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HYPERBOLIC VARIANTS OF PONCELET'S THEOREM

by

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ABSTRACT

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In 1813, J. Poncelet proved his beautiful theorem in projective geometry, Poncelet's Closure Theorem, which states that: if C and D are two smooth conics in general position, and there is an n-gon inscribed in C and circumscribed around D, then for any point of C, there exists an n-gon, also inscribed in C and circumscribed around D, which has this point for one of its vertices.

There are some formulae related to Poncelet's Theorem, in which introduce relations between two circles' data (their radii and the distance between their centres), when there is a bicentric n-gon between them. In Euclidean geometry, for example, we have Chapple's and Fuss's Formulae.

We introduce a proof that Poncelet's Theorem holds in hyperbolic geometry. Also, we present hyperbolic Chapple's and Fuss's Formulae, and more general, we prove a Euclidean general formula, and two version of hyperbolic general formulae, which connect two circles' data, when there is an embedded bicentric n-gon between them. We formulate a conjecture that the Euclidean formulae should appear as a factor of the lowest order terms of a particular series expansion of the hyperbolic formulae.

Moreover, we define a three-manifold X, constructed from $n = 3$ case of Poncelet's Theorem, and prove that X can be represented as the union of two disjoint solid tori, we also prove that X is Seifert fibre space.

Contents

Author's declaration

I, Amal Alabdullatif, declare that this thesis and the work presented in it are my own and has been generated by me as the result of my own original research.

Hyperbolic Variants of Poncelet's Theorem

I confirm that:

1. This work was done wholly or mainly while in candidature for a research degree at this University;
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Introduction

"One of the most important and also most beautiful theorems in classical projective geometry is that of Poncelet, concerning closed polygons, which are inscribed in one conic and circumscribed about another. " These words were written in 1977 by Griffiths and Harris, and perfectly describe Poncelet's Theorem [21]. Poncelet's Closure Theorem, or more commonly called Poncelet's Theorem, is one of the most interesting and beautiful theorems in mathematics. It has a very simple formulation but it is extremely difficult to prove. There are various proofs of this theorem; most of them are not elementary (Poncelet, Jacobi, Griffiths and Harris etc.).

The French mathematician Jean-Victor Poncelet proved this theorem, during his captivity in Saratov, Russia, in 1813, after the Napoleon's war against the country. The first proof of Poncelet was analytic. However, in 1822, Poncelet chose to publish another purely geometric proof in his book "Treatise on the Projective Properties of Figures". Suppose that two ellipses are given in the plane, together with a closed polygonal line inscribed in one of them and circumscribed around the other one, then, Poncelet's Theorem states that infinitely many such closed polygonal lines exist and every point of the first ellipse is a vertex of such a polygon. Besides, all these polygons have the same number of sides [4].

Later, using the addition theorem for elliptic functions, Jacobi gave another proof of the theorem in 1828, in his proof, Poncelet's Theorem is equivalent to the addition theorems for elliptic curves. Another proof of Poncelet's Theorem, in a modern algebro-geometrical manner, was provided by Griffiths and Harris in 1977. This also presented an interesting generalisation of Poncelet's Theorem in the three-dimensional case, considering polyhedral surfaces both inscribed and circumscribed around two quadrics [15]. Also, in [15], a generalisation of Poncelet's

Theorem to higher-dimensional spaces was introduced. In 2008, Flatto proved Poncelet's Theorem in the hyperbolic plane using dynamical system [19].

Moreover, many mathematicians have proposed various methods of proving Euclidean Poncelet's Theorem. In 1991, Lion [32] proved results about the maximum or minimum of the length of an embedded n-gon inscribed in an ellipse or circumscribed around it, respectively. Combining these, he obtained a new proof of Poncelet's Theorem on homofocal ellipses and embedded n-gons. Also, Valley [51], in 2012, involved vector bundles techniques to propose a proof of Poncelet's Theorem. Later, in 2014, Halbeisen and Hungerbuhler [22] gave an elementary proof of Poncelet's Theorem by showing that it is a purely combinatorial consequence of Pascal's Theorem.

Furthermore, several publications have appeared in recent years discussing Poncelet's Theorem. In 1994, King [31] Discussed a property of some invariant measures with applications to conic sections, geometric set-inclusion and number theory. More over, Cieslak and Szczygielska [10] in 2008 showed that each oval and a natural number $n \geq 3$ generate an annulus which possesses the Poncelet's Theorem property. A necessary and sufficient condition of existence of circuminscribed n-gons in an annulus is given. In 2012, Weir and Wessel [52] proved Poncelet's Theorem for triangles, they presented the conditions for a line through two points on a conic C to be tangent to a conic D; then, showed the conditions for the existence of a Poncelet's triangle. Recently, Schwartz and Tabachnikov [42], 2016, demonstrated that the locus of the centers of mass of the family of Poncelet's polygons, inscribed into a conic C and circumscribed about a conic D, is a conic homothetic to C.

The prehistory of Poncelet's Theorem is connected to special formulae related to the geometry of n-gons. These include Chapple's and Fuss's formulae. Chapple's Formula [4], was derived in 1746 by William Chapple and published in the English periodical *Miscellanea curiosa mathematica*. According to this formula, there is a triangle circumscribing a circle D with radius r_E and inscribed in a circle C with radius R_E, if and only if $d_E^2 = R_E^2 - 2r_E R_E$, where d_E is the distance between the circles' centres. Fuss's Formula [4], obtained by Nicolaus Fuss in 1797, relating the quantities of two circles when there is a bicentric quadrilateral between them. It states that $(R_E^2 - d_E^2)^2 = 2r_E^2(R_E^2 + d_E^2)$ is satisfied if and only if there is a bicentric quadrilateral between two circles. The relationship between Poncelet's Theorem and Chapple's and Fuss's formulae was first introduced by Jacobi, as Poncelet did not recognize it previously [4]. Moreover, in 1827, Steiner derived additional formulae for bicentric embedded pentagons, hexagons and octagons [4]. Besides, Chaundy introduced many formulae, in 1923, also did Kerawala, in 1947, in a simple form. Indeed, there is a general analytical expression, presented by Richelot in 1830 and Kerawala in 1947, using Jacobi's elliptic function, connecting the data of two circles when there is a bicentric n-gon between them [53].

An important event occurred in the early nineteenth century; Bolyai and Lobachevsky discovered the hyperbolic geometry, which is a kind of non-Euclidean geometry. The difference between the parallel axiom in Euclidean geometry and hyperbolic geometry introduces different facts in each geometry, which leads to different trigonometric identities. Thus, because a smaller area of the hyperbolic plane can be seen as more Euclidean, we have a conjecture that the Euclidean formulae will appear as a factor of the lowest order terms of the general hyperbolic formulae.

The first chapter starts by giving an overview of Poncelet's Theorem in the Euclidean plane. After that, we discuss the differences between the Euclidean geometry and the hyperbolic geometry, and introduce some of their facts, which make it different when dealing with each one. As well as, we introduce a brief introduction about some formulae related to Poncelet's Theorem in the Euclidean plane. Those formulae are satisfied, when there is an embedded n-gon inscribed in one circle and circumscribed around another circle. They give nice relations between the circumscribing circle's radius R_E, the inscribed circle's radius r_E and the distance between the circumscribed circle's center and the inscribed circle's center d_E. We cover Chapple's, Fuss's and Steiner's formulae. Also, we present a general formula in the Euclidean plane, that depends on the Jacobi's elliptic function.

The second chapter describes the models of the hyperbolic geometry, where the most action of this work takes place. We present Klein disk model and Poincaré disk model. Then, we introduce a proof that Poncelet's Theorem holds in the hyperbolic plane for ellipses. Our proof uses the fact that, in Klein model, every hyperbolic line is a Euclidean line inside the unit disk, also, uses two lemmas showing that in Klein model, every hyperbolic circle and hyperbolic ellipse are Euclidean ellipses. We follow that by showing that every Euclidean ellipse in the unit disk is a hyperbolic ellipse in Klein model.

We start the third chapter, by introducing general formulae for special cases, where the circles are concentric with embedded and non-embedded n-gon between them, in Euclidean and hyperbolic geometry. After that, we present the hyperbolic analogues to the Euclidean formulae (Chapple's, Fuss's formulae), which are related to Poncelet's Theorem in the hyperbolic geometry. Like the Euclidean case, those formulae relating the circumscribed circle's radius R, the inscribed circle's radius r and the distance d between the centers of the two circles, when there is a triangle, a quadrilateral, respectively inscribed in one circle and circumscribed around the other. To prove that, we apply two lemmas, give a relation between the inscribed circle's radius and the interior angles of a triangle, a quadrilateral, respectively circumscribed this circle. Firstly, we prove the analogues to Chapple's Formula in the hyperbolic geometry and compare it with the Euclidean one. Then, the analogues to Fuss's Formula in the hyperbolic geometry is presented, following by discussing the connection between Fuss's Formula in the Euclidean geometry and the analogues to it in the hyperbolic geometry. In these two comparison, we see the Euclidean formula as a factor of the lowest order terms of the hyperbolic one, after taking the limits as the circumscribed circle's radius R approaches 0. As well as, by plotting on Maple, we can see that the relations between r, d in the hyperbolic formulae are closed to their relations in the Euclidean formulae when R is very small.

In the fourth chapter, we first introduce a general method, that helps later to prove general expressions connecting the circles' data of a bicentric embedded n-gon in Euclidean and hyperbolic geometries. We next apply this method to present a general formula relating the quantities of two circles, when there is an embedded n-gon between them, on the Euclidean plane. Later, we demonstrate two general formulae in the hyperbolic plane. The first one, by applying a lemma, connecting the inscribed circle's radius with the interior angles of an embedded n-gon, circumscribed this circle. The other general formula in the hyperbolic plane is proved by manipulating the general method, which introduced at the beginning of the chapter. In the last section, we present a conjecture from observations of the hyperbolic general formulae (following the results of Chapple's and Fuss's comparison) showing that, if we consider the hyperbolic general formula for an embedded n-gon inscribed in one circle and circumscribed the other, then we can write that formula as $f_n(R, d, r) = 0$, by using the expressions $\cosh(R) \simeq 1 + \frac{R^2}{2}$, $\sinh(R) \simeq R$, for R small. Then, when we write $f_n(R, d, r) = \sum_{i=1}^{k} g_i(R, d, r)$, the lowest order non-zero term g_i has a Euclidean equivalent as a factor. This conjecture may help in proving Euclidean formulae, using hyperbolic facts.

Finally, in the last chapter, we present Poncelet's Theorem from a different perspective, since we define a three-dimensional manifold X constructed according to the Euclidean Poncelet's Theorem,

$$X = \{(x, y, z) \mid x \neq y \neq z \neq x\} \subseteq S^1 \times S^1 \times S^1$$

in which $(x, y, z) \in X$ represents vertices of a triangle inscribed in S^1 and circumscribing another circle. We prove that X is orientable, non-compact manifold. Next, we define the configuration space of n distinct points in a topological space, and show that Poncelet's three-manifold X is nothing but a configuration space of three points on a circle. Moreover, we prove that X is disconnected and can be represented as disjoint union of two solid tori. More over, we define Seifert fibre space SFS and prove that Poncelet's three-manifold is a Seifert fibre space. This will help in determine what do the fibres that come from Poncelet's look like, and also to investigate if the orbits coming from Poncelet's are geodesics, in the natural three-manifold metric on X. Lastly, we define a three-dimensional manifold X_R constructed according to hyperbolic Poncelet's Theorem,

$$X_R = \{(x, y, z) \mid x \neq y \neq z \neq x\} \subseteq C \times C \times C$$

in which $(x, y, z) \in X_R$ represents vertices of a triangle inscribed in C and circumscribed another circle. Comparing X and X_R raises a question about the relation between the hyperbolic orbits and the Euclidean orbits, when R approaches 0. We introduce an example, showing that the hyperbolic coordinates of the triangle's vertices, converge to the Euclidean coordinates, when the triangle is isosceles and R approaches 0, which gives a clue that the hyperbolic orbits converge to the Euclidean orbits when $R \longrightarrow 0$.

Chapter 1

Poncelet's Theorem Background

In this chapter, we present a brief history about Poncelet's Theorem in the Euclidean plane. Then we discuss the differences between the Euclidean geometry and the hyperbolic geometry, which allows us to deal with Poncelet's Theorem in the hyperbolic geometry in a different way. We follow this discussion, by presenting a brief introduction to formulae related to Poncelet's Theorem in the Euclidean plane. To be more specific, when an embedded n-gon is inscribed in a circle and circumscribed around other, there is a relationship between the circumscribed circle's radius, the inscribed circle's radius and the distance between the circles' centres. Those formulae include Chapple's Formula for $n = 3$, Fuss's Formula for $n = 4$ and Steiner's Formulae for $n = 5, 6, 8$. We also take a look at a general formula in the Euclidean geometry, which is generated using Jacobi elliptic function.

1.1 Poncelet's Theorem in the Euclidean Geometry

This section presents a brief history of Poncelet's Theorem in the Euclidean geometry.

Suppose that C and D are two concentric circles with D inside C. If we can draw a triangle T inscribed in C and circumscribed around D, then it is obvious that from any point on C, we can draw a triangle inscribed in C and circumscribed around D, because the triangle T can be rotated around their common centre

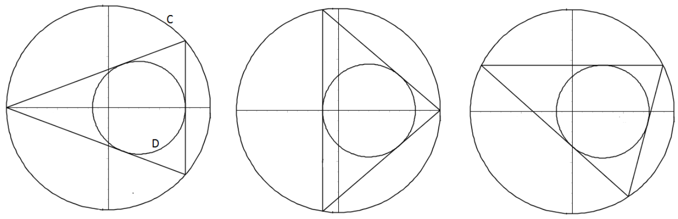

Figure 1.1: Euclidean Poncelet's Theorem

within the annular ring bounded by C and D. The interesting fact is that the same situation still holds for non-concentric circles, and more generally, for two ellipses with an n-gon, such that $n \geq 3$. Moreover, this was proven true for conics in the projective geometry. It also has a higher-dimensional generalization [15].

This beautiful theorem, see Figure 1.1, called Poncelet's Closure Theorem, or Poncelet's Theorem, some times it is called the Porism of Poncelet, as it does not occur in general, but when it does, it occurs for infinitely many cases. The French mathematician, Jean-Victor Poncelet, proved this theorem during his captivity in Russia, in Saratov in 1813. The proof was published in 1822 in his book "Treatise on the Projective Properties of Figures". [19]

Poncelet's own articulation of Closure Theorem stated that: "If any polygon is at the same time inscribed in a conic and circumscribed about another conic, then there are an infinity of such polygons with the same property with respect to the two curves; or rather, all those polygons which one would try to describe at will, under these conditions, will close by themselves on these curves. And conversely, if it happens that, while trying to inscribe arbitrarily in a conic a polygon whose sides will touch another, this polygon does not close by itself, it would necessarily be impossible that there are others which do have that property." [4]

Poncelet's Theorem is one of the most interesting theorems in mathematics, because despite its easy formulation, it is difficult to prove. Many mathematicians have been inspired by Poncelet's Theorem, which encourages them to undertake extra study. There is a substantial collection of literature that discusses different proofs of the theorem and its generalisation, particularly, from the period just perior to the beginning of the twentieth century. [4]

The following theorem introduces the real case of Poncelet's Theorem.

Theorem 1.1.1. *(The Real Case of Poncelet's Theorem) [19]*
Let C and D be two disjoint ellipses in the Euclidean plane, with D inside C. Suppose there is an n-gon inscribed in C and circumscribed around D, then for any other point of C, there exists an n-gon, inscribed in C and circumscribed around D, which has this point for one of its vertices. In addition, all these n-gons have the same number of sides.

An n-gon with vertices $v_1, ..., v_n$ is inscribed in C and circumscribed around D if the points $v_1, ..., v_n$ lie in C and the lines determined by the pairs of consecutive points $(v_1, v_2), (v_2, v_3), ...(v_n, v_1)$ are tangent to D [19].

Also, as we mentioned above, Poncelet's Theorem works for conics in the complex projective plane P_2.

Theorem 1.1.2. *(P_2-Version of Poncelet's Theorem)[19]*
Let C and D be two smooth conics in general position. Suppose there is an n-gon inscribed in C and circumscribed around D, then for any point of C, there exists an n-gon, also inscribed in C and circumscribed around D, which has this point for one of its vertices. In addition, all these n-gons have the same number of sides.

Recall that the conic in the projective plane P_2 [19] is a curve with equation $Q(x) = 0$, such that $Q(x) = \sum a_{ij}x_ix_j$ is a quadratic form in $x = (x_1, x_2, x_3)$. A conic is considered smooth if it has a tangent line at each of its points. In addition, the conics C and D are in general position if they intersect in four points [19].

There are several proofs of this theorem, and most of them are complicated. During his captivity in Russia, Poncelet gave an analytic proof for his theorem in 1813. However, in 1822, Poncelet chose to publish another purely geometric proof in his "Treatise on the Projective Properties of Figures". Poncelet's proof, which is complicated, reduces the theorem to two circles [19].

Indeed, Poncelet's Theorem is deduced as a corollary of a much more general theorem, namely Poncelet's General Theorem, in which he considered $n + 1$ conics of a pencil in the projective plane. If there exists an n-gon with vertices lying on the first of these conics and each side touching one of the other n conics, then infinitely many such n-gons exist. Such n-gons are called Poncelet's polygons. It is the case of Poncelet's Theorem when all the other n conics in the General Theorem coincide [15].

Following that, another proof given by Jacobi in 1828, using the addition theorem for elliptic functions, in his proof, Poncelet's Theorem is equivalent to the addition theorem for elliptic curves [15].

In 1977, Griffiths and Harris introduced another proof of Poncelet's Theorem through a modern algebra-geometric way. Here, generalization of Poncelet's Theorem to the three-dimensional case was presented, by consider polyhedral surfaces both inscribed and circumscribed around two quadrics [15].

However, more than one century before the Griffiths and Harris generalization, Darboux, in 1870, proved a generalization of Poncelet's Theorem in three-dimensional space [15].

Moreover, Poncelet's Theorem was generalized to higher-dimensional spaces in [15].

1.2 Between the Euclidean and the Hyperbolic Worlds

Euclidean Geometry is the study of the flat space with curvature 0. It was named after Euclid, a Greek mathematician who lived in 300 BC, in his Elements (the famous book by Euclid), five axioms of geometry were given, which form the foundation of the Euclidean geometry. The fifth axiom, which is also known as the Parallel Axiom, states that:

If a straight line falls on two straight lines in such a manner that the interior angles on the same side are together less than two right angles, then the straight lines, if produced indefinitely, meet on that side on which are the angles less than the two right angles [45].

Despite the clarity of the axiom, arguments raged among mathematicians and led to the establishment of the Non-Euclidean geometry, where the hyperbolic geometry is a kind of it. It was first introduced by Janos Bolyai and Nikolai Lobachevsky, at the beginning of the nineteenth century. The hyperbolic geometry is the study of a saddle-shaped space with a negative curvature. The Parallel Axiom in the hyperbolic geometry is given as follows:

Given a straight line and a point not on the line, there exists an infinite number of straight lines through the point parallel to the original line [40].

In this work, we are dealing with Poncelet's Theorem for ellipses and circles in the hyperbolic plane. When dealing with the hyperbolic geometry, the situation will be different as the difference between the parallel axioms leads to various facts in each geometry. Two parallel lines are equidistant in the Euclidean geometry, which cannot be true in the hyperbolic geometry. Moreover, many Euclidean triangle facts are not true in the hyperbolic plane. For example, in the hyperbolic geometry, the sum of the triangle's angles is less than π, and the triangles with the same angles have the same areas. Also, the similarities of the Euclidean triangles cannot be applied hyperbolically, as there are no similar triangles in the hyperbolic geometry, all similar triangles are congruent.

Proceeding from these differences, different trigonometric identities for hyperbolic triangles were proven. These hyperbolic identities help to prove some hyperbolic theorems that may not be proved in the Euclidean world. In chapter 4, we can see that the first general hyperbolic expression which connects the data of two circles (the circumscribed circle's radius, the inscribed circle's radius and the distance between the circles' centres) when there is a bicentric n-gon between them, has been proven in a direct geometrical manner using hyperbolic trigonometric identities, whereas, we cannot use similar method in the Euclidean plane.

Non-Euclidean geometry has many advantages, it opened up the geometry and revealed an active field of research, with many applications in science and art. For example, as a description of the space-time, Einstein's general theory of relativity applies non-Euclidean geometry.

An interesting point in dealing with hyperbolic features is the fact that whenever we focus on a smaller and smaller areas of the hyperbolic plane, we can see it as more and more Euclidean. This may help us to prove the Euclidean formulae using the hyperbolic facts, by taking the limits of the hyperbolic formula when the circumscribed circle radius approaches 0 and concentrating on the lowest order terms of the formula. We believe that the Euclidean formula should appear as a factor of these lowest order terms.

1.3 Formulae Related to Poncelet's Theorem in the Euclidean Plane

This section takes a look at the prehistory of Poncelet's Theorem. The majority of information here comes from [4]. In the Euclidean geometry, there are special formulae that are relevant to Poncelet's Theorem for values of n, where n is the number of sides of an embedded n-gon. Those formulae relating the quantities of two circles when there is an embedded n-gon inscribed in one circle and circumscribed around the other, where R_E is the circumscribing circle's radius, r_E is the inscribed circle's radius and d_E is the distance between the circumcentre and the incentre. We cover Chapple's Formula for $n = 3$, Fuss's Formula for $n = 4$ and Steiner's Formulae for $n = 5, 6, 8$, as well as a general formula in the Euclidean geometry, which depends on Jacobi's elliptic function.

Definition 1.3.1. *A bicentric n-gon is an n-gon which has both a circumscribed circle (which touches each vertex) and an inscribed circle (which is tangent to each side).*

1.3.1 Euclidean Chapple's Formula

We start with Chapple's Formula which relating the quantities of two circles when there is a bicentric triangle between them.

Theorem 1.3.2. *(Chapple's Formula)*
Let C and D be two disjoint circles in the Euclidean plane, with D inside C. There is a bicentric triangle between the circles if and only if

$$d_E^2 = R_E^2 - 2r_E R_E$$

This formula was presented in 1746, by William Chapple, in an article in the English periodical *Miscellanea curiosa mathematica* [9]. No earlier appearance of the formula is known. It remained unrecognised by most mathematician as did the majority of Chapple's work. The study of Chapple's paper was first raised by Mackay in 1887. Later, in his "Vorlesungen", Cantor used this fact in 1907. From that time, the formula was referred to as Chapple's. In some papers, this formula is called (Euler Formula) because some nineteenth century authors attributed this formula to Euler, where he proved it in 1765.

The proof of this formula is clear and can be done in a geometric way using some Euclidean facts. For example, the similarity of the Euclidean triangles, the fact that the interior angles of a triangle add up to 2π, and also using intersecting chords theorem, which states that when two chords intersect each other inside a circle, the products of their segments are equal [44].

1.3.2 Euclidean Fuss's Formula

In 1797, Fuss's Formula was presented by Nicolaus Fuss, this formula introduces a relationship between the quantities of two circles when there is a bicentric quadrilateral between them.

Theorem 1.3.3. *(Fuss's Formula)*
Let C and D be two disjoint circles in the Euclidean plane, with D inside C, then there is a bicentric quadrilateral between them if and only if

$$(R_E^2 - d_E^2)^2 = 2r_E^2(R_E^2 + d_E^2)$$

Fuss's Formula is proved in a simple geometric way similar to Chapple's Formula, and also using some Euclidean facts, which include the premises that the opposite angles of a cyclic quadrilateral are supplementary, the interior angles of a triangle add up to 2π, the holdings of Pythagorean Theorem and also the intersecting chords theorem, which states that when two chords intersect each other inside a circle, the products of their segments are equal [44].

1.3.3 Other Euclidean Formulae

In the second volume of "Crelle's Jurnal für die reine und angewandte Mathematik" in 1827, Steiner proposed other formulae for bicentric embedded pentagons, hexagons and octagons, however, no proof is presented for any of them.
For a bicentric embedded pentagon the formula is

$$r_E(R_E-d_E) = (R_E+d_E)\sqrt{(R_E - r_E + d_E)(R_E - r_E - d_E)}+(R_E+d_E)\sqrt{2R_E(R_E - r_E - d_E)}$$

for a bicentric embedded hexagon the formula is

$$3(R_E^2 - d_E^2)^4 = 4r_E^2(R_E^2 + d_E^2)(R_E^2 - d_E^2)^2 + 16r_E^4 d_E^2 R_E^2$$

and the formula of a bicentric embedded octagon is given as follow

$$8r_E^2[(R_E^2-d_E^2)^2-r_E^2(R_E^2+d_E^2)]\{(R_E^2+d_E^2)[(R_E^2-d_E^2)^4+4r_E^4d_E^2R_E^2]-8r_E^2d_E^2R_E^2(R_E^2-d_E^2)^2\}$$
$$= [(R_E^2 - d_E^2)^4 - 4r_E^4 d_E^2 R_E^2]^2$$

In 1923, Chaundy presented formulae for $n = 3, 4, 5, 6, 7, 8, 9, 10, 12, 14, 16, 18, 20$,

whereas the expression for $n = 11$ is derived by Richelot in 1830. Moreover, many formulae were established by Kerawala in 1947 in a simple form [53].

In fact, there is a general analytical expression relating the circumscribed circle's radius R_E, the inscribed circle's radius r_E, and the distance between the circumscribed circle's center and the inscribed circle's center d_E for a bicentric n-gon, Define

$$a = \frac{1}{(R_E + d_E)}$$

$$b = \frac{1}{(R_E - d_E)}$$

$$c = \frac{1}{r_E}$$

Since r_E, R_E and d_E are positive quantities with $d_E < R_E$, $0 < a < b$, let

$$\lambda = 1 + \frac{2c^2(a^2 - b^2)}{a^2(b^2 - c^2)}$$

$$\omega = \cosh^{-1}(\lambda)$$

and define the elliptic modulus k via $k^2 = 1 - e^{-2\omega}$.

Thus, the condition for a Euclidean n-gon to be bicentric is

$$sc\left(\frac{K(k)}{n}, k\right) = \frac{(c\sqrt{b^2 - a^2} + b\sqrt{c^2 - a^2})}{a(b + c)}$$

where $sc(x, k)$ is a Jacobi elliptic function and $K(k)$ is a complete elliptic integral of the first kind, this general formula was given by Richelot in 1830 and Kerawala in 1947 [53].

Chapter 2

Poncelet's Theorem in the Hyperbolic Geometry

This chapter explains the hyperbolic geometry, and describes the models of the hyperbolic geometry where most of the action of this work takes place. These models are Klein disk model and Poincaré disk model. Then, we introduce two lemmas to show that in Klein model, every hyperbolic circle and hyperbolic ellipse are Euclidean ellipses. We follow that by showing that the opposite direction is also right, this means that every Euclidean ellipse in the unit disk is a hyperbolic ellipse in Klein model. After that, we introduce the hyperbolic version of Poncelet's Theorem, which is the main goal of this chapter, we prove it depending on the fact that in Klein model, every hyperbolic line is a Euclidean line inside the unit disk, also, using the two lemmas mentioned above.

2.1 Models of the Hyperbolic Geometry

The hyperbolic geometry [40] is a type of non-Euclidean geometry where for a hyperbolic line L and a point z in a hyperbolic plane, not on L, there are at least two distinct lines through z, which do not intersect L. Two hyperbolic lines are said to be parallel if they have no common points. Two models of the hyperbolic plane, Klein disk model and Poincaré disk model, are used in this work.

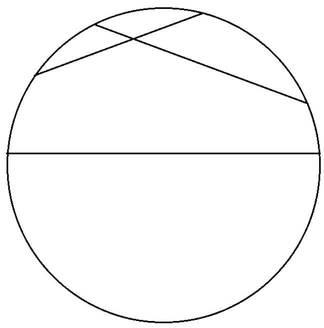

Figure 2.1: Lines in Klein Model

2.1.1 Klein Disk Model

In Klein disk model of the hyperbolic plane [24], the hyperbolic plane is the bounded open disk $K = \{z \in C||z| < 1\}$ in the complex plane C determined by the Euclidean unit circle S^1. The points of this model are the Euclidean points within the disk. Furthermore, the hyperbolic lines are defined as the open Euclidean chords in this model, see Figure 2.1.

One point of caution when using this model is that the angles in the hyperbolic plane are distorted from the Euclidean angles when represented in Klein disk. This model is therefore called non-conformal or angle distorting.

The hyperbolic distance [26] in this model can be given by the following formula: If $z = (x, y), w = (a, b)$ are the Euclidean coordinates of two points z, w, in the unit disk, the hyperbolic distance between them in Klein model is:

$$d_K(z, w) = \operatorname{arccosh}\left[\frac{1 - xa - yb}{\sqrt{(1 - x^2 - y^2)(1 - a^2 - b^2)}}\right] \tag{2.1}$$

The isometries in this geometry are projective transformations of the plane that preserve the unit disk [49]. Recall that the projective transformation T [19] in the projective plane P_2 can be defined as a function $T : P_2 \to P_2$ such that $T(p) = A(p)$, where A is a 3×3 non-singular matrix and $p = (x, y, z)$. This bijection maps lines to lines (but does not necessarily preserve parallelism). Although the

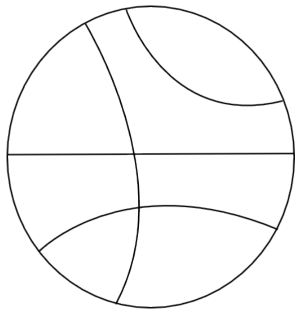

Figure 2.2: Lines in Poincaré Disk Model

projective transformations preserve incidence, they do not preserve sizes or angles. They take conic section to conic section [46].

2.1.2 Poincaré Disk Model

On the other hand, the hyperbolic plane in Poincaré disk model [1] is also the bounded open disk $D = \{z \in C ||z| < 1\}$ in the complex plane C. The points of this model are the Euclidean points within the disk. Furthermore, the hyperbolic lines in this hyperbolic plane consist of all segments of circles contained within the disk that are orthogonal to the boundary of the disk, plus all the Euclidean diameters of the disk, see Figure 2.2.

The hyperbolic distance in this model [26] can be given by the following formula: If z, w are two points in the unit disk, the hyperbolic distance between them in Poincaré disk model is:

$$d_D(z, w) = \text{arccosh}\left[1 + 2\frac{|z - w|^2}{(1 - |z|^2)(1 - |w|^2)}\right]$$

where $|.|$ denotes the usual Euclidean distance

In Poincaré model, the angles between the hyperbolic lines can be measured directly as the Euclidean ones, therefore, it is also referred to as a conformal model. All isometrics within Poincaré model are Möbius transformations [1], where Möbius transformation in the complex plane C is defined as a function $m : C \to C$ of the form $m(z) = \frac{az+b}{cz+d}$ where $a, b, c, d \in C$ and $ad - bc \neq 0$.

Recall that a hyperbolic circle Q in the hyperbolic plane is defined as a set of points $y \in D$, such that $d_D(x, y) = r$ where x is the hyperbolic centre of Q, $r > 0$ is the hyperbolic radius of Q and $d_D(x, y)$ represents the hyperbolic distance in the hyperbolic Poincaré disk D [1].

Every hyperbolic circle in Poincaré model is a Euclidean circle and conversely, all the Euclidean circles in the unit disk are a hyperbolic circles in Poincaré model. However, the centres and radii in general will be different in the hyperbolic version, compared with the Euclidean version [1].

Theorem 2.1.1. *[1] Every Möbius transformation takes circles in C to circles in C.*

2.1.3 Hyperbolic Trigonometric Identities

Now, we present some hyperbolic trigonometric identities which will be used to prove some formulae in hyperbolic geometry related to Poncelet's Theorem.

Theorem 2.1.2. *(Hyperbolic Trigonometric Identities)[35], [29]*
Let x, y be any real numbers, then

1. $$\cosh^2(x) - \sinh^2(x) = 1$$

2. $$\operatorname{sech}^2(x) = 1 - \tanh^2(x)$$

3. $$\cosh(x \pm y) = \cosh(x)\cosh(y) \pm \sinh(x)\sinh(y)$$

4. $$\cosh(2x) = \sinh^2(x) + \cosh^2(x) = 2\sinh^2(x) + 1 = 2\cosh^2(x) - 1$$

5. $$\sinh(2x) = 2\sinh(x)\cosh(x)$$

In the following two theorems, we present some identities for hyperbolic triangles, which will be used later.

Theorem 2.1.3. *[2] (The Hyperbolic Sine and Cosine Rules)*
Let T be any hyperbolic triangle and let a, b, and c be the hyperbolic lengths of its sides, let α, β and γ be its interior angles, where α is the interior angle opposite the side of hyperbolic length a, β is the interior angle opposite the side of hyperbolic length b, and γ is the interior angle opposite the side of hyperbolic length c. The laws are given as follow
the hyperbolic law of cosines I

$$\cosh(a) = \cosh(b)\cosh(c) - \sinh(c)\sinh(b)\cos(\alpha) \tag{2.2}$$

the hyperbolic law of cosines II

$$\cosh(c) = \frac{\cos(\alpha)\cos(\beta) + \cos(\gamma)}{\sin(\alpha)\sin(\beta)} \tag{2.3}$$

the hyperbolic law of sines

$$\frac{\sinh(a)}{\sin(\alpha)} = \frac{\sinh(b)}{\sin(\beta)} = \frac{\sinh(c)}{\sin(\gamma)} \tag{2.4}$$

Theorem 2.1.4. *[2] (Right-angled Triangle)*
Let T be a right hyperbolic triangle and let a, b and c be the hyperbolic lengths of its sides, let α , β and $\frac{\pi}{2}$ be its interior angles, where α is the interior angle opposite the side of hyperbolic length a, β is the interior angle opposite the side of hyperbolic length b, and $\frac{\pi}{2}$ is the interior angle opposite the side of hyperbolic length c. Then we have the following

1. $$\cosh(c) = \cosh(a)\cosh(b)$$

2. $$\tanh(b) = \sinh(a)\tan(\beta)$$

3.

$$\sinh(b) = \sinh(c)\sin(\beta)$$

4.

$$\tanh(a) = \tanh(c)\cos(\beta)$$

5.

$$\cos(\alpha) = \cosh(a)\sin(\beta)$$

2.2 Poncelet's Theorem in the Hyperbolic Geometry

In this section, we wish to provide an alternative proof to [19], which show that Poncelet's Theorem holds in the hyperbolic plane. We start with two calculations to show that the hyperbolic circles and the hyperbolic ellipses in Klein model are Euclidean ellipses in the unit disk, we use these lemmas later to prove Poncelet's Theorem in the hyperbolic plane. Also, we prove the opposite direction, that every Euclidean ellipse in the unit disk is a hyperbolic ellipse in Klein model. At the end of this chapter, we introduce the hyperbolic version of Poncelet's Theorem and its proof.

We begin this section by proving the following lemma which shows that every hyperbolic circle in Klein model is a Euclidean ellipse. This fact was introduced previously by Busemann and Kelly in [6].

Lemma 2.2.1. *In Klein model of the hyperbolic plane, every hyperbolic circle is a Euclidean ellipse.*

Proof. To prove this lemma, we start with the definition of the hyperbolic circle. Then, using the distance formula in Klein model (2.1) in this definition and doing some calculations with simplifications to get the equation of the hyperbolic circle in Klein model, which we can see that it is a Euclidean ellipse's equation.

Let $w = a + ib \in K$ be the hyperbolic centre of the hyperbolic circle C and $r > 0$

be its hyperbolic radius. The equation of the hyperbolic circle C is given as follow

$$d_K(z,w) = r$$

where $z = x + iy$ is an arbitrary point on the circle C. From the definition of the hyperbolic distance in Klein model, we get that

$$\operatorname{arccosh}\left[\frac{1 - ax - by}{\sqrt{(1 - x^2 - y^2)(1 - a^2 - b^2)}}\right] = r$$

By taking cosh of both sides, squaring and rearranging

$$(1 - ax - by)^2 = (1 - x^2 - y^2)(1 - |w|^2)\cosh^2(r)$$

Then, by solving the equation and setting $1 - |w|^2 = W$, from which we see that

$$1 - 2ax + a^2x^2 - 2by + 2abxy + b^2y^2 = (1 - x^2 - y^2)W\cosh^2(r)$$

Rearranging the equation

$$[a^2 + W\cosh^2(r)]x^2 + 2abxy + [b^2 + W\cosh^2(r)]y^2 - 2ax - 2by + 1 - W\cosh^2(r) = 0 \tag{2.5}$$

We know that the conic section described by the equation of the form

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0 \text{ with } A, B, C \text{ not all zero}$$

represents a Euclidean ellipse if $B^2 - 4AC < 0$ [13].
Thus, (2.5) is an equation of a Euclidean ellipse provided that,

$$4a^2b^2 - 4[a^2 + W\cosh^2(r)][b^2 + W\cosh^2(r)] < 0 \tag{2.6}$$

By simplifying, we see that

$$4a^2b^2 - 4[a^2 + W\cosh^2(r)][b^2 + W\cosh^2(r)] = -4W\cosh^2(r)[a^2 + b^2 + W\cosh^2(r)]$$

is less than 0.
So, the hyperbolic circle's equation is nothing but the Euclidean ellipse's equation. As a result, we can see that every hyperbolic circle in Klein model is a Euclidean ellipse. □

In the following, we show that every hyperbolic ellipse in Klein model is a Euclidean ellipse in the unit disk by using similar way as the previous lemma. Furthermore, we use these lemmas later to prove the hyperbolic version of Poncelet's Theorem

Lemma 2.2.2. *In Klein model of the hyperbolic plane, every hyperbolic ellipse is a Euclidean ellipse.*

Proof. To prove this lemma, we define the hyperbolic ellipse. Then, using the distance formula in Klein model (2.1) in this definition, and doing some calculations with simplifications to get the equation of the hyperbolic ellipse in Klein model, which we can see that it is a Euclidean ellipse's equation.
Let $w = a + ib, v = p + iq$ be the foci of a hyperbolic ellipse, so the equation of the hyperbolic ellipse is given by

$$d_K(z, w) + d_K(z, v) = c$$

where $z = x + iy$ is an arbitrary point on the ellipse and $c > 0$ is a constant.
We need

$$d_K(w, v) < c \tag{2.7}$$

in order for the ellipse to be non empty.

From the definition of the hyperbolic distance in Klein model, the equation of the hyperbolic ellipse is given by

$$\operatorname{arccosh}\left[\frac{1 - xa - yb}{\sqrt{(1 - x^2 - y^2)(1 - a^2 - b^2)}}\right] + \operatorname{arccosh}\left[\frac{1 - xp - yq}{\sqrt{(1 - x^2 - y^2)(1 - p^2 - q^2)}}\right] = c$$

Set $1-|z|^2 = Z, 1-|w|^2 = W$ and $1-|v|^2 = V$, then the hyperbolic ellipse equation is

$$\operatorname{arccosh}\left[\frac{1-xa-yb}{\sqrt{ZW}}\right] = c - \operatorname{arccosh}\left[\frac{1-xp-yq}{\sqrt{ZV}}\right]$$

by taking cosh of both sides and using identity 3, Theorem 2.1.2, we find that

$$\frac{1-xa-yb}{\sqrt{ZW}} = \cosh(c)\left[\frac{1-xp-yq}{\sqrt{ZV}}\right] - \sinh(c)\sinh\left(\operatorname{arccosh}\left[\frac{1-xp-yq}{\sqrt{ZV}}\right]\right)$$

However, from identity 1, Theorem 2.1.2, we find that

$$\sinh(y) = \sqrt{\cosh^2(y)-1}$$

set $y = \operatorname{arccosh}(x)$, then

$$\sinh(\operatorname{arccosh}(x)) = \sqrt{(\cosh(\operatorname{arccosh} x))^2 - 1} = \sqrt{x^2-1} \tag{2.8}$$

so,

$$\frac{1-xa-yb}{\sqrt{ZW}} = \cosh(c)\left[\frac{1-xp-yq}{\sqrt{ZV}}\right] - \sinh(c)\sqrt{\frac{(1-xp-yq)^2}{ZV} - 1}$$

which means that

$$\sinh(c)\sqrt{\frac{(1-xp-yq)^2}{ZV} - 1} = \cosh(c)\left[\frac{1-xp-yq}{\sqrt{ZV}}\right] - \left[\frac{1-xa-yb}{\sqrt{ZW}}\right]$$

Squaring both sides

$$\sinh^2(c)\left[\frac{(1-xp-yq)^2}{ZV} - 1\right] = \cosh^2(c).\left[\frac{(1-xp-yq)^2}{ZV}\right]$$

$$-2\cosh(c).\left[\frac{(1-xa-yb)(1-xp-yq)}{Z\sqrt{WV}}\right] + \left[\frac{(1-xa-yb)^2}{ZW}\right]$$

By multiplying both sides of the equation by ZWV, we find that

$$\sinh^2(c)(1-xp-yq)^2W - \sinh^2(c)ZWV = \cosh^2(c)(1-xp-yq)^2W$$

$$-2\cosh(c)(1-xa-yb)(1-xp-yq)\sqrt{WV} + (1-xa-yb)^2V$$

Rearranging the equation and using identity 1, Theorem 2.1.2

$$(1-xp-yq)^2W+\sinh^2(c)ZWV-2\cosh(c)(1-xa-yb)(1-xp-yq)\sqrt{WV}+(1-xa-yb)^2V = 0$$

From which we see that

$$W(1-2px-2qy+2pqxy+p^2x^2+q^2y^2)+WV\sinh^2(c)(1-x^2-y^2)-2\sqrt{WV}\cosh(c)$$

$$(1-(a+p)x-(b+q)y+(pb+aq)xy+apx^2+bqy^2)+V(1-2ax-2by+2abxy+a^2x^2+b^2y^2) = 0$$

So, the equation of the hyperbolic ellipse is given by

$$[p^2W-S-2paR+a^2V]x^2+[q^2W-S-2qbR+b^2V]y^2+[2pqW-2(pb+qa)R+2abV]xy$$

$$+[-2pW+2(a+p)R-2aV]x+[-2qW+2(b+q)R-2bV]y+[W+S-2R+V] = 0 \tag{2.9}$$

where $S = WV\sinh^2(c), R = \sqrt{WV}\cosh(c)$.
As mentioned at the previous proof, the equation 2.9 is a Euclidean ellipse if the quantity

$$[2pqW-2(pb+qa)R+2abV]^2-4[p^2W-S-2paR+a^2V][q^2W-S-2qbR+b^2V]$$

less than 0 [13].
By solving the left hand side, we find that it equals

$$8pqabWV + 4R^2(pb+qa)^2 - 4p^2b^2WV + 4p^2WS - 4a^2q^2WV + 4a^2VS$$

$$-16paqbR^2 - 8paRS + 4q^2WS + 4b^2VS - 8qbRS - 4S^2$$

Simplifying, the quantity becomes

$$-8pqabS + 4p^2b^2S + 4a^2q^2S + 4(p^2+q^2)WS + 4(a^2+b^2)VS - 8(pa+qb)SR - 4S^2$$

After dividing by $4S$, we find that the hyperbolic ellipse is a Euclidean ellipse if

$$(pb-aq)^2 - 2(pa+qb)R + (p^2+q^2)W + (a^2+b^2)V < S$$

i.e. if

$$(pb-aq)^2 - 2(pa+qb)R + (p^2+q^2)W + (a^2+b^2)V + WV < R^2$$

From the condition of the distance between the foci of the hyperbolic ellipse (2.7)

$$d_K(w,v) < c$$

$$\frac{1-ap-bq}{\sqrt{WV}} < \cosh(c)$$

Squaring both sides and rearranging

$$(1-ap-bq)^2 < WV\cosh^2(c) = R^2$$

So, the hyperbolic ellipse is a Euclidean ellipse if:

$$(pb-aq)^2 - 2(pa+qb)R + (p^2+q^2)W + (a^2+b^2)V + WV \leq (1-ap-bq)^2$$

i.e if

$$-4pbaq - 2paR - 2qbR - 2p^2a^2 - 2b^2q^2 + 2ap + 2bq \leq 0$$

Simplifying

$$-(ap+bq)^2 + (ap+bq) \leq (ap+bq)R$$

Dividing the inequality by $(pa + bq)$ and squaring both sides, we need that

$$(ap + bq)^2 - 2(ap + bq) + 1 \leq R^2$$

i.e $(ap + bq - 1)^2 \leq R^2$
which is the inequality (2.7)
So, every hyperbolic ellipse in Klein model is a Euclidean ellipse. □

At this point, one question arises, whether the opposite direction of the previous lemma true.
The next proposition reveals that for every Euclidean ellipse in the unit disk, we can deduce the hyperbolic values of it as a hyperbolic ellipse.

Proposition 2.2.1. *Every Euclidean ellipse in the unit disk is a hyperbolic ellipse in Klein model.*

Proof. To prove the proposition, we firstly introduce the equation of the ellipse as a Euclidean ellipse, then as a hyperbolic ellipse. Compare these equations and write the hyperbolic values of the hyperbolic ellipse in terms of the Euclidean values of the Euclidean ellipse. After that, by taking the maximum and minimum values of the Euclidean figures, we see that the hyperbolic values still work as values of a hyperbolic ellipse.
To simplify calculations, we know that the isometries in Klein model are projective transformations which preserve the ellipses [19]. So, without loss of generality, set $-a, a$ the foci of the Euclidean ellipse in the unit disk which is symmetric about the origin, so the equation of the Euclidean ellipse can be given by

$$d(z, a) + d(z, -a) = r$$

where $z = x + iy$ is an arbitrary point on the ellipse, $r > 0$ is a constant and $d(z, a)$ is the Euclidean distance between z and a.
From the definition of the Euclidean distance, the equation of the Euclidean ellipse is given by

$$\sqrt{(x-a)^2 + y^2} + \sqrt{(x+a)^2 + y^2} = r$$

Squaring both sides and rearranging

$$(x-a)^2+y^2+(x+a)^2+y^2-r^2=-2\sqrt{((x-a)^2+y^2)((x+a)^2+y^2)}$$

Again, squaring both sides and rearranging

$$4x^4+4a^4+8x^2a^2+4y^4+r^4-4y^2r^2+8x^2y^2-4x^2r^2+8a^2y^2-4a^2r^2$$

$$=4x^4-8a^2x^2+8x^2y^2+4a^4+8a^2y^2+4y^4$$

From which we see that

$$r^4=4(x^2r^2+a^2r^2+y^2r^2-4a^2x^2)$$

Dividing both sides in the equation by $4r^2$

$$\frac{r^2}{4}=x^2+a^2+y^2-\frac{4a^2x^2}{r^2}$$

So the equation of the Euclidean ellipse in the unit disk which is symmetric about the origin is given as follow

$$(1-\frac{4a^2}{r^2})x^2+y^2=\frac{r^2}{4}-a^2 \tag{2.10}$$

On the other hand, we deduce the equation of that ellipse as a hyperbolic ellipse in Klein model. By reflection, the foci of the hyperbolic ellipse will be on the real line.

Let $p,-p$ be the foci of this hyperbolic ellipse which is centred at the origin in Klein model, so the equation of the hyperbolic ellipse can be given by

$$d_K(z,p)+d_K(z,-p)=t$$

where $z=x+iy$ is an arbitrary point on the ellipse and $t>0$ is a constant.

From the definition of the hyperbolic distance in Klein model, the equation of the hyperbolic ellipse is given by

$$\operatorname{arccosh}\left[\frac{1-xp}{\sqrt{(1-x^2-y^2)(1-p^2)}}\right]+\operatorname{arccosh}\left[\frac{1+xp}{\sqrt{(1-x^2-y^2)(1-p^2)}}\right]=t$$

By taking cosh of both sides, using identity 3, Theorem 2.1.2, and using the relation (2.8) that $\sinh(\operatorname{arccosh}(x))=\sqrt{x^2-1}$, we find that

$$\left(\frac{1-xp}{\sqrt{(1-x^2-y^2)(1-p^2)}}\right)\left(\frac{1+xp}{\sqrt{(1-x^2-y^2)(1-p^2)}}\right)$$

$$+\sqrt{\left(\frac{(1-xp)^2}{(1-x^2-y^2)(1-p^2)}-1\right)\left(\frac{(1+xp)^2}{(1-x^2-y^2)(1-p^2)}-1\right)}=\cosh(t)$$

Rearranging and setting $T=\cosh(t)$

$$T-\frac{1-x^2p^2}{(1-x^2-y^2)(1-p^2)}=$$

$$\frac{\sqrt{((1-xp)(1+xp))^2+(1-x^2-y^2)^2(1-p^2)^2-2(1-x^2-y^2)(1-p^2)(1+x^2p^2)}}{(1-x^2-y^2)(1-p^2)}$$

Multiplying both sides by $(1-x^2-y^2)(1-p^2)$, and sequaring

$$T^2(1-x^2-y^2)^2(1-p^2)^2+(1-x^2p^2)^2-2T(1-x^2-y^2)(1-p^2)(1-x^2p^2)$$

$$=(1-x^2p^2)^2+(1-x^2-y^2)^2(1-p^2)^2-2(1-x^2-y^2)(1-p^2)(1+x^2p^2)$$

Simplifying the equation and dividing both sides by $(1-x^2-y^2)(1-p^2)$

$$T^2(1-x^2-y^2)(1-p^2)-2T(1-x^2p^2)=(1-x^2-y^2)(1-p^2)-2(1+x^2p^2)$$

Rearranging and dividing by $1 - p^2$

$$(T^2 - 1)(1 - x^2 - y^2) + \frac{2(1 + x^2p^2) - 2T(1 - x^2p^2)}{1 - p^2} = 0$$

Dividing by $T^2 - 1$ and rearranging

$$1 - x^2 - y^2 + \frac{2x^2p^2}{(1 - p^2)(T - 1)} - \frac{2}{(1 - p^2)(T + 1)} = 0$$

Thus, the equation of the hyperbolic ellipse in Klein model, which is symmetric about the origin is given by

$$\left(1 - \frac{2p^2}{(1 - p^2)(T - 1)}\right) x^2 + y^2 = 1 - \frac{2}{(1 - p^2)(T + 1)} \tag{2.11}$$

Comparing the Euclidean equation of the ellipse (2.10) and the hyperbolic equation of it (2.11), we see that

$$\frac{4a^2}{r^2} = \frac{2p^2}{(1 - p^2)(T - 1)} \tag{2.12}$$

$$\frac{r^2}{4} - a^2 = 1 - \frac{2}{(1 - p^2)(T + 1)} \tag{2.13}$$

Now, we try to find the hyperbolic values p, t in terms of the Euclidean values a, r, from (2.13)

$$(T + 1)\left(1 - \frac{r^2}{4} + a^2\right) = \frac{2}{1 - p^2}$$

$$T\left(1 - \frac{r^2}{4} + a^2\right) + \left(1 - \frac{r^2}{4} + a^2\right) = \frac{2}{1 - p^2}$$

$$T = \frac{2}{(1 - p^2)(1 - \frac{r^2}{4} + a^2)} - 1 \tag{2.14}$$

Using (2.14) in (2.12)

$$\frac{4a^2}{r^2} = \frac{2p^2}{(1 - p^2)\left(\frac{2}{(1-p^2)\left(1-\frac{r^2}{4}+a^2\right)} - 2\right)}$$

Rearranging

$$\frac{4a^2}{r^2} = \frac{p^2(1-\frac{r^2}{4}+a^2)}{1-(1-p^2)(1-\frac{r^2}{4}+a^2)}$$

$$\frac{4a^2}{r^2}\left(1-(1-p^2)\left(1-\frac{r^2}{4}+a^2\right)\right) = p^2\left(1-\frac{r^2}{4}+a^2\right)$$

$$\frac{4a^2}{r^2} - \frac{4a^2}{r^2}\left(1-\frac{r^2}{4}+a^2\right) + \frac{4a^2p^2}{r^2}\left(1-\frac{r^2}{4}+a^2\right) = p^2\left(1-\frac{r^2}{4}+a^2\right)$$

$$p^2\left(1-\frac{r^2}{4}+a^2\right)\left(\frac{4a^2}{r^2}-1\right) = \frac{4a^2}{r^2}\left(1-\frac{r^2}{4}+a^2-1\right) = a^2\left(\frac{4a^2}{r^2}-1\right)$$

From which we see that

$$p^2 = \frac{a^2}{1-\frac{r^2}{4}+a^2} \tag{2.15}$$

Compensating in (2.14)

$$T = \frac{2}{\left(1-\frac{a^2}{1-\frac{r^2}{4}+a^2}\right)\left(1-\frac{r^2}{4}+a^2\right)} - 1$$

From which we see that

$$T = \frac{2}{1-\frac{r^2}{4}} - 1 = \frac{1+\frac{r^2}{4}}{1-\frac{r^2}{4}} = \cosh(t) \tag{2.16}$$

We know that the minimum value of a equals 0, and also the minimum value of r approaches 0 as $r > 0$ which from (2.15) and (2.16) implies that $p = 0$ and t approaches 0, which are the minimum values of p and t as values of hyperbolic ellipse. In addition to that, the maximum value of a approaches 1 as the ellipse lies within the unit disk, in this case r approaches 2 which from (2.15) and (2.16) implies that p approaches 1, t approaches ∞ which still values of a hyperbolic ellipse in Klein model. As a result of that, whatever the Euclidean values of the Euclidean ellipse in the unit disk, we can get the hyperbolic values of it as a hyperbolic ellipse. Thus, every Euclidean ellipse in the unit disk is a hyperbolic ellipse in Klein model. □

Now, we introduce the hyperbolic version of Poncelet's Theorem. As we mentioned in the introduction, Flatto [19] proved hyperbolic Poncelet's Theorem using dynamical system in 2008. However, we present a new proof of this theorem depending on the fact of the hyperbolic lines in the Klein model, that they are Euclidean lines inside the unit disk, and the previous two lemmas.

Theorem 2.2.2. *(The Hyperbolic Version of Poncelet's Theorem)*
Let C and D be two disjoint hyperbolic ellipses, with D inside C. Suppose there is a hyperbolic n-gon inscribed in C and circumscribed about D, then for any other point of C, there exists a hyperbolic n-gon, inscribed in C and circumscribed around D, which has this point for one of its vertices. In addition, all these n-gons have the same number of sides.

Proof. Let C and D be two disjoint hyperbolic ellipses, with D inside C. Suppose there is a hyperbolic n-gon P inscribed in C and circumscribed around D.
We know from the definition of Klein model, that the hyperbolic line is defined as a Euclidean line inside the unit circle, also in Lemma 2.2.1 and Lemma 2.2.2, we have proved that the hyperbolic circles and the hyperbolic ellipses in Klein model are Euclidean ellipses. And as we know that there is an isomorphism between Poincaré disk model and Klein model [26], where Poincaré disk model is conformal. Therefore, when a hyperbolic line is tangent to a hyperbolic circle or ellipse, then the corresponding Euclidean line is tangent to the corresponding Euclidean ellipse. As a result of that, the n-gon P inscribed in a Euclidean ellipse and circumscribed a Euclidean ellipse.
Since the Poncelet's Theorem holds for the Euclidean ellipses, Theorem 1.1.1, therefore, the Poncelet's Theorem will hold for the hyperbolic circles and the hyperbolic ellipses in the hyperbolic plane. □

Chapter 3

Some Formulae Related to Poncelet's Theorem in the Hyperbolic Geometry

The purpose of this chapter is to prove the hyperbolic analogues to the Euclidean formulae (Chapple's and Fuss's Formulae), which is arising from Poncelet's Theorem. Those formulae are satisfied, when there is an embedded n-gon inscribed in one circle and circumscribed around other circle. We start by proving general formulae for the special cases, when the circles are concentric, in both Euclidean and hyperbolic geometry, where the n-gon embedded or non-embedded. After that, we prove the analogues to Chapple's Formula in the hyperbolic geometry. We follow that by discussing the connection between Chapple's Formula in the Euclidean geometry and the analogues to it in the hyperbolic geometry. Then, we prove the analogues to Fuss's Formula in the hyperbolic geometry too, and discuss the relation between Fuss's formulae in both Euclidean and hyperbolic geometries when R approaches 0. By figuring the relation between the formulae, we aim to take the advantage of the hyperbolic geometry to get information in the Euclidean geometry.

Remark 3.0.1. *we use a standard way to introduce some proofs in Poincaré disk model in this chapter and the following chapter. Doing that by assuming that the circumcircle C is centred at the origin O with radius R and the inscribed circle*

D is centred at the point on the positive real axis, with hyperbolic distance d, and radius r. Then, set the n-gon so that one of its vertices is at the same line with the circumscribed circle's centre and the inscribed circle's centre.

It suffices to prove the formulae in this convenient case where the calculations become easier, because we know that Poncelet's Theorem holds in the hyperbolic plane, Theorem 2.2.2. Also, all isometrics within this model are Möbius transformations which take circles to circles, Theorem 2.1.1.

3.1 Special Cases when d=0

We know from Section 1.3, that there are special Euclidean formulae related to Poncelet's Theorem, which relating the data of two circles when there is a bicentric embedded n-gon inscribed in one circle and circumscribed around the other. In this section, we prove general formulae for the special cases, where circles are concentric (the circumscribed circle's centre and the inscribed circle's centre coincide), with a bicentric n-gon between them. In this situation, the n-gon is regular, because the n-gon's sides touch the inscribed circle in their midpoints, hence, from the congruency of the triangles, the n-gon is regular. Firstly, we prove a general formula in the Euclidean plane for embedded n-gons, then turn to prove a general formula in the hyperbolic plane, also for hyperbolic embedded n-gons. Then, we give a general formula for any non-embedded bicentric n-gon in the Euclidean plane, then in the hyperbolic plane.

3.1.1 Embedded n-gon's Formulae

We introduce formulae relating the radii of two concentric circles when there is a regular embedded n-gon between them, in the Euclidean and the hyperbolic planes.
We begin by defining the embedded n-gon.

Definition 3.1.1. *(Embedded n-gon) It means a simple n-gon which consists of straight non-intersecting sides that are joined pair-wise to form closed path.*

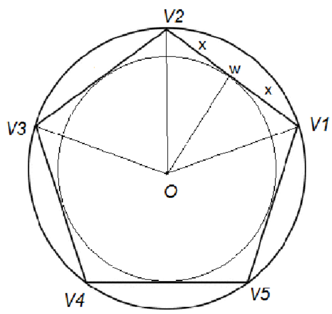

Figure 3.1: Euclidean Regular Embedded n-gon

The following theorem gives a formula relating the radii of two concentric circles when there is a bicentric embedded n-gon between them in the Euclidean plane.

Theorem 3.1.2. *Let C and D be two disjoint concentric circles in the Euclidean plane, with D inside C. Let R_E denote the radius of C and r_E denote the radius of D. Assume that there is a bicentric embedded n-gon between them, then the relation between R_E and r_E satisfies the following*

$$r_E = \cos\left(\frac{\pi}{n}\right) R_E \tag{3.1}$$

Proof. We prove this formula directly by using trigonometric identities and Pythagorean Theorem. Assume that C and D are two disjoint concentric circles with D inside C in the Euclidean plane. Suppose there is a bicentric embedded n-gon P between them, then, P is a regular polygon. Let v_j denote vertices of P such that, $1 \leqslant j \leqslant n$. Let a be the Euclidean length of its side. Assume that w is the point on the side v_1v_2 where the perpendicular from the two circles' centre O meets the side, so $|Ow| = r_E$.

Set $|v_1w| = |v_2w| = x$, and apply the law of cosine to the triangle v_1Ov_2

$$\cos\left(\frac{2\pi}{n}\right) = \frac{2R_E^2 - (2x)^2}{2R_E^2}$$

$$2R_E^2 \cos\left(\frac{2\pi}{n}\right) = 2R_E^2 - 4x^2 \tag{3.2}$$

Using Pythagorean Theorem on the triangle v_1Ow

$$x^2 = R_E^2 - r_E^2$$

Using that in (3.2)

$$2R_E^2 \cos\left(\frac{2\pi}{n}\right) = 2R_E^2 - 4(R_E^2 - r_E^2)$$

$$2R_E^2 \cos\left(\frac{2\pi}{n}\right) = -2R_E^2 + 4r_E^2 = 2(2r_E^2 - R_E^2)$$

which means that

$$2r_E^2 = R_E^2 \left(\cos\left(\frac{2\pi}{n}\right) + 1\right)$$

But $\cos(2x) = 2\cos^2(x) - 1$, we find that

$$r_E = \cos\left(\frac{\pi}{n}\right) R_E$$

which is the desired formula. □

Now, we turn to the hyperbolic plane and give a formula relating the radii of two concentric circles when there is a bicentric embedded n-gon between them in the hyperbolic plane.

Theorem 3.1.3. *Let C and D be two disjoint concentric circles in the hyperbolic plane, with D inside C. Let R denote the radius of C, and r denote the radius of D. Assume that there is a bicentric embedded n-gon between them, then, the relation between R and r can be given as follow*

$$\tanh(r) = \cos\left(\frac{\pi}{n}\right) \tanh(R) \tag{3.3}$$

Proof. We can prove this formula directly by using a hyperbolic right triangle identity 4, Theorem 2.1.4.

Assume that C and D are two disjoint concentric circles with D inside C in

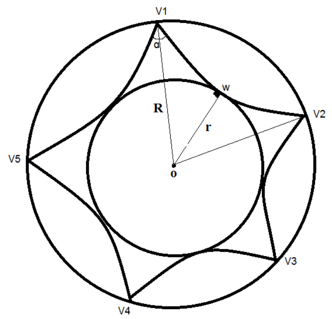

Figure 3.2: Hyperbolic Regular Embedded n-gon

Poincaré disk. Suppose there is a bicentric hyperbolic embedded n-gon P between them, so, P is a regular n-gon. Let v_j be vertices of P such that $1 \leqslant j \leqslant n$, see Figure 3.2. Assume that w is the point on the n-gon's side where the perpendicular from the circles' centre O meets the side, thus the hyperbolic distance $p(Ow) = r$. Using identity 4, Theorem 2.1.4, in the right triangle wOv_1,

$$\tanh(r) = \cos\left(\frac{\pi}{n}\right)\tanh(R)$$

□

For example, when $n = 3$, the polygon is triangle, then, the relations between the inscribed circle's radius r and the circumscribed circle's radius R are:
$R = 2r$ (in the Euclidean plane) and $\tanh(R) = 2\tanh(r)$ (in the hyperbolic plane).

3.1.2 Non-Embedded n-gon's Formulae

We now prove formulae relating the radii of two concentric circles when there is a regular non-embedded n-gon of any kind between them, in the Euclidean and the hyperbolic planes.

The regular non-embedded n-gon is defined as follow

Definition 3.1.4. *[16](Regular non-embedded n-gon)*
A regular non-embedded n-gon $\{n/q\}$, with n, q positive integers, $n > 2$, is a figure

formed by connecting with lines every qth point out of n regularly spaced points lying on a circumference, which divides the circumference into n equal parts. It is sometimes called, a regular n-gram.

This definition for both Euclidean and hyperbolic non-embedded n-gon. There are potentially many different ones, when $q = 1$, we have the regular embedded polygon. The number q is called the density of the non-embedded n-gon and without loss of generality, we take $q < n/2$.

Also, to still have n-gon with a closed path, (n, q) should be relatively prime which means that $gcd(n, q) = 1$, where gcd denotes the greatest common divisor.

The following theorem gives a formula connecting the radii of two concentric circles, when there is a bicentric non-embedded n-gon between them, in the Euclidean plane.

Theorem 3.1.5. *Let C and D be two disjoint concentric circles in the Euclidean plane, with D inside C. Let R_E denote the radius of C and r_E denote the radius of D. Assume their is a bicentric non-embedded n-gon between them, then, the relation between R_E and r_E can be given as follow*

$$r_E = \cos\left(\frac{q\pi}{n}\right) R_E \tag{3.4}$$

where q is the density of the non-embedded n-gon.

Proof. Assume that C and D are two disjoint concentric circles with D inside C in the Euclidean plane. Suppose that there is a bicentric non-embedded n-gon P between them, with density q, so P is regular. Let v_j be vertices of P such that $1 \leqslant j \leqslant n$, and let a denote the n-gon's side, see Figure 3.3.
Assume that w is the point on the side a, where the perpendicular from the circles' centre O meets the side, thus $|Ow| = r_E$.
As the n-gon is regular, w sits at the middle of a, as a result of that

$$r_E = \frac{1}{2} R_E |1 + e^{i2\pi(\frac{q}{n})}|$$

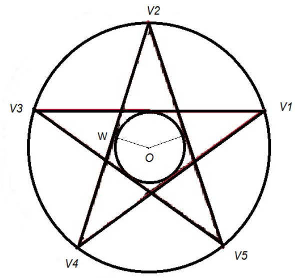

Figure 3.3: Euclidean Regular Non-embedded n-gon

Applying the Euclidean distance definition

$$r_E = \frac{1}{2} R_E \sqrt{\left(1 + \cos\left(\frac{2\pi q}{n}\right)\right)^2 + \sin^2\left(\frac{2\pi q}{n}\right)}$$

After applying the trigonometric identities and rearranging, we find that

$$\cos\left(\frac{q\pi}{n}\right) = \frac{r_E}{R_E}$$

which is the desired formula. □

As well as, we get a formula relating the radii of two concentric circles, when there is a bicentric non-embedded n-gon between them in the hyperbolic plane.

Theorem 3.1.6. *Let C and D be two disjoint concentric circles in the hyperbolic plane, with D inside C. Let R denote the radius of C and r denote the radius of D. Assume that their is a bicentric non-embedded n-gon between them, then the relation between R and r can be given as follow*

$$\tanh(r) = \cos\left(\frac{q\pi}{n}\right) \tanh(R) \tag{3.5}$$

Proof. we can get the formula by using hyperbolic right triangle identities.
Assume that C and D are two disjoint concentric circles with D inside C in

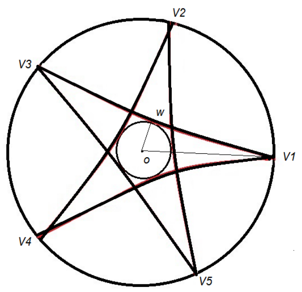

Figure 3.4: Hyperbolic Regular Non-embedded n-gon

Poincaré disk. Suppose that there is a bicentric hyperbolic non-embedded n-gon P between them with density q. Thus, P is regular. Let v_j be vertices of P, such that, $1 \leqslant j \leqslant n$. Let a be the side of the polygon, see Figure 3.4. Assume that w is the point on the side a where the perpendicular from the circles' centre O meets the side, so the hyperbolic distance $p(Ow) = r$, where r is the radius of the inscribed circle D. It is obvious that the angle v_1Ow equals $\left(\frac{q\pi}{n}\right)$, then by using identity 4, Theorem 2.1.4, on the right triangle v_1Ow, we get the desired formula

$$\cos\left(\frac{q\pi}{n}\right) = \frac{\tanh(r)}{\tanh(R)}$$

□

At this point, we can see that for these simple cases when the circles are concentric, when we look at the relation (3.3) between the hyperbolic radii R and r of two concentric hyperbolic circles when there is a bicentric non-embedded n-gon between them

$$\tanh(r) = \cos\left(\frac{\pi}{n}\right)\tanh(R)$$

and let $R \longrightarrow 0$, we can see that $\lim_{R\to 0} \frac{\tanh(R)}{R} = 1$, which means $\tanh(R) \simeq R$, as well $\tanh(r) \simeq r$ for very small R and r, and the relation will be close to

$$r = \cos\left(\frac{\pi}{n}\right) R$$

which is the same relation (3.1) in the Euclidan plane.
That gives a clue for the conjecture in Section 4.4, that Euclidean formulae should appear as a factor of hyperbolic formulae when R approaches 0.

3.2 Chapple's Formula in the Hyperbolic Geometry

In this section, we prove Chapple's Formula in the hyperbolic plane. This formula determines a relation between the circumscribed circle's radius, the inscribed circle's radius and the distance between the two circles' centres, when there is a hyperbolic bicentric triangle between them. We know that Euclidean Chapple's Formula states that when C and D are two disjoint circles in the Euclidean plane, with D inside C, there is a bicentric triangle between the circles if and only if

$$d_E^2 = R_E^2 - 2r_E R_E$$

As we mentioned in the first chapter, the proof of the Euclidean Chapple's Formula was clear, using some Euclidean facts such as the similarity of the triangles, the fact that the interior angles of a triangle add up to 2π and also using the intersecting chords theorem. Whereas, in the hyperbolic geometry, these facts can not be used.

To prove the hyperbolic version, we need the following nice theorem which relating the hyperbolic radius of the inscribed circle with the interior angles of the triangle.

Theorem 3.2.1. *[2] The radius r of the inscribed circle of a triangle T is given by*

$$\tanh^2(r) = \frac{\cos^2(\alpha) + \cos^2(\beta) + \cos^2(\gamma) + 2\cos(\alpha)\cos(\beta)\cos(\gamma) - 1}{2(1+\cos(\alpha))(1+\cos(\beta))(1+\cos(\gamma))} \tag{3.6}$$

where α, β and γ are the interior angles of the triangle T

Proof. This theorem was proved, by using some hyperbolic trigonometric iden-

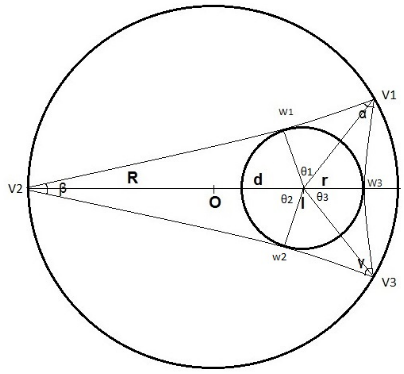

Figure 3.5: Hyperbolic Chapple's Formula

tities, with hyperbolic law of cosine on the triangle to obtain a relation, next, rearranging this relation, by using right triangle identities. □

Now, we prove Chapple's Formula in the hyperbolic plane, by using the previous theorem, and doing some calculations using the hyperbolic trigonometric identities to deduce the formula.

Theorem 3.2.2. *(Chapple's Formula in the Hyperbolic Plane)*
Let C and D be two disjoint circles in the hyperbolic plane, with D inside C, suppose that T is a hyperbolic triangle between them. Let R denote the radius of C, r denote the radius of D and d be the distance between the circles' centres, then we have the following relation between R, r and d.

$$\tanh(r) = \frac{\tanh(R+d)\left(\cosh^2(R)\left(\sinh^2(r)-1\right)+\cosh^2(r+d)\right)}{\cosh^2(r+d)-\cosh^2(R)\cosh^2(r)} \tag{3.7}$$

Proof. To make the proof easier, we use a convenient case where the triangle is isosceles. Then, using the hyperbolic triangle identities to find the interior angles of the triangle in terms of R, r and d. After that, we use the previous theorem and doing some calculations using the hyperbolic trigonometric functions to deduce the formula.

We shall use the standard method mentioned in Remark 3.0.1, and state a triangle

T with vertices v_j, $1 \leqslant j \leqslant 3$. Let a_j be the hyperbolic length of its adjacent side, and let α, β and γ be its interior angles, respectively. Set the triangle so that one of its vertices say v_2, is at the same line with the circumscribed circle's centre and the inscribed circle's centre, thus, the triangle line of symmetry passes through the centres of the circles, as a result of that, the triangle is isosceles, then $\alpha = \gamma$. Assume that w_j is the point on the side a_j where the perpendicular from the inscribed circle's centre I meets the side; see Figure 3.5.

By using Theorem 3.2.1 in this situation when the triangle is isosceles, we find that

$$\tanh^2(r) = \frac{2\cos^2(\alpha) + \cos^2(\beta) + 2\cos^2(\alpha)\cos(\beta) - 1}{2(1+\cos(\alpha))^2(1+\cos(\beta))}$$

Hence

$$\tanh^2(r) = \frac{2\cos^2(\alpha)(\cos(\beta)+1) + (\cos(\beta)+1)(\cos(\beta)-1)}{2(1+\cos(\alpha))^2(1+\cos(\beta))}$$

After simplification,

$$\tanh^2(r) = \frac{2\cos^2(\alpha) + \cos(\beta) - 1}{2(1+\cos(\alpha))^2}$$

which means that

$$\tanh^2(r) = \frac{2(\cos^2(\alpha) - 1) + \cos(\beta) + 1}{2(1+\cos(\alpha))^2}$$

After some simplification,

$$\tanh^2(r) = \frac{\cos(\alpha) - 1}{\cos(\alpha) + 1} + \frac{\cos(\beta) + 1}{2(\cos(\alpha) + 1)^2} \tag{3.8}$$

Thus, we need to find $\cos(\alpha), \cos(\beta)$ in terms to solve (3.8).

Now, doing some calculations to find $\cos(\alpha)$,
the triangle $v_1 I w_3$ is a right triangle, so from 2, Theorem 2.1.4,

$$\tanh(r) = \sinh\left(\frac{a_3}{2}\right)\tan\left(\frac{\alpha}{2}\right) \tag{3.9}$$

Using the relation 1, Theorem 2.1.4, on the triangle $v_1 O w_3$

$$\cosh\left(\frac{a_3}{2}\right) = \frac{\cosh(R)}{\cosh(d+r)} \tag{3.10}$$

together with the identity, $\cosh^2(x) - \sinh^2(x) = 1$

$$\sinh^2\left(\frac{a_3}{2}\right) = \cosh^2\left(\frac{a_3}{2}\right) - 1 = \frac{\cosh^2(R)}{\cosh^2(d+r)} - 1$$

we obtain from (3.9) that

$$\tan\left(\frac{\alpha}{2}\right) = \frac{\tanh(r)\cosh(d+r)}{\sqrt{\cosh^2(R) - \cosh^2(d+r)}}$$

and we know that

$$\cos(x) = \frac{1}{\sqrt{1+\tan^2(x)}}$$

So,

$$\cos\left(\frac{\alpha}{2}\right) = \sqrt{\frac{\cosh^2(R) - \cosh^2(d+r)}{\cosh^2(R) - \cosh^2(d+r)\,\mathrm{sech}^2(r)}} \tag{3.11}$$

by using double-angle identity, $\cos(\alpha) = 2\cos^2(\frac{\alpha}{2}) - 1$, we find that

$$\cos(\alpha) = 2\frac{\cosh^2(R) - \cosh^2(d+r)}{\cosh^2(R) - \cosh^2(d+r)\,\mathrm{sech}^2(r)} - 1$$

Thus,

$$\cos(\alpha) = \frac{\cosh^2(R) - 2\cosh^2(d+r) + \cosh^2(d+r)\,\mathrm{sech}^2(r)}{\cosh^2(R) - \cosh^2(d+r)\,\mathrm{sech}^2(r)} \tag{3.12}$$

Turning now to find $\cos(\beta)$,
the triangle $v_2 I w_1$ is a right triangle, so applying the relation 3, Theorem 2.1.4,

$$\sin\left(\frac{\beta}{2}\right) = \frac{\sinh(r)}{\sinh(R+d)} \tag{3.13}$$

So

$$\cos^2\left(\frac{\beta}{2}\right) = \frac{\sinh^2(R+d) - \sinh^2(r)}{\sinh^2(R+d)} \tag{3.14}$$

Now, by using (3.12) and (3.14) in (3.8) with rearranging, we find that the first part of the right hand side of (3.8) is

$$\frac{\cos(\alpha)-1}{\cos(\alpha)+1} = \frac{2(\cosh^2(d+r)\,\mathrm{sech}^2(r) - \cosh^2(d+r))}{2(\cosh^2(R) - \cosh^2(d+r))}$$

but by using identity 2, Theorem 2.1.2, we find that

$$\frac{\cos(\alpha)-1}{\cos(\alpha)+1} = \frac{-\cosh^2(d+r)\tanh^2(r)}{\cosh^2(R) - \cosh^2(d+r)}$$

(3.15)

Also, after calculation, we find that the second part of the right hand side of (3.8) equals

$$\frac{\cos(\beta)+1}{2(\cos(\alpha)+1)^2} = \frac{(\cosh^2(R+d) - \cosh^2(r))(\cosh^2(R)\cosh^2(r) - \cosh^2(d+r))^2}{4\cosh^4(r)\sinh^2(R+d)(\cosh^2(R) - \cosh^2(d+r))^2} \tag{3.16}$$

Then from (3.15) and (3.16) in (3.8) with rearranging

$$\frac{\tanh^2(r)\cosh^2(R)}{\cosh^2(R) - \cosh^2(d+r)} = \frac{(\cosh^2(R+d) - \cosh^2(r))(\cosh^2(R)\cosh^2(r) - \cosh^2(d+r))^2}{4\cosh^4(r)\sinh^2(R+d)(\cosh^2(R) - \cosh^2(d+r))^2}$$

after arranging

$$\tanh^2(r) = \frac{(\cosh^2(R+d) - \cosh^2(r))(\cosh^2(R)\cosh^2(r) - \cosh^2(d+r))^2}{4\cosh^2(R)\cosh^4(r)\sinh^2(R+d)(\cosh^2(R) - \cosh^2(d+r))} \tag{3.17}$$

which presents a relation between the circles' data, when there is a triangle between them.

We make it more simpler, by using Theorem 4.1.1 from Chapler 4, we find that the following relation satisfies, when there is a bicentric triangle between two circles

$$2\tan(\theta_1) + \tan(\theta_2) - \tan^2(\theta_1)\tan(\theta_2) = 0 \tag{3.18}$$

where θ_i denotes the angle $v_i I w_i$, $1 \leqslant i \leqslant 3$, such that, from the hyperbolic right triangle identities,

$$\tan(\theta_1) = \frac{\sqrt{\cosh^2(R) - \cosh^2(r+d)}}{\sinh(r)\cosh(R)}$$

$$\tan(\theta_2) = \frac{\sqrt{\tanh^2(R+d) - \tanh^2(r)}}{\tanh(r)}$$

Using these information in (3.18) with squaring and rearranging, we find that,

$$\tanh^2(r) = \frac{\tanh^2(R+d)\left(\cosh^2(R)(\sinh^2(r) - 1) + \cosh^2(r+d)\right)^2}{\left(\cosh^2(R)\cosh^2(r) - \cosh^2(r+d)\right)^2} \tag{3.19}$$

which means that

$$\tanh(r) = \frac{\tanh(R+d)\left(\cosh^2(R)(\sinh^2(r) - 1) + \cosh^2(r+d)\right)}{\cosh^2(r+d) - \cosh^2(R)\cosh^2(r)} \tag{3.20}$$

is the desired formula, as $\tanh(r) > 0$. □

3.3 Comparing the Hyperbolic Chapple's Formula with the Euclidean one

In this section, we introduce a relation between Chapple's Formulae (the hyperbolic and the Euclidean) by using numerical calculations.

There is evidence that a very small area of the hyperbolic plane appears more Euclidean.
Suppose that T is a hyperbolic triangle circumscribed by a circle of radius R in a very small area of the hyperbolic plane. Let α, β and γ be its interior angles, and

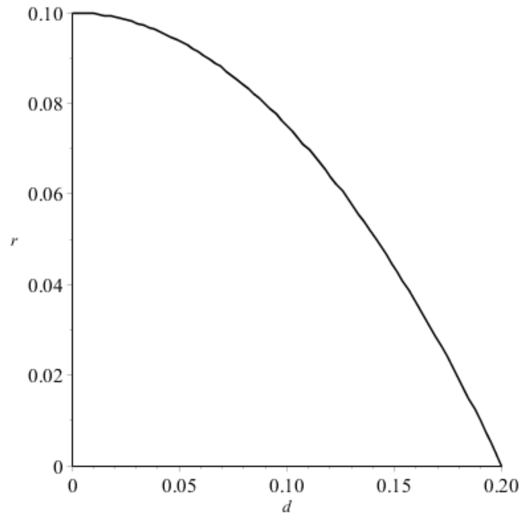

Figure 3.6: Euclidean Chapple's Formula when $R = .20$

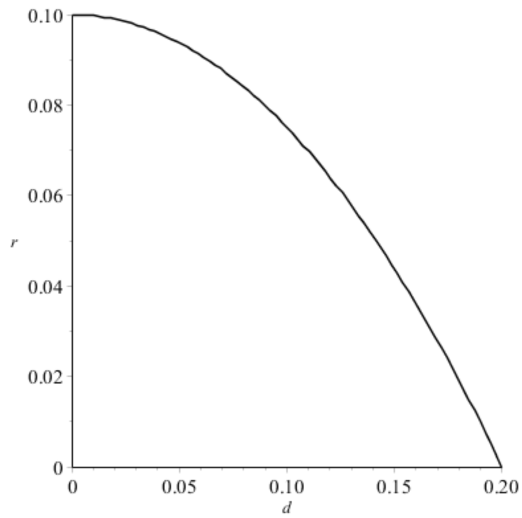

Figure 3.7: Hyperbolic Chapple's Formula when $R = .20$

let A denote the area of that triangle, when letting $R \longrightarrow 0$, we can see $A \longrightarrow 0$ as a consequence, because T lies in a very small area. However, from Gauss-Bonnet Theorem

$$A = \pi - (\alpha + \beta + \gamma)$$

as A is getting close to 0, we find that $\alpha + \beta + \gamma$ is getting close to π.
So, T approaches sort of a Euclidean triangle, then, if we look at a very small piece of the hyperbolic plane, it starts to look more and more Euclidean.

We therefore, have a hypothesis that if we look at hyperbolic Poncelet's Theorem on a very small circle with fixed small R, the relationship between the inscribed circle's radius r and the distance between the circles' centres d, should start looking like the Euclidean relationship. Our observation depends on numerical calculations that there is a relationship between hyperbolic Chapple's Formula and Euclidean one, as Maple shows that if we take a very small circumscribed circle with radius R, and look at the relation we get between r and d, we can see that the hyperbolic version is getting very close to the Euclidean one, as R is close to 0, see Figures 3.6 and 3.7.
Also, we know from the Taylor series expressions that:

$$\sinh(x) = x + \frac{x^3}{3!} + \frac{x^5}{5!} + ...$$

$$\cosh(x) = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + ...$$

$$\tanh(x) = x - \frac{x^3}{3} + \frac{2x^5}{15} + ...$$

So, when we take the hyperbolic Chapple's Formula

$$\tanh(r) = \frac{\tanh(R+d)\left(\cosh^2(R)(\sinh^2(r) - 1) + \cosh^2(r+d)\right)}{\cosh^2(r+d) - \cosh^2(R)\cosh^2(r)} \tag{3.21}$$

we can write it as

$$\cosh^2(r+d)\sinh(r-R-d) - \cosh(r)\cosh^2(R)(\sinh(r)\cosh(r+R+d) - \sinh(R+d)) = 0 \tag{3.22}$$

Let $R \longrightarrow 0$, we can see that $\lim_{R\to 0} \frac{\cosh(R)}{1+\frac{R^2}{2}} = 1$, $\lim_{R\to 0} \frac{\sinh(R)}{R} = 1$, which means that $\cosh(R) \simeq 1 + \frac{R^2}{2}$, $\sinh(R) \simeq R$, for very small R, as well, this apply to r and d,
with some calculations, we can see that hyperbolic Chapple's Formula closes to the relation

$$r\left(r^2 + R^2 - (r+d)^2\right) + (R+d)\left(r^2 - R^2 + (r+d)^2\right) = 0 \tag{3.23}$$

for R sufficient small.
Doing some calculations on the lowest order terms, we can see that it equals to

$$-(R^2 - 2Rr - d^2)(d + R + r) \tag{3.24}$$

where $(R^2 - 2Rr - d^2) = 0$ is the Euclidean Chapple's Formula.
So, the Euclidean Chapple Formula appears as a factor of the lowest order terms of the analogous hyperbolic, where R approaches 0, this leads to the conjecture in Section 4.4.

3.4 Fuss's Formula in the Hyperbolic Geometry

In this section, we introduce Fuss's Formula for quadrilaterals in the hyperbolic plane. We know that Euclidean Fuss's Formula states that if C and D are two disjoint circles in the Euclidean plane, with D inside C, such that R_E denotes the radius of C, r_E denotes the radius of D and d_E denotes the distance between the circles' centres, then there is a bicentric quadrilateral between them if and only if

$$(R_E^2 - d_E^2)^2 = 2r_E^2(R_E^2 + d_E^2)$$

As we mentioned in the second chapter, the proof of Euclidean Fuss's Formula uses some Euclidean facts, such as Pythagorean Theorem, the opposite angles of a cyclic quadrilateral are supplementary, the fact that the interior angles of a triangle add up to 2π and also using intersecting chords theorem, however, these facts can not be used in the hyperbolic geometry.

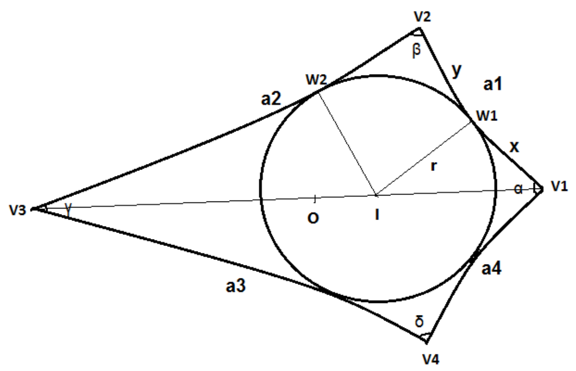

Figure 3.8: Hyperbolic Kite

We need the following lemma, which relating the radius of the inscribed circle with the interior angles of a kite. The proof of this lemma has the same basic structure as Theorem 3.2.1.
We define the hyperbolic kite, as we use it in the proof to make the calculations easier, by setting the quadrilateral in such way to be a kite.

Definition 3.4.1. *(Hyperbolic Kite)*
A hyperbolic kite is a quadrilateral with two pairs of equal length adjacent sides, see Figure 3.8.

Lemma 3.4.1. *The radius r of the inscribed circle of a hyperbolic kite K is given by*

$$\tanh^2(r) = \frac{\cos^2(\frac{\alpha}{2}) + \cos^2(\beta) + \cos^2(\frac{\gamma}{2}) + 2\cos(\frac{\alpha}{2})\cos(\beta)\cos(\frac{\gamma}{2}) - 1}{(1+\cos(\beta))(1+\cos(\beta)+2\cos(\frac{\alpha}{2})\cos(\frac{\gamma}{2}))} \tag{3.25}$$

where α, β and γ are the interior angles of the kite K such that α and γ are the angles between the two equal length sides.

Proof. To prove this lemma, we firstly use some hyperbolic trigonometric identities with hyperbolic law of cosine on the triangle $v_1v_2v_3$ to obtain a relation, then rearranging this relation and using trigonometric identities and right triangle identities on it.

Let K be a hyperbolic kite in Poincaré disk D, with vertices v_j, $1 \leqslant j \leqslant 4$, see Figure 3.8. Let a_j be the hyperbolic length of its adjacent side and let α, β, γ and δ be its interior angles, respectively. Because it is a kite, $a_1 = a_4$, $a_2 = a_3$ and $\beta = \delta$. Assume that w_i is the point on the side a_i, where the perpendicular from the inscribed circle's centre I meets the side, thus, the hyperbolic distance $p(Iw_i) = r$.

Set $p(v_1w_1) = x$ and $p(v_2w_1) = y$, by using identity 3, Theorem 2.1.2, and the hyperbolic law of cosines II, Theorem 2.1.3, on the triangle $v_1v_2v_3$, we get that

$$\cosh(x+y) = \cosh(x)\cosh(y) + \sinh(x)\sinh(y) = \frac{\cos(\frac{\alpha}{2})\cos(\beta) + \cos(\frac{\gamma}{2})}{\sin(\frac{\alpha}{2})\sin(\beta)}$$

Rearranging

$$\cos\left(\frac{\alpha}{2}\right)\cos(\beta)+\cos\left(\frac{\gamma}{2}\right)-\sin\left(\frac{\alpha}{2}\right)\sin(\beta)\sinh(x)\sinh(y) = \sin\left(\frac{\alpha}{2}\right)\sin(\beta)\cosh(x)\cosh(y)$$

Squaring both sides and using trigonometric identities

$$\left(\cos\left(\frac{\alpha}{2}\right)\cos(\beta) + \cos\left(\frac{\gamma}{2}\right) - \sin\left(\frac{\alpha}{2}\right)\sinh(x)\sin(\beta)\sinh(y)\right)^2 =$$

$$\sin^2\left(\frac{\alpha}{2}\right)(1+\sinh^2(x))\sin^2(\beta)(1+\sinh^2(y))$$

Using the identity $\cos^2(x) + \sin^2(x) = 1$ and rearranging

$$\left(\cos\left(\frac{\alpha}{2}\right)\cos(\beta) + \cos\left(\frac{\gamma}{2}\right) - \sin\left(\frac{\alpha}{2}\right)\sinh(x)\sin(\beta)\sinh(y)\right)^2 =$$

$$\left(\left(1-\cos^2\left(\frac{\alpha}{2}\right)\right) + \sin^2\left(\frac{\alpha}{2}\right)\sinh^2(x)\right)\left((1-\cos^2(\beta)) + (\sin^2(\beta)\sinh^2(y)\right) \tag{3.26}$$

By using trigonometric identities, we find that

$$\sin(\beta) = (1+\cos(\beta))\tan\left(\frac{\beta}{2}\right) \tag{3.27}$$

also, by using identity 2, Theorem 2.1.4, on the right triangle v_2Iw_1, we can see that

$$\tanh(r) = \sinh(y)\tan\left(\frac{\beta}{2}\right) \tag{3.28}$$

(3.27) and (3.28) yield that

$$\sin(\beta)\sinh(y) = (1+\cos(\beta))\tanh(r) \tag{3.29}$$

On the other hand, using identity (3.28) on the right triangle $v_1 I w_1$ with the identity

$$\tan(\alpha) = \frac{\sin(\alpha)}{\cos(\alpha)}$$

yield that

$$\sin\left(\frac{\alpha}{2}\right)\sinh(x) = \tanh(r)\cos\left(\frac{\alpha}{2}\right) \tag{3.30}$$

Substitution from (3.29) and (3.30) in (3.26)

$$\left(\left(\cos\left(\frac{\alpha}{2}\right)\cos(\beta)+\cos\left(\frac{\gamma}{2}\right)\right)-\left(\tanh(r)\cos\left(\frac{\alpha}{2}\right)\right)((1+\cos(\beta))\tanh(r)\right)^2 =$$

$$\left(\left(1-\cos^2\left(\frac{\alpha}{2}\right)\right)+\left(\tanh^2(r)\cos^2\left(\frac{\alpha}{2}\right)\right)\right)\left(\left(1-\cos^2(\beta)\right)+(1+\cos(\beta))^2\tanh^2(r)\right)$$

Squaring and simplifying

$$\cos^2\left(\frac{\alpha}{2}\right)\cos^2(\beta)+\cos^2\left(\frac{\gamma}{2}\right)+2\cos\left(\frac{\alpha}{2}\right)\cos(\beta)\cos\left(\frac{\gamma}{2}\right)$$

$$-2\cos\left(\frac{\alpha}{2}\right)(1+\cos(\beta))\left(\cos\left(\frac{\alpha}{2}\right)\cos(\beta)+\cos\left(\frac{\gamma}{2}\right)\right)\tanh^2(r) = \left(1-\cos^2\left(\frac{\alpha}{2}\right)\right)\left(1-\cos^2(\beta)\right)$$

$$+\tanh^2(r)\cos^2\left(\frac{\alpha}{2}\right)\left(1-\cos^2(\beta)\right)+(1+\cos(\beta))^2\tanh^2(r)\left(1-\cos^2\left(\frac{\alpha}{2}\right)\right)$$

After simplification, we find that

$$\tanh^2(r) = \frac{\cos^2(\frac{\alpha}{2})+\cos^2(\beta)+\cos^2(\frac{\gamma}{2})+2\cos(\frac{\alpha}{2})\cos(\beta)\cos(\frac{\gamma}{2})-1}{(1+\cos(\beta))(1+\cos(\beta)+2\cos(\frac{\alpha}{2})\cos(\frac{\gamma}{2}))}$$

which is the desired formula. □

Now, we try to prove Fuss's Formula in the hyperbolic plane, by using the previous lemma, and doing some calculations using the hyperbolic trigonometric functions to deduce the formula.

Theorem 3.4.2. *(Fuss's Formula in the Hyperbolic Geometry)*
Let C and D be two disjoint circles in the hyperbolic plane, with D inside C, suppose that K is a hyperbolic quadrilateral between them. Let R denote the radius of C, r denote the radius of D and d be the distance between the circles' centres. Then, we have the following relation between R, r and d

$$\tanh^4(r) = \frac{\left(s^2(R-d)-s^2(r)\right)^2\left(s^2(R+d)-s^2(r)\right)^2}{c^4(r)\Big(\left(c^4(R)+s^4(d)\right)\left(s^2(R-d)-s^2(r)\right)\left(s^2(R+d)-s^2(r)\right)-s^4(r)c^4(R)s^4(d)\Big)} \tag{3.31}$$

where $s(x)=\sinh(x)$ *and* $c(x)=\cosh(x)$

Proof. To make the proof easier, we use a convenient case where the quadrilateral is a kite, then, using the hyperbolic triangle identities to find the interior angles of the kite, in terms of R, r and d. After that, we use the previous lemma and doing some calculations, using the hyperbolic trigonometric functions to deduce the formula.

We shall use the standard way, Remark 3.0.1, and state a quadrilateral K with vertices v_j, $1 \leqslant j \leqslant 4$. Let a_j be the hyperbolic lengths of its adjacent sides, and let α ,β, γ and δ be its interior angles, respectively. Set the quadrilateral so that one of its vertices say v_1 is at the same line with the circumscribed circle's centre and the inscribed circle's centre. Thus, the quadrilateral line of symmetry passes through the centres of the circles, as a result of that the quadrilateral is a kite, then $a_1 = a_4$, $a_2 = a_3$ and $\beta = \delta$. Assume that w_i is the point on the side a_i where the perpendicular from the inscribed circle's centre I meets the side, see Figure 3.8.

We now need to find $\cos\left(\frac{\alpha}{2}\right)$, $\cos(\beta)$ and $\cos\left(\frac{\gamma}{2}\right)$ to use it in (3.25), which lead us to the desired formula.
By using identity 3, Theorem 2.1.4, on the right triangle $v_1 I w_1$, we can see that

$$\sin\left(\frac{\alpha}{2}\right) = \frac{\sinh(r)}{\sinh(R-d)} \tag{3.32}$$

So, $\cos(\frac{\alpha}{2}) = \sqrt{1 - \frac{\sinh^2(r)}{\sinh^2(R-d)}}$

$$\cos\left(\frac{\alpha}{2}\right) = \frac{\sqrt{\sinh^2(R-d) - \sinh^2(r)}}{\sinh(R-d)} \tag{3.33}$$

Also, to find $\cos\left(\frac{\gamma}{2}\right)$, we use the same identity on the right triangle $v_3 I w_2$, then, we can see that

$$\sin\left(\frac{\gamma}{2}\right) = \frac{\sinh(r)}{\sinh(R+d)} \tag{3.34}$$

thus,

$$\cos\left(\frac{\gamma}{2}\right) = \frac{\sqrt{\sinh^2(R+d) - \sinh^2(r)}}{\sinh(R+d)} \tag{3.35}$$

Now, turning on to find $\cos(\beta)$

$$\cos(\beta) = \cos\left(\frac{\alpha+\gamma}{2}\right) = \cos\left(\frac{\alpha}{2}\right)\cos\left(\frac{\gamma}{2}\right) - \sin\left(\frac{\alpha}{2}\right)\sin\left(\frac{\gamma}{2}\right)$$

so, from (3.32), (3.33), (3.34) and (3.35) we find that

$$\cos(\beta) = \frac{\sqrt{\left(\sinh^2(R+d) - \sinh^2(r)\right)\left(\sinh^2(R-d) - \sinh^2(r)\right)} - \sinh^2(r)}{\sinh(R+d)\sinh(R-d)} \tag{3.36}$$

Using the equations (3.32), (3.34) and (3.36) in (3.25), to find Fuss's Formula in the hyperbolic plane.

After rearranging and simplification, the relation will be

$$\tanh^4(r) = \frac{\left(s^2(R-d) - s^2(r)\right)^2\left(s^2(R+d) - s^2(r)\right)^2}{c^4(r)\Big(\left(c^4(R) + s^4(d)\right)\left(s^2(R-d) - s^2(r)\right)\left(s^2(R+d) - s^2(r)\right) - s^4(r)c^4(R)s^4(d)\Big)} \tag{3.37}$$

where $s(x) = \sinh(x)$ and $c(x) = \cosh(x)$. □

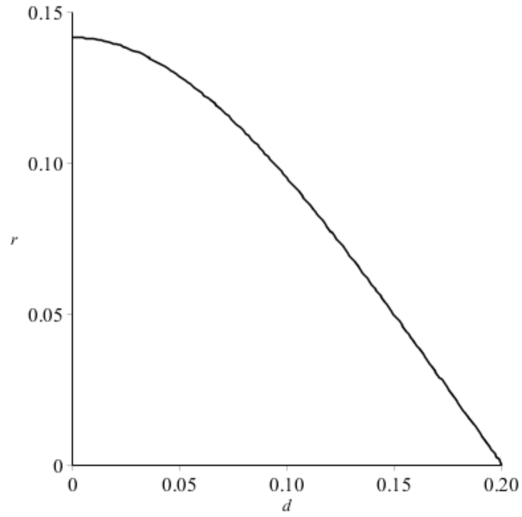

Figure 3.9: Euclidean Fuss's Formula when $R = .20$

3.5 The Relation Between the Hyperbolic and the Euclidean Fuss's Formulae

Completing on what we have done with Chapple's Formula, in this section we compare the hyperbolic Fuss's Formula and the Euclidean one, using numerical calculations. As what have been discussed, we have a fact that the very small piece of the hyperbolic plane looks more Euclidean, therefore, when we take a very small circumscribed circle's radius R, and consider the $2D$ curve which comes out by plotting the relation between d and r in the hyperbolic Fuss's Formula with fixed small R, it can be seen that this curve is very close to the curve that expresses the relation between r and d in the Euclidean Fuss's Formula. That gives us a numerical evidence about the relation between the hyperbolic Fuss's Formula and the Euclidean one.

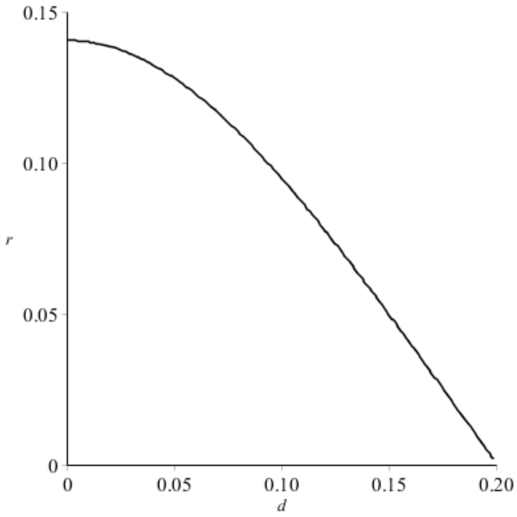

Figure 3.10: Hyperbolic Fuss's Formula when $R = .20$

On the other hand, when we take hyperbolic Fuss's Formula,

$$\tanh^4(r) = \frac{\left(s^2(R-d) - s^2(r)\right)^2 \left(s^2(R+d) - s^2(r)\right)^2}{c^4(r)\Big(\left(c^4(R) + s^4(d)\right)\left(s^2(R-d) - s^2(r)\right)\left(s^2(R+d) - s^2(r)\right) - s^4(r)c^4(R)s^4(d)\Big)} \tag{3.38}$$

where $s(x) = \sinh(x)$ and $c(x) = \cosh(x)$, it can be written as

$$s^4(r)\Big(\left(c^4(R) + s^4(d)\right)\left(s^2(R-d) - s^2(r)\right)\left(s^2(R+d) - s^2(r)\right) - s^4(r)c^4(R)s^4(d)\Big) - \left(s^2(R-d) - s^2(r)\right)^2 \left(s^2(R+d) - s^2(r)\right)^2 = 0 \tag{3.39}$$

Let $R \longrightarrow 0$, we find that $\lim_{R\to 0} \frac{\cosh(R)}{1+\frac{R^2}{2}} = 1$, $\lim_{R\to 0} \frac{\sinh(R)}{R} = 1$, so $\cosh(R) \simeq 1 + \frac{R^2}{2}$, $\sinh(R) \simeq R$, for R sufficient small.

with some calculations, we can see that Fuss's Formula closes to the relation

$$r^4\left((1+d^4)\left((R-d)^2 - r^2\right)\left((R+d)^2 - r^2\right) - r^4d^4\right) - \left(\left((R-d)^2 - r^2\right)\left((R+d)^2 - r^2\right)\right)^2 \tag{3.40}$$

with doing some calculations, on the lowest order terms we can see that it equals to

$$\left((R^2-d^2)^2+r^4-2r^2(R^2+d^2)\right)\left((R^2-d^2)^2-2r^2(R^2+d^2)\right) \tag{3.41}$$

where $(R^2-d^2)^2-2r^2(R^2+d^2)=0$ is the Euclidean Fuss's Formula.
Thus, the hyperbolic Fuss's Formula include the analogues Euclidean formula as a factor of its lowest order terms, when R approaches 0, this presents a further evidence of the conjecture in Section 4.4.

3.6 Future Work

In this chapter, we introduce and prove hyperbolic Chapple Formula which satisfies when there is a hyperbolic triangle inscribed in one circle and circumscribed around another circle. We need to know the conditions for the hyperbolic triangle to be bicentric, also, if hyperbolic Chapple Formula satisfies for two circles, is there a bicentric triangle between these circles. On the same way, we present hyperbolic Fuss Formula which satisfies when there is a hyperbolic quadrilateral inscribed in one circle and circumscribed around another circle. However, we require to find the conditions for the hyperbolic quadrilateral to be bicentric, besides, if hyperbolic Fuss Formula satisfies for two circles, is there a bicentric quadrilateral between them. On the other hand, future work will also involve finding formulae connecting the data of two circles when there is a bicentric embedded and non-embedded pentagon between them and investigate if there is any relation between these formulae.

Chapter 4

General Formulae for Embedded n-gons

In the first section of the previous chapter, we have proved general formulae relating the quantities of two circles, when there is a bicentric n-gon inscribed in one circle and circumscribed around the other in the Euclidean and the hyperbolic planes for the special cases when circles are concentric.

Now, we work more generally to introduce general formulae, in the Euclidean and the hyperbolic planes for non-concentric circles, which will give relations between the circles' data, when there is a bicentric embedded n-gon inscribed in one circle and circumscribed around the other. At the beginning, we present a general method, which helps later to prove general expressions connecting the circles' data of a bicentric embedded n-gon in the Euclidean and the hyperbolic geometries.

Then, by using this method, we introduce a general formula relating the quantities of two circles, when there is an embedded n-gon between them, in the Euclidean plane. We follow that by demonstrating two general formulae in the hyperbolic plane, one by applying a lemma, which introduces a relation between the radius of the inscribed circle and the interior angles of an embedded n-gon circumscribed it, and the other by manipulating the general method, which was introduced at the beginning of this chapter.
At the last section, we present and discuss a conjecture, that if we consider the

hyperbolic general formula for an embedded n-gon which is inscribed in one circle and circumscribed around other circle, we can write this formula as a functon in R, d, r, by using the expressions $\cosh(R) \simeq 1 + \frac{R^2}{2}, \sinh(R) \simeq R$ for R small, then, by taking the lowest order non zero terms, we should see the Euclidean equivalent as a factor.

4.1 General Construction

In this section, we prove a general method, which can be used to determine a relation between circles' data in general, when there is an embedded n-gon between them. This method get a direction, to prove a general Euclidean formula, and a general hyperbolic formula.

To prove this method, we use the basic facts that the measure of a full angle equals 2π, and $\tan(\pi) = 0$. Then, we apply the identity of the tangent of a sum [5].
Now, we explain the identity of the tangent of a sum, as we use it in proving the general construction.
Firstly, we define the elementary symmetric polynomials, which denoted by $\sigma_1, \sigma_2, ..., \sigma_n$ in the variables $X_1, X_2, ..., X_n$
they are defined as follow,

$$\sigma_k(X_1, X_2, ..., X_n) = \sum_{1 \leq j_1 < j_2 < \cdots < j_k \leq n} X_{j_1} X_{j_2} ... X_{j_k}$$

where $k = 1, 2, ..., n$.
The polynomial σ_k is called the elementary symmetric polynomial of degree k [48], where

$$\sigma_1(X_1, X_2, ..., X_n) = \sum_{j=1}^{n} X_j$$

$$\sigma_n(X_1, X_2, ..., X_n) = X_1 X_2 ... X_n$$

Now, the identity of the tangent of a sum [5], states that:

$$\tan(\sum_{i=1}^{n}\theta_i) = \frac{\sum_{j\ odd}(-1)^{\frac{(j-1)}{2}}\sigma_j}{1+\sum_{j\ even}(-1)^{\frac{j}{2}}\sigma_j} \tag{4.1}$$

$$= \frac{\sigma_1 - \sigma_3 + \sigma_5 ....}{1 - \sigma_2 + \sigma_4 ....}$$

where $\sigma_1, \sigma_2, ..., \sigma_n$ are the elementary symmetric polynomials in the variables $\tan\theta_1, \tan\theta_2, ..., \tan\theta_n$, so that $\sigma_k = \sigma_k(\tan\theta_1, \tan\theta_2, ..., \tan\theta_n)$.

Next, we present the first theorem, in this chapter, which introduces a general construction, using the previous concepts. In this theorem the elementary symmetric polynomials $\sigma_1, \sigma_2, ..., \sigma_n$ in the variables

$$\tan(\theta_i)$$

for $i = 1, 2, 3, ..., n$, where θ_i are the angles v_iIw_i as described in Figure 4.1.

Theorem 4.1.1. *Let P be an embedded n-gon circumscribed a circle D, then:*

$$\sum_{j=1\ odd}^{n}(-1)^{\frac{(j+1)}{2}}\sigma_j = \sigma_1 - \sigma_3 + \sigma_5 ... \pm \sigma_n = 0 \tag{4.2}$$

when the n-gon has an odd number of sides n, and

$$\sum_{j=1\ odd}^{n-1}(-1)^{\frac{(j+1)}{2}}\sigma_j = \sigma_1 - \sigma_3 + \sigma_5 ... \pm \sigma_{n-1} = 0 \tag{4.3}$$

when the n-gon has an even number of sides n.

Proof. Let P be an embedded n-gon circumscribed a circle D, with vertices v_i, $1 \leqslant i \leqslant n$. Assume that w_i is the point on the polygon's side, where the perpendicular from the inscribed circle's centre I meets the side. Denote θ_i for the angles v_iIw_i, where $1 \leqslant i \leqslant n$, see Figure 4.1.

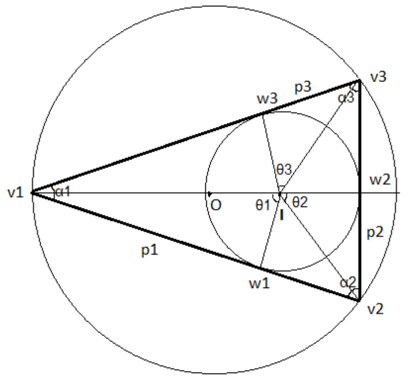

Figure 4.1: Euclidean Embedded n-gon

The triangles v_iIw_{i-1} and v_iIw_i are congruent, thus,

$$2(\theta_1 + \theta_2 + ... + \theta_n) = 2\pi$$

which means,

$$\tan(\sum_{i=1}^{n} \theta_i) = 0 \tag{4.4}$$

From the identity of the tangent of a sum (4.1), we find the desired results. □

4.2 General Formula for Euclidean Embedded n-gon

From the Previous studies, there is a general analytical expression, presented by Richelot in 1830 and Kerawala in 1947, using Jacobi's elliptic function, connecting the data of two circles when there is a bicentric n-gon between them [53]. Also, Nohara and Arimoto [37] in 2012, showed the necessary and sufficient condition for bicentric n-gons which are circumscribed and inscribed by two circles using Jacobian elliptic functions. Moreover, the formulae for a bicentric triangle, quadrilateral and pentagon were presented and the fact that these formulae are the necessary and sufficient conditions for bicentrical n-gons were also presented.

In this section, we use Theorem 4.1.1, in the Euclidean plane, to find a formula connecting the data of two circles, when their is a bicentric embedded n-gon between them. For this formula, we use the elementary symmetric polynomials $\sigma_1, \sigma_2, ..., \sigma_n$ in the variable $p_1, p_2, ..., p_n$, where p_i denotes the geodesic $v_i w_i$ of the n-gon, also, we denotes α_i to represent the interior angle of the n-gon, $i = 1, 2, ..., n$, see Figure 4.1.

Theorem 4.2.1. *Let C and D be two disjoint circles in the Euclidean plane, with D inside C. Let R_E denote the radius of C, r_E denote the radius of D and d_E denote the distance between the two circles' centres. Assume that there is a bicentric embedded n-gon between them, then the relation between R_E, r_E and d_E can be given by using the following equations:*

$$r_E^{n-1} = \frac{\sum_{j=3\, odd}^{n} (-1)^{\frac{(j+1)}{2}} r_E^{n-j} \sigma_j}{\sigma_1} \tag{4.5}$$

when the n-gon has an odd number of sides n, and

$$r_E^{n-2} = \frac{\sum_{j=3\, odd}^{n-1} (-1)^{\frac{(j+1)}{2}} r_E^{n-(j+1)} \sigma_j}{\sigma_1} \tag{4.6}$$

when the n-gon has an even number of sides n, such that σ_j is the elementary symmetric polynomial in $p_1, p_2, ..., p_n$
where

$$p_1 = \sqrt{(R_E - d_E)^2 - r_E^2}$$

$$p_2 = 2R_E \cos\left(\frac{\alpha_1}{2}\right) - p_1$$

$$p_i = 2R_E \cos\left(\left(\sum_{j=2}^{i-1} (-1)^j \alpha_j\right) - \left(\frac{\alpha_1}{2}\right)\right) - p_{i-1}$$

when i is an odd number $\geqslant 3$,

$$p_i = 2R_E \cos\left(\left(\sum_{j=2}^{i-1}(-1)^{j+1}\alpha_j\right) + \left(\frac{\alpha_1}{2}\right)\right) - p_{i-1}$$

when i is an even number $\geqslant 4$,
such that,

$$\tan\left(\frac{\alpha_i}{2}\right) = \frac{r_E}{p_i}$$

Proof. To prove this theorem, we use the previous theorem, with some trigonometric identities, to get a relation between R_E, r_E and d_E.

Let P be a Euclidean embedded n-gon inscribed in C and circumscribed around D with vertices v_j, $1 \leqslant j \leqslant n$. We shall use the standard way mentioned in Remark 3.0.1, and let a_j be the Euclidean length of its sides v_jv_{j+1}, and let α_j be its interior angle. Assume that w_j is the point on the side a_j where the perpendicular from the inscribed circle's centre I meets the side. We denote θ_j for the angle v_jIw_j and set p_j for the geodesics v_jw_j where $1 \leqslant j \leqslant n$, see Figure 4.1.
From Theorem 4.1.1,

$$\sum_{j=1\ odd}^{n}(-1)^{\frac{(j+1)}{2}}\sigma_j = \sigma_1 - \sigma_3 + \sigma_5 ... \pm \sigma_n = 0 \tag{4.7}$$

when the n-gon has an odd number of sides n, and

$$\sum_{j=1\ odd}^{n-1}(-1)^{\frac{(j+1)}{2}}\sigma_j = \sigma_1 - \sigma_3 + \sigma_5 ... \pm \sigma_{n-1} = 0 \tag{4.8}$$

when the n-gon has an even number of sides n,
such that the elementary symmetric polynomials $\sigma_1, \sigma_2, ..., \sigma_n$ are in the variables $\tan(\theta_i)$,

thus

$$\sum_{j=1}^{n}\tan(\theta_j)-\sum_{1\leq j_1<j_2<j_3\leq n}\tan(\theta_{j_1})\tan(\theta_{j_2})\tan(\theta_{j_3})+...\pm\tan(\theta_1)\tan(\theta_2)...\tan(\theta_n)=0 \tag{4.9}$$

when the n-gon has an odd number of sides n, and

$$\sum_{j=1}^{n}\tan(\theta_j)-\sum_{1\leq j_1<j_2<j_3\leq n}\tan(\theta_{j_1})\tan(\theta_{j_2})\tan(\theta_{j_3})+...\pm\tan(\theta_1)\tan(\theta_2)...\tan(\theta_{n-1})=0 \tag{4.10}$$

when the n-gon has an even number of sides n.

From the triangles $v_i I w_i$,

$$\tan(\theta_i)=\frac{p_i}{r_E}$$

we can see that

$$r_E^{n-1}=\frac{r_E^{n-3}\sum_{1\leq j_1<j_2<j_3\leq n}p_{j_1}p_{j_2}p_{j_3}-...\mp p_1p_2...p_n}{\sum_{j=1}^{n}p_j} \tag{4.11}$$

when the n-gon has an odd number of sides n, and

$$r_E^{n-1}=\frac{r_E^{n-3}\sum_{1\leq j_1<j_2<j_3\leq n}p_{j_1}p_{j_2}p_{j_3}-...\mp r_E(p_1p_2...p_{n-1})}{\sum_{j=1}^{n}p_j} \tag{4.12}$$

when the n-gon has an even number of sides n.
Assume that σ_j is the elementary symmetric polynomials in $p_1,p_2,...,p_n$.
This means that

$$r_E^{n-1}=\frac{\sum_{j=3\,odd}^{n}(-1)^{\frac{(j+1)}{2}}r_E^{n-j}\sigma_j}{\sigma_1} \tag{4.13}$$

when the n-gon has an odd number of sides n, and

$$r_E^{n-2}=\frac{\sum_{j=3\,odd}^{n-1}(-1)^{\frac{(j+1)}{2}}r_E^{n-(j+1)}\sigma_j}{\sigma_1} \tag{4.14}$$

when the n-gon has an even number of sides n.

Turning now to define p_i in respect of R_E, r_E, d_E
Using the Pythagorean Theorem on the triangle $v_1 I w_1$,

$$p_1 = \sqrt{(R_E - d_E)^2 - r_E^2}$$

Also, as we see, that $p_2 = a_1 - p_1$

Applying the law of cosine on the triangle $v_1 O v_2$ to find a_1

$$\cos\left(\frac{\alpha_1}{2}\right) = \frac{-R_E^2 + R_E^2 + a_1^2}{2R_E a_1}$$

$$a_1 = 2R_E \cos\left(\frac{\alpha_1}{2}\right)$$

So,

$$p_2 = 2R_E \cos\left(\frac{\alpha_1}{2}\right) - \sqrt{(R_E - d_E)^2 - r_E^2}$$

By the same way, $p_3 = a_2 - p_2$

Applying the law of cosine on the triangle $v_2 O v_3$

$$a_2 = 2R_E \cos\left(\alpha_2 - \frac{\alpha_1}{2}\right)$$

Thus,

$$p_3 = 2R_E\left(\cos\left(\alpha_2 - \frac{\alpha_1}{2}\right) - \cos\left(\frac{\alpha_1}{2}\right)\right) + \sqrt{(R_E - d_E)^2 - r_E^2}$$

Also, $p_4 = a_3 - p_3$
So,

$$p_4 = 2R_E\left(\cos\left(\alpha_3 - \alpha_2 + \frac{\alpha_1}{2}\right) - \cos\left(\alpha_2 - \frac{\alpha_1}{2}\right) + \cos\left(\frac{\alpha_1}{2}\right)\right) - \sqrt{(R_E - d_E)^2 - r_E^2}$$

By the same way,

$$p_i = 2R_E \cos\left(\left(\sum_{j=2}^{i-1} (-1)^j \alpha_j\right) - \left(\frac{\alpha_1}{2}\right)\right) - p_{i-1}$$

when i is an odd number $\geqslant 3$, and

$$p_i = 2R_E \cos\left(\left(\sum_{j=2}^{i-1}(-1)^{j+1}\alpha_j\right) + \left(\frac{\alpha_1}{2}\right)\right) - p_{i-1}$$

when i is an even number $\geqslant 4$.
where, from the triangles $v_i I w_i$

$$\tan\left(\frac{\alpha_i}{2}\right) = \frac{r_E}{p_i}$$

□

When $n = 3$, the general formula will be

$$r^2 = \frac{p_1 p_2 p_3}{p_1 + p_2 + p_3}$$

where

$$p_1 = \sqrt{(R_E - d_E)^2 - r_E^2}$$

$$p_2 = \frac{(R_E + d_E)\sqrt{(R_E - d_E)^2 - r_E^2}}{R_E - d_E}$$

$$p_3 = \frac{((R_E^2 - d_E^2)^2 + 4R_E r_E^2 d_E)\sqrt{(R_E - d_E)^2 - r_E^2}}{(R_E^2 - d_E^2)^2 - 4R_E r_E^2 d_E}$$

After calculations, the Euclidean general formula when $n = 3$ is

$$R_E^2 - d_E^2 - 2R_E r_E = 0$$

which is the Euclidean Chapple Formula.
Also, when $n = 4$, it will be the Euclidean Fuss's Formula. This general formula presents a relation between circles' data, when there is an embedded n-gon between them, in a clear geometrical way, instead of using Richelot's Formula which depends on Jacobi's elliptic functions.

4.3 General Formulae for Hyperbolic Embedded n-gon

In this section, we prove two general formulae in the hyperbolic plane. The first one is introduced by applying a lemma, which determines a relation between the radius of the inscribed circle and the interior angles of an embedded n-gon circumscribed it. The other formula is proved by using the general method, which was introduced at the beginning of this chapter.

In the following lemma, we prove a general formula relating the radius of the inscribed circle with the interior angles of an embedded n-gon circumscribed it. In this lemma, the angle θ denotes the angle v_1Iv_2, and α_j denotes the interior angle of the n-gon, see Figure 4.2. In the proof, we use the same structure as proof of Theorem 3.2.1 [2].

Lemma 4.3.1. *The radius r of the inscribed circle of a hyperbolic embedded n-gon P is given by*

$$\tanh^2(r) = 1 - \frac{\sin^2(\theta)}{\cos^2(\frac{\alpha_1}{2}) + \cos^2(\frac{\alpha_2}{2}) + 2\cos(\frac{\alpha_1}{2})\cos(\frac{\alpha_2}{2})\cos(\theta)} \tag{4.15}$$

where α_1 and α_2 are the first two interior angles of the n-gon P and θ can be represented as follows

$$\theta = \pi - \left(\arcsin\left(\frac{\cos\left(\frac{\alpha_2}{2}\right)}{\cosh(r)}\right)\right)$$

when $n = 3$,

$$\theta = \pi - \left(\arcsin\left(\frac{\cos\left(\frac{\alpha_2}{2}\right)}{\cosh(r)}\right) + \arcsin\left(\frac{\cos\left(\frac{\alpha_3}{2}\right)}{\cosh(r)}\right)\right)$$

when $n = 4$,

$$\theta = \pi - \left(\arcsin\left(\frac{\cos\left(\frac{\alpha_2}{2}\right)}{\cosh(r)}\right) + 2\sum_{j=3}^{\frac{n+1}{2}} \arcsin\left(\frac{\cos\left(\frac{\alpha_j}{2}\right)}{\cosh(r)}\right)\right)$$

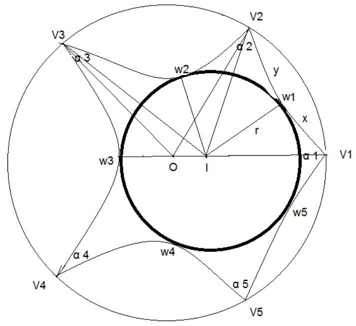

Figure 4.2: Hyperbolic n-gon 1

when the n-gon has an odd number of sides n, and

$$\theta = \pi - \left(\arcsin\left(\frac{\cos\left(\frac{\alpha_2}{2}\right)}{\cosh(r)} \right) + 2\sum_{j=3}^{\frac{n}{2}} \arcsin\left(\frac{\cos\left(\frac{\alpha_j}{2}\right)}{\cosh(r)} \right) + \arcsin\left(\frac{\cos\left(\frac{\alpha_{\left(\frac{n+2}{2}\right)}}{2}\right)}{\cosh(r)} \right) \right)$$

when the n-gon has an even number of sides n.

Proof. To prove this lemma, we use a convenient case where the n-gon is symmetric, then, we use some hyperbolic trigonometric identities with hyperbolic law of cosine on the triangle $v_1 I v_2$ to obtain a relation between the first two interior angles of the n-gon and the angle θ, which represents the angle $v_1 I v_2$. Then, rearranging this relation and using trigonometric identities, and right triangle identities in it, after that, we write the angle θ in terms of the interior angles of the n-gon.

We shall use the standard way mentioned at the beginning of the previous chapter, Remark 3.0.1, and let P be a hyperbolic embedded n-gon inscribed in C and circumscribed around D, with vertices v_j, $1 \leqslant j \leqslant n$, see Figure 4.2. Let a_j be the hyperbolic length of its side $v_j v_{j+1}$, and let α_j be its interior angle.

Set the n-gon so that one of its vertices say v_1 is at the same line with the circumscribed circle's centre and the inscribed circle's centre. Thus, the n-gon line of

symmetry passes through the centres of the circles, as a result of that, the n-gon is symmetric, so $\alpha_2 = \alpha_n$, $\alpha_3 = \alpha_{n-1}$,....
Assume that w_j is the point on the side a_j where the perpendicular from the inscribed circle's centre I meets the side, thus, the hyperbolic distance $p(Iw_j) = r$.

Set $p(v_1w_1) = x$ and $p(v_2w_1) = y$, by using identity 3, Theorem 2.1.2, and the hyperbolic law of cosines II, Theorem 2.1.3, on the triangle v_1Iv_2, we get that

$$\cosh(x+y) = \cosh(x)\cosh(y) + \sinh(x)\sinh(y) = \frac{\cos(\frac{\alpha_1}{2})\cos\left(\frac{\alpha_2}{2}\right) + \cos(\theta)}{\sin(\frac{\alpha_1}{2})\sin\left(\frac{\alpha_2}{2}\right)}$$

Rearranging

$$\cos\left(\frac{\alpha_1}{2}\right)\cos\left(\frac{\alpha_2}{2}\right) + \cos(\theta) - \sin\left(\frac{\alpha_1}{2}\right)\sin\left(\frac{\alpha_2}{2}\right)\sinh(x)\sinh(y)$$

$$= \sin\left(\frac{\alpha_1}{2}\right)\sin\left(\frac{\alpha_2}{2}\right)\cosh(x)\cosh(y)$$

Squaring both sides and using the trigonometric identities

$$\left(\cos\left(\frac{\alpha_1}{2}\right)\cos\left(\frac{\alpha_2}{2}\right) + \cos(\theta) - \sin\left(\frac{\alpha_1}{2}\right)\sinh(x)\sin\left(\frac{\alpha_2}{2}\right)\sinh(y)\right)^2 =$$

$$\sin^2\left(\frac{\alpha_1}{2}\right)(1+\sinh^2(x))\sin^2\left(\frac{\alpha_2}{2}\right)(1+\sinh^2(y))$$

Using the identity $\cos^2(x) + \sin^2(x) = 1$ and rearranging

$$\left(\cos\left(\frac{\alpha_1}{2}\right)\cos\left(\frac{\alpha_2}{2}\right) + \cos(\theta) - \sin\left(\frac{\alpha_1}{2}\right)\sinh(x)\sin\left(\frac{\alpha_2}{2}\right)\sinh(y)\right)^2 =$$

$$\left(\left(1-\cos^2\left(\frac{\alpha_1}{2}\right)\right) + \sin^2\left(\frac{\alpha_1}{2}\right)\sinh^2(x)\right)\left(\left(1-\cos^2\left(\frac{\alpha_2}{2}\right)\right) + \sin^2\left(\frac{\alpha_2}{2}\right)\sinh^2(y)\right) \tag{4.16}$$

By using identity 2, Theorem 2.1.4, on the right triangle v_1Iw_1, we can see that

$$\tanh(r) = \sinh(x)\tan\left(\frac{\alpha_1}{2}\right) \tag{4.17}$$

the equation (4.17) with the identity

$$\tan(\alpha) = \frac{\sin(\alpha)}{\cos(\alpha)}$$

yield that

$$\sin\left(\frac{\alpha_1}{2}\right)\sinh(x) = \tanh(r)\cos\left(\frac{\alpha_1}{2}\right) \tag{4.18}$$

and the same for y, $\frac{\alpha_2}{2}$,

$$\sin\left(\frac{\alpha_2}{2}\right)\sinh(y) = \tanh(r)\cos\left(\frac{\alpha_2}{2}\right) \tag{4.19}$$

Substitution from (4.18) and (4.19) in (4.16)

$$\left(\left(\cos\left(\frac{\alpha_1}{2}\right)\cos\left(\frac{\alpha_2}{2}\right)+\cos(\theta)\right)-\left(\tanh(r)\cos\left(\frac{\alpha_1}{2}\right)\right)\left(\tanh(r)\cos\left(\frac{\alpha_2}{2}\right)\right)\right)^2 =$$

$$\left(\left(1-\cos^2\left(\frac{\alpha_1}{2}\right)\right)+\left(\tanh^2(r)\cos^2\left(\frac{\alpha_1}{2}\right)\right)\right)\left(\left(1-\cos^2\left(\frac{\alpha_2}{2}\right)\right)+\tanh^2(r)\cos^2\left(\frac{\alpha_2}{2}\right)\right)$$

Squaring and simplifying

$$\cos^2\left(\frac{\alpha_1}{2}\right)\cos^2\left(\frac{\alpha_2}{2}\right)+\cos^2(\theta)+2\cos\left(\frac{\alpha_1}{2}\right)\cos\left(\frac{\alpha_2}{2}\right)\cos(\theta)-2\cos^2\left(\frac{\alpha_1}{2}\right)\cos^2\left(\frac{\alpha_2}{2}\right)\tanh^2(r)$$

$$-2\cos\left(\frac{\alpha_1}{2}\right)\cos\left(\frac{\alpha_2}{2}\right)\cos(\theta)\tanh^2(r) = \left(1-\cos^2\left(\frac{\alpha_1}{2}\right)\right)\left(1-\cos^2\left(\frac{\alpha_2}{2}\right)\right)$$

$$+\tanh^2(r)\cos^2\left(\frac{\alpha_1}{2}\right)\left(1-\cos^2\left(\frac{\alpha_2}{2}\right)\right)+\tanh^2(r)\cos^2\left(\frac{\alpha_2}{2}\right)\left(1-\cos^2\left(\frac{\alpha_1}{2}\right)\right)$$

After simplification, we find that

$$\tanh^2(r) = \frac{\cos^2(\frac{\alpha_1}{2})+\cos^2\left(\frac{\alpha_2}{2}\right)+\cos^2(\theta)+2\cos(\frac{\alpha_1}{2})\cos\left(\frac{\alpha_2}{2}\right)\cos(\theta)-1}{\cos^2(\frac{\alpha_1}{2})+\cos^2\left(\frac{\alpha_2}{2}\right)+2\cos(\frac{\alpha_1}{2})\cos\left(\frac{\alpha_2}{2}\right)\cos(\theta)}$$

By simplifying

$$\tanh^2(r) = 1+\frac{\cos^2(\theta)-1}{\cos^2(\frac{\alpha_1}{2})+\cos^2(\frac{\alpha_2}{2})+2\cos(\frac{\alpha_1}{2})\cos(\frac{\alpha_2}{2})\cos(\theta)} \tag{4.20}$$

which is the desired formula.

Now, trying to find the angle θ in terms of the other internal angles of the n-gon. When n is odd, $\theta = \pi - v_2Iw_{(\frac{n+1}{2})}$,
however,

$$v_2Iw_{(\frac{n+1}{2})} = v_2Iw_2 + w_2Iw_3 + ... + w_{(\frac{n-1}{2})}Iw_{(\frac{n+1}{2})}$$

We can see that $w_{j-1}Iw_j = 2(v_jIw_j)$ from the congruence of the triangles $w_{j-1}Iv_j$ and v_jIw_j.
By using identity 5, Theorem 2.1.4, on the triangle v_jIw_j, we find that

$$\sin(v_jIw_j) = \frac{\cos\left(\frac{\alpha_j}{2}\right)}{\cosh(r)}$$

thus, $v_jIw_j = \arcsin\left(\frac{\cos\left(\frac{\alpha_j}{2}\right)}{\cosh(r)}\right)$, then

$$v_2Iw_{(\frac{n+1}{2})} = \arcsin\left(\frac{\cos\left(\frac{\alpha_2}{2}\right)}{\cosh(r)}\right) + 2\arcsin\left(\frac{\cos\left(\frac{\alpha_3}{2}\right)}{\cosh(r)}\right) + ... + 2\arcsin\left(\frac{\cos\left(\frac{\alpha_{(\frac{n+1}{2})}}{2}\right)}{\cosh(r)}\right)$$

so,

$$\theta = \pi - \left(\arcsin\left(\frac{\cos\left(\frac{\alpha_2}{2}\right)}{\cosh(r)}\right) + 2\arcsin\left(\frac{\cos\left(\frac{\alpha_3}{2}\right)}{\cosh(r)}\right) + ... + 2\arcsin\left(\frac{\cos\left(\frac{\alpha_{(\frac{n+1}{2})}}{2}\right)}{\cosh(r)}\right)\right)$$

when the n-gon has an odd number of sides n.

On the other hand, when n is even, $\theta = \pi - v_2Iv_{(\frac{n+2}{2})}$,
but

$$v_2Iv_{(\frac{n+2}{2})} = v_2Iw_2 + w_2Iw_3 + ... + w_{(\frac{n}{2})}Iv_{(\frac{n+2}{2})}$$

We can see that, $w_{j-1}Iw_j = 2(v_jIw_j)$ from the congruence of the triangles $w_{j-1}Iv_j$ and v_jIw_j
By using identity 5, Theorem 2.1.4, on the triangle v_jIw_j, we find that

$$\sin(v_jIw_j) = \frac{\cos\left(\frac{\alpha_j}{2}\right)}{\cosh(r)}$$

So, $v_j I w_j = \arcsin\left(\frac{\cos(\frac{\alpha_j}{2})}{\cosh(r)}\right)$, then

$$v_2 I v_{(\frac{n+2}{2})} = \arcsin\left(\frac{\cos\left(\frac{\alpha_2}{2}\right)}{\cosh(r)}\right) + 2\arcsin\left(\frac{\cos\left(\frac{\alpha_3}{2}\right)}{\cosh(r)}\right) + ... + \arcsin\left(\frac{\cos\left(\frac{\alpha_{(\frac{n+2}{2})}}{2}\right)}{\cosh(r)}\right)$$

thus,

$$\theta = \pi - \left(\arcsin\left(\frac{\cos\left(\frac{\alpha_2}{2}\right)}{\cosh(r)}\right) + 2\arcsin\left(\frac{\cos\left(\frac{\alpha_3}{2}\right)}{\cosh(r)}\right) + ... + \arcsin\left(\frac{\cos\left(\frac{\alpha_{(\frac{n+2}{2})}}{2}\right)}{\cosh(r)}\right)\right)$$

when the n-gon has an even number of sides n. □

The previous lemma presents a general formula relating the radius of the inscribed circle with the interior angles of an embedded n-gon circumscribed it, which we use it in the following theorem, and write every interior angle in terms of R, r and d, to find a general formula in the hyperbolic plane in which relating the quantities of two circles when there is a bicentric embedded n-gon inscribed in one circle and circumscribed around the other.

Theorem 4.3.1. *Let C and D be two disjoint circles in the hyperbolic plane, with D inside C. Let R denote the radius of C, r denote the radius of D and d denote the distance between the two circles' centres. Assume that their is a bicentric embedded n-gon between them, then, the relation between R, r and d can be given by*

$$\tanh^2(r) = 1 - \frac{\sin^2(\theta)}{\cos^2(\frac{\alpha_1}{2}) + \cos^2(\frac{\alpha_2}{2}) + 2\cos(\frac{\alpha_1}{2})\cos(\frac{\alpha_2}{2})\cos(\theta)} \tag{4.21}$$

such that,

$$\cos\left(\frac{\alpha_1}{2}\right) = \frac{\sqrt{\sinh^2(R-d) - \sinh^2(r)}}{\sinh(R-d)} \tag{4.22}$$

$$\tan\left(\frac{\alpha_2}{2}\right) = \frac{\tanh(r)}{\sinh\left(\operatorname{arccosh}\left(\frac{\cosh^2(R)+\sinh^2(R)\cos^2\left(\frac{\alpha_1}{2}\right)}{\cosh^2(R)-\sinh^2(R)\cos^2\left(\frac{\alpha_1}{2}\right)}\right) - \operatorname{arccosh}\left(\frac{\cosh(R-d)}{\cosh(r)}\right)\right)} \tag{4.23}$$

$$\tan\left(\frac{\alpha_i}{2}\right) = \frac{\tanh(r)}{\sinh(p_i)} \tag{4.24}$$

as p_i defined as follow

$$p_i = \operatorname{arccosh}\left(\frac{\cosh^2(R) + \sinh^2(R)\cos^2\left(\left(\sum_{j=2}^{i-1}(-1)^j\alpha_j\right) - \left(\frac{\alpha_1}{2}\right)\right)}{\cosh^2(R) - \sinh^2(R)\cos^2\left(\left(\sum_{j=2}^{i-1}(-1)^j\alpha_j\right) - \left(\frac{\alpha_1}{2}\right)\right)}\right) - p_{i-1} \tag{4.25}$$

when $i \geqslant 3$ is an odd number, and

$$p_i = \operatorname{arccosh}\left(\frac{\cosh^2(R) + \sinh^2(R)\cos^2\left(\left(\sum_{j=2}^{i-1}(-1)^{j-1}\alpha_j\right) + \left(\frac{\alpha_1}{2}\right)\right)}{\cosh^2(R) - \sinh^2(R)\cos^2\left(\left(\sum_{j=2}^{i-1}(-1)^{j-1}\alpha_j\right) + \left(\frac{\alpha_1}{2}\right)\right)}\right) - p_{i-1} \tag{4.26}$$

when $i \geqslant 4$ is an even number.

Proof. We use a convenient case, where the n-gon is symmetric, next, we use some hyperbolic trigonometric identities to obtain $\cos\left(\frac{\alpha_1}{2}\right)$, $\cos\left(\frac{\alpha_2}{2}\right)$ and $\cos(\theta)$ in terms of R, r and d. Then using those relations in the relation (4.15) to find the desired formula.

We use the standard way, mentioned at the beginning of the previous chapter, Remark 3.0.1, and let P be a hyperbolic embedded n-gon inscribed in C and circumscribed around D, with vertices v_j, $1 \leqslant j \leqslant n$, see Figure 4.2. Let a_j be the hyperbolic length of its side v_jv_{j+1}, and α_j be its interior angle. Assume that w_j is the point on the side a_j where the perpendicular from the inscribed circle's centre I meets the side, set p_j for the geodesics v_jw_j, where $1 \leqslant j \leqslant n$. From the similarity of the triangles $w_{j-1}Iv_j$ and v_jIw_j, we can see that, $w_{j-1}v_j = p_j$. Because we set the n-gon so that one of its vertices say v_1 is at the same line with the circumscribed circle's centre and the inscribed circle's centre, the n-gon line of symmetry passes through the centres of the circles, as a result of that, the n-gon is symmetric, so, $\alpha_2 = \alpha_n$, $\alpha_3 = \alpha_{n-1}$,....

By using the hyperbolic relation 3, Theorem 2.1.4, on the right triangle $v_1 I w_1$, we find that

$$\sin\left(\frac{\alpha_1}{2}\right) = \frac{\sinh(r)}{\sinh(R-d)}$$

so,

$$\cos\left(\frac{\alpha_1}{2}\right) = \frac{\sqrt{\sinh^2(R-d) - \sinh^2(r)}}{\sinh(R-d)} \quad (4.27)$$

Turning now to find the angle $\frac{\alpha_2}{2}$ in terms of α_1, using the hyperbolic law of cosine on the triangle $v_1 O v_2$

$$\cosh(R) = \cosh(R)\cosh(a_1) - \sinh(R)\sinh(a_1)\cos\left(\frac{\alpha_1}{2}\right)$$

Rearranging and using identity 1, Theorem 2.1.2

$$\cosh(R)\left(1 - \cosh(a_1)\right) = -\sinh(R)\cos\left(\frac{\alpha_1}{2}\right)\sqrt{\cosh^2(a_1) - 1}$$

Squaring both sides

$$\cosh^2(R)\left(1 + \cosh^2(a_1) - 2\cosh(a_1)\right) = \sinh^2(R)\cosh^2(a_1)\cos^2\left(\frac{\alpha_1}{2}\right) - \sinh^2(R)\cos^2\left(\frac{\alpha_1}{2}\right)$$

Solving the equation for $\cosh(a_1)$

$$\cosh(a_1) =$$

$$\frac{2\cosh^2(R) \pm \sqrt{4\cosh^4(R) - 4(\cosh^2(R) - \sinh^2(R)\cos^2\left(\frac{\alpha_1}{2}\right))(\cosh^2(R) + \sinh^2(R)\cos^2\left(\frac{\alpha_1}{2}\right))}}{2\left(\cosh^2(R) - \sinh^2(R)\cos^2\left(\frac{\alpha_1}{2}\right)\right)}$$

so,

$$\cosh(a_1) = \frac{\cosh^2(R) \pm \sinh^2(R)\cos^2\left(\frac{\alpha_1}{2}\right)}{\cosh^2(R) - \sinh^2(R)\cos^2\left(\frac{\alpha_1}{2}\right)}$$

We have two possibility, when we take the minus, we get that $\cosh(a_1) = 1$ which means $a_1 = 0$, but $a_1 > 0$, contradiction, thus,

$$a_1 = \operatorname{arccosh}\left(\frac{\cosh^2(R) + \sinh^2(R)\cos^2\left(\frac{\alpha_1}{2}\right)}{\cosh^2(R) - \sinh^2(R)\cos^2\left(\frac{\alpha_1}{2}\right)}\right) \tag{4.28}$$

but from the right triangle $v_1 I w_1$

$$\cosh(R-d) = \cosh(r)\cosh(p_1)$$

from that

$$p_1 = \operatorname{arccosh}\left(\frac{\cosh(R-d)}{\cosh(r)}\right) \tag{4.29}$$

from 4.28 and 4.29

$$p_2 = a_1 - p_1 = \operatorname{arccosh}\left(\frac{\cosh^2(R) + \sinh^2(R)\cos^2\left(\frac{\alpha_1}{2}\right)}{\cosh^2(R) - \sinh^2(R)\cos^2\left(\frac{\alpha_1}{2}\right)}\right) - \operatorname{arccosh}\left(\frac{\cosh(R-d)}{\cosh(r)}\right) \tag{4.30}$$

In the right triangle $v_2 I w_1$, we find that

$$\tanh(r) = \sinh(p_2)\tan\left(\frac{\alpha_2}{2}\right)$$

So, we can write α_2 as a function of R, r, d and α_1

$$\tan\left(\frac{\alpha_2}{2}\right) = \frac{\tanh(r)}{\sinh\left(\operatorname{arccosh}\left(\frac{\cosh^2(R)+\sinh^2(R)\cos^2\left(\frac{\alpha_1}{2}\right)}{\cosh^2(R)-\sinh^2(R)\cos^2\left(\frac{\alpha_1}{2}\right)}\right) - \operatorname{arccosh}\left(\frac{\cosh(R-d)}{\cosh(r)}\right)\right)} \tag{4.31}$$

Now, finding the angle α_3, in terms of α_1 and α_2, by following the same steps as α_2, using the hyperbolic law of cosine on the triangle $v_2 O v_3$

$$\cosh(R) = \cosh(R)\cosh(a_2) - \sinh(R)\sinh(a_2)\cos\left(\alpha_2 - \frac{\alpha_1}{2}\right)$$

as the previous,

$$a_2 = \operatorname{arccosh}\left(\frac{\cosh^2(R) + \sinh^2(R)\cos^2\left(\alpha_2 - \frac{\alpha_1}{2}\right)}{\cosh^2(R) - \sinh^2(R)\cos^2\left(\alpha_2 - \frac{\alpha_1}{2}\right)}\right) \tag{4.32}$$

but we know that $p_3 = a_2 - p_2$, so from (4.32) and (4.30)

$$p_3 = c^{-1}\left(\frac{c^2(R)+s^2(R)\cos^2\left(\alpha_2-\frac{\alpha_1}{2}\right)}{c^2(R)-s^2(R)\cos^2\left(\alpha_2-\frac{\alpha_1}{2}\right)}\right) - c^{-1}\left(\frac{c^2(R)+s^2(R)\cos^2\left(\frac{\alpha_1}{2}\right)}{c^2(R)-s^2(R)\cos^2\left(\frac{\alpha_1}{2}\right)}\right) + c^{-1}\left(\frac{c(R-d)}{c(r)}\right) \tag{4.33}$$

where $c(x) = \cosh(x), s(x) = \sinh(x)$ and $c^{-1}(x) = \operatorname{arccosh}(x)$.

In the right triangle v_3Iw_2, we find that

$$\tanh(r) = \sinh(p_3)\tan\left(\frac{\alpha_3}{2}\right)$$

Thus, we can write α_3 as a function of R, r, d, α_1 and α_2

$$\tan\left(\frac{\alpha_3}{2}\right) = \frac{\tanh(r)}{s\left(c^{-1}\left(\frac{c^2(R)+s^2(R)\cos^2\left(\alpha_2-\frac{\alpha_1}{2}\right)}{c^2(R)-s^2(R)\cos^2\left(\alpha_2-\frac{\alpha_1}{2}\right)}\right) - c^{-1}\left(\frac{c^2(R)+s^2(R)\cos^2\left(\frac{\alpha_1}{2}\right)}{c^2(R)-s^2(R)\cos^2\left(\frac{\alpha_1}{2}\right)}\right) + c^{-1}\left(\frac{c(R-d)}{c(r)}\right)\right)} \tag{4.34}$$

by the same way of exploring, we can write every angle α_j as a function of R, r, d and α_i where $1 \leqslant i < j$ as the following

$$\tan\left(\frac{\alpha_i}{2}\right) = \frac{\tanh(r)}{\sinh(p_i)} \tag{4.35}$$

as p_i defined as follow

$$p_i = \cosh^{-1}\left(\frac{\cosh^2(R)+\sinh^2(R)\cos^2\left(\left(\sum_{j=2}^{i-1}(-1)^j\alpha_j\right)-\left(\frac{\alpha_1}{2}\right)\right)}{\cosh^2(R)-\sinh^2(R)\cos^2\left(\left(\sum_{j=2}^{i-1}(-1)^j\alpha_j\right)-\left(\frac{\alpha_1}{2}\right)\right)}\right) - p_{i-1} \tag{4.36}$$

when $i \geqslant 3$ is an odd number, and

$$p_i = \cosh^{-1}\left(\frac{\cosh^2(R) + \sinh^2(R)\cos^2\left(\left(\sum_{j=2}^{i-1}(-1)^{j-1}\alpha_j\right) + \left(\frac{\alpha_1}{2}\right)\right)}{\cosh^2(R) - \sinh^2(R)\cos^2\left(\left(\sum_{j=2}^{i-1}(-1)^{j-1}\alpha_j\right) + \left(\frac{\alpha_1}{2}\right)\right)}\right) - p_{i-1} \tag{4.37}$$

when $i \geqslant 4$ is an even number.

Finally, using the equations (4.27), (4.31)and (4.35) in the relation (4.21) gives a general formula in the hyperbolic plane in which relating the quantities of two circles R, r, d, when there is a bicentric embedded n-gon inscribed in one circle and circumscribed around the other, where R is the circumscribed circle's radius, r is the inscribed circle's radius and d is the distance between the centres of the two circles. □

In the following theorem, we work at the same way as the Euclidean general formula to present another formula for the hyperbolic bicentric n-gon. To prove this formula, we use the elementary symmetric polynomials $\sigma_1, \sigma_2, ..., \sigma_n$ in the variable $\tanh(p_1), \tanh(p_2), ..., \tanh(p_n)$, where p_i denote the geodesic $v_i w_i$ of the n-gon, see Figure 4.3, also, we denote α_i to represent the interior angle of the n-gon, $i = 1, 2, ..., n$

Theorem 4.3.2. *Let C and D be two disjoint circles in the hyperbolic plane, with D inside C. Let R denote the radius of C, r denote the radius of D and d denote the distance between the two circles' centres. Assume that their is a bicentric embedded n-gon between them, then the relation between R, r and d, can be given by using the following equations:*

$$\sinh^{n-1}(r) = \frac{\sum_{j=3\ odd}^{n}(-1)^{\frac{(j+1)}{2}}\sinh^{n-j}(r)\sigma_j}{\sigma_1} \tag{4.38}$$

when the n-gon has an odd number of sides n, and

$$\sinh^{n-2}(r) = \frac{\sum_{j=3\,odd}^{n-1}(-1)^{\frac{(j+1)}{2}}\sinh^{n-(j+1)}(r)\sigma_j}{\sigma_1} \tag{4.39}$$

when the n-gon has an even number of sides n,
such that σ_j are the elementary symmetric polynomials in $\tanh(p_1), \tanh(p_2), ..., \tanh(p_n)$,
where

$$p_1 = \operatorname{arccosh}\left(\frac{\cosh(R-d)}{\cosh(r)}\right)$$

$$p_2 = \operatorname{arccosh}\left(\frac{\cosh^2(R)+\sinh^2(R)\cos^2\left(\frac{\alpha_1}{2}\right)}{\cosh^2(R)-\sinh^2(R)\cos^2\left(\frac{\alpha_1}{2}\right)}\right) - \operatorname{arccosh}\left(\frac{\cosh(R-d)}{\cosh(r)}\right)$$

$$p_i = \operatorname{arccosh}\left(\frac{\cosh^2(R)+\sinh^2(R)\cos^2\left(\left(\sum_{j=2}^{i-1}(-1)^j\alpha_j\right)-\left(\frac{\alpha_1}{2}\right)\right)}{\cosh^2(R)-\sinh^2(R)\cos^2\left(\left(\sum_{j=2}^{i-1}(-1)^j\alpha_j\right)-\left(\frac{\alpha_1}{2}\right)\right)}\right) - p_{i-1} \tag{4.40}$$

when $i \geqslant 3$ is an odd number, and

$$p_i = \operatorname{arccosh}\left(\frac{\cosh^2(R)+\sinh^2(R)\cos^2\left(\left(\sum_{j=2}^{i-1}(-1)^{j-1}\alpha_j\right)+\left(\frac{\alpha_1}{2}\right)\right)}{\cosh^2(R)-\sinh^2(R)\cos^2\left(\left(\sum_{j=2}^{i-1}(-1)^{j-1}\alpha_j\right)+\left(\frac{\alpha_1}{2}\right)\right)}\right) - p_{i-1} \tag{4.41}$$

when $i \geqslant 4$ is an even number,
such that,

$$\tan\left(\frac{\alpha_i}{2}\right) = \frac{\tanh(r)}{\sinh(p_i)}$$

Proof. We use Theorem 4.1.1, to get a relation between the angles $\theta_1, \theta_2, ..., \theta_n$, using the identity of the tangent of a sum. Then we apply hyperbolic trigonometric

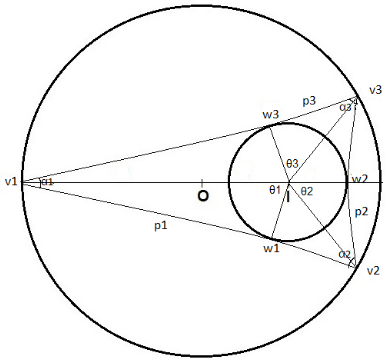

Figure 4.3: Hyperbolic Polygon 2

functions to express this relation in terms of $\tanh(p_i)$ and $\sinh(r)$, where $1 \leqslant i \leqslant n$. We use the standard way, mentioned at the beginning of the previous chapter, Remark 3.0.1, and let P be a hyperbolic embedded n-gon inscribed in C and circumscribed around D, with vertices v_j, let α_j be its interior angle, $1 \leqslant j \leqslant n$, see Figure 4.3. Assume that w_j is the point on the side a_j where the perpendicular from the inscribed circle's centre I meets the side, set p_j for the geodesics $v_j w_j$ and the angle θ_j be the angle $v_j I w_j$, where $1 \leqslant j \leqslant n$.
From Theorem 4.1.1,

$$\sum_{j=1}^{n} \tan(\theta_j) - \sum_{1 \leq j_1 < j_2 < j_3 \leq n} \tan(\theta_{j_1}) \tan(\theta_{j_2}) \tan(\theta_{j_3}) + ... \pm \tan(\theta_1) \tan(\theta_2) ... \tan(\theta_n) = 0 \tag{4.42}$$

when the n-gon has an odd number of sides n, and

$$\sum_{j=1}^{n} \tan(\theta_j) - \sum_{1 \leq j_1 < j_2 < j_3 \leq n} \tan(\theta_{j_1}) \tan(\theta_{j_2}) \tan(\theta_{j_3}) + ... \pm \tan(\theta_1) \tan(\theta_2) ... \tan(\theta_{n-1}) = 0 \tag{4.43}$$

when the n-gon has an even number of sides n.
Set $s(r) = \sinh(r), t(p_i) = \tanh(p_i)$

From the triangles v_iIw_i,

$$\tan(\theta_i) = \frac{t(p_i)}{s(r)}$$

By using this relation, we can see that

$$s^{n-1}(r) = \frac{s^{n-3}(r) \sum_{1\leq j_1<j_2<j_3\leq n} t(p_{j_1})t(p_{j_2})t(p_{j_3}) - ... \mp t(p_1)t(p_2)...t(p_n)}{\sum_{j=1}^{n} t(p_j)} \tag{4.44}$$

when the n-gon has an odd number of sides n, and

$$s^{n-1}(r) = \frac{s^{n-3}(r) \sum_{1\leq j_1<j_2<j_3\leq n} t(p_{j_1})t(p_{j_2})t(p_{j_3}) - ... \mp s(r)t(p_1)t(p_2)...t(p_n)}{\sum_{j=1}^{n} t(p_j)} \tag{4.45}$$

when the n-gon has an even number of sides n.
Then, from the elementary symmetric polynomials definition, we find that

$$\sinh^{n-1}(r) = \frac{\sum_{j=3\ odd}^{n} (-1)^{\frac{(j+1)}{2}} \sinh^{n-j}(r)\sigma_j}{\sigma_1} \tag{4.46}$$

when the n-gon has an odd number of sides n, and

$$\sinh^{n-2}(r) = \frac{\sum_{j=3\ odd}^{n-1} (-1)^{\frac{(j+1)}{2}} \sinh^{n-(j+1)}(r)\sigma_j}{\sigma_1} \tag{4.47}$$

when the n-gon has an even number of sides n.
Using identity 2, Theorem 2.1.4, on the right triangle v_iIw_i,

$$\tan\left(\frac{\alpha_i}{2}\right)\sinh(p_i) = \tanh(r)$$

$1 \leqslant i \leqslant n$, where p_i is proved in Theorem 4.3.1 □

4.4 The Conjecture of the Hyperbolic General Formula

In this section, we present a corollary that the hyperbolic general constructions can be written as a polynomial which gives a value of 0 when the case of Poncelet's. Then, we formulate a conjecture that the Euclidean formula appears as a factor of the lowest order terms of a particular series expansion of the hyperbolic general formula.

From the previous comparison of Chapple's and Fuss's Formulae, in the Euclidean and hyperbolic planes, we see that the hyperbolic Chapple's Formula can be written as

$$\cosh^2(r+d)\sinh(r-R-d)-\cosh(r)\cosh^2(R)(\sinh(r)\cosh(r+R+d)-\sinh(R+d)) = 0 \tag{4.48}$$

As well the hyperbolic Fuss's Formula written as

$$s^4(r)\Big(\big(c^4(R)+s^4(d)\big)\big(s^2(R-d)-s^2(r)\big)\big(s^2(R+d)-s^2(r)\big)-s^4(r)c^4(R)s^4(d)\Big)$$
$$-\big(s^2(R-d)-s^2(r)\big)^2\big(s^2(R+d)-s^2(r)\big)^2=0 \tag{4.49}$$

which are polynomials in $\cosh(R), \sinh(R), \cosh(r), \sinh(r), \cosh(d), \sinh(d)$. Also, when we look at the first hyperbolic general formula, Theorem 4.3.1

$$\tanh^2(r) = 1 - \frac{\sin^2(\theta)}{\cos^2(\frac{\alpha_1}{2}) + \cos^2(\frac{\alpha_2}{2}) + 2\cos(\frac{\alpha_1}{2})\cos(\frac{\alpha_2}{2})\cos(\theta)} \tag{4.50}$$

It can be written as

$$\cosh^2(r)\sin^2(\theta) = \cos^2\left(\frac{\alpha_1}{2}\right) + \cos^2\left(\frac{\alpha_2}{2}\right) + 2\cos\left(\frac{\alpha_1}{2}\right)\cos\left(\frac{\alpha_2}{2}\right)\cos(\theta) \tag{4.51}$$

Squaring the equation,

$$\cosh^4(r)\sin^4(\theta)+2\sin^2(\theta)\left(2\cos^2\left(\frac{\alpha_1}{2}\right)\cos^2\left(\frac{\alpha_2}{2}\right)-\cosh^2(r)\left(\cos^2\left(\frac{\alpha_1}{2}\right)+\cos^2\left(\frac{\alpha_2}{2}\right)\right)\right)$$

$$+\left(\cos^2\left(\frac{\alpha_1}{2}\right)-\cos^2\left(\frac{\alpha_2}{2}\right)\right)^2=0 \tag{4.52}$$

By replacing $\sin(\theta), \cos\left(\frac{\alpha_1}{2}\right), \cos\left(\frac{\alpha_2}{2}\right)$ by their values as in Theorem 4.3.1, we get also a polynomial in $\cosh(R), \sinh(R), \cosh(r), \sinh(r), \cosh(d), \sinh(d)$.
These lead to the following corollary.

Corollary 4.4.1. *For $n \geqslant 3$, there exists a polynomial f_n in*

$$\cosh(R), \sinh(R), \cosh(r), \sinh(r), \cosh(d), \sinh(d)$$

such that if we have two disjoint circles C and D in the hyperbolic plane, with D inside C, with an embedded Poncelet's n-gon between them, such that, R_0 denotes the radius of C, r_0 denotes the radius of D and d_0 denotes the distance between the two circles' centres, then

$$f_n(\cosh(R_0), \sinh(R_0), \cosh(r_0), \sinh(r_0), \cosh(d_0), \sinh(d_0)) = 0$$

When R approaches 0, also r and d do, then by taking the approximations

$$\cosh(R) \simeq 1 + \frac{R^2}{2}$$

$$\sinh(R) \simeq R$$

and expand Chapple's and Fuss's Formulae, we find from Section 3.3 that the lowest order terms of hyperbolic Chapple's Formula after expansion is

$$-(R^2 - 2Rr - d^2)(d + R + r) \tag{4.53}$$

where $(R^2 - 2Rr - d^2) = 0$ is the Euclidean Chapple's Formula.
Also, from Section 3.5 the lowest order terms of hyperbolic Fuss's Formula after expansion is

$$\left((R^2-d^2)^2 + r^4 - 2r^2(R^2+d^2)\right)\left((R^2-d^2)^2 - 2r^2(R^2+d^2)\right) \tag{4.54}$$

where $(R^2 - d^2)^2 - 2r^2(R^2 + d^2) = 0$ is the Euclidean Fuss's Formula.

As well as, when we look at the first hyperbolic general formula and take $n = 3$,

$$\sin(\theta) = \frac{\cos\left(\frac{\alpha_2}{2}\right)}{\cosh(r)}$$

So, (4.52) can be written as

$$\cos^4\left(\frac{\alpha_2}{2}\right) + 2\cos^2\left(\frac{\alpha_2}{2}\right)\left(2\cos^2\left(\frac{\alpha_1}{2}\right)\cos^2\left(\frac{\alpha_2}{2}\right)\operatorname{sech}^2(r) - \left(\cos^2\left(\frac{\alpha_1}{2}\right) + \cos^2\left(\frac{\alpha_2}{2}\right)\right)\right.$$
$$\left. + \left(\cos^2\left(\frac{\alpha_1}{2}\right) - \cos^2\left(\frac{\alpha_2}{2}\right)\right)^2 = 0 \qquad (4.55)\right.$$

which represents the general formula when $n = 3$, such that

$$\cos\left(\frac{\alpha_1}{2}\right) = \frac{\sqrt{\sinh^2(R-d) - \sinh^2(r)}}{\sinh(R-d)} \qquad (4.56)$$

$$\tan\left(\frac{\alpha_2}{2}\right) = \frac{\sinh(r)\left(\sinh^2(R-d) + \sinh^2(R)\sinh^2(r)\right)}{\left(\sinh^2(R)\cosh^2(r) - \sinh^2(d)\right)\sqrt{\sinh^2(R-d) - \sinh^2(r)}} \qquad (4.57)$$

Compensation of 4.56 and 4.57 in 4.55, and take the limit when R approaches 0, so $\cosh(R) \simeq 1 + \frac{R^2}{2}$, and $\sinh(R) \simeq R$,
with some calculations, we can see that the lowest order terms of this general formula, when $n = 3$ is closed to

$$(R^2 - 2rR - d^2)(R^2 + 2rR - d^2)(R^2 - 2rd - d^2)(R^2 + 2rd - d^2) \qquad (4.58)$$

where $R^2 - 2rR - d^2 = 0$ is the Euclidean Chapple Formula.

Furthermore, when we take the second general hyperbolic formula, Theorem 4.3.2, when $n = 3$,

$$\sinh^2(r) = \frac{\tanh(p_1)\tanh(p_2)\tanh(p_3)}{\tanh(p_1) + \tanh(p_2) + \tanh(p_3)} \qquad (4.59)$$

If the triangle is isosceles, the formula will be

$$\sinh^2(r) = \frac{\tanh(p_1)\tanh^2(p_2)}{\tanh(p_1) + 2\tanh(p_2)} \qquad (4.60)$$

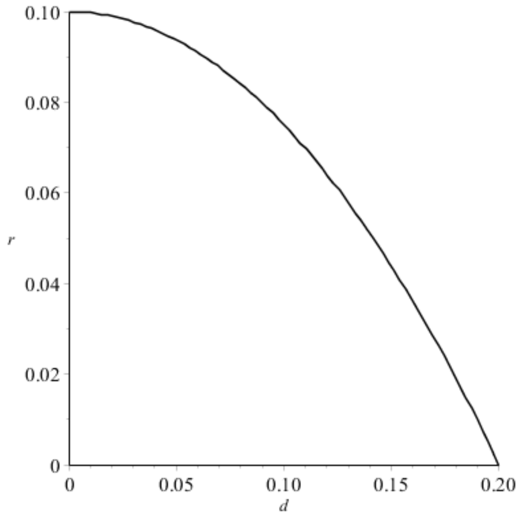

Figure 4.4: Euclidean Chapple's Formula when $R = .20$

where,

$$\tanh(p_1) = \frac{\sqrt{\cosh^2(R-d) - \cosh^2(r)}}{\cosh(R-d)} \tag{4.61}$$

$$\tanh(p_2) = \frac{(\sinh^2(R)\cosh^2(r) - \sinh^2(d))\sqrt{\cosh^2(R-d) - \cosh^2(r)}}{\cosh(R-d)\sinh^2(d) - \sinh(R)\cosh^2(r)(\sinh(d) - \cosh(R)\sinh(R-d))} \tag{4.62}$$

Usinh 4.61 and 4.62, in 4.60, and take the limit when R going to 0, we find that the lowest order terms of this general formula, when $n = 3$, is closed to

$$-(R^2 - 2rR - d^2)(R-d)^2(R^2 + 2rR - d^2) \tag{4.63}$$

where $R^2 - 2rR - d^2 = 0$ is the Euclidean Chapple Formula.

On the other hand, Maple shows us, that if we take a very small circumscribed circle with radius R, and look at the relation between r and d, we can see that the hyperbolic general expressions, when $n = 3$, Theorem 4.3.1 and Theorem 4.3.2, are getting very close to the Euclidean Chapple's as R is closed to 0, see Figures 4.4, 4.5 and 4.6.

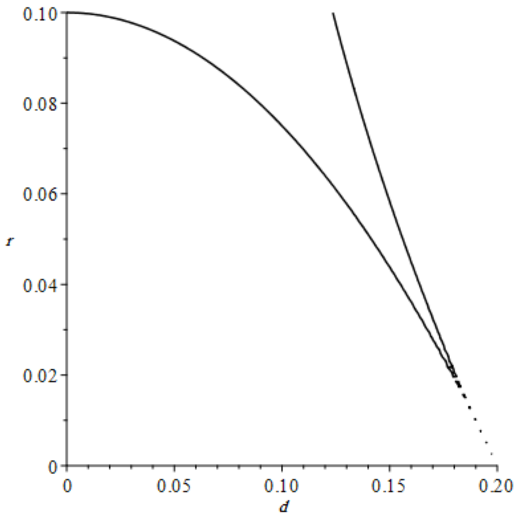

Figure 4.5: First Hyperbolic General Formula when $n = 3$, $R = .20$

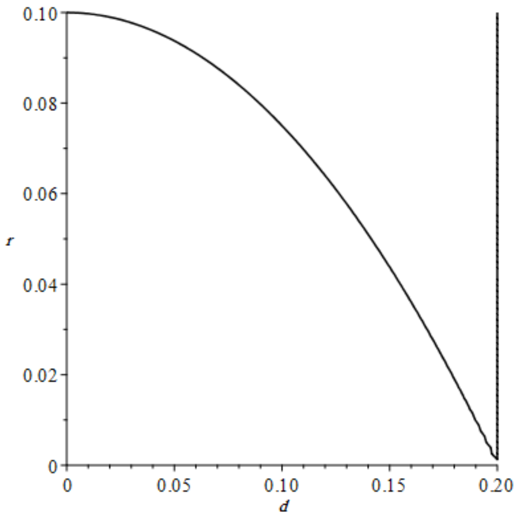

Figure 4.6: Second Hyperbolic General Formula when $n = 3$, $R = .20$

More general, when we look at the second general hyperbolic formula, Theorem 4.3.2, when R approaches 0, we notice that this formula and all the definitions of its components, are closed to the analogous to them in the Euclidean general formula, Theorem 4.2.1, for every sufficient small R.

These observations lead us to the next conjecture, which may help in proving Euclidean formulae, using the hyperbolic geometry.

Conjecture 4.4.2. *Let C and D be two disjoint circles in the hyperbolic plane, with D inside C. Let R denote the radius of C, r denote the radius of D and d denote the distance between the two circles' centres. Assume that their is a bicentric embedded n-gon between them, then, we can write the hyperbolic general formula as $f_n(R, r, d) = 0$, by using the expressions*

$$\cosh(x) \simeq 1 + \frac{x^2}{2}$$

$$\sinh(x) \simeq x$$

for R small, $x = R, r, d, R + d, ..., R + d + r$,
such that

$$f_n(R, r, d) = \sum_{i=1}^{k} g_i(R, r, d)$$

where $g_i(R, r, d)$ is homogeneous of order i,
then the lowest order non zero g_i has the Euclidean version as a factor.

4.5 Future Work

This chapter introduces and proves general formulae which satisfy when there is a hyperbolic embedded n-gon inscribed in one circle and circumscribed around another circle. Further research will be required to know the conditions for a hyperbolic n-gon to be bicentric, also, when these general hyperbolic formulae satisfy for two circles, is there a bicentric n-gon between these circles. Several other questions remain to be addressed, for instance, investigating if there are formulae connecting the data of two circles when there is a bicentric non-embedded n-gon between them in Euclidean and hyperbolic geometry and how they do relate. Also, finding the connection between them and the general formulae of embedded n-gon. Furthermore, we find that the Euclidean formulae appear as a factor of the lowest order terms of the hyperbolic formulae, we need to understand the other factors of the lowest order terms and if there is a relation between these factors and the number of sides of the n-gon.

Chapter 5

Poncelet's Three-Manifold X

In this chapter, we look at Poncelet's Theorem from other point of view. We have an idea based on Poncelet's Theorem 1.1.1 in the Euclidean plane and we study the special case when we have two circles and $n = 3$. If there is a triangle with vertices (a, b, c) inscribed in the unit circle S^1, it will circumscribed other circle D with radius r [50], such that $2r + d^2 = 1$ Theorem 1.3.2, where d is the distance between the circles' centres. When we get the triangle orbit, where Poncelet's Theorem realized, we get triples of points each represents vertices of a triangle circumscribing the same circle D. Nevertheless, there is only one circle inscribed in any triangle and touches each of the three sides of the triangle [50]. Therefore, we imagine the loop that contains of all vertices of triangles which inscribed in S^1 and circumscribed the same circle D. And then we define Poncelet's three-manifold X which contains of all triples of vertices of triangles that inscribed in S^1. So, we start this chapter by defining Poncelet's loop, which is constructed from all vertices of triangles circumscribing the same circle and inscribing in S^1 with direction x to y to z to x. Next, we define a three-dimensional manifold X, which is constructed from Euclidean Poncelet's Theorem. Define

$$X = \{(x, y, z) \mid x \neq y \neq z \neq x\} \subseteq S^1 \times S^1 \times S^1$$

where $(x, y, z) \in X$ represents vertices of a triangle inscribed in S^1 and circumscribing other circle. We prove that X is orientable non-compact three-manifold. Then, we define the configuration space of n distinct points on a topological space, and show that the Poncelet's three-manidold X is a configuration space of three points on a circle. Moreover, we prove that X is disconnected and can be represented as a disjoint union of two solid tori. After that, we define the Seifert fibre space SFS, and prove that Poncelet's three-manifold X is a Seifert fibre space where the fibers are the Poncelet's orbits. Because every Seifert fibre space admits geometric structure [43], this will help to study if the fibres that come from Poncelet's Theorem are geodesics with respect to the natural geometry of X. Lastly, we define a three-dimensional manifold X_R, which is constructed from the hyperbolic Poncelet's Theorem. Define

$$X_R = \{(x, y, z) \mid x \neq y \neq z \neq x\} \subseteq C \times C \times C$$

where $(x, y, z) \in X_R$ represents vertices of a hyperbolic triangle inscribed in a circle C with radius R, and circumscribed other circle. Then we present an example showing that the coordinates of hyperbolic isosceles triangle converge to the Euclidean coordinates, when R approaches 0, which gives a clue that the hyperbolic orbits converge to the Euclidean orbits, when R approaches 0.

5.1 Poncelet's Three-Manifold X

Take a point (a, b, c) which represents vertices of a triangle T, then get the triangle orbit. From Poncelet's Theorem, we can get triples of points each represents vertices of a triangle circumscribed the same circle as T. However, we know that there is only one circle C inscribes in any triangle T, and touches each of the three sides of the triangle [50], this leads us to the following definition.

Definition 5.1.1. *(Poncelet's Loop)*
Let $(a, b, c) = (e^{i\theta}, e^{i\alpha(\theta)}, e^{i\beta(\theta)})$, *where* $0 \leqslant \theta, \alpha(\theta), \beta(\theta) < 2\pi$ *are vertices of a triangle* T. *Define*

$$A_C = \{(e^{i\theta}, e^{i\alpha(\theta)}, e^{i\beta(\theta)}) \mid 0 \leqslant \theta, \alpha(\theta), \beta(\theta) < 2\pi\} \tag{5.1}$$

to be a loop contains of all vertices of triangles which have the same inscribed circle C.

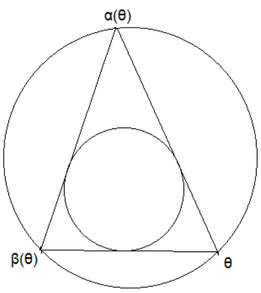

Figure 5.1: Poncelet's Orbit

Now we define Poncelet's three-manifold X which is deduced from Euclidean Poncelet's Theorem.

Definition 5.1.2. *(Poncelet's Three-Manifold)*
Poncelet's Theorem allows us to define a fibre structure on the three-dimensional manifold

$$X = \{(x, y, z) \mid x \neq y \neq z \neq x\} \subseteq S^1 \times S^1 \times S^1$$

where $(x, y, z) \in X$ *represents vertices of a triangle inscribed in* S^1.

Now, we introduce a lemma, which be used in proving that X is a Seifert fibre space

Lemma 5.1.1. *1. Every point in* X *lies on some loop like (5.1).*

2. Every point of X *lies on only one such loop.*

Proof. Let $(a, b, c) \in X$, from the definition of Poncelet's three-manifold; we find that it represents vertices of a triangle circumscribed by S^1. But we know that every triangle has an inscribed circle C [50]. Thus, from the definition of Poncelet's loop, $(a, b, c) \in A_C$, which proves the first part.

To prove 2, suppose $(\acute{a}, \acute{b}, \acute{c}) \in X$ such that $(\acute{a}, \acute{b}, \acute{c})$ belongs to two loops A_C and $A_{\acute{C}}$, so, $(\acute{a}, \acute{b}, \acute{c}) \in A_C$ and $(\acute{a}, \acute{b}, \acute{c}) \in A_{\acute{C}}$. From Poncelet's three- manifold definition, they are vertices of a triangle circumscribed the circles C and $\acute{C}$, however, there should be one circle C inscribed in any triangle [50]. Thus, $C = \acute{C}$ and $(\acute{a}, \acute{b}, \acute{c})$ belongs to just one loop.

As a result of the uniqueness of the inscribed circle for any triangle [50], we can see that $A_C \cap A_{\acute{C}} = \emptyset$ for every circle $\acute{C}$ different from C, then $X = \bigcup A_C$, where A_C is an embeded copy of S^1, which means that X can be represented as a disjoint union of fibres. □

The two following theorems prove that the Poncelet three-manifold is orientable and non-compact.

Theorem 5.1.3. *The Poncelet's three-manifold X is orientable.*

Proof. As the Poncelet three-manifold X is an open sub-manifold of the orientable manifold $S^1 \times S^1 \times S^1$, so it is orientable itself. □

Remark 5.1.4. *The Poncelet's three-manifold X is non-compact.*

Proof. To prove that, let $(x, y, x_n) \in X$, any infinite sequence of points in X such that $x_n \longrightarrow x$, where $x_n, x \in S^1$.
However, $(x, y, x) \notin X$, because

$$X = \{(x, y, z) \mid x \neq y \neq z \neq x\} \subseteq S^1 \times S^1 \times S^1$$

Thus, the infinite sequence of points sampled from the Poncelet three-manifold X get arbitrarily close to point $(x, y, x) \notin X$, that prove that X is non-compact. □

We present the following lemma, which represents the Cartesian coordinates of the Euclidean triangle's vertices which is inscribed in a circle and circumscribed about other circle. We use this lemma later to show that hyperbolic coordinates of the isosceles triangle converge to the Euclidean coordinates of isosceles triangle, when R approaches 0.

Lemma 5.1.2. *[38](Circular Path of the Triangle's Vertices)*
Let T be a triangle inscribed in a circle C with radius R and circumscribed a circle D, with radius r, then the vertices (a,b,c) of the triangle T can be given by the following Cartesian coordinates

$$a(\theta) = \big(r\cos(\theta) + d\cos^2(\theta) + W\sin(\theta), r\sin(\theta) + d\cos(\theta)\sin(\theta) - W\cos(\theta)\big)$$

$$b(\theta) = \big(r\cos(\theta) + d\cos^2(\theta) - W\sin(\theta), r\sin(\theta) + d\cos(\theta)\sin(\theta) + W\cos(\theta)\big)$$

$$c(\theta) = \left(\frac{R(2dR - (R^2+d^2)\cos(\theta))}{R^2+d^2-2dR\cos(\theta)}, \frac{(d^2-R^2)R\sin(\theta)}{R^2+d^2-2dR\cos(\theta)}\right)$$

$0 \leqslant \theta < 2\pi$
such that $W = \sqrt{R^2 - (r + d\cos(\theta))^2}$, where d is the distance between the circumscribed circle's centre and the inscribed circle's centre.

Proof. Because that the isometries of the Euclidean plane take circles to circles, then without loss of generality, it can be assumed that C and D are the circumscribed circle and the inscribed circle of a triangle T with vertices (a,b,c), such that, $O = [0,0]$ and $I = [d,0]$ are the centres of them, respectively.

The equations of the circumscribed circle and the inscribed circle are thus $C: x^2 + y^2 = R^2$, $D: (x-d)^2 + y^2 = r^2$, where $0 < r \leqslant \frac{R}{2}$. Assume the point where one side of the triangle touches D, then get the gradient of this side as it is vertical on the inscribed circle radius. after that, calculate the intersection of the circle C and this line to find a and b. Finally, the two tangents from a and b to D, which are different from the line, intersect in the third vertex c on C.

Set L for the line carrying a and b, since L has to be tangent to D, it touches D at the point $(d + r\cos(\theta), r\sin(\theta))$. Then, the gradient of L is $k = \frac{-\cos(\theta)}{\sin(\theta)}$, as a result, L is given by

$$L : x\cos(\theta) + y\sin(\theta) = r + d\cos(\theta)$$

with $\theta \in [0, 2\pi)$. The points a and b are the intersections of L and C, so by solving the two equations, we find that a and b are given by

$$a(\theta) = \big(r\cos(\theta) + d\cos^2(\theta) + W\sin(\theta), r\sin(\theta) + d\cos(\theta)\sin(\theta) - W\cos(\theta)\big) \tag{5.2}$$

$$b(\theta) = \big(r\cos(\theta) + d\cos^2(\theta) - W\sin(\theta), r\sin(\theta) + d\cos(\theta)\sin(\theta) + W\cos(\theta)\big) \tag{5.3}$$

where $W = \sqrt{R^2 - (r + d\cos(\theta))^2}$

The two tangents from a and b to D, which are different from the line L, intersect in third vertex $c \in C$, where

$$c(\theta) = \left(\frac{R(2dR - (R^2 + d^2)\cos(\theta))}{R^2 + d^2 - 2dR\cos(\theta)}, \frac{(d^2 - R^2)R\sin(\theta)}{R^2 + d^2 - 2dR\cos(\theta)}\right) \tag{5.4}$$

□

5.2 Poncelet's Three-Manifold as a configuration space

In this section, we define the configuration space, and show that the Poncelet's three-manifold X is a configuration space of three points on a circle. In addition to that, we prove that X is disconnected and can be expressed as disjoint union of two solid tori, that will help in determine the geometry of the three-manifold X.

Definition 5.2.1. *[12] (Configuration Space)*
The configuration space of n distinct labeled points on a topological space X, denoted $F_n(X)$, is the space $F_n(X) = \{(x_1, \cdots, x_n) \in X^n : x_i \neq x_j, \forall\ i \neq j\}$

Corollary 5.2.2. *As the Poncelet's three-manifold*

$$X = \{(x, y, z) \mid x \neq y \neq z \neq x\} \subseteq S^1 \times S^1 \times S^1$$

can be seen as three points on S^1, which are free to move, then it is a configuation space of three points on a circle, $F_3(S^1)$

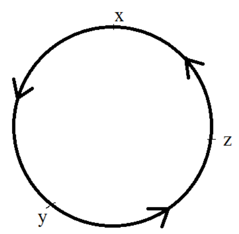

Figure 5.2: Counter Clockwise Moving

In the following lemma, we prove that Poncelet's three-manifold X is disconnected.

Lemma 5.2.1. *The Poncelet's three-manifold X is disconnected, which means it can be decomposed to disjoint union of non empty subsets.*

Proof. In this proof, we label three points x, y, z on the circle S^1 as vertices of a triangle. If we head clockwise from x, we either encounter y or z first. These two possibilities give rise to the two components of X.

Firstly, take x, y, z three points on S^1, moving counter clockwise, produces the triple (x, y, z), Figure 5.2. The other different triples corresponding to the same triangle with moving in same direction are (y, z, x) and (z, x, y).

On the other side, moving clockwise, produces the triple (x, z, y), Figure 5.3. The other different triples corresponding to the same triangle with moving in same direction are (z, y, x) and (y, x, z). Those six possibilities (x, y, z), (y, z, x), (z, x, y), (x, z, y) (z, y, x) and (y, x, z) are the all possibilities for triples as we have three points. That means X can be expressed as a disjoint union of two components, clockwise orbit component and counter clockwise orbit component. □

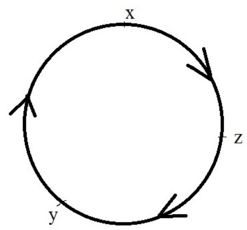

Figure 5.3: Clockwise Moving

We present the next lemma, as we use it in Theorem 5.2.3, to prove that the Poncelet's three-manifold X can be presented as a union of two disjoint solid tori.

Lemma 5.2.2. *[11] Suppose G is a topological group and $F_k(G)$ denotes the configuration space of k points in G. Then, there is a natural homeomorphism*

$$F_k(G) \cong G \times F_{k-1}(G \setminus \{e\})$$

where $\{e\}$ is the set of a single point.

Proof. This lemma was proved in [11] by defining a homeomorphism h such that

$$h : F_k(G) \to G \times F_{k-1}(G \setminus \{e\})$$

where

$$h(g_1, \cdots, g_k) = (g_1, g_1^{-1}g_2, \cdots, g_1^{-1}g_k)$$

□

Theorem 5.2.3. *The Poncelet's three-manifold X, can be expressed as the union of two disjoint solid tori.*

Proof. We prove this theorem, by applying lemma 5.2.2 on Poncelet's three-manifold X.

From that, we see that

$$F_3(S^1) = S^1 \times F_2(S^1 \setminus \{e\})$$

But $S^1 \setminus \{e\}$ is homeomorphic to R (and also to the interval $(0, 1)$),
thus,

$$F_2(S^1 \setminus \{e\}) = R \times F_1(R \setminus \{e\})$$

Finally, we note that for any space X, $F_1(X) = X$,
hence, putting these all together, this space we care about is

$$X = F_3(S^1) = S^1 \times (0, 1) \times [(0, 1) \cup (1, 2)]$$

which is homeomorphic to two disjoint copies of $S^1 \times (0, 1)^2$. As $(0, 1)^2$ is homeomorphic to an open disc of radius 1, each copy of $S^1 \times (0, 1)^2$ is the interior of $S^1 \times D^2$. So, we can see X as the union of two disjoint solid tori. This theorem should help in determine which geometry does the three-manifold X have. □

5.3 Seifert Fibre Space

This section presents a definition of Seifert fibre space, for that, we firstly define the fibred solid torus, and the fiber preserving homeomorphesim.

Definition 5.3.1. *[43](fibred solid torus)*
A fibred solid torus is a solid torus with a foliation by circles, which is finitely covered by a trivial fibred solid torus. Where the trivial fibred solid torus is defined as $D^2 \times S^1$ with the product foliation by circles. Thus, the fibres are the circles $y \times S^1$, for y in D^2.

Also, we can define *a fibred solid torus* in other way [27]. Let (μ, ν) be a pair of relatively prime integers, and let D^2 be the unit disk defined in polar coordinates as $D^2 = \{(r, \theta) : 0 \leqslant r \leqslant 1\}$. A fibred solid torus of type (μ, ν) is the quotient of the cylinder $D^2 \times I$, via the identification $((r, \theta), 1) = ((r, \theta + 2\pi\frac{\nu}{\mu}), 0)$. The fibres are the images of the arcs $\{x\} \times I$.

Thus, the core arises from the identification of $\{0\} \times I$, can meets the meridianal disk $D^2 \times 0$ once. Every other fibre meets $D^2 \times 0$ exactly μ times. Up to fibre preserving homeomorphism, we may assume that $\mu > 0$ and $0 < \nu < \frac{\mu}{2}$. If $\mu > 1$, we call the core an exceptional fibre, otherwise we call the core a regular fibre.

Definition 5.3.2. *[27](Fibre Preserving Homeomorphism)*
If M and N are Seifert fibre spaces, then a fibre preserving homeomorphism from M to N is a homeomorphism from M to N, that takes each fibre of M onto a fibre of N.

Definition 5.3.3. *[43] (The Seifert Fibre Space (SFS))*
The Seifert fibre space is a three-dimensional manifold M, with a decomposition of M into disjoint circles, called fibres, such that each circle has a neighbourhood in M which is a union of fibres and is homeomorphic to a fibred solid torus via a fibre preserving homeomorphism.

In the next section, we prove that Poncelet's three-manifold X is a Seifert fibre space, with one exceptional fibre of order 3, that is the case when the triangle is equilateral. This exceptional fibre is the core of the quotient $(D^2 \times I)/\sim$ such that $((r, \theta), 1) \sim ((r + \frac{2\pi}{3}), 0)$. When we take an equilateral triangle T with vertices (a, b, c) inscribed in S^1 and circumscribed a circle D; by Chapple's Formula, the circles are concentric. Then when we get the triangle orbits, it can be rotated within the circular ring bounded by S^1 and D. As a result, when the equilateral triangle T going the third way around, it get back to the same triangle.

5.4 Poncelet's Three-Manifold X is SFS

The purpose of this section is to prove that the three-dimensional manifold X is Seifert fibre space SFS. As we know, every SFS admits a geometric structure [43]. Thus, proving that X is SFS, should help in determine the fibres, that come out from Euclidean Poncelet's construction, and determine if these fibres are geodesic in respect of the geometry of X.

In the following theorem, we give a proof that the three-dimensional manifold X is SFS. To do that, we prove that this manifold decomposes into a disjoint union of fibres, such that each fibre has a tubular neighbourhood, that forms a fibred solid torus.

Theorem 5.4.1. *The three-dimensional manifold*

$$X = \{(x, y, z) \mid x \neq y \neq z \neq x\} \subseteq S^1 \times S^1 \times S^1$$

is SFS, with one exceptional fibre contains of all triples which are vertices of equilateral triangles.

Proof. To prove that X is SFS, we should prove that it decomposes into a disjoint union of curves called fibres, such that each fibre has a tubular neighbourhood that forms a fibred solid torus. We start with the regular fibres, following that by the special case when the fibre is exceptional.

Firstly, from lemma 5.1.1, we can see that X can be realized as a disjoint union of circular fibers. These fibres are the loops A_C such that every loop contains of all vertices of triangles which have the same inscribed circle.

Now, to prove that each fibre has a tubular neighbourhood, Let A_C be the loop contains of all vertices of triangles, which are not equilateral, and have the inscribed circle C with radius r, and set $(a, b, c) \in A_C$. Make a small variation to the radius of the inscribed circle r, and take $\epsilon > 0$ small enough. We can see that $(\acute{a}, \acute{b}, c)$ are vertices of a triangle circumscribed a circle with radius $\acute{r}$, such that

$$\mid r - \acute{r} \mid < \epsilon \tag{5.5}$$

Assume that $\acute{a} = a$, $\acute{b} = b$. then the triangle with vertices $\acute{a}, \acute{b}, c$ is the same as the triangle with vertices a, b, c. However, we know that there is only one circle inscribes in any triangle and touches each of the three sides of the triangle [50]. As a result of that, $r = \acute{r}$, but that contradict 5.5. Thus, $\acute{a} \neq a$, $\acute{b} \neq b$.

Look at the disks D^2 around the loop A_C which centre at the points $(a, b, c) \in A_C$, where $0 \leqslant \theta < 2\pi$. We can see that for any two points $(\acute{a}, \acute{b})$ and $(\acute{\acute{a}}, \acute{\acute{b}})$ in those disks (with keeping c fixed as it does not depend on the inscribed circle radius Theorem 5.1.2, $\acute{a} \neq \acute{\acute{a}}$ and $\acute{b} \neq \acute{\acute{b}}$ as $\acute{r} \neq \acute{\acute{r}}$, where $\acute{r}$, $\acute{\acute{r}}$ are the radii of the inscribed circles of the triangles with vertices $(\acute{a}, \acute{b}, c)$ and $(\H{a}, \H{b}, c)$, respectively. That is because, if $\acute{a} = \H{a}$ and $\acute{b} = \H{b}$, the triangle with vertices $(\acute{a}, \acute{b}, c)$ coincident with the triangle with vertices $(\H{a}, \H{b}, c)$. Which means that $\acute{r} = \H{r}$, and that contradict what we set. This gives us a tubular neighbourhood for the loop A_C, contains of all loops $A_{\acute{C}}$ such that $\mid \acute{r} - r \mid < \epsilon$, where $r, \acute{r}$ are the radii of the circles $C, \acute{C}$ respectively.

We now showing that the tubular neighbourhood of the regular fiber A_C is homeomorphic to a fibred solid torus. For that, we define a function between this tubular neighbourhood and a fibred solid torus $D^2 \times S^1$ and prove that it is bijection (injective and surjective), continuous and the inverse function is also continuous.
Let us define a function $h : T \to D^2 \times S^1$, such that $h(A_{\acute{D}}) = (\acute{a}, \acute{b}) \times \theta$ for all $0 \leqslant \theta < 2\pi$, where T denotes the tubular neighbourhood of the regular loop A_C. We would like to prove that h is homeomorphism.

To prove that h is bijection, we should prove that it is injective and surjective function.

Firstly, Let $A_{\acute{D}}$ and $A_{\H{D}}$ be two loops in T, such that $h(A_{\acute{D}}) = h(A_{\H{D}})$ so, $(\acute{a}, \acute{b}) = (\H{a}, \H{b})$, which means that $(\acute{a}, \acute{b}, c)$ and $(\H{a}, \H{b}, c)$ are vertices of same triangle. As a result, they belong to the same loop. Then, $A_{\acute{D}} = A_{\H{D}}$, which prove that h is an injective function. Moreover, for each point $(\acute{a}, \acute{b}) \times \theta \in D^2 \times S^1$, the triple $(\acute{a}, \acute{b}, c)$ represents vertices of a triangle. Furthermore, from the Poncelet's loop definition, they belong to a loop $A_C \in T$. So, for every point $(\acute{a}, \acute{b}) \times \theta$, there is a loop A_C, such that $h(A_C) = (\acute{a}, \acute{b}) \times \theta$ which proves that h is a surjective function.

To prove the continuity at a point, we should prove that for any neighbourhood of its image, the pre image is again a neighbourhood of that point. Let $(\H{a}, \H{b}) \in N$ where N is a neighbourhood of the point $(\acute{a}, \acute{b}) \in D^2$. Then, $(\acute{a}, \acute{b}, c)$ and $(\H{a}, \H{b}, c)$ are vertices of triangles circumscribed circles with radii $\acute{r}$, $\H{r}$, respectively, such that $\mid \H{r} - \acute{r} \mid < \epsilon$, where $\epsilon > 0$ small enough. Then, $(\H{a}, \H{b}, c) \in A_{\H{C}}$, in the neighbourhood of $A_{\acute{C}}$, such that $(\acute{a}, \acute{b}, c) \in A_{\acute{C}}$. So, for any neighbourhood of the point $(\acute{a}, \acute{b}) \times \theta$, the pre image is a set of loops $\{A_{\H{C}}\}$ which make a tubular neighbourhood for the loop $A_{\acute{C}}$, and that satisfied for all the points in $D^2 \times S^1$. As a result of that, the function h is continuous.

To prove the continuity of the inverse function h^{-1}, let $A_{\acute{\acute{C}}} \in T$ is a loop in the neighbourhood of $A_{\acute{C}}$. Then, $A_{\acute{\acute{C}}}$ contains of all vertices of the triangles circumscribed a circle with radius $\acute{\acute{r}}$, such that $\mid \acute{\acute{r}} - \acute{r} \mid < \epsilon$, where $\epsilon > 0$ small enough, and $\acute{r}$ is the radius of the inscribed circle of the triangles of the loop $A_{\acute{C}}$. As a result of the previous discussion of the tubular neighbourhood, the point $(\acute{\acute{a}}, \acute{\acute{b}}, c) \in A_{\acute{\acute{C}}}$ is located at the neighbourhood of the point $(\acute{a}, \acute{b}, c) \in A_{\acute{C}}$. So, any neighbourhood of the loop $A_{\acute{C}}$ has an image contains of the points $(\acute{\acute{a}}, \acute{\acute{b}})$ in the neighbourhood of the point $(\acute{a}, \acute{b})$, where $0 \leqslant \theta < 2\pi$. This happen for all loops $A_{\acute{C}}$. So the function h^{-1} is also continuous.
From the previous, h is a homeomorphism between the tubular neighbourhood T and the fibred solid torus $D^2 \times S^1$, which takes every fibre A_C to a torus fibre. That completes the part of the proof for the regular fibres.

Turning now to prove that the exceptional fibre satisfies the conditions of SFS. Recall that the exceptional fibre of the Poncelet's three-manifold X contains of all triples which are vertices of equilateral triangles. So, in this three-manifold the exceptional fibre closes up in third way around but the other fibres keep going around it. However, if we start at a point and take equilateral triangle, when we spin it the third way around, we get the same triangle but with different point by interchanging the coordinates. Thus, It should go all the way around to get back to exactly the same point.

Now, we prove that this exceptional fiber has a tubular neighbourhood that forms a fibred solid torus.
To prove that it has a tubular neighbourhood, Let A_D be the loop contains of all vertices of equilateral triangles, and have the inscribed circle D with radius $\frac{1}{2}$, and set $(1, e^{\left(\frac{2\pi i}{3}\right)}, e^{\left(\frac{4\pi i}{3}\right)}) \in A_D$. Make a small variation to the inscribed circle and take $\epsilon > 0$ small enough, so that $0 < d < \epsilon$ is the distance between the S^1 centre and the centre of a new circle C. We can see that $(1, b, c)$ are vertices of a triangle circumscribed C. Assume that $b = e^{\left(\frac{2\pi i}{3}\right)}$, $c = e^{\left(\frac{4\pi i}{3}\right)}$, then the triangle with vertices $1, b, c$ coincides on the equilateral triangle. However, we know that there is only one circle inscribes in any triangle and touches each of the three sides of the triangle [50]. As a result of that, $C = D$, which contradict what we set. So, $b \neq e^{\left(\frac{2\pi i}{3}\right)}$, $c \neq e^{\left(\frac{4\pi i}{3}\right)}$.

Look at the disks D^2 around the loop A_D, which centre at the points

$$(e^{i\alpha}, e^{i(\alpha+\frac{2\pi}{3})}, e^{i(\alpha+\frac{4\pi}{3})}) \in A_D$$

where $0 \leqslant \alpha < 2\pi$. We can see that, for any two points (b, c) and $(\acute{b}, \acute{c})$ in those disks, $b \neq \acute{b}$ and $c \neq \acute{c}$ as $d \neq \acute{d}$, where d , $\acute{d}$ are the distances between the S^1 center and the centres of the inscribed circles of the triangles with vertices (a, b, c) and $(a, \acute{b}, \acute{c})$, respectively. That because, if $b = \acute{b}$ and $c = \acute{c}$, the triangle with vertices (a, b, c) coincidences with the triangle with vertices $(a, \acute{b}, \acute{c})$. Which means that $d = \acute{d}$ and that contradict what we set. This gives us a tubular neighbourhood for the loop A_D, contains of all loops A_C, such that $0 < d < \epsilon$.

We now showing that the tubular neighbourhood of the exceptional fibre A_D is homeomorphic to a fibred solid torus. For that we define a function between this tubular neighbourhood and a fibred solid torus $D^2 \times S^1$ and prove that it is bijection (injective and surjective), continuous and the inverse function is also continuous.
Let us define a function $f : T \to D^2 \times S^1$, such that $f(A_D) = (0, 0) \times \theta$ and $f(A_C) = (b, c) \times \theta$, for all $0 \leqslant \theta < 2\pi$, where T denotes the tubular neighbourhood of the exceptional loop A_D. We would like to prove that f is homeomorphism.

To prove that f is bijection, we should prove that it is injective and surjective function. Firstly, Let A_C and $A_{\acute{C}}$ be two loops in T such that $f(A_C) = f(A_{\acute{C}})$, so, $(b, c) = (\acute{b}, \acute{c})$, which means that (a, b, c) and $(a, \acute{b}, \acute{c})$ are vertices of the same triangle. As a result, they belong to the same loop. Then, $A_C = A_{\acute{C}}$, which prove that f is an injective function. Moreover, for each point $(b, c) \times \theta \in D^2 \times S^1$, the triple (a, b, c) represents vertices of a triangle. Furthermore, from the Poncelet's loop definition, they belong to a loop $A_C \in T$. So, for every point $(b, c) \times \theta$, there is a loop A_C, such that $f(A_C) = (b, c) \times \theta$, which proves that f is a surjective function.

To prove the continuity at a point, we should prove that for any neighbourhood of its image the pre image is again a neighbourhood of that point. Let $(\acute{b}, \acute{c}) \in N$ where N is a neighbourhood of the point $(b, c) \in D^2$. Then, (a, b, c) and $(a, \acute{b}, \acute{c})$ are vertices of triangles circumscribed circles such that $0 < d, \acute{d} < \epsilon$ where $d, \acute{d}$

are the distances between the S^1 centre and the circles centres, respectively and $\epsilon > 0$ small enough. Then, $(a, \acute{b}, \acute{c}) \in A_{\acute{C}}$ in the neighbourhood of A_C such that $(a, b, c) \in A_C$. So, for any neighbourhood of the point $(b, c) \times \theta$ the pre image is a set of loops $\{A_{\acute{C}}\}$ which make a tubular neighbourhood for the loop A_C and that satisfied for all the points in $D^2 \times S^1$. As a result of that, the function h is a continuous function.

To prove the continuity of the inverse function f^{-1}, let $A_{\acute{C}} \in T$ is a loop in the neighbourhood of A_C. Then, $A_{\acute{C}}$ contains of all vertices of the triangles circumscribed a circle with distance $\acute{d}$ between its centre and the S^1 centre, such that $0 < \acute{d} < \epsilon$, where $\epsilon > 0$ small enough. As a result of the previous discussion of the tubular neighbourhood, the point $(a, \acute{b}, \acute{c}) \in A_{\acute{C}}$ is located at the neighbourhood of the point $(a, b, c) \in A_C$. So, any neighbourhood of the loop A_C has an image contains of the points $(\acute{b}, \acute{c})$ in the neighbourhood of the point (b, c), where $0 \leqslant \theta < 2\pi$. This happen for all loops A_C. So the function f^{-1} is also continuous.

Then f is a homeomorphism between the tubular neighbourhood T of the exceptional fiber and the fibred solid torus $D^2 \times S^1$, which takes every fibre A_C to a torus fiber.
From those two parts, the proof is completed, that Poncelet's three-manifold X is SFS. □

5.5 Hyperbolic Poncelet's Three-Manifold X_R

This section defines hyperbolic Poncelet's three-manifold X_R, which is deduced from the hyperbolic Poncelet's Theorem, when $n = 3$, it should have the same characteristic as Poncelet's three-manifold X, which is discussed at the previous sections, as it has the same way of construction, from triples of points on a circle, that are vertices of a triangle. Also, we present an example shows that when R approaches 0, the coordinates of a hyperbolic isosceles triangle converge to the Euclidean coordinates, which presents an evidence that the hyperbolic fibres converge to the Euclidean fibres when R approachs 0

Hyperbolic Poncelet's three-manifold is defined as follow

Definition 5.5.1. *(Hyperbolic Poncelet's Three-Manifold)*
Hyperbolic Poncelet's Theorem allows us to define a fibre structure on the three-dimensional manifold

$$X_R = \{(x, y, z) \mid x \neq y \neq z \neq x\} \subseteq C \times C \times C$$

where $(x, y, z) \in X_R$ represents vertices of a triangle inscribed in a circle C with radius R.

When we look at the hyperbolic Poncelet's three-manifold for radius R, we get a collection of orbits, as well the Euclidean Poncelet's three-manifold X presents a collection of orbits. We have a general question that as R approaches 0, do the hyperbolic orbits converge to the Euclidean ones.

We have an explicit example showing that the hyperbolic coordinates of an isosceles triangle converge to the Euclidean coordinates, when R approaches 0, which we think it is true for all triangles, as when the hyperbolic isosceles triangle moves slowly around the inscribed circle, where R is very small, the hyperbolic vertices should be close to the Euclidean vertices as smaller area of hyperbolic plane looks more Euclidean.

Example 5.5.2. *Let T be a triangle in Poincaré disk model, which inscribed in a circle C, with radius R, and circumscribed about a circle D, with radius r, assume that the circumscribed circle is centred at the origin O and the inscribed circle is centred at the point on the positive real axis, with hyperbolic distance d. Let v_j, $1 \leqslant j \leqslant 3$, be the vertices of T, and set the triangle, so that one of its vertices, say v_2, is at the same line with the circumscribed circle's centre and the inscribed circle's centre. Thus, the triangle line of symmetry passes through the centres of the circles, as a result of that, the triangle is an isosceles triangle, $v_1 = Re^{i\theta}$, $v_2 = -R$ and $v_3 = Re^{-i\theta}$; see Figure 5.4.*
By using hyperbolic trigonometric functions, we find that

$$\cos(\theta) = \frac{\tanh(r + d)}{\tanh(R)}, \sin(\theta) = \frac{\sqrt{\tanh^2(R) - \tanh^2(r + d)}}{\tanh(R)}$$

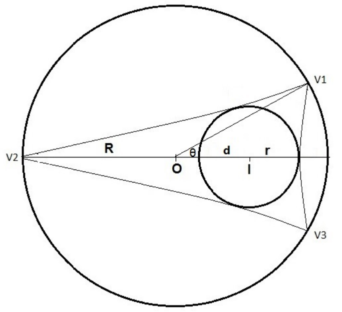

Figure 5.4: Hyperbolic Isosceles Triangle

Letting $R \longrightarrow 0$*, so,* $\lim_{R\to 0} \frac{\tanh(R)}{R} = 1$*, which means that* $\tanh(R) \simeq R$*, where* R *is sufficient small.*
Applying the limit, we find that

$$\cos(\theta) \simeq \frac{r+d}{R}$$

$$\sin(\theta) \simeq \frac{\sqrt{R^2-(r+d)^2}}{R}$$

Then

$$v_1 = Re^{i\theta} \simeq r + d + i\sqrt{R^2-(r+d)^2}$$

where R, r, d *are sufficient small, which is the same as the Euclidean coordinates in same situation, where from Theorem 5.1.2, when we choose* $\theta = 0$*, we can see that a vertice* b *of the Euclidean isosceles triangle, can be given as the following:*

$$b = (r+d, +\sqrt{R^2-(r+d)^2})$$

As well, the other vertices of the hyperbolic triangle T *will close to the Euclidean vertices, when* R *approaches* 0*.*

This gives a clue, that the hyperbolic fibre is close the Euclidean one, when R approaches 0.

5.6 Future Work

In this chapter, we define a three-manifold X, which is constructed from the Euclidean Poncelet's Theorem. We show that X can be represented as a disjoint union of two solid tori and we prove that this three-manifold is a Seifert fibre space. So, X admits geometric structure, which we thought that it is a hyperbolic geometry. Because it is SFS, we need to understand its fibres and how they appear, also we need to determine if these fibres, which come from X as a Seifert fibre space are geodesics in the natural metric of Poncelet's three-manifold X

On the other hand, we define the hyperbolic three-manifold X_R which is constructed from the hyperbolic Poncelet's Theorem and present an example, that the vertices of an isosceles hyperbolic triangle, is converging to the Euclidean vertices.
There is an issue raises from comparing the hyperbolic orbits with the Euclidean orbits, as when we take a triple of points on the circle S^1, and look at its Euclidean orbit, as vertices of a Euclidean triangle circumscribed a circle, and take the same triple of points on a circle with radius R, for some values of R, and look at its hyperbolic orbit, as vertices of hyperbolic triangle circumscribed a circle, what is the relation between these orbits.
Our example shows that the hyperbolic vertices of an isosceles triangle converge to the Euclidean vertices as $R \longrightarrow 0$, consequently, we have a thought that the hyperbolic orbits converge to the Euclidean orbits when R approaches 0, as a smaller and smaller area of hyperbolic geometry, appears more and more Euclidean.

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