5.98
0.10	0.29	1.39	3.06	5.11	7.46
1.00	1.17	2.31	3.49	4.59	5.54
5.00	1.22	1.68	1.94	2.09	2.17
10.00	1.02	1.20	1.26	1.27	1.27

TABLE 37
DA = 0.1, DB = -0.1, DK = 0.1

K	BETA 2.0	3.0	4.0	5.0	6.0
0.01	-0.01	0.84	2.67	5.09	8.00
0.10	1.07	2.43	4.35	6.62	9.17
1.00	1.77	2.99	4.19	5.24	6.14
5.00	1.54	1.93	2.13	2.24	2.29
10.00	1.23	1.33	1.35	1.34	1.32

TABLE 38
DA = 0.1, DB = 0.0, DK = -0.1

K	BETA 1.0	2.0	3.0	4.0	5.0
0.01	0.09	0.82	2.14	3.99	6.34
0.10	0.10	0.76	1.87	3.37	5.21
1.00	0.12	0.65	1.41	2.29	3.19
5.00	0.12	0.46	0.80	1.09	1.31
10.00	0.12	0.35	0.54	0.68	0.78

TABLE 39
DA = 0.1, DB = 0.0, DK = 0.0

K	BETA 1.0	2.0	3.0	4.0	5.0
0.01	0.45	1.61	3.33	5.55	8.26
0.10	0.45	1.47	2.88	4.65	6.73
1.00	0.41	1.12	1.97	2.88	3.76
5.00	0.32	0.68	1.00	1.25	1.44
10.00	0.26	0.48	0.65	0.76	0.85

TABLE 40
DA = 0.1, DB = 0.0, DK = 0.1

K	BETA 1.0	2.0	3.0	4.0	5.0
0.01	1.00	2.56	4.63	7.19	10.21
0.10	0.97	2.31	3.99	5.98	8.26
1.00	0.81	1.65	2.54	3.44	4.30
5.00	0.57	0.90	1.18	1.40	1.56
10.00	0.45	0.61	0.74	0.84	0.90

TABLE 41
DA = 0.1, DB = 0.1, DK = -0.1

K	BETA 1.0	2.0	3.0	4.0	5.0
0.01	0.79	4.36	6.55	8.90	11.66
0.10	0.63	2.44	3.59	5.01	6.78
1.00	0.32	0.78	1.24	1.87	2.59
5.00	0.10	0.21	0.40	0.62	0.84
10.00	0.05	0.10	0.22	0.35	0.47

TABLE 42
DA = 0.1, DB = 0.1, DK = 0.0

K	BETA 1.0	2.0	3.0	4.0	5.0
0.01	1.13	5.28	7.91	10.65	13.79
0.10	0.98	3.19	4.64	6.30	8.28
1.00	0.57	1.17	1.69	2.35	3.06
5.00	0.21	0.33	0.52	0.74	0.94
10.00	0.11	0.17	0.28	0.41	0.52

TABLE 43
DA = 0.1, DB = 0.1, DK = 0.1

K	BETA 1.0	2.0	3.0	4.0	5.0
0.01	1.68	6.33	9.37	12.47	15.93
0.10	1.50	4.06	5.76	7.63	9.77
1.00	0.94	1.61	2.16	2.81	3.50
5.00	0.38	0.47	0.65	0.85	1.03
10.00	0.20	0.24	0.34	0.46	0.56

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TABLE 44

DA = -0.1, DB = -0.1, DK = -0.1

K	BETA 2.0	3.0	4.0	5.0	6.0
0.01	-1.29	-0.66	1.80	5.60	10.58
0.10	-0.39	0.57	2.81	6.08	10.27
1.00	0.80	1.66	2.86	4.34	6.13
5.00	0.62	0.95	1.39	1.97	2.70
10.00	0.35	0.55	0.86	1.29	1.85

TABLE 45

DA = -0.1, DB = -0.1, DK = 0.0

K	BETA 2.0	3.0	4.0	5.0	6.0
0.01	-2.37	-2.29	-0.36	2.90	7.33
0.10	-1.33	-0.85	0.91	3.65	7.30
1.00	0.15	0.75	1.65	2.79	4.19
5.00	0.33	0.63	1.03	1.57	2.24
10.00	0.19	0.39	0.67	1.08	1.61

TABLE 46

DA = -0.1, DB = -0.1, DK = 0.1

K	BETA 2.0	3.0	4.0	5.0	6.0
0.01	-3.12	-3.51	-2.06	0.72	4.65
0.10	-1.96	-1.88	-0.56	1.72	4.89
1.00	-0.26	0.14	0.80	1.68	2.77
5.00	0.15	0.40	0.77	1.26	1.90
10.00	0.10	0.27	0.53	0.90	1.41

TABLE 47

DA = -0.1, DB = 0.0, DK = -0.1

K	BETA 1.0	2.0	3.0	4.0	5.0
0.01	1.22	3.37	6.57	10.90	16.37
0.10	1.15	3.06	5.83	9.52	14.20
1.00	0.96	2.14	3.57	5.32	7.42
5.00	0.69	1.21	1.80	2.53	3.45
10.00	0.55	0.90	1.33	1.92	2.67

TABLE 48

DA = -0.1, DB = 0.0, DK = 0.0

K	BETA 1.0	2.0	3.0	4.0	5.0
0.01	0.54	2.14	4.79	8.51	13.38
0.10	0.52	1.92	4.15	7.27	11.36
1.00	0.45	1.29	2.37	3.73	5.41
5.00	0.36	0.81	1.37	2.06	2.93
10.00	0.31	0.65	1.08	1.64	2.36

TABLE 49

DA = -0.1, DB = 0.0, DK = 0.1

K	BETA 1.0	2.0	3.0	4.0	5.0
0.01	0.16	1.27	3.41	6.62	10.94
0.10	0.16	1.11	2.88	5.51	9.07
1.00	0.15	0.71	1.52	2.57	3.89
5.00	0.15	0.53	1.05	1.71	2.53
10.00	0.15	0.47	0.88	1.42	2.11

TABLE 50

DA = -0.1, DB = 0.1, DK = -0.1

K	BETA 1.0	2.0	3.0	4.0	5.0
0.01	2.48	9.55	15.32	21.59	28.94
0.10	2.24	7.49	12.02	17.25	23.55
1.00	1.66	3.97	5.99	8.37	11.21
5.00	1.19	2.08	2.96	4.05	5.39
10.00	1.04	1.71	2.45	3.37	4.53

TABLE 51

DA = -0.1, DB = 0.1, DK = 0.0

K	BETA 1.0	2.0	3.0	4.0	5.0
0.01	1.64	8.16	13.33	18.93	25.58
0.10	1.43	6.10	10.02	14.60	20.21
1.00	1.02	2.88	4.53	6.44	8.68
5.00	0.76	1.57	2.42	3.47	4.74
10.00	0.71	1.38	2.11	3.00	4.12

TABLE 52

DA = -0.1, DB = 0.1, DK = 0.1

K	BETA 1.0	2.0	3.0	4.0	5.0
0.01	1.11	7.15	11.78	16.80	22.81
0.10	0.93	5.09	8.47	12.49	17.48
1.00	0.60	2.14	3.38	4.90	6.79
5.00	0.47	1.19	2.01	3.01	4.24
10.00	0.46	1.12	1.84	2.72	3.80

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TABLE 53
DA = 0.0, DB = -0.1, DK = -0.1

K	BETA 2.0	3.0	4.0	5.0	6.0
0.01	-3.91	-6.15	-7.43	-8.26	-8.84
0.10	-2.54	-3.69	-4.28	-4.63	-4.87
1.00	-0.54	-0.58	-0.54	-0.48	-0.43
5.00	-0.02	0.01	0.02	0.03	0.04
10.00	0.03	0.05	0.06	0.06	0.06

TABLE 54
DA = 0.0, DB = -0.1, DK = 0.0

K	BETA 2.0	3.0	4.0	5.0	6.0
0.01	-4.04	-6.33	-7.65	-8.50	-9.10
0.10	-2.57	-3.72	-4.29	-4.64	-4.87
1.00	-0.51	-0.51	-0.45	-0.38	-0.33
5.00	0.04	0.07	0.09	0.09	0.09
10.00	0.09	0.11	0.10	0.10	0.09

TABLE 55
DA = 0.0, DB = -0.1, DK = 0.1

K	BETA 2.0	3.0	4.0	5.0	6.0
0.01	-3.92	-6.27	-7.62	-8.49	-9.10
0.10	-2.38	-3.51	-4.07	-4.41	-4.63
1.00	-0.31	-0.29	-0.21	-0.14	-0.09
5.00	0.16	0.17	0.16	0.16	0.15
10.00	0.18	0.17	0.15	0.13	0.12

TABLE 56
DA = 0.0, DB = 0.0, DK = -0.1

K	BETA 1.0	2.0	3.0	4.0	5.0
0.01	0.13	0.14	0.14	0.14	0.14
0.10	0.13	0.13	0.13	0.13	0.14
1.00	0.10	0.10	0.10	0.10	0.10
5.00	0.05	0.03	0.02	0.02	0.01
10.00	0.04	0.02	0.01	0.01	0.00

TABLE 57
DA = 0.0, DB = 0.0, DK = 0.0

K	BETA 1.0	2.0	3.0	4.0	5.0
0.01	0.00	0.00	0.00	0.00	0.00
0.10	0.00	0.00	0.01	0.01	0.01
1.00	0.01	0.00	0.00	0.00	0.00
5.00	0.00	0.00	0.00	0.00	0.00
10.00	0.00	0.00	0.00	0.00	0.00

TABLE 58
DA = 0.0, DB = 0.0, DK = 0.1

K	BETA 1.0	2.0	3.0	4.0	5.0
0.01	0.11	0.11	0.11	0.11	0.11
0.10	0.10	0.11	0.11	0.11	0.11
1.00	0.08	0.08	0.08	0.08	0.08
5.00	0.04	0.03	0.02	0.01	0.01
10.00	0.03	0.01	0.01	0.00	0.00

TABLE 59
DA = 0.0, DB = 0.1, DK = -0.1

K	BETA 1.0	2.0	3.0	4.0	5.0
0.01	0.98	5.60	7.66	8.82	9.56
0.10	0.81	3.55	4.44	4.89	5.15
1.00	0.47	1.06	1.03	0.97	0.90
5.00	0.26	0.26	0.22	0.19	0.17
10.00	0.22	0.20	0.18	0.16	0.14

TABLE 60
DA = 0.0, DB = 0.1, DK = 0.0

K	BETA 1.0	2.0	3.0	4.0	5.0
0.01	0.74	5.44	7.55	8.73	9.49
0.10	0.57	3.28	4.16	4.59	4.84
1.00	0.28	0.80	0.74	0.66	0.59
5.00	0.12	0.14	0.11	0.10	0.10
10.00	0.10	0.11	0.11	0.10	0.10

TABLE 61
DA = 0.0, DB = 0.1, DK = 0.1

K	BETA 1.0	2.0	3.0	4.0	5.0
0.01	0.75	5.54	7.69	8.89	9.66
0.10	0.58	3.27	4.13	4.55	4.79
1.00	0.27	0.73	0.65	0.56	0.49
5.00	0.08	0.08	0.06	0.05	0.05
10.00	0.04	0.06	0.06	0.07	0.07

SENSITIVITY OF XP POLICIES IN THE WEIBULL CASE: MODEL II

TABLE 62
DA = 0.1, DB = -0.1, DK = -0.1

K	BETA 2.0	3.0	4.0	5.0	6.0
0.01	-3.36	-4.62	-4.30	-2.86	-0.50
0.10	-1.78	-1.69	-0.51	1.38	3.85
1.00	0.13	0.75	1.41	1.99	2.49
5.00	0.62	1.01	1.29	1.48	1.61
10.00	0.66	0.89	1.01	1.06	1.08

TABLE 63
DA = 0.1, DB = -0.1, DK = 0.0

K	BETA 2.0	3.0	4.0	5.0	6.0
0.01	-2.63	-3.53	-2.81	-0.96	1.82
0.10	-1.01	-0.53	1.02	3.27	6.06
1.00	0.73	1.49	2.21	2.81	3.28
5.00	0.95	1.30	1.54	1.68	1.76
10.00	0.88	1.05	1.12	1.14	1.14

TABLE 64
DA = 0.1, DB = -0.1, DK = 0.1

K	BETA 2.0	3.0	4.0	5.0	6.0
0.01	-1.73	-2.31	-1.24	1.00	4.19
0.10	-0.08	0.74	2.63	5.19	8.27
1.00	1.41	2.28	3.03	3.61	4.04
5.00	1.28	1.58	1.75	1.85	1.89
10.00	1.09	1.19	1.22	1.21	1.20

TABLE 65
DA = 0.1, DB = 0.0, DK = -0.1

K	BETA 1.0	2.0	3.0	4.0	5.0
0.01	0.09	0.90	2.54	4.99	8.26
0.10	0.09	0.77	2.05	3.88	6.24
1.00	0.09	0.46	0.94	1.43	1.91
5.00	0.09	0.34	0.61	0.84	1.03
10.00	0.09	0.30	0.47	0.60	0.69

TABLE 66
DA = 0.1, DB = 0.0, DK = 0.0

K	BETA 1.0	2.0	3.0	4.0	5.0
0.01	0.44	1.74	3.85	6.79	10.53
0.10	0.42	1.52	3.19	5.39	8.07
1.00	0.37	0.96	1.56	2.13	2.63
5.00	0.29	0.58	0.82	1.02	1.19
10.00	0.24	0.43	0.58	0.69	0.76

TABLE 67
DA = 0.1, DB = 0.0, DK = 0.1

K	BETA 1.0	2.0	3.0	4.0	5.0
0.01	1.00	2.74	5.28	8.65	12.83
0.10	0.94	2.42	4.43	6.95	9.93
1.00	0.78	1.55	2.25	2.85	3.34
5.00	0.55	0.82	1.03	1.19	1.32
10.00	0.43	0.58	0.69	0.77	0.82

TABLE 68
DA = 0.1, DB = 0.1, DK = -0.1

K	BETA 1.0	2.0	3.0	4.0	5.0
0.01	0.77	6.59	10.95	15.37	20.33
0.10	0.59	3.89	6.01	8.25	10.86
1.00	0.28	0.96	1.22	1.53	1.88
5.00	0.10	0.19	0.33	0.51	0.68
10.00	0.05	0.10	0.20	0.31	0.42

TABLE 69
DA = 0.1, DB = 0.1, DK = 0.0

K	BETA 1.0	2.0	3.0	4.0	5.0
0.01	1.06	7.53	12.44	17.42	22.90
0.10	0.86	4.59	7.08	9.67	12.60
1.00	0.49	1.34	1.72	2.11	2.50
5.00	0.21	0.34	0.48	0.64	0.80
10.00	0.11	0.17	0.27	0.38	0.47

TABLE 70
DA = 0.1, DB = 0.1, DK = 0.1

K	BETA 1.0	2.0	3.0	4.0	5.0
0.01	1.57	8.61	14.05	19.51	25.47
0.10	1.32	5.43	8.26	11.15	14.37
1.00	0.86	1.82	2.30	2.73	3.13
5.00	0.39	0.50	0.62	0.77	0.91
10.00	0.22	0.25	0.34	0.43	0.52

We may conclude from these observations that in practical situations it is advisable to estimate α as accurately as possible; if there is any doubt about the value of β it is safer to underestimate than to overestimate.

Model II

Tables 44 to 70 show that the robustness characteristics of model I are also evident in model II, i.e. model II is mostly sensitive to α and positive errors in β. In some of the cases considered Δ approaches 30 percent, in contrast to model I where no value exceeded 20 percent. Thus, $\underline{x}_p$ policies are less robust in the model II case.

The combination $K = 0.01$, $\hat{\alpha} = \alpha$, $\hat{\beta} = 0.9\beta$ (Tables 53, 54 and 55) yields some Δ values in the region of minus 10 per cent, which means that policies exist which achieve an expected cost at most 90 per cent of that corresponding to the 'best' $\underline{x}_p$ policy, suggesting that $\underline{x}_p$ policies are possibly not highly efficient for small K with model II.

The sensitivity analysis presented above is only a partial analysis, since only errors of 10 per cent have been considered. For a more complete study, other failure distributions, and combinations of other δ values should be examined. However, because of the amount of computation that this leads to, such an analysis is only practical in special cases. The attempt here has been to give an overall picture in the limited space available.

Chapter 7

Comparison of Periodic, Mean Residual Life, and x_p Inspection Policies

7.1 Discussion of Efficiency

In the previous chapters three families of suboptimal inspection policies have been considered and in each instance a method has been given for computing the best policy in the Weibull case. In this section a comparison is made of these three families of policies.

The obvious yardstick by which to judge a suboptimal policy is the optimal policy: a natural measure of efficiency being the ratio of the expected costs due to each. Since in every case the method for finding the best suboptimal policy consists of minimising the expected cost numerically, calculating the best policy simultaneously determines the corresponding expected cost. However, the algorithms described in Chapter 2 for computing the optimal policy do not enable us to evaluate the minimum expected cost associated with it.

Now the expected cost for models I and II can be written as

$$\sum_{j=1}^{\infty} \left\{ c_1 \bar{F}_{j-1} + c_2 x_j \left(\bar{F}_{j-1} - \bar{F}_j \right) \right\} - c_2 E(T)$$

and

$$\sum_{j=1}^{\infty} \left\{ c_1 \bar{F}_{j-1} + c_2 (x_j - x_{j-1}) \left(\bar{F}_{j-1} - \bar{F}_j \right) \right\} ,$$

respectively, where $\bar{F}_j = \bar{F}(x_j)$. Note that the terms in the above series are never negative, so that by computing the first n inspection times of the optimal policy, $x_1^*, \ldots, x_n^*$, and summing the series to n terms, a lower bound on the minimum expected cost can be obtained. This lower bound can be made as sharp as we please by taking n to be sufficiently large. The results presented in this Chapter were obtained by taking n to be the smallest integer satisfying

$$\bar{F}(x_n) < 10^{-10} \quad .$$

Tables 71 and 72 give the expected costs incurred in the Weibull case by the best periodic, mean residual life and $\underline{x}_p$ policies as well as the minimum expected cost for models I and II respectively. The efficiencies can be calculated from these tables and are given in Tables 73 and 74. From Tables 73 and 74 it is clear that none of the three families of policies could be described as inefficient in the Weibull case for the values of K and β considered. For model I there is little to choose between the mean residual life and $\underline{x}_p$ policies both of which are superior to the periodic policy, especially when K is small and $\beta \geqslant 2$. For model II however the mean residual life policy is clearly the best of the three, the $\underline{x}_p$ policy offering little improvement over the periodic policy particularly when K is small. The mean residual life and periodic policies are generally more efficient for model II than model I, whereas for the $\underline{x}_p$ policy this is not the case.

It is important to remember that efficiency is not the only criterion to be taken into account when the choice of an inspection policy is made: a good policy should also be simple to compute and practical to use (although it is at least arguable that with the current availability of high speed computers, the ease of computation is not of great importance).

Of the policies considered the $\underline{x}_p$ is the simplest to compute, since the expression for the mean number of inspections takes on a particularly simple form; the periodic policy also poses no real problems, in contrast to the mean residual life policy which (in the

Table 71

Expected costs (measured in units of αc_2) due to periodic, mean residual life, $\underline{x}_p$ and optimal inspection policies in the Weibull case; model I

K \ β	1.0	1.5	2.0	2.5	3.0	3.5	4.0	
0.01	0.1482	0.1396	0.1381	0.1382	0.1386	0.1391	0.1396	per
	0.1482	0.1349	0.1265	0.1199	0.1144	0.1097	0.1055	mrl
	0.1482	0.1360	0.1282	0.1216	0.1158	0.1106	0.1059	$\underline{x}_p$
	0.1482	0.1342	0.1246	0.1169	0.1105	0.1050	0.1003	opt
0.05	0.3504	0.3269	0.3227	0.3227	0.3238	0.3249	0.3261	per
	0.3504	0.3145	0.2905	0.2716	0.2557	0.2422	0.2305	mrl
	0.3504	0.3163	0.2937	0.2754	0.2600	0.2467	0.2352	$\underline{x}_p$
	0.3504	0.3138	0.2891	0.2699	0.2542	0.2409	0.2296	opt
0.1	0.5162	0.4787	0.4710	0.4707	0.4723	0.4741	0.4757	per
	0.5162	0.4603	0.4228	0.3939	0.3706	0.3514	0.3353	mrl
	0.5162	0.4625	0.4268	0.3987	0.3755	0.3558	0.3390	$\underline{x}_p$
	0.5162	0.4599	0.4223	0.3934	0.3701	0.3508	0.3343	opt
0.5	1.3577	1.2362	1.1914	1.1628	1.1051	1.0462	0.9949	per
	1.3577	1.2018	1.1048	1.0355	0.9822	0.9392	0.9035	mrl
	1.3577	1.2029	1.1037	1.0308	0.9743	0.9289	0.8917	$\underline{x}_p$
	1.3577	1.2007	1.1006	1.0278	0.9716	0.9267	0.8899	opt
1.0	2.1462	1.9429	1.8397	1.7455	1.6577	1.5846	1.5250	per
	2.1462	1.9066	1.7628	1.6623	1.5861	1.5259	1.4768	mrl
	2.1462	1.9052	1.7560	1.6507	1.5718	1.5102	1.4608	$\underline{x}_p$
	2.1462	1.9034	1.7539	1.6489	1.5703	1.5090	1.4599	opt
5.0	7.0907	6.5043	6.1789	5.9640	5.8141	5.7044	5.6209	per
	7.0907	6.4849	6.1595	5.9524	5.8083	5.7023	5.6211	mrl
	7.0907	6.4768	6.1443	5.9347	5.7906	5.6854	5.6053	$\underline{x}_p$
	7.0907	6.4762	6.1437	5.9343	5.7903	5.6852	5.6051	opt

Table 72

Expected costs (measured in units of αc_2) due to periodic, mean residual life, $\underline{x}_p$ and optimal inspection policies in the Weibull case; model II

K \ β	1.0	1.5	2.0	2.5	3.0	3.5	4.0	
0.01	0.2051	0.1950	0.1933	0.1934	0.1940	0.1947	0.1954	per
	0.2051	0.1901	0.1804	0.1727	0.1661	0.1604	0.1553	mrl
	0.2051	0.1948	0.1927	0.1924	0.1927	0.1931	0.1936	$\underline{x}_p$
	0.2051	0.1897	0.1794	0.1711	0.1640	0.1577	0.1523	opt
0.05	0.4731	0.4500	0.4460	0.4462	0.4476	0.4492	0.4508	per
	0.4731	0.4394	0.4183	0.4019	0.3881	0.3762	0.3657	mrl
	0.4731	0.4486	0.4423	0.4404	0.4399	0.4399	0.4401	$\underline{x}_p$
	0.4731	0.4390	0.4176	0.4008	0.3867	0.3745	0.3637	opt
0.1	0.6851	0.6513	0.6454	0.6457	0.6476	0.6499	0.6521	per
	0.6851	0.6370	0.6089	0.5881	0.5712	0.5570	0.5447	mrl
	0.6851	0.6483	0.6374	0.6332	0.6313	0.6303	0.6299	$\underline{x}_p$
	0.6851	0.6366	0.6075	0.5855	0.5674	0.5518	0.5380	opt
0.5	1.6931	1.6013	1.5813	1.5804	1.5853	1.5914	1.5969	per
	1.6931	1.5784	1.5277	1.5009	1.4852	1.4753	1.4686	mrl
	1.6931	1.5870	1.5429	1.5181	1.5018	1.4903	1.4816	$\underline{x}_p$
	1.6931	1.5741	1.5133	1.4740	1.4455	1.4231	1.4047	opt
1.0	2.5805	2.4272	2.3828	2.3711	2.3638	2.3456	2.3235	per
	2.5805	2.4020	2.3244	2.2823	2.2556	2.2365	2.2217	mrl
	2.5805	2.4041	2.3213	2.2708	2.2358	2.2099	2.1897	$\underline{x}_p$
	2.5805	2.3941	2.3016	2.2452	2.2069	2.1791	2.1579	opt
5.0	7.7789	7.2509	6.9774	6.7962	6.6691	6.5762	6.5057	per
	7.7789	7.2333	6.9573	6.7838	6.6633	6.5746	6.5067	mrl
	7.7789	7.2228	6.9368	6.7597	6.6390	6.5515	6.4852	$\underline{x}_p$
	7.7789	7.2206	6.9349	6.7583	6.6381	6.5508	6.4848	opt

Table 73

Efficiency of periodic, mean residual life and $\underline{x}_p$ policies in the Weibull case; model I

K \ β	1.0	1.5	2.0	2.5	3.0	3.5	4.0	
0.01	100.0	96.13	90.22	84.59	79.73	75.49	71.85	per
	100.0	99.48	98.50	97.50	96.59	95.72	95.07	mrl
	100.0	98.68	97.19	96.13	95.42	94.94	94.71	$\underline{x}_p$
0.05	100.0	95.99	89.59	83.64	78.51	74.15	70.41	per
	100.0	99.78	99.52	99.37	99.41	99.46	99.61	mrl
	100.0	99.21	98.43	98.00	97.77	97.65	97.62	$\underline{x}_p$
0.1	100.0	96.07	89.66	83.58	78.36	73.99	70.28	per
	100.0	99.91	99.88	99.87	99.87	99.83	99.70	mrl
	100.0	99.44	98.95	98.67	98.56	98.59	98.61	$\underline{x}_p$
0.5	100.0	97.13	92.38	88.39	87.92	88.58	89.45	per
	100.0	99.91	99.62	99.26	98.92	98.67	98.49	mrl
	100.0	99.82	99.72	99.71	99.72	99.76	99.80	$\underline{x}_p$
1.0	100.0	97.97	95.34	94.47	94.73	95.23	95.73	per
	100.0	99.83	99.50	99.19	99.00	98.89	98.86	mrl
	100.0	99.91	99.88	99.89	99.90	99.92	99.94	$\underline{x}_p$
5.0	100.0	99.57	99.43	99.50	99.59	99.66	99.72	per
	100.0	99.87	99.74	99.70	99.69	99.70	99.72	mrl
	100.0	99.99	99.99	99.99	99.99	100.0	100.0	$\underline{x}_p$

Table 74

Efficiency of periodic, mean residual life and $\underline{x}_p$ policies in the Weibull case; model II

K \ β	1.0	1.5	2.0	2.5	3.0	3.5	4.0	
0.01	100.0	97.28	92.81	88.47	84.54	81.00	77.94	per
	100.0	99.79	99.45	99.07	98.74	98.32	98.07	mrl
	100.0	97.38	93.10	88.93	85.11	81.67	78.67	$\underline{x}_p$
0.05	100.0	97.56	93.63	89.83	86.39	83.37	80.68	per
	100.0	99.91	99.83	99.73	99.64	99.55	99.45	mrl
	100.0	97.86	94.42	91.01	87.91	85.13	82.64	$\underline{x}_p$
0.1	100.0	97.74	94.13	90.68	87.62	84.91	82.50	per
	100.0	99.94	99.77	99.56	99.33	99.07	98.77	mrl
	100.0	98.20	95.31	92.47	89.88	87.55	85.41	$\underline{x}_p$
0.5	100.0	98.30	95.70	93.27	91.18	89.42	87.96	per
	100.0	99.73	99.06	98.21	97.33	96.46	95.65	mrl
	100.0	99.19	98.08	97.10	96.25	95.49	94.81	$\underline{x}_p$
1.0	100.0	98.64	96.59	94.69	93.36	92.90	92.87	per
	100.0	99.67	99.02	98.37	97.84	97.43	97.13	mrl
	100.0	99.58	99.15	98.87	98.71	98.61	98.55	$\underline{x}_p$
5.0	100.0	99.58	99.39	99.44	99.54	99.61	99.68	per
	100.0	99.82	99.68	99.62	99.62	99.64	99.66	mrl
	100.0	99.97	99.97	99.98	99.99	99.99	99.99	$\underline{x}_p$

Weibull case at least) is somewhat intractable.

When it comes to implementation, no policy could be simpler than the periodic policy, since in practice the inspection intervals would be rounded off to the nearest convenient number of time units. In the periodic case where the inspection intervals are all the same, this rounding off only has to be done once.

Taking the above considerations and the results of Chapter 6 into account, we may conclude that $\underline{x}_p$ policies are probably the most attractive of the nonperiodic policies considered, since they are robust, efficient, and easy to compute.

7.2 Further developments

The preceding chapters of this thesis have been devoted to a problem which comes under the general heading of Inspection Problems. The results in these chapters could be extended to cover a wider class of problems, for example one could consider the case when the true state of the system is not revealed exactly by inspection. This type of problem is likely to arise in the medical field where tests can sometimes result in "false positives" or "false negatives", Put in general terms we might observe a random variable X whose probability distribution depended in some way on the state of the system, the problem we have considered being the special case when X reveals the true state of the system with probability 1.

Yet another class of problems arises if we contemplate systems which can be in more than two states. As a simple case we would consider systems having the two states E_0 and E_1 representing normal working and failure respectively, and in addition a third state E_2 representing some intermediate form of failure. In these multistate problems one can predict difficulties in finding a reasonable model for the times between transition from state to state.

The models that have been considered in this thesis are in some sense crude and it would be naive to suggest that they apply directly to a large number of live problems. However, they have succeeded in showing that computing optimal inspection policies is no simple matter, and by considering suboptimal policies the calculations can be made much easier. It is likely that suboptimal policies, particularly the $\underline{x}_p$ policy with its attractive Markov property, will be valuable tools in solving inspection problems associated with more elaborate models.

References

Abramovitz, M. and Stegun, I.A. (1965). Handbook of mathematical functions. Dover publications, New York.

Aitchison, J. and Brown, J.A.C. (1957). The lognormal distribution. Cambridge University Press.

Barlow, R.E. and Hunter, L.C. (1960). Mathematical models for system reliability, Part II. The Sylvania Technologist, 8, 55-65.

Barlow, R.E., Hunter, L.C. and Proschan, F. (1963). Optimum checking procedures. J. SIAM, 11, 1078-1095.

Brender, D.M. (1963). A surveillance model for recurrent events. IBM Watson Research Centre report.

Bryson, M.C. and Siddiqui, M.M. (1969). Some criteria for ageing. JASA, 64, 1472-1483.

Davis, D.J. (1952). An analysis of some failure data. JASA, 47, 113-150.

Derman, C. (1961). On minimax surveillance schedules. Naval Research Logistics Quarterly, 8, 415-419.

Drenick, R.F. (1960). The failure law of complex equipment. J. SIAM, 8, 125-149.

Duncan, A.J. (1956). The economic design of $\bar{X}$ charts used to maintain current control of a process. JASA, 51, 228-242.

Epstein, B. (1958). The exponential distribution and its role in life testing. Industrial Quality Control, 15 (6), 2-7.

Finney, D.J. (1941). On the distribution of a variate whose logarithm is normally distributed. JRSS Supplement, 7, 155-161.

Fisher, R.A. (1922). On the interpretation of χ^2 from contingency tables and the calculation of P. JRSS (A), 85, 87-94.

Flett, T.M. (1966). Mathematical analysis. McGraw-Hill.

Freudenthal, A.M. (1960). Prediction of fatique life. J. Appl. Phys., 31, 2196-2198.

Johnson, N.L. and Kotz, S. (1970). Continuous univariate distributions. Wiley, New York.

Kao, J.H.K. (1956). A new life-quality measure for electron tubes. IRE Transactions on Reliability and Quality Control, PGRQC-7, 1-11.

Kao, J.H.K. (1958). Computer methods for estimating Weibull parameters in reliability studies. IRE Transactions on Reliability and Quality Control, PGRQC-13, 15-22.

Karlin, S., Proschan, F. and Barlow, R.E. (1961). Moment inequalities of Polya frequency functions. Pacific J. Math., 11, 1023-1033.

Keller, J.B. (1974). Optimum checking schedules for systems subject to random failure. Management Science, 21, 256-260.

Knopp, K. (1947). Theory and application of infinite series. Haffner, New York.

Mann, N.R., Schafer, R.E. and Singpurwalla, N.D. (1974). Methods for statistical analysis of reliability and life data. Wiley, New York.

Munford, A.G. and Shahani, A.K. (1973). A nearly optimal inspection policy. Operational Res. Quart., 23, 373-379.

Munford, A.G. and Shahani, A.K. (1974). An inspection policy in the Weibull case. Operational Res. Quart., 24, 453-458.

Saunders, S.C. (1968). On the determination of a safe life for distributions classified by failure rate. Technometrics, 10, 361-377.

Savage, I.R. (1956). Cycling. Naval Research Logistics Quarterly, 3, 163-175.

Watson, G.S. and Leadbetter, M.R. (1964). Hazard analysis I. Biometrika, 51, 175-184.

Watson, G.S. and Leadbetter, M.R. (1964). Hazard analysis II. Sankhyā (A), 26, 101-116.

Watson, G.S. and Wells, W.T. (1961). On the possibility of improving the mean useful life of items by eliminating those with short lives. Technometrics, 3, 281-298.

Weibull, W. (1951). A statistical distribution function of wide applicability. J. Appl. Mech., 18, 293-297.

Zelen, M. and Dannemiller, M.C. (1961). The robustness of life testing procedures derived from the exponential distribution. Technometrics, 3, 29-49.