EQUIVARIANT K-HOMOLOGY FOR HYPERBOLIC REFLECTION GROUPS

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ABSTRACT. We compute the equivariant K-homology of the classifying space for proper actions, for cocompact 3-dimensional hyperbolic reflection groups. This coincides with the topological K-theory of the reduced C^*-algebra associated to the group, via the Baum-Connes conjecture. We show that, for any such reflection group, the associated K-theory groups are torsion-free. This means that we can complete previous computations with rational coefficients to get results with integral coefficients. On the way, we establish an efficient criterion for checking torsion-freeness of K-theory groups, which can be applied far beyond the scope of the present paper.

1. INTRODUCTION

For a discrete group Γ, a general problem is to compute $K_*(C_r^*\Gamma)$, the topological K-theory of the reduced C^*-algebra of Γ. The Baum-Connes Conjecture predicts that this functor can be determined, in a homological manner, from the complex representation rings of the finite subgroups of Γ. This viewpoint led to general recipes for computing the rational topological K-theory $K_*(C_r^*\Gamma) \otimes \mathbb{Q}$ of groups, through the use of Chern characters (see for instance Lück and Oliver [16] and Lück [13], [15], as well as related earlier work of Adem [1]). When Γ has small homological dimension, one can sometimes even give completely explicit formulas for the rational topological K-theory, see for instance Lafont, Ortiz, and Sánchez-García [11] for the case where Γ is a 3-orbifold group.

On the other hand, performing integral calculations for these K-theory groups is much harder. For 2-dimensional crystallographic groups, such calculations have been done in M. Yang's thesis [25]. This was subsequently extended to the class of cocompact planar groups by Lück and Stamm [18], and to certain higher dimensional dimensional crystallographic groups by Davis and Lück [4] (see also Langer and Lück [12]). For 3-dimensional groups, Lück [14] completed this calculation for the semi-direct product $\mathrm{Hei}_3(\mathbb{Z}) \rtimes \mathbb{Z}_4$ of the 3-dimensional integral Heisenberg group with a specific action of the cyclic group $\mathbb{Z}_4$. Some further computations were completed by Isely [7] for groups of the form $\mathbb{Z}^2 \rtimes \mathbb{Z}$; by Rahm [22] for the class of Bianchi groups; by Pooya and Valette [21] for solvable Baumslag-Solitar groups; and by Flores, Pooya and Valette [5] for lamplighter groups of finite groups.

When the result was that the K-theory groups are torsion-free, like in the above quoted papers [14] and [4], then so far, this was considered as an ad-hoc computational result. In the present paper, we can however explain such lack of torsion with a new criterion (Theorem 4), which can be checked very efficiently (we do this

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for the Heisenberg semidirect product group of [14] in Appendix C, and for the crystallographic groups of [4] in Appendix D).

The main purpose of the present paper is to establish the following formula for the integral K-theory groups of cocompact 3-dimensional hyperbolic reflection groups. We note that, by a celebrated result of Andre'ev [2], there is a simple algorithm that inputs a Coxeter group Γ, and decides whether or not there exists a hyperbolic polyhedron $P_\Gamma \subset \mathbb{H}^3$ which generates Γ. In particular, given an arbitrary Coxeter group, one can easily verify if it is covered by the following Main Theorem.

Main Theorem. *Let Γ be a cocompact 3-dimensional hyperbolic reflection group, generated by reflections in the side of a hyperbolic polyhedron $\mathcal{P} \subset \mathbb{H}^3$. Then*

$$K_0(C_r^*(\Gamma)) \cong \mathbb{Z}^{cf(\Gamma)} \quad \textit{and} \quad K_1(C_r^*(\Gamma)) \cong \mathbb{Z}^{cf(\Gamma)-\chi(\mathcal{C})},$$

where the integers $cf(\Gamma), \chi(\mathcal{C})$ can be explicitly computed from the combinatorics of the polyhedron $\mathcal{P}$.

Let us briefly describe the contents of the paper. In Section 2, we provide background material on hyperbolic reflection groups, topological K-theory, and the Baum-Connes Conjecture. We also introduce our main tool, the Atiyah-Hirzebruch type spectral sequence. In Section 3, we use the spectral sequence to show that the K-theory groups we are interested in coincide with the Bredon homology groups $\mathrm{H}_0^{\mathfrak{Fin}}(\Gamma;\, R_\mathbb{C})$ and $\mathrm{H}_1^{\mathfrak{Fin}}(\Gamma;\, R_\mathbb{C})$ respectively. We also explain, using the Γ-action on $\mathbb{H}^3$, why the homology group $\mathrm{H}_1^{\mathfrak{Fin}}(\Gamma;\, R_\mathbb{C})$ is torsion-free. In contrast, showing that $\mathrm{H}_0^{\mathfrak{Fin}}(\Gamma;\, R_\mathbb{C})$ is torsion-free is much more difficult. In Section 4, we give a linear algebraic proof, inspired by the "representation ring splitting" technique of [22], and establishing a novel criterion for proving torsion-freedom of $\mathrm{H}_0^{\mathfrak{Fin}}(\Gamma;\, R_\mathbb{C})$ for any collection of groups Γ with specified types of finite subgroups. Note that our proven lack of torsion in the Bredon homology is not a property shared by all discrete groups acting on hyperbolic 3-space: for example, 2-torsion occurs in $\mathrm{H}_0^{\mathfrak{Fin}}(\Gamma;\, R_\mathbb{C})$ for a Bianchi group Γ whenever it has a 2-dihedral subgroup $C_2 \times C_2$ [22].

Finally, in Section 5, we provide an explicit formula for the rank of the Bredon homology groups (and hence for the K-groups we are interested in), in terms of the combinatorics of the polyhedron $\mathcal{P}$. Our paper concludes with two fairly long Appendices, which contain all the character tables and induction homomorphisms used in our proofs, making our results explicit and self-contained.

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2. Background Material

2.1. **K-theory and the Baum-Connes Conjecture.** Associated to a discrete group Γ, one has $C_r^*\Gamma$, the *reduced C^*-algebra* of Γ. This algebra is defined to be the closure, in the operator norm, of the linear span of the image of the regular representation $\lambda : \Gamma \to B(l^2(\Gamma))$ of Γ on the Hilbert space $l^2(\Gamma)$ of square-summable complex valued functions on Γ. This algebra encodes various analytic properties of the group Γ [20].

For a C^*-algebra A, one can define the *topological K-theory* groups $K_*(A) := \pi_{*-1}(GL(A))$, which are the homotopy groups of the space $GL(A)$ of invertible matrices with entries in A. Due to Bott periodicity, there are canonical isomorphisms $K_*(A) \cong K_{*+2}(A)$, and thus it is sufficient to consider $K_0(A)$ and $K_1(A)$.

In the special case where $A = C_r^*\Gamma$, the *Baum-Connes Conjecture* predicts that there is a canonical isomorphism $K_n^\Gamma(X) \to K_n(C_r^*(\Gamma))$, where X is a model for $\underline{E}\Gamma$ (the classifying space for Γ-actions with isotropy in the family of finite subgroups), and $K_*^\Gamma(-)$ is the equivariant K-homology functor. The Baum-Connes conjecture has been verified for many classes of groups. We refer the interested reader to the monograph by Mislin and Valette [20] or the survey article by Lück and Reich [17] for more information about these topics.

2.2. **Hyperbolic reflection groups.** By a d-dimensional *hyperbolic polyhedron*, we mean the region of $\mathbb{H}^d$ enclosed by a given finite number of (geodesic) hyperplanes, that is, the intersection of a collection of half-spaces associated to the hyperplanes. Let $\mathcal{P} \subset \mathbb{H}^d$ be a polyhedron such that all the interior angles between faces are of the form π/m_{ij} where $m_{ij} \geq 2$ an integer (although some pairs of faces may be disjoint). Let $\Gamma = \Gamma_{\mathcal{P}}$ the associated Coxeter group generated by the reflections in the hyperplanes containing the faces of $\mathcal{P}$.

The Γ-space $\mathbb{H}^d$ is then a model of $\underline{E}\Gamma$ with fundamental domain $\mathcal{P}$. This is a strict fundamental domain, that is, no further points of $\mathcal{P}$ are identified under the group action, and hence $\mathcal{P} = \Gamma\backslash\mathbb{H}^d$ (we will use left-action notation). Recall that Γ admits the following Coxeter presentation

$$\Gamma = \langle s_1, \ldots, s_n \mid (s_i s_j)^{m_{ij}} \rangle \tag{1}$$

where n is the number of distinct hyperplanes enclosing $\mathcal{P}$, s_i denotes the reflection on the i^{th} face, and $m_{ij} \geq 2$ are integers such that: $m_{ii} = 1$ for all i, and, if $i \neq j$, the corresponding faces meet with interior angle π/m_{ij}. We will write $m_{ij} = \infty$ if the corresponding faces do not intersect.

Note that $\mathcal{P}$ may not be a CW or simplicial complex with the natural structure given by vertices, edges, faces, etc (for example, an ideal triangle in $\mathbb{H}^2$ or the region enclosed by two distinct hyperplanes). However, if $\mathcal{P}$ has finite volume, it is a simplicial complex except from possibly ideal vertices at infinity. In such a case (i.e. $\mathcal{P}$ non-compact) we can obtain a cocompact model of $\underline{E}\Gamma$ by equivariantly removing some horoballs in $\mathbb{H}^d$ (see §4 in [8]). Hence a fundamental domain of the resulting action would be a copy of $\mathcal{P}$ with all ideal vertices truncated.

For the rest of this article, $d = 3$, $\mathcal{P}$ is compact, and X is $\mathbb{H}^3$ with the Γ-action described above.

2.3. **Cell structure of the orbit space.** Let $J = \{1, \ldots, n\}$ and write $\langle S \rangle$ for the subgroup generated by a subset $S \subset \Gamma$. At a vertex of $\mathcal{P}$, the concurrent faces (a minimum of 3) must generate a reflection group acting on the 2-sphere, hence it must be a spherical triangle group and in particular it forces the number of incident faces to be exactly three. The only finite Coxeter groups with 3 generators are the triangle groups $\Delta(2, 2, m)$ for some $m \geq 2$, $\Delta(2, 3, 3)$, $\Delta(2, 3, 4)$ and $\Delta(2, 3, 5)$, where we use the notation

$$\Delta(p, q, r) = \left\langle s_1, s_2, s_3 \mid s_1^2, s_2^2, s_3^2, (s_1 s_2)^p, (s_1 s_3)^q, (s_2 s_3)^r \right\rangle. \tag{2}$$

Starting from our compact polyhedron $\mathcal{P}$, we obtain a Γ-CW-structure on $X = \mathbb{H}^3$ with:

- one orbit of 3-cells, with trivial stabilizer;
- n orbits of 2-cells (faces) with stabilisers $\langle s_i \rangle \cong \mathbb{Z}/2$ $(i = 1, \ldots, n)$;
- one orbit of 1-cells (edges) per (unordered) pair $i, j \in J$ with $m_{ij} \neq \infty$, with stabilizer $\langle s_i, s_j \rangle \cong D_{m_{ij}}$ dihedral;
- one orbit of 0-cells (vertices) per (unordered) triple $i, j, k \in J$ with $\langle s_i, s_j, s_k \rangle$ finite, with stabilizer $\langle s_i, s_j, s_k \rangle \cong \Delta(m_{ij}, m_{ik}, m_{jk})$.

We introduce the following notation (after having fixed an order on the Coxeter generators, or equivalently, on the faces of the polyhedron) for the simplices of $\mathcal{P}$:

$$\begin{aligned} &f_i && \text{(faces)},\\ &e_{ij} &= f_i \cap f_j \quad & \text{(edges)},\\ &v_{ijk} &= f_i \cap f_j \cap f_k = e_{ij} \cap e_{ik} \cap e_{jk} \quad & \text{(vertices)}, \end{aligned} \tag{3}$$

whenever the intersections are non-empty, that is, whenever $m_{ij} \neq \infty$, respectively when $\{m_{ij}, m_{ik}, m_{jk}\}$ equals $\{2, 2, m\}$ for some $m \geq 2$, $\{2, 3, 3\}$, $\{2, 3, 4\}$ or $\{2, 3, 5\}$.

2.4. **A spectral sequence.** We ultimately want to compute the K-theory groups of the reduced C^*-algebra of Γ via the Baum-Connes conjecture. Note that the conjecture holds for these groups: Coxeter groups have the Haagerup property [3] and hence satisfy Baum-Connes [6]. Therefore, it suffices to compute the equivariant K-homology groups $K_*^\Gamma(X)$, since X is a model of $\underline{E}\Gamma$. In turn, these groups can be obtained by the Bredon homology of X and an equivariant Atiyah-Hirezebruch spectral sequence coming from the inclusion of the skeleta of the Γ-CW-complex X [19]. The second page of this spectral sequence is given by the Bredon homology groups (cf. [19])

$$E^2_{p,q} = \begin{cases} \mathrm{H}_p^{\mathfrak{Fin}}(\Gamma;\, R_\mathbb{C}) & q \text{ even},\\ 0 & q \text{ odd}. \end{cases} \tag{4}$$

The coefficients of the Bredon homology groups (a functor from the orbit category with respect to the family of finite subgroups of Γ, to $\mathbb{Z}$-modules) are given by the complex representation ring of the cell stabilizers, which are finite subgroups. In order to simplify notation, we will often write H_p to denote $\mathrm{H}_p^{\mathfrak{Fin}}(\Gamma;\, R_\mathbb{C})$.

Before defining Bredon homology, we note that $\dim(X) = 3$ already implies, by homological algebra arguments, the following.

Proposition 1. *There are short exact sequences*

$$0 \longrightarrow \operatorname{coker}(d^3_{3,0}) \longrightarrow K_0^\Gamma(X) \longrightarrow H_2 \longrightarrow 0$$

and

$$0 \longrightarrow H_1 \longrightarrow K_1^\Gamma(X) \longrightarrow \ker(d^3_{3,0}) \longrightarrow 0\,.$$

(Here $d^3_{3,0}\colon E^3_{3,0} = H_3 \longrightarrow E^3_{0,2} = H_2$ is the differential on the E^3-page.)

Proof. This follows at once from [19, Theorem 5.29] but, for completeness, we give a direct proof based on the Atiyah-Hirzebruch spectral sequence. Write K_n for $K_n^\Gamma(X)$. Firstly, note that the E^2-page is concentrated in the $0 \leq p \leq 3$ columns (since $\dim(X) \leq 3$), and in the q even rows —see (4) above.

Secondly, recall that the bidegree of the differentials $d^k_{p,q}$ on the E^k-page is $(-k, k-1)$. Thus the only non-trivial differentials are $d^3_{3,q}$, $q \geq 0$ even, from $E^3_{3,q} = H_3$ to $E^3_{0,q+2} = H_0$ (since $E^3 = E^2$). All in all, $E^\infty = E^2$ except $E^\infty_{3,q} = \ker(d^3_{3,q})$ and $E^\infty_{0,q} = \operatorname{coker}(d^3_{3,q})$, that is,

$$E^\infty_{p,q} = \begin{cases} H_p & p = 1,2 \text{ and } q \text{ even,} \\ \operatorname{coker}(d^3_{3,0}) & p = 0 \text{ and } q \text{ even,} \\ \ker(d^3_{3,0}) & p = 3 \text{ and } q \text{ even,} \\ 0 & \text{otherwise.} \end{cases}$$

Now consider the filtration $\ldots \subset F_{p-1,q+1} \subset F_{p,q} \subset \ldots$ of K_{p+q}, where each $F_{p,q} = \operatorname{Im}\left(K^\Gamma_{p+q}(X^p) \to K^\Gamma_{p+q}(X)\right)$, the image of the map induced in K-homology by the inclusion of the p-skeleton in X. In particular, $F_{p,q} = 0$ for $p < 0$ and $F_{p,q} = K_{p+q}$ for $p \geq \dim(X) = 3$. The quotient $F_{p,q}/F_{p-1,q+1}$ is isomorphic to $E^\infty_{p,q} = E^2_{p,q}$, as given above. Therefore we have, on one hand, the filtration:

$$0 = F_{-1,2} = F_{0,1} \subset F_{1,0} = F_{2,-1} \subset F_{3,-2} = K_1,$$

with only two non-trivial quotients isomorphic to $E^\infty_{1,0} = H_1$ and $E^\infty_{3,-2} = \ker(d^3_{3,0})$. On the other hand, we have the filtration:

$$0 = F_{-1,1} \subset F_{0,0} = F_{1,-1} \subset F_{2,-2} = F_{3,-3} = K_0,$$

with non-trivial quotients isomorphic to $E^\infty_{0,0} = \operatorname{coker}(d^3_{3,0})$ and $E^\infty_{2,-2} = H_2$. These results combined together give the short exact sequences above. □

2.5. **Bredon Homology.** To shorten notation, write Γ_e for $\operatorname{stab}_\Gamma(e)$. The Bredon homology groups in (4) can be defined as the homology groups of the following chain complex (recall that X is a model of $\underline{E}\Gamma$)

$$\ldots \longrightarrow \bigoplus_{e \in I_d} R_{\mathbb{C}}(\Gamma_e) \xrightarrow{\partial_d} \bigoplus_{e \in I_{d-1}} R_{\mathbb{C}}(\Gamma_e) \longrightarrow \ldots, \tag{5}$$

where I_d is a set of orbit representatives of d-cells ($d \geq 0$), and ∂_d is defined via the geometric boundary map and induction between representation rings, as follows. If ge' is in the boundary of e ($e \in I_d$, $e' \in I_{d-1}$, $g \in \Gamma$), then ∂ restricted to $R_{\mathbb{C}}(\Gamma_e) \to R_{\mathbb{C}}(\Gamma_{e'})$ is given by the composition

$$R_{\mathbb{C}}(\Gamma_e) \xrightarrow{\text{ind}} R_{\mathbb{C}}(\Gamma_{ge'}) \xrightarrow{\cong} R_{\mathbb{C}}(\Gamma_{e'}),$$

where the first map is the induction homomorphism of representation rings associated to the subgroup inclusion $\Gamma_e \subset \Gamma_{ge'}$, and the second is the isomorphism induced by conjugation $\Gamma_{ge'} = g\Gamma_{e'}g^{-1}$. Finally, we add a sign depending on a chosen (and thereafter fixed) orientation on the faces of $\mathcal{P}$. The value $\partial_d(e)$ equals the sum of these maps over all boundary cells of e.

Since $\mathcal{P}$ is a strict fundamental domain, we can choose the faces of $\mathcal{P}$ as orbit representatives and thus g (as above) is always the identity. We will implicitly make this assumption from now on.

3. Analyzing the Bredon chain complex for Γ

Let $S = \{s_i : 1 \leq i \leq n\}$ be the set of Coxeter generators and $J = \{1, \ldots, n\}$. Let

$$0 \longrightarrow \mathcal{C}_3 \xrightarrow{\partial_3} \mathcal{C}_2 \xrightarrow{\partial_2} \mathcal{C}_1 \xrightarrow{\partial_1} \mathcal{C}_0 \longrightarrow 0 \tag{6}$$

be the Bredon chain complex associated to X (since X is 3-dimensional).

Next we analyse each differential in the chain complex above. Recall that for a finite group G the complex representation ring $R_{\mathbb{C}}(G)$ is defined as the free abelian group with basis the set of irreducible representations of G (the ring structure is not relevant in this setting). Hence $R_{\mathbb{C}}(G) \cong \mathbb{Z}^{c(G)}$, where we write $c(G)$ for the set of conjugacy classes in G.

3.1. **Analysis of ∂_3.** Let G be a finite group with irreducible representations $\rho_1, \dots, \rho_m$ of degree $n_1, \dots, n_m$, and τ the only representation of the trivial subgroup $\{1_G\} \leq G$. Then τ induces the regular representation in G:

$$\mathrm{Ind}_{\{1_G\}}^{G}(\tau) = n_1\rho_1 + \ldots + n_m\rho_m \, . \tag{7}$$

Lemma 1. *Let X be a Γ-CW-complex with finite stabilizers, and $k \in \mathbb{N}$. If there is a unique orbit of k-cells and this orbit has trivial stabiliser, then $H_k = 0$, provided that $\partial_k \neq 0$.*

Proof. The Bredon module $\mathcal{C}_k$ equals $R_{\mathbb{C}}(\langle 1 \rangle) \cong \mathbb{Z}$ with generator τ, the trivial representation. Then $\partial_k(\tau) \neq 0$ implies $\ker(\partial_k) = 0$ and therefore the corresponding homology group vanishes. □

From the lemma we immediately conclude that $H_3 = 0$ if $\partial_3 \neq 0$. The case $\partial_3 = 0$ occurs if and only if all boundary faces are pairwise identified, which cannot happen since $\mathcal{P}$ is a strict fundamental domain (the group acts by reflections on the faces). We conclude that $H_3 = 0$ and, using Proposition 1, we obtain:

Proposition 2. *We have $K_1^{\Gamma}(X) \cong H_1$, and there is a short exact sequence*

$$0 \longrightarrow H_0 \longrightarrow K_0^{\Gamma}(X) \longrightarrow H_2 \longrightarrow 0 \, .$$

3.2. **Analysis of ∂_2.** Let f be a face of $\mathcal{P}$ and $e \in \partial f$ an edge. Suppose, using the notation in (3), that $f = f_i$ and $e = e_{ij}$. Then we have a map $R_{\mathbb{C}}(\langle s_i \rangle) \to R_{\mathbb{C}}(\langle s_i, s_j \rangle)$ induced by inclusion. Recall that $\langle s_i \rangle \cong C_2$ and $\langle s_i, s_j \rangle \cong D_{m_{ij}}$. A straightforward analysis (see Appendix A for character tables, and Appendix B for induction homomorphism notation and calculations) shows that

$$\begin{aligned} \rho_1 \uparrow &= \chi_1 + \widehat{\chi_4} + \sum \phi_p \, , \\ \rho_2 \uparrow &= \chi_2 + \widehat{\chi_3} + \sum \phi_p \, , \end{aligned}$$

if $i < j$, or

$$\begin{aligned} \rho_1 \uparrow &= \chi_1 + \widehat{\chi_3} + \sum \phi_p \, , \\ \rho_2 \uparrow &= \chi_2 + \widehat{\chi_4} + \sum \phi_p \, , \end{aligned}$$

if $j < i$. That is, as a map of free abelian groups $\mathbb{Z}^2 \to \mathbb{Z}^{c(D_{m_{ij}})}$,

$$\begin{aligned} (a,b) &\mapsto \pm(a, b, \widehat{b}, \widehat{a}, a+b, \ldots, a+b) \quad \text{or} \\ (a,b) &\mapsto \pm(a, b, \widehat{a}, \widehat{b}, a+b, \ldots, a+b), \end{aligned}$$

Using the analysis above, we can now show

Theorem 1. *For any compact $\mathcal{P}$, we have that $H_2 = 0$.*

From this theorem and Proposition 2, we immediately obtain

Corollary 1. *$K_0^\Gamma(X) = H_0$ and $K_1^\Gamma(X) = H_1$.*

Proof of Theorem. Fix an orientation on the polyhedron $\mathcal{P}$, and consider the induced orientations on the faces. At an edge we have two incident faces f_i and f_j with opposite orientations so without loss of generality we have, as a map of free abelian groups,

$$\begin{array}{rcl} R_{\mathbb{C}}(\langle s_i\rangle) \oplus R_{\mathbb{C}}(\langle s_j\rangle) \cong \mathbb{Z}^2 \oplus \mathbb{Z}^2 & \to & \mathbb{Z}^{c(D_{m_{ij}})} \cong R_{\mathbb{C}}(\langle s_i, s_j\rangle) \\ (a, b \,|\, c, d) & \mapsto & (a-c, b-d, \widehat{a-d}, \widehat{b-c}, S, \dots, S) \end{array} \tag{8}$$

where $S = a + b - c - d$, and the elements with a hat $\widehat{}$ appear only when m_{ij} is even. Note that we use vertical bars '|' for clarity, to separate elements coming from different representation rings.

By the preceding analysis, $\partial_2(x) = 0$ implies that, for each $i, j \in J$, if the faces f_i and f_j meet, then

(1) $a_i = a_j$ and $b_i = b_j$, if m_{ij} is odd, and
(2) $a_i = a_j = b_i = b_j$, if m_{ij} is even.

Suppose that $f_1, \dots, f_n$ are the faces of $\mathcal{P}$. Let $x = (a_1, b_1| \dots |a_n, b_n) \in \mathcal{C}_2$ be an element in $\mathrm{Ker}(\partial_2)$. Note that $\partial\mathcal{P}$ is connected (since P is homeomorphic to $\mathbb{D}^3$), so by (1) and (2) above, we have that $a_1 = \dots = a_n$ and $b_1 = \dots = b_n$. Since the stabilizer of a vertex is a spherical triangle group, there is an even m_{ij}, which also forces $a = b$. Therefore we have $x = (a, a| \dots |a, a)$ so $x = \partial_3(a)$ (note that the choice of orientation above forces all signs to be positive), and this gives $\ker(\partial_2) \subseteq \mathrm{im}(\partial_3)$, which suffices to prove equality and hence the vanishing of the second homology group. ☐

3.3. **Analysis of ∂_1.** Let $e = e_{ij}$ be an edge and $v = v_{ijk} \in \partial e$ a vertex, using the notation in (3). We study all possible induction homomorphisms $R_{\mathbb{C}}(\langle s_i, s_j\rangle) \to R_{\mathbb{C}}(\langle s_i, s_j, s_k\rangle)$ in Appendix B and conclude that H_1 is torsion-free, as follows.

Theorem 2. *There is no torsion in H_1.*

Proof. Consider the Bredon chain complex

$$\mathcal{C}_2 \xrightarrow{\partial_2} \mathcal{C}_1 \xrightarrow{\partial_1} \mathcal{C}_0.$$

To prove that $H_1 = \ker(\partial_1)/\mathrm{im}(\partial_2)$ is torsion-free, it suffices to prove that $\mathcal{C}_1/\mathrm{im}(\partial_2)$ is torsion-free. Let $\alpha \in \mathcal{C}_1$ and $0 \neq k \in \mathbb{Z}$ such that $k\alpha \in \mathrm{im}(\partial_2)$. We shall prove that $\alpha \in \mathrm{im}(\partial_2)$.

Since $k\alpha \in \mathrm{im}(\partial_2)$, we can find $\beta \in \mathcal{C}_2$ with $\partial_2(\beta) = k\alpha$. Suppose that $\mathcal{P}$ has n faces, and write $\beta = (a_1, b_1| \dots |a_n, b_n) \in \mathcal{C}_2$, using vertical bars '|' to separate elements coming from different representation rings. We shall see that one can find a 1-chain β', homologous to β, and with every entry of β' a multiple of k.

At an edge e_{ij}, the differential ∂_2 takes the form (cf. §3.2)

$$\begin{array}{ccl} R_{\mathbb{C}}(\langle s_i\rangle) \oplus R_{\mathbb{C}}(\langle s_j\rangle) & \longrightarrow & R_{\mathbb{C}}(\langle s_i, s_j\rangle) \\ (a, b \,|\, a', b') & \mapsto & \pm(a-a', b-b', \widehat{b-a'}, \widehat{a-b'} \,|\, S, \dots, S), \end{array} \tag{9}$$

where $S = a + b - a' - b'$, and the elements with a hat $\widehat{}$ appearing only when m_{ij} is even. Since every entry of $\partial_2(\beta)$ is a multiple of k, using (9), we have that for every pair of intersecting faces f_i and f_j,

$$a_i \equiv a_j \pmod{k} \quad \text{and} \quad b_i \equiv b_j \pmod{k}.$$

Equation (9) also shows that $\mathbf{1}_{\partial P} = (1,1|\dots|1,1)$, the formal sum over all generators of representation rings of face stabilizers of $\partial P \in \mathcal{C}_2$, is in the kernel of ∂_2. In particular, setting $\beta' = \beta - a_1\mathbf{1}_{\partial P}$, we see that $\partial_2(\beta') = \partial_2(\beta) = \alpha$ and we can assume without loss of generality that β' satisfies $a'_1 \equiv 0 \pmod{k}$.

Let us consider the coefficients for the 1-chain β'. For every face f_j intersecting f_1, we have $a'_1 - a'_j \equiv 0 \pmod{k}$, which implies $a'_j \equiv 0 \pmod{k}$. Since $\partial\mathcal{P}$ is connected, repeating this argument we have $a'_i \equiv 0 \pmod{k}$ for all i. In addition, there are even m_{ij} (the stabilizer of a vertex is a spherical triangle group), and hence (9) also gives $a'_i - b'_j \equiv 0 \pmod{k}$, which implies $b'_j \equiv 0 \pmod{k}$. Exactly the same argument as above gives then $b'_i \equiv 0 \pmod{k}$ for all i.

Since all coefficients of β' are divisible by k, we conclude that $\alpha = \partial_2(\beta'/k) \in \operatorname{im}(\partial_2)$, as desired. □

We note that a similar method of proof can be used (in many cases) to show that H_0 is torsion-free. This approach is carried out in [10].

Corollary 2. *Let $cf(\Gamma)$ be the number of conjugacy classes of elements of finite order in Γ, and $\chi(\mathcal{C})$ the Euler characteristic of the Bredon chain complex (6). Then we have*

$$H_1 \cong \mathbb{Z}^{cf(\Gamma)-\chi(\mathcal{C})}.$$

Proof. The Euler characteristic of a chain complex coincides with the alternating sum of the ranks of the homology groups

$$\chi(\mathcal{C}) = \operatorname{rank}(H_0) - \operatorname{rank}(H_1) + \operatorname{rank}(H_2) - \operatorname{rank}(H_3).$$

Since $H_3 = H_2 = 0$, we have $\operatorname{rank}(H_1) = \operatorname{rank}(H_0) - \chi(\mathcal{C})$, and $\operatorname{rank}(H_0) = cf(\Gamma)$ [19]. Since H_1 is torsion-free (Theorem 2), the result follows. □

Note that both $cf(\Gamma)$ and $\chi(\mathcal{C})$ can be obtained directly from the geometry of the polyhedron $\mathcal{P}$ or, equivalently, from the Coxeter integers m_{ij}. We show this explicitly in Section 5.

Remark 1. Our results agree with a previous article by three of the authors [11], where we computed the rank of the Bredon homology for groups Γ with a cocompact, 3-manifold model X of the classifying space $\underline{E}\Gamma$. Firstly, note that Proposition 1 coincides with [11, Lemma 3], and, with respect to the vanishing of H_3, Proposition 2 follows from [11, Lemma 7]. The rank of $H_2(X)$ is given in [11, Corollary 14] by $\beta_2(Y)$ if X/Γ is a closed oriented 3-manifold, or $s + t' + 2t + \beta_2(Y) - 1$ otherwise. Here Y is the union of the closures of all interior faces of X/Γ along with all the non-dihedral boundary components, s is the number of orientable non-odd dihedral components, t the number of orientable odd dihedral components (see [11] for definitions), t' is the number of orientable, odd, connected components in ∂Y, and $\beta_2()$ indicates the second Betti number. In our case, $X/G = \mathcal{P}$ has a unique boundary component, which must be dihedral (all edges stabilizers are of the form $D_{m_{ij}}$ with $m_{ij} \geq 2$), and there are no interior faces. Therefore $Y = \emptyset$, $t' = 0$ and either $s = 0$ and $t = 1$ (if all m_{ij} are odd), or $s = 0$ and $t = 1$ (if there is at least one m_{ij} even). Picking any vertex v on ∂P, it has stabilizer which is a finite triangle group, and hence at least one m_{ij} equals 2. Thus we have indeed $s = 0$ and $t = 1$ and we conclude that $\operatorname{rank}(H_2) = 0$, as expected. The rank of H_0 coincides with the number of conjugacy classes of elements of finite order in Γ. This number can be deduced from the 1-skeleton of a model of $\underline{E}\Gamma$, as explained in [11, §3.2], or in

§5.1 below. Finally, the rank of H_1 is obtained, in both [11] and this article, from the rank of the other homology groups, and the Euler characteristic of the chain complex (5) (or (1) in [11]), which equals the alternating sum of the number of conjugacy classes in the stabilizers of Γ-orbits of cells.

To complete the computation of the Bredon homology, and hence of the equivariant K-homology, all that remains is the compute the torsion subgroup of H_0. We will show that in fact H_0 is also torsion-free.

Theorem 3. *There is no torsion in* H_0*.*

We postpone the proof to Section 4 below. An immediate consequence of Theorem 3 is

Corollary 3. $K_0^\Gamma(X)$ *is torsion-free of rank* $cf(\Gamma)$*.*

Combining Corollaries 1, 2, and 3 immediately yields the above Main Theorem. Moreover, in Section 5, we will give a formula for $cf(\Gamma)$ and $\chi(\mathcal{C})$ from the geometry of the polyhedron.

4. No torsion in H_0

In this section, we give a proof of Theorem 3 inspired by the representation ring splitting technique of [22]. For this purpose, we establish a criterion for $\mathrm{H}_0^{\mathfrak{Fin}}(\Gamma;\, R_\mathbb{C})$ to be torsion-free which is quite efficient to check (by elementary linear algebra), and which is satisfied for Γ a hyperbolic Coxeter group. The verification of that criterion relies on simultaneous base transformations of the representation rings, bringing the induction homomorphisms into the desired form, as carried out in Appendices A and B. An alternative, geometric, proof for Theorem 3 is outlined for hyperbolic Coxeter groups, and completed for a specific collection of hyperbolic Coxeter groups, in [10].

Definition 1. The *vertex block* of a given vertex v in a Bredon chain complex differential matrix ∂_1 consists of all the blocks of ∂_1 that are representing maps induced (on complex representation rings from $\Gamma_e \to \Gamma_v$) by edges e adjacent to v.

Therefore, denoting by n_0 the rank of $R_\mathbb{C}(\Gamma_v)$, if v is adjacent to edges e_1, e_2 and e_3, with corresponding representation rings of rank n_1, n_2 and n_3, then its vertex block is a submatrix of ∂_1 (identified with its matrix representation after fixing a basis) of size $n_1+n_2+n_3$ times n_0.

Our proof of Theorem 3 consists of checking on the vertex blocks the following criterion that guarantees that $\mathrm{H}_0^{\mathfrak{Fin}}(\Gamma;\, R_\mathbb{C})$ is torsion-free.

Theorem 4. *If there exists a base transformation such that all minors in all vertex blocks are in the set* $\{-1,0,1\}$*, then* $\mathrm{H}_0^{\mathfrak{Fin}}(\Gamma;\, R_\mathbb{C})$ *is torsion-free.*

For the proof of this criterion, we will use a simple linear algebra lemma, which we prove again here for the sake of completeness, even though it was known to linear algebraists before the authors re-discovered it.

Lemma 2. *For square matrices* M *and* B*, we have*

$$\det\begin{pmatrix} M & 0 \\ * & B\end{pmatrix} = \det(M)\cdot\det(B).$$

Proof. Writing $M = (a_{i,j})$ and denoting by $M^{i,j}$ the (i,j)-minor of M (that is, the determinant of the block obtained by omitting the i-th row and the j-th column), we have

$$\det \begin{pmatrix} M & 0 \\ * & B \end{pmatrix} = \sum_{j=1}^{\#M} a_{1,j}(-1)^{1+j} M^{1,j} \cdot \det(B),$$

by iterating the development of the determinant by minors $(\#M - 1)$ times, and making use of the zero block each time. □

Proof of Theorem 4. As a generality on Smith Normal Forms, known already to Smith [24], we note that the elementary divisors of a matrix A can be computed (up to multiplication by a unit) as $\alpha_i = \frac{d_i(A)}{d_{i-1}(A)}$, where $d_i(A)$ (called i-th determinant divisor) equals the greatest common divisor of all $i \times i$ minors of the matrix A, and $d_0(A) := 1$.

Let us use the notation

$$\text{pre-rank}(\partial_1) := \operatorname{rank}_{\mathbb{Z}} C_1 - \operatorname{rank}_{\mathbb{Z}} \ker \partial_1,$$

where C_1 is the module of 1-chains in the Bredon chain complex.

Then $\mathrm{H}_0^{\mathfrak{Fin}}(\Gamma;\, R_{\mathbb{C}})$ is torsion-free if and only if $\alpha_i = \pm 1$ for all $1 \leq i \leq \text{pre-rank}(\partial_1)$, which, by the remark above, follows from finding an $(i \times i)$-minor in the Bredon chain complex differential matrix ∂_1 with value ± 1, for each $1 \leq i \leq \text{pre-rank}(\partial_1)$. Let us show this is indeed the case, by induction on i.

Induction basis. For $i = 1$, we observe that there are vertices with adjacent edges for the action of Γ, hence there are non-zero vertex blocks. As by assumption all the entries in the vertex blocks are in the set $\{-1, 0, 1\}$, there exists an entry of value ± 1.

Inductive step. Let $2 \leq i \leq \text{pre-rank}(\partial_1)$, and assume that there exists an $(i-1) \times (i-1)$-minor of ∂_1 of value ± 1. We have to find an $i \times i$-minor of ∂_1 of value ± 1. Let B' be the $(i-1) \times (i-1)$-block of ∂_1, whose determinant is ± 1 according to the inductive hypothesis. As $i \leq \text{pre-rank}(\partial_1)$, there exists a vertex block V with the following property : After suitable base transformation, which puts V into the upper left corner of ∂_1, the block B' can be extended to an $i \times i$-block

$$B'' = \det \begin{pmatrix} M & 0 \\ * & B \end{pmatrix},$$

with B a square sub-block of B', and M a square sub-block of V. Here, B is completely disjoint with V, and therefore we get the zero block in the upper right corner of B'' (note that the vertex blocks have been constructed to contain all entries from adjacent edges, so the remainders of their rows are zero). In case that B' is already disjoint with V, we simply have $B = B'$, and M is then a single entry of V. Otherwise, we construct B as the maximal square sub-block of B' that has all of its rows and columns outside rows and columns in which V is present. Then M is constructed by taking the intersection of B' and V, and extending it to a square block of size $n - \text{size}(B)$ inside V. Lemma 2 yields $\det(B'') = \det(M) \cdot \det(B)$. Now $\det(M) \in \{-1, 0, 1\}$ by the assumption that there is no torsion in the vertex blocks; and as $n \leq \text{pre-rank}(\partial_1)$, we can reach $\det(B'') \neq 0$ and hence $\det(M) \neq 0$. For B as a sub-block of B', $\det(B') \in \{-1, 1\}$ entails $\det(B) \in \{-1, 1\}$. This implies $\det(B'') = \pm 1$ and we are done. □

Using Theorem 4, the lack of non-trivial torsion in H_0 is a consequence of the following result, whose proof depends on the simultaneous base transformations in Appendix B.

Proposition 3. *For a system of finite subgroups of types $A_5 \times C_2$, S_4, $S_4 \times C_2$, $\Delta(2,2,2) = (C_2)^3$ and $\Delta(2,2,m) = C_2 \times D_m$ for $m \geq 3$ as vertex stabilizers, with their three 2-generator Coxeter subgroups as adjacent edge stabilizers, there is a simultaneous base transformation such that all vertex blocks have all their minors contained in the set $\{-1, 0, 1\}$.*

Proof. We apply the base transformation specified in Appendix A. Then we already see that all of the induced maps (Appendix B, all Tables referenced in this proof can be found there) have all of their entries in the set $\{-1, 0, 1\}$. Next, we assemble the vertex blocks from the three vertex-edge-adjacency induced maps for any given vertex stabilizer type. By Tables 24 and 25, the vertex block of a stabilizer of type D_m for $m \geq 3$ odd consists of

$$\text{two blocks } \pm \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 \end{pmatrix} \text{ and one block } \pm \begin{pmatrix} \text{identity matrix of size } \frac{m+3}{2} \\ 0 \end{pmatrix}.$$

By Lemma 2, the columns which are concentrated in one entry ± 1 cannot increase the absolute value of the determinant of a block which is being extended into them. We conclude that all minors are in $\{0, \pm 1\}$.

For $m \geq 6$ even, but not a power of 2, Tables 26 and 27 yield the following vertex block, where each matrix block is specified up to orientation sign (we make this assumption from now on),

$D_m \times C_2$	$D_m \hookrightarrow D_m \times C_2$							$D_2 \hookrightarrow D_m \times C_2$				$D_2 \hookrightarrow D_m \times C_2$			
$\rho_1 \otimes \chi_1 \downarrow$	1	0	0	0	0	0	0	1	0	0	0	1	0	0	0
$\rho_1 \otimes (\chi_2 - \chi_1) \downarrow$	0	1	0	1	0	0	0	0	0	0	1	0	0	0	1
$\rho_1 \otimes (\chi_3 - \chi_2) \downarrow$	0	0	1	-1	0	0	0	0	0	0	0	0	0	0	0
$\rho_1 \otimes (\chi_4 - \chi_1) \downarrow$	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0
$\rho_1 \otimes (\phi_1 - \chi_3 - \chi_1) \downarrow$	0	0	0	1	1	0	0	0	0	0	0	0	0	0	0
$\vdots$ $\rho_1 \otimes (\phi_p - \phi_{p-1}) \downarrow$ $\vdots$	0	0	0	0	0	$\ddots$	0	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\rho_1 \otimes (\phi_{\frac{m}{2}-1} - \phi_{\frac{m}{2}-2}) \downarrow$	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0
$(\rho_2 - \rho_1) \otimes \chi_1 \downarrow$								0	1	0	0	0	1	0	0
$(\rho_2 - \rho_1) \otimes (\chi_2 - \chi_1) \downarrow$								0	0	1	0	0	0	1	0
$(\rho_2 - \rho_1) \otimes (\chi_3 - \chi_2) \downarrow$								0	0	0	0	0	0	0	0
$(\rho_2 - \rho_1) \otimes (\chi_4 + \chi_3 - \chi_2 - \chi_1) \downarrow$	**0**							0	0	0	0	0	0	0	0
$(\rho_2 - \rho_1) \otimes (\phi_1 - \chi_2 - \chi_1) \downarrow$								0	0	0	0	0	0	0	0
$\vdots$ $\rho_1 \otimes (\phi_p - \phi_{p-1}) \downarrow$ $\vdots$								$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\rho_1 \otimes (\phi_{\frac{m}{2}-1} - \phi_{\frac{m}{2}-2}) \downarrow$								0	0	0	0	0	0	0	0

By Lemma 2, we can ignore the rows and columns which have at most one entry ± 1, and reduce the above vertex block to the finite block

$$\begin{pmatrix} 1 & 0 & 1 & 0 & \pm 1 & \pm 1 \\ 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \end{pmatrix},$$

for which we can easily check that it has all its minors in $\{0, \pm 1\}$.

For $m \geq 4$ a power of 2, Tables 28 and 29 yield the following vertex block,

$D_m \times C_2$	$D_m \hookrightarrow D_m \times C_2$							$D_2 \hookrightarrow D_m \times C_2$				$D_2 \hookrightarrow D_m \times C_2$			
$\rho_1 \otimes \chi_1 \downarrow$	1	0	0	0	0	0	0	1	0	0	0	1	0	0	0
$\rho_1 \otimes (\chi_2 - \chi_1) \downarrow$	0	1	0	1	0	0	0	0	0	0	1	0	0	0	1
$\rho_1 \otimes (\chi_3 - \chi_1) \downarrow$	0	1	1	0	0	0	0	0	0	0	0	0	0	0	0
$\rho_1 \otimes (\chi_4 - \chi_2) \downarrow$	0	-1	0	0	0	0	0	0	0	0	0	0	0	0	0
$\rho_1 \otimes (\phi_1 - \chi_2 - \chi_1) \downarrow$	0	0	1	0	1	0	0	0	0	0	0	0	0	0	0
$\vdots$ $\rho_1 \otimes (\phi_p - \phi_{p-1}) \downarrow$ $\vdots$	0	0	0	0	0	$\ddots$	0	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$(\phi_{\frac{m}{2}-1} - \phi_{\frac{m}{2}-2}) \downarrow$	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0
$(\rho_2 - \rho_1) \otimes \chi_1 \downarrow$								0	1	0	0	0	1	0	0
$(\rho_2 - \rho_1) \otimes (\chi_2 - \chi_1) \downarrow$								0	0	1	0	0	0	1	0
$(\rho_2 - \rho_1) \otimes (\chi_3 - \chi_1) \downarrow$								0	0	0	0	0	0	0	0
$(\rho_2 - \rho_1) \otimes (\chi_4 + \chi_3 - \chi_2 - \chi_1) \downarrow$				**0**				0	0	0	0	0	0	0	0
$(\rho_2 - \rho_1) \otimes (\phi_1 - \chi_2 - \chi_1) \downarrow$								0	0	0	0	0	0	0	0
$\vdots$ $\rho_1 \otimes (\phi_p - \phi_{p-1}) \downarrow$ $\vdots$								$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\rho_1 \otimes (\phi_{\frac{m}{2}-1} - \phi_{\frac{m}{2}-2}) \downarrow$								0	0	0	0	0	0	0	0

By Lemma 2, we can ignore the rows and columns which have at most one entry ± 1, and reduce the above vertex block to the finite block

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \pm 1 & 0 & \pm 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & \pm 1 & 0 & \pm 1 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix},$$

for which we can easily check that it has all its minors in $\{0, \pm 1\}$. For the finitely many remaining stabilizer types, we can proceed case-by-case: we input each vertex block into a computer routine which computes all of its minors (such a routine is straightforward to implement and takes approximately two seconds per vertex block on a standard computer; the third author's implementation is available at http://math.uni.lu/~rahm/vertexBlocks/). Note that for the groups under consideration, the matrix rank of the vertex block is at most 7, so the 8×8-minors are all zero, and it is enough to compute the $n \times n$-minors for $n \leq 7$. □

Corollary 4. *For any Coxeter group Γ having a system of finite subgroups of types $\Delta(2,2,2) = (C_2)^3$, $\Delta(2,2,m) = C_2 \times D_m$ for $m \geq 3$, S_4, $S_4 \times C_2$ or $A_5 \times C_2$ as vertex stabilizers, we have that the Bredon homology group $\mathrm{H}_0^{\mathfrak{Fin}}(\Gamma;\, R_{\mathbb{C}})$ is torsion-free.*

Remark 2. Note that to use the criterion in Theorem 4 for proving that $\mathrm{H}_n^{\mathfrak{Fin}}(\Gamma;\, R_{\mathbb{C}})$ is torsion-free for a group Γ, the following should be taken into account:

(a) The proof of the theorem, stated for $n = 0$, implicitly uses the fact that $H_0(\underline{\mathrm{B}}\Gamma; \mathbb{Z})$ is always torsion-free. To extend the theorem to $n > 0$, one should either work with groups for which $H_n(\underline{\mathrm{B}}\Gamma; \mathbb{Z})$ is torsion-free (such as Coxeter groups), or use the splitting described in section 7 of [22] in order to make a statement only about the part of $\mathrm{H}_n^{\mathfrak{Fin}}(\Gamma;\, R_{\mathbb{C}})$ complementary to $H_n(\underline{\mathrm{B}}\Gamma; \mathbb{Z})$.

(b) Before trying to prove the hypothesis in the theorem using base transformations (as in Appendix A), which can become quite laborious, one should construct the vertex blocks without any base transformation and compute their elementary divisors. If there exists a suitable simultaneous base transformation which proves the criterion, then those elementary divisors must be in the set $\{-1, 0, 1\}$.

5. $cf(\Gamma)$ and $\chi(\mathcal{C})$ from the geometry of $\mathcal{P}$

Let Γ be the reflection group of the compact 3-dimensional hyperbolic polyhedron $\mathcal{P}$. In this section, we compute the number of conjugacy classes of elements of finite order of Γ, $cf(\Gamma)$, and the Euler characteristic of the Bredon chain complex (5), $\chi(\mathcal{C})$, from the geometry of the polyhedron $\mathcal{P}$. This makes our Bredon homology and equivariant K-theory results, in particular the above Main Theorem, explicit.

5.1. **Conjugacy classes of elements of finite order.** We now give an algorithm to calculate $cf(\Gamma)$, the number of conjugacy classes of elements of finite order in the Coxeter group Γ. We know that each element of finite order can be conjugated to one which stabilizes one of the k-dimensional faces of the polyhedron, for some $k \in \{0, 1, 2\}$. Of course, the only element which stabilizes *all* faces is the identity element. Let us set that aside, and consider the non-identity elements, to which we associate the integer k. We now count the elements according to the integer k, in descending order.

<u>Case $k = 2$:</u> These are the conjugacy classes represented by the canonical generators of the Coxeter group Γ. The number of these is given by the total number $|\mathcal{P}^{(2)}|$ of facets of the polyhedron $\mathcal{P}$.

<u>Case $k = 1$:</u> These elements are edge stabilizers which are not conjugate to the stabilizer of a face. We first note that there are some possible conjugacies between edge stabilizers. Geometrically, these occur when there is a geodesic $\gamma \subset \mathbb{H}^3$ whose projection onto the fundamental domain $\mathcal{P}$ covers multiple edges inside the 1-skeleton $\mathcal{P}^{(1)}$. A detailed analysis of when this can happen is given in [9]. Following the description in that paper, we decompose the 1-skeleton into equivalence classes of edges, where two edges are equivalent if there exists a geodesic whose projection passes through both edges. Denote by $[\mathcal{P}^{(1)}]$ the set of equivalence classes of edges, and note that each equivalence class $[e]$ has a well defined group associated to it, which is just the dihedral group Γ_e stabilizing a representative edge. We can thus count the conjugacy classes in the corresponding dihedral group, and discard the three conjugacy classes already accounted for (the conjugacy class of the two canonical generators counted in case $k = 2$, as well as the identity). Thus the contribution from finite elements of this type is given by

$$\sum_{[e]\in[\mathcal{P}^{(1)}]} (c(\Gamma_e) - 3).$$

(Recall that $c(D_m)$, the number of conjugacy classes in a dihedral group of order $2m$, is $m/2 + 3$ if m even, and $(m - 1)/2 + 2$ if m is odd.)

<u>Case $k = 0$:</u> Finally, we consider the contribution from the elements in the vertex stabilizers *which have not already been counted.* That is to say, for each vertex $v \in \mathcal{P}^{(0)}$, we count the conjugacy classes of elements in the corresponding 3-generated spherical triangle group, which cannot be conjugated into one of the canonical 2-generated special subgroups. This number, $\bar{c}(\Gamma_v)$, depends only on the isomorphism

type of the spherical triangle group Γ_v, see Table 1. The contribution from these types of finite elements is thus

$$\sum_{v\in\mathcal{P}^{(0)}} \bar{c}(\Gamma_v)\,.$$

Γ_v	$c(\Gamma_v)$	$\bar{c}(\Gamma_v)$
$\Delta(2,2,m)$	$2\,c(D_m)$	$c(D_m)-3$
$\Delta(2,3,3)$	5	1
$\Delta(2,3,4)$	10	3
$\Delta(2,3,5)$	10	5

TABLE 1. Number of conjugacy classes in spherical triangle groups. The left column is the total number (cf. Appendix A), and the right column the number of those not conjugated into one of the three canonical 2-generated special subgroups.

Combining all these, we obtain the desired (combinatorial) formula for the number of conjugacy classes of elements of finite order inside the group Γ:

$$cf(\Gamma) = 1 + |\mathcal{P}^{(2)}| + \sum_{[e]\in[\mathcal{P}^{(1)}]} (c(\Gamma_e)-3) + \sum_{v\in\mathcal{P}^{(0)}} \bar{c}(\Gamma_v)\,.$$

5.2. **Euler characteristic.** The Euler characteristic of the Bredon chain complex can be easily calculated from the stabilizers of the various faces of the polyhedron $\mathcal{P}$, according to the formula:

$$\chi(\mathcal{C}) = \sum_{f\in\mathcal{P}} (-1)^{\dim(f)} \dim(R_{\mathbb{C}}(\Gamma_f))\,.$$

Depending on the dimension of the faces, we know exactly what the dimension of the complex representation ring is (the number of conjugacy classes in the stabilizer):

- for the 3-dimensional face (the interior), the stabilizer is trivial, so there is a 1-dimensional complex representation ring;
- for the 2-dimensional faces, the stabilizer are $\mathbb{Z}_2$, and there is a 2-dimensional complex representation ring;
- for the 1-dimensional faces e, the stabilizers are dihedral groups, and there is a $c(\Gamma_e)$-dimensional complex representation ring;
- for the 0-dimensional faces v, the stabilizers are spherical triangle groups, and there is a $c(\Gamma_v)$-dimensional complex representation ring.

Putting these together, we obtain

$$\chi(\mathcal{C}) = -1 + 2|\mathcal{P}^{(2)}| - \sum_{e\in\mathcal{P}^{(1)}} c(\Gamma_e) + \sum_{v\in\mathcal{P}^{(0)}} c(\Gamma_v)\,.$$

All in all, we have a more explicit version of the above Main Theorem, from the geometry of the polyhedron $\mathcal{P}$.

Main Theorem (explicit). *Let* Γ *be a cocompact* 3*-dimensional hyperbolic reflection group, generated by reflections in the side of a hyperbolic polyhedron* $\mathcal{P} \subset \mathbb{H}^3$*. Then* $K_0(C_r^*(\Gamma))$ *is a torsion-free abelian group of rank*

$$cf(\Gamma) = 1 + |\mathcal{P}^{(2)}| + \sum_{[e]\in[\mathcal{P}^{(1)}]} (c(\Gamma_e) - 3) + \sum_{v\in\mathcal{P}^{(0)}} \bar{c}(\Gamma_v)\,,$$

and $K_1(C_r^*(\Gamma))$ *is a torsion-free abelian group of rank*

$$cf(\Gamma) - \chi(\mathcal{C}) = 2 - |\mathcal{P}^{(2)}| + \sum_{[e]\in[\mathcal{P}^{(1)}]} (c(\Gamma_e) - 3) + \sum_{e\in\mathcal{P}^{(1)}} c(\Gamma_e) - \sum_{v\in\mathcal{P}^{(0)}} (c(\Gamma_v) - \bar{c}(\Gamma_v))\,,$$

and the values $c(\Gamma_v)$ *and* $\bar{c}(\Gamma_v)$ *can be obtained from Table 1.*

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Appendix A. Character tables and base transformations

In this Appendix, we list the character tables of all the groups involved in the Bredon chain complex (6), that is, finite Coxeter subgroups of Γ up to rank three. This serves as a reference for the main text, and fixes the notation. In the character tables below, rows correspond to irreducible representations, and columns to representatives of conjugacy classes, written in term of the Coxeter generators $s_1, \ldots, s_n$ in a fixed Coxeter presentation of Γ, as in (1).

In addition, for each character table, we apply elementary row operations to obtain the transformed tables needed for Appendix B, which are in turn used in our proof that H_0 is torsion-free (Section 4). Although the rows of the transformed tables do no more consist of irreducible characters, it is easy to check that they still constitute bases of the complex representation rings.

Note that, for consistency across subgroups, we will pay attention to the order of the Coxeter generators in a subgroup. We write e for the identity element in Γ.

A.1. **Rank 0.** This is the trivial group, with character table given below.

	e
τ	1

Table 2. Character table of the trivial group.

A.2. **Rank 1.** A rank 1 Coxeter group is a cyclic group of order 2. Write s_i for its Coxeter generator, then its character table is Table 3 below.

C_2	e	s_i
ρ_1	1	1
ρ_2	1	-1

Table 3. Character table of $\langle s_i \rangle \cong C_2$.

A.3. **Rank 2.** A finite rank 2 Coxeter group with Coxeter generators s_i and s_j is a dihedral group of order $m = m_{ij} \geq 2$,

$$D_m = \langle s_i, s_j \mid s_i^2 = s_j^2 = (s_i s_j)^m \rangle \,. \tag{10}$$

The character table of this group is given in Table 4 below, where $0 \leq r \leq m-1$, p varies between 1 and $m/2-1$ if m is even or $(m-1)/2$ if m is odd, and the hat $\widehat{\ }$ denotes a character which appear only when m is even. In order to be consistent,

D_m	$(s_is_j)^r$	$s_j(s_is_j)^r$
χ_1	1	1
χ_2	1	-1
$\widehat{\chi_3}$	$(-1)^r$	$(-1)^r$
$\widehat{\chi_4}$	$(-1)^r$	$(-1)^{r+1}$
ϕ_p	$2\cos\left(\frac{2\pi pr}{m}\right)$	0

TABLE 4. Character table of $< s_i, s_j >\cong D_m$, where $i < j$.

we assume the Coxeter generators are ordered so that $i < j$. If $j < i$, then the character table is identical except that the third and fourth rows (the characters $\widehat{\chi_3}$ and $\widehat{\chi_4}$) are interchanged, since $(s_js_i)^r = (s_is_j)^{-r}$ and $s_i(s_js_i)^r = s_j(s_is_j)^{-r-1}$.

For the case $m = 2$, that is, $D_2 = C_2 \times C_2$, we will sometimes use the notation coming from the character table of C_2 (Table 3) instead. (Recall that the irreducible characters of a direct product $G \times H$ are obtained from the irreducible characters of G and H as $\rho_i \otimes \tau_j$, where $(\rho_i \otimes \tau_j)(g,h) = \rho_i(g) \cdot \tau_j(h)$.) This gives the notation and characters in Table 5, which are equivalent to Table 4 with $\rho_1 \otimes \rho_1 = \chi_1$, $\rho_1 \otimes \rho_2 = \chi_4$, $\rho_2 \otimes \rho_1 = \chi_3$ and $\rho_2 \otimes \rho_2 = \chi_2$. As before, we assume $i < j$, or, if $j < i$, the third and fourth rows (characters) must be interchanged.

$C_2 \times C_2$	e	s_i	s_j	s_is_j
$\rho_1 \otimes \rho_1$	1	1	1	1
$\rho_1 \otimes \rho_2$	1	1	-1	-1
$\rho_2 \otimes \rho_1$	1	-1	1	-1
$\rho_2 \otimes \rho_2$	1	-1	-1	1

TABLE 5. Alternative character table of $\langle s_i, s_j \rangle \cong D_2 = C_2 \times C_2$, $i < j$.

We now give the base transformations of the character table of D_m needed later, shown in Tables 6, 7 and 8.

A.4. **Rank 3.** A finite rank 3 Coxeter subgroup is one of the spherical triangle groups $\Delta(2,2,m)$, with $m \geq 2$, $\Delta(2,3,3)$, $\Delta(2,3,4)$, or $\Delta(2,3,5)$, using the notation in (2), or, more compactly, the Coxeter diagrams in Figure 1.

In these diagrams, the vertices represent Coxeter generators and the edges are labelled by m_{ij}, with the conventions: no edge if $m_{ij} = 2$, and no label if $m_{ij} = 3$.

D_2	e	s_is_j	s_i	s_j
$\sum \chi_i$	4	0	0	0
$\chi_2+\chi_3$	2	0	-2	0
χ_3	1	-1	-1	1
$\chi_3+\chi_4$	2	-2	0	0

TABLE 6. Base transformation of the character table of D_2.

D_m	e	s_i	s_is_j	$(s_is_j)^r$	$(s_is_j)^{\frac{m-1}{2}}$
$\chi_1+\chi_2+2\sum_{p=1}^{\frac{m-1}{2}}\phi_p$	$2m$	0	0	$\dots$	0
$\chi_2+\sum_{p=1}^{\frac{m-1}{2}}\phi_p$	m	-1	0	$\dots$	0
$\sum_{p=1}^{\frac{m-1}{2}}\phi_p$	$m-1$	0	-1	$\dots$	-1
$\vdots$ $\sum_{p=k}^{\frac{m-1}{2}}\phi_p$ $\vdots$	$m-2k+1$	$\vdots$	$a_{k,1}$	$a_{k,r}$	$a_{k,\frac{m-1}{2}}$
$\phi_{\frac{m-1}{2}}$	2	0	b_1	b_r	$b_{\frac{m-1}{2}}$

TABLE 7. Base transformation of the character table of D_m, $m \geq 3$ odd. Here, $a_{k,r} := \sum_{p=k}^{\frac{m-1}{2}} 2\cos(\frac{2\pi pr}{m})$, $b_r := 2\cos(\frac{\pi(m-1)r}{m})$, and $1 < k, r < \frac{m-1}{2}$.

D_m	e	s_i	s_is_j	$(s_is_j)^r$	$(s_is_j)^{\frac{m}{2}}$	$s_js_is_j$
$\sum_{p=1}^{4}\chi_p+2\sum_{p=1}^{\frac{m}{2}-1}\phi_p$	$2m$	0	0	$\dots$	0	0
$\chi_2+\chi_3+\sum_{p=1}^{\frac{m}{2}-1}\phi_p$	m	0	0	$\dots$	0	-2
$\chi_3+\sum_{p=1}^{\frac{m}{2}-1}\phi_p$	$m-1$	1	-1	$\dots$	-1	-1
$\chi_2+\chi_4+\sum_{p=1}^{\frac{m}{2}-1}\phi_p$	m	-2	0	$\dots$	0	0
$\sum_{p=1}^{\frac{m}{2}-1}\phi_p$	$m-2$	0	0	$-1-(-1)^r$	$-1-(-1)^{\frac{m}{2}}$	0
$\vdots$ $\sum_{p=k}^{\frac{m}{2}-1}\phi_p$ $\vdots$	$m-2k$	$\vdots$	$a_{k,1}$	$a_{k,r}$	$2\sum_{p=k}^{\frac{m}{2}-1}(-1)^p$	0
$\phi_{\frac{m}{2}-1}$	2	0	b_1	b_r	$2(-1)^{\frac{m}{2}-1}$	0

TABLE 8. Base transformation of the character table of D_m, $m \geq 4$ even. Here, $a_{k,r} := \sum_{p=k}^{\frac{m}{2}-1} 2\cos(\frac{2\pi pr}{m})$, $b_r := 2\cos(\frac{\pi(m-2)r}{m})$, $1 < k < \frac{m}{2}-1$ and $1 < r < \frac{m}{2}$.

FIGURE 1. Coxeter diagrams of rank 3 spherical Coxeter groups.

A.4.1. $\Delta(2,2,2)$. This triangle group is isomorphic to $C_2 \times C_2 \times C_2$ and we have irreducible characters $\rho_{abc} := \rho_a \otimes \rho_b \otimes \rho_c$, $a, b, c \in \{1, 2\}$, from Table 3, listed in Table 9 below, where $\rho_a \otimes \rho_b \otimes \rho_c(x) = \rho_a(x_1) \cdot \rho_b(x_2) \cdot \rho_c(x_3)$ for all $x =$

$(x_1, x_2, x_3) \in C_2 \times C_2 \times C_2$. Here we assume that we have ordered the Coxeter

$C_2 \times C_2 \times C_2$	e	s_i	s_j	s_k	s_is_j	s_is_k	s_js_k	$s_is_js_k$
$\rho_{111} := \rho_1 \otimes \rho_1 \otimes \rho_1$	1	1	1	1	1	1	1	1
$\rho_{112} := \rho_1 \otimes \rho_1 \otimes \rho_2$	1	1	1	−1	1	−1	−1	−1
$\rho_{121} := \rho_1 \otimes \rho_2 \otimes \rho_1$	1	1	−1	1	−1	1	−1	−1
$\rho_{122} := \rho_1 \otimes \rho_2 \otimes \rho_2$	1	1	−1	−1	−1	−1	1	1
$\rho_{211} := \rho_2 \otimes \rho_1 \otimes \rho_1$	1	−1	1	1	−1	−1	1	−1
$\rho_{212} := \rho_2 \otimes \rho_1 \otimes \rho_2$	1	−1	1	−1	−1	1	−1	1
$\rho_{221} := \rho_2 \otimes \rho_2 \otimes \rho_1$	1	−1	−1	1	1	−1	−1	1
$\rho_{222} := \rho_2 \otimes \rho_2 \otimes \rho_2$	1	−1	−1	−1	1	1	1	−1

TABLE 9. Character table of $\langle s_i, s_j, s_k\rangle \cong \Delta(2,2,2) = C_2 \times C_2 \times C_2$, $i < j < k$, from Table 3.

generators s_i, s_j, s_k so that $i < j < k$. Finally, the base transformation of the character table of $\Delta(2,2,2)$ needed later is shown in Table 10.

$C_2 \times C_2 \times C_2$	e	s_i	s_j	s_k	s_is_j	s_is_k	s_js_k	$s_is_js_k$
ρ_{111}	1	1	1	1	1	1	1	1
$\rho_{112} - \rho_{111}$	0	0	0	−2	0	−2	−2	−2
$\rho_{121} - \rho_{111}$	0	0	−2	0	−2	0	−2	−2
$\rho_{122} - \rho_{121}$	0	0	0	−2	0	−2	2	2
$\rho_{211} - \rho_{111}$	0	−2	0	0	−2	−2	0	−2
$\rho_{212} - \rho_{211}$	0	0	0	−2	0	2	−2	2
$\rho_{221} - \rho_{121}$	0	−2	0	0	2	−2	0	2
$\rho_{222} - \rho_{221}$	0	0	0	−2	0	2	2	−2

TABLE 10. Base transformation of the character table of $\langle s_i, s_j, s_k\rangle \cong \Delta(2,2,2) = C_2 \times C_2 \times C_2$, $i < j < k$.

A.4.2. $\Delta(2,2,m)$ *with* $m > 2$. This group is isomorphic to $C_2 \times D_m$, and has Coxeter presentation

$$\Delta(2,2,m) = \left\langle s_i, s_j, s_k \mid s_i^2, s_j^2, s_k^2, (s_is_j)^2, (s_is_k)^2, (s_js_k)^m \right\rangle ,$$

where we have sorted the Coxeter generators s_j and s_k such that $j < k$ (the generator s_i is uniquely determined from the presentation). As a direct product of two groups, the character table of this group can be obtained from those of C_2 (Table 3) and D_m (Table 4). This is shown on Table 11, where T_{D_m} is the matrix of entries of the character table of D_m (Table 4). As explained before, if $k < j$ one needs to swap the characters χ_3 and χ_4, that is, swap $\rho_1 \otimes \chi_3$ and $\rho_1 \otimes \chi_4$, and $\rho_2 \otimes \chi_3$ and $\rho_2 \otimes \chi_4$.

The corresponding base transformations for $\Delta(2,2,m) \cong C_2 \times D_m$ are given in Table 12 (m odd), Table 13 ($m \geq 6$ even not a power of 2), and Table 14 ($m \geq 4$ a power of 2).

$\Delta(2,2,m)$	$(s_js_k)^r$	$s_k(s_js_k)^r$	$s_i(s_js_k)^r$	$s_is_k(s_js_k)^r$
$\rho_1\otimes\chi_1$				
$\rho_1\otimes\chi_2$				
$\rho_1\otimes\widehat{\chi_3}$	T_{D_m}		T_{D_m}	
$\rho_1\otimes\widehat{\chi_4}$				
$\rho_1\otimes\phi_p$				
$\rho_2\otimes\chi_1$				
$\rho_2\otimes\chi_2$				
$\rho_2\otimes\widehat{\chi_3}$	T_{D_m}		$-T_{D_m}$	
$\rho_2\otimes\widehat{\chi_4}$				
$\rho_2\otimes\phi_p$				

TABLE 11. Character table of $\Delta(2,2,m)$, $m=m_{jk}>2$, $j<k$.

$D_m\times C_2$	e	s_i	$(s_is_j)^r$	α	αs_i	$\alpha(s_is_j)^r$
$\rho_1\otimes\chi_1$	1	1	1	1	1	1
$\rho_1\otimes(\chi_2-\chi_1)$	0	-2	0	0	-2	0
$\rho_1\otimes(\phi_1-\chi_2-\chi_1)$	0	0	b_r	0	0	b_r
$\rho_1\otimes(\phi_p-\phi_{p-1})$,	0	0	$a_{p,r}$	0	0	$a_{p,r}$
$(\rho_2-\rho_1)\otimes\chi_1$	0	0	1	-2	-2	-2
$(\rho_2-\rho_1)\otimes(\chi_2-\chi_1)$	0	0	0	0	4	0
$(\rho_2-\rho_1)\otimes(\phi_1-\chi_2-\chi_1)$	0	0	0	0	-4	$-2b_r$
$(\rho_2-\rho_1)\otimes(\phi_p-\phi_{p-1})$,	0	0	0	0	0	$-2a_{p,r}$

TABLE 12. Base transformation of the character table of $D_m\times C_2$ for $m\geq 3$ odd. Here $a_{p,r}:=2\cos(\frac{2\pi pr}{m})-2\cos(\frac{2\pi(p-1)r}{m})$, $b_r:=2\cos(\frac{2\pi r}{m})-2$, $2\leq p\leq\frac{m-1}{2}$ and $1\leq r\leq m-1$.

A.4.3. $\Delta(2,3,3)$. This triangle group has Coxeter presentation

$$\Delta(2,3,3)=\left\langle s_i,s_j,s_k \mid s_i^2,s_j^2,s_k^2,(s_is_j)^3,(s_is_k)^2,(s_js_k)^3\right\rangle . \tag{11}$$

This group is, in standard notation, A_3, and it is isomorphic to the symmetric group S_4, with Coxeter generators $s_i=(1\ 2)$, $s_j=(2\ 3)$ and $s_k=(3\ 4)$, for instance. Then we have conjugacy classes (cycle types in S_4) represented, in terms of the Coxeter generators, by $s_i=(1\ 2)$, $s_is_j=(1\ 3\ 2)$, $s_is_js_k=(1\ 4\ 3\ 2)$ and $s_is_k=(1\ 2)(3\ 4)$. In particular, we have the character table for $\Delta(2,3,3)$ shown in Table 15 below. In this case, the Coxeter generators are unique up to conjugation since $\mathrm{Out}(S_4)$ is trivial, hence the choice of s_i and s_k does not affect the character table.

The required base transformation of the character table Table 15 is given in Table 16.

A.4.4. $\Delta(2,3,4)$. This triangle group has Coxeter presentation

$$\Delta(2,3,4)=\left\langle s_i,s_j,s_k \mid s_i^2,s_j^2,s_k^2,(s_is_j)^3,(s_is_k)^2,(s_js_k)^4\right\rangle , \tag{12}$$

and the Coxeter generators are uniquely determined from the presentation. In standard notation, this is the finite Coxeter group B_3 (or C_3). This group is

$D_m \times C_2$	s_i	$(s_is_j)^r$	$s_js_is_j$	αs_i	$\alpha(s_is_j)^r$	$\alpha s_js_is_j$
$\rho_1 \otimes \chi_1$	1	1	1	1	1	1
$\rho_1 \otimes (\chi_2 - \chi_1)$	-2	0	-2	-2	0	-2
$\rho_1 \otimes (\chi_3 - \chi_2)$	2	c_r	0	2	c_r	0
$\rho_1 \otimes (\chi_4 - \chi_1)$	-2	c_r	0	-2	c_r	0
$\rho_1 \otimes (\phi_1 - \chi_3 - \chi_1)$	-2	b_r	0	-2	b_r	0
$\vdots$ $\rho_1 \otimes (\phi_p - \phi_{p-1})$ $\vdots$	0	$a_{p,r}$	0	0	$a_{p,r}$	0
$(\rho_2 - \rho_1) \otimes \chi_1$	0	0	0	-2	-2	-2
$(\rho_2 - \rho_1) \otimes (\chi_2 - \chi_1)$	0	0	0	4	0	4
$(\rho_2 - \rho_1) \otimes (\chi_3 - \chi_2)$	0	0	0	-4	$-2c_r$	0
$(\rho_2 - \rho_1) \otimes (\chi_4 + \chi_3 - \chi_2 - \chi_1)$	0	0	0	0	$-4c_r$	0
$(\rho_2 - \rho_1) \otimes (\phi_1 - \chi_2 - \chi_1)$	0	0	0	0	$-2(b_r + c_r)$	0
$(\rho_2 - \rho_1) \otimes (\phi_p - \phi_{p-1})$	0	0	0	0	$-2a_{p,r}$	0

TABLE 13. Base transformation of the character table of $D_m \times C_2$ for $m \geq 6$ even, not a power of 2. Here $a_{p,r} := 2\cos(\frac{2\pi pr}{m}) - 2\cos(\frac{2\pi(p-1)r}{m})$, $b_r := 2\cos(\frac{2\pi r}{m}) - (-1)^r - 1$, $c_r = (-1)^r - 1$ and $1 < p, r < \frac{m}{2}$.

$D_m \times C_2$	s_i	$(s_is_j)^r$	$s_js_is_j$	αs_i	$\alpha(s_is_j)^r$	$\alpha s_js_is_j$
$\rho_1 \otimes \chi_1$	1	1	1	1	1	1
$\rho_1 \otimes (\chi_2 - \chi_1)$	-2	0	-2	-2	0	-2
$\rho_1 \otimes (\chi_3 - \chi_2)$	0	c_r	-2	0	c_r	-2
$\rho_1 \otimes (\chi_4 - \chi_1)$	0	c_r	2	0	c_r	2
$\rho_1 \otimes (\phi_1 - \chi_3 - \chi_1)$	0	b_r	0	0	b_r	0
$\vdots$ $\rho_1 \otimes (\phi_p - \phi_{p-1})$ $\vdots$	0	$a_{p,r}$	0	0	$a_{p,r}$	0
$(\rho_2 - \rho_1) \otimes \chi_1$	0	0	0	-2	-2	-2
$(\rho_2 - \rho_1) \otimes (\chi_2 - \chi_1)$	0	0	0	4	0	4
$(\rho_2 - \rho_1) \otimes (\chi_3 - \chi_2)$	0	0	0	0	$-2c_r$	4
$(\rho_2 - \rho_1) \otimes (\chi_4 + \chi_3 - \chi_2 - \chi_1)$	0	0	0	0	$-4c_r$	0
$(\rho_2 - \rho_1) \otimes (\phi_1 - \chi_2 - \chi_1)$	0	0	0	0	$-2b_r$	0
$(\rho_2 - \rho_1) \otimes (\phi_p - \phi_{p-1})$	0	0	0	0	$-2a_{p,r}$	0

TABLE 14. Base transformation of the character table of $D_m \times C_2$ for $m \geq 4$ even a power of 2. Here $a_{p,r} := 2\cos(\frac{2\pi pr}{m}) - 2\cos(\frac{2\pi(p-1)r}{m})$, $b_r := 2\cos(\frac{2\pi r}{m}) - 2$, $c_r = (-1)^r - 1$ and $1 < p, r < \frac{m}{2}$.

isomorphic to $S_4 \times C_2$ with Coxeter generators, for instance, $s_i = (1\ 2)\alpha$, $s_j = (1\ 3)\alpha$ and $s_k = (1\ 2)(3\ 4)\alpha$, where α is the generator of the C_2 factor. We can choose representatives of the conjugacy classes in terms of the Coxeter generators, for

$\Delta(2,3,3)$	e	s_i	$s_i s_j$	$s_i s_j s_k$	$s_i s_k$
ξ_1	1	1	1	1	1
ξ_2	1	-1	1	-1	1
ξ_3	2	0	-1	0	2
ξ_4	3	1	0	-1	-1
ξ_5	3	-1	0	1	-1

TABLE 15. Character table of $\langle s_i, s_j, s_k\rangle \cong \Delta(2,3,3) \cong S_4$.

S_4	e	s_i	$s_i s_j$	$s_i s_j s_k$	$s_i s_k$
ξ_1	1	1	1	1	1
$\widetilde{\xi_2} := \xi_2 - \xi_1$	0	-2	0	-2	0
$\widetilde{\xi_3} := \xi_3 - \xi_2 - \xi_1$	0	0	-3	0	0
$\widetilde{\xi_4} := \xi_4 - \xi_3 - \xi_1$	0	0	0	-2	-4
$\widetilde{\xi_5} := \xi_5 - \xi_4 - \xi_2 + \xi_1$	0	0	0	4	0

TABLE 16. Base transformation of the character table of $\Delta(2,3,3) \cong S_4$.

example,

$$\begin{aligned} e &\sim e & \alpha &\sim (s_i s_j s_k)^3 \\ (1\ 2) &\sim s_i s_k & (1\ 2)\alpha &\sim s_i \\ (1\ 2\ 3) &\sim s_i s_j & (1\ 2\ 3)\alpha &\sim s_i s_j s_k \\ (1\ 2\ 3\ 4) &\sim s_j s_k & (1\ 2\ 3\ 4)\alpha &\sim s_i (s_j s_k)^2 \\ (1\ 2)(3\ 4) &\sim (s_j s_k)^2 & (1\ 2)(3\ 4)\alpha &\sim s_k \end{aligned}$$

where '$\sim$' means 'conjugate'. Using this representatives, and the fact that $\Delta(2,3,4)$ is a direct product, the character table of this group can be written as in Table 17 below, where T_{S_4} is the matrix of coefficients of the character table of S_4 (Table 15).

The base transformation needed is given in Table 18, where $\{\widetilde{\xi_i}\}$ is the transformed basis of the character table of S_4, Table 16.

A.4.5. $\Delta(2,3,5)$. This group has Coxeter presentation

$$\Delta(2,3,5) = \left\langle s_i, s_j, s_k \mid s_i^2, s_j^2, s_k^2, (s_i s_j)^3, (s_i s_k)^2, (s_j s_k)^5 \right\rangle \tag{13}$$

and it is isomorphic to $A_5 \times C_2$ with Coxeter generators $s_i = (1\ 2)(3\ 5)\alpha$, $s_j = (1\ 2)(3\ 4)\alpha$, $s_k = (1\ 5)(2\ 3)\alpha$, for example. In standard notation, this is the exceptional finite Coxeter group H_3.

Remark 3. These Coxeter generators are *not* unique, since $\mathrm{Out}(A_5 \times C_2) \cong C_2$. There are then two sets of Coxeter generators up to conjugation, the other one given by conjugation by a suitable $g \in S_5 \setminus A_5$, for instance, conjugating by $g = (2\ 4)$: $s_i' = (1\ 4)(3\ 5)\alpha$, $s_j' = (1\ 4)(2\ 3)\alpha$, $s_k' = (1\ 5)(3\ 4)\alpha$. In the first case $s_j s_k =$

$S_4 \times C_2$	e	$s_i s_k$	$s_i s_j$	$s_j s_k$	$(s_j s_k)^2$	$(s_i s_j s_k)^3$	s_i	$s_i s_j s_k$	$s_i(s_j s_k)^2$	s_k
$\rho_1 \otimes \xi_1$										
$\rho_1 \otimes \xi_2$										
$\rho_1 \otimes \xi_3$			T_{S_4}					T_{S_4}		
$\rho_1 \otimes \xi_4$										
$\rho_1 \otimes \xi_5$										
$\rho_2 \otimes \xi_1$										
$\rho_2 \otimes \xi_2$										
$\rho_2 \otimes \xi_3$			T_{S_4}					$-T_{S_4}$		
$\rho_2 \otimes \xi_4$										
$\rho_2 \otimes \xi_5$										

TABLE 17. Character table of $\langle s_i, s_j, s_k \rangle \cong \Delta(2,3,4) = S_4 \times C_2$.

$S_4 \times C_2$	(1)	(12)	(123)	(1234)	(12)(34)	α	$\alpha(12)$	$\alpha(123)$	$\alpha(1234)$	$\alpha(12)(34)$
α_1	1	1	1	1	1	1	1	1	1	1
α_2	0	-2	0	-2	0	0	-2	0	-2	0
α_3	0	0	-3	0	0	0	0	-3	0	0
α_4	0	-2	0	4	-4	2	0	2	6	-2
α_5	0	0	0	4	0	0	0	0	4	0
α_6	0	2	0	2	0	-2	0	-2	0	-2
α_7	0	-4	0	0	0	0	0	0	4	0
α_8	0	0	0	0	0	0	0	6	0	0
α_9	0	0	0	6	-4	4	0	4	-6	8
α_{10}	0	0	0	0	0	0	0	0	-8	0

where $\alpha_1 := \rho_1 \otimes \xi_1$, $\alpha_2 := \rho_1 \otimes \widetilde{\xi_2}$, $\alpha_3 := \rho_1 \otimes \widetilde{\xi_3}$, $\alpha_4 := \rho_1 \otimes (\widetilde{\xi_4} + 2\widetilde{\xi_5}) - (\rho_2 \otimes \xi_1 - \rho_1 \otimes \xi_2)$, $\alpha_5 := \rho_1 \otimes \widetilde{\xi_5}$, $\alpha_6 := \rho_2 \otimes \xi_1 - \rho_1 \otimes \xi_2$, $\alpha_7 := \rho_2 \otimes \xi_2 - \rho_2 \otimes \xi_1 + \rho_1 \otimes (\xi_5 - \xi_4)$, $\alpha_8 := (\rho_2 - \rho_1) \otimes \widetilde{\xi_3}$, $\alpha_9 := \rho_2 \otimes (\widetilde{\xi_4} + 2\widetilde{\xi_5}) + (\rho_1 - \rho_2) \otimes (\xi_1 + \xi_2)$, $\alpha_{10} := (\rho_2 - \rho_1) \otimes \widetilde{\xi_5}$.

TABLE 18. Base transformation of the character table of $\langle s_i, s_j, s_k \rangle \cong \Delta(2,3,4) = S_4 \times C_2$.

(1 3 4 2 5), conjugated to (1 2 3 4 5), and in the second case $s'_j s'_k = (1\ 3\ 2\ 4\ 5)$, conjugated to (1 3 2 4 5), which represent different conjugacy classes in A_5.

A character table for the alternating group A_5 reads as follows.

A_5	e	(1 2 3)	(1 2)(3 4)	(1 2 3 4 5)	(1 3 4 5 2)
ξ_1	1	1	1	1	1
ξ_2	4	1	0	-1	-1
ξ_3	5	-1	1	0	0
ξ_4	3	0	-1	$\frac{1+\sqrt{5}}{2}$	$\frac{1-\sqrt{5}}{2}$
ξ_5	3	0	-1	$\frac{1-\sqrt{5}}{2}$	$\frac{1+\sqrt{5}}{2}$

If we call T_{A_5} the matrix of coefficients of this table then the character table of $\Delta(2,3,5)$ is given by Table 19, where we have used the following representatives of the conjugacy classes in terms of the Coxeter generators ('$\sim$' means 'conjugate')

$$\begin{aligned} e &\sim e & \alpha &\sim (s_i s_j s_k)^5 \\ (1\ 2\ 3) &\sim s_i s_j & (1\ 2\ 3)\alpha &\sim s_i (s_j s_k)^2 \\ (1\ 2)(3\ 4) &\sim s_i s_k & (1\ 2)(3\ 4)\alpha &\sim s_i \\ (1\ 2\ 3\ 4\ 5) &\sim s_j s_k & (1\ 2\ 3\ 4\ 5)\alpha &\sim s_i s_j s_k \\ (1\ 2\ 3\ 5\ 4) &\sim (s_i s_j s_k)^4 & (1\ 2\ 3\ 5\ 4)\alpha &\sim s_j s_i s_k \end{aligned}$$

$A_5 \times C_2$	e	$s_i s_j$	$s_i s_k$	$s_j s_k$	$(s_i s_j s_k)^4$	$(s_i s_j s_k)^5$	$s_i (s_j s_k)^2$	s_i	$s_i s_j s_k$	$s_j s_i s_k$
$\rho_1 \otimes \xi_1$										
$\rho_1 \otimes \xi_2$										
$\rho_1 \otimes \xi_3$			T_{A_5}					T_{A_5}		
$\rho_1 \otimes \xi_4$										
$\rho_1 \otimes \xi_5$										
$\rho_2 \otimes \xi_1$										
$\rho_2 \otimes \xi_2$										
$\rho_2 \otimes \xi_3$			T_{A_5}					$-T_{A_5}$		
$\rho_2 \otimes \xi_4$										
$\rho_2 \otimes \xi_5$										

TABLE 19. Character table of $\langle s_i, s_j, s_k \rangle \cong \Delta(2,3,5) = A_5 \times C_2$.

Remark 4. With the non-conjugate set of Coxeter generators s'_i, s'_j, s'_k (see Remark 3), we get the analogous set of representatives except swapping the obvious ones, that is, $s_j s_k \sim (s'_i s'_j s'_k)^4$, $(s_i s_j s_k)^4 \sim s'_j s'_k$, $(s_i s_j s_k)^5 s_j s_k \sim s'_i s'_j s'_k$ and $s_i s_j s_k \sim (s'_i s'_j s'_k)^5 s'_j s'_k$. On the character table this amounts to interchanging $\rho_i \otimes \xi_4$ with $\rho_i \otimes \xi_5$ for $i = 1$ and 2.

For the base transformation required, first note that the character table for the alternating group A_5 above can be transformed as follows,

A_5	e	$(1\ 2\ 3)$	$(1\ 2)(3\ 4)$	$(1\ 2\ 3\ 4\ 5)$	$(1\ 3\ 4\ 5\ 2)$
ξ_1	1	1	1	1	1
$\widetilde{\xi_2} := \xi_2 - 4\xi_1$	0	-3	-4	-5	-5
$\widetilde{\xi_3} := \xi_3 - \xi_2 - \xi_1$	0	-3	0	0	0
$\widetilde{\xi_4} := \xi_4 - \xi_2 + \xi_1$	0	0	0	$\frac{5+\sqrt{5}}{2}$	$\frac{5-\sqrt{5}}{2}$
$\widetilde{\xi_5} := \xi_5 + \xi_4 - \xi_1 - \xi_3$	0	0	-4	0	0

and then

A_5	e	$(1\ 2\ 3)$	$(1\ 2)(3\ 4)$	$(1\ 2\ 3\ 4\ 5)$	$(1\ 3\ 4\ 5\ 2)$
ξ_1	1	1	1	1	1
$\widetilde{\widetilde{\xi_2}} := \widetilde{\xi_2} - \widetilde{\xi_3} - \widetilde{\xi_5}$	0	0	0	-5	-5
$\widetilde{\xi_3}$	0	-3	0	0	0
$\widetilde{\xi_4}$	0	0	0	$\frac{5+\sqrt{5}}{2}$	$\frac{5-\sqrt{5}}{2}$
$\widetilde{\xi_5}$	0	0	-4	0	0

With this notation, the required base transformation is given in Table 20.

$A_5 \times C_2$	e	s_is_j	s_is_k	s_js_k	$(s_is_js_k)^4$	$(s_is_js_k)^5$	$s_i(s_js_k)^2$	s_i	$s_is_js_k$	$s_js_is_k$
β_1	1	1	1	1	1	1	1	1	1	1
β_2	0	0	0	$\sqrt{5}$	$-\sqrt{5}$	0	0	0	$\sqrt{5}$	$-\sqrt{5}$
β_3	0	-3	0	0	0	0	-3	0	0	0
β_4	0	0	0	$\frac{5+\sqrt{5}}{2}$	$\frac{5-\sqrt{5}}{2}$	0	0	0	$\frac{5+\sqrt{5}}{2}$	$\frac{5-\sqrt{5}}{2}$
β_5	0	0	-4	0	0	4	4	0	4	4
β_6	0	0	0	0	0	-2	-2	-2	-2	-2
β_7	0	0	0	0	0	0	0	0	10	10
β_8	0	0	0	0	0	0	6	0	0	0
β_9	0	0	0	0	0	0	0	0	$-5-\sqrt{5}$	$-5+\sqrt{5}$
β_{10}	0	0	0	0	0	-8	-8	0	-8	-8

where $\beta_1 := \rho_1 \otimes \xi_1$, $\beta_2 := \rho_1 \otimes \widetilde{\widetilde{\xi_2}} + 2\rho_1 \otimes \widetilde{\xi_4}$, $\beta_3 := \rho_1 \otimes \widetilde{\xi_3}$, $\beta_4 := \rho_1 \otimes \widetilde{\xi_4}$, $\beta_5 := \rho_1 \otimes \widetilde{\xi_5} - 2(\rho_2 - \rho_1) \otimes \xi_1$, $\beta_6 := (\rho_2 - \rho_1) \otimes \xi_1$, $\beta_7 := (\rho_2 - \rho_1) \otimes \widetilde{\widetilde{\xi_2}}$, $\beta_8 := (\rho_2 - \rho_1) \otimes \widetilde{\xi_3}$, $\beta_9 := (\rho_2 - \rho_1) \otimes \widetilde{\xi_4}$, $\beta_{10} := (\rho_2 - \rho_1) \otimes (\widetilde{\xi_5} + 4\xi_1)$.

TABLE 20. Base transformation of the character table of $\langle s_i, s_j, s_k \rangle \cong \Delta(2,3,5) = A_5 \times C_2$.

APPENDIX B. INDUCTION HOMOMORPHISMS

In this Appendix, we compute all possible induction homomorphisms $R_{\mathbb{C}}(H) \to R_{\mathbb{C}}(G)$ appearing in the Bredon chain complex (6). That is, G is a finite Coxeter subgroup of Γ of rank n generated by $n \leq 3$ of the Coxeter generators (1), and H is a subgroup of G generated by a subset of exactly $n-1$ of those Coxeter generators.

We give explicit induction homomorphisms with respect to the standard character tables, and also with respect to the transformed bases in Appendix A for rank 3 subgroups, as this is needed for the simultaneous base transformation argument in the proof that $\mathrm{H}_0^{\mathfrak{Fin}}(\Gamma; R_{\mathbb{C}})$ is torsion-free (Section 4).

We implicitly use the character tables and notation in Appendix A, and Frobenius reciprocity [23], throughout this Appendix. We also note that Frobenius reciprocity extends linearly:

Lemma 3. *If H is a subgroup of a finite group G and ϕ and η, respectively τ and π, are representations of G, respectively H, then*

$$(\phi\downarrow + \xi\downarrow | \tau + \pi) = (\phi + \xi | \tau\uparrow + \pi\uparrow).$$

Proof.

$$\begin{aligned}(\phi\downarrow + \xi\downarrow | \tau + \pi) &= \frac{1}{|H|}\sum_{h\in H}(\phi\downarrow + \xi\downarrow)(h)\cdot\overline{(\tau+\pi)(h)}\\ &= \frac{1}{|H|}\sum_{h\in H}(\phi\downarrow\cdot\overline{\tau} + \xi\downarrow\cdot\overline{\tau} + \phi\downarrow\cdot\overline{\pi} + \xi\downarrow\cdot\overline{\pi})(h)\end{aligned}$$

which by Frobenius reciprocity on irreducible characters equals

$$\frac{1}{|G|}\sum_{g\in G}(\phi\cdot\overline{\tau}\uparrow + \xi\cdot\overline{\tau}\uparrow + \phi\cdot\overline{\pi}\uparrow + \xi\cdot\overline{\pi}\uparrow)(h) = (\phi + \xi | \tau\uparrow + \pi\uparrow). \qquad \square$$

B.1. **Rank 1.** The only induction homomorphism in this case is

$$R_{\mathbb{C}}(\{e\}) \to R_{\mathbb{C}}(\langle s_i \rangle)$$

which must the regular representation $\tau \mapsto \rho_1 + \rho_2$ shown, in terms of free abelian groups, in Figure 2.

$$\begin{array}{rcl} R_{\mathbb{C}}(\{e\}) & \to & R_{\mathbb{C}}(C_2) \\ \mathbb{Z} & \to & \mathbb{Z}^2 \\ a & \mapsto & (a,a) \end{array}$$

FIGURE 2. Induction homomorphism from $H = \{e\}$ to $G = \langle s_i \rangle \cong C_2$.

B.2. **Rank 2.** In this case G is a dihedral group with the presentation

$$G = \langle s_i, s_j \mid s_i^2 = s_j^2 = (s_i s_j)^m \rangle ,$$

where $m = m_{ij}$ and we assume $i < j$. Consider first the case $H = \langle s_i \rangle$. The characters of D_m (Table 5) restricted to H are (note that $s_i = s_j(s_i s_j)^{m-1}$)

$D_m \downarrow$	e	s_i
χ_1	1	1
χ_2	1	-1
$\widehat{\chi_3}$	1	-1
$\widehat{\chi_4}$	1	1
ϕ_p	2	0

Multiplying with the rows of the character table of $H \cong C_2$ (Table 3) we obtain the induced representations

$$\begin{array}{rcl} \rho_1 \uparrow & = & \chi_1 + \widehat{\chi_4} + \sum \phi_p, \\ \rho_2 \uparrow & = & \chi_2 + \widehat{\chi_3} + \sum \phi_p. \end{array}$$

The other case is when $H = \langle s_j \rangle$. This is analogous, but note that, in order to keep the notation consistent with Table 4, the characters χ_3 and χ_4 must be interchanged in the even case. Specifically, we have now $s_j = s_j(s_i s_j)^0$ and hence

$D_m \downarrow$	e	s_j
χ_1	1	1
χ_2	1	-1
$\widehat{\chi_3}$	1	1
$\widehat{\chi_4}$	1	-1
ϕ_p	2	0

and therefore

$$\begin{array}{rcl} \rho_1 \uparrow & = & \chi_1 + \widehat{\chi_3} + \sum \phi_p, \\ \rho_2 \uparrow & = & \chi_2 + \widehat{\chi_4} + \sum \phi_p. \end{array}$$

All in all, as maps of free abelian groups, we have that the induction homomorphisms $R_{\mathbb{C}}(H) \to R_{\mathbb{C}}(G)$ shown in Figure 3.

B.3. **Rank 3.** We compute each case individually.

$$\begin{array}{rcll} R_{\mathbb{C}}(H) & \to & R_{\mathbb{C}}(G) & \\ \mathbb{Z}^2 & \to & \mathbb{Z}^{c(D_m)} & \\ (a,b) & \mapsto & (a,b,\widehat{b},\widehat{a},a+b,\dots,a+b) & \text{for } H=\langle s_i\rangle, \\ (a,b) & \mapsto & (a,b,\widehat{a},\widehat{b},a+b,\dots,a+b) & \text{for } H=\langle s_j\rangle, \end{array}$$

FIGURE 3. Induction homomorphisms from H to $G = \langle s_i, s_j\rangle \cong D_m$, $m = m_{ij}$ and $i < j$.

B.3.1. $G = \Delta(2,2,2)$. This group is isomorphic to $C_2 \times C_2 \times C_2$, and has Coxeter generators s_i, s_j, s_k where $i < j < k$. We compute the induction homomorphisms for the Coxeter subgroups $\langle s_i, s_j\rangle$, $\langle s_i, s_k\rangle$ and $\langle s_j, s_k\rangle$, all three direct factors of G and isomorphic to $C_2 \times C_2$. Using the bases of $R_{\mathbb{C}}(C_2 \times C_2)$ and $R_{\mathbb{C}}(\Delta(2,2,2))$ induced from C_2 (Tables 5 and 9), we immediately obtain the induction homomorphisms shown, as maps of free abelian groups, shown in Figure 4.

Remark 5. Recall how to the induction homomorphism works from a group A to direct product $A \times B$: if ρ is a representation of A then $\mathrm{Ind}_A^{A\times B}(\rho) = \rho \otimes r_B$, where r_B is the regular representation of B.

$$\begin{array}{rcll} R_{\mathbb{C}}(H) & \to & R_{\mathbb{C}}(G) & \\ \mathbb{Z}^4 & \to & \mathbb{Z}^8 & \\ (a,b,c,d) & \mapsto & (a,a,b,b,c,c,d,d) & \text{for } H=\langle s_i,s_j\rangle, \\ (a,b,c,d) & \mapsto & (a,b,a,b,c,d,c,d) & \text{for } H=\langle s_i,s_k\rangle, \\ (a,b,c,d) & \mapsto & (a,b,c,d,a,b,c,d) & \text{for } H=\langle s_j,s_k\rangle. \end{array}$$

FIGURE 4. Induction homomorphisms from H to $G = \langle s_i, s_j, s_k\rangle \cong \Delta(2,2,2) = C_2 \times C_2 \times C_2$, and $i < j < k$.

On the other hand, with respect to the transformed bases (Tables 6 and 10 in Appendix A), the induction homomorphisms take the form shown in Tables 21, 22, and 23. Note that, for the transformed bases, we show the restricted characters, and the induced map, in two adjacent tables separated by double vertical bars.

B.3.2. $G = \Delta(2,2,m)$, $m > 2$. This group is isomorphic to $C_2 \times D_m$ with Coxeter presentation

$$\Delta(2,2,m) = \left\langle s_i, s_j, s_k \mid s_i^2, s_j^2, s_k^2, (s_is_j)^2, (s_is_k)^2, (s_js_k)^m \right\rangle ,$$

We have three relevant Coxeter subgroups H, which we treat separately. In each case, we restrict the characters of G (Table 11) to the subgroup H and then use Frobenius reciprocity to write the induced characters of H into G in terms of the characters of G.

<u>Case 1: $H = \langle s_i, s_j\rangle \cong D_2 = C_2 \times C_2$.</u>

The elements e, s_i, s_j and s_is_j of H are obtained from the 1st, 3rd, 2nd and 4th

$\langle s_j, s_k\rangle \hookrightarrow C_2 \times C_2 \times C_2$	e	s_j	s_k	s_js_k	$(\cdot\vert\sum\chi_i)$	$(\cdot\vert\chi_2+\chi_3)$	$(\cdot\vert\chi_3+\chi_4)$	$(\cdot\vert\chi_4)$
$\rho_{111}\downarrow$	1	1	1	1	1	0	0	0
$\rho_{112}-\rho_{111}\downarrow$	0	0	−2	−2	0	1	0	0
$\rho_{121}-\rho_{111}\downarrow$	0	−2	0	−2	0	0	1	1
$\rho_{122}-\rho_{121}\downarrow$	0	0	−2	2	0	1	0	−1
$\rho_{211}-\rho_{111}\downarrow$	0	0	0	0	0	0	0	0
$\rho_{212}-\rho_{211}\downarrow$	0	0	−2	−2	0	1	0	0
$\rho_{221}-\rho_{121}\downarrow$	0	0	0	0	0	0	0	0
$\rho_{222}-\rho_{221}\downarrow$	0	0	−2	2	0	1	0	−1

TABLE 21. Restricted characters and induced map $\langle s_j, s_k\rangle \hookrightarrow \Delta(2,2,2) = C_2 \times C_2 \times C_2$.

$\langle s_i, s_k\rangle \hookrightarrow C_2 \times C_2 \times C_2$	e	s_i	s_k	s_is_k	$(\cdot\vert\sum\chi_i)$	$(\cdot\vert\chi_2+\chi_3)$	$(\cdot\vert\chi_3+\chi_4)$	$(\cdot\vert\chi_4)$
$\rho_{111}\downarrow$	1	1	1	1	1	0	0	0
$\rho_{112}-\rho_{111}\downarrow$	0	0	−2	−2	0	1	0	0
$\rho_{121}-\rho_{111}\downarrow$	0	0	0	0	0	0	0	0
$\rho_{122}-\rho_{121}\downarrow$	0	0	−2	−2	0	1	0	0
$\rho_{211}-\rho_{111}\downarrow$	0	−2	0	−2	0	0	1	1
$\rho_{212}-\rho_{211}\downarrow$	0	0	−2	2	0	1	0	−1
$\rho_{221}-\rho_{121}\downarrow$	0	−2	0	−2	0	0	1	1
$\rho_{222}-\rho_{221}\downarrow$	0	0	−2	2	0	1	0	−1

TABLE 22. Restricted characters and induced map $\langle s_i, s_k\rangle \hookrightarrow \Delta(2,2,2) = C_2 \times C_2 \times C_2$.

$\langle s_i, s_j\rangle \hookrightarrow C_2 \times C_2 \times C_2$	e	s_i	s_j	s_is_j	$(\cdot\vert\sum\chi_i)$	$(\cdot\vert\chi_2+\chi_3)$	$(\cdot\vert\chi_3+\chi_4)$	$(\cdot\vert\chi_4)$
$\rho_{111}\downarrow$	1	1	1	1	1	0	0	0
$(\rho_{112}-\rho_{111})\downarrow$	0	0	0	0	0	0	0	0
$(\rho_{121}-\rho_{111})\downarrow$	0	0	−2	−2	0	1	0	0
$(\rho_{122}-\rho_{121})\downarrow$	0	0	0	0	0	0	0	0
$(\rho_{211}-\rho_{111})\downarrow$	0	−2	0	−2	0	0	1	1
$(\rho_{212}-\rho_{211})\downarrow$	0	0	0	0	0	0	0	0
$(\rho_{221}-\rho_{121})\downarrow$	0	−2	0	2	0	0	1	0
$(\rho_{222}-\rho_{221})\downarrow$	0	0	0	0	0	0	0	0

TABLE 23. Restricted characters and induced map $D_2 \cong \langle s_i, s_j\rangle \hookrightarrow \Delta(2,2,2) = C_2 \times C_2 \times C_2$.

column of Table 11 for r equals $0, 0, n-1$ and $n-1$ respectively, giving the restrictions (by abuse of notation we indicate the restricted character with the same symbols):

$C_2 \times D_m \downarrow$	e	s_i	s_j	$s_i s_j$
$\rho_1 \otimes \chi_1$	1	1	1	1
$\rho_1 \otimes \chi_2$	1	1	-1	-1
$\rho_1 \otimes \widehat{\chi_3}$	1	1	-1	-1
$\rho_1 \otimes \widehat{\chi_4}$	1	1	1	1
$\rho_1 \otimes \phi_p$	2	2	0	0
$\rho_2 \otimes \chi_1$	1	-1	1	-1
$\rho_2 \otimes \chi_2$	1	-1	-1	1
$\rho_2 \otimes \widehat{\chi_3}$	1	-1	-1	1
$\rho_2 \otimes \widehat{\chi_4}$	1	-1	1	-1
$\rho_2 \otimes \phi_p$	2	-2	0	0

Now we use Frobenius reciprocity (multiply the rows of the character tables of H with those of $G \downarrow H$ above, and divide by $|H| = 4$) to obtain the coefficients of the induced irreducible representations of H in G in terms of the irreducible representations of G. If $i < j$ then the character table of H is the one given in Table 5, but if $j < i$ we should interchange the second and third characters (rows) in that table. This gives

$$\begin{aligned}
\rho_1 \otimes \rho_1 \uparrow &= \rho_1 \otimes \chi_1 + \widehat{\rho_1 \otimes \chi_4} + \textstyle\sum \rho_1 \otimes \phi_p \\
\rho_1 \otimes \rho_2 \uparrow &= \rho_1 \otimes \chi_2 + \widehat{\rho_1 \otimes \chi_3} + \textstyle\sum \rho_1 \otimes \phi_p \\
\rho_2 \otimes \rho_1 \uparrow &= \rho_2 \otimes \chi_1 + \widehat{\rho_2 \otimes \chi_4} + \textstyle\sum \rho_2 \otimes \phi_p \\
\rho_2 \otimes \rho_2 \uparrow &= \rho_2 \otimes \chi_2 + \widehat{\rho_2 \otimes \chi_3} + \textstyle\sum \rho_2 \otimes \phi_p
\end{aligned}$$

if $i < j$, and

$$\begin{aligned}
\rho_1 \otimes \rho_1 \uparrow &= \rho_1 \otimes \chi_1 + \widehat{\rho_1 \otimes \chi_4} + \textstyle\sum \rho_1 \otimes \phi_p \\
\rho_1 \otimes \rho_2 \uparrow &= \rho_2 \otimes \chi_1 + \widehat{\rho_2 \otimes \chi_4} + \textstyle\sum \rho_2 \otimes \phi_p \\
\rho_2 \otimes \rho_1 \uparrow &= \rho_1 \otimes \chi_2 + \widehat{\rho_1 \otimes \chi_3} + \textstyle\sum \rho_1 \otimes \phi_p \\
\rho_2 \otimes \rho_2 \uparrow &= \rho_2 \otimes \chi_2 + \widehat{\rho_2 \otimes \chi_3} + \textstyle\sum \rho_2 \otimes \phi_p
\end{aligned}$$

if $j < i$. Equivalently, as a map of free abelian groups, we have the homomorphisms shown in Figure 5.

$$\begin{array}{rcll}
R_{\mathbb{C}}(H) & \to & R_{\mathbb{C}}(G) & \\
\mathbb{Z}^4 & \to & \mathbb{Z}^{2 \cdot c(D_m)} & \\
(a,b,c,d) & \mapsto & (a, b, \widehat{b}, \widehat{a}, a+b, \ldots, a+b,\ c, d, \widehat{d}, \widehat{c}, c+d, \ldots, c+d) & \text{if } i < j, \\
(a,b,c,d) & \mapsto & (a, c, \widehat{c}, \widehat{a}, a+c, \ldots, a+c,\ b, d, \widehat{d}, \widehat{b}, b+d, \ldots, b+d) & \text{if } j < i.
\end{array}$$

FIGURE 5. Induction homomorphisms from $H = \langle s_i, s_j \rangle \cong C_2 \times C_2$ to $G = \langle s_i, s_j, s_k \rangle \cong \Delta(2,2,m) = C_2 \times D_m$, $m = m_{jk} > 2$, and $j < k$.

Case 2: $H = \langle s_i, s_k \rangle \cong C_2 \times C_2$.

Restricting Table 11 to H (1st, 3rd, 2nd and 4th column for k equals 0) we get

$C_2 \times D_m \downarrow$	e	s_i	s_k	$s_i s_k$
$\rho_1 \otimes \chi_1$	1	1	1	1
$\rho_1 \otimes \chi_2$	1	1	-1	-1
$\rho_1 \otimes \widehat{\chi_3}$	1	1	1	1
$\rho_1 \otimes \widehat{\chi_4}$	1	1	-1	-1
$\rho_1 \otimes \phi_p$	2	2	0	0
$\rho_1 \otimes \chi_1$	1	-1	1	-1
$\rho_1 \otimes \chi_2$	1	-1	-1	1
$\rho_1 \otimes \widehat{\chi_3}$	1	-1	1	-1
$\rho_1 \otimes \widehat{\chi_4}$	1	-1	-1	1
$\rho_1 \otimes \phi_p$	2	-2	0	0

Via Frobenius reciprocity we obtain, assumming first $i < k$,

$$\begin{array}{rcll} & & \rho_1 \otimes \_ & \rho_2 \otimes \_ \\ \rho_1 \otimes \rho_1 \uparrow & = & \rho_1 \otimes \chi_1 + \widehat{\rho_1 \otimes \chi_3} + \sum \rho_1 \otimes \phi_p & \\ \rho_1 \otimes \rho_2 \uparrow & = & \rho_1 \otimes \chi_2 + \widehat{\rho_1 \otimes \chi_4} + \sum \rho_1 \otimes \phi_p & \\ \rho_2 \otimes \rho_1 \uparrow & = & & \rho_2 \otimes \chi_1 + \widehat{\rho_2 \otimes \chi_3} + \sum \rho_2 \otimes \phi_p \\ \rho_2 \otimes \rho_2 \uparrow & = & & \rho_2 \otimes \chi_2 + \widehat{\rho_2 \otimes \chi_4} + \sum \rho_2 \otimes \phi_p \end{array}$$

If $k < i$, the calculation is the same but we must interchange again the 2nd and 3rd generators. All in all, we have the homomorphisms shown in Figure 6 as maps between free abelian groups.

$$\begin{array}{rcll} R_{\mathbb{C}}(H) & \to & R_{\mathbb{C}}(G) & \\ \mathbb{Z}^4 & \to & \mathbb{Z}^{2 \cdot c(D_m)} & \\ (a,b,c,d) & \mapsto & (a,b,\widehat{a},\widehat{b},a+b,\ldots,a+b,\ c,d,\widehat{c},\widehat{d},c+d,\ldots,c+d) & \text{if } i<k, \\ (a,b,c,d) & \mapsto & (a,c,\widehat{a},\widehat{c},a+c,\ldots,a+c,\ b,d,\widehat{b},\widehat{d},b+d,\ldots,b+d) & \text{if } k<i. \end{array}$$

FIGURE 6. Induction homomorphisms from $H = \langle s_i, s_k \rangle \cong C_2 \times C_2$ to $G = \langle s_i, s_j, s_k \rangle \cong \Delta(2,2,m) = C_2 \times D_m$, $m = m_{jk} > 2$, and $j < k$.

Case 3: $H = \langle s_j, s_k \rangle \cong D_m$

In this case the condition $j < k$ already holds. Restricting Table 11 to H (the first two columns)

$C_2 \times D_m \downarrow$	$(s_j s_k)^r$	$s_k(s_j s_k)^r$
$\rho_1 \otimes \chi_1$		
$\rho_1 \otimes \chi_2$		
$\rho_1 \otimes \widehat{\chi_3}$	T_m	
$\rho_1 \otimes \widehat{\chi_4}$		
$\rho_1 \otimes \phi_p$		
$\rho_2 \otimes \chi_1$		
$\rho_2 \otimes \chi_2$		
$\rho_2 \otimes \widehat{\chi_3}$	T_m	
$\rho_2 \otimes \widehat{\chi_4}$		
$\rho_2 \otimes \phi_p$		

where T_m are the coefficients of the character table of D_m as in Table 4. This immediately gives

$$\begin{array}{rcll} \chi_i \uparrow & = & \rho_1 \otimes \chi_i + \rho_2 \otimes \chi_i & \text{for all } i, \text{ and} \\ \phi_p \uparrow & = & \rho_1 \otimes \phi_i + \rho_2 \otimes \phi_p & \text{for all } p. \end{array}$$

Equivalently, as a map of free abelian groups, this induction homomorphism is the one given in Figure 7.

$$\begin{array}{rcl} R_{\mathbb{C}}(H) & \to & R_{\mathbb{C}}(G) \\ \mathbb{Z}^{c(D_m)} & \to & \mathbb{Z}^{2 \cdot c(D_m)} \\ (a, b, \widehat{c}, \widehat{d}, r_1, \ldots, r_N) & \mapsto & (a, b, \widehat{c}, \widehat{d}, r_1, \ldots, r_N,\ a, b, \widehat{c}, \widehat{d}, r_1, \ldots, r_N). \end{array}$$

FIGURE 7. Induction homomorphisms from $H = \langle s_j, s_k \rangle \cong D_m$ to $G = \langle s_i, s_j, s_k \rangle \cong \Delta(2,2,m) = C_2 \times D_m$, $m = m_{jk} > 2$, and $j < k$.

Finally, we compute the induction homomorphisms with respect to the transformed basis (Tables 6, 7, 8, 12, 13, 14 in Appendix A), summarising the results in Tables 24 to 29.

B.3.3. $G = \Delta(2,3,3)$. This group is isomorphic to the symmetric group S_4 with Coxeter presentation

$$\Delta(2,3,3) = \left\langle s_i, s_j, s_k \mid s_i^2, s_j^2, s_k^2, (s_i s_j)^3, (s_i s_k)^2, (s_j s_k)^3 \right\rangle \qquad (14)$$

and we assume $i < k$. We have again three relevant Coxeter subgroups H.

<u>Case 1: $H = \langle s_i, s_j \rangle \cong D_3$</u>

The expanded character table for D_3 (from Table 4), assuming first $i < j$, is

D_3	e	s_i	s_j	$s_i s_j$	$s_j s_i$	$s_i s_j s_i$
χ_1	1	1	1	1	1	1
χ_2	1	-1	-1	1	1	-1
ϕ_1	2	0	0	-1	-1	0

There are 3 conjugacy classes, $\{e\}$, $\{s_i, s_j, s_i s_j s_i = s_j s_i s_j\}$ and $\{s_i s_j, s_j s_i\}$, which remain unchanged if we swap s_i and s_j, hence if $j < i$ the table stays the same and we do not have to treat those two cases separately. The character table of G

$D_m \hookrightarrow D_m \times C_2$	s_i	$(s_i s_j)^r$	α_1	α_2	α_3	β_k
$\rho_1 \otimes \chi_1 \downarrow$	1	1	1	0	0	0
$\rho_1 \otimes (\chi_2 - \chi_1) \downarrow$	-2	0	0	1	0	0
$\rho_1 \otimes (\phi_1 - \chi_2 - \chi_1) \downarrow$	0	b_r	0	0	1	0
$\vdots\ \rho_1 \otimes (\phi_p - \phi_{p-1}) \downarrow\ \vdots$	0	$a_{p,r}$	0	0	0	$\delta_{p,k}$
$(\rho_2 - \rho_1) \otimes \chi_1 \downarrow$	0	0	0	0	0	0
$(\rho_2 - \rho_1) \otimes (\chi_2 - \chi_1) \downarrow$	0	0	0	0	0	0
$(\rho_2 - \rho_1) \otimes (\phi_1 - \chi_2 - \chi_1) \downarrow$	0	0	0	0	0	0
$\vdots\ (\rho_2 - \rho_1) \otimes (\phi_p - \phi_{p-1}) \downarrow\ \vdots$	0	0	0	0	0	0

where $\alpha_1 := \sum_{\ell=1}^{2} \chi_\ell + 2 \sum_{\ell=1}^{(m-1)/2} \phi_\ell$, $\alpha_2 := \chi_2 + \sum_{\ell=1}^{(m-1)/2} \phi_\ell$, $\alpha_3 := \sum_{\ell=1}^{(m-1)/2} \phi_\ell$,

$$\beta_k := \sum_{\ell=k}^{(m-1)/2} \phi_\ell.$$

TABLE 24. Restricted characters and map induced by $D_m \hookrightarrow \Delta(2,2,m) = D_m \times C_2$ for $m \geq 3$ odd. Here $a_{p,r} := 2\cos(\frac{2\pi pr}{m}) - 2\cos(\frac{2\pi(p-1)r}{m})$, $b_r := 2\cos(\frac{2\pi r}{m}) - 2$, $\delta_{p,k}$ the Kronecker delta, $2 \leq p, k \leq \frac{m-1}{2}$ and $0 \leq r \leq \frac{m-1}{2}$.

$D_2 \hookrightarrow D_m \times C_2$	e	s_i	α	αs_i	$\sum \chi_i$	$\chi_2 + \widehat{\chi_3}$	$\widehat{\chi_3}$	$\widehat{\chi_3} + \widehat{\chi_4}$
$\rho_1 \otimes \chi_1 \downarrow$	1	1	1	1	1	0	0	0
$\rho_1 \otimes (\chi_2 - \chi_1) \downarrow$	0	-2	0	-2	0	0	0	1
$\rho_1 \otimes (\phi_1 - \chi_2 - \chi_1) \downarrow$	0	0	0	0	0	0	0	0
$\vdots\ \rho_1 \otimes (\phi_p - \phi_{p-1}) \downarrow\ \vdots$	0	0	0	0	0	0	0	0
$(\rho_2 - \rho_1) \otimes \chi_1 \downarrow$	0	0	-2	-2	0	1	0	0
$(\rho_2 - \rho_1) \otimes (\chi_2 - \chi_1) \downarrow$	0	0	0	4	0	0	1	0
$(\rho_2 - \rho_1) \otimes (\phi_1 - \chi_2 - \chi_1) \downarrow$	0	0	0	0	0	0	0	0
$\vdots\ (\rho_2 - \rho_1) \otimes (\phi_p - \phi_{p-1}) \downarrow\ \vdots$	0	0	0	0	0	0	0	0

TABLE 25. Restricted characters and map induced by the two inclusions $D_2 \hookrightarrow \Delta(2,2,m) = D_m \times C_2$, for $m \geq 3$ odd. Here $2 \leq p \leq \frac{m-1}{2}$.

(Table 15) restricted to H consists on the 1st, 2nd, 2nd, 3rd, 3rd, 2nd columns (since $s_i \sim s_j$, $s_j s_i \sim s_i s_j$ and $s_i s_j s_i \sim s_i$):

$S_4 \downarrow$	e	s_i	s_j	$s_i s_j$	$s_j s_i$	$s_i s_j s_i$
ξ_1	1	1	1	1	1	1
ξ_2	1	-1	-1	1	1	-1
ξ_3	2	0	0	-1	-1	0
ξ_4	3	1	1	0	0	1
ξ_5	3	-1	-1	0	0	-1

$D_m \hookrightarrow D_m \times C_2$	s_i	$(s_is_j)^r$	$s_js_is_j$	α_1	α_2	α_3	α_4	α_5	β_k
$\rho_1 \otimes \chi_1 \downarrow$	0	0	1	1	0	0	0	0	0
$\rho_1 \otimes (\chi_2 - \chi_1) \downarrow$	-2	0	-2	0	1	0	1	0	0
$\rho_1 \otimes (\chi_3 - \chi_2) \downarrow$	0	c_r	-2	0	0	1	-1	0	0
$\rho_1 \otimes (\chi_4 - \chi_1) \downarrow$	-2	c_r	0	0	0	0	1	0	0
$\rho_1 \otimes (\phi_1 - \chi_3 - \chi_1) \downarrow$	-2	b_r	0	0	0	0	1	1	0
$\vdots$ $\rho_1 \otimes (\phi_p - \phi_{p-1}) \downarrow$ $\vdots$	0	$a_{p,r}$	0	0	0	0	0	0	$\delta_{p,k}$
$(\rho_2 - \rho_1) \otimes \chi_1 \downarrow$									
$(\rho_2 - \rho_1) \otimes (\chi_2 - \chi_1) \downarrow$									
$(\rho_2 - \rho_1) \otimes (\chi_3 - \chi_2) \downarrow$		**0**				**0**			
$(\rho_2 - \rho_1) \otimes (\chi_4 + \chi_3 - \chi_2 - \chi_1) \downarrow$									
$(\rho_2 - \rho_1) \otimes (\phi_1 - \chi_2 - \chi_1) \downarrow$									
$\vdots$ $(\rho_2 - \rho_1) \otimes (\phi_p - \phi_{p-1}) \downarrow$ $\vdots$									

where $\alpha_1 := \sum_{\ell=1}^{4} \chi_\ell + 2 \sum_{p}^{\frac{m}{2}-1} \phi_p$, $\alpha_2 := \chi_2 + \chi_3 + \sum_{\ell=1}^{\frac{m}{2}-1} \phi_\ell$, $\alpha_3 := \chi_3 + \sum_{\ell=1}^{\frac{m}{2}-1} \phi_\ell$,

$\alpha_4 := \chi_2 + \chi_4 + \sum_{\ell=1}^{\frac{m}{2}-1} \phi_\ell$, $\alpha_5 := \sum_{\ell=1}^{\frac{m}{2}-1} \phi_\ell$, $\beta_k := \sum_{\ell=k}^{\frac{m}{2}-1} \phi_\ell$.

TABLE 26. Restricted characters and map induced by the inclusion $D_m \hookrightarrow \Delta(2,2,m) = D_m \times C_2$, for $m \geq 6$ even, not a power of 2. Here $a_{p,r} := 2\cos(\frac{2\pi pr}{m}) - 2\cos(\frac{2\pi(p-1)r}{m})$, $b_r := 2\cos(\frac{2\pi r}{m}) - (-1)^r - 1$, $c_r := (-1)^r - 1$, $\delta_{p,k}$ the Kronecker delta, $2 \leq p, k \leq \frac{m}{2} - 1$, and $0 \leq r \leq \frac{m}{2}$.

$D_2 \hookrightarrow D_m \times C_2$	e	$s_js_is_j$	α	$\alpha s_js_is_j$	$\sum \chi_i$	$\chi_2 + \widehat{\chi_3}$	$\widehat{\chi_3}$	$\widehat{\chi_3} + \widehat{\chi_4}$
$\rho_1 \otimes \chi_1 \downarrow$	1	1	1	1	1	0	0	0
$\rho_1 \otimes (\chi_2 - \chi_1) \downarrow$	0	-2	0	-2	0	0	0	1
$\rho_1 \otimes (\chi_3 - \chi_2) \downarrow$	0	0	0	0	0	0	0	0
$\rho_1 \otimes (\chi_4 - \chi_1) \downarrow$	0	0	0	0	0	0	0	0
$\rho_1 \otimes (\phi_1 - \chi_3 - \chi_1) \downarrow$	0	0	0	0	0	0	0	0
$\vdots$ $\rho_1 \otimes (\phi_p - \phi_{p-1}) \downarrow$ $\vdots$	0	0	0	0	0	0	0	0
$(\rho_2 - \rho_1) \otimes \chi_1 \downarrow$	0	0	-2	-2	0	1	0	0
$(\rho_2 - \rho_1) \otimes (\chi_2 - \chi_1) \downarrow$	0	0	0	4	0	0	1	0
$(\rho_2 - \rho_1) \otimes (\chi_3 - \chi_2) \downarrow$	0	0	0	0	0	0	0	0
$(\rho_2 - \rho_1) \otimes (\chi_4 + \chi_3 - \chi_2 - \chi_1) \downarrow$	0	0	0	0	0	0	0	0
$(\rho_2 - \rho_1) \otimes (\phi_1 - \chi_2 - \chi_1) \downarrow$	0	0	0	0	0	0	0	0
$\vdots$ $(\rho_2 - \rho_1) \otimes (\phi_p - \phi_{p-1}) \downarrow$ $\vdots$	0	0	0	0	0	0	0	0

TABLE 27. Restricted characters and map induced by the two inclusions $D_2 \hookrightarrow \Delta(2,2,m) = D_m \times C_2$, for $m \geq 2$ even, not a power of 2. Here $2 \leq p \leq \frac{m}{2} - 1$. Our choice on the generators is $(12)(34) \mapsto s_js_is_j$ and $(12) \mapsto \alpha$; any other choice yields an equivalent matrix.

Multiplying rows and dividing by $|H| = 6$ we obtain

$$\begin{aligned} \chi_1 \uparrow &= \xi_1 + \xi_4 \\ \chi_2 \uparrow &= \xi_2 + \xi_5 \\ \phi_1 \uparrow &= \xi_3 + \xi_4 + \xi_5 \end{aligned}$$

$D_m \hookrightarrow D_m \times C_2$	s_i	$(s_i s_j)^r$	$s_j s_i s_j$	α_1	α_2	α_3	α_4	α_5	β_k
$\rho_1 \otimes \chi_1 \downarrow$	1	1	1	1	0	0	0	0	0
$\rho_1 \otimes (\chi_2 - \chi_1) \downarrow$	-2	0	-2	0	1	0	1	0	0
$\rho_1 \otimes (\chi_3 - \chi_2) \downarrow$	0	c_r	-2	0	1	1	0	0	0
$\rho_1 \otimes (\chi_4 - \chi_1) \downarrow$	0	c_r	2	0	-1	0	0	0	0
$\rho_1 \otimes (\phi_1 - \chi_3 - \chi_1) \downarrow$	0	b_r	0	0	0	1	0	1	0
$\vdots$ $\rho_1 \otimes (\phi_p - \phi_{p-1}) \downarrow$ $\vdots$	0	$a_{p,r}$	0	0	0	0	0	0	$\delta_{p,k}$
$(\rho_2 - \rho_1) \otimes \chi_1 \downarrow$	**0**			**0**					
$(\rho_2 - \rho_1) \otimes (\chi_2 - \chi_1) \downarrow$									
$(\rho_2 - \rho_1) \otimes (\chi_3 - \chi_2) \downarrow$									
$(\rho_2 - \rho_1) \otimes (\chi_4 + \chi_3 - \chi_2 - \chi_1) \downarrow$									
$(\rho_2 - \rho_1) \otimes (\phi_1 - \chi_2 - \chi_1) \downarrow$									
$\vdots$ $(\rho_2 - \rho_1) \otimes (\phi_p - \phi_{p-1}) \downarrow$ $\vdots$									

where $\alpha_1 := \sum_{\ell=1}^{4} \chi_\ell + 2 \sum_{p}^{\frac{m}{2}-1} \phi_p$, $\alpha_2 := \chi_2 + \chi_3 + \sum_{\ell=1}^{\frac{m}{2}-1} \phi_\ell$, $\alpha_3 := \chi_3 + \sum_{\ell=1}^{\frac{m}{2}-1} \phi_\ell$,

$\alpha_4 := \chi_2 + \chi_4 + \sum_{\ell=1}^{\frac{m}{2}-1} \phi_\ell$, $\alpha_5 := \sum_{\ell=1}^{\frac{m}{2}-1} \phi_\ell$, $\beta_k := \sum_{\ell=k}^{\frac{m}{2}-1} \phi_\ell$.

TABLE 28. Restricted characters and map induced by the inclusion $D_m \hookrightarrow \Delta(2,2,m) = D_m \times C_2$, for $m \geq 4$ a power of 2. Here $a_{p,r} := 2\cos(\frac{2\pi pr}{m}) - 2\cos(\frac{2\pi(p-1)r}{m})$, $b_r := 2\cos(\frac{2\pi r}{m}) - 2$, $c_r := (-1)^r - 1$, $\delta_{p,k}$ the Kronecker delta, $2 \leq p,k \leq \frac{m}{2} - 1$ where $1 < r < \frac{m}{2}$ and $1 < k < \frac{m}{2} - 1$.

$D_2 \hookrightarrow D_m \times C_2$	e	s_i	α	αs_i	$\sum \chi_i$	$\chi_2 + \widehat{\chi_3}$	$\widehat{\chi_3}$	$\widehat{\chi_3} + \widehat{\chi_4}$
$\rho_1 \otimes \chi_1 \downarrow$	1	1	1	1	1	0	0	0
$\rho_1 \otimes (\chi_2 - \chi_1) \downarrow$	0	-2	0	-2	0	0	0	1
$\rho_1 \otimes (\chi_3 - \chi_1) \downarrow$	0	0	0	0	0	0	0	0
$\rho_1 \otimes (\chi_4 - \chi_2) \downarrow$	0	0	0	0	0	0	0	0
$\rho_1 \otimes (\phi_1 - \chi_2 - \chi_1) \downarrow$	0	0	0	0	0	0	0	0
$\vdots$ $\rho_1 \otimes (\phi_p - \phi_{p-1}) \downarrow$ $\vdots$	0	0	0	0	0	0	0	0
$(\rho_2 - \rho_1) \otimes \chi_1 \downarrow$	0	0	-2	-2	0	1	0	0
$(\rho_2 - \rho_1) \otimes (\chi_2 - \chi_1) \downarrow$	0	0	0	4	0	0	1	0
$(\rho_2 - \rho_1) \otimes (\chi_3 - \chi_1) \downarrow$	0	0	0	0	0	0	0	0
$(\rho_2 - \rho_1) \otimes (\chi_4 + \chi_3 - \chi_2 - \chi_1) \downarrow$	0	0	0	0	0	0	0	0
$(\rho_2 - \rho_1) \otimes (\phi_1 - \chi_2 - \chi_1) \downarrow$	0	0	0	0	0	0	0	0
$\vdots$ $(\rho_2 - \rho_1) \otimes (\phi_p - \phi_{p-1}) \downarrow$ $\vdots$	0	0	0	0	0	0	0	0

TABLE 29. Restricted characters and map induced by the two inclusions $D_2 \hookrightarrow \Delta(2,2,m) = D_m \times C_2$, for $m \geq 4$ a power of 2. Here $2 \leq p \leq \frac{m}{2} - 1$. Our choice on the generators is $(12)(34) \mapsto s_i$ and $(12) \mapsto \alpha$, and any other choice yields an equivalent matrix.

Equivalently, we have the map of free abelian groups shown in Figure 8.

Case 2: $H = \langle s_j, s_k \rangle \cong D_3$

$$\begin{array}{rcl} R_{\mathbb{C}}(H) & \to & R_{\mathbb{C}}(G) \\ \mathbb{Z}^3 & \to & \mathbb{Z}^5 \\ (a,b,c) & \mapsto & (a,b,c,a+c,b+c). \end{array}$$

FIGURE 8. Induction homomorphism from $H = \langle s_i, s_j \rangle \cong D_3$ or $H = \langle s_j, s_k \rangle \cong D_3$ to $G = \langle s_i, s_j, s_k \rangle \cong \Delta(2,3,3) = S_4$, and $i < k$.

This case is completely analogous to the previous one, so we obtain the same map, also shown in Figure 8.

Case 3: $H = \langle s_i, s_k \rangle \cong D_2 = C_2 \times C_2$

The character table of G (Table 15) restricted to H consists on the 1st, 2nd, 2nd, 5th columns:

$S_4 \downarrow$	e	s_i	s_k	$s_i s_k$
ξ_1	1	1	1	1
ξ_2	1	-1	-1	1
ξ_3	2	0	0	2
ξ_4	3	1	1	-1
ξ_5	3	-1	-1	-1

Multiplying the rows with the characters of H (Table 5) and dividing by $|H| = 4$ we obtain (note that $i < k$ already holds)

$$\begin{array}{rcl} \rho_1 \otimes \rho_1 \uparrow & = & \xi_1 + \xi_3 + \xi_4 \\ \rho_1 \otimes \rho_2 \uparrow & = & \xi_4 + \xi_5 \\ \rho_2 \otimes \rho_1 \uparrow & = & \xi_4 + \xi_5 \\ \rho_2 \otimes \rho_2 \uparrow & = & \xi_2 + \xi_3 + \xi_5 \end{array}$$

Equivalently, this is the homomorphism of abelian groups shown in Figure 9. Note that this is the first time that of an induction homomorphism with nontrivial kernel.

$$\begin{array}{rcl} R_{\mathbb{C}}(H) & \to & R_{\mathbb{C}}(G) \\ \mathbb{Z}^4 & \to & \mathbb{Z}^5 \\ (a,b,c,d) & \mapsto & (a,d,a+d,a+b+c,b+c+d). \end{array}$$

FIGURE 9. Induction homomorphism from $H = \langle s_i, s_k \rangle \cong C_2 \times C_2$ to $G = \langle s_i, s_j, s_k \rangle \cong \Delta(2,3,3) = S_4$, and $i < k$.

Now we give the induction homomorphisms with respect to the transformed bases (Tables 6, 7, 16 in Appendix A), summarised in Table 30.

B.3.4. $G = \Delta(2,3,4)$. This group is isomorphic to $S_4 \times C_2$ with Coxeter presentation

$$\Delta(2,3,4) = \left\langle s_i, s_j, s_k \mid s_i^2, s_j^2, s_k^2, (s_i s_j)^3, (s_i s_k)^2, (s_j s_k)^4 \right\rangle .$$

The three relevant induction homomorphism are as follows.

Case 1: $H = \langle s_i, s_k \rangle \cong D_2 = C_2 \times C_2$

$D_2 \hookrightarrow S_4$	(1)	(1 2)	(1 2)(3 4)	$(\widetilde{\xi_i}\vert\sum\chi_i)$	$(\widetilde{\xi_i}\vert\chi_2+\widehat{\chi_3})$	$(\widetilde{\xi_i}\vert\widehat{\chi_3})$	$(\widetilde{\xi_i}\vert\widehat{\chi_4})$
$\widetilde{\xi_1}\downarrow$	1	1	1	1	0	0	0
$\widetilde{\xi_2}\downarrow$	0	−2	0	0	1	0	0
$\widetilde{\xi_3}\downarrow$	0	0	0	0	0	0	0
$\widetilde{\xi_4}\downarrow$	0	0	−4	0	0	1	1
$\widetilde{\xi_5}\downarrow$	0	0	0	0	0	0	0

$D_3 \hookrightarrow S_4$	(1)	(12)	(123)	$(\widetilde{\xi_i}\vert 2\phi_1+\sum\chi_i)$	$(\widetilde{\xi_i}\vert\chi_2+\phi_1)$	$(\widetilde{\xi_i}\vert\phi_1)$
$\widetilde{\xi_1}\downarrow$	1	1	1	1	0	0
$\widetilde{\xi_2}\downarrow$	0	−2	0	0	1	0
$\widetilde{\xi_3}\downarrow$	0	0	−3	0	0	1
$\widetilde{\xi_4}\downarrow$	0	0	0	0	0	0
$\widetilde{\xi_5}\downarrow$	0	0	0	0	0	0

where $\widetilde{\xi_1} := \xi_1$, $\widetilde{\xi_2} := \xi_2 - \xi_1$, $\widetilde{\xi_3} := \xi_3 - \xi_2 - \xi_1$, $\widetilde{\xi_4} := \xi_4 - \xi_3 - \xi_1$, $\widetilde{\xi_5} := \xi_5 - \xi_4 - \xi_2 + \xi_1$.

TABLE 30. Restricted characters and map induced by the inclusions of D_2 and D_3 into S_4. The second inclusion $\langle(23),(34)\rangle \cong D_3 \hookrightarrow S_4$ induces the same map as the first one, because $(23) \sim (12)$ and $(234) \sim (123)$.

The character table of G (Table 17) restricted to H consists on the 1st, 7th, 10th and 2nd columns:

$S_4 \times C_2 \downarrow$	e	s_i	s_k	$s_i s_k$
$\rho_1 \otimes \xi_1$	1	1	1	1
$\rho_1 \otimes \xi_2$	1	−1	1	−1
$\rho_1 \otimes \xi_3$	2	0	2	0
$\rho_1 \otimes \xi_4$	3	1	−1	1
$\rho_1 \otimes \xi_5$	3	−1	−1	−1
$\rho_2 \otimes \xi_1$	1	−1	−1	1
$\rho_2 \otimes \xi_2$	1	1	−1	−1
$\rho_2 \otimes \xi_3$	2	0	−2	0
$\rho_2 \otimes \xi_4$	3	−1	1	1
$\rho_2 \otimes \xi_5$	3	1	1	−1

Suppose first that $i < k$. Multiplying these rows with the rows of Table 5 we deduce that (note the shortcut in notation)

$$\begin{array}{rcll} & & \rho_1 \otimes \_ & \rho_2 \otimes \_ \\ \rho_1 \otimes \rho_1 \uparrow & = & \xi_1 + \xi_3 + \xi_4 + & \xi_4 + \xi_5 \\ \rho_1 \otimes \rho_2 \uparrow & = & \xi_4 + \xi_5 + & \xi_2 + \xi_3 + \xi_5 \\ \rho_2 \otimes \rho_1 \uparrow & = & \xi_2 + \xi_3 + \xi_5 + & \xi_4 + \xi_5 \\ \rho_2 \otimes \rho_2 \uparrow & = & \xi_4 + \xi_5 + & \xi_1 + \xi_3 + \xi_4 \end{array}$$

On the other hand, if $k < i$, then we should interchange the 2nd and 3rd generators. All in all, we have the homomorphisms of free abelian groups shown in Figure 10.

Case 2: $H = \langle s_i, s_j \rangle \cong D_3$

$$\begin{array}{rcll} R_{\mathbb{C}}(H) & \to & R_{\mathbb{C}}(G) & \\ \mathbb{Z}^4 & \to & \mathbb{Z}^{10} & \\ (a,b,c,d) & \mapsto & (a,c,a+c,a+b+d,b+c+d,d,b,b+d,a+c+d,a+b+c) & \text{if } i<k, \\ (a,b,c,d) & \mapsto & (a,b,a+b,a+c+d,b+c+d,d,c,c+d,a+b+d,a+b+c) & \text{if } k<i. \end{array}$$

FIGURE 10. Induction homomorphism from $H = \langle s_i, s_k\rangle \cong C_2 \times C_2$ to $G = \langle s_i, s_j, s_k\rangle \cong \Delta(2,3,4) = S_4 \times C_2$.

The characters of G (Table 17) restricted to H consists on the 1st, 7th, 7th, 3rd, 3rd, 7th columns:

$S_4 \times C_2 \downarrow$	e	s_i	s_j	s_is_j	s_js_i	$s_is_js_i$
$\rho_1 \otimes \xi_1$	1	1	1	1	1	1
$\rho_1 \otimes \xi_2$	1	−1	−1	1	1	−1
$\rho_1 \otimes \xi_3$	2	0	0	−1	−1	0
$\rho_1 \otimes \xi_4$	3	1	1	0	0	1
$\rho_1 \otimes \xi_5$	3	−1	−1	0	0	−1
$\rho_2 \otimes \xi_1$	1	−1	−1	1	1	−1
$\rho_2 \otimes \xi_2$	1	1	1	1	1	1
$\rho_2 \otimes \xi_3$	2	0	0	−1	−1	0
$\rho_2 \otimes \xi_4$	3	−1	−1	0	0	−1
$\rho_2 \otimes \xi_5$	3	1	1	0	0	1

Multiplying these rows with the rows of character table of D_3 above we deduce that (recall that this table is fixed by interchanging the Coxeter generators)

$$\begin{array}{rcll} & & \rho_1 \otimes \_ & \rho_2 \otimes \_ \\ \chi_1 \uparrow & = & \xi_1 + \xi_4 + & \xi_2 + \xi_5 \\ \chi_1 \uparrow & = & \xi_2 + \xi_5 + & \xi_1 + \xi_4 \\ \phi_1 \uparrow & = & \xi_3 + \xi_4 + \xi_5 + & \xi_3 + \xi_4 + \xi_5 \end{array}$$

or, equivalently, the linear map shown in Figure 11.

$$\begin{array}{rcl} R_{\mathbb{C}}(H) & \to & R_{\mathbb{C}}(G) \\ \mathbb{Z}^3 & \to & \mathbb{Z}^{10} \\ (a,b,c) & \mapsto & (a,b,c,a+c,b+c,b,a,c,b+c,a+c). \end{array}$$

FIGURE 11. Induction homomorphism from $H = \langle s_i, s_j\rangle \cong D_3$ to $G = \langle s_i, s_j, s_k\rangle \cong \Delta(2,3,4) = S_4 \times C_2$.

Case 3: $H = \langle s_j, s_k\rangle \cong D_4$

First we expand the character table of D_4, assuming $j < k$,

D_4	e	s_j	s_k	s_js_k	s_ks_j	$s_js_ks_j$	$s_ks_js_k$	$s_js_ks_js_k$
χ_1	1	1	1	1	1	1	1	1
χ_2	1	−1	−1	1	1	−1	−1	1
χ_3	1	−1	1	−1	−1	1	−1	1
χ_4	1	1	−1	−1	−1	−1	1	1
ϕ_1	2	0	0	0	0	0	0	−2

Note that if $k < j$ we should interchange the characters χ_3 and χ_4 in order to maintain the notation consistent. The characters of G restricted to H are the 1st, 7th, 10th, 4th, 4th, 10th, 7th, 5th columns of Table 17:

$S_4 \times C_2 \downarrow$	e	s_2	s_3	s_2s_3	s_3s_2	$s_2s_3s_2$	$s_3s_2s_3$	$s_2s_3s_2s_3$
$\rho_1 \otimes \xi_1$	1	1	1	1	1	1	1	1
$\rho_1 \otimes \xi_2$	1	−1	1	−1	−1	1	−1	1
$\rho_1 \otimes \xi_3$	2	0	2	0	0	2	0	2
$\rho_1 \otimes \xi_4$	3	1	−1	−1	−1	−1	1	−1
$\rho_1 \otimes \xi_5$	3	−1	−1	1	1	−1	−1	−1
$\rho_2 \otimes \xi_1$	1	−1	−1	1	1	−1	−1	1
$\rho_2 \otimes \xi_2$	1	1	−1	−1	−1	−1	1	1
$\rho_2 \otimes \xi_3$	2	0	−2	0	0	−2	0	2
$\rho_2 \otimes \xi_4$	3	−1	1	−1	−1	1	−1	−1
$\rho_2 \otimes \xi_5$	3	1	1	1	1	1	1	−1

Multiplying these rows with the rows of the character table of D_4 above we deduce that

$$\begin{array}{rcll} & & \rho_1 \otimes \_ & \rho_2 \otimes \_ \\ \chi_1 \uparrow & = & \xi_1 + \xi_3 + & \xi_5 \\ \chi_2 \uparrow & = & \xi_5 + & \xi_1 + \xi_3 \\ \chi_3 \uparrow & = & \xi_2 + \xi_3 + & \xi_4 \\ \chi_4 \uparrow & = & \xi_4 + & \xi_2 + \xi_3 \\ \phi_1 \uparrow & = & \xi_4 + \xi_5 + & \xi_4 + \xi_5 \end{array}$$

The computation in the case $k < j$ is identical, but interchanging χ_3 and χ_4. All in all, we have the induction homomorphisms shown, as maps between abelian groups, in Figure X

$$\begin{array}{rcll} R_{\mathbb{C}}(H) & \to & R_{\mathbb{C}}(G) & \\ \mathbb{Z}^5 & \to & \mathbb{Z}^{10} & \\ (a,b,c,d,e) & \mapsto & (a,c,a+c,d+e,b+e,b,d,b+d,c+e,a+e) & \text{if } j<k, \\ (a,b,c,d,e) & \mapsto & (a,d,a+d,c+e,b+e,b,c,b+c,d+e,a+e) & \text{if } k<j. \end{array}$$

FIGURE 12. Induction homomorphism from $H = \langle s_j, s_k \rangle \cong D_4$ to $G = \langle s_i, s_j, s_k \rangle \cong \Delta(2,3,4) = S_4 \times C_2$.

On the other hand, the induction homomorphisms with respect to the transformed bases (Tables 7, 8, 18 in Appendix A) are summarised in Tables 31, 32 and 33.

$D_2 \hookrightarrow S_4 \times C_2$	(1)	$\alpha(12)$	$\alpha(12)(34)$	(34)	$(\cdot\|\sum \chi_i)$	$(\cdot\|\chi_2 + \widehat{\chi_3})$	$(\cdot\|\widehat{\chi_3} + \widehat{\chi_4})$	$(\cdot\|\widehat{\chi_4})$
$\alpha_1 \downarrow$	1	1	1	1	1	0	0	0
$\alpha_2 \downarrow$	0	−2	0	−2	0	1	0	0
$\alpha_3 \downarrow$	0	0	0	0	0	0	0	0
$\alpha_4 \downarrow$	0	0	−2	−2	0	0	1	1
$\alpha_5 \downarrow$	0	0	0	0	0	0	0	0
$\alpha_6 \downarrow$	0	0	−2	2	0	0	1	0
$\alpha_7 \downarrow$	0	0	0	−4	0	0	0	1
$\alpha_8 \downarrow$	0	0	0	0	0	0	0	0
$\alpha_9 \downarrow$	0	0	0	0	0	0	0	0
$\alpha_{10} \downarrow$	0	0	0	0	0	0	0	0

TABLE 31. Restricted characters and map induced by the inclusion of D_2 into $S_4 \times C_2$. Here $\alpha_1, \ldots, \alpha_{10}$ are as in Table 18.

$D_4 \hookrightarrow S_4 \times C_2$	(1)	$\alpha(13)$	(13)(24)	$\alpha(12)(34)$	(1432)
$\alpha_1 \downarrow$	1	1	1	1	1
$\alpha_2 \downarrow$	0	−2	0	0	−2
$\alpha_3 \downarrow$	0	0	0	0	0
$\alpha_4 \downarrow$	0	0	−4	−2	4
$\alpha_5 \downarrow$	0	0	0	0	4
$\alpha_6 \downarrow$	0	0	0	−2	2
$\alpha_7 \downarrow$	0	0	0	0	0
$\alpha_8 \downarrow$	0	0	0	0	0
$\alpha_9 \downarrow$	0	0	0	0	0
$\alpha_{10} \downarrow$	0	0	0	0	0

$D_4 \hookrightarrow S_4 \times C_2$	$(\cdot\|2\phi_1 + \sum \chi_i)$	$(\cdot\|\chi_2 + \widehat{\chi_3} + \phi_1)$	$(\cdot\|\widehat{\chi_3} + \phi_1)$	$(\cdot\|\widehat{\chi_4} + \chi_2)$	$(\cdot\|\phi_1)$
$\alpha_1 \downarrow$	1	0	0	0	0
$\alpha_2 \downarrow$	0	0	0	1	0
$\alpha_3 \downarrow$	0	0	0	0	0
$\alpha_4 \downarrow$	0	1	0	−1	1
$\alpha_5 \downarrow$	0	0	−1	0	0
$\alpha_6 \downarrow$	0	1	0	0	0
$\alpha_7 \downarrow$	0	0	0	0	0
$\alpha_8 \downarrow$	0	0	0	0	0
$\alpha_9 \downarrow$	0	0	0	0	0
$\alpha_{10} \downarrow$	0	0	0	0	0

TABLE 32. Restricted characters (top) and map induced by the inclusion of D_4 into $S_4 \times C_2$ (bottom). Here $\alpha_1, \ldots, \alpha_{10}$ are as in Table 18.

B.3.5. $G = \Delta(2,3,5)$. This group is isomorphic to $A_5 \times C_2$ with Coxeter presentation

$$\Delta(2,3,5) = \left\langle s_i, s_j, s_k \mid s_i^2, s_j^2, s_k^2, (s_is_j)^3, (s_is_k)^2, (s_js_k)^5 \right\rangle .$$

We have again three relevant induction homomorphisms.

Case 1: $H = \langle s_i, s_k \rangle \cong D_2 = C_2 \times C_2$

The characters of G (Table 19) restricted to H consists on the 1st, 8th, 8th, 3rd

$D_3 \hookrightarrow S_4 \times C_2$	(1)	$\alpha(12)$	(123)	$(\cdot\|2\phi_1 + \sum \chi_i)$	$(\cdot\|\chi_2 + \phi_1)$	$(\cdot\|\phi_1)$
$\alpha_1 \downarrow$	1	1	1	1	0	0
$\alpha_2 \downarrow$	0	-2	0	0	1	0
$\alpha_3 \downarrow$	0	0	-3	0	0	1
$\alpha_4 \downarrow$	0	0	0	0	0	0
$\alpha_5 \downarrow$	0	0	0	0	0	0
$\alpha_6 \downarrow$	0	0	0	0	0	0
$\alpha_7 \downarrow$	0	0	0	0	0	0
$\alpha_8 \downarrow$	0	0	0	0	0	0
$\alpha_9 \downarrow$	0	0	0	0	0	0
$\alpha_{10} \downarrow$	0	0	0	0	0	0

TABLE 33. Restricted characters and map induced by the inclusion of D_3 into $S_4 \times C_2$. Here $\alpha_1, \ldots, \alpha_{10}$ are as in Table 18.

columns

$A_5 \times C_2 \downarrow$	e	s_i	s_k	$s_i s_k$
$\rho_1 \otimes \xi_1$	1	1	1	1
$\rho_1 \otimes \xi_2$	4	0	0	0
$\rho_1 \otimes \xi_3$	5	1	1	1
$\rho_1 \otimes \xi_4$	3	-1	-1	-1
$\rho_1 \otimes \xi_5$	3	-1	-1	-1
$\rho_2 \otimes \xi_1$	1	-1	-1	1
$\rho_2 \otimes \xi_2$	4	0	0	0
$\rho_2 \otimes \xi_3$	5	-1	-1	1
$\rho_2 \otimes \xi_4$	3	1	1	-1
$\rho_2 \otimes \xi_5$	3	1	1	-1

Suppose first $i < k$. Multiplying these rows with the rows of Table 5 we obtain

$$\begin{array}{rcll} & & \rho_1 \otimes \_ & \rho_2 \otimes \_ \\ \rho_1 \otimes \rho_1 \uparrow & = & \xi_1 + \xi_2 + 2\xi_3 + & \xi_2 + \xi_3 + \xi_4 + \xi_5 \\ \rho_1 \otimes \rho_2 \uparrow & = & \xi_2 + \xi_3 + \xi_4 + \xi_5 + & \xi_2 + \xi_3 + \xi_4 + \xi_5 \\ \rho_2 \otimes \rho_1 \uparrow & = & \xi_2 + \xi_3 + \xi_4 + \xi_5 + & \xi_2 + \xi_3 + \xi_4 + \xi_5 \\ \rho_2 \otimes \rho_2 \uparrow & = & \xi_2 + \xi_3 + \xi_4 + \xi_5 + & \xi_1 + \xi_2 + 2\xi_3 \end{array}$$

If $k < i$ we must interchange the 2nd and 3rd generators, but note that we obtain the same map. All in one, we have one induction map, given as a homomorphism of free abelian groups in Figure 13.

$$\begin{array}{rcl} R_{\mathbb{C}}(H) & \to & R_{\mathbb{C}}(G) \\ \mathbb{Z}^4 & \to & \mathbb{Z}^{10} \\ (a,b,c,d) & \mapsto & (a, b+c+d, 2a+b+c+d, b+c+d, b+c+d, \\ & & d, a+b+c, a+b+c+2d, a+b+c, a+b+c). \end{array}$$

FIGURE 13. Induction homomorphism from $H = \langle s_i, s_k \rangle \cong C_2 \times C_2$ to $G = \langle s_i, s_j, s_k \rangle \cong \Delta(2,3,5) = A_5 \times C_2$.

Case 2: $H = \langle s_i, s_j \rangle \cong D_3$

The characters of G (Table 19) restricted to H consists on the 1st, 8th, 8th, 2nd, 2nd and 8th columns:

$A_5 \times C_2 \downarrow$	e	s_i	s_j	$s_i s_j$	$s_j s_i$	$s_i s_j s_i$
$\rho_1 \otimes \xi_1$	1	1	1	1	1	1
$\rho_1 \otimes \xi_2$	4	0	0	1	1	0
$\rho_1 \otimes \xi_3$	5	1	1	-1	-1	1
$\rho_1 \otimes \xi_4$	3	-1	-1	0	0	-1
$\rho_1 \otimes \xi_5$	3	-1	-1	0	0	-1
$\rho_2 \otimes \xi_1$	1	-1	-1	1	1	-1
$\rho_2 \otimes \xi_2$	4	0	0	1	1	0
$\rho_2 \otimes \xi_3$	5	-1	-1	-1	-1	-1
$\rho_2 \otimes \xi_4$	3	1	1	0	0	1
$\rho_2 \otimes \xi_5$	3	1	1	0	0	1

Multiplying these rows with the rows of the character table of D_3 (which is independent of whether $i < j$ or $j < i$)

$$\begin{array}{rcll} & & \rho_1 \otimes \_ & \rho_2 \otimes \_ \\ \chi_1 \uparrow & = & \xi_1 + \xi_2 + \xi_3 + & \xi_2 + \xi_4 + \xi_5 \\ \chi_2 \uparrow & = & \xi_2 + \xi_4 + \xi_5 + & \xi_1 + \xi_2 + \xi_3 \\ \phi_1 \uparrow & = & \xi_2 + 2\xi_3 + \xi_4 + \xi_5 + & \xi_2 + 2\xi_3 + \xi_4 + \xi_5 \end{array}$$

This is then the map of free abelian groups shown in Figure 14.

$$\begin{array}{rcl} R_{\mathbb{C}}(H) & \to & R_{\mathbb{C}}(G) \\ \mathbb{Z}^3 & \to & \mathbb{Z}^{10} \\ (a,b,c) & \mapsto & (a, a+b+c, a+2c, b+c, b+c, \\ & & b, a+b+c, b+2c, a+c, a+c). \end{array}$$

FIGURE 14. Induction homomorphism from $H = \langle s_i, s_j \rangle \cong D_3$ to $G = \langle s_i, s_j, s_k \rangle \cong \Delta(2,3,5) = A_5 \times C_2$.

Case 3: $H = \langle s_j, s_k \rangle \cong D_5$

First we expand the character table for D_5 (from Table 4)

D_5	e	s_j	s_k	$s_j s_k$	$s_k s_j$	$s_j s_k s_j$	$s_k s_j s_k$	$s_j s_k s_j s_k$	$s_k s_j s_k s_j$	$s_j s_k s_j s_k s_j$
χ_1	1	1	1	1	1	1	1	1	1	1
χ_2	1	-1	-1	1	1	-1	-1	1	1	-1
ϕ_1	2	0	0	$\varphi - 1$	$\varphi - 1$	0	0	φ	φ	0
ϕ_2	2	0	0	φ	φ	0	0	$\varphi - 1$	$\varphi - 1$	0

where φ is the golden ratio $\frac{1+\sqrt{5}}{2}$, and we have used $2\cos\left(\frac{2\pi}{5}\right) = \varphi - 1$ and $2\cos\left(\frac{4\pi}{5}\right) = \varphi$. Note that this table is independent of interchanging s_i with s_j. Next we restrict the characters of G (Table 19) to H, that is, the 1st, 8th, 8th, 4th,

$D_2 \hookrightarrow A_5 \times C_2$	(1)	(12)(35)α	(15)(23)α	(13)(25)
$\beta_1 \downarrow$	1	1	1	1
$\beta_2 \downarrow$	0	0	0	0
$\beta_3 \downarrow$	0	0	0	0
$\beta_4 \downarrow$	0	0	0	0
$\beta_5 \downarrow$	0	0	0	-4
$\beta_6 \downarrow$	0	-2	-2	0
$\beta_7 \downarrow$	0	0	0	0
$\beta_8 \downarrow$	0	0	0	0
$\beta_9 \downarrow$	0	0	0	0
$\beta_{10} \downarrow$	0	0	0	0

$D_2 \hookrightarrow A_5 \times C_2$	$(\cdot\vert\sum\chi_i)$	$(\cdot\vert\chi_2+\chi_3)$	$(\cdot\vert\chi_3+\chi_4)$	$(\cdot\vert\chi_4)$
$\beta_1 \downarrow$	1	0	0	0
$\beta_2 \downarrow$	0	0	0	0
$\beta_3 \downarrow$	0	0	0	0
$\beta_4 \downarrow$	0	0	0	0
$\beta_5 \downarrow$	0	0	0	1
$\beta_6 \downarrow$	0	1	1	0
$\beta_7 \downarrow$	0	0	0	0
$\beta_8 \downarrow$	0	0	0	0
$\beta_9 \downarrow$	0	0	0	0
$\beta_{10} \downarrow$	0	0	0	0

TABLE 34. Restricted characters (top) and map induced by the inclusion of D_2 into $A_5 \times C_2$ (bottom).

4th, 8th, 8th, 5th, 5th and 8th columns.

$A_5 \times C_2 \downarrow$	e	s_j	s_k	s_js_k	s_ks_j	$s_js_ks_j$	$s_ks_js_k$	$s_js_ks_js_k$	$s_ks_js_ks_j$	$s_js_ks_js_ks_j$
$\rho_1 \otimes \xi_1$	1	1	1	1	1	1	1	1	1	1
$\rho_1 \otimes \xi_2$	4	0	0	-1	-1	0	0	-1	-1	0
$\rho_1 \otimes \xi_3$	5	1	1	0	0	1	1	0	0	1
$\rho_1 \otimes \xi_4$	3	-1	-1	φ	φ	-1	-1	$-\varphi+1$	$-\varphi+1$	-1
$\rho_1 \otimes \xi_5$	3	-1	-1	$-\varphi+1$	$-\varphi+1$	-1	-1	φ	φ	-1
$\rho_2 \otimes \xi_1$	1	-1	-1	1	1	-1	-1	1	1	-1
$\rho_2 \otimes \xi_2$	4	0	0	-1	-1	0	0	-1	-1	0
$\rho_2 \otimes \xi_3$	5	-1	-1	0	0	-1	-1	0	0	-1
$\rho_2 \otimes \xi_4$	3	1	1	φ	φ	1	1	$-\varphi+1$	$-\varphi+1$	1
$\rho_2 \otimes \xi_5$	3	1	1	$-\varphi+1$	$-\varphi+1$	1	1	φ	φ	1

Remark 6. With a non-conjugated choice of Coxeter generators, it would be the 5th instead of the 4th column, or, equivalently, swapping $\rho_i \otimes \xi_4$ with $\rho_i \otimes \xi_5$ for $i = 1$ and 2.

Multiplying these rows with the rows of character table of D_5 above, and using that $\varphi^2 - \varphi = 1$, we obtain

$$\begin{array}{lcll} & & \rho_1 \otimes \_ & \rho_2 \otimes \_ \\ \chi_1 \uparrow & = & \xi_1 + \xi_3 + & \xi_4 + \xi_5 \\ \chi_2 \uparrow & = & \xi_4 + \xi_5 + & \xi_1 + \xi_3 \\ \phi_1 \uparrow & = & \xi_2 + \xi_3 + \xi_4 + & \xi_2 + \xi_3 + \xi_4 \\ \phi_2 \uparrow & = & \xi_2 + \xi_3 + \xi_5 + & \xi_2 + \xi_3 + \xi_5 \end{array}$$

This gives the homomorphism of free abelian groups in Figure 15.

We finish by giving the same induction homomorphisms but this time with respect to the transformed bases (Tables 6, 7, 20 in Appendix A) in Tables 34, 35 and 36.

$$\begin{array}{rcl} R_{\mathbb{C}}(H) & \to & R_{\mathbb{C}}(G) \\ \mathbb{Z}^4 & \to & \mathbb{Z}^{10} \\ (a,b,c,d) & \mapsto & (a, c+d, a+c+d, b+c, b+d, \\ & & b, c+d, b+c+d, a+c, a+d). \end{array}$$

Figure 15. Induction homomorphism from $H = \langle s_j, s_k \rangle \cong D_5$ to $G = \langle s_i, s_j, s_k \rangle \cong \Delta(2,3,5) = A_5 \times C_2$.

$D_3 \hookrightarrow A_5 \times C_2$	(1)	$(12)(34)\alpha$	(123)	$(\cdot \mid \sum \chi_i + 2\phi_1)$	$(\cdot \mid \chi_2 + \phi_1)$	$(\cdot \mid \phi_1)$
$\beta_1 \downarrow$	1	1	1	1	0	0
$\beta_2 \downarrow$	0	0	0	0	0	0
$\beta_3 \downarrow$	0	0	-3	0	0	1
$\beta_4 \downarrow$	0	0	0	0	0	0
$\beta_5 \downarrow$	0	0	0	0	0	0
$\beta_6 \downarrow$	0	-2	0	0	1	0
$\beta_7 \downarrow$	0	0	0	0	0	0
$\beta_8 \downarrow$	0	0	0	0	0	0
$\beta_9 \downarrow$	0	0	0	0	0	0
$\beta_{10} \downarrow$	0	0	0	0	0	0

Table 35. Restricted characters and map induced by the inclusion of D_3 into $A_5 \times C_2$.

$D_5 \hookrightarrow A_5 \times C_2$	(1)	(12345)	(12354)	$(12)(34)\alpha$	$(\cdot \mid \sum \chi_i + 2\sum \phi_i)$	$(\cdot \mid \chi_2 + \sum \phi_i)$	$(\cdot \mid \sum \phi_i)$	$(\cdot \mid \phi_2)$
$\beta_1 \downarrow$	1	1	1	1	1	0	0	0
$\beta_2 \downarrow$	0	$\sqrt{5}$	$-\sqrt{5}$	0	0	0	0	1
$\beta_3 \downarrow$	0	0	0	0	0	0	0	0
$\beta_4 \downarrow$	0	$\frac{5+\sqrt{5}}{2}$	$\frac{5-\sqrt{5}}{2}$	0	0	0	-1	0
$\beta_5 \downarrow$	0	0	0	0	0	0	0	0
$\beta_6 \downarrow$	0	0	0	-2	0	1	0	0
$\beta_7 \downarrow$	0	0	0	0	0	0	0	0
$\beta_8 \downarrow$	0	0	0	0	0	0	0	0
$\beta_9 \downarrow$	0	0	0	0	0	0	0	0
$\beta_{10} \downarrow$	0	0	0	0	0	0	0	0

Table 36. Restricted characters and map induced by the inclusion of D_5 into $A_5 \times C_2$.

We show that $\mathrm{H}_0^{\mathfrak{Fin}}(\Gamma;\, R_{\mathbb{C}})$ is torsion-free for Γ the Heisenberg semidirect product group of Lück's paper.

Appendix C. The Heisenberg semidirect product group

In Tables 37 and 38, we transform the character tables of all the non-trivial finite subgroups of the Heisenberg semidirect product group, as specified by Lück [14].

In Tables 39, 40 and 41, we compute all possible induction homomorphisms $R_{\mathbb{C}}(H) \to R_{\mathbb{C}}(G)$ appearing in any possible Bredon chain complex.

Cyclic group of order 2 with generator s.

$$\left(\begin{array}{c|cc} C_2 & e & s \\ \hline \rho_1 & 1 & 1 \\ \rho_2 & 1 & -1 \end{array}\right) \mapsto \left(\begin{array}{c|cc} C_2 & e & s \\ \hline \rho_1+\rho_2 & 2 & 0 \\ \rho_2 & 1 & -1 \end{array}\right)$$

TABLE 37. Character table of C_2

Cyclic group of order 4 with generator s, we let $i^2 = -1$.

$$\left(\begin{array}{c|cccc} C_4 & e & s & s^2 & s^3 \\ \hline \rho_1 & 1 & 1 & 1 & 1 \\ \rho_2 & 1 & -1 & 1 & -1 \\ \rho_3 & 1 & i & -1 & -i \\ \rho_4 & 1 & -i & -1 & i \end{array}\right) \mapsto \left(\begin{array}{c|cccc} C_4 & e & s & s^2 & s^3 \\ \hline \rho_1 & 1 & 1 & 1 & 1 \\ \rho_2-\rho_1 & 0 & -2 & 0 & -2 \\ \rho_3-\rho_1 & 0 & i-1 & -2 & -i-1 \\ \rho_4-\rho_3 & 0 & -2i & 0 & 2i \end{array}\right)$$

TABLE 38. Character table of C_4

The only non-trivial inclusion of a cyclic group of order 2 into a cyclic group of order 4:

$C_2 \hookrightarrow C_4$	e	s^2	$(\cdot\|\rho_1+\rho_2)$	$(\cdot\|\rho_2)$
$\rho_1 \downarrow$	1	1	1	0
$(\rho_2-\rho_1)\downarrow$	0	0	0	0
$(\rho_3-\rho_1)\downarrow$	0	-2	0	1
$(\rho_4+\rho_3-\rho_2-\rho_1)\downarrow$	0	0	0	0

TABLE 39. The only non-trivial inclusion $C_2 \hookrightarrow C_4$: $s \mapsto s^2$.

The trivial inclusion of a cyclic group of order 2 into a cyclic group of order 4:

$C_2 \hookrightarrow C_4$	e	e	$(\cdot\|\rho_1+\rho_2)$	$(\cdot\|\rho_2)$
$\rho_1 \downarrow$	1	1	1	0
$(\rho_2-\rho_1)\downarrow$	0	0	0	0
$(\rho_3-\rho_1)\downarrow$	0	0	0	0
$(\rho_4+\rho_3-\rho_2-\rho_1)\downarrow$	0	0	0	0

TABLE 40. The trivial inclusion $C_2 \hookrightarrow C_4$: $s \mapsto e$.

The inclusion of the trivial group into a cyclic group of order 4:

$C_2 \hookrightarrow C_4$	e	$(\cdot\|\tau)$
$\rho_1 \downarrow$	1	1
$(\rho_2-\rho_1)\downarrow$	0	0
$(\rho_3-\rho_1)\downarrow$	0	0
$(\rho_4+\rho_3-\rho_2-\rho_1)\downarrow$	0	0

TABLE 41. The inclusion of the trivial group $\{1\} \hookrightarrow C_4$

Obviously, any concatenation of copies of the three matrices given in Tables 39, 40 and 41 yields a matrix with all of its minors contained in the set $\{-1, 0, 1\}$. For the inclusions into cyclic groups of order 2, we proceed analogously, only it is then even simpler to compute the induced matrices. *Hence by Theorem 4,* $\mathrm{H}_0^{\mathfrak{Fin}}(\Gamma;\, R_{\mathbb{C}})$ *is torsion-free for* Γ *the Heisenberg semidirect product group of Lück's article* [14].

Appendix D. Crystallographic groups

Davis and Lück [4] consider the semidirect product of $\mathbb{Z}^n$ with the cyclic p-group $\mathbb{Z}/p$, where the action of $\mathbb{Z}/p$ on $\mathbb{Z}^n$ is given by an integral representation, which is assumed to act freely on the complement of zero. The action of this semidirect product group Γ on $\underline{E}\Gamma \cong \mathbb{R}^n$ is crystallographic, with $\mathbb{Z}^n$ acting by lattice translations, and $\mathbb{Z}/p$ acting with a single fixed point. In particular, all cell stabilizers are trivial except for one orbit of vertices of stabilizer type $\mathbb{Z}/p$. So all maps in the Bredon chain complex are induced by the trivial representation, and we can easily apply Theorem 4 to see that $\mathrm{H}_0^{\mathfrak{Fin}}(\Gamma;\, R_{\mathbb{C}})$ is torsion-free for Γ.