University of Southampton
Faculty of Mathematical Sciences

Counting Homotopy Types of Certain Gauge Groups.

by

Tyrone Cutler

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ABSTRACT

FACULTY OF MATHEMATICAL SCIENCES

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Abstract

Two original studies into the homotopy theory of gauge groups are presented. In the first the number of homotopy types of $Sp(3)$-gauge groups over S^4 are counted, obtaining exact odd-primary information and best possible 2-local bounds. This method follows a thorough examination of the commutator product $Sp(1) \wedge Sp(3) \to Sp(3)$ to draw its conclusions.

In the second work the homotopy types of $U(n)$-gauge groups over S^4 and $\mathbb{C}P^2$ are examined. Homotopy decompositions of the $U(n)$-gauge groups over S^4 are given in terms of certain $SU(n)$ and $PU(n)$-gauge groups and these are then use these to count the number of $U(2)$-, $U(3)$- and $U(5)$-gauge groups over S^4.

Over CP^2 the problem is delicate. Numerous results for the $U(n)$-gauge groups are given for general values of n, including certain homotopy decompositions and p-local properties. In the final section the previous results are applied to the case of $U(2)$ to obtain the most complete statements.

The final part of the thesis is a survey article detailing the history of the homotopy theory of gauge groups. Influential papers and turning points in the subject are discuss, and the survey ends with an outlook on possible future directions for research.

Contents

Declaration of Authorship.

I, Tyrone Cutler, declare that the thesis entitled *Counting Homotopy Types of Certain Gauge Groups* and the work presented in the thesis are both my own, and have been generated by me as the result of my own original research. I confirm that:

1. this work was done wholly or mainly while in candidature for a research degree at this University;
2. where any part of this thesis has previously been submitted for a degree or any other qualification at this University or any other institution, this has been clearly stated;
3. where I have consulted the published work of others, this is always clearly attributed;
4. where I have quoted from the work of others, the source is always given. With the exception of such quotations, this thesis is entirely my own work;
5. I have acknowledged all main sources of help;
6. where the thesis is based on work done by myself jointly with others, I have made clear exactly what was done by others and what I have contributed myself;

Signed

Date

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1 Introduction.

Gauge groups have a rich history in algebraic topology and in particular there have been many papers dedicated to studying G-gauge groups over certain simply connected 4-manifolds X, for G a compact Lie group. As the rank of G and the topological complexity of X increases the problem of enumerating the distinct homotopy types of G-gauge groups over X generally becomes much more challenging and there is still much that is not known in many cases.

In this thesis I present two original studies into the homotopy theory of such gauge groups. In the first I enumerate the homotopy types of $Sp(3)$-gauge groups over S^4. Even in the context of the relatively simple base space S^4, gauge groups associated with the 21-dimensional, rank 3 Lie group $Sp(3)$ have not previously been studied. My approach to the problem is to study the commutator $Sp(1) \wedge Sp(3) \to Sp(3)$ and infer from it information about the number of distinct homotopy types amongst the $Sp(3)$-gauge groups. I obtain complete odd-primary information and provide best possible bounds on the 2-local information.

In the second work included in the thesis I study the homotopy types of $U(n)$-gauge groups over S^4 and $\mathbb{C}P^2$. Most of the previous studies of gauge groups over S^4 and $\mathbb{C}P^2$ that appear in the literature have been of gauge groups associated with simply connected, compact Lie groups. Despite the fact that $U(n)$-gauge groups over Riemann surfaces have been extensively studied, they have not previously been studied in the context of 4-dimensional base space.

Working over S^4 I provide homotopy decompositions of the $U(n)$-gauge groups in terms of certain $SU(n)$- and $PU(n)$-gauge groups. I then use these homotopy decompositions to count the number of distinct homotopy types of $U(2)$-, $U(3)$- and $U(5)$-gauge groups based on pre-existing knowledge of the homotopy theoretic properties of the $SU(n)$- and $PU(n)$-gauge groups.

Moving on to look at $U(n)$-gauge groups over $\mathbb{C}P^2$ the problem becomes a lot more delicate. The $U(n)$-gauge groups are now doubly indexed by the first and second Chern classes of the principal bundles to which they are associated. A non-trivial first Chern class represents a new twisting to the bundle structure whose effects on the homotopy type of the associated gauge group has not previously been examined. In this context not even the based gauge groups are well understood.

I provide homotopy decompositions of these $U(n)$-gauge groups in certain cases and study their p-local behaviour. Results are stated throughout the paper for general values of n, and these are then applied in the context of $U(2)$-gauge groups to obtain the most complete results.

The final section of this thesis is comprised of a survey article detailing the history of the application of algebraic topological methods to gauge theory and gauge groups. I discuss the important rôle that the topology of gauge groups has played in Yang-Mills theory and how this has influenced concurrent mathematical directions. I also review the connections that the study of the homotopy types of gauge groups has to homotopy commutativity in compact Lie groups, as well as its connections to self-equivalence groups and infinite dimensional Lie theory. I end the article with an examination of the current mathematical trends and an outlook on future directions of research such as higher gauge theory.

The organisation of the thesis is as follows. Each of the three articles is essentially presented separately. Each contains a brief abstract providing explicit details of its particular results, and an introduction providing the details of its unique background and motivation. Since the articles are presented in this manner a more detailed account of their structures is given in the introduction fo the article itself.

Appearing first, before any of the studies, is a section containing the background information, definitions and notation necessary to best present the later research. I define the category of principal G-bundles that I will adopt and discuss the basic properties of its objects and morphisms. We are led through this to a brief review of classifying spaces for topological groups, which feature prominently in the later work. Finally we give the definition of a *gauge group* and describe some of their basic algebraic and geometric properties.

The second section of the thesis is where the detailed work begins with the study of the homotopy types of $Sp(3)$-gauge groups over S^4 and this is followed in section 4 by the work on the homotopy types of $U(n)$-gauge groups over S^4 and $\mathbb{C}P^2$. The final section of the thesis contains the survey article.

2 The Homotopy Theory of Principal Bundles.

There are many good texts that deal with the material in this section and the basic reference is Husemoller's classic textbook [100]. Another wonderful book dedicated to the topic of bundles is Steenrod's text [177] although its notation is somewhat dated. For more modern but briefer texts that deal with related material we recommend are tom Dieck's Algebraic Topology [198] and Ralph Cohen's web notes [52].

In the following we assume that all (topological) spaces have the homotopy types of CW-complexes and each such space possesses a distinguised base point. We denote by $\mathcal{T}$ the category of such such spaces and use $h\mathcal{T}$ to denote its associated homotopy category. A map is assumed to be continuous although not necessarily based. For spaces X, Y we denote by $[X, Y]$ the set of homotopy classes of pointed maps $X \to Y$ and by $[X, Y]_{free}$ the set of free homotopy classes $X \to Y$.

Definition 2.1 *A **bundle** is a map $p : E \to B$ called the **projection**. The space E is said to be the **total space** and the space B is called the **base space**. For each point $b \in B$, the inverse image $F_b = p^{-1}(b) \subseteq E$ is called the **fibre** over b.*

This is possibly the loosest definition that captures the basic essence of what we mean by a *bundle*. If $p : E \to B$ is a bundle as defined above then one shoud think of the underlying set of E as the union of the fibres $F_b = p^{-1}(b)$ parametised by the points $b \in B$. The topology on E serves to glue the fibres together and, depending on the topological complexity of E, this may happen in an extremely non-trivial manner.

Definition 2.2 *Given bundles $p : E \to B$, $p' : E' \to B'$ a **bundle morphism** $\theta = (f, \bar{f})$ from p to p' is a pair of maps $f : B \to B'$, $\bar{f} : E \to E'$ such that the diagram*

$$\begin{tikzcd} E \arrow[r, "\bar{f}"] \arrow[d, "p"'] & E' \arrow[d, "p'"] \\ B \arrow[r, "f"] & B' \end{tikzcd} \tag{2.1}$$

is strictly commutative. We denote this $\theta = (f, \bar{f}) : p \Rightarrow p'$. If $\varphi = (g, \bar{g}) : p' \Rightarrow p''$ is a second bundle morphism then the composition $\varphi \circ \theta = (g \circ f, \bar{g} \circ \bar{f}) : p \Rightarrow p''$ is defined. A morphism $\theta = (f, \bar{f}) : p \Rightarrow p'$ is said to be an ***isomorphism*** *if there is a morphism $\varphi = (g, \bar{g}) : p \to p''$ such that $\varphi \circ \theta = (id_E, id_B) = id_p$ and $\theta \circ \varphi = (id_{E'}, id_{B'}) = id_{p'}$.*

It is easy to see from the definitions that morphisms of bundles have good compositional properties and there are obvious identity morphisms. Furthermore the cartesian product in $\mathcal{T}$ furnishes Bun with a product. Thus the collection of bundles as defined above forms a category Bun, which is simply the morphism category $Mor(\mathcal{T})$. As is well known this caegory has a rich structure and in the following we shall refine the basic notion of a bundle by requiring additional structure on the fundamental objects.

Example 2.3 *The following provide some typical examples of bundles that arise frequently in application.*

1. *Given spaces B, F the* ***product*** *or* ***trivial*** *bundle over B with fibre F is the projection*
$$pr_B : B \times F \to B \tag{2.2}$$
onto the first factor. The fibre over $b \in B$ is $F_b = \{b\} \times F \cong F$.

2. *Given a space B the identity id_B define a bundles with base and total space equal to B. The fibre over b is the one point space $F_b = \{b\}$. If $p : E \to B$ is any bundle then the map p defines a unique morphism $P = (p, id_B) : p \Rightarrow id_B$. Hence id_B is a terminal object in the category $Bun(B)$ defined below in (x.x).*

3. *Given a bundle $p : E \to B$ and a map $f : A \to B$, then the* ***pullback*** *$f^*E = \{(a, e) \in A \times E \,|\, f(a) = p(b)\}$ becomes a bundle over A with projection $f^*p : f^*E \to A$ induced by the projection on the first factor. If $f : A \subseteq B$ is a subspace inclusion then we write $f^*E = E|_A$ and refer to it as the* ***restriction*** *of E.*

Definition 2.4 *Let $p : E \to B$ be a bundle. A cross section s of p is a morphism $s : id_B \Rightarrow p$. More precisely it is a map $s : B \to E$ such that $p \circ s = id_B$. We denote the space of cross sections of a given bundle by either $\Gamma(p)$ or $\Gamma(E)$ with the subspace topology of $Map(B, E)$.*

One refinement we may choose to make to the broad definition is to restrict attention to the slice category $Bun(B)$ of bundles over a chosen space B. This is the full subcategory of Bun generated by the objects $p : E \to B$ with base space B. If the context is clear we shall write $p \Rightarrow_B P'$ to indicate a morphism in this category and we say that it is a morphism **over** B. The product of $p : E \to B$ and $p' : E' \to B$ in $Bun(B)$ is the fibre product, formed by pullback over B and denoted $E \times_B E'$. Note that there is little loss of generality in restricting to this subcategory since if $F : p \Rightarrow p'$ is a bundle morphism as in (4.23), then once a pullback f^*E' is chosen there is unique factorisation $p \Rightarrow f^*(p') \Rightarrow p'$, with $p \Rightarrow f^*(p')$ a morphism of bundles over B.

Definition 2.5 *Let F be a space. A bundle $p : E \to B$ is said to have* ***typical fibre*** *F if there exists a homeomorphism $F \cong p^{-1}(b)$ for each $b \in B$.*

The primary example of a bundle with typical fibre F is the product bundle (2.2). The bundles in which we shall be interested share this simplicity only *locally*. Globally they may have extremely non-trivial topologies.

Definition 2.6 *Let $p : E \to B$ be a bundle. Then*

1. *$p : E \to B$ is **trivial** with fibre F if there is a bundle isomorphism $\varphi : p \overset{\cong}{\Rightarrow} pr_B$ over B with the trivial bundle with fibre F. The bundle isomorphism φ is said to be a **trivialisation**.*

2. *$p : E \to B$ is **trivial over an open subset** $U \subseteq B$ if the restriction $E|_U = p^{-1}(U)$ is trivial.*

3. *$p : E \to B$ is **locally trivial** with fibre F if there exists an open covering $\mathcal{U} = (U_i)_{i \in \mathcal{I}}$ of B such that $E|_{U_i}$ is trivial with fibre F over each member $U_i \in \mathcal{U}$. Such a covering $\mathcal{U}$ is said to be a **trivialising open cover**.*

The reader should bare in mind that our particular choice of category of topological spaces streamlines many of the following results and statements. In particular any open cover of a paracompact space is numerable, and since CW complexes are paracompact it follows that all the bundles we study will be numerable (see [198] for definitions of these terms and proofs of the statements).

The product bundle $pr_B : B \times F \to B$ is a fibration, and so any locally trivial bundle $p : E \to B$ is locally a fibration. That the local trivialisations may be pieced together sufficiently consistently to guarantee that it satisfies a stronger property is the content of the next proposition, originally proved by Dold.

Proposition 2.7 *[65] If $p : E \to B$ is locally trivial then it is a fibration.* ▮

Note that surjectivity is implicit in the definition of a locally trivial map. Whilst a fibration $p : E \to B$ need not be surjective, it will surject onto each connected component of B with which its image has non-trivial intersection [**?**]. Therefore we lose no information by assuming from the start that B is connected and when discussing such properties of bundles and locally trivial maps we shall always make this tacit assumption.

Proposition 2.8 *The pullback of a locally trivial bundle is locally trivial.* ▮

We are now in the position to specialise further and shall shortly give a definition of a *principal* bundle. First we provide some context for the definition.

Definition 2.9 *Let G be a topological group. A right **action** of G on a space X is a map $\mu : X \times G \to X$, denoted $\mu(x, g) = xg$ for $x \in X$, $g \in G$, satisfying*

1. *$(xg)h = x(gh)$, $\forall x \in X,\ g, h \in G$.*

2. *$x1 = x$, $\forall x \in X$, where $1 = 1_G$ is the identity element of G.*

*A space X equiped with a G-action μ is said to be a G-**space**. The G-action is said to be **free** if for any $x \in X$, the equation $xg = x$ implies that $g = 1_G$.*

*If X is a G-space then the relation on X given by $x \sim y \Leftrightarrow y = xg$ for some $g \in G$ is seen to be an equivalence relation. We denote the identification space $X/G = X/\sim$ and refer to it as the **orbit space**.*

*A map $\theta : X \to Y$ between G-spaces X, Y is said to be G-**equivariant** or a G-**map** if $\theta(xg) = \theta(x)g, \forall x \in X, g \in G$. We denote the space of G-maps by G-$Map(X, Y)$, with the subspace topology inherited from $Map(X, Y)$. If $\theta : X \to Y$ is a G-map then we denote by $\theta/G : X/G \to Y/G$ the map of orbit spaces induced by the composition $X \xrightarrow{\theta} Y \to Y/G$ and call it the G-**quotient of** θ.*

A left G-action is defined in an analogous manner, and if X is a left G-space the two concepts are related by defining a right action $x \cdot g = g^{-1}x$ for $x \in X, g \in G$.

To bridge the gap with our definition of a bundle we consider the trivial bundle $B \times G \to B$ with fibre G. The group G acts freely on this space from the right $(b, g)h = (b, gh)$ and the quotient by this action is $B \times * \cong B$. This prompts the following definition.

Definition 2.10 *Let G be a topological group and let $p : P \to B$ be a bundle with P a (right) G-space. Then $p : P \to B$ is said to be a **principal** G-**bundle** or simply a G-**bundle** if*

1. *$p : P \to B$ is locally trivial with fibre G such that a trivialising open cover $\mathcal{U} = (U_i)_{i\in\mathcal{I}}$ may be chosen with each bundle isomorphism $\varphi_i : P|_{U_i} \Rightarrow U_i \times G$ a G-map.*

2. *$p(e \cdot g) = p(e), \forall e \in P, g \in G$.*

For a given topological group G the collection of G-bundles forms a full subcategory $Prin_G$ of Bun. With this definition a morphism $\theta : p \Rightarrow p'$ is supplied by a G-equivariant map $\theta : P \to P'$ which naturally induces a map $B \to B'$ of base spaces. We consider also the full subcategory $Prin_G(B)$ of $Bun(B)$ consisting of G-bundles $P \to B$ and their morphisms over B.

We shall consider the implications of our definition after stating the following well-known technical lemma which will be useful in the discussion. We shall assume from here on out that we have fixed a chosen topological group G.

Lemma 2.11 *[100] The following hold*

1. *Let G be a topological group and X a G-space. Then the canonical quotient $p : X \to X/G$ is an open map.*

2. *A locally trivial map $p : E \to B$ is open.* ∎

Proof The first statement is trivial. For the second, let $V \subseteq E$ be open. To show that $p(V)$ is open it will be sufficient to prove that each $b \in p(V)$ is an interior point. Thus let $b \in p(V)$ and let $U \subseteq E$ be an open neighbourhood of b over which p is trivial by a homeomorphism $\varphi : p^{-1}(U) \cong U \times F$ for some space F. Then $p^{-1}(U) \cap V \subseteq E$ is open and $pr_1 \circ \varphi(p^{-1}(U) \cap V) \subseteq B$ is an open neighbourhood of b contained inside $p(V)$, since φ is a homeomorphism and pr_1 is an open map. ∎

Let $p : P \to B$ be a principal G-bundle. Then the definition shows that p factors through the quotient map $q : P \to P/G$ and induces a bijection $\bar{p} : P/G \to B$. By Lemma 2.11 both the quotient q and projection p are open maps and it follows from this that the induced map $\bar{p}$ is a homeomorphism which in fact defines a G-bundle isomorphism $\theta : q \overset{\cong}{\Rightarrow} p$ that fixes P.

It is also apparent from the definition that G acts freely on P. Therefore we have an identification of the G-bundle $p : P \to B$ with the free right G-space P. Note, however, that not every free action will yield a principal G-bundle, the condition of local triviality being a special property.

Proposition 2.12 *Let $p : P \to B$ be a principal G-bundle and $f : A \to B$ a map. Then the pullback $f^*(p) : f^*P \to B$ is a principal G-bundle and the canonical map $f^*P \to P$ defines a morphism of principal G-bundles covering f. If $f = f_0 \simeq f_1 = f' : A \to B$ are homotopic maps, then there is an isomophism of principal G-bundles $f_0^*P \cong f_1^*P$ over A.*

Proof The pullback may be defined canonically as the space

$$f^*P = \{(a, e) \in A \times P \,|\, f(a) = p(e))\} \subseteq A \times P \tag{2.3}$$

with the subspace topology. The bundle projection $q : f^*P \to A$ is supplied by projecting onto the first fact. If $\varphi : p^{-1}(U) \cong U \times G$ trivialises P over an open set $U \subseteq B$, then f^*P is trivial over the open set $f^{-1}(U) \subseteq A$ by the morphism induced by $1 \times \varphi$. Hence f^*P is a locally trivial bundle. It inherits a G-action from P by $(a, e)g = (a, eg)$ which is seen to be free. Since $q((a, e)g) = q(a, eg) = q(a, e)$ the pullback is seen to be a principal G-bundle. The map $f^*P \to P$ is G-equivariant and covers f by construction. The statement that homotopic maps induce homotopic pullbacks is proven in [198] Theorem 14.3.1. ∎

The following is now an immediate consequence of the homotopic invariance of pullbacks.

Corollary 2.13 *If $B \simeq *$ is contractible, then every principal G-bundle $p : P \to B$ is trivial.* ∎

From the proposition we also get information about the morphisms in $Prin_G$ and $Prin_G(B)$.

Proposition 2.14 *Let X be a free G-space and $p : P \to B$ be principal G-bundle. If $\theta : X \to P$ is a G-equivariant map such that the induced map $\theta/G : X/G \to B$ is a homeomorphism, then $\theta : X \to P$ is a homeomorphism. Thus $X \to X/G$ is a principal G-bundle.*

Proof Write $f = \theta/G$ for the induced homeomorphism. Then by Proposition 2 the pullback $f^*P \to X/G$ is a principal G-bundle and the map $f^*P \to P$ is an isomorphism of G-bundles over f. Since the G-action on X is free, the induced map $X \to f^*P$ is seen to be a homeomorphism over X/G. Thus $X \to X/G$ is furnished with a G-bundle structure. Composition gives that $\theta : X \to f^*P \to P$ is a homeomorphism. ∎

From this we immediately get the following.

Corollary 2.15 *Every morphism in $Prin_G(B)$ is an isomorphism.* ∎

We make tacit use of the next proposition later on in this text. Its statement summarises the nice properties possesed by Lie groups.

Proposition 2.16 ([36] Theorem I.4.3) *Let G be a topological group and $H \leq G$ a subgroup. Then the quotient map $q : G \to G/H$ is a principal H-bundle if and only if there exists an open neighbourhood $U \subseteq G/H$ of the coset $1_G \cdot H$ and a section $s : U \to G$ of q over U. In particular, if G is a Lie group and $H \subseteq G$ is a closed subgroup, then $G \to G/H$ is a principal H-bundle.* ∎

Let X be a free G-space and define

$$C(X) = \{(x, xg) \in X \times X \,|\, x \in X, g \in G\}. \tag{2.4}$$

The set function

$$\tau_X : C(X) \to G, \qquad (x, xg) \mapsto g \tag{2.5}$$

is called the **translation map** of the G-action.

Proposition 2.17 *Let X be a free G-space such that the quotient $q : X \to X/G$ is a locally trivial map. Then the translation map $\tau_X : C(X) \to G$ is continuous.*

Proof First note that if q is trivial, then $X \cong X/G \times G$ and the translation map is continuous since it is given by $((xG, g), (xG, h)) \mapsto g^{-1}h$. Assume then that q is only locally trivial and let $\mathcal{U} = ((U_i, \varphi_i))_{i\in\mathcal{I}}$ be a trivialising open cover of X/G. Note that for each $i \in \mathcal{I}$ the restriction of τ_X to $C(q^{-1}(U_i))$ factors

$$\tau_X|_{q^{-1}(U_i)} : C(q^{-1}(U_i)) \overset{C(\varphi_i)}{\cong} C(U_i \times G) \xrightarrow{\tau_{U_i\times G}} G \tag{2.6}$$

where $C(\varphi_i)$ is induced by $\varphi_i \times \varphi_i$ and $\tau_{U_i\times G}$ is the translation map of the trivial restriction. Thus $\tau_X|_{q^{-1}(U_i)}$ is continuous. If $U_i, U_j \in \mathcal{U}$ have non-empty intersection then the two restrictions $\tau_X|_{q^{-1}(U_i)}$, $\tau_X|_{q^{-1}(U_j)}$ agree on $C(q^{-1}(U_i \cap U_j))$, as is seen from equivariance. Since $C(X) = C(\cup_i q^{-1}(U_i)) = \cup_i C(q^{-1}(U_i))$ we may use the pasting lemma [153] to complete the proof. ∎

The translation function is important in the more general study of group actions [199] and its presence here streamlines many key results.

Proposition 2.18 *A principal G-bundle $p : P \to B$ is trivial if and only if it has a section.*

Proof It is obvious that if $P \cong B \times G$ is trivial then it has a section. Conversely let $s : B \to P$ be a section of p. Then the map $B \times G \to P$, $(b, g) \mapsto s(b)g$ defines a G-bundle morphism over B with inverse $P \to B \times G$, $e \mapsto (p(e), \tau_P(s(p(e)), e)$. ∎

The importance of principal G-bundles in the larger setting of the theory of bundles cannot be overstated and their rigid structures underlie many of the interesting examples of general bundles that one may wish to study as we shall now see.

Definition 2.19 *Let $p : P \to B$ be a principal G-bundle and F a left G-space. The* ***Borel construction*** *on P by F is the bundle $p_F : P \times_G F \to B \to B$ with*

$$P \times_G F = \frac{P \times G}{(e, x) \sim (eg, g^{-1}x)} \tag{2.7}$$

and the map $p_F : P \times_G F \to B$ induced by p and projection onto the first factor.

The Borel construction is defined in such a manner that it inherits many nice features from its parent G-bundle.

Proposition 2.20 *For $p : P \to B$, and F as in the definition above the Borel construction $p_F : P \times_G F \to B$ yields a locally trivial bundle with fibre F.*

Proof Let $\varphi : p^{-1}(U) \cong U \times G$ trivialise P over an open subset $U \subseteq B$. Then since φ is G-equivariant it induces a bundle isomorphism

$$p_F^{-1}(U) = p^{-1}(U) \times_G F \overset{\varphi}{\cong} (U \times G) \times_G F = U \times (G \times_G F) \cong U \times F \tag{2.8}$$

which proves local triviality. The fibre over a point $b \in B$ is $p_F^{-1}(b) \times_G F \cong G \times_G F \cong F$. ∎

In such circumstances we call $p_F : P \times_G F \to B$ the **associated fibre bundle** and refer to G as the **structure** group of the bundle. It is not difficult to show the following useful homeomorphisms.

Proposition 2.21 *There are canonical isomorphisms*

1. $X \times_G G \cong X$ *and* $G \times_G Y \cong Y$.
2. $(X \times_G \times Y) \times_H \times Z \cong X \times_G \times (Y \times_H \times Z)$.
3. $(f^* P) \times_G F \cong f^*(P \times_G F)$ ∎

The next result is key and will make several appearances in the remainder of this section.

Proposition 2.22 *Let $p : P \to B$ be a principal G-bundle and F a left G-space. Then there is a natural bijection G-$Map(P, F) \leftrightarrow \Gamma(P \times_G F)$. That is, for a left G-space F, sections of $p_F : P \times_G F \to B$ are in one-to-one correspondence with maps $f : P \to F$ satisfying $f(eg) = g^{-1} f(e)$.*

Proof Let $f : P \to F$ be G-equivariant and define $\widehat{s_f} : P \to P \times_G F$, $e \mapsto (e, f(e))$. By the G-equivariance of f it follows that $\widehat{s_f}$ gives a well-defined bundle morphism $p \Rightarrow p_F$ over B. It is readily seen to be constant on the G-orbits in P, and so factors through the projection $p : P \to B$ to give a bundle morphism $s_f : id_B \Rightarrow p_F$, that is, a section $s_f \in \Gamma(P \times_G F)$.

To show the converse observe that the following square is a pullback

$$\begin{tikzcd} P \times F \arrow[r, "q"] \arrow[d, "pr_1"] & P \times_G F \arrow[d, "p_F"] \\ P \arrow[r, "p"] & B \end{tikzcd} \tag{2.9}$$

where q is the quotient. Thus a section $s \in \Gamma(P \times_G F)$ induces a section $\varphi : P \to P \times F$ of the projection pr_1 which obviously has the form $\varphi = (id_P \times f) \circ \Delta$ for some map $f : P \to F$ determined by the condition $q \circ \varphi = s \circ p$. Since $(eg, f(eg)) = s(p(eg)) = s(p(e)) = (e, f(e)) \in P \times_G F$ is independent of $g \in G$ it follows that $f(eg) = g^{-1}f(e)$ so that f is G-equivaraiant. It is continuous by construction. ∎

The next example yields a canonically associated bundle of groups from a given G-bundle. It will play a key rôle in the sequel.

Example 2.23 (The Adjoint Bundle) *Let $p : P \to B$ be a principal G-bundle. We make G into a left G-space through the adjoint action*

$$Ad : G \to Aut(G), \qquad g \mapsto \left[Ad_g : h \mapsto ghg^{-1}\right]. \tag{2.10}$$

The Borel constuction then yields the ***adjoint bundle***

$$Ad(P) = P \times_G G \to B \tag{2.11}$$

This is not a principal G-bundle, but rather a bundle of groups over B. The total space is the quotient $P \times G/\left((e,h) \sim (eg, g^{-1}hg)\right)$.

Before we come to give a definition of a *gauge group* and discuss their key properties we shall make a short detour into the theory of classifying spaces sine they will play a fundmanetal part in later developments.

Proposition 2.24 *Let $p : P \to B$ be a principal G-bundle with $P \simeq *$ contractible. Then for all spaces A the assignment is bijective $\Phi : [A, B] \to Prin_G(A)/equiv$, $f \mapsto f^*P$ is bijective, where $Prin_G(A)/equiv$ is the set of isomorphism classes of principal G-bundles over A.*

Proof In view of Proposition 2.12 we get that he function Φ is injective by construction and we only need to show that it is surjective. To this end let $q : Q \to A$ be a G-bundle and form the Borel space $Q \times_G P = Q \times P/\left((e,x) \sim (eg, xg)\right)$ as a bundle over A with fibre P. Since the fibre is contractible the projection $q_P : Q \times_G P \to A$ is a homotopy equivalence and we can choose a homotopy section $s' : A \to Q \times_G P$ and a homotopy $H : id_A \simeq q_F \circ s'$. Using H and the homotopy lifting property of q_F (c.f. Propositions 2.7 and 2.20) we can find a homotopy $s' \simeq s$ with $s : A \to Q \times_G P$ a strict section satisfying $q_F \circ s = id_A$. By Proposition 2.22 this section is equivalent to a G-equivariant map $\theta : Q \to P$ covering $f = \theta/G : A \to B$. Hence $Q \cong f^*P$ and Φ is a surjection. ∎

Call a principal G-bundle with contractible total space **universal**. The previous result shows that if such a bundle exists then it is unique up to equivalence. It was an early triumph of Milnor [144] that these bundles do indeed exist.

Proposition 2.25 *Let G be a topological group. Then there exists a principal G-bundle $\pi_G : EG \to BG$ which is universal in the sense that any principal G-bundle $p : P \to B$ admits a bundle morphism $\theta_P : p \Rightarrow \pi_G$ that is unique up to G-homotopy such that the correspondence $Prin_G(B)/equiv \to [B, BG]$ defined by sending p to the homotopy class of $\theta_P/G : B \to BG$ is a bijection.*

We often abuse notation and refer to a particular choice of bundle $\pi_G : EG \to BG$ as *the* universal principal G-bundle and call the base space BG the **classifying space** of G. As specified by the proposition the universal G-bundle is in fact a certain isomorphism class of principal G-bundles with weakly contractible total spaces. It follows that the spaces EG, BG are specified up to, at least, homotopy equivalence so no confusion should arise. Several authors have constructed both specific and generic fuctorial models for universal G-bundles $TopGro \to Prin$ [66], [135], [141], [144], [165], Segal:1968, [178].

The functorality may be understood in our interpretation as follows. Let $\alpha : G \to H$ be a continuous homomorphism, defining a left G-action on H by $g \cdot h = \alpha(g)h$ for $g \in G$, $h \in H$. Form the borel construction $EG \times_G H$ and note that this space has a canonical right H-action, $(e, h) \cdot k = (e, hk)$ which is easily seen to be free. The quotient by this action is $(EG \times_G H)/H \cong EG \times_G (H/H) \cong EG \times_G * \cong BG$. The local trivialisations of $\pi_G : EG \to BG$ provide trivialisations for the free H-action so that the orbit map $\pi_\alpha : EG \times_G H \to BG$ becomes a principal H-bundle. By proposition 2.24 the H-bundle π_α is classified by a unique homotopy class of maps $B\alpha : BG \to BH$. In this manner we define a functor $TopGro \xrightarrow{B(-)} h\mathcal{T}$ to the homotopy category of spaces. The compatibility with composition of continous homomorphisms $\alpha : G \to H$, $\beta : H \to K$ follows from the isomorphisms

$$B(\beta \circ \alpha)^* EK \cong EG \times_G K \cong (EG \times_G H) \times_H K \cong (B\alpha^* EH) \times_H K \cong B\beta^*(B\alpha^* EK) \quad (2.12)$$

Note that the homotopy fibre of $B\alpha : BG \to BH$ is the quotient space G/H.

The terminology for the following definition comes from the special case in which $\alpha : G \leq H$ is a subgroup inclusion.

Definition 2.26 *Let $p : P \to B$ be a principal H-bundle and $\alpha : G \to H$ a continuous homomorphism. Then p is said to have a α-**reduction of structure** if one of the following two conditions is met.*

1. *There exists a principal G-bundle $p' : P' \to B$ and an isomorphism of principal H-bundles $\theta : P \cong P' \times_G H$ over B.*

2. *A classifying map $f : B \to BH$ for p factors up to homotopy $f \simeq B\alpha \circ f'$ for some map $f' : B \to BG$.*

*If the homomorphism α is understood then we say that p admits a G-**reduction of structure**.*

We show that the two conditions of the definition are equivalent.

Proof Consider the chain of ismorphisms

$$P \cong P' \times_G H \cong (f'^* EG) \times_G H \cong f'^*(EG \times_G H) \cong f'^*(B\alpha^* EH) \cong (B\alpha \circ f')^* EH. \quad (2.13)$$

Starting from the data of the G-bundle p' and H-isomorphism θ one lets f' classify P' and reads from left to right to prove the implication (1)⇒(2). Conversely, starting with the data of the map f', one defines $p' : P' \to B$ to be the G-bundle classified by f' and reads from right to left to prove the implication (2)⇒(1). ∎

At this point we have gathered enough background to have arrived at this section's raison d'être.

Definition 2.27 *Let $p : P \to B$ be a principal G-bundle. The* ***automorphism group*** *$Aut(P)$ of the bundle is the group (under composition) of all principal G-bundle isomorphisms $p \overset{\cong}{\Rightarrow} p$. The* ***gauge group*** *$\mathcal{G}(P)$ is the group (under composition) of all principal G-bundle isomorphisms $p \overset{\cong}{\Rightarrow}_B p$ over B. The* ***based gauge group*** *$\mathcal{G}_*(P)$ is the subgroup of $\mathcal{G}(P)$ consisting of those bundle automorphisms which restrict to the identity on the fibre over a chosen basepoint.*

From Proposition 2.14 we see that the gauge group $\mathcal{G}(P)$ is the group of all G-equivariant maps $P \to P$ that cover the identity on B. It inherits a topology from $Map(P, P)$ under which it become a topological group.

Proposition 2.28 *Let $p : P \to B$ be a principal G-bundle. Then there are isomorphisms of topological groups*

$$\mathcal{G}(P) \cong G\text{-}Map(P, G) \cong \Gamma(Ad(P)) \tag{2.14}$$

where G acts on itself (from the right) through the adjoint action $h \cdot g = g^{-1}hg$, and the adjoint bundle $Ad(P)$ is defined as in Example 2.10.

Proof The map $\mathcal{G}(P) \to G\text{-}Map(P, G)$ sending $f : P \to P$ to the composition

$$\bar{f} : P \xrightarrow{\Delta} C(P) \xrightarrow{1\times f} C(P) \xrightarrow{\tau_P} G \tag{2.15}$$

is continuous and well-defined. Since the maps in $\mathcal{G}(P)$ are G-equivariant it is seen to be a homomorphism. The inverse is given by sending $\bar{f} : P \to G$ to the composition

$$f : P \xrightarrow{\Delta} P \times P \xrightarrow{1\times \bar{f}} P \times G \xrightarrow{\mu} P \tag{2.16}$$

where $\mu : P \times G \to P$ is the G-action.

The correspondence of the second claimed isomorphism is set up by Proposition 2.22 and it is not readily seen to be a homomorphism with repect to the point products on each of the spaces. The fact of its continuity is more delicate and we refer the reader to [63] Theorem 4.1 for complete details of its proof. ∎

Each of the three representations has certain advantages for studying different aspects of the gauge group. For example the last makes the existence of inverses clear, and applying Proposition 2.22 yields the following corollary.

Corollary 2.29 *Let $p : P \to B$ be a principal G-bundle if either*

1. *P is trivial, or*
2. *G is abelian.*

then $\mathcal{G}(P) \cong Map(B, G)$.

Theorem 2.30 *The assignment $P \mapsto \mathcal{G}(P)$ produces a contravariant functor $Prin_G \to TopGroup$ into the category of topological groups.*

Proof Let $p : P \to B$, $q : Q \to C$ be principal G-bundles and $F = (f, \hat{f}) : q \Rightarrow q$ a morphism in the category. Then F induces a continuous map $f^* : \mathcal{G}(Q) \cong G\text{-}Map(Q, G) \to G\text{-}Map(P, G) \cong \mathcal{G}(P)$. This map is compatible with the pointwise product stucture and so is a homomorphism. If F is an isomorphism of principal G-bundles then f is G-equivariant homeomorphism and thus f^* is an isomorphism of topological groups. ∎

Thus if one working with a particular model $p : P \to B$ in a given G-bundle isomomorphism class, then it makes good sense to consider its gauge group as a representative of a certain isomorphism class of topological groups.

We have seen that the gauge group $\mathcal{G}(P)$ and the based subgroup $\mathcal{G}_*(P)$ of a fixed G-bundle $p : P \to B$ are well-defined topological groups. It follows that both these groups have classifying spaces and there is a map classifying the subgroup inclusion. This theme has been explored by Gottlieb [80], Gottlieb:1972 as well as Atiyah and Bott [14]. There is also a brief account contained in Husemoller's text [100]. To state their results we need to introduce some notation.

Let $p : P \to B$ be a principal G-bundle classified by a map $f : B \to BG$ into the base space of the univeral G-bundle $\pi_G : EG \to BG$. Define

$$B\mathcal{G}(P) = Map^f(B, BG) \tag{2.17}$$

to be the path component of the map f in the space of maps $B \to BG$. Let $\pi : G\text{-}Map^F(P, EG) \to Map(B, BG)$ be the map sending a G-map to its G-quotient, $\theta \mapsto \theta/G$ and define

$$E\mathcal{G}(P) = G\text{-}Map^f(P, EG) = \pi^{-1}(Map^f(B, BG)) \tag{2.18}$$

to be the space of all G-equivariant maps $\theta : P \to EG$ such that $\theta/G \simeq f$. Finally let $\pi_{\mathcal{G}(P)} : E\mathcal{G}(P) \to B\mathcal{G}(P)$ be the restriction of π. Its continuitity is verified in [80] Lemma 1.

Proposition 2.31 *With notation as above, if B is compact and G is a Lie group, then the bundle $\pi_{\mathcal{G}(P)} : E\mathcal{G}(P) \to B\mathcal{G}(P)$ is a universal principal $\mathcal{G}(P)$-bundle.*

Proof The proof requires several steps. Firstly one must produce a free action of $\mathcal{G}(P)$ on $\Sigma\mathcal{G}(P)$ and show that the projection $\pi_{\mathcal{G}(P)}$ coincides with the $\mathcal{G}(P)$-quotient map. Secondly one needs to verify that the quotient map is locally trivial and so indeed defines a principal $\mathcal{G}(P)$-bundle. Lastly it must be shown that the bundle so defined is universal. Let us begin.

For the first step we define a right action of $\mathcal{G}(P)$ on $E\mathcal{G}(P) = G\text{-}Map^f(P, EG)$ by precomposition, $\theta \cdot \alpha = \theta \circ \alpha$ for $\theta \in G\text{-}Map^f(P, EG)$, $\alpha \in \mathcal{G}(P)$. It follows that $\pi_{\mathcal{G}(P)}(\varphi \cdot \alpha) = (\varphi \circ \alpha)/G = \varphi/G \circ \alpha/G = \varphi/G = \pi_{\mathcal{G}(P)}(\varphi)$ so that $\pi_{\mathcal{G}(P)}$ is constant on the $\mathcal{G}(P)$-orbits. The action is readily seen to be free using the representation $sg(P) \cong G\text{-}Map(P, G)$ and the freeness of the G-action on EG.

Now let $\theta, \varphi \in G\text{-}Map^f(P, EG)$ be two maps such that $\pi_{\mathcal{G}(P)}(\theta) = \pi_{\mathcal{G}(P)}(\varphi)$, that is, $\theta/G = \varphi/G$. Consider the composition

$$\alpha = \alpha(\theta, \varphi) : P \xrightarrow{\Delta} P \times P \hookrightarrow C(P) \xrightarrow{\theta \times \varphi} C(EG) \xrightarrow{\tau_{EG}} G \tag{2.19}$$

where the first map is the diagonal, the second the inclusion and the third is well defined since for each $e \in P$ it holds that $\theta(e), \varphi(e) \in \pi_G^{-1}(f(p(e)))$. The map α so defined is continuous and satisfies $\alpha(\theta,\varphi)(eg) = g^{-1}(\alpha(\theta,\varphi)(e))g$ and so by Proposition 2.22 defines a gauge transformation $\alpha \in \mathcal{G}(P)$. This shows that the fibre of $\pi_{\mathcal{G}(P)}$ is $\mathcal{G}(P)$. Furthermore note that the assignment $(\theta,\varphi) \mapsto \alpha(\theta,\varphi)$ produces a continuous map $\alpha : C(G\text{-}Map^f(P, EG)) \to \mathcal{G}(P)$ which serves as the translation function for the $\mathcal{G}(P)$-action. Since B is compact and G is a Lie group it follows that $\mathcal{G}(P)$ carries the structure of a Lie group [218] and from this it follows from [163] Theorem 4.1 that continuity of the tranlation function is a sufficient condition for local triviality.

It only remains to show that $\pi_{(P)}$ is universal and for this we appeal to Proposition 2.24 so that the proof will be completed by showing that $G\text{-}Map^f(P, EG)$ is contractible. To show this we will use Proposition 2.22 and equivalently show that $\Gamma(P \times_G EG)$ is contractible. From the projection $p_{EG} : P \times_G EG \to B$ wWe get an induced fibration

$$\Gamma(P \times_G EG) \to Map(B, P \times_G EG) \xrightarrow{p_{EG*}} Map(B, B) \tag{2.20}$$

where we identify the strict fibre of p_{EG*} as the space of sections. Since $EG \simeq *$ the map p_{EG} is a homotopy equivalence and therefore so is p_{EG*}. Hence $\Gamma(P \times_G EG) \simeq *$ and we are complete. ∎

Gottlieb proves a slightly more general version of theorem in [79], foregoing the assumptions of compact B and Lie G, but the statement here will be sufficient for our applications.

In a similar manner we may define

$$B\mathcal{G}_*(P) = Map^f_*(B, BG) \tag{2.21}$$

to be the path component of f in the space of based maps $B \to BG$ (obviously we must choose f itself to be based for this definition to make sense, but since BG is connected [144] we may always homotope f to a based map [136] so this will not be restrictive.), as well as

$$E\mathcal{G}_*(P) = G\text{-}Map^f_*(P, EG) = \pi^{-1}(Map^f_*(B, BG)) \tag{2.22}$$

to be the subspace of pointed G-maps in $G\text{-}Map^f(P, EG)$, and let $\pi_{\mathcal{G}_*(P)} : E\mathcal{G}_*(P) \to B\mathcal{G}_*(P)$ be the obvious restriction. In a similar manner to the proceeding proposition we may now prove the following.

Proposition 2.32 *For B compact and G a Lie group, with notation as above, the bundle $\pi_{\mathcal{G}_*(P)} : E\mathcal{G}_*(P) \to B\mathcal{G}_*(P)$ is a universal principal $\mathcal{G}_*(P)$-bundle.* ∎

This particular choice for the universal bundle has many nice features. One such feature follows from the fact that evaluation maps are fibrations [80].

Corollary 2.33 *There is a fibration sequence*

$$B\mathcal{G}_*(P) \to B\mathcal{G}(P) \xrightarrow{ev} BG \tag{2.23}$$

where $ev : B\mathcal{G}(P) = Map^f(B, BG) \to BG$ is evaluation at a chosen basepoint $* \in B$. ∎

We refer to the fibration sequence appearing in this corollary as the **evaluation fibration sequence** of $\mathcal{G}(P)$.

3 The Homotopy Types of $Sp(3)$-Gauge Groups.

Abstract

Let $\mathcal{G}_k$ denote the gauge group of the principal $Sp(3)$-bundle over S^4 with first symplectic Pontryagin class $k \in H^4S^4 = \mathbb{Z}$. We show that if there is a homotopy equivalence $\mathcal{G}_k \simeq \mathcal{G}_l$ then $(84, k) = (84, l)$ and provide a partial converse that if $(168, k) = (168, l)$ then there is a local homotopy equivalence $\mathcal{G}_k \simeq \mathcal{G}_l$ after rationalisation or localisation at any prime. This follows from a thorough examination of the commutator map $c : Sp(1) \wedge Sp(3) \to Sp(3)$ and we determine its order to be either $4 \cdot 3 \cdot 7 = 84$ or $8 \cdot 3 \cdot 7 = 168$. In particular we give exact information on its odd-primary order and obtain strict bounds of 4 or 8 for its 2-local order.

3.1 Introduction

If G is a compact, connected Lie group and $P \xrightarrow{p} X$ is a principal G-bundle then the gauge group $\mathcal{G}(P)$ of P consists of all G-equivariant self maps $P \to P$ that cover the identity on X. Crabb and Sutherland [61] have shown that if X is a finite complex and G is as above, then the number of homotopy types amongst all the gauge groups of principal G-bundles over X is finite. This is in spite of the fact that the number of G-bundle-isomorphism classes over X is generally infinite.

Another interesting mathematical facet derives through the work of Whitehead [213] and Lang [130] who demonstrated a basic relationship between the homotopy types of certain gauge groups and the order of certain commutators on G. Now the study of the homotopy commutativity of Lie groups and H-spaces has a rich history in algebraic topology. If G is a compact Lie group then the commutator map $c : G \wedge G \to G$ has finite order in the homotopy set $[G \wedge G, G]$ [103] and this order serves as a basic measure of the non-commutativity of the group G. The calculation of the order of c is a difficult problem and has been posed by both Arkowitz and Lin [131].

Due to application in the fields of physics and geometry an important class of gauge groups has emerged as those of simply connected, simple compact Lie groups over simply connected 4-manifolds. There exist in the literature many positive results enumerating the homotopy types of these objects. For instance, Kono [125] has shown that there are precisely 6 distinct homotopy types amongst the gauge groups of $SU(2)$-bundles over S^4. This result was later extended to count the number of distinct homotopy types of the $SU(2)$- [127] and $SU(3)$- [90] gauge groups over simply connected 4-manifolds, as well as the number of p-local homotopy types of $SU(5)$-gauge groups over S^4 [192].

Of more relevance to the current paper is Theriault's calculation [187] of the number of homotopy types of $Sp(2)$-gauge groups over S^4. Let $\mathcal{G}_k(Sp(2))$ denote the gauge group of the principal $Sp(2)$-bundle over S^4 with first symplectic Pontryagin class $k \in H^4S^4 = \mathbb{Z}$ and let (m, n) denote the greatest common divisor of the integers m, n. His result states that if there is a homotopy equivalence $\mathcal{G}_k(Sp(2)) \simeq \mathcal{G}_l(Sp(2))$ then $(40, k) = (40, l)$, and he also gives a partial converse that if $(40, k) = (40, l)$ then there is a local homotopy equivalence $\mathcal{G}_k(Sp(2)) \simeq \mathcal{G}_l(Sp(2))$ when localised rationally or at any prime.

In this paper we add to his result by studying the homotopy types of $Sp(3)$-gauge groups over S^4. Our intention is to push the current boundaries of unstable techniques. These results should be of interest to physicists who study $Sp(n)$-gauge theories as extensions of the standard $SU(2) = Sp(1)$-gauge theory [214], [28]. For notational ease we denote by $\mathcal{G}_k$ the gauge group of the principal $Sp(3)$-bundle over S^4 with first symplectic Pontryagin class $k \in H^4S^4$. We prove the following theorem.

Theorem 3.1 *The following hold.*

1. *If there is a homotopy equivalence $\mathcal{G}_k \simeq \mathcal{G}_l$ then $(84, k) = (84, l)$.*
2. *If $(168, k) = (168, l)$ then there is a local homotopy equivalence $\mathcal{G}_k \simeq \mathcal{G}_l$ after rationalisation or localisation at any prime.*

It is our method that provides a link with the commutator map $c : Sp(1) \wedge Sp(3) \to Sp(3)$ and the theorem is obtained, in part, by a direct calculation of its order.

Theorem 3.2 *The order of the commutator $c : Sp(1) \wedge Sp(3) \to Sp(3)$ is either $4 \cdot 3 \cdot 7 = 84$ or $8 \cdot 3 \cdot 7 = 168$.*

The discrepenacy occurs only in the 2-component of the order of c, which we bound to be either 4 or 8. The effect of this is that all the odd primary information contained in Theorem 3.1 is exact and the only possible refinement is in the 2-component of the second statement.

The lack of precision in the 2-component is unfortunate but understandable. The problem of calculating the order of the commutators on Lie groups is delicate and becomes increasingly intricate as the rank of the group grows. Exact answers have only appeared in the literature so far for a handful of low rank Lie groups. For Lie groups of higher rank one can only hope to obtain exact odd-primary information and strict bounds on the 2-local order using current methods. The odd-primary information we give on the order of $c : Sp(1) \wedge Sp(3) \to Sp(3)$ is exact and the 2-local bounds are the best possible short of an exact answer.

3.2 Preliminaries.

Let G be a connected, compact Lie group and $P \xrightarrow{p} X$ a principal G-bundle over a connected, finite complex X. Let $\mathcal{G}(P)$ denote the gauge group of P and $\mathcal{G}_*(P)$ the based gauge group consisting of those gauge transformations of P that reduce to the identity on the fibre. Let BG be the classifying space for G and $EG \to BG$ be the universal G-bundle with contractible total space. Then it is well known [100] that the isomorphism classes of principal G-bundles over X are in one-to-one correspondence with the homotopy classes of maps $X \to BG$ via the prescription $f \mapsto f^*EG$ which forms the pullback bundle $f^*EG \to X$.

Choose a map $f : X \to BG$ that classifies the bundle $P \xrightarrow{p} X$. Then Gottlieb [80] has shown the existence of homotopy equivalences

$$B\mathcal{G}(P) \simeq Map^f(X, BG), \qquad B\mathcal{G}_*(P) \simeq Map^f_*(X, BG) \tag{3.1}$$

where $B\mathcal{G}(P)$ and $B\mathcal{G}_*(P)$ are the classifying spaces of $\mathcal{G}(P)$ and $\mathcal{G}_*(P)$ respectively, $Map^f(X, BG)$ denotes the path component of the map f in the space of free maps $X \to BG$ and $Map^f_*(X, BG)$ denotes the path component of f in the space of based maps $X \to BG$.

Using the representations (4.8) leads to the evaluation fibration for $\mathcal{G}(P)$

$$\ldots \to \mathcal{G}(P) \to G \xrightarrow{\delta} B\mathcal{G}_*(P) \to B\mathcal{G}(P) \xrightarrow{ev} BG \tag{3.2}$$

where $ev : B\mathcal{G}(P) \simeq Map^f(X, BG) \to BG$ is given by evaluation at the basepoint of X. In (4.9) we have made explicit the homotopy equivalence $G \simeq \Omega BG$ and denoted the fibration connection map by $\delta : G \to B\mathcal{G}_*(P)$. This map shall play a prominent rôle in the following.

Now apply this construction to the case $G = Sp(n)$ and $X = S^4$. The components of $Map(S^4, BSp(n))$ are indexed by the integers; the kth component corresponding to the principal $Sp(n)$-bundle P_k with first Pontryagin class $q_1(P_k) = k \in H^4S^4 = \mathbb{Z}$. Write $\mathcal{G}(P_k) = \mathcal{G}_k$ so that we have

$$B\mathcal{G}_k \simeq Map^k(S^4, BSp(n)), \qquad B\mathcal{G}_{*k} \simeq Map^k_*(S^4, BSp(n)). \tag{3.3}$$

We may make a further identification using the homotopy equivalences that are induced by the co-H-structure on S^4

$$B\mathcal{G}_*(P_k) \simeq Map^k_*(S^4, BSp(n)) = \Omega^4_k BSp(n) \simeq \Omega^4_0 BSp(n) \simeq \Omega^3_0 Sp(n), \tag{3.4}$$

where $\Omega^r_n(-) = Map^n_*(S^r, -)$ denotes the component of the degree n map in the r-fold loop functor. With all this in place the evaluation fibration of the kth principal $Sp(n)$-bundle $P_k \to S^4$ becomes

$$\ldots \to \mathcal{G}_k \to Sp(n) \xrightarrow{\delta_k} \Omega^3_0 Sp(n) \to B\mathcal{G}_k \xrightarrow{ev} BSp(n) \tag{3.5}$$

The important rôle played by the connecting map δ_k has already been observed. In particular the gauge group $\mathcal{G}_k$ appears in the fibration sequence (4.10) as the homotopy fibre of δ_k. Thus one way to understand the homotopy type of $\mathcal{G}_k$ is through study of the map δ_k. This approach is facilitated somewhat by the following observations.

Theorem 3.3 (Lang [130], Whitehead [213]) *The triple adjoint of δ_k is the generalised Samelson product*

$$\langle k \cdot \epsilon_3, id_{Sp(n)} \rangle : S^3 \wedge Sp(n) \to Sp(n) \tag{3.6}$$

where $\epsilon_3 \in \pi_3 Sp(n) = \mathbb{Z}$ is a generator. ∎

Since the Samelson product is bilinear if follows that $\langle k \cdot \epsilon_3, id_{Sp(n)} \rangle = k \cdot \langle \epsilon_3, id_{Sp(n)} \rangle$, where the right hand side is the k-fold sum of δ_1 in the abelian group $[S^3 \wedge Sp(3), Sp(3)]$. This immediately leads to the following.

Proposition 3.4 *In the abelian group $[Sp(n), \Omega^3_0 Sp(n)]$ there is equality*

$$\delta_k \simeq k \cdot \delta_1 \quad \blacksquare \tag{3.7}$$

Thus the focal object becomes the connecting map δ_1. A further important feature of δ_1 is that it has finite order [103]. It now follows from Proposition 3.4 that a key step in determining the number of distinct homotopy types amongst the gauge groups $\mathcal{G}_k$ is determining the order of δ_1, or, equivalently, by Theorem 4.5, the order of the Samelson product $\langle \epsilon_3, id_{Sp(n)} \rangle$, which is to be the focus of the next section.

3.3 The Order of the Commutator $c : S^3 \wedge Sp(3) \to Sp(3)$.

From Theorem 4.5 we know that the adjoint of δ_1 is the Samelson product $\langle \epsilon_3, id_{Sp(3)} \rangle$. Make the identification $Sp(1) \cong S^3$ and let $i_{3,1} : S^3 \cong Sp(1) \hookrightarrow Sp(3)$ be the subgroup inclusion so that we may take $i_{3,1} = \epsilon_3$ as the generator of $\pi_3 Sp(3)$. Now $\langle \epsilon_3, id_{Sp(3)} \rangle$ is given by the commutator product $c : S^3 \wedge Sp(3) \to Sp(3)$, defined to be the unique map extending

$$c' : S^3 \times Sp(3) \to Sp(3), \qquad (x, y) \mapsto i_{3,1}(x) y i_{3,1}(x)^{-1} y^{-1} \tag{3.8}$$

over the smash.

The commutator map c has been studied by various authors, and, in particular, Bott [31] has examined Samelson products in the classical groups. Using the fact that the subgroup embeddings of $Sp(1)$ and $Sp(2)$ into $Sp(3)$ may be chosen such that the former two groups commute strictly in $Sp(3)$ he has shown that there is a map θ making the following diagram commute up to homotopy

$$\begin{tikzcd} S^3 \wedge Sp(3) \arrow[r, "c"] \arrow[d, "1\wedge p"'] & Sp(3) \arrow[d, equal] \\ S^{14} \arrow[r, "\theta"] & Sp(3) \end{tikzcd} \tag{3.9}$$

where $p : Sp(3) \to S^{14} \cong Sp(3)/Sp(2)$ is the projection of the fibration $Sp(2) \xrightarrow{i} Sp(3) \xrightarrow{p} S^{14}$.

We shall identify the map θ shortly, in Proposition 3.5. It is useful first to make some definitions. Let $i_{n,m} : Sp(m) \to Sp(n)$ denote the $(4m+2)$-connected subgroup inclusion. In general we let $\epsilon_k^{(n)}$ denote a generator of $\pi_k Sp(n)$, when it is understood, and for $k \leq 4n+1$ define $\epsilon_k^{(n+1)} = i_{n+1,n*}\epsilon_k^{(n)}$. Thus $\epsilon_3^{(n)} = i_{n,1}$. Since we are mainly dealing with $\pi_* Sp(3)$ we will drop the superscripts (3) on the generators when no confusion can arise and denote them $\epsilon_k \in \pi_k Sp(3)$.

The homotopy groups of low rank symplectic groups have been calculated by Mimura and Toda [149], [150] and in the following we shall use their results freely. We shall also make use of the results and notation in Toda's book [197] without further comment.

Proposition 3.5 *The map θ is a generator of $\pi_{14} Sp(3) = \mathbb{Z}_{2\cdot 7!} = \mathbb{Z}_{10080}$ and there is a homotopy*

$$c \simeq \epsilon_{14} \circ \Sigma^3 p. \tag{3.10}$$

Proof Consider the Samelson product $\langle \epsilon_3, \epsilon_{11} \rangle \in \pi_{14} Sp(3)$. This product has been calculated by Bott [31] who has demonstrated that

$$\langle \epsilon_3, \epsilon_{11} \rangle = 5! \cdot \epsilon_{14}. \tag{3.11}$$

Now using the diagram (3.9) we have that

$$\langle \epsilon_3, \epsilon_{11} \rangle = c \circ (1 \wedge \epsilon_{11}) = \theta \circ (1 \wedge p) \circ (1 \wedge \epsilon_{11}) = \theta \circ (1 \wedge (p_* \epsilon_{11})) \tag{3.12}$$

where the first equality is the definition of the Samelson product. From the homotopy exact sequence of the fibration $Sp(2) \xrightarrow{i} Sp(3) \xrightarrow{p} S^{14}$ we get the following short exact sequence

$$\begin{tikzcd} 0 \arrow[r] & \pi_{11} Sp(3) \arrow[r, "p_*"] \arrow[d, equal] & \pi_{11} S^{11} \arrow[r, "\Delta"] \arrow[d, equal] & \pi_{10} Sp(2) \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & \mathbb{Z} \arrow[r, "\times 5!"] & \mathbb{Z} \arrow[r] & \mathbb{Z}_{120} \arrow[r] & 0. \end{tikzcd} \tag{3.13}$$

In this sequence the map p_* is injective since $\pi_{11}Sp(2) = \mathbb{Z}_2$ is torsion whilst $\pi_{11}Sp(3) = \mathbb{Z}$ is free abelian. The connecting map Δ is surjective since $\pi_{10}Sp(3) = 0$. Using this we identify

$$p_*\epsilon_{11} = 5! \cdot id_{S^{11}}. \tag{3.14}$$

Now combining equations (3.11) and (3.12) we get

$$\langle \epsilon_3, \epsilon_{11} \rangle = 5! \cdot \theta = 5! \cdot \epsilon_{14} \tag{3.15}$$

and we conclude that up to multiplication by a unit

$$\theta = \epsilon_{14}. \tag{3.16}$$

By redefining ϵ_{14} if necessary we may assume that (3.16) is strict. With this the homotopy $c \simeq \epsilon_{14} \circ \Sigma^3 p$ follows from the definition of θ in diagram (3.9). ∎

Note that the order of $\langle \epsilon_3, \epsilon_{11} \rangle$ is 84 and so provides a lower bound on the order of c. On the other hand, since $\theta = \epsilon_{14}$ generates $\pi_{14}Sp(3)$, Lemma 3.5 sets an extreme upper bound. These observations give us the following.

Corollary 3.6 *The order of $c : S^3 \wedge Sp(3) \to Sp(3)$ is divisible by $84 = 2^2 \cdot 3 \cdot 7$ and is not more than $10080 = 2^4 \cdot 3^2 \cdot 5 \cdot 7$.* ∎

Of course we expect the actual order of the commutator to be somewhere between the two extremes. To determine it exactly we shall use local homotopy theory to work one prime at a time. Note that ϵ_{14} is trival at primes > 7 so the following is immediate from Proposition 3.5.

Corollary 3.7 *After localisation at any prime $p > 7$, the commutator map $c : S^3 \wedge Sp(3) \to Sp(3)$ becomes null-homotopic.* ∎

We may therefore limit attention to the primes 2, 3, 5 and 7 in the following. We shall be interested in the skeletal filtration of $Sp(3)$ and the order of c restricted to the various skeleta. Before proceeding we pause to review the cellular structure of $Sp(3)$.

The integral cohomology of $Sp(3)$ is torsion-free and well known to be an exterior algebra on classes x_i of degree i

$$H^*Sp(3) = \Lambda(x_3, x_7, x_{11}). \tag{3.17}$$

Reduced mod p these classes have the following relations under the action of the mod p Steenrod algebra

$$\begin{array}{lll} Sq^4 x_3 = 0 & \mathcal{P}^1_{(3)} x_3 = -x_7 & \mathcal{P}^1_{(5)} x_3 = x_{11} \\ Sq^4 x_7 = x_{11} & \mathcal{P}^1_{(3)} x_7 = 0 & \mathcal{P}^1_{(5)} x_7 = 0 \\ Sq^4 x_{11} = 0 & \mathcal{P}^1_{(3)} x_{11} = 0 & \mathcal{P}^1_{(5)} x_{11} = 0 \end{array} \tag{3.18}$$

where the first column is for $p = 2$, the second for $p = 3$ and the third for $p = 5$. All other primary operations are trivial.

Recall that the symplectic quasi-projective space Q_n is a generating complex for $Sp(n)$. For $n \leq 3$ these spaces have the following cellular structures

$$Q_1 = S^3, \qquad Q_2 = S^3 \cup_{\nu'} e^7, \qquad Q_3 = S^3 \cup_{\nu'} e^7 \cup_{\gamma_2} e^{11} \tag{3.19}$$

where $\nu' \in \pi_6 S^3 = \mathbb{Z}_{12}$ is the Blakers-Massey generator and $\gamma_2 : S^{10} \to Q_2$ is some map lifting $\nu_7 : S^{10} \to S^7$ through the map $Q_2 \to S^7$ that pinches to the top cell. That Q_n is a generating complex for $Sp(n)$ means that there is a map $\iota_n : Q_n \to Sp(n)$ that induces an isomorphism $QH^*Sp(n) \cong H^*Q_n$ on the module of indecomposables.

Now applying this to $Sp(3)$ we see that it has a cellular structure

$$\begin{array}{rcl} Sp(3) & \simeq & S^3 \cup_{\nu'} e^7 \cup e^{10} \cup e^{11} \cup e^{14} \cup e^{18} \cup e^{21} \\ & & \{x_3,\, x_7,\, x_3x_7,\, x_{11},\, x_3x_{11},\, x_7x_{11},\, x_3x_7x_{11}\} \end{array} \tag{3.20}$$

The action (3.18) of the Steenrod algebra gives some information as to how the cells are attached. More is obtained by studying a theorem of Miller [142] who gives a stable decomposition.

Theorem 3.8 (Miller [142]) *There is a stable splitting*

$$Sp(3) \simeq_{(S)} Q_3 \vee \left(S^{10} \cup_{\nu_{10}} e^{14} \cup e^{18}\right) \vee S^{21} \qquad \blacksquare \tag{3.21}$$

Let X_i denote the i-skeleton of $Sp(3)$. Then the previous discussion identifies the bottom two skeleta $X_3 = S^3$ and $X_7 = Q_2$. Since the inclusion $Sp(2) \xrightarrow{i} Sp(3)$ is 10-connected and induces an isomorphism on homology in degree 10 we get that $X_{10} = Sp(2)$.

We obtain the attaching map for the 11-cell using the homotopy commutative diagram

$$\begin{tikzcd} S^{10} \arrow[r, "\gamma_2"] \arrow[d, "E"] & Q_2 \arrow[r] \arrow[d, "\iota"] & Q_3 \arrow[r, "q"] \arrow[d, "\iota"] & S^{11} \arrow[d, equal] \\ \Omega S^{11} \arrow[r, "\Delta"] & Sp(2) \arrow[r, "i"] & Sp(3) \arrow[r, "p"] & S^{11} \end{tikzcd} \tag{3.22}$$

in which the top row is the cofibration sequence deriving from the cellular structure of Q_3 and the bottom row is a fibration sequence. The map E is the suspension and the map Δ is the fibration connecting map. We apply π_{10} to (3.22) to get

$$\begin{tikzcd} \pi_{10}S^{10} = \mathbb{Z} \arrow[r, "\gamma_{2*}"] \arrow[d, "\cong"', "E_*"] & \pi_{10}Q_2 \arrow[r] \arrow[d, "\iota_*"] & \pi_{10}Q_3 \arrow[r] \arrow[d] & 0 \\ \pi_{11}S^{11} \arrow[r, "\Delta"] & \pi_{10}Sp(2) = \mathbb{Z}_{120} \arrow[r] \arrow[d] & 0 = \pi_{10}Sp(3) & \\ & 0 & & \end{tikzcd} \tag{3.23}$$

The rows and columns of this diagram are exact. The bottom row is the homotopy exact sequence of a fibration and the top row and the columns are Blakers-Massey exact sequences of cofibrations. The attaching map for the 11-cell is seen to be $\bar{\gamma}_2 = \iota \circ \gamma_2 : S^{10} \to X_{10} = Sp(2)$

which generates $\pi_{10}Sp(2)$ and gives us $X_{11} = Sp(2) \cup_{\bar{\gamma}_2} e^{11}$. Note that by construction the map $p' : Sp(2) \cup_{\bar{\gamma}_2} e^{11} \to S^{11}$ that pinches to the top cell extends the pinch map $q : Q_3 \to S^{11}$ and can be taken to be a restriction of the fibre map $p : Sp(3) \to S^{11}$.

The upshot of all this is the following.

Corollary 3.9 *There is a splitting*

$$S^3 \wedge X_{11} \simeq (S^3 \wedge Q_3) \vee S^{13} \tag{3.24}$$

Proof Mimura [147] showed that $S^3 \wedge Sp(2) \simeq (S^3 \wedge Q_2) \vee S^{13}$. Precisely he showed that the attaching map $\varphi_9 : S^9 \to Q_2$ for the top cell of $Sp(2)$ becomes trivial after two suspensions. Now consider the following homotopy commutative diagram in which each row and column is a cofiber sequence.

$$\begin{tikzcd}
* \arrow[r] \arrow[d] & S^9 \arrow[r, equal] \arrow[d, "\varphi_9"] & S^9 \arrow[d, "j\circ\varphi_9"] \\
S^{10} \arrow[r, "\gamma_2"] \arrow[d, equal] & Q_2 \arrow[r, "j"] \arrow[d, "\iota"] & Q_3 \arrow[d] \\
S^{10} \arrow[r, "\bar{\gamma}_2"] & Sp(2) \arrow[r, "i'"] & Sp(2) \cup_{\bar{\epsilon}_{10}} e^{11} = X_{11}.
\end{tikzcd} \tag{3.25}$$

This diagram displays the 11-skeleton X_{11} as the cofibre of the map $j \circ \varphi_9$. Since φ_9 becomes trivial after two suspensions, the same is true for the composite $j \circ \varphi_9$. Thus we obtain

$$S^3 \wedge X_{11} \simeq S^3 \wedge \left(Q_3 \cup_{j\circ\varphi_9} e^{10}\right) \simeq (S^3 \wedge Q_3) \cup_{\Sigma^3(j\circ\varphi_9)} e^{13} \simeq \left(S^3 \wedge Q_3\right) \vee S^{13} \quad \blacksquare \tag{3.26}$$

3.3.1 The 2-Local Order of c.

In this section we obtain bounds on the 2-local order of c. For this purpose all spaces and groups will be localised at the prime 2 throughout. Recall that $\pi_{14}Sp(3) = \mathbb{Z}_{32}$ with generator ϵ_{14} and $\pi_{11}Sp(3) = \mathbb{Z}_{(2)}$ with generator ϵ_{11}. We shall need the following.

Lemma 3.10 (Oguchi [160]) *The following relation holds in the 2-local homotopy module $\pi_*Sp(3)$.*

$$4 \cdot \epsilon_{14} = \pm\epsilon_{11} \circ \nu_{11} \quad \blacksquare \tag{3.27}$$

The main result of this section is Proposition 3.14. Its proof hinges on several key lemmas.

Lemma 3.11 *The composite $\nu_8 \circ q : Q_3 \to S^8$ is null-homotopic, where $q : Q_3 \to S^{11}$ is the pinch map to the top cell.*

Proof Consider the homotopy commutative diagram of cofiber sequences

$$\begin{tikzcd}
* \arrow[r] \arrow[d] & S^{10} \arrow[r, equal] \arrow[d, "\gamma_2"] & S^{10} \arrow[d, "r_2\circ\gamma_2"] \\
S^3 \arrow[r] \arrow[d, equal] & Q_2 \arrow[r, "r_2"] \arrow[d] & S^7 \arrow[d] \\
S^3 \arrow[r] & Q_3 \arrow[r, "r'"] & Q_3/S^3
\end{tikzcd} \tag{3.28}$$

in which both bottom left-hand horizontal maps are the inclusions of the bottom cell and r_2 and r' are the canonical collapse maps. The space $Q_3/S^3 \simeq S^7 \cup_{r_2 \circ \gamma_2} e^{11}$ is seen to have cohomology

$$\widetilde{H}^i(Q_3/S^3;\mathbb{Z}_2) = \begin{cases} \mathbb{Z}_2\{x_7\} & i=7 \\ \mathbb{Z}_2\{x_{11}\} & i=11 \\ 0 & \text{otherwise} \end{cases}, \qquad Sq^4 x_7 = x_{11}. \tag{3.29}$$

Since Sq^4 detects the stable class $\nu \in \pi_3^s$, we infer that $r_2 \circ \gamma_2 \simeq a \cdot \nu_7$ with $a \equiv 1 \pmod 2$. If necessary we may redefine the maps r_2, γ_2, to obtain that

$$r_2 \circ \gamma_2 = \nu_7 \tag{3.30}$$

and $Q_3/S^3 \simeq S^7 \cup_{\nu_7} e^{11}$. Now extend (3.28) downwards to get a homotopy commutative diagram

$$\begin{tikzcd} S^3 \arrow[r] \arrow[d] & Q_3 \arrow[r, "r'"] \arrow[d, "q"] & S^7 \cup_{\nu_7} e^{11} \arrow[d] \\ * \arrow[r] \arrow[d] & S^{11} \arrow[r, equal] \arrow[d, "\Sigma\gamma_2"] & S^{11} \arrow[d, "\nu_8"] \\ S^4 \arrow[r] & \Sigma Q_2 \arrow[r, "\Sigma r_2"] & S^8. \end{tikzcd} \tag{3.31}$$

This tells us that

$$\nu_8 \circ q \simeq \Sigma r_2 \circ \Sigma\gamma_2 \circ q \simeq * \tag{3.32}$$

since $\Sigma\gamma_2$ and q are adjacent maps in a cofiber sequence. ∎

Lemma 3.12 *The composite map* $S^3 \wedge Q_3 \xrightarrow{1\wedge\iota} S^3 \wedge Sp(3) \xrightarrow{c} Sp(3)$ *has order 4.*

Proof First note that the order of $c \circ (1 \wedge \iota)$ must be at least 4. This lower bound is provided by the Samelson product $\langle \epsilon_3, \epsilon_{11} \rangle = c \circ (\epsilon_3 \wedge \epsilon_{11})$ which we will show factors through the inclusion $1 \wedge \iota$.

The smash $\epsilon_3 \wedge \epsilon_{11}$ is map $S^{14} \to S^3 \wedge Sp(3)$ and as such factors through the 14-skeleton which, by Corollary 3.9, has the form $S^3 \wedge X_{11} \simeq \Sigma^3 Q_3 \vee S^{13}$. Write $c \simeq \epsilon_{14} \circ (1 \wedge p)$ using Lemma 3.5 and recall that the S^{13} factor comes from $S^3 \wedge Sp(2) \simeq \Sigma^3 Q_2 \vee S^{13}$ and is projected trivially to S^{14} by the restriction of $1 \wedge p$. Therefore there must exist a map e such that $\langle \epsilon_3, \epsilon_{11} \rangle$ factors

$$\langle \epsilon_3, \epsilon_{11} \rangle : S^{14} \xrightarrow{e} S^3 \wedge Q_3 \xrightarrow{1\wedge\iota} S^3 \wedge Sp(3) \xrightarrow{c} Sp(3). \tag{3.33}$$

It was noted just after Lemma 3.5 that $\langle \epsilon_3, \epsilon_{11} \rangle$ has order $84 = 4 \cdot 3 \cdot 7$. Therefore equation (3.33) requires that $c \circ (1 \wedge \iota)$ have order at least 4 after localisation at 2.

We shall now show that 4 is also an upper bound for the order of $c \circ (1 \wedge \iota)$. From Lemma 3.5 and the homotopy commutative square

$$\begin{tikzcd} Q_3 \arrow[r, "\iota"] \arrow[d, "q"] & Sp(3) \arrow[d, "p"] \\ S^{11} \arrow[r, equal] & S^{11} \end{tikzcd} \tag{3.34}$$

we get

$$c \circ (1 \wedge \iota) \simeq \epsilon_{14} \circ \Sigma^3 p \circ \Sigma^3 \iota \simeq \epsilon_{14} \circ \Sigma^3 q. \tag{3.35}$$

Then Lemma 3.10 gives us

$$4 \cdot (c \circ (1 \wedge \iota)) \simeq 4 \cdot (\epsilon_{14} \circ \Sigma^3 q) = (\epsilon_{14} \circ \Sigma^3 q) \circ \underline{4} = (4 \cdot \epsilon_{14}) \circ \Sigma^3 q \simeq \pm \epsilon_{11} \circ \nu_{11} \circ \Sigma^3 q \tag{3.36}$$

and from Lemma 3.11 we get $\nu_{11} \circ \Sigma^3 q \simeq *$. It follows that

$$4 \cdot (c \circ (1 \wedge \iota)) \simeq \pm \epsilon_{11} \circ \big(\nu_{11} \circ \Sigma^3 q\big) \simeq * \tag{3.37}$$

and the order $c \circ (1 \wedge \iota)$ is no more than 4.

We have now shown that the order of $c \circ (1 \wedge \iota)$ is no less than 4 and no more than 4. Therefore it is exactly 4. ∎

Corollary 3.13 *The restriction of c to the 16-skeleton $S^3 \wedge X_{11} \simeq (S^3 \wedge Q_3) \vee S^{13}$ has order 4.*

Proof It was shown in Corollary 3.9 that $S^3 \wedge X_{11} \simeq (S^3 \wedge Q_3) \vee S^{13}$. Since the restriction $1 \wedge p| : S^{13} \to S^{14}$ is obviously trivial the order of $c|_{S^3 \wedge X_{11}}$ is the same as that of $c|_{S^3 \wedge Q_3}$, which we have just shown to be 4. ∎

Proposition 3.14 *After localisation at 2 the order of the commutator $c : S^3 \wedge Sp(3) \to Sp(3)$ is either 4 or 8.*

Proof In Lemma 3.12 it was shown that the composite map $\nu_{11} \circ (1 \wedge q) : S^3 \wedge Q_3 \to S^{11}$ is trivial. Moreover, it was noted in Corollary 3.13 that the restriction of $1 \wedge p$ to the S^{13} wedge-summand of $S^3 \wedge X_{11} \simeq (S^3 \wedge Q_3) \vee S^{13}$ is trivial. Therefore since $(1 \wedge p) \circ (1 \wedge \iota) \simeq 1 \wedge q$ we get that

$$\nu_{11} \circ (1 \wedge p)|_{S^3 \wedge X_{11}} \simeq * \tag{3.38}$$

and there exists an extension $\zeta : (S^3 \wedge Sp(3)) \,/\, (S^3 \wedge X_{11}) \simeq (S^{17} \cup_{\nu_{17}} e^{21}) \vee S^{24} \to S^{11}$ of the map $\nu_{11} \circ (1 \wedge p)$ through the cofiber of the inclusion that makes the following diagram homotopy commutative.

$$\begin{tikzcd}
(S^3 \wedge Q_3) \vee S^{13} \simeq S^3 \wedge X_{11} \arrow[r] \arrow[d] & * \arrow[d] & \\
S^3 \wedge Sp(3) \arrow[r, "\nu_{11} \circ (1 \wedge p)"] \arrow[d, "r"] & S^{11} \arrow[r, "\epsilon_{11}"] \arrow[d, equal] & Sp(3) \arrow[d, equal] \\
(S^{17} \cup_{\nu_{17}} e^{21}) \vee S^{24} \arrow[r, "\zeta"] & S^{11} \arrow[r, "\epsilon_{11}"] & Sp(3)
\end{tikzcd} \tag{3.39}$$

Note that the composite of the maps in the middle line satisfies $\epsilon_{11} \circ \nu_{11} \circ (1 \wedge p) \simeq (4 \cdot \epsilon_{14}) \circ (1 \wedge p) \simeq \pm 4 \cdot c$. Therefore

$$4 \cdot c \simeq \pm \epsilon_{11} \circ \zeta \circ r. \tag{3.40}$$

Now ζ is completely determined by its restictions to each of the spaces in the wedge. If we can determine the maximum order of each of these restrictions then we can determine the maximum order of ζ and, through (3.40), obtain an upper bound on the order of $4 \cdot c$.

Let $\zeta_1 = \zeta|_{S^{17}\cup_{\nu_{17}} e^{21}}$ and $\zeta_2 = \zeta|_{S^{24}}$. Then we have

$$\zeta_1 \in [S^{17} \cup_{\nu_{17}} e^{21}, S^{11}] \tag{3.41}$$
$$\zeta_2 \in \pi_{24} S^{11} = \mathbb{Z}_2 \oplus \mathbb{Z}_2 \tag{3.42}$$

and we immediately see that ζ_2 has order at most 2 so it only remains to determine the maximum order of ζ_1. To this end we introduce the cofibration sequence

$$\ldots \to S^{20} \xrightarrow{\nu_{17}} S^{17} \xrightarrow{j} S^{17} \cup_{\nu_{17}} e^{21} \xrightarrow{\rho} S^{21} \to \ldots \tag{3.43}$$

and get the exact Puppe sequence

$$\begin{tikzcd}
\pi_{20}S^{11} \arrow[d, equal] & \pi_{17}S^{11} \arrow[l, "\nu_{17}^*"'] \arrow[d, equal] & {[S^{17} \cup_{\nu_{17}} e^{21}, S^{11}]} \arrow[l, "j^*"'] & \pi_{21}S^{11} \arrow[l, "\rho^*"'] \arrow[d, equal] & \pi_{18}S^{11} \arrow[l, "\nu_{18}^*"'] \arrow[d, equal] \\
{\mathbb{Z}_2\{\nu_{11}^3\} \oplus \mathbb{Z}_2\{\mu_{11}\} \oplus \mathbb{Z}_2\{\eta_{11}\circ\epsilon_{12}\}} & \mathbb{Z}_2\{\nu_{11}^2\} & & {\mathbb{Z}_2\{\sigma_{11}\circ\nu_{18}\} \oplus \mathbb{Z}_2\{\eta_{11}\circ\mu_{12}\}} & \mathbb{Z}_{16}\{\sigma_{11}\}.
\end{tikzcd} \tag{3.44}$$

Here we see that $\nu_{17}^* : \pi_{17}S^{11} \to \pi_{20}S^{11}$ is monic, taking ν_{11}^2 to ν_{11}^3. This means that j^* must be trivial. We also get that $\nu_{18}^* : \pi_{18}S^{11} \to \pi_{21}S^{11}$ takes the generator σ_{11} onto the $\mathbb{Z}_2$ summand generated by $\sigma_{11} \circ \nu_{18}$. The end result of all this is an isomorphism

$$\rho^* : [S^{17} \cup_{\nu_{17}} e^{21}, S^{11}] \cong \mathbb{Z}_2\{\eta_{11} \circ \mu_{12}\} \subseteq \pi_{21}S^{11}. \tag{3.45}$$

As $\zeta_1 \in [S^{17} \cup_{\nu_{17}} e^{21}, S^{11}]$ we get that its maximal order is 2. Thus the maximal order of ζ is also 2. Therefore the $\pm$ in (3.40) is inconsequential and we may conclude that

$$8 \cdot c \simeq \epsilon_{11} \circ (2 \cdot \zeta) \circ r \simeq *. \tag{3.46}$$

It follows that the order of c is bounded above by 8. We already have a lower bound of 4, so the order of c must be either 4 or 8. ∎

Remark It is readily verified that the composition $\epsilon_{11} \circ \eta_{11} \circ \mu_{12} \in \pi_{24}Sp(3)$ is non-trivial. Thus Proposition 3.14 does not obviously improve in any way.

3.3.2 The 3-Local Order of c.

We now calculate the 3-local order of c. All spaces and maps are to be localised at the prime 3 throughout this section. We conclude this section with Proposition 3.18, the proof of which will depend on a string of lemmas.

Lemma 3.15 *The following relations hold in the 3-local homotopy modules $\pi_*Sp(2)$ and $\pi_*Sp(3)$.*

1. $\epsilon_7^{(2)} \in \langle i_{2,1}, \alpha_1(3), 3 \cdot id_{S^6}\rangle$
2. $\epsilon_7^{(2)} \circ \alpha_1(7) = \epsilon_{10}^{(2)}$

3. $\epsilon_7^{(2)} \circ \alpha_2(7) = \epsilon_{14}^{(2)}$

4. $\epsilon_7^{(3)} \circ \alpha_1(7) = 0$

5. $\epsilon_7^{(3)} \circ \alpha_2(7) = 3 \cdot \epsilon_{14}^{(3)}$

Proof The bracket of item 1) arises from the null-compositions

$$S^6 \xrightarrow{3 \cdot id_{S^6}} S^6 \xrightarrow{\alpha_1(3)} S^3 \xrightarrow{i_{2,1}} Sp(2). \tag{3.47}$$

Since the mapping cone of $\alpha_1(3)$ is the generating complex Q_2 we may take the canonical inclusion $\iota : Q_2 \hookrightarrow Sp(2)$ as the extension of $i_{3,2}$. We may also take the generator ϵ'_7 of $\pi_7 Q_2$ as the colifting of the degree 3 map $3 : S^7 \to S^7$. The composition $\iota \circ \epsilon'_7$ is then a member of the bracket and since ι is a 9-connected map this composition is exactly the generator of $\pi_7 Sp(2)$. The indeterminacy is $3 \cdot \pi_7 Sp(2)$. This shows that

$$\epsilon_7^{(2)} \in \langle i_{2,1}, \alpha_1(3), 3 \cdot id_{S^6} \rangle \subseteq \pi_7 Sp(2) / \left(3 \cdot \pi_7 Sp(2)\right). \tag{3.48}$$

We now use this result to simultaneously prove Item 2) and Item 3). For $i = 1, 2$ we have

$$\epsilon_7^{(2)} \circ \alpha_i(7) \in \langle i_{2,1}, \alpha_1(3), 3 \cdot id_{S^6} \rangle \circ \alpha_i(7) = -i_{2,1} \circ \langle \alpha_1(3), 3 \cdot id_{S^6}, \alpha_i(7) \rangle. \tag{3.49}$$

Note that $\alpha_2(3) = \langle \alpha_1(3), 3 \cdot id_{S^6}, \alpha_1(7) \rangle$ with no indeterminacy and $\alpha_3(3) = \langle \alpha_1(3), 3 \cdot id_{S^6}, \alpha_2(7) \rangle$ with no indeterminacy [197]. Turning to the homotopy exact sequence of $S^3 \xrightarrow{i_{2,1}} Sp(2) \xrightarrow{p_2} S^7$ we see that $i_{2,1*} : \pi_r S^3 \to \pi_r Sp(2)$ is an isomorphism for $r = 10, 14$. Thus, up to sign,

$$\epsilon_7^{(2)} \circ \alpha_1(7) = i_{2,1*} \alpha_2(3) = \epsilon_{10}^{(2)} \tag{3.50}$$

$$\epsilon_7^{(2)} \circ \alpha_2(7) = i_{2,1*} \alpha_3(3) = \epsilon_{14}^{(2)} \tag{3.51}$$

with no indeterminacy.

Finally we turn to Items 4) and 5). The generator $\epsilon_7^{(3)}$ is stable and satisfies $\epsilon_7^{(3)} = i_{3,2*} \epsilon_7^{(2)}$. Thus for $i = 1, 2$ it also satisfies

$$\epsilon_7^{(3)} \circ \alpha_i(7) = i_{3,2*} \left(\epsilon_7^{(2)} \circ \alpha_i(7) \right) = i_{3,2*} \epsilon_{6+4i}^{(2)} \tag{3.52}$$

For $i = 1$ this element is trivial as $\pi_{10} Sp(3) = 0$. For $i = 2$, however, the homotopy exact sequence of $Sp(2) \xrightarrow{i_{3,2}} Sp(3) \xrightarrow{p} S^{11}$ gives a short exact sequence

$$0 \to \pi_{14} Sp(2) = \mathbb{Z}_3 \xrightarrow{i_{3,2*}} \pi_{14} Sp(3) = \mathbb{Z}_9 \xrightarrow{p_*} \pi_{14} S^{11} = \mathbb{Z}_3 \to 0 \tag{3.53}$$

from which we see that

$$\epsilon_7^{(3)} \circ \alpha_2(7) = i_{3,2*} \epsilon_{14}^{(2)} = 3 \cdot \epsilon_{14}^{(3)} \tag{3.54}$$

as was claimed. ∎

Lemma 3.16 *After localisation at 3 the composite $c \circ (1 \wedge \iota) : S^3 \wedge Q_3 \to Sp(3)$ has order 3.*

Proof By (3.18) Q_3 has the cell structure $S^3 \cup_{\alpha_1(3)} e^7 \cup e^{11}$ with the 3- and 7-cells linked by the Steenrod operation $\mathcal{P}^1$ which detects the stable class α_1. No primary operation connects the 7- and 11-cells so therefore

$$Q_3/S^3 \simeq S^7 \vee S^{11}. \tag{3.55}$$

Now following the methods of [94] it is possible to construct a secondary cohomology operation Θ based on the null composition $K(\mathbb{Z},3) \xrightarrow{\beta\mathcal{P}^1} K(\mathbb{Z}_3,8) \xrightarrow{\mathcal{P}^1} K(\mathbb{Z}_3,12)$ and evaluate it on $\Sigma\mathbb{C}P^5$ as $\Theta(\sigma x) = \sigma x^4$ where x generates $H^2\mathbb{C}P^5$ and σ is the supension. The cofiber sequence $\mathbb{H}P^2 \to Q_3 \to \Sigma\mathbb{C}P^5$ may then be used to evaluation it on Q_3 as $\Theta(x_3) = x_{11}$. This operation is obviously related to the Liulevicius-Shimada-Yamanoshita stable operation $\mathcal{R}$ which is based on the relation $2\mathcal{P}^2\beta + \beta\mathcal{P}^1\mathcal{P}^1 - 2\mathcal{P}^1\beta\mathcal{P}^1 = 0$. Since $\mathcal{R}$ detects the stable class α_2 [94], we see that Θ detects $\alpha_2(3) \in \pi_{10}S^3$. Thus we identify the connecting map in the cofiber sequence

$$S^3 \xrightarrow{j} Q_3 \xrightarrow{q} S^7 \vee S^{11} \xrightarrow{(\alpha_1(4),\alpha_2(4))} S^4 \to \dots \tag{3.56}$$

Now suspend the cofiber sequence (3.56) and apply the functor $[-, Sp(3)]$ to get

$$0 \xleftarrow{j^*} [S^3 \wedge Q_3, Sp(3)] \xleftarrow{q^*} \pi_{14}Sp(3) = \mathbb{Z}_9 \xleftarrow{\alpha_2(7)^*} \pi_7 Sp(3) = \mathbb{Z}_{(3)} \leftarrow \dots \tag{3.57}$$

Here we have used that $\pi_6 Sp(3) = 0$ and $\pi_{10}Sp(3) = 0$. This gives us an isomorphism $[S^3 \wedge Q_3, Sp(3)] \cong \mathrm{coker}(\alpha_2(7)^*)$. Recalling from Lemma 3.15 that $\epsilon_7^{(3)} \circ \alpha_2(7) = 3 \cdot \epsilon_{14}^{(3)}$ we get

$$[S^3 \wedge Q_3, Sp(3)] \cong \mathbb{Z}_9/\mathbb{Z}_3 \cong \mathbb{Z}_3 \tag{3.58}$$

and since $c \circ (1 \wedge \iota)$ is an element in this set, if it is non-trivial then it has order 3. To show $c \circ (1 \wedge \iota)$ is non-trivial we need only recall that the non-trivial Samelson product $\langle \epsilon_3, \epsilon_{11} \rangle$ factors through this map. ∎

Lemma 3.17 *The following diagram is homotopy commutative.*

$$\begin{tikzcd} S^3 \wedge Sp(3) \arrow[r, "c"] \arrow[d, "1\wedge p"] & Sp(3) \arrow[d, "p"] \\ S^{14} \arrow[r, "\alpha_1(11)"] & S^{11}. \end{tikzcd} \tag{3.59}$$

Proof Since $\pi_{13}Sp(2) = 0$ we see that $p_* : \pi_{14}Sp(3) \to \pi_{14}S^{11}$ is onto and $p_*\epsilon_{14} = \alpha_1(11)$. From here the homotopy commutativity of (3.59) is a straightforward consequence of Lemma 3.5. We get

$$p \circ c \simeq p \circ (\epsilon_{14} \circ (1 \wedge p)) \simeq \alpha_1(11) \circ (1 \wedge p) \tag{3.60}$$

which is exactly the claim. ∎

We are now ready to tackle the main result of this section. We remind the reader that all spaces and groups are localised at 3 throughout.

Proposition 3.18 *After localisation at 3 the commutator map* $c : S^3 \wedge Sp(3) \to Sp(3)$ *has order* 3.

Proof From Lemma 3.17 we find that $p \circ (3 \cdot c) \simeq *$, since $\alpha_1(11)$ has order 3. Thus there is a lift $\widehat{(3 \cdot c)} : S^3 \wedge Sp(3) \to Sp(2)$ satisfying

$$i \circ \widehat{(3 \cdot c)} \simeq (3 \cdot c) \tag{3.61}$$

where $i : Sp(2) \to Sp(3)$ is the inclusion. The proof shall proceed from here by examining the exact Puppe sequences surounding the various cofibring and fibrings and producing a further lift of the map $\widehat{(3 \cdot c)}$ through the connecting map $\Delta : \Omega S^{11} \to Sp(2)$. From this we shall conclude that $3 \cdot c$ itself must be trivial.

We seek to calculate the homotopy set $[S^3 \wedge Sp(3), Sp(2)]$. The cofibration sequence $(S^3 \wedge Q_3) \vee S^{13} \xrightarrow{j_{16}} S^3 \wedge Sp(3) \xrightarrow{q_{16}} \left(S^{17} \cup_{\alpha_1(17)} e^{21}\right) \vee S^{24}$ yields an exact sequence of abelian groups

$$\cdots \leftarrow [\left(S^3 \wedge Q_3\right) \vee S^{13}, Sp(2)] \xleftarrow{j_{16}^*} [S^3 \wedge Sp(3), Sp(2)] \xleftarrow{q_{16}^*} [\left(S^{17} \cup_{\alpha_1(17)} e^{21}\right) \vee S^{24}, Sp(2)] \leftarrow \dots \tag{3.62}$$

We evaluate the group on the right. We have

$$[\left(S^{17} \cup_{\alpha_1(17)} e^{21}\right) \vee S^{24}, Sp(2)] = [S^{17} \cup_{\alpha_1(17)} e^{21}, Sp(2)] \oplus \pi_{24} Sp(2) \tag{3.63}$$

and we must calculate both groups. Firstly we tackle $\pi_{24} Sp(2)$ using the homotopy exact sequence of $S^3 \to Sp(2) \to S^7$ to get the exact sequence

$$\dots \to \pi_{24} S^3 \to \pi_{24} Sp(2) \to \pi_{24} S^7 \to \dots \tag{3.64}$$

We have $\pi_{24} S^3 = 0$ and $\pi_{24} S^7 = 0$ so by exactness

$$\pi_{24} Sp(2) = 0. \tag{3.65}$$

For the other factor on the right-hand side of equation (3.63) we use the exact sequence

$$\cdots \leftarrow \pi_{17} Sp(2) \leftarrow [S^{17} \cup_{\alpha_1(17)} e^{21}, Sp(2)] \leftarrow \pi_{21} Sp(2) \leftarrow \dots \tag{3.66}$$

that is induced by the cofibring $S^{17} \hookrightarrow S^{17} \cup_{\alpha_1(17)} e^{21} \to S^{21}$ derived from the cellular structure of the 2-cell complex. We have $\pi_{17} Sp(2) = 0$ and $\pi_{21} Sp(2) = 0$ so the exactness of (3.66) gives us

$$[S^{17} \cup_{\alpha_1(17)} e^{21}, Sp(2)] = 0. \tag{3.67}$$

This means that the map j_{16}^* appearing in the sequence (3.62) is monic and any map $S^3 \wedge Sp(3) \to Sp(2)$ is determined by its restriction to the 16-skeleton.

We come to the group $[(S^3 \wedge Q_3) \vee S^{13}, Sp(2)] = [S^3 \wedge Q_3, Sp(2)] \oplus \pi_{13} Sp(2)$. We have for the second factor $\pi_{13} Sp(2) = 0$, and for the first the exact sequence

$$\begin{tikzcd} \pi_6 Sp(2) \arrow[d, equal] & {[S^3 \wedge Q_3, Sp(2)]} \arrow[l] \arrow[d, equal] & \pi_{10} Sp(2) \oplus \pi_{14} Sp(2) \arrow[l] \arrow[d, equal] & \pi_7 Sp(2) \arrow[l, "\alpha_1(7)^* \oplus \alpha_2(7)^*"'] \arrow[d, equal] \\ 0 & \mathrm{coker}(\alpha_1(7)^* \oplus \alpha_2(7)^*) & \mathbb{Z}_3 \oplus \mathbb{Z}_3 & \mathbb{Z} \end{tikzcd} \tag{3.68}$$

that comes from the cofibration $S^6 \to S^3 \wedge Q_3 \xrightarrow{r} S^{10} \vee S^{14}$. Since $\pi_6 Sp(2) = 0$ we have $[S^3 \wedge Q_3, Sp(2)] \cong \text{coker}(\alpha_1(7)^* \oplus \alpha_2(7)^*)$. Lemma 3.15 tells us that $\epsilon_7^{(2)} \circ \alpha_1(7) = \epsilon_{10}^{(2)}$ and $\epsilon_7^{(2)} \circ \alpha_2(7) = \epsilon_{14}^{(2)}$ from which it follows that

$$[S^3 \wedge Q_3, Sp(2)] \cong \text{coker}(\alpha_1(7)^* \oplus \alpha_2(7)^*) \cong \mathbb{Z}_3. \tag{3.69}$$

Now define maps $\lambda : S^3 \wedge Q_3 \to Sp(2)$ and $\tilde{\lambda} : S^3 \wedge Q_3 \to \Omega S^{11}$ to be the compositions

$$\lambda : S^3 \wedge Q_3 \xrightarrow{r} S^{10} \vee S^{14} \xrightarrow{pinch} S^{10} \xrightarrow{\epsilon_{10}^{(2)}} Sp(2) \tag{3.70}$$

$$\tilde{\lambda} : S^3 \wedge Q_3 \xrightarrow{r} S^{10} \vee S^{14} \xrightarrow{pinch} S^{10} \xrightarrow{E} \Omega S^{11} \tag{3.71}$$

where E is the suspension map. It is easy to see that λ generates $[S^3 \wedge Q_3, Sp(2)]$. Moreover, if $\Delta : \Omega S^{11} \to Sp(2)$ denotes the connecting map of the fibration $Sp(2) \to Sp(3) \to S^{11}$ then

$$\Delta \circ \tilde{\lambda} = \Delta \circ E \circ (pinch) \circ r \simeq \epsilon_{10}^{(2)} \circ (pinch) \circ r = \lambda \tag{3.72}$$

and $\tilde{\lambda}$ is seen to be a lift of λ through Δ.

Now according to Yamaguchi [221] the quaternionic Stiefel manifold $Sp(3)/Sp(1)$ is 3-regular. Thus there is a 3-local equivalence $Sp(3)/Sp(1) \simeq S^7 \times S^{11}$ which furthermore may be chosen such that the following diagram commutes up to homotopy

$$\begin{tikzcd} Q_3 \arrow[r, "\iota"] \arrow[d, "r"] & Sp(3) \arrow[d, "R"] \\ S^7 \vee S^{11} \arrow[r, hook] & S^7 \times S^{11} \simeq Sp(3)/Sp(1) \end{tikzcd} \tag{3.73}$$

where R is the canonical projection.

Now define maps

$$\Lambda : S^3 \wedge Sp(3) \xrightarrow{1 \wedge R} S^3 \wedge \left(S^7 \times S^{11}\right) \xrightarrow{pinch} S^3 \wedge S^7 = S^{10} \xrightarrow{\epsilon_{10}^{(2)}} Sp(2) \tag{3.74}$$

$$\widetilde{\Lambda} : S^3 \wedge Sp(3) \xrightarrow{1 \wedge R} S^3 \wedge \left(S^7 \times S^{11}\right) \xrightarrow{pinch} S^3 \wedge S^7 = S^{10} \xrightarrow{E} \Omega S^{11}. \tag{3.75}$$

Then firstly

$$\Delta \circ \widetilde{\Lambda} \simeq \Lambda \tag{3.76}$$

so that $\widetilde{\Lambda}$ is a lift of Λ through Δ, and secondly, using the homotopy commutativity of (3.73) we get

$$\Lambda \circ \iota \simeq \lambda : S^3 \wedge Q_3 \to Sp(2). \tag{3.77}$$

Recall that any map $S^3 \wedge Sp(3) \to Sp(2)$ was seen to be determined up to homotopy by its restriction to $S^3 \wedge Q_3$. Therefore since λ generates $[S^3 \wedge Q_3, Sp(2)]$, the map Λ must be a generator of $[S^3 \wedge Sp(3), Sp(2)]$. We already have a map $\widehat{(3 \cdot c)} \in [S^3 \wedge Sp(3), Sp(2)]$ lifting 3 times the commutator so by the fact that Λ generates the group there exists an integer a such that

$$\widehat{(3 \cdot c)} \simeq a \cdot \Lambda. \tag{3.78}$$

We may now use equation (3.76) to write

$$\widehat{(3 \cdot c)} \simeq a \cdot (\Delta \circ \widetilde{\Lambda}) \tag{3.79}$$

from which it follows that

$$3 \cdot c \simeq i \circ \widehat{(3 \cdot c)} \simeq a \cdot (i \circ \Delta \circ \widetilde{\Lambda}) \simeq * \tag{3.80}$$

is trivial since i and Δ are adjacent maps in a fibration sequence.

Now we have seen previously using the Samelson product $\langle \epsilon_3, \epsilon_{11} \rangle$ that c is non-trivial and we have just demonstrated that $3 \cdot c$ is trivial. Therefore the order of c is exactly 3. ∎

3.3.3 The 5-Local Order of c.

The Lie group $Sp(3)$ is quasi-5-regular [148]. This means that there is a 5-local homotopy equivalence

$$\psi : B_1^2 \times S^7 \simeq Sp(3) \tag{3.81}$$

where B_1^2 is an S^3-bundle over S^{11} with characteristic element $\alpha_1(3) \in \pi_{10} S^3$. The Lie group $Sp(2)$, on the other hand is completely regular at the prime 5 and there is a 5-local homotopy equivalence

$$\psi' : S^3 \times S^7 \simeq Sp(2). \tag{3.82}$$

We work locally during this section and all spaces and groups are to be localised at 5.

Lemma 3.19 *The following relation holds in the 5-local homotopy module $\pi_* Sp(3)$.*

$$\epsilon_7 \circ \alpha_1(7) = \epsilon_{14} \in \pi_{14} Sp(3). \tag{3.83}$$

Proof Given the homotopy equivalence ψ in equation (3.81) it follows that for the generator $\epsilon_7 \in \pi_7 Sp(3)$ we may take the composition

$$\epsilon_7 : S^7 \overset{in}{\hookrightarrow} B_1^2 \times S^7 \xrightarrow{\psi} Sp(3). \tag{3.84}$$

We also have

$$\pi_{14} Sp(3) \cong \pi_{14} \left(B_1^2 \times S^7 \right) \cong \pi_{14} B_2^1 \oplus \pi_{14} S^7 \cong \pi_{14} B_2^1 \oplus \mathbb{Z}_5\{\alpha_1(7)\}. \tag{3.85}$$

Since we know that the 5-component of $\pi_{14} Sp(3) = \mathbb{Z}_5$ we get

$$\pi_{14} B_2^1 = 0 \tag{3.86}$$

$$\pi_{14} Sp(3) \cong \pi_{14} S^7 = \mathbb{Z}_5\{\alpha_1(7)\}. \tag{3.87}$$

From this observation and the definition (3.84) we get the lemma. ∎

Proposition 3.20 *After localisation at 5 the commutator map $c : S^3 \wedge Sp(3) \to Sp(3)$ becomes null-homotopic.*

Proof Recall the local homotopy equivalences (3.81), (3.88) and denote by $\psi_1 : B_1^2 \to Sp(3)$, $\psi_2 : S^7 \to Sp(3)$, $\psi_1' : S^3 \to Sp(2)$ $\psi_2' : S^7 \to Sp(2)$ the restrictions of ψ and ψ' to each of the factors in the products. By their construction in [151] the maps ψ, ψ' are recovered completely as the pointwise products

$$\psi(x,y) = \psi_1(x)\cdot\psi_2(y) \tag{3.88}$$

$$\psi'(x,y) = \psi_1'(x)\cdot\psi_2'(y). \tag{3.89}$$

Note that since $\pi_7 Sp(2) \cong \pi_7 Sp(3)$, the map ψ_2 factors up to homotopy as

$$\psi_2 : S^7 \xrightarrow{\psi_2'} Sp(2) \xrightarrow{i} Sp(3). \tag{3.90}$$

Now let $\mu : Sp(3) \times S^{11} \to Sp(3)$ denote the natual action of $Sp(3)$ on $S^{11} \cong Sp(3)/Sp(2)$, let $m : Sp(3) \times Sp(3) \to Sp(3)$ denote the group multiplication and consider the following diagram

$$\begin{tikzcd}
B_1^2 \times S^7 \arrow[r, "\psi_1\times\psi_2"'] \arrow[rrr, bend left, "\psi"] \arrow[d, "pr_1"] & Sp(3)\times Sp(2) \arrow[r, "1\times i"'] \arrow[d, "pr_1"] & Sp(3)\times Sp(3) \arrow[r, "m"'] \arrow[d, "1\times p"] & Sp(3) \arrow[d, "p"] \\
B_1^2 \arrow[r, "\psi_1"] \arrow[rrr, bend right, "p'"'] & Sp(3) \arrow[r, "in_1"] & Sp(3)\times S^{11} \arrow[r, "\mu"] & S^{11}
\end{tikzcd} \tag{3.91}$$

where $p' : B_1^2 \to S^{11}$ is defined to be the composition of the maps along the bottom row. The properties of the action μ [12] ensure that the right-hand square homotopy commutes whilst the middle and left-hand squares homotopy commute by construction. It follows that the whole diagram is homotopy commutative. The observations (3.88) and (3.90) show that the composite along the top row is ψ. In particular (3.91) serves to verify the homotopy commutativity of the following diagram

$$\begin{tikzcd}
B_1^2 \times S^7 \arrow[r, "\psi", "\simeq"'] \arrow[d, "pr_1"'] & Sp(3) \arrow[d, "p"] \\
B_1^2 \arrow[r, "p'"] & S^{11}.
\end{tikzcd} \tag{3.92}$$

Note that as it is defined the map p' is seen to induce an isomorphism $p'_* : H_{11}B_1^2 \cong H_{11}S^{11}$. From this we may conclude that its homotopy fibre is homotopy equivalent to S^3.

Now B_1^2 has a cell-structure [105]

$$B_1^2 \simeq S^3 \cup_{\alpha_1(3)} e^{11} \cup_\varphi e^{14} \tag{3.93}$$

where φ is the attaching map for the 14-cell. The map $\alpha_1(3)$ is stable but the methods of [55] may be used to show that the top cell splits off the complex after a single suspension. There results a homotopy equivalence $e : \Sigma B_1^2 \simeq (S^4 \cup_{\alpha_1(4)} e^{12}) \vee S^{15}$ which may be chosen so that the following diagram becomes homotopy commutative

$$\begin{tikzcd}
\Sigma B_1^2 \arrow[r, "e", "\simeq"'] \arrow[d, "\Sigma p'"'] & \left(S^4 \cup_{\alpha_1(4)} e^{12}\right) \vee S^{15} \arrow[r, "pinch"] & S^4 \cup_{\alpha_1(4)} e^{12} \arrow[d, "q'"] \\
S^{12} \arrow[rr, equal] & & S^{12}
\end{tikzcd} \tag{3.94}$$

where q' is the quotient map that appears in the cofiber sequence

$$S^4 \to S^4 \cup_{\alpha_1(4)} e^{12} \xrightarrow{q'} S^{12} \xrightarrow{\alpha_1(5)} S^5 \to \dots \tag{3.95}$$

Now form the diagram

$$\begin{tikzcd}
S^3 \wedge (B_1^2 \times S^7) \arrow[rrr, "1\wedge\psi", "\simeq"'] \arrow[d, "1\wedge pr_1"'] & & & S^3 \wedge Sp(3) \arrow[r, "c"] \arrow[d, "1\wedge p"] & Sp(3) \arrow[dd, equal] \\
S^3 \wedge B_1^2 \arrow[rrr, "1\wedge p'"] \arrow[d, "1\wedge e"', "\simeq"] & & & S^{14} \arrow[d, equal] & \\
\left(S^6 \cup_{\alpha_1(6)} e^{14}\right) \vee S^{17} \arrow[r, "pinch"] & S^6 \cup_{\alpha_1(6)} e^{14} \arrow[rr, "q'"] & & S^{14} \arrow[r, "\epsilon_{14}"] & Sp(3).
\end{tikzcd} \tag{3.96}$$

The square on the right-hand side of the diagram is (3.9) from Lemma 3.5. The top left hand square is the triple suspension of (3.92) and the bottom left-hand square is the double suspension of (3.94). We have seen the homotopy commuativity of each individual square and so conclude that the entire diagram is homotopy commutative.

The fact that $1 \wedge \psi$ is a homotopy equivalance now gives us a string of implications

$$c \simeq * \Longleftrightarrow c \circ (1 \wedge \psi) \simeq * \Longleftrightarrow \epsilon_{14} \circ q' \circ (pinch) \circ e \circ (1 \wedge pr_1) \simeq *. \tag{3.97}$$

Furthermore since e is also a homotopy equivlence we also have

$$\epsilon_{14} \circ q' \simeq * \Longrightarrow \epsilon_{14} \circ q' \circ (pinch) \circ e \circ (1 \wedge pr_1) \simeq *. \tag{3.98}$$

It follows that we will be able to conclude that c is null homotopic if we are able to prove that $\epsilon_{14} \circ q'$ is null homotopic.

Now it was shown in Lemma 3.19 that $\epsilon_{14} = \epsilon_7 \circ \alpha_1(7)$. Therefore

$$\epsilon_{14} \circ q' \simeq \epsilon_7 \circ (\alpha_1(7) \circ q'). \tag{3.99}$$

But $\alpha_1(7) \circ q'$ is null homotopic since they are adjacent maps in the cofiber sequence (3.95). Therefore

$$\epsilon_{14} \circ q' \simeq \epsilon_7 \circ (\alpha_1(7) \circ q') \simeq * \tag{3.100}$$

and we conclude from the implications (3.97), (3.98) that

$$c \simeq *. \quad \blacksquare \tag{3.101}$$

3.3.4 The 7-Local Order of c.

Proposition 3.21 *After localisation at 7 the commutator map $c : S^3 \wedge Sp(3) \to Sp(3)$ has order 7.*

Proof A lower bound for the 7-component of the order of c is obtained from the Samelson product $\langle \epsilon_3, \epsilon_{11} \rangle$. We also obtain an upper bound of 7 from the order of the map ϵ_{14} in Lemma 3.5. Since the lower bound matches the upper bound exactly, we conclude that the order of c is exactly 7 when localised at 7. $\blacksquare$

3.4 Proof of Theorems 3.1 and 3.2

Proof of 3.2 We combine Propositions 3.14, 3.18, 3.20 and 3.21 together with the observation 3.7 that c is trivial at all primes ≥ 11. ∎

Now having established the order of the commutator map c - and thus the connecting map δ_1 - we shall need one more lemma, due to Theriault. If Y is an H-space and k is an integer, we denote the kth power map $k : Y \to Y$.

Lemma 3.22 (Theriault [187]) *Let X, Y be H-spaces and $f : X \to Y$ a map of finite order m. Let F_k be the homotopy fibre of $k \circ f : X \to Y$. If $(m, k) = (m, k')$ then there is a homotopy equivalence $F_k \simeq F_{k'}$ after rationalisation of localisation at any prime.* ∎

Proof of 3.1 The first statement is an observation originally due to Sutherland [181]. In the homotopy sequence of the evaluation fibration (4.10) Theorem 4.5 identifies $\delta_{k*} : \pi_{11}Sp(3) \to \pi_{14}Sp(3)$ with the Samelson product $\langle k \cdot \epsilon_3, \epsilon_{11}\rangle$. Since $\pi_{10}Sp(3) = 0$ we use Bott's result [31] to get

$$\pi_{11}B\mathcal{G}_k = \mathrm{coker}(\delta_{k*}) = \mathbb{Z}/120(84, k)\mathbb{Z}. \tag{3.102}$$

Therefore if $\mathcal{G}_k \simeq \mathcal{G}_l$ it must be that $(84, k) = (84, l)$.

To prove the second statement we use Proposition 3.4 and get for the connecting map $\delta_k \simeq k \cdot \delta_1$. Since δ_1 is the triple adjoint of the commutator c, Theorem 3.2 gives its order as either $4 \cdot 3 \cdot 7 = 84$ or $8 \cdot 3 \cdot 7 = 168$. In either case $168 \cdot \delta_1 \simeq *$ so we now free the fact that $\mathcal{G}_k$ is the homotopy fibre of δ_k to apply Lemma 3.22 and get a local homotopy equivalence $\mathcal{G}_k \simeq \mathcal{G}_l$ after rationalisation or localisation at any prime whenever $(168, k) = (168, l)$. ∎

4 The Homotopy Types of $U(n)$-Gauge Groups Over S^4 and $\mathbb{C}P^2$.

Abstract

The homotopy types of $U(n)$-gauge groups over the two most fundamental 4-manifolds S^4 and $\mathbb{C}P^2$ are studied. We give homotopy decompositions of the $U(n)$-gauge groups over S^4 in terms of certain $SU(n)$- and $PU(n)$-gauge groups and use these decompositions to enumerate the homotopy types of the U(2)-, U(3)- and U(5)-gauge groups. Over $\mathbb{C}P^2$ we provide bounding results on the number of homotopy types of $U(n)$-gauge groups, provide p-local decompositions and give homotopy decompositions of certain $U(n)$-gauge groups in terms of certain $SU(n)$-gauge groups. Applications are then given to count the number of homotopy types of $U(2)$-gauge groups over $\mathbb{C}P^2$.

4.1 Introduction.

If G is a topological group and $P \xrightarrow{p} X$ is a principal G-bundle over a space X then a natural object to study is the gauge group $\mathcal{G}(P)$ of the bundle. This is the group, under composition, of fibrewise G-equivariant maps $E \to E$ that cover the identity on X.

The study of the homotopy types of certain gauge groups has been a topic of much recent attention and perhaps the most interesting examples arise when G is a subgroup of one of the linear groups. In particular, the cases $G = U(n)$ with X a Riemann surface and $G = SU(n)$ with X a simply connected 4-manifold have applications to geometry [68], [70] and physics [43], [173]. Sutherland [181] and Theriault [189], [193] have both contributed valuable work towards the understanding of the first problem whilst Kono [127] and Theriault [190], [192] have provided complete solutions of the second for certain small values of n.

In all the cases cited the number of principal G-bundle-isomorphism classes over X is countably infinite, yet, in a key paper, Crabb and Sutherland [61] have demonstrated that if G is a compact, connected Lie group and X is a finite complex, then the number of distinct homotopy types amongst all the gauge groups of principal G-bundles over X is finite. Consideration of the problem, on the other hand, shows that this number seems to be proportional to the topological and geometric complexity of the space X and the group G. As cells are added to X and the rank of G grows, the number of distinct homotopy types is generally observed to grow.

The case when $X = S^4$ and $G = SU(n) \subseteq U(n)$ was first tackled by Kono [125] and is a natural problem to study for many reasons. In particular any $U(n)$-bundle over S^4 has a reduction of structure to an $SU(n)$-bundle and it is logical to first study the gauge group associated to the second, simpler, object. The extension of this problem to X being any simply connected 4-manifolds is perhaps less natural as in this case the first Chern class of a principal $U(n)$-bundle does not automatically vanish and is in fact the primary obstruction to it having an $SU(n)$-reduction of structure.

The problem for $U(n)$-bundles over a simply connected 4-manifold X differs from that for $U(n)$-bundles over surfaces or $SU(n)$-bundles over simply connected 4-manifolds due to the presence of extra low-dimensional topological information. Specifically, there are now two independent obstructions to a $U(n)$-bundle being trivial, namely the first and second Chern classes $(c_1, c_2) \in H^2(X;\mathbb{Z}) \oplus H^4(X;\mathbb{Z})$, which may be used to index the $U(n)$-bundle-isomorphism classes over X. This double-index makes the problem much more delicate than, say, the study of $SU(n)$-bundles over X. The first Chern class represents a twisting of the bundle over the 2-skeleton $X_2 \simeq \vee S^2$ whilst the second Chern class corresponds to the non-triviality of the bundle over the closed 4-cell $e^4 \subseteq X$. In this context the problem has not previously been studied.

The goal of this paper is to determine the homotopy types of $U(n)$-gauge groups over the 4-sphere S^4 and the complex projective plane $\mathbb{C}P^2$, the two most fundamental examples of simply-connected 4-manifolds.

First we examine the homotopy types of the gauge groups belonging to $U(n)$-bundles over S^4. Since any such $U(n)$-bundle has a reduction of structure to an $SU(n)$-bundle, one may hope that the homeomorphism $U(n) \cong S^1 \times SU(n)$ is somehow reflected in the topology of the gauge groups of these bundles and we show that this is indeed the case.

For $G = U(n), SU(n), PU(n)$ let $\mathcal{G}_k(S^4, G)$ denote the gauge group of the principal G-bundle over S^4 with second Chern class $k \in H^4(S^4;\mathbb{Z})$.

Theorem 4.1 *The following statements hold.*

1. *For $n \geq 3$ the gauge group $\mathcal{G}_k(S^4, U(n))$ is a trivial $\mathcal{G}_k(S^4, SU(n))$-bundle over S^1 and*

there is an isomorphism of principal $\mathcal{G}_k(S^4, SU(n))$-bundles $\mathcal{G}_k(S^4, U(n)) \cong \mathcal{G}_k(S^4, SU(n)) \times S^1$ over S^1.

2. *For $n = 2$ and $k = 2l$, the gauge group $\mathcal{G}_{2l}(S^4, U(2))$ is a trivial $\mathcal{G}_{2l}(S^4, SU(2))$-bundle over S^1 and there is an isomorphism of principal $\mathcal{G}_{2l}(S^4, SU(2))$-bundles $\mathcal{G}_{2l}(S^4, U(2)) \cong \mathcal{G}_{2l}(S^4, SU(2)) \times S^1$ over S^1.*

3. *For n=2 and $k = (2l + 1)$ the gauge group $\mathcal{G}_{2l+1}(S^4, U(2))$ is a trivial S^1-bundle over $\mathcal{G}_{2l+1}(S^4, PU(2))$ and there is an isomorphism of principal S^1-bundles $\mathcal{G}_{2l+1}(S^4, U(2)) \cong S^1 \times \mathcal{G}_{2l+1}(S^4, PU(2))$ over $\mathcal{G}_{2l+1}(S^4, PU(2))$.*

It was shown by Kono in [125] that $\mathcal{G}_k(S^4, SU(2)) \simeq \mathcal{G}_l(S^4, SU(2))$ if and only if $(12, k) = (12, l)$ and shown by Kamiyama, Kishimoto, Kono and Tsukuda in [111] that $\mathcal{G}_k(S^4, PU(2)) \simeq \mathcal{G}_l(S^4, PU(2))$ if and only if $(12, k) = (12, l)$. Likewise, it was shown by Hamanaka and Kono in [90] (see also [190]) that $\mathcal{G}_k(S^4, SU(3)) \simeq \mathcal{G}_l(S^4, SU(3))$ if and only if $(k, 24) = (l, 24)$ and shown by Theriault in [192] that $\mathcal{G}_k(S^4, SU(5)) \simeq \mathcal{G}_l(S^4, SU(5))$ when rationalised or localised at any prime if and only if $(k, 120) = (l, 120)$. Therefore Theorem 4.1 immediately gives the following corollary.

Corollary 4.2 *The following statements hold.*

1. *There is a homotopy equivalence $\mathcal{G}_k(S^4, U(2)) \simeq \mathcal{G}_l(S^4, U(2))$ if and only if $(12, k) = (12, l)$.*

2. *There is a homotopy equivalence $\mathcal{G}_k(S^4, U(3)) \simeq \mathcal{G}_l(S^4, U(3))$ if and only if $(k, 24) = (l, 24)$.*

3. *There is a homotopy equivalence $\mathcal{G}_k(S^4, U(5)) \simeq \mathcal{G}_l(S^4, U(5))$ when rationalised or localised at any prime if and only if $(k, 120) = (l, 120)$.* ∎

Turning now to the complex projective plane it has a cell structure $\mathbb{C}P^2 = S^2 \cup_\eta e^4$, where η is the Hopf map. In this case it is not automatic for a $U(n)$-bundle to have a reduction of structure to an $SU(n)$-bundle and the study of the homotopy types of $U(n)$-gauge groups over $\mathbb{C}P^2$ is a new and entirely unexplored area. If $c_1 = 0$ then the bundle does have a reduction of structure to an $SU(n)$-bundle and there is some hope that this simplification will be reflected in the topology of its gauge group. If $c_1 \neq 0$ then there is a new twisting to the bundle whose effects are previously unstudied. In fact when c_1 is nontrivial the problem is very intricate and a complete solution to the problem is beyond the reach of current techniques. In this case not even the homotopy types of the based gauge groups, consisting of those bundle automorphisms that restrict to the identity on the fibre, are well understood.

Let $\mathcal{G}^{(k,l)}(\mathbb{C}P^2, U(n))$ be the gauge group of the $U(n)$-bundle over $\mathbb{C}P^2$ with first and second Chern classes $(c_1, c_2) = (k, l) \in H^2(\mathbb{C}P^2) \oplus H^4(\mathbb{C}P^2)$ and let $\mathcal{G}^{(k,l)}_*(\mathbb{C}P^2, U(n))$ be the based gauge group associated to the same bundle. Also let $\mathcal{G}^{(k,l)}(\mathbb{C}P^2, PU(n))$ be the gauge group of the $PU(n)$-bundle with the indicated characteristic classes and let $\mathcal{G}^l(\mathbb{C}P^2, SU(n))$ be the gauge group of the $SU(n)$-bundle over $\mathbb{C}P^2$ with second Chern class $c_2 = l \in H^4(\mathbb{C}P^2)$.

Many results for general $U(n)$-gauge groups are stated and proved in section 4.5. These are then refined and applied to the case of $U(2)$ to produce our most complete results. We prove the following.

Theorem 4.3 *Let k, l be integers. Then the following hold:*

1. *There is a homotopy equivalence*

$$\mathcal{G}^{(k,l)}(\mathbb{C}P^2, U(2)) \simeq \begin{cases} \mathcal{G}^{(0,l')}(\mathbb{C}P^2, U(2)), & k \text{ even} \\ \mathcal{G}^{(1,l')}(\mathbb{C}P^2, U(2)), & k \text{ odd} \end{cases} \tag{4.1}$$

for a suitable integer l'.

2. *There is a homotopy equivalence*

$$\mathcal{G}^{(k,l)}(\mathbb{C}P^2, U(2)) \simeq \mathcal{G}^{(k,l+12)}(\mathbb{C}P^2, U(2)). \tag{4.2}$$

3. *The gauge group $\mathcal{G}^{(0,l)}(\mathbb{C}P^2, U(2))$ is a trivial $\mathcal{G}^l(\mathbb{C}P^2, SU(2))$-bundle over S^1 and there is an isomorphism of principal bundles*

$$\mathcal{G}^{(0,l)}(\mathbb{C}P^2, U(2)) \cong S^1 \times \mathcal{G}^l(\mathbb{C}P^2, SU(2)). \tag{4.3}$$

4. *When localised away from 2 there is a product splitting*

$$\mathcal{G}^{(1,l)}(\mathbb{C}P^2, U(2)) \simeq S^1 \times \mathcal{G}^{4l-1}(\mathbb{C}P^2, SU(2)). \tag{4.4}$$

5. *For any integer values of l, l' it holds that*

$$\mathcal{G}^{(0,l)}(\mathbb{C}P^2, U(2)) \not\simeq \mathcal{G}^{(1,l')}(\mathbb{C}P^2, U(2)). \tag{4.5}$$

6. *For any integer values of l, l' it holds for the based gauge groups that*

$$\mathcal{G}_*^{(0,l)}(\mathbb{C}P^2, U(2)) \not\simeq \mathcal{G}_*^{(1,l')}(\mathbb{C}P^2, U(2)). \tag{4.6}$$

The $SU(2)$-gauge groups $\mathcal{G}^l(\mathbb{C}P^2, SU(2))$ appear in 4.3.3 and 4.3.4. These gauge groups were studied by Kono and Tsukuda in [127] and their Theorem 1.2 may be applied to yield the following corollary.

Corollary 4.4 *There is a homotopy equivalence $\mathcal{G}^{(0,l)}(\mathbb{C}P^2, U(2)) \simeq \mathcal{G}^{(0,l')}(\mathbb{C}P^2, U(2))$ if and only if $(6, l) = (6, l')$. When localised away from 2 there is a homotopy equivalence $\mathcal{G}^{(1,l)}(\mathbb{C}P^2, U(2)) \simeq \mathcal{G}^{(1,l')}(\mathbb{C}P^2, U(2))$ if and only if $gcd(4l-1, 6) = gcd(4l'-1, 6)$. In particular, when localised at an odd prime $p \geq 5$, the gauge group $\mathcal{G}^{(1,l)}$ has the trivial homotopy type*

$$\mathcal{G}^{(1,l)}(\mathbb{C}P^2, U(2)) \simeq U(2) \times Map^*(\mathbb{C}P^2, U(2)). \quad \blacksquare \tag{4.7}$$

A similar statement can be made for $U(3)$-gauge groups using the material in section 4.5 and the information on $SU(3)$-gauge groups contained in [190]. We leave its formulation to the interested reader.

What we show leaves us just one step short of a complete classification of the $U(2)$-gauge groups over $\mathbb{C}P^2$. We partition the homotopy types into the two non-intersecting sets $\{\mathcal{G}^{(0,l)}(\mathbb{C}P^2, U(2))\}$ and $\{\mathcal{G}^{(1,l)}(\mathbb{C}P^2, U(2))\}$ and give a full classification of the homotopy types in the first set and a full classifaction of the odd-primary homotopy types in the second. The discrepancy that remains is in the second set. In particular 4.3.2 shows that there are at most 4 distinct homotopy types amongst the 2-local gauge groups $\mathcal{G}^{(1,l)}(\mathbb{C}P^2, U(2))$. In section 4.6.3 we examine the low dimensional homotopy of their classifying spaces and show that there are at least 2 distinct homotopy types. It is undecided whether the actual number of distinct homotopy types of these gauge groups is 2, 3 or 4. A complete integral statement would be the most desirable resolution.

The layout of the paper is as follows. In section 4.2 we present preliminary information and set up notation for gauge groups. We examine the relationship between $U(n)$, $SU(n)$ and $PU(n)$ and give a classification of their principal bundles over S^4 and $\mathbb{C}P^2$. In section 4.3 we study the homotopy types of $U(n)$-gauge groups over S^4 and prove Theorem 4.1. Section 4.5 is dedicated to examining the homotopy types of $U(n)$-gauge groups over $\mathbb{C}P^2$. This section is broken into three subsections. In the first we study the connecting maps for certain evaluation fibrations, in the second we give p-local decompositions of the $U(n)$-gauge groups and in the third we examine the relationship between certain $U(n)$- and $SU(n)$-gauge groups. Finally in section 4.6.1 we apply what we have collected to the case of $U(2)$-gauge groups over $\mathbb{C}P^2$. In 4.6.2 we study the homotopy types of the based $U(2)$-gauge groups and their classifying spaces and in 4.6.3 we study the homotopy types of the full $U(2)$-gauge groups and complete the proof of Theorem 4.3.

4.2 Preliminaries.

Let G be a connected, compact Lie group and $P \xrightarrow{p} X$ a principal G-bundle over a connected, finite complex X. Let $\mathcal{G}(P)$ denote the gauge group of P and $\mathcal{G}_*(P)$ the based gauge group consisting of those gauge transformations of P that reduce to the identity on the fibre. Let BG be the classifying space for G and $EG \to BG$ be the universal G-bundle with contractible total space. Then it is well known that the isomorphism classes of principal G-bundles over X are in one-to-one correspondence with the homotopy classes of maps $X \to BG$ via the prescription $f \mapsto f^*EG$ which forms the pullback bundle $f^*EG \to X$.

Choose a map $f : X \to BG$ that classifies the bundle $P \xrightarrow{p} X$. Then Gottlieb [80] has shown the existence of homotopy equivalences

$$B\mathcal{G}(P) \simeq Map^f(X, BG), \qquad B\mathcal{G}_*(P) \simeq Map^f_*(X, BG) \tag{4.8}$$

where $B\mathcal{G}(P)$ and $B\mathcal{G}_*(P)$ are the classifying spaces of $\mathcal{G}(P)$ and $\mathcal{G}_*(P)$ respectively, $Map^f(X, BG)$ denotes the path component of f in the space of free maps $X \to BG$ and $Map^f_*(X, BG)$ denotes the path component of f in the space of based maps $X \to BG$.

Using the representations (4.8) leads to the evaluation fibration for $\mathcal{G}(P)$

$$\ldots \to \mathcal{G}(P) \to G \xrightarrow{\delta} B\mathcal{G}_*(P) \to B\mathcal{G}(P) \xrightarrow{ev} BG \tag{4.9}$$

where $ev : B\mathcal{G}(P) \simeq Map^f(X, BG) \to BG$ is given by evaluation at the basepoint of X. In (4.9) we have made explicit the homotopy equivalence $G \simeq \Omega BG$ and denoted the fibration connection map by $\delta : G \to B\mathcal{G}_*(P)$. This map shall play a prominent rôle in the following.

The situation with which we shall primarily be concerned is where $X = S^4$ or $X = \mathbb{C}P^2$ and $G = U(n)$, however we shall also have need to consider $SU(n)$- and $PU(n)$-bundles over these spaces and their gauge groups. For this purpose we shall now briefly discuss the relationship between these groups and set up the notation required in later sections. In the following we shall always assume that $n \geq 2$. The special case of $U(1)$ will be treated seperately.

The special unitary group $SU(n)$ is related to $U(n)$ by the fibration sequence

$$SU(n) \xrightarrow{j} U(n) \xrightarrow{det} S^1 \tag{4.10}$$

which is homotopically split by the inclusion $S^1 \cong U(1) \hookrightarrow U(n)$. There results a homeomorphism $U(n) \cong SU(n) \times S^1$ although this map does not respect the H-space structures.

The projective unitary group $PU(n)$ is the quotient of $U(n)$ by its centre, which is comprised of the diagonal matrices $\{\lambda I_n | \lambda \in S^1\} \cong S^1$. Equivalently, it is the quotient of $SU(n)$ by its centre $\mathbb{Z}_n$ of nth roots of unity along the diagonal matrices. This is displayed in the following commutative diagram of fibrations which defines the projection maps π, ρ, and the centre inclusion Δ

$$\begin{tikzcd}
\mathbb{Z}_n \arrow[r, "\delta"] \arrow[d] & S^1 \arrow[r, "n"] \arrow[d, "\Delta"] & S^1 \arrow[d, equal] \\
SU(n) \arrow[r, "j"] \arrow[d, "\rho"] & U(n) \arrow[r, "det"] \arrow[d, "\pi"] & S^1 \arrow[d] \\
PU(n) \arrow[r, equal] \arrow[d] & PU(n) \arrow[r] \arrow[d] & * \arrow[d] \\
K(\mathbb{Z}_n, 1) \arrow[r, "\delta"] & BS^1 \arrow[r, "n"] & BS^1.
\end{tikzcd} \tag{4.11}$$

The maps labelled n are the degree n self maps on the Eilenberg-Mac Lane spaces and δ is the Bockstein connecting map.

Something to take note of at this point is that the groups $U(n)$, $SU(n)$ and $PU(n)$ all share the same higher dimensional homotopy. In particular there are homotopy equivalences

$$\Omega^k SU(n) \overset{\Omega^k j}{\simeq} \Omega^k U(n) \overset{\Omega^k \pi}{\simeq} \Omega^k PU(n), \qquad \text{for } k \geq 2. \tag{4.12}$$

The last thing to address is the classification of principal $U(n)$-bundles over $X = S^4$ and $X = \mathbb{C}P^2$ that shall lead to our labelling of the components of the mapping spaces in (4.8). For this we need to enumerate the elements of the homotopy set $[X, BU(n)]$ and as in each case X is a 4-dimensional complex it suffices to examine the homotopy classes of mappings of X into the 4^{th} Postnikov section of $BU(n)$. For each $n \geq 2$, $BU(n)$ has a 4^{th} section

$$(c_1, c_2) : BU(n) \to BU(n)[4] = K(\mathbb{Z}, 2) \times K(\mathbb{Z}, 4) \tag{4.13}$$

where $c_1 \in H^2(BU(n))$, $c_2 \in H^4(BU(n))$ are the first and second Chern classes, respectively, and this leads to isomophisms of sets

$$c_2 : [S^4, BU(n)] \xrightarrow{\cong} H^4(S^4) = \mathbb{Z} \tag{4.14}$$

$$(c_1, c_2) : [\mathbb{C}P^2, BU(n)] \xrightarrow{\cong} H^2(\mathbb{C}P^2) \oplus H^4(\mathbb{C}P^2) = \mathbb{Z} \oplus \mathbb{Z}. \tag{4.15}$$

It follows that over S^4, isomorphism classes of principal $U(n)$-bundles are indexed by the integers in $H^4(S^4) = \mathbb{Z}$ corresponding to their second Chern classes and that over $\mathbb{C}P^2$, isomorphism classes of principal $U(n)$-bundles are indexed by the pair of integers in $H^2(\mathbb{C}P^2) \oplus H^4(\mathbb{C}P^2) = \mathbb{Z} \oplus \mathbb{Z}$ corresponding to their first and second Chern classes.

A similar classification applies to $SU(n)$- and $PU(n)$-bundles over S^4 and $\mathbb{C}P^2$.

4.3 Homotopy Types of $U(n)$-Gauge Groups Over S^4.

Let $n \geq 2$ and let $P_k \xrightarrow{p_k} S^4$ be the principal $U(n)$-bundle with second chern class $k \in H^4(S^4)$. Applying the classifying functor to the fibration sequence (4.10), it is easily seen that any classifying map $f_k : S^4 \to BU(n)$ for P_k lifts to BSU(n) and there results a reduction of structure $P_k \cong \tilde{P}_k \times_{SU(n)} U(n)$ with $\tilde{P}_k \xrightarrow{\tilde{p}_k} S^4$ an $SU(n)$-bundle with second Chern class $c_2(\tilde{P}_k) = k \in H^4(S^4)$.

We now consider the gauge groups $\mathcal{G}_k(S^4, U(n)) = \mathcal{G}(P_k)$ of P_k and $\mathcal{G}_k(S^4, SU(n)) = \mathcal{G}(\tilde{P}_k)$ of $\tilde{P}_k$. In this section we shall be working only over the base space S^4, so for convenience we use the notation $\mathcal{G}_k^{U(n)} = \mathcal{G}_k(S^4, U(n))$ and $\mathcal{G}_k^{SU(n)} = \mathcal{G}_k(S^4, SU(n))$ since no confusion may arise. In subsequent sections we shall revert to the notation introduced previously as we shall have need to differentiate between the gauge groups of bundles over different spaces.

From equation (4.8) we get a model for the classifying spaces,

$$B\mathcal{G}_k^{U(n)} = Map^k(S^4, BU(n)) \tag{4.16}$$

$$B\mathcal{G}_k^{SU(n)} = Map^k(S^4, BSU(n)) \tag{4.17}$$

which leads to the evaluation fibration sequences of equation (4.9)

$$\ldots \to U(n) \xrightarrow{\delta_k^U} B\mathcal{G}_{*k}^{U(n)} \xrightarrow{i^U} B\mathcal{G}_k^{U(n)} \xrightarrow{e^U} BU(n) \tag{4.18}$$

$$\ldots \to SU(n) \xrightarrow{\delta_k^{SU}} B\mathcal{G}_{*k}^{SU(n)} \xrightarrow{i^{SU}} B\mathcal{G}_k^{SU(n)} \xrightarrow{e^{SU}} BSU(n), \tag{4.19}$$

where we label the connecting maps δ_k^U, δ_k^{SU}, respectively.

Using (4.8) again we obtain a string of homotopy equivalences for the classifying spaces of the based gauge groups

$$B\mathcal{G}_{*k}^{U(n)} = Map_*^k(S^4, BU(n)) = \Omega_k^4 BU(n) \simeq \Omega_k^3 U(n) \simeq \Omega_0^3 U(n) \tag{4.20}$$

and similarly

$$B\mathcal{G}_{*k}^{SU(n)} \simeq \Omega_0^3 SU(n). \tag{4.21}$$

These spaces appear in the evaluation fibrations (4.18), (4.19) and with respect to these homotopy equivalences we have the following result due to Whitehead and Lang.

Theorem 4.5 (Lang [130], [213]) *The triple adjoint of δ_k^G is the generalised Samelson product*

$$\langle k \cdot \epsilon_3, id_G \rangle : S^3 \wedge G \to G \tag{4.22}$$

where $\epsilon_3 \in \pi_3(G) = \mathbb{Z}$ is a generator. ∎

Note that this theorem applies for $G = U(n)$, $G = SU(n)$ and also for $G = PU(n)$.

4.4 Proof of Theorem 4.1.

We begin by relating the two fibration sequences (4.18), (4.19). In cohomology the map $Bj^* : H^4(BU(n)) \to H^4(BSU(n))$ is epic and takes the second Chern class c_2 isomophically between the two groups. It thus induces a map between the components of the mapping spaces appearing in (4.16). There results a map $\widehat{Bj} : B\mathcal{G}_k^{SU(n)} \to B\mathcal{G}_k^{U(n)}$ which in turn, as $Bj \circ e^U(f) = Bj(f(*)) = e^{SU} \circ \widehat{Bj}(f)$ for $f \in B\mathcal{G}_k^{U(n)}$, gives a map between the evaluation fibrations. Given the homotopy equivalences (4.20), (4.21) it is clear that the induced map of fibres is the homotopy equivalence $\Omega^3 j$.

We remark at this stage that for $n = 1$ the problem is trivial: $U(1) = S^1$ is abelian and $BS^1 = K(\mathbb{Z}, 2)$ is an Eilenberg-Mac Lane space. Over S^4 there is only the trivial $U(1)$-bundle and with regards to its gauge group we have $B\mathcal{G} \simeq Map^0(S^4, K(\mathbb{Z}, 2)) \simeq K(\mathbb{Z}, 2)$ and $B\mathcal{G}_* \simeq Map^0_*(S^4, K(\mathbb{Z}, 2)) \simeq *$ so that $\mathcal{G} = Map(S^4, S^1) \simeq S^1$ and $\mathcal{G}_* \simeq *$. With this simple result out of the way we shall henceforth only consider the values $n \geq 2$, and we tacitly assume this is so in all of the following.

We assemble all this information into a commutative diagram of fibrations

$$\begin{tikzcd}
* \arrow[r] \arrow[d] & S^1 \arrow[r, equal] \arrow[d, "\epsilon"] & S^1 \arrow[d, "*"] \\
\Omega_0^3 SU(n) \arrow[r, "i^{SU}"] \arrow[d, "\Omega^3 j", "\simeq"'] & B\mathcal{G}_k^{SU(n)} \arrow[r, "e^{SU}"] \arrow[d, "\widehat{Bj}"] \arrow[dr, phantom, "(*)"] & BSU(n) \arrow[d, "Bj"] \\
\Omega_0^3 U(n) \arrow[r, "i^U"] \arrow[d] & B\mathcal{G}_k^{U(n)} \arrow[r, "e^U"] \arrow[d] & BU(n) \arrow[d, "Bdet"] \\
* \arrow[r] & K(\mathbb{Z},2) \arrow[r, equal] & K(\mathbb{Z},2).
\end{tikzcd} \tag{4.23}$$

The homotopy equivalence $\Omega^3 j$ serves to identify the square labelled $(*)$ as a homotopy pullback. This in turn identifies the contractible space situated in the top left corner of the diagram, and from this it follows that the fibre of $\widehat{Bj} : B\mathcal{G}_k^{SU(n)} \to B\mathcal{G}_k^{U(n)}$ has the homotopy type of S^1. Recalling that $BS^1 = K(\mathbb{Z}, 2)$, the fact that the fibration sequence down the right hand side of the diagram is principal induces the map $B\mathcal{G}_k^{U(n)} \to K(\mathbb{Z}, 2)$. The triviality of the connecting map $S^1 \to BSU(n)$ on the right hand side of the diagram follows from the connectivity of $BSU(n)$.

Now extend the homotopy pullback diagram (4.23) upwards. Each column becomes a fibration sequence and in particular there is a fibring

$$\ldots \to \mathcal{G}_k^{SU(n)} \xrightarrow{\Omega\widehat{Bj}} \mathcal{G}_k^{U(n)} \to S^1 \xrightarrow{\epsilon} B\mathcal{G}_k^{SU(n)} \to \ldots \tag{4.24}$$

in which $\mathcal{G}_k^{U(n)}$ appears as the homotopy fibre of a map $\epsilon \in \pi_1(B\mathcal{G}_k^{SU(n)})$.

Proof of 4.1.1 For $n \geq 3$ we have $\pi_1(BSU(n)) = 0$ and $\pi_1(\Omega^3 SU(n)) = \pi_4(SU(n)) = 0$. Therefore it follows from the middle row of (4.23) that $\pi_1(B\mathcal{G}_k^{SU(n)}) = 0$ and the map ϵ in (4.24) is null-homotopic. The result now follows from the general theory of principal bundles. ∎

We now focus on the case $n = 2$.

Lemma 4.6 *The following hold:*

$$\pi_1\left(B\mathcal{G}_k^{SU(2)}\right) = \mathbb{Z}_2 \tag{4.25}$$

$$\pi_1\left(B\mathcal{G}_k^{U(2)}\right) = \begin{cases} 0 & k \text{ odd} \\ \mathbb{Z}_2 & k \text{ even} \end{cases} \tag{4.26}$$

$$\pi_2\left(B\mathcal{G}_k^{SU(2)}\right) = \mathbb{Z}_2 \tag{4.27}$$

$$\pi_2\left(B\mathcal{G}_k^{U(2)}\right) = \mathbb{Z} \oplus \mathbb{Z}_2 \tag{4.28}$$

Proof The statements (4.25), (4.27) follow easily from the homotopy sequence of the evaluation fibration (4.19) for $B\mathcal{G}_k^{SU(2)}$. For the other groups we use Lang's Theorem 4.22 to get that the map induced in homotopy by the connecting map $\delta_k^U : U(2) \to \Omega_0^3 U(2)$ is given by the Samelson product

$$\pi_r(U(2)) \to \pi_{r+3}(U(2)), \qquad \alpha \mapsto \langle k\epsilon_3, \alpha\rangle = k\langle \epsilon_3, \alpha\rangle \tag{4.29}$$

where $\epsilon_3 \in \pi_3(U(2)) = \mathbb{Z}$ is a generator. Bott [31] has calculated the value of this product for $r = 1$ and shown that if $\epsilon_1 \in \pi_1(U(2)) = \mathbb{Z}$ is a generator, then $\langle \epsilon_3, \epsilon_1\rangle$ generates $\pi_4(U(2)) = \mathbb{Z}_2$.

In the fibre sequence (4.18) we have $\pi_1(BU(2)) = 0$ from which it follows that

$$\pi_1(B\mathcal{G}_k^{U(2)}) = \mathrm{coker}(\delta_k^U) = \mathrm{coker}\left(\pi_1(U(2)) \xrightarrow{k\langle\epsilon_3,-\rangle} \pi_4(U(2))\right) = \mathbb{Z}_{gcd(2,k)} \tag{4.30}$$

and we get (4.26).

Again in (4.18) we have $\pi_3(BU(2)) = 0$ so we obtain a short exact sequence

$$0 \to \pi_5(U(n)) = \mathbb{Z}_2 \to \pi_2(B\mathcal{G}_k^{U(2)}) \to \ker(\delta_k^U) \to 0. \tag{4.31}$$

By (4.30) $\ker \delta_k^U$ is either $\mathbb{Z}$ or $2\mathbb{Z}$ so this sequence must split to give (4.28). ∎

This lemma implies that there are exactly two isomorphism classes of principal $\mathcal{G}_k^{SU(2)}$-bundles over S^1, one of which is trivial and the other of which is represented by the generator of $\pi_1(B\mathcal{G}_k^{SU(2)}) = \mathbb{Z}_2$. Owing to the fibre sequence (4.24), the gauge group $\mathcal{G}_k^{U(2)}$ must belong to one of these, and to which is decided by the homotopy class of the map ϵ.

Proof of 4.1.2 Apply the functor π_1 to the homotopy pullback diagram (4.23). There are two cases to consider and the information from the previous lemma allows us to fill in the groups.

$$
\begin{tikzcd}
 & & \pi_2(B\mathcal{G}_k^{U(2)}) \arrow[d] & \\
 & 0 \arrow[d] & \pi_1(S^1) = \mathbb{Z} \arrow[d, "\epsilon_*"] & \\
0 \arrow[r] & \mathbb{Z}_2 \arrow[r, "\cong"] \arrow[d, "\cong"] & \pi_1(B\mathcal{G}_k^{SU(2)}) \arrow[r] \arrow[d] & 0 \\
\mathbb{Z} \arrow[r, "\delta_k^U"] & \mathbb{Z}_2 \arrow[r] \arrow[d] & 0 & \\
 & 0 & &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & & \pi_2(B\mathcal{G}_k^{U(2)}) \arrow[d] & \\
 & 0 \arrow[d] & \pi_1(S^1) = \mathbb{Z} \arrow[d, "\epsilon_*"] & \\
0 \arrow[r] & \mathbb{Z}_2 \arrow[r, "\cong"] \arrow[d, "\cong"] & \pi_1(B\mathcal{G}_k^{SU(2)}) \arrow[r] \arrow[d, "\widehat{Bj}_*"] & 0 \\
0 \arrow[r] & \mathbb{Z}_2 \arrow[r, "\cong"] \arrow[d] & \pi_1(B\mathcal{G}_k^{U(2)}) \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 &
\end{tikzcd}
\tag{4.32}
$$

k odd

k even

For k odd the map ϵ_* is an epimorphism onto a non-trivial group so that ϵ itself must be essential. For k even it follows from the exactness of the diagram that $\widehat{Bj}_*$ is an isomorphism and the map $\epsilon = \epsilon_*(1) = 0$ is trivial. The statements of the theorem now follow by arguing as in the proof of Theorem 4.1.1. ∎

There is more to say about the case $n = 2$ and $k = (2l + 1)$. For this we return to some generality and introduce the projective unitary group $PU(n)$. We have the homotopy equivalences $\Omega^k SU(n) \simeq \Omega^k U(n) \simeq \Omega^k PU(n)$ of equation (4.12) and we may use these to follow the same steps that we did for the creation of the diagram (4.23) and build homotopy pullback diagrams

$$
\begin{tikzcd}
* \arrow[r] \arrow[d] & K(\mathbb{Z},2) \arrow[r, equal] \arrow[d] & K(\mathbb{Z},2) \arrow[d] \\
\Omega_0^3 U(n) \arrow[r, "i^U"] \arrow[d, "\simeq"', "\Omega^3\pi"] & B\mathcal{G}_k^{U(n)} \arrow[r, "e^U"] \arrow[d, "\widehat{B\pi}"] \arrow[dr, phantom, "(*)"] & BU(n) \arrow[d, "B\pi"] \\
\Omega_0^3 PU(n) \arrow[r, "i^{PU}"] \arrow[d] & B\mathcal{G}_k^{PU(n)} \arrow[r, "e^{PU}"] \arrow[d, "\eta"] & BPU(n) \arrow[d, "\chi"] \\
* \arrow[r] & K(\mathbb{Z},3) \arrow[r, equal] & K(\mathbb{Z},3)
\end{tikzcd}
\qquad
\begin{tikzcd}
* \arrow[r] \arrow[d] & K(\mathbb{Z}_n,1) \arrow[r, equal] \arrow[d] & K(\mathbb{Z}_n,1) \arrow[d] \\
\Omega_0^3 SU(n) \arrow[r, "i^{SU}"] \arrow[d, "\simeq"', "\Omega^3\rho"] & B\mathcal{G}_k^{SU(n)} \arrow[r, "e^{SU}"] \arrow[d, "\widehat{B\rho}"] \arrow[dr, phantom, "(*)"] & BSU(n) \arrow[d, "B\rho"] \\
\Omega_0^3 PU(n) \arrow[r, "i^{PU}"] \arrow[d] & B\mathcal{G}_k^{PU(n)} \arrow[r, "e^{PU}"] \arrow[d, "\theta"] & BPU(n) \arrow[d, "\omega_2"] \\
* \arrow[r] & K(\mathbb{Z}_n,2) \arrow[r, equal] & K(\mathbb{Z}_n,2).
\end{tikzcd}
\tag{4.33}
$$

The bottom rows of these diagrams exist for the same reason as before and we use them to define the map θ from the class $\omega_2 \in H^2(BPU_n; \mathbb{Z}_n)$ and the map η from the integral class $\chi = \delta\omega_2$.

Fix $n = 2$. Then $SU(2) = Spin(3)$ is the 2-connected cover of $PU(2) = SO(3)$. Kamiyama, Kishimoto, Kono and Tsukuda [111] have examined the gauge groups of $SO(3)$-

bundes over S^4 and calculated enough Samelson products in $SO(3)$ so as to obtain the low-dimensional homotopy groups of $B\mathcal{G}_k^{PU(2)}$.

Lemma 4.7 (Kamiyama, Kishimoto, Kono, Tsukuda [111])

$$\pi_1(B\mathcal{G}_k^{PU(2)}) = \begin{cases} 0 & k \text{ odd} \\ \mathbb{Z}_2 & k \text{ even} \end{cases} \tag{4.34}$$

$$\pi_2(B\mathcal{G}_k^{PU(2)}) = \begin{cases} \mathbb{Z}_2 & k \text{ odd} \\ \mathbb{Z}_4 & k \equiv 2 \text{ mod } 4 \\ \mathbb{Z}_2 \oplus \mathbb{Z}_2 & k \equiv 0 \text{ mod } 4 \end{cases} \quad \blacksquare \tag{4.35}$$

We now have enough information to complete the proof of 4.1.

Proof of 4.1.3 For $k = (2l+1)$, Lemma 4.7 and the Hurewicz Theorem give $H_1(B\mathcal{G}_{2l+1}^{PU(2)}) = 0$ and $H_2(B\mathcal{G}_{2l+1}^{PU(2)}) = \mathbb{Z}_2$. Furthermore, rational homotopy shows us that $H_3(B\mathcal{G}_{2l+1}^{PU(2)})$ is torsion. From universal coefficients it now follows that

$$H^1(B\mathcal{G}_{2l+1}^{PU(2)}) = 0, \qquad H^2(B\mathcal{G}_{2l+1}^{PU(2)}) = 0, \qquad H^3(B\mathcal{G}_{2l+1}^{PU(2)}) = \mathbb{Z}_2. \tag{4.36}$$

Consider then the Serre spectral sequence with integral coefficients for the evaluation fibration of $B\mathcal{G}_{2l+1}^{PU(2)}$. Using the cohomology groups above it follows from connectivity that there is an exact sequence

$$\begin{tikzcd} 0 \arrow[r] & H^2(\Omega_0^3 PU(2)) \arrow[r, "\tau"] \arrow[d, equal] & H^3(BPU(2)) \arrow[r, "e^{PU*}"] \arrow[d, equal] & H^3(B\mathcal{G}_{2l+1}^{PU(2)}) \arrow[r] \arrow[d, equal] & H^3(\Omega_0^3 PU(2)) \arrow[r] & \dots \\ & \mathbb{Z}_2 & \mathbb{Z}_2 & \mathbb{Z}_2 & & \end{tikzcd} \tag{4.37}$$

The transgression τ must be injective in degree 2 by exactness and so must be an isomorphism. The map $e^{PU*} : H^3(BPU(2)) \to H^3(B\mathcal{G}_{2l+1}^{PU(2)})$ is therefore trivial. Returning to the diagrams (4.33) we see that the classifying map $\eta : B\mathcal{G}_{2l+1}^{PU(2)} \to K(\mathbb{Z}, 3)$ factors through the map $\chi : BPU(2) \to K(\mathbb{Z}, 3)$ as

$$\eta = \chi \circ e^{PU} = e^{PU*}(\chi) \tag{4.38}$$

Since e^{PU*} is trivial in degree 3 cohomology, so is the map $\eta = e^{PU*}(\chi)$. Thus the fibration sequence $S^1 \to \mathcal{G}_{2l+1}^{U(2)} \to \mathcal{G}_{2l+1}^{PU(2)}$ is classified by the trivial map $\Omega\eta \simeq *$ and splits. $\blacksquare$

Remark In the case that $k = 2l$ it is not hard to see that there is a nontrivial relationship between the gauge groups $\mathcal{G}_{2l}^{SU(2)}$ and $\mathcal{G}_{2l}^{PU(2)}$, and between $\mathcal{G}_{2l}^{U(2)}$ and $\mathcal{G}_{2l}^{PU(2)}$. The maps η, θ are essential in these cases and the bundle structure is twisted.

Remark The proof of Theorem 4.1.3 actually allows for a slightly stronger conclusion to be drawn. It is shown that $\eta : B\mathcal{G}_k^{PU(2)} \to K(\mathbb{Z}, 3)$ is trivial for odd k. Since this map classifies the principal fibration $\widehat{B\pi} : B\mathcal{G}_k^{U(2)} \to B\mathcal{G}_k^{PU(2)}$ it is possible to use the principal action to construct a homotopy equivalence $B\mathcal{G}_k^{U(2)} \simeq B\mathcal{G}_k^{PU(2)} \times K(\mathbb{Z}, 2)$. Thus the claimed splitting actually happens on the level of classifying spaces.

4.5 Homotopy Types of $U(n)$-Gauge Groups Over $\mathbb{C}P^2$.

Let $E_{(k,l)} \xrightarrow{p_{(k,l)}} \mathbb{C}P^2$ be the principal $U(n)$-bundle over $\mathbb{C}P^2$ with Chern classes $(c_1, c_2) = (k, l) \in H^2(\mathbb{C}P^2) \oplus H^4(\mathbb{C}P^2)$ and let $\mathcal{G}^{(k,l)}(\mathbb{C}P^2, U(n))$ denote the gauge group of this bundle. We assume that $n \geq 2$. From (4.8) there are homotopy equivalences

$$B\mathcal{G}^{(k,l)}(\mathbb{C}P^2, U(n)) \simeq Map^{(k,l)}(\mathbb{C}P^2, BU(n)), \quad B\mathcal{G}_*^{(k,l)}(\mathbb{C}P^2, U(n)) \simeq Map_*^{(k,l)}(\mathbb{C}P^2, BU(n)) \tag{4.39}$$

and these spaces sit in the evaluation fibration

$$\ldots \to \mathcal{G}^{(k,l)}(\mathbb{C}P^2, U(n)) \to U(n) \xrightarrow{\lambda_{(k,l)}} B\mathcal{G}_*^{(k,l)}(\mathbb{C}P^2, U(n)) \to B\mathcal{G}^{(k,l)}(\mathbb{C}P^2, U(n)) \xrightarrow{e} BU(n) \tag{4.40}$$

where $\lambda_{(k,l)} : U(n) \to B\mathcal{G}_*^{(k,l)}(\mathbb{C}P^2, U(n))$ denotes the fibration connecting map.

Now the cellular structure of $\mathbb{C}P^2$ gives rise to a cofiber sequence

$$S^3 \xrightarrow{\eta} S^2 \xrightarrow{i} \mathbb{C}P^2 \xrightarrow{q} S^4 \to \ldots \tag{4.41}$$

where η is the Hopf map. This sequence comes furished with a coaction $c : \mathbb{C}P^2 \to \mathbb{C}P^2 \vee S^4$ produced by pinching a sphere out of the top cell. Application of the the functor $Map_*(-, BU(n))$ to (4.41) yields a principal fibring

$$\Omega^4 BU(n) \xrightarrow{q^*} Map_*(\mathbb{C}P^2, BU(n)) \xrightarrow{i^*} \Omega^2 BU(n) \tag{4.42}$$

and the coaction induces a homotopy action

$$\mu : Map_*(\mathbb{C}P^2, BU(n)) \times \Omega^4 BU(n) \to Map_*(\mathbb{C}P^2, BU(n)), \quad (f, \omega_l) \mapsto \nabla \circ (f \vee \omega_l) \circ c. \tag{4.43}$$

In (4.42) we have identified $Map_*(S^r, BU(n)) = \Omega^r BU(n)$ and used the homeomorphism $Map_*(\mathbb{C}P^2 \vee S^4, BU(n)) \cong Map_*(\mathbb{C}P^2, BU(n)) \times \Omega^4 BU(n)$.

In cohomology the coaction induces the map $c^* : H^*(\mathbb{C}P^2) \oplus H^*(S^4) \to H^*(\mathbb{C}P^2)$ which in degree 4 satisfies $c^*(l \cdot x^2, l' \cdot s_4) = l \cdot x^2 + l' \cdot q^* s_4 = (l + l') \cdot x^2$, where $x \in H^2(\mathbb{C}P^2)$, $s_4 \in H^4(S^4)$ are integral generators. From this it is clear that after taking components there results a map

$$\mu : Map_*^{(k,l)}(\mathbb{C}P^2, BU(n)) \times \Omega^4_{l'} BU(n) \to Map_*^{(k,l+l')}(\mathbb{C}P^2, BU(n)). \tag{4.44}$$

For each nonnegative integer r fix a loop $\omega_r \in \Omega^4_r BU(n)$ and extend this to all integers by using the loop inverse to define $\omega_{-r} = -\omega_r$. Then with $B\mathcal{G}_*^{(k,l)}(\mathbb{C}P^2, BU(n)) \simeq Map_*^{(k,l)}(\mathbb{C}P^2, BU(n))$ there results, for each triple of integers k, l, r, a map $S_r : B\mathcal{G}_*^{(k,l)}(\mathbb{C}P^2, BU(n)) \to B\mathcal{G}_*^{(k,l+r)}(\mathbb{C}P^2, BU(n))$ defined by

$$S_r(f) = \mu(f, \omega_r) : \mathbb{C}P^2 \xrightarrow{c} \mathbb{C}P^2 \vee S^4 \xrightarrow{f \vee \omega_r} BU(n) \vee BU(n) \xrightarrow{\nabla} BU(n). \tag{4.45}$$

That this map is well defined follows from the properties of the coaction [12]. Moreover it is straightforward to show using these properties that S_r is a homotopy equivalence with inverse S_{-r} [132]. Using these maps we get the following.

Proposition 4.8 *For each integer $k \in \mathbb{Z}$ and any $l, l' \in \mathbb{Z}$ there are homotopy equivalences*

$$B\mathcal{G}_*^{(k,l)}(\mathbb{C}P^2, BU(n)) \simeq B\mathcal{G}_*^{(k,l')}(\mathbb{C}P^2, BU(n)). \quad \blacksquare \tag{4.46}$$

This proposition cannot be upgraded to a statement about the classifying space of the full gauge group due to its reliance on the coaction c. We still are able to get the following result, however.

Theorem 4.9 *With respect to the evaluation fibrations, for each pair of integers (k, l) and any integer r, there are fibre homotopy equivalences*

$$B\mathcal{G}^{(k,l)}(\mathbb{C}P^2, U(n)) \simeq B\mathcal{G}^{(k+rn, l+kr(n-1)+1/2n(n-1)r^2)}(\mathbb{C}P^2, U(n)) \tag{4.47}$$

over $BU(n)$. These restrict to homotopy equivalences

$$B\mathcal{G}_*^{(k,l)}(\mathbb{C}P^2, U(n)) \simeq B\mathcal{G}_*^{(k+rn, l+kr(n-1)+1/2n(n-1)r^2)}(\mathbb{C}P^2, U(n)). \tag{4.48}$$

Proof Let $E(\eta) \xrightarrow{\eta} \mathbb{C}P^2$ denote the canonical line bundle with $c_1(\eta) = x \in H^2(\mathbb{C}P^2)$ a generator. Now let E be a $U(n)$-bundle over $\mathbb{C}P^2$, $r \in \mathbb{Z}$ and form the tensor product $E \otimes \eta^r$, where

$$\eta^r = \begin{cases} \otimes^r \eta & r > 0 \\ \epsilon^1 & r = 0 \\ \otimes^{-r} \bar{\eta} & r < 0 \end{cases} \tag{4.49}$$

with ϵ^1 denoting the trivial line bundle and $\bar{\eta}$ denoting the conjugate bundle. Then $E \otimes \eta^r$ is again a $U(n)$-bundle and calculating with the Chern character shows that

$$c_1(E \otimes \eta^r) = c_1(E) + (rn) \cdot x \tag{4.50}$$

$$c_2(E \otimes \eta^r) = c_2(E) + r(n-1) \cdot c_1(E) \cup x + \frac{1}{2} r^2 n(n-1) \cdot x^2. \tag{4.51}$$

We use the tensor product with η^r as above to define a map

$$T_r = ((-) \otimes \eta^r) : Map^{(k,l)}(\mathbb{C}P^2, BU(n)) \to Map^{(k+rn, l+kr(n-1)+1/2n(n-1)r^2)}(\mathbb{C}P^2, BU(n)) \tag{4.52}$$

as follows. Let $f : \mathbb{C}P^2 \to BU(n)$ represent the $U(n)$-bundle E. Then $E \otimes \eta^r$ may be represented by the composite

$$T_r(f) = T_r(E) : \mathbb{C}P^2 \xrightarrow{\Delta} \mathbb{C}P^2 \times \mathbb{C}P^2 \xrightarrow{f \times \eta^r} BU(n) \times BU(1) \xrightarrow{m_n} BU(n) \tag{4.53}$$

where m_n is the H-action of $BU(1)$ on $BU(n)$ induced by the tensor product and η^r here is the composite $\mathbb{C}P^2 \xrightarrow{\Delta} \Pi^r \mathbb{C}P^2 \xrightarrow{\Pi^r \eta} \Pi^r BU(1) \xrightarrow{m_1 \circ (1 \times m_1) \circ \cdots \circ (1 \times \cdots \times 1 \times m_1)} BU(1)$ (with η replaced by $\bar{\eta}$ if $r < 0$, and $\eta^0 = *$ is defined to be the constant map).

If $f \in Map^{(k,l)}(\mathbb{C}P^2, BU(n))$, then it is seen from (4.50) that $T_r(f)$ lies is the correct component and everything is well-defined. We take (4.53) as the definition of T_r.

We may now use elementary properties of the tensor product to determine some of the features of T_r. We abuse notation by blurring the distinction between a bundle and a representing map for it. It follows from the associativity of the tensor product that

$$T_{-r} \circ T_r(E) = (E \otimes \eta^r) \otimes \eta^{-r} \cong E \otimes (\eta^r \otimes \eta^{-r}) \cong E \otimes \epsilon^1 \cong E \tag{4.54}$$

which demonstates that T_r has a homotopy inverse and is a homotopy equivalence.

Since we may choose the map classifying η to be basepoint preserving and we may assume that the action m_n is strictly unital in the sense that $m_n(x,*) = x$ for all $x \in BU(n)$ and $m_n(*, z) = B\Delta(z)$ for all $z \in BU(1)$, we see the compatibility with the evaluation fibration,

$$e \circ T_r(f) = m_n \circ (f \times (r\eta)) \circ \Delta(*) = m_n(f(*), *) = f(*) = e(f). \tag{4.55}$$

Hence by [65] the map T_r is a fibre homotopy equivalence and induces a homotopy equivalence of fibres $B\mathcal{G}_*^{(k,l)}(\mathbb{C}P^2, U(n)) \simeq B\mathcal{G}_*^{(k+rn,l+kr(n-1)+1/2n(n-1)r^2)}(\mathbb{C}P^2, U(n))$. ∎

Before moving on we record one further lemma which will be of use in our applications. In the literature, for simply connected G, a standard technique is to take components in the fibration (4.42) and attempt to factor the connecting map δ through the fibre inclusion $q^* : \Omega_0^3 G \to Map_*(\mathbb{C}P^2, BG)$. Here the problem is different. Once components are taken it is not even clear what the homotopy fibre of i^* actually is. What allows for progress to be made is the action μ; its presence makes the map i^* into what Zabrodsky [222] calls a weakly principal fibration.

Let $i^* : Map_*^{(k,l)}(\mathbb{C}P^2, BU(n)) \to \Omega_k^2 BU(n)$ be the map induced by $i : S^2 \hookrightarrow \mathbb{C}P^2$ and let $X(k,l;n)$ denote the homotopy fibre of this map. For any space X the map μ produces an operation of the homotopy set $[X, \Omega_0^3 U(n)]$ on $[X, Map_*^{(k,l)}(\mathbb{C}P^2, BU(n))]$. We denote this action by the symbol $+$ in the following.

Proposition 4.10 *There is a homotopy equivalence*

$$X(k,l;n) \simeq \Omega_0^4 BU(n) \simeq \Omega_0^3 SU(n). \tag{4.56}$$

Moreover, in the resulting homotopy fibration sequence

$$\Omega_0^4 BU(n) \xrightarrow{j} Map_*^{(k,l)}(\mathbb{C}P^2, BU(n)) \xrightarrow{i^*} \Omega_k^2 BU(n) \tag{4.57}$$

we may identify the fibre inclusion j as the map given by

$$j(\omega) = f_{(k,l)} + \omega \tag{4.58}$$

where $f_{(k,l)} \in Map^{(k,l)}(\mathbb{C}P^2, BU(n))$ is a chosen basepoint map.

Proof Consider the following homotopy commutative diagram in which each row and column is a cofiber sequence, ν is the suspension coproduct and the space C_c is the mapping cone of the coaction c

$$\begin{tikzcd} S^2 \arrow[r, equal] \arrow[d, "i"] & S^2 \arrow[r] \arrow[d, "in_1 \circ i"] & * \arrow[d] \\ \mathbb{C}P^2 \arrow[r, "c"] \arrow[d, "q"] \arrow[dr, phantom, "\text{(hpo)}"] & \mathbb{C}P^2 \vee S^4 \arrow[r, "r"] \arrow[d, "q \vee 1"] & C_c \arrow[d, "\gamma"] \\ S^4 \arrow[r, "\nu"] & S^4 \vee S^4 \arrow[r, "(1,-1)"] & S^4. \end{tikzcd} \tag{4.59}$$

The fact that the diagram homotopy commutes follows from the properties of the coaction and so the square labelled (hpo) is a homotopy pushout by construction. From this one infers that the induced map of cofibers $\gamma : C_c \xrightarrow{\simeq} S^4$ is a homotopy equivalence. Moreover, this identification makes it possible to identify the map r as $r = (q, -1) : \mathbb{C}P^2 \vee S^4 \to S^4$.

Now with the identifications of $C_c \simeq S^4$ and $r = (q, -1)$ in place, apply $Map_*(-, BU(n))$ to the diagram and use (4.44) to replace the map induced by the coaction c with the action μ. The result of this is the following diagram in which each column and row is a fibration sequence, and the square labelled (hpb) is a homotopy pullback

$$\begin{tikzcd}
* \arrow[r] \arrow[d] & \Omega^4 BU(n) \arrow[r, equal] \arrow[d, "in_2"] & \Omega^4 BU(n) \arrow[d, "q^*"] \\
\Omega^4 BU(n) \arrow[r, "(q^*,-1)"] \arrow[d, equal] & Map_*(\mathbb{C}P^2, BU(n)) \times \Omega^4 BU(n) \arrow[r, "\mu"] \arrow[d, "pr_1"] \arrow[dr, phantom, "\text{(hpb)}"] & Map_*(\mathbb{C}P^2, BU(n)) \arrow[d, "i^*"] \\
\Omega^4 BU(n) \arrow[r, "q^*"] & Map_*(\mathbb{C}P^2, BU(n)) \arrow[r, "i^*"] & \Omega^2 BU(n).
\end{tikzcd} \tag{4.60}$$

We wish to take components of the mapping spaces appearing in this diagram. The process of doing so, however, will make it difficult to identify the fibres and fibre inclusions. Let us reduce the problem first by forming the homotopy pullback $E(i^*, i^*)$ of i^* with itself and examining the induced map $\theta : Map_*^{(k,0)}(\mathbb{C}P^2, BU(n)) \times \Omega^4_l BU(n) \to E(i^*, i^*)$ in the diagram

$$\begin{tikzcd}
Map_*^{(k,0)}(\mathbb{C}P^2, BU(n)) \times \Omega^4_l BU(n) \arrow[dr, dashed, "\theta"] \arrow[drr, bend left, "\mu"] \arrow[ddr, bend right, "pr_1"'] & & \\
& E(i^*, i^*) \arrow[r, "t"] \arrow[d, "s"] \arrow[dr, phantom, "\text{(hpb)}"] & Map_*^{(k,l)}(\mathbb{C}P^2, BU(n)) \arrow[d, "i^*"] \\
& Map_*^{(k,0)}(\mathbb{C}P^2, BU(n)) \arrow[r, "i^*"] & \Omega^2_k BU(n).
\end{tikzcd} \tag{4.61}$$

We claim that θ may be chosen to be a weak equivalence. Indeed, consider the cubical diagram

$$\begin{tikzcd}
E(i^*, i^*) \arrow[r, "t"] \arrow[d, "s"] \arrow[ddrr, dashed, "\phi"] & Map_*^{(k,l)}(\mathbb{C}P^2, BU(n)) \arrow[d, "i^*"] \arrow[ddrr, hook] & & \\
Map_*^{(k,0)}(\mathbb{C}P^2, BU(n)) \arrow[r, "i^*"] \arrow[ddrr, hook] & \Omega^2_k BU(n) \arrow[ddrr, hook] & & \\
& & Map_*(\mathbb{C}P^2, BU(n)) \times \Omega^4 BU(n) \arrow[r, "\mu"] \arrow[d, "pr_1"] & Map_*(\mathbb{C}P^2, BU(n)) \arrow[d, "i^*"] \\
& & Map_*(\mathbb{C}P^2, BU(n)) \arrow[r, "i^*"] & \Omega^2 BU(n).
\end{tikzcd} \tag{4.62}$$

The front square is the homotopy pullback square in the diagram (4.60). The hooked diagonal maps are the inclusions of the path components and $\phi : E(i^*, i^*) \to Map_*(\mathbb{C}P^2, BU(n)) \times \Omega^4 BU(n)$ is a map induced by the homotopy pullback square.

Now, since $pr_1 \circ \phi \simeq in \circ s : E(i^*, i^*) \to Map_*^{(k,0)}(\mathbb{C}P^2, BU(n)) \hookrightarrow Map_*(\mathbb{C}P^2, BU(n))$ it must be that ϕ factors

$$\phi : E(i^*, i^*) \to Map_*^{(k,0)}(\mathbb{C}P^2, BU(n)) \times \Omega^2 BU(n) \hookrightarrow Map_*(\mathbb{C}P^2, BU(n)) \times \Omega^4 BU(n). \tag{4.63}$$

Similarly, we argue that since $\mu \circ \phi \simeq in \circ t : E(i^*, i^*) \to Map_*^{(k,l)}(\mathbb{C}P^2, BU(n)) \hookrightarrow Map_*(\mathbb{C}P^2, BU(n))$ it must be that ϕ factors

$$\phi : E(i^*, i^*) \xrightarrow{\phi'} Map_*^{(k,0)}(\mathbb{C}P^2, BU(n)) \times \Omega^2_l BU(n) \xhookrightarrow{in} Map_*(\mathbb{C}P^2, BU(n)) \times \Omega^4 BU(n) \tag{4.64}$$

for some map ϕ'.

Now let us bring back the map $\theta : Map_*^{(k,0)}(\mathbb{C}P^2, BU(n)) \times \Omega^4_l BU(n) \to E(i^*, i^*)$. The composite $\phi \circ \theta$ may be chosen up to homotopy to be the inclusion $in : Map_*^{(k,0)}(\mathbb{C}P^2, BU(n)) \times \Omega^4_l BU(n) \hookrightarrow Map_*(\mathbb{C}P^2, BU(n)) \times \Omega^4 BU(n)$. Assume choices are made for θ and ϕ such that this is accomplished. Now factor ϕ through ϕ' to obtain a self-map $\psi = \phi' \circ \theta : Map_*^{(k,0)}(\mathbb{C}P^2, BU(n)) \times \Omega^4_l BU(n) \to Map_*^{(k,0)}(\mathbb{C}P^2, BU(n)) \times \Omega^4_l BU(n)$ satisfying

$$in \circ \psi \simeq in : Map_*^{(k,0)}(\mathbb{C}P^2, BU(n)) \times \Omega^4_l BU(n) \to Map_*(\mathbb{C}P^2, BU(n)) \times \Omega^4 BU(n). \tag{4.65}$$

Using this property is it easy to see that θ and ϕ' are mutually inverse homotopy equivalences. In short, we have shown that the following diagram is a homotopy pullback

$$\begin{tikzcd} Map_*^{(k,0)}(\mathbb{C}P^2, BU(n)) \times \Omega^4_l BU(n) \arrow[r, "\mu"] \arrow[d, "pr_1"] \arrow[dr, phantom, "\text{(hpb)}"] & Map_*^{(k,l)}(\mathbb{C}P^2, BU(n)) \arrow[d, "i^*"] \\ Map_*^{(k,0)}(\mathbb{C}P^2, BU(n)) \arrow[r, "i^*"] & \Omega^2_k BU(n). \end{tikzcd} \tag{4.66}$$

Finally we come to take fibres of the maps in this square. It is easy to identify the homotopy fibre of the projection pr_1 on the left-hand side of the square as well as its fibre inclusion. It is also not difficult to identify the homotopy fibre of μ along the top of the diagram and its fibre inclusion. Extending (4.66) with this information yields

$$\begin{tikzcd} * \arrow[r] \arrow[d] & \Omega^4_l BU(n) \arrow[r, "\simeq", "e"'] \arrow[d, "v"] & X(k,l;n) \arrow[d, "j'"] \\ \Omega^4_l BU(n) \arrow[r, "u"] \arrow[d, "\simeq"] & Map_*^{(k,0)}(\mathbb{C}P^2, BU(n)) \times \Omega^4_l BU(n) \arrow[r, "\mu"] \arrow[d, "pr_1"] \arrow[dr, phantom, "\text{(hpb)}"] & Map_*^{(k,l)}(\mathbb{C}P^2, BU(n)) \arrow[d, "i^*"] \\ X(k,0;n) \arrow[r] & Map_*^{(k,0)}(\mathbb{C}P^2, BU(n)) \arrow[r, "i^*"] & \Omega^2_k BU(n) \end{tikzcd} \tag{4.67}$$

where $u : \Omega^4_l BU(n) \to Map_*^{(k,0)}(\mathbb{C}P^2, BU(n)) \times \Omega^4_l BU(n)$ is defined by $u(\omega) = (f_{(k,l)} - \omega, \omega)$ for a chosen basepoint map $f_{(k,l)} \in Map_*^{(k,l)}(\mathbb{C}P^2, BU(n))$, $v : \Omega^4_l BU(n) \to Map_*^{(k,0)}(\mathbb{C}P^2, BU(n)) \times$

$\Omega^4_l BU(n)$ is defined by $v(\omega) = (f_{(k,l)} - \omega_0, \omega)$ for a chosen basepoint loop $\omega_0 \in \Omega^4_l BU(n)$ and e is defined by the diagram.

The homotopy equivalences between the fibres come from the bottom-right hand square being a homotopy pullback. This gives us the homotopy type of $X(k, l; n)$. Take the homotopy equivalence on the top row as an identification $e : \Omega^4_l BU(n) \simeq X(k, l)$ to give a homotopy fibration sequence

$$\Omega^4_l BU(n) \xrightarrow{j''} Map^{(k,l)}_*(\mathbb{C}P^2, BU(n)) \xrightarrow{i^*} \Omega^2 BU(n) \tag{4.68}$$

where $j'' \simeq j' \circ e \simeq \mu \circ v$ is given by

$$j''(\omega) = \mu \circ v(\omega) = \mu(f_{(k,l)} - \omega_0, \omega) = f_{(k,l)} + (\omega - \omega_0). \tag{4.69}$$

Finally we choose the homotopy equivalence $\Omega^4_0 BU(n) \simeq \Omega^4_l BU(n)$ to be given by $\omega \mapsto \omega + \omega_0$ and define j in (4.57) to be the composite of j'' with this map. We get

$$j(\omega) = (f_{(k,l)} + \omega + \omega_0) - \omega_0 = f_{(k,l)} + \omega. \quad \blacksquare \tag{4.70}$$

Before moving on we comment on the homotopy types of $U(1)$-gauge groups over $\mathbb{C}P^2$. Since $BU(1) = K(\mathbb{Z}, 2)$ we have $[\mathbb{C}P^2, BU(1)] = H^2\mathbb{C}P^2 = \mathbb{Z}$ so $U(1)$-bundle isomorphism classes are in correspondence with the integers. Using a theorem of Thom [196], however, we get from equation (4.8) that $B\mathcal{G}^k(\mathbb{C}P^2, U(1)) \simeq Map^k(\mathbb{C}P^2, BU(1)) \simeq BU(1)$ and $B\mathcal{G}^k_*(\mathbb{C}P^2, U(1)) \simeq Map^k_*(\mathbb{C}P^2, BU(1)) \simeq *$ so there is a unique homotopy type amongst the classifying spaces of these gauge groups. It follows that $\mathcal{G}^k(\mathbb{C}P^2, U(1)) \simeq S^1$ and $\mathcal{G}^k(\mathbb{C}P^2, U(1)) \simeq *$. For the remainder of the paper we will always assume that $n \geq 2$.

4.5.1 A Decomposition of the Connecting Map $\lambda_{(k,l)}$.

For each pair of integers (k, l) we have the evaluation fibration (4.40). There is also the evaluation fibration for the gauge group of a $U(n)$-bundle over S^4 with second Chern class $l \in H^4(S^4) = \mathbb{Z}$

$$\ldots \to \mathcal{G}^l(S^4, U(n)) \to U(n) \xrightarrow{\delta_l} \Omega^3_0 U(n) \to B\mathcal{G}^l(S^4, U(n)) \xrightarrow{e} BU(n). \tag{4.71}$$

The connecting map $\delta_l : U(n) \to \Omega^3_0 U(n)$ of this sequence is more amenable to study than $\lambda_{(k,l)}$ since by Theorem 4.5 it is adjoint to the Samelson product $\langle \epsilon_3, id_{U(n)} \rangle : S^3 \wedge U(n) \to U(n)$. The purpose of this section is to relate the two maps $\lambda_{(k,l)}$, δ_l in such a way as to allow more easily obtained information about δ_l to be passed to $\lambda_{(k,l)}$. For the following recall the action $\mu : Map^{(k,l)}_*(\mathbb{C}P^2, BU(n)) \times \Omega^4_{l'} BU(n) \to Map^{(k,l+l')}_*(\mathbb{C}P^2, BU(n))$ and the notation $+$ for the induced operation of the homotopy set $[X, \Omega^3_0 U(n)]$ on $[X, Map^{(k,l)}_*(\mathbb{C}P^2, BU(n))]$.

Theorem 4.11 *In the homotopy set $[U(n), Map^{(k,l)}_*(\mathbb{C}P^2, BU(n))]$ there is equality*

$$\lambda_{(k,l)} = \lambda_{(k,0)} + \delta_l. \tag{4.72}$$

Proof Since $BU(n)$ is connected the components of the unbased mapping space $Map(\mathbb{C}P^2 \vee S^4, BU(n))$ are in one-to-one correspondence with those of the based mapping space $Map_*(\mathbb{C}P^2 \vee S^4, BU(n))$. Thus we may consider the map of evaluation fibrations induced by c in the diagram

$$\begin{tikzcd} U(n) \arrow[r, "\Delta_{(k,0;l)}"] \arrow[d, equal] & Map_*^{(k,0;l)}(\mathbb{C}P^2 \vee S^4, BU(n)) \arrow[r] \arrow[d, "c^*"] & Map^{(k,0;l)}(\mathbb{C}P^2 \vee S^4, BU(n)) \arrow[r, "e"] \arrow[d, "c^*"] & BU(n) \arrow[d, equal] \\ U(n) \arrow[r, "\lambda_{(k,l)}"] & Map_*^{(k,l)}(\mathbb{C}P^2, BU(n)) \arrow[r] & Map^{(k,l)}(\mathbb{C}P^2, BU(n)) \arrow[r, "e"] & BU(n) \end{tikzcd} \tag{4.73}$$

which we use to define the fibration connecting map $\Delta_{(k,0;l)}$ belonging to the top row. Here we label the components of the mapping space on the top row using the triple of integers $(k, 0; l)$ according to $Map_*^{(k,0;l)}(\mathbb{C}P^2 \vee S^4, BU(n)) \cong Map_*^{(k,0)}(\mathbb{C}P^2, BU(n)) \times \Omega_l^4 BU(n)$.

We wish to identify the connecting map $\Delta_{(k,0;l)}$ under the homeomorphism

$$\theta = (in_0^*, in_1^*) \circ \Delta : Map_*(\mathbb{C}P^2 \vee S^4, BU(n)) \cong Map_*(\mathbb{C}P^2, BU(n)) \times \Omega^4 BU(n) \tag{4.74}$$

where $in_0 : \mathbb{C}P^2 \to \mathbb{C}P^2 \vee S^4$ and $in_1 : S^4 \to \mathbb{C}P^2 \vee S^4$ are the wedge inclusions. These inclusions induce natural maps $in_0^* : Map^{(k,0;l)}(\mathbb{C}P^2 \vee S^4, BU(n)) \to Map^{(k,0)}(\mathbb{C}P^2, BU(n))$ and $in_1^* : Map^{(k,0;l)}(\mathbb{C}P^2 \vee S^4, BU(n)) \to Map^l(S^4, BU(n))$ which are compatible with all three evaluation fibrations and produce factorisations

$$\begin{tikzcd} U(n) \arrow[r, "\Delta_{(k,0;l)}"] \arrow[d, equal] & Map^{(k,0;l)}(\mathbb{C}P^2 \vee S^4, BU(n)) \arrow[d, "in_0^*"] \\ U(n) \arrow[r, "\lambda_{(k,0)}"] & Map^{(k,0)}(\mathbb{C}P^2, BU(n)) \end{tikzcd} \qquad \begin{tikzcd} U(n) \arrow[r, "\Delta_{(k,0;l)}"] \arrow[d, equal] & Map^{(k,0;l)}(\mathbb{C}P^2 \vee S^4, BU(n)) \arrow[d, "in_0^*"] \\ U(n) \arrow[r, "\delta_l"] & \Omega_l^4 BU(n). \end{tikzcd} \tag{4.75}$$

Since the homeomorphism θ sends $f \in Map^{(k,0;l)}(\mathbb{C}P^2, BU(n))$ to the pair $(f \circ in_0, f \circ in_1) \in Map_*^{(k,0)}(\mathbb{C}P^2, BU(n)) \times \Omega_l^4 BU(n)$ the above factorisations show that for $x \in U(n)$ we have $\theta\left(\Delta_{(k,0;l)}(x)\right) = (\Delta_{(k,0;l)}(x) \circ in_0, \Delta_{(k,0;l)}(x) \circ in_1) = (\lambda_{(k,0)}(x), \delta_l(x))$.

Now let us return to the diagram (4.73) and use the homeomorphism θ to alter the left-most square which will now appear as

$$\begin{tikzcd} U(n) \arrow[r, "(\lambda_{(k,0)}{,}\delta_l)"] \arrow[d, equal] & Map_*^{(k,0)}(\mathbb{C}P^2, BU(n)) \times \Omega_l BU(n) \arrow[d, "\mu"] \\ U(n) \arrow[r, "\lambda_{(k,l)}"] & Map_*^{(k,l)}(\mathbb{C}P^2, BU(n)). \end{tikzcd} \tag{4.76}$$

To identify the action μ appearing here we simply note that the inverse to θ is the map

$$\theta^{-1} = \nabla \circ (- \vee -) : Map_*(\mathbb{C}P^2, BU(n)) \times \Omega^4 BU(n) \cong Map_*(\mathbb{C}P^2 \vee S^4, BU(n)) \tag{4.77}$$

which sends the pair $(f, g) \in Map_*^{(k,0)}(\mathbb{C}P^2, BU(n)) \times \Omega_l^4 BU(n)$ to the composite $\nabla \circ (f \vee g) : \mathbb{C}P^2 \vee S^4 \to BU(n)$. Thus $c^*\left(\theta^{-1}(f, g)\right) = c^*(\nabla \circ (f \vee g)) = \nabla \circ (f \vee g) \circ c = \mu(f, g) = f + g$. Combining (4.76) and (4.77) gives us the claimed equality of homotopy classes

$$\lambda_{(k,l)} = \mu(\lambda_{(k,0)}, \delta_l) = \lambda_{(k,0)} + \delta_l. \quad \blacksquare \tag{4.78}$$

Now we have noted that the connecting map $\delta_l : U(n) \to \Omega_0^3 U(n)$ is adjoint to a Samelson product in $U(n)$. It follows from the linearity of the Samelson product that $\delta_l \simeq l \cdot \delta_1$. It is also true [103] that δ_1 is rationally trivial and has finite order in the group $[U(n), \Omega_0^3 U(n)]$. Denote the order of δ_1 by $|\delta_1|$. Then for each fixed integer k, the following corollary gives an upper bound on the number of homotopy types amongst the gauge groups $\mathcal{G}^{(k,l)}(\mathbb{C}P^2, U(n))$.

Corollary 4.12 *For each pair of integers (k,l) there is a homotopy equivalence*

$$\mathcal{G}^{(k,l)}(\mathbb{C}P^2, U(n)) \simeq \mathcal{G}^{(k,l+|\delta_1|)}(\mathbb{C}P^2, U(n)). \tag{4.79}$$

Proof Recall that $\mathcal{G}^{(k,l)}(\mathbb{C}P^2, U(n))$ is the homotopy fibre of the connecting map $\lambda_{(k,l)}$, which, according to Theorem 4.11, has a decomposition $\lambda_{(k,l)} = \lambda_{(k,0)} + \delta_l$. We may further decompose this map by focusing on the map $\delta_l : U(n) \to \Omega_0^3 U(n)$. Consider the degree l map $l : S^4 \to S^4$. This map induces a map of evaluation fibrations

$$\begin{tikzcd} U(n) \arrow[r, "\delta_1"] \arrow[d, equal] & \Omega_1^3 U(n) \arrow[r] \arrow[d, "l^*"] & Map^1(S^4, BU(n)) \arrow[r, "e"] \arrow[d, "l^*"] & BU(n) \arrow[d, equal] \\ U(n) \arrow[r, "\delta_l"] & \Omega_l^3 U(n) \arrow[r] & Map^l(S^4, BU(n)) \arrow[r, "e"] & BU(n). \end{tikzcd} \tag{4.80}$$

The adjoint of the left-most square of this ladder is the diagram

$$\begin{tikzcd} S^3 \wedge U(n) \arrow[r, "\widetilde{\delta_1}"] & U(n) \arrow[d, equal] \\ S^3 \wedge U(n) \arrow[u, "l\wedge 1"] \arrow[r, "\widetilde{\delta_l}"] & U(n) \end{tikzcd} \tag{4.81}$$

which displays a factorisation of the adjoint map $\widetilde{\delta_l} = \widetilde{\delta_1} \circ (l \wedge 1) = l \cdot \widetilde{\delta_1}$. Adjointing back this equality becomes

$$\delta_l = l \cdot \delta_1 \in [U(n), \Omega_0^3 U(n)]. \tag{4.82}$$

Now the coaction $c : \mathbb{C}P^2 \to \mathbb{C}P^2 \vee S^4$ satisfies $(c \vee 1) \circ c = (1 \vee \nu) \circ c$, where $\nu : S^4 \to S^4 \vee S^4$ is the coproduct. This means that for any space X and maps $f : \mathbb{C}P^2 \to X$, $\alpha, \beta : S^4 \to X$ we have in our notation $f + (\alpha + \beta) = (f + \alpha) + \beta$. Applying this to the case at hand gives us the decomposition

$$\lambda_{(k,l)} = \lambda_{(k,0)} + \delta_l = \lambda_{(k,0)} + l \cdot \delta_1. \tag{4.83}$$

Thus if $|\delta_1|$ is the order of $|\delta_1|$ then

$$\lambda_{(k,l+|\delta_1|)} = \lambda_{(k,0)} + (l + |\delta_1|)\delta_1 = (\lambda_{(k,0)} + l \cdot \delta_1) + |\delta_1| \cdot \delta_1 = \lambda_{(k,0)} + l \cdot \delta_1 = \lambda_{(k,l)}. \tag{4.84}$$

Thus there is a homotopy $\lambda_{(k,l)} \simeq \lambda_{(k,l+|\delta_1|)}$ and since the gauge groups $\mathcal{G}^{(k,l)}(\mathbb{C}P^2, U(2))$ and $\mathcal{G}^{(k,l+|\delta_1|)}(\mathbb{C}P^2, U(2))$ are the homotopy fibres of $\lambda_{(k,l)}$ and $\lambda_{(k,l+||\delta_1|)}$, respectively, they share a common homotopy type. ∎

4.5.2 p-Local Decompositions of $\mathcal{G}^{(k,l)}(\mathbb{C}P^2, U(n))$.

Given the close relationship between the groups $U(n)$, $SU(n)$ and $PU(n)$ displayed in the fibration diagram (4.11), it is not surprising that the group $U(n)$ has particularly nice properties when localised at certain primes. The purpose of this section is to formalise this statement and examine how much of this behaviour is transferred to the $U(n)$-gauge groups over $\mathbb{C}P^2$.

Lemma 4.13 *For each pair of integers $k, l \in \mathbb{Z}$ there is a homotopy equivalence*

$$Map_*^{(k,l)}(\mathbb{C}P^2, BU(n)) \simeq Map_*^{(k,l)}(\mathbb{C}P^2, BPU(n)). \tag{4.85}$$

Proof Start with the fibration sequence $S^1 \xrightarrow{\Delta} U(n) \xrightarrow{\pi} PU(n)$, and the induced fibring of classifying spaces

$$\ldots \to BS^1 \xrightarrow{B\Delta} BU(n) \xrightarrow{B\pi} BPU(n) \xrightarrow{\chi} K(\mathbb{Z}, 3) \tag{4.86}$$

where χ is the map defined in diagram (4.33). Now consider the fibre sequence of functors $\Omega^4(-) \xrightarrow{q^*} Map_*(\mathbb{C}P^2, -) \xrightarrow{i^*} \Omega^2(-)$ and its product with the fibration (4.86). The result of this is the following homotopy commutative diagram in which each row and column is a fibre sequence.

$$\begin{tikzcd}
* \arrow[r] \arrow[d] & Map_*(\mathbb{C}P^2, BS^1) \arrow[r, "i^*", "\simeq"'] \arrow[d, "\widehat{B\Delta}"] & \Omega^2 BS^1 \simeq \mathbb{Z} \arrow[r] \arrow[d, "\Omega^2 B\Delta"] & * \arrow[d] \\
\Omega^4 BU(n) \arrow[r, "q^*"] \arrow[d, "\Omega^4 B\pi", "\simeq"'] & Map_*(\mathbb{C}P^2, BU(n)) \arrow[r, "i^*"] \arrow[d, "\widehat{B\pi}"] \arrow[dr, phantom, "(*)"] & \Omega^2 BU(n) \arrow[r] \arrow[d, "\Omega^2 B\pi"] & \Omega^3 BU(n) \arrow[d, "\Omega^3 B\pi", "\simeq"'] \\
\Omega^4 PU(n) \arrow[r, "q^*"] \arrow[d] & Map_*(\mathbb{C}P^2, BPU(n)) \arrow[r, "i^*"] \arrow[d, "\widehat{\chi}\simeq *"] & \Omega^2 BPU(n) \arrow[r] \arrow[d, "\Omega^2\chi\simeq *"] & \Omega^3 BPU(n) \arrow[d, "\Omega^3\chi\simeq *"] \\
* \arrow[r] & Map_*(\mathbb{C}P^2, K(\mathbb{Z},3)) \arrow[r, "i^*", "\simeq"'] & \Omega^2 K(\mathbb{Z},3) \simeq S^1 \arrow[r] & \mathbb{Z}
\end{tikzcd} \tag{4.87}$$

In the third column the map $\Omega^2\chi : \Omega PU(n) \to S^1$ is trivial as it factors through a map $\mathbb{Z}_n \to S^1$, something that may be seen by considering diagram (4.11). A consequence of this is that $\Omega^2 B\pi$ admits a section $s : \Omega^2 BPU(n) \to \Omega^2 BU(n)$, from which we then deduce that $\Omega^3 B\pi$ and $\Omega^4 B\pi$ are homotopy equivalences. By an application of Thom's theorem [196] we find that the top-most and bottom-most maps i^* are also homotopy equivalences and we conclude that the square labelled $(*)$ is a homotopy pullback.

It follows that the fibration $\widehat{B\pi}$ in the second column is the pullback of a trivial principal fibration and that its classifying map $\widehat{\chi} \simeq *$ is also trivial. There results a section $\widehat{s} : Map_*(\mathbb{C}P^2, BPU(n)) \to Map_*(\mathbb{C}P^2, BU(n))$ of $\widehat{B\pi}$ and using the induced principal action of $Map_*(\mathbb{C}P^2, BS^1) \simeq \mathbb{Z}$ on $Map_*(\mathbb{C}P^2, BU(n))$ we add may add the maps $\widehat{s}$ and $\widehat{B\Delta}$ to get a homotopy equivalence

$$Map_*(\mathbb{C}P^2, BPU(n)) \times \mathbb{Z} \simeq Map_*(\mathbb{C}P^2, BU(n)). \tag{4.88}$$

To complete the proof we need only take components on each side of this equation. ∎

An easy consequence of this theorem is the following very useful homotopy pullback.

Lemma 4.14 *The labelled square appearing in the following homotopy commutative diagram is a homotopy pullback.*

$$\begin{tikzcd}
{*} \arrow[r] \arrow[d] & BS^1 \arrow[r, equal] \arrow[d, "\Omega\xi"] & BS^1 \arrow[d, "B\Delta"] \\
B\mathcal{G}_*^{(k,l)}(\mathbb{C}P^2, U(n)) \arrow[r] \arrow[d, "\simeq"', "\widehat{B\pi}|"] & B\mathcal{G}^{(k,l)}(\mathbb{C}P^2, U(n)) \arrow[r, "e^U"] \arrow[d, "\widehat{B\pi}"] \arrow[dr, phantom, "(hpb)"] & BU(n) \arrow[d, "B\pi"] \\
B\mathcal{G}_*^{(k,l)}(\mathbb{C}P^2, PU(n)) \arrow[r] \arrow[d] & B\mathcal{G}^{(k,l)}(\mathbb{C}P^2, PU(n)) \arrow[r, "e^{PU}"] \arrow[d, "\xi"] & BPU(n) \arrow[d, "\chi"] \\
{*} \arrow[r] & K(\mathbb{Z}, 3) \arrow[r, equal] & K(\mathbb{Z}, 3).
\end{tikzcd} \tag{4.89}$$

Proof The map $B\pi : BU(n) \to BPU(n)$ induces the map $\widehat{B\pi} : B\mathcal{G}^{(k,l)}(\mathbb{C}P^2, U(n)) = Map^{(k,l)}(\mathbb{C}P^2, BU(n)) \to Map^{(k,l)}(\mathbb{C}P^2, BPU(n)) = B\mathcal{G}^{(k,l)}(\mathbb{C}P^2, PU(n))$. The restriction of $\widehat{B\pi}$ to the fibre is the homotopy equivalence $\widehat{B\pi}| : Map_*^{(k,l)}(\mathbb{C}P^2, BU(n)) \simeq Map_*^{(k,l)}(\mathbb{C}P^2, BPU(n))$ that was demonstrated in Lemma 4.13. It follows that (hpb) is a homotopy pullback, implying that the homotopy fibre of $\widehat{B\pi}$ is that of $B\pi$, which is the Eilenberg-Mac Lane space BS^1. This displays $\widehat{B\pi}$ as the homotopy pullback of the principal fibration $B\pi$ along e^{PU}. Thus $\widehat{B\pi}$ too is a principal fibration, classified by some map $\xi : B\mathcal{G}^{(k,l)}(\mathbb{C}P^2, PU(n)) \to K(\mathbb{Z}, 3)$ and this completes the construction of (4.89). ∎

The main application of this lemma is the following theorem, which is the main result of this section. It is of interest in its own right, however, and the reader should bear in mind the accidental isomorphism $PU(2) \cong SO(3)$ when we study $U(2)$-gauge groups in a subsequent section. In this case the lemma says the the classifying spaces of the based $U(2)$- and $SO(3)$-gauge groups are of the same homotopy type.

Proposition 4.15 *Let p be a prime not dividing n. Then after localisation at p the following diagram becomes homotopy commutative*

$$\begin{tikzcd}
S^1 \arrow[r] \arrow[d, "\Delta"] & {*} \arrow[r] \arrow[d] & BS^1 \arrow[r, equal] \arrow[d] & BS^1 \arrow[d, "B\Delta"] \\
U(n) \arrow[r, "\lambda_{(k,l)}"] \arrow[d, "q"] & B\mathcal{G}_*^{(k,l)}(\mathbb{C}P^2, U(n)) \arrow[r] \arrow[d, "\simeq"] & B\mathcal{G}^{(k,l)}(\mathbb{C}P^2, U(n)) \arrow[r] \arrow[d, "\widehat{B\pi}"] & BU(n) \arrow[d, "B\pi"] \\
SU(n) \arrow[r, "\delta_{l'}"] & B\mathcal{G}_*^{l'}(\mathbb{C}P^2, SU(n)) \arrow[r] & B\mathcal{G}^{l'}(\mathbb{C}P^2, SU(n)) \arrow[r] & BSU(n)
\end{tikzcd} \tag{4.90}$$

where

$$l' = \begin{cases} 2nl - (n-1)k^2, & n \text{ even} \\ 2nl - \frac{(n-1)}{2}k^2, & n \text{ odd.} \end{cases} \tag{4.91}$$

Proof Let p is a prime not dividing n and localise at p. Then the canonical projection $\rho : SU(n) \to PU(n)$ becomes a homotopy equivalence which we shall use to identify these groups and their classifying spaces. From Lemma 4.13 there is an integral homotopy equivalence $B\mathcal{G}_*^{(k,l)}(\mathbb{C}P^2, U(n)) \simeq B\mathcal{G}_*^{(k,l)}(\mathbb{C}P^2, PU(n))$ and we use the map induced by ρ to identify $B\mathcal{G}^{(k,l)}(\mathbb{C}P^2, PU(n)) = Map^{(k,l)}(\mathbb{C}P^2, BPU(n)) \simeq B\mathcal{G}^{l'}(\mathbb{C}P^2, SU(n)) = Map^{l'}(\mathbb{C}P^2, BSU(n))$, and likewise $B\mathcal{G}_*^{(k,l)}(\mathbb{C}P^2, SU(n)) \simeq B\mathcal{G}_*^{l'}(\mathbb{C}P^2, SU(n))$. Note that we have used different integer labels for the components on each side of the equivalence. This a point to which we shall shortly return.

The diagram (4.89) appearing in Lemma 4.14 now appears as

$$\begin{tikzcd}
S^1 \arrow[r] \arrow[d, "\Delta"] & * \arrow[r] \arrow[d] & BS^1 \arrow[r, equal] \arrow[d] & BS^1 \arrow[d, "B\Delta"] \\
U(n) \arrow[r] \arrow[d, "\pi"] & B\mathcal{G}_*^{(k,l)}(\mathbb{C}P^2, U(n)) \arrow[r] \arrow[d, "\simeq"] & B\mathcal{G}^{(k,l)}(\mathbb{C}P^2, U(n)) \arrow[r] \arrow[d, "\widehat{B\pi}"] & BU(n) \arrow[d, "B\pi"] \\
SU(n) \arrow[r, "\delta_{l'}"] & B\mathcal{G}_*^{l'}(\mathbb{C}P^2, SU(n)) \arrow[r] & B\mathcal{G}^{l'}(\mathbb{C}P^2, SU(n)) \arrow[r] & BSU(n).
\end{tikzcd} \tag{4.92}$$

At this stage we have left the notation of (4.89) in place: the maps on the diagram are not canonical projections and must be carefully calculated. The bottom row of the diagram now appears as the evaluation fibration of the gauge group of an $SU(n)$-bundle over $\mathbb{C}P^2$.

We next identify the integer l'. The action of $B\Delta$ on the first two integral Chern classes was calculated by Woodward in [220] where it was shown that

$$B\Delta^* c_1 = nx \tag{4.93}$$

$$B\Delta^* c_2 = \frac{n(n-1)}{2} x^2 \tag{4.94}$$

where $x \in H^2 BS^1$ is a generator. Using this he proves that the image of $B\pi^* : H^4 BPU(n) \to H^4 BU(n)$ is generated by

$$\begin{cases} -(n-1)c_1^2 + 2nc_2, & n \text{ even} \\ -\frac{(n-1)}{2}c_1^2 + 2nc_2, & n \text{ odd} \end{cases} \tag{4.95}$$

and so we obtain the action of $B\pi^*$ on the Pontryagin class $p_1 \in H^4 BPU(n)$

$$B\pi^* p_1 = \begin{cases} -(n-1)c_1^2 + 2nc_2, & n \text{ even} \\ -\frac{(n-1)}{2}c_1^2 + 2nc_2, & n \text{ odd.} \end{cases} \tag{4.96}$$

With this the value of l' is now clear. If $f : \mathbb{C}P^2 \to BU(2)$ satisfies $f^* c_1 = k \cdot x$, $f^* c_2 = l \cdot x^2$ then

$$\begin{aligned} \widehat{B\pi}(f)^* p_1 &= (B\pi \circ f)^* p_1 \\ &= f^* \left(\begin{cases} -(n-1)c_1^2 + 2nc_2, & n \text{ even} \\ -\frac{(n-1)}{2}c_1^2 + 2nc_2, & n \text{ odd} \end{cases} \right) \\ &= \begin{cases} (2nl - (n-1)k^2) \cdot x^2, & n \text{ even} \\ \left(2nl - \frac{(n-1)}{2}k^2\right) \cdot x^2, & n \text{ odd} \end{cases} \end{aligned} \tag{4.97}$$

and we have

$$l' = \begin{cases} 2nl - (n-1)k^2, & n \text{ even} \\ 2nl - \frac{(n-1)}{2}k^2, & n \text{ odd}. \end{cases} \tag{4.98}$$

With this in place it only remains to study the map π in (4.92). Consider the diagram

$$\begin{tikzcd} S^1 \times SU(n) \arrow[r, "n \cdot i + j"] \arrow[d, "pr_2"] & U(n) \arrow[d, "\pi"] \\ SU(n) \arrow[r, "\rho"] & PU(n). \end{tikzcd} \tag{4.99}$$

where $i : S^1 = U(1) \to U(n)$ is the inclusion. We claim that this diagram homotopy commutes. Indeed, the map π is a homomorphism so

$$\pi \circ (n \cdot i + j) = n \cdot (\pi \circ i) + \pi \circ j \simeq * + \rho \circ pr_2 \simeq \rho \circ pr_2. \tag{4.100}$$

Note that since $(n,p) = 1$, the map $n \cdot i + j$ on the top row is a mod p homotopy equivalence with inverse $(1/n \cdot det) \times q : U(n) \to S^1 \times SU(n)$ where q, i appear in the fibration sequence $S^1 \xrightarrow{i} U(n) \xrightarrow{q} U(n)/S^1 \cong SU(n)$.

The map we seek to identify is $\rho^{-1} \circ \pi : U(n) \to SU(n)$ and using (4.99) we may write this as

$$\rho^{-1} \circ \pi \simeq pr_2 \circ (n \cdot i + j)^{-1} \simeq pr_2 \circ (1/n \cdot det \times q) = q. \tag{4.101}$$

In short, up to equivalence we may simply replace the map π in (4.92) with the projection q. This information is presented, with a slight abuse of notation, in the diagram (4.90). ∎

Corollary 4.16 *Let p be a prime not dividing n. Then after localisation at p there is a homotopy equivalence*

$$\mathcal{G}^{(k,l)}(\mathbb{C}P^2, U(n)) \simeq \mathcal{G}^{l'}(\mathbb{C}P^2, SU(n)) \times S^1 \tag{4.102}$$

where

$$l' = \begin{cases} 2nl - (n-1)k^2, & n \textit{ even} \\ 2nl - \frac{(n-1)}{2}k^2, & n \textit{ odd}. \end{cases} \tag{4.103}$$

Proof Loop diagram (4.90) of Proposition 4.15 to obtain

$$\begin{tikzcd} \mathcal{G}_*^{(k,l)}(\mathbb{C}P^2, U(n)) \arrow[r] \arrow[d, "\simeq"] & \mathcal{G}^{(k,l)}(\mathbb{C}P^2, U(n)) \arrow[r] \arrow[d, "\Omega\widehat{B\pi}"] & U(n) \arrow[d, "\pi"] \\ \mathcal{G}_*^{l'}(\mathbb{C}P^2, SU(n)) \arrow[r] \arrow[d] & \mathcal{G}^{l'}(\mathbb{C}P^2, SU(n)) \arrow[r] \arrow[d, "\epsilon"] & SU(n) \arrow[d, "*"] \\ * \arrow[r] & BS^1 \arrow[r, equal] & BS^1. \end{tikzcd} \tag{4.104}$$

Since $SU(n)$ is 2-connected the connecting map $SU(n) \to BS^1$ of the right-hand vertical fibration is trivial. The connecting map $\epsilon : \mathcal{G}^{l'}(\mathbb{C}P^2, SU(n)) \to BS^1$ factors through this trivial map so it too is trivial. This observation induces the product decomposition $\mathcal{G}^{(k,l)}(\mathbb{C}P^2, U(n)) \simeq \mathcal{G}^{l'}(\mathbb{C}P^2, SU(n)) \times S^1$. ∎

Remark It is interesting to observe that the splitting of Corollary 4.16 actually happens on the level of classifying spaces. Indeed, localised at a prime p not dividing n the map $\chi : BPU(n) \to K(\mathbb{Z}, 3)$ becomes trivial since it factors through the contractible space $K(\mathbb{Z}_n, 2)_{(p)} \simeq K((\mathbb{Z}_n)_{(p)}, 2) \simeq *$. Using the principal action we get a local homotopy equivalence $BS^1 \times BSU(n) \simeq BS^1 \times BPU(n) \simeq BU(n)$ (see [101]) which induces a local homotopy equivalence $Map(\mathbb{C}P^2, BS^1) \times Map(\mathbb{C}P^2, BSU(n)) \simeq Map(\mathbb{C}P^2, SU(n))$. The inverse is not easily identified, however, and it is difficult to correctly match components.

4.6 The Relationship Between $B\mathcal{G}^{(0,l)}(\mathbb{C}P^2, U(n))$ and $B\mathcal{G}^l(\mathbb{C}P^2, SU(n))$.

Consider the bundles $E_{(0,l)} \to \mathbb{C}P^2$ with trivial first Chern class. These bundles restrict to trivial bundles over S^2 so therefore $\mathcal{G}(E_{(0,l)}|_{S^2}) \simeq \mathcal{G}(S^2 \times U(n)) \simeq Map(S^2, U(n))$. This observation hints that the gauge groups $\mathcal{G}(E_{(0,l)})$ of the unrestricted bundles may have a particularly simple structure and this is indeed the case. In this section we examine the toplogies of these gauge groups and uncover their relationship with certain $SU(n)$-gauge groups. Our first result relates to the classifying spaces of the based groups.

Lemma 4.17 *For each integer l there is a homotopy equivalence*

$$B\mathcal{G}^{(0,l)}_*(\mathbb{C}P^2, U(n)) \simeq B\mathcal{G}^l_*(\mathbb{C}P^2, SU(n)). \tag{4.105}$$

Proof Start with the fibration sequence $SU(n) \xrightarrow{j} U(n) \xrightarrow{det} S^1$. Note that this sequence is split by the inclusion $S^1 = U(1) \hookrightarrow U(n)$ which, being itself a group homomorphism, induces a splitting of the fibring of classifying spaces

$$BSU(n) \xrightarrow{Bj} BU(n) \xrightarrow{Bdet} BS^1. \tag{4.106}$$

Now consider the fibre sequence of functors $\Omega^4(-) \xrightarrow{q^*} Map_*(\mathbb{C}P^2, -) \xrightarrow{i^*} \Omega^2(-)$ and take its product with the fibration (4.106). The result of this is a homotopy commutative diagram in which each row and column is a fibre sequence

$$\begin{tikzcd}
* \arrow[r] \arrow[d] & * \arrow[r] \arrow[d] & * \arrow[d] \\
\Omega^4 BSU(n) \arrow[r, "q^*"] \arrow[d, "\Omega^4 Bj", "\simeq"'] & Map_*(\mathbb{C}P^2, BSU(n)) \arrow[r, "i^*"] \arrow[d, "Bj_*"] \arrow[dr, phantom, "(*)"] & \Omega^2 BSU(n) \arrow[d, "\Omega^2 Bj"] \\
\Omega^4 BU(n) \arrow[r, "q^*"] \arrow[d] & Map_*(\mathbb{C}P^2, BU(n)) \arrow[r, "i^*"] \arrow[d, "Bdet_*"] & \Omega^2 BU(n) \simeq \mathbb{Z} \times \Omega^2 BSU(n) \arrow[d, "\Omega^2 Bdet"] \\
* \arrow[r] & Map_*(\mathbb{C}P^2, BS^1) \arrow[r, "i^*", "\simeq"'] & \Omega^2 BS^1 \simeq \mathbb{Z}.
\end{tikzcd} \tag{4.107}$$

By a result of Thom [196] $Map_*(\mathbb{C}P^2, BS^1) \simeq Map_*(\mathbb{C}P^2, K(\mathbb{Z}, 2)) \simeq K(H^2(\mathbb{C}P^2), 0) = K(H^2(S^2), 0)$ so the map i^* along the bottom of the diagram is a homotopy equivalence. Since also $\Omega^4 Bj$ on the left-hand side of the diagram is a homotopy equivalence it follows that the square $(*)$ is a homotopy pullback.

Restricting the downwards arrow on the right-hand side of $(*)$ to the basepoint component converts it into a homotopy equivalence $\Omega^2 Bj : \Omega^2 BSU(n) \simeq \Omega_0^2 BU(n)$ and i^* takes the component $Map_*^{(0,l)}(\mathbb{C}P^2, BU(n))$ into $\Omega_0^2 BU(n)$. Since homotopy classes $\mathbb{C}P^2 \to BSU(n)$ are classified by their action on the second Chern class, the map Bj_* sends the component $Map_*^l(\mathbb{C}P^2, BSU(n))$ into $Map_*^{(0,l)}(\mathbb{C}P^2, BU(n))$.

From these observations we obtain, through restriction, another homotopy pullback, here labelled (hpb),

$$\begin{tikzcd}
\Omega_l^4 BSU(n) \arrow[r, "q^*"] \arrow[d, "\Omega_l^4 Bj", "\simeq"'] & Map_*^l(\mathbb{C}P^2, BSU(n)) \arrow[r, "i^*"] \arrow[d, "Bj_*|", "\simeq"'] \arrow[dr, phantom, "\text{(hpb)}"] & \Omega^2 BSU(n) \arrow[d, "\Omega_0^2 Bj", "\simeq"'] \\
\Omega_l^4 BU(n) \arrow[r, "q^*"] & Map_*^{(0,l)}(\mathbb{C}P^2, BU(n)) \arrow[r, "i^*"] & \Omega_0^2 BU(n)
\end{tikzcd} \tag{4.108}$$

The restriction $Bj_*|$ on the left-hand side of the square is clearly displayed here as a homotopy equivalence. ∎

The previous lemma leads immediately to the next and the pair is subsequently used to relate the classifying spaces of the full $U(n)$- and $SU(n)$-gauge groups.

Lemma 4.18 *The square labelled (hpb) in the following diagram is a homotopy pullback*

$$\begin{tikzcd}
* \arrow[r] \arrow[d] & S^1 \arrow[r, equal] \arrow[d, "\nu"] & S^1 \arrow[d, "*"] \\
B\mathcal{G}_*^l(\mathbb{C}P^2, SU(n)) \arrow[r] \arrow[d, "Bj_*", "\simeq"'] & B\mathcal{G}^l(\mathbb{C}P^2, SU(n)) \arrow[r, "e^{SU}"] \arrow[d, "\widehat{Bj}"] \arrow[dr, phantom, "(hpb)"] & BSU(n) \arrow[d, "Bj"] \\
B\mathcal{G}_*^{(0,l)}(\mathbb{C}P^2, U(n)) \arrow[r] \arrow[d] & B\mathcal{G}^{(0,l)}(\mathbb{C}P^2, U(n)) \arrow[r, "e^U"] \arrow[d, "\xi"] & BU(n) \arrow[d, "Bdet"] \\
* \arrow[r] & BS^1 \arrow[r, equal] & BS^1.
\end{tikzcd} \tag{4.109}$$

Proof Introduce the evaluation fibrations for each of the gauge groups $\mathcal{G}^{(0,l)}(\mathbb{C}P^2, U(n))$ and $\mathcal{G}^l(\mathbb{C}P^2, SU(n))$ and use the map $Bj : BSU(n) \to BU(n)$ to induce the maps $\widehat{Bj}$ and Bj_* that appear vertically in (4.109). From Lemma 4.17 we know that Bj_* is a homotopy equivalence and we use this to identify the labelled square as a homotopy pullback. In turn this identifies the homotopy fibre of $\widehat{Bj} : B\mathcal{G}^l(\mathbb{C}P^2, SU(n)) \to B\mathcal{G}^{(0,l)}(\mathbb{C}P^2, U(n))$ as S^1.

Now the fibration $S^1 \to BSU(n) \xrightarrow{Bj} BU(n)$ is principal, induced by the map $Bdet = c_1 : BU(n) \to BS^1$ which is split by the inclusion $Bi : BS^1 \to BU(n)$. This is reflected by the null homotopic connecting map $S^1 \to BSU(n)$ and by the fact that the principal homotopy action of the sequence is trivial. The principal action of the downwards middle sequence is then the pullback of a trivial action and is thus itself trivial. This shows the existence of a map $\xi : B\mathcal{G}^{(0,l)}(\mathbb{C}P^2, U(n)) \to BS^1$ classifying the fibration [98] which we take as the definition of the map appearing in (4.109). Defining ν to be the connecting map for the fibration sequence completes the proof of the statement. ∎

Theorem 4.19 *The gauge group $\mathcal{G}^{(0,l)}(\mathbb{C}P^2, U(n))$ is a trivial $\mathcal{G}^l(\mathbb{C}P^2, SU(n))$-bundle over S^1 and there exists an isomorphism of principal bundles*

$$\mathcal{G}^{(0,l)}(\mathbb{C}P^2, U(n)) \cong S^1 \times \mathcal{G}^l(\mathbb{C}P^2, SU(n)). \tag{4.110}$$

Proof Focus on the middle column of diagram (4.109). Looping this column upwards gives a fibration sequence

$$\mathcal{G}^l(\mathbb{C}P^2, SU(n)) \xrightarrow{\Omega\widehat{Bj}} \mathcal{G}^{(0,l)}(\mathbb{C}P^2, U(n)) \xrightarrow{\Omega\xi} S^1 \xrightarrow{\nu} B\mathcal{G}^l(\mathbb{C}P^2, SU(n)). \tag{4.111}$$

We shall show that the connecting map $\nu : S^1 \to B\mathcal{G}^l(\mathbb{C}P^2, SU(n))$ must be null-homotopic by demonstrating that $\pi_1(B\mathcal{G}^l(\mathbb{C}P^2, SU(n))) = 0$. This then gives the desired splitting.

To this end we note that $\pi_1(BSU(n)) = 0$ so (4.109) gives an epimorphism $\pi_1(B\mathcal{G}^l_*(\mathbb{C}P^2, SU(n))) \to \pi_1(B\mathcal{G}^l(\mathbb{C}P^2, SU(n)))$. It follows that we may conclude the proof if we can show that $\pi_1(B\mathcal{G}^l_*(\mathbb{C}P^2, SU(n))) = 0$. Using the action of $\Omega^4 BSU(n)$ we obtain a homotopy equivalence $B\mathcal{G}^l_*(\mathbb{C}P^2, SU(n)) = Map^l(\mathbb{C}P^2, BSU(n)) \simeq Map^0_*(\mathbb{C}P^2, BSU(n))$ and we find this latter space in the fibration sequence $\Omega^4_0 BSU(n) \xrightarrow{q^*} Map^0_*(\mathbb{C}P^2, BSU(n)) \xrightarrow{i^*} \Omega^2 BSU(n)$ which is induced by the cofibring $S^3 \xrightarrow{\eta} S^2 \xrightarrow{i} \mathbb{C}P^2 \xrightarrow{q} S^4$. From the long exact homotopy sequence of this fibration we obtain the exact sequence

$$\pi_4(BSU(n)) = \mathbb{Z} \xrightarrow{\eta^*} \pi_5(BSU(n)) \xrightarrow{q^*} \pi_1(Map^0_*(\mathbb{C}P^2, BSU(n))) \xrightarrow{i^*} 0 = \pi_3(BSU(n)). \tag{4.112}$$

For $n \geq 3$ we have $\pi_5(BSU(n)) = 0$ and in this case $\pi_1(Map^0_*(\mathbb{C}P^2, BSU(n))) = 0$ must be the trivial group. When $n = 2$ we have $BSU(2) \simeq BS^3$, $\pi_4(BS^3) = \mathbb{Z}$ generated by the inclusion of the bottom cell $i : S^4 \hookrightarrow BS^3$ and $\pi_5(BS^3) = \mathbb{Z}_2$ generated by $i \circ \eta$. Thus $\eta^* : \pi_4(BS^3) \xrightarrow{\eta^*} \pi_5(BS^3)$ is onto and once again we have $\pi_1(Map^0_*(\mathbb{C}P^2, BSU(2))) = 0$. From the proceeding comments we are now able to conclude the stated result. ∎

4.6.1 Homotopy Types of $U(2)$-Gauge Groups Over $\mathbb{C}P^2$.

In this section we give application for the various theorems we have collected by studying the homotopy types of $U(2)$-gauge groups over $\mathbb{C}P^2$. We first examine the homotopy types of the based $U(2)$-gauge groups over $\mathbb{C}P^2$ and their classifying spaces. The results collected here are then used to complete the proof of Theorem 4.3 in section 4.6.3.

4.6.2 The Based Gauge Groups.

We focus here on the classifying spaces of the based gauge groups. Using Theorem 4.9 and Propostion 4.8 we see that there are at most two distinct types, with representatives $B\mathcal{G}^{(0,0)}_*$ and $B\mathcal{G}^{(1,0)}_*$. The main result of this section is that these two spaces do in fact represent distinct homotopy types. The proof proceeds by an investigation of the low-dimensional homotopy groups of each of the types.

This result is in strong contrast to previous cases that appear in the literature. For instance when studying the gauge groups associated to any simply connected, compact lie group over a simply connected 4-manifold, the analogue of Proposition 4.8 is enough to ensure that there is a unique homotopy type amongst the classifying spaces of the based gauge groups.

Lemma 4.20

$$\pi_1 B\mathcal{G}_*^{(0,0)}(\mathbb{C}P^2, U(2)) = 0 \tag{4.113}$$

$$\pi_1 B\mathcal{G}_*^{(1,0)}(\mathbb{C}P^2, U(2)) = \mathbb{Z}_2 \tag{4.114}$$

Proof We first show (4.113). We identify $B\mathcal{G}_*^{(0,0)}(\mathbb{C}P^2, U(2)) \simeq Map_*^{(0,0)}(\mathbb{C}P^2, BU(2))$. Then we have a fibre sequence

$$\ldots \to \Omega^2 S^3 \xrightarrow{\eta^*} \Omega_0^3 S^3 \xrightarrow{q^*} Map_*^{(0,0)}(\mathbb{C}P^2, BU_2) \xrightarrow{i^*} \Omega S^3 \tag{4.115}$$

which is the restriction of the sequence (4.42) to the basepoint components. Examining the homotopy sequence of this fibration in low dimensions leads to the exact sequence

$$\pi_2 B\mathcal{G}_*^{(0,0)}(\mathbb{C}P^2, U(2)) \xrightarrow{i^*} \pi_3(S^3) = \mathbb{Z} \xrightarrow{\eta^*} \pi_4(S^3) = \mathbb{Z}_2 \xrightarrow{q^*} \pi_1 B\mathcal{G}_*^{(0,0)}(\mathbb{C}P^2, U(2)) \to 0 \tag{4.116}$$

with q^* onto since $\pi_1 \Omega S^3 = \pi_2 S^3 = 0$. As $\pi_3 S^3 = \mathbb{Z}$ is generated by the identity id_{S^3} and $\pi_4 S^3 = \mathbb{Z}_2$ is generated by the Hopf map η, the middle map of this sequence is in fact onto and it must be that $\pi_1 B\mathcal{G}_*^{(0,0)} = 0$, proving (4.113).

Turning now to (4.114) we first recall that $B\mathcal{G}^{(1,0)}(\mathbb{C}P^2, U(2)) \simeq B\mathcal{G}^{(1,1)}(\mathbb{C}P^2, U(2))$ by Proposition 4.8 so to prove the statement it will be sufficient to caluclate $\pi_1 B\mathcal{G}^{(1,1)}(\mathbb{C}P^2, U(2))$. We have a fibring

$$\Omega_0^3 S^3 \xrightarrow{j} Map_*^{(1,1)}(\mathbb{C}P^2, BU_2) \xrightarrow{i^*} \Omega S^3 \tag{4.117}$$

granted by Proposition 4.10 but in this case there is no easy identification of the fibration connecting map. In any case the homotopy exact sequence shows us that $\pi_1 Map_*^{(1,1)}(\mathbb{C}P^2, BU_2)$ is a quotient of $\pi_1 \Omega_0^3 S^3 = \mathbb{Z}_2$ and so it is either trivial or $\mathbb{Z}_2$. We will show it is the latter.

Consider the $U(2)$-bundle $U(2) \hookrightarrow SU(3) \to \mathbb{C}P^2$ obtained by letting $SU(3)$ act on $\mathbb{C}P^2$ in the standard way and let $f : \mathbb{C}P^2 \to BU(2)$ be a classifying map for this bundle. Then a simple calculation with the Serre spectral sequence shows that this bundle has Chern classes $(f^*c_1, f^*c_2) = (x, x^2)$ so that f is an element of $Map^{(1,1)}(\mathbb{C}P^2, BU(2)) = B\mathcal{G}^{(1,1)}(\mathbb{C}P^2, U(2))$.

Now the homotopy fibration sequence

$$SU(3) \to \mathbb{C}P^2 \xrightarrow{f} BU(2) \tag{4.118}$$

gives rise to a short exact sequence of homotopy groups

$$0 \to \pi_5 SU(3) = \mathbb{Z} \to \pi_5 \mathbb{C}P^2 = \mathbb{Z} \xrightarrow{f_*} \pi_5 BU(2) = \mathbb{Z}_2 \to 0 \tag{4.119}$$

Here $\pi_6 BU(2) = \pi_5 U(2) = \mathbb{Z}_2$ so the left-hand map is monic and $\pi_4 SU(3) = 0$ so f_* is epic.

Now $\pi_5 \mathbb{C}P^2$ is generated by the quotient map $\gamma : S^5 \to \mathbb{C}P^2$, so (4.119) shows that $f_*(\gamma) = f \circ \gamma = \gamma^*(f)$ is a generator, $\tilde{\epsilon}_5$, of $\pi_5 BU(2)$. It follows that γ induces a map of evaluation fibrations as in the diagram

$$\begin{tikzcd}
\ldots \arrow[r] & U(2) \arrow[r, "\lambda_{(1,1)}"] \arrow[d, equal] & B\mathcal{G}_*^{(1,1)}(\mathbb{C}P^2, U(2)) \arrow[r] \arrow[d, "\gamma^*"] & B\mathcal{G}^{(1,1)}(\mathbb{C}P^2, U(2)) \arrow[r, "e"] \arrow[d, "\gamma^*"] & BU(2) \arrow[d, equal] \\
\ldots \arrow[r] & U(2) \arrow[r, "\delta'"] & \Omega_0^4 U(2) \arrow[r] & Map^{\tilde{\epsilon}_5}(S^5, BU(2)) \arrow[r] & BU(2).
\end{tikzcd} \tag{4.120}$$

Here we see a factorisation of the connecting map δ' of the bottom sequence through $\lambda_{(1,1)}$.

By the work of Lang [130] the adjoint of δ' is the Samelson product $\langle \epsilon_4, id_{U(2)} \rangle : S^4 \wedge U(2) \to U(2)$, where ϵ_4 generates $\pi_4 U(2) = \mathbb{Z}_2$ and is adjoint to $\tilde{\epsilon}_5$. In degree one homotopy we then have $\delta'_* : \pi_1 U(2) \to \pi_5 U(2)$ taking a generator $\epsilon_1 \in \pi_1 U(2)$ to the Samelson product $\langle \epsilon_4, \epsilon_1 \rangle$.

We claim that this Samelson product is non-trivial and in fact is a generator of the group. Indeed, if $\epsilon_3 \in \pi_3 U(2) = \mathbb{Z}$ is a generator, then $\langle \epsilon_3, \epsilon_1 \rangle = \epsilon_4$ generates $\pi_4 U(2) = \mathbb{Z}_2$ [31]. Now using that the inclusion $j : S^3 = SU(2) \hookrightarrow U(2)$ gives an isomorphism on homotopy in degree greater than 1 we have a second identification of the generators as $\epsilon_3 = j$ and $\epsilon_4 = \langle \epsilon_3, \epsilon_1 \rangle = j_* \eta$. From this we get that

$$\langle \epsilon_4, \epsilon_1 \rangle = \langle j_* \eta, \epsilon_1 \rangle = \langle \epsilon_3, \epsilon_1 \rangle \circ (\eta \wedge 1) = j \circ \eta \circ \Sigma \eta \tag{4.121}$$

is the generator of $\pi_5 U(2) = \mathbb{Z}_2$ and the Samelson product is non-trivial.

It follows that $\delta'_* : \pi_1 U(2) \to \pi_5 U(2)$ is onto and the factorisation in the left-hand square of (4.120) shows that $\pi_1 B\mathcal{G}_*^{(1,1)}(\mathbb{C}P^2, U(2))$ cannot be the trivial group. We now know that $\pi_1 B\mathcal{G}_*^{(1,1)}(\mathbb{C}P^2, U(2))$ is a non-trivial quotient of $\mathbb{Z}_2$. It must therefore be $\mathbb{Z}_2$. ∎

We see from the lemma that there are in fact two distinct homotopy types amongst these classifying spaces. Since the homotopic information presented in Lemma 4.20 is retained after looping, the distinction is passed onto the based gauge groups they classify as well. We record this in the following statement.

Theorem 4.21 *There are exactly two different homotopy types amongst the classifying spaces of based gauge groups of $U(2)$-bundles over $\mathbb{C}P^2$. A similar statement holds for the the based gauge groups they classify. In particular*

$$B\mathcal{G}_*^{(0,0)}(\mathbb{C}P^2, U(2)) \not\simeq B\mathcal{G}_*^{(1,0)}(\mathbb{C}P^2, U(2)) \tag{4.122}$$

and

$$\mathcal{G}_*^{(0,0)}(\mathbb{C}P^2, U(2)) \not\simeq \mathcal{G}_*^{(1,0)}(\mathbb{C}P^2, U(2)). \quad ∎ \tag{4.123}$$

It turns out the distinction between these classifying spaces is solely due to 2-local phenomena. The odd primary homotopy types of these based gauge groups are much more understandable.

Proposition 4.22 *Localised away from 2 there is a single homotopy type amongst the classifying spaces of based gauge groups of $U(2)$-bundles over $\mathbb{C}P^2$. A Similar statement holds for the the based gauge groups they classify. In particular there are local homotopy equivalences $B\mathcal{G}_*^{(0,0)}(\mathbb{C}P^2, U(2)) \simeq B\mathcal{G}_*^{(1,0)}(\mathbb{C}P^2, U(2))$ and $\mathcal{G}_*^{(1,0)}(\mathbb{C}P^2, U(2)) \simeq \mathcal{G}_*^{(0,0)}(\mathbb{C}P^2, U(2)) \simeq Map_*(\mathbb{C}P^2, U(2))$.*

Proof Once 2 is inverted we have two chains of homotopy equivalences,

$$B\mathcal{G}_*^{(0,0)}(\mathbb{C}P^2, U(2)) \simeq Map_*^{(0,0)}(\mathbb{C}P^2, BU(2)) \simeq Map_*^{(0,0)}(\mathbb{C}P^2, BPU(2)) \simeq Map_*^{0}(\mathbb{C}P^2, BS^3) \tag{4.124}$$

$$B\mathcal{G}_*^{(1,0)}(\mathbb{C}P^2, U(2)) \simeq Map_*^{(1,0)}(\mathbb{C}P^2, BU(2)) \simeq Map_*^{(1,0)}(\mathbb{C}P^2, BPU(2)) \simeq Map_*^{0}(\mathbb{C}P^2, BS^3) \tag{4.125}$$

which follow from Lemma 4.13 and Proposition 4.15. Combining these two chains yields the first statement.

With this established it is now immediate that

$$\mathcal{G}_*^{(1,0)}(\mathbb{C}P^2, U(2)) \simeq \Omega B\mathcal{G}_*^{(1,0)}(\mathbb{C}P^2, U(2)) \simeq \Omega B\mathcal{G}_*^{(0,0)}(\mathbb{C}P^2, U(2)) \simeq \mathcal{G}_*^{(0,0)}(\mathbb{C}P^2, U(2)). \tag{4.126}$$

Moreover, since $Map_*^{(0,0)}(\mathbb{C}P^2, BU(2))$ is the component containing the constant map it holds that

$$\begin{aligned}\mathcal{G}_*^{(0,0)}(\mathbb{C}P^2, U(2)) &\simeq \Omega B\mathcal{G}_*^{(0,0)}(\mathbb{C}P^2, U(2)) \simeq \Omega Map_*^{(0,0)}(\mathbb{C}P^2, BU(2)) \\ &\simeq Map_*^{(0,0)}(\Sigma\mathbb{C}P^2, BU(2)) \simeq Map_*^{(0,0)}(\mathbb{C}P^2, \Omega BU(2)) \\ &\simeq Map_*^{(0,0)}(\mathbb{C}P^2, U(2))\end{aligned} \tag{4.127}$$

Thus

$$\mathcal{G}_*^{(1,0)}(\mathbb{C}P^2, U(2)) \simeq \mathcal{G}_*^{(0,0)}(\mathbb{C}P^2, U(2)) \simeq Map_*^{(0,0)}(\mathbb{C}P^2, U(2)). \quad \blacksquare \tag{4.128}$$

For later use we record the following.

Lemma 4.23

$$\pi_2 B\mathcal{G}_*^{(0,0)} = \mathbb{Z} \tag{4.129}$$

$$\pi_2 B\mathcal{G}_*^{(1,0)} = \mathbb{Z} \oplus \mathbb{Z}_2 \tag{4.130}$$

Proof We first show (4.129) using the homotopy exact sequence of the fibring $\Omega_0^3 S^3 \xrightarrow{q^*} Map_*^{(0,0)}(\mathbb{C}P^2, BU_2) \xrightarrow{i^*} \Omega S^3$. After the appropriate identifications we get

$$\pi_4 S^3 \xrightarrow{\eta^*} \pi_5 S^3 \xrightarrow{q^*} \pi_2 Map_*^{(0,0)}(\mathbb{C}P^2, BU_2) \xrightarrow{i^*} \pi_3 S^3 \xrightarrow{\eta^*} \pi_4 S^4. \tag{4.131}$$

Now $\pi_4 S^3 = \mathbb{Z}_2$ is generated by η and $\pi_5 S^3 = \mathbb{Z}_2$ is generated by η^2 so the left hand η^* is an isomorphism and we get $\pi_2 Map_*^{(0,0)}(\mathbb{C}P^2, BU_2) = \ker(\eta^* : \pi_3 S^3 \to \pi_4 S^3)$. From Lemma 4.20 we know that $\ker(\eta^* : \pi_3 S^3 \to \pi_4 S^3) = 2\mathbb{Z}$ and so we are able to conclude (4.129).

Now we shall show (4.130). We begin by using the map induced by $i : S^2 \hookrightarrow \mathbb{C}P^2$ to compare the evaluation fibration sequence of the gauge group $\mathcal{G}^{(1,0)}(\mathbb{C}P^2, U(2))$ with that of the gauge group $\mathcal{G}^1(S^2, U(2))$ belonging to the restricted bundle. The result of this is the following homotopy commutative diagram

$$\begin{tikzcd}
* \arrow[r] \arrow[d] & \Omega_0^3 S^3 \arrow[r, equal] \arrow[d, "j"] & \Omega_0^3 S^3 \arrow[r] \arrow[d] & * \arrow[d] \\
U(2) \arrow[r, "\lambda_{(1,0)}"] \arrow[d, equal] & Map_*^{(1,0)}(\mathbb{C}P^2, BU(2)) \arrow[r] \arrow[d, "i^*"] \arrow[dr, phantom, "\text{(hpb)}"] & B\mathcal{G}^{(1,0)}(\mathbb{C}P^2, U(2)) \arrow[r, "e"] \arrow[d, "i^*"] & BU(2) \arrow[d, equal] \\
U(2) \arrow[r, "\delta_1"] & \Omega S^3 \arrow[r] & B\mathcal{G}^1(S^2, U(2)) \arrow[r] & BU(2).
\end{tikzcd} \tag{4.132}$$

By construction the square labelled (hpb) is a homotopy pullback. In this diagram we have used the homotopy fibration sequence

$$\Omega_0^3 S^3 \xrightarrow{j} Map_*^{(1,0)}(\mathbb{C}P^2, BU(2)) \xrightarrow{i^*} \Omega S^3 \tag{4.133}$$

supplied by Proposition 4.10 to identify the spaces appearing in the top line of the diagram. The point of which to take note is the factorisation of the connecting map δ_1 of the bottom evaluation sequence through both i^* and $\lambda_{(1,0)}$.

Now it was shown in Lemma 4.20 that $\pi_1 Map_*^{(1,0)}(\mathbb{C}P^2, BU(2)) = \mathbb{Z}_2$, and argued there also that the map $j_* : \pi_1\Omega_0^3 S^3 \xrightarrow{\cong} \pi_1 Map_*^{(1,0)}(\mathbb{C}P^2, BU(2))$ is an ismorphism. Shortly we shall show that the map $j_* : \pi_2\Omega_0^3 S^3 \to \pi_2 Map_*^{(1,0)}(\mathbb{C}P^2, BU(2))$ is injective. If we assume this for now and examine the homotopy sequence of the fibration (4.133) we find a short exact sequence

$$0 \to \pi_2\Omega_0^3 S^3 \xrightarrow{j_*} \pi_2 Map_*^{(1,0)}(\mathbb{C}P^2, BU(2)) \xrightarrow{i^*} \pi_2\Omega S^3 \to 0 \tag{4.134}$$

which must split for algebraic reasons since $\pi_2\Omega S^3 = \pi_3 S^3 = \mathbb{Z}$ is free abelian. This then allows us to get (4.130) by concluding that

$$\pi_2 Map_*^{(1,0)}(\mathbb{C}P^2, BU(2)) \cong \pi_2\Omega S^3 \oplus \pi_2\Omega_0^3 S^3 \cong \mathbb{Z} \oplus \mathbb{Z}_2. \tag{4.135}$$

We shall now provide the details of the argument for $j_* : \pi_2\Omega_0^3 S^3 \to \pi_2 Map_*^{(1,0)}(\mathbb{C}P^2, BU(2))$ being injective. We have seen that the Samelson product $\langle \epsilon_1, \epsilon_3\rangle$ is non-trivial and that the map $\delta_{1*} : \pi_3 U(2) \to \pi_3\Omega S^3 \cong \pi_4 U(2)$ is given the assignment $\alpha \mapsto \langle \epsilon_1, \alpha\rangle$. It must therefore be that δ_{1*} is a non-trivial surjection $\pi_3 U(2) = \mathbb{Z} \to \pi_4 U(2) = \mathbb{Z}_2$. We have seen in (4.132) that δ_1 factors through $i^* : Map_*^{(1,0)}(\mathbb{C}P^2, BU(2)) \to \Omega S^3$ so it must be that i^* induces a non-trivial surjection in this degree. Consequently the connecting map of the sequence $\Delta : \pi_3\Omega S^3 \to \pi_2\Omega^3 S^3$ must be trivial and the following map of the sequence, $j_* : \pi_2\Omega_0^3 S^3 \to \pi_2 Map_*^{(1,0)}(\mathbb{C}P^2, BU(2))$, must be injective. ∎

4.6.3 The Full Gauge Groups.

Moving on now to the full gauge gauge groups we first use Theorem 4.9 to reduce the enumeration problem to a more manageble size. Applying its statement in the case of $n = 2$ yields the following.

Proposition 4.24 *For all integers, k, l, r there are homotopy equivalences*

$$B\mathcal{G}^{(k,l)}(\mathbb{C}P^2, U(2)) \simeq B\mathcal{G}^{(k+2r, l+kr+r^2)}(\mathbb{C}P^2, U(2)) \tag{4.136}$$

and consequently also homotopy equivalences

$$\mathcal{G}^{(k,l)}(\mathbb{C}P^2, U(2)) \simeq \mathcal{G}^{(k+2r, l+kr+r^2)}(\mathbb{C}P^2, U(2)). \quad \blacksquare \tag{4.137}$$

Now assume given a particular gauge group $\mathcal{G}^{(k,l)}(\mathbb{C}P^2, U(2))$. Write $k = 2k' + \epsilon$ with $\epsilon \in \{0, 1\}$ and take $r = -k'$ in the formula of Proposition 4.24 to obtain a homotopy equivalence

$$B\mathcal{G}^{(k,l)}(\mathbb{C}P^2, U(2)) \simeq B\mathcal{G}^{(\epsilon, l-k'^2-\epsilon k')}(\mathbb{C}P^2, U(2)) \tag{4.138}$$

with $\epsilon = 0$ or 1. We see that in studying the homotopy type of any given gauge group $\mathcal{G}^{(k,l)}(\mathbb{C}P^2, U(2))$ we can always find a homotopy equivalent object with $k = 0$ or 1. This observation reduces the problem of studying the homotopy types of all possible $U(2)$-gauge groups over $\mathbb{C}P^2$ to just the study of the the homotopy types of the gauge groups

$$\mathcal{G}^{(0,l)}(\mathbb{C}P^2, U(2)) \qquad\qquad \mathcal{G}^{(1,l)}(\mathbb{C}P^2, U(2)) \tag{4.139}$$

and their classifying spaces as l ranges over the integers. In the following we shall always therefore assume that k is an integer mod 2.

We next obtain an upper bound on the number of homotopy types in each class. We apply Corollary 4.12 and feed in the information from Theorem 4.1, that $|\delta_1| = 12$, to get the following.

Proposition 4.25 *For each pair of integers k, l, there is a homotopy equivalence*

$$\mathcal{G}^{(k,l)}(\mathbb{C}P^2, U(2)) \simeq \mathcal{G}^{(k,l+12)}(\mathbb{C}P^2, U(2)). \qquad \blacksquare \tag{4.140}$$

This proposition gives an upper bound that is not neccessarily met. Indeed there are fewer distinct homotopy types amongst the gauge groups in the first class. For these objects we have already solved the problem completely.

Proposition 4.26 *There is a homotopy equivalence $\mathcal{G}^{(0,l)}(\mathbb{C}P^2, U(2)) \simeq \mathcal{G}^{(0,l')}(\mathbb{C}P^2, U(2))$ if and only if $gcd(6, l) = gcd(6, l')$.*

Proof From Theorem 4.19 we obtain a decomposition

$$\mathcal{G}^{(0,l)}(\mathbb{C}P^2, U(2)) \simeq S^1 \times \mathcal{G}^l(\mathbb{C}P^2, SU(2)). \tag{4.141}$$

Now the gauge groups of principal $SU(2)$-bundles over $\mathbb{C}P^2$ have been studied by Kono and Tsukuda in [127]. In fact they give results relating to the homotopy types of $SU(2)$-bundles over any simply connected 4-manifold. To apply their results we simply note that the signature of the intersection form on $\mathbb{C}P^2$ is $+1$, so their Theorem 1.2 applies to give $\mathcal{G}^l(\mathbb{C}P^2, SU(2)) \simeq \mathcal{G}^{l'}(\mathbb{C}P^2, SU(2))$ if and only if $\gcd(6, l) = \gcd(6, l')$. Combining this information with the splitting in equation (4.141) completes the proof. $\blacksquare$

Turning now towards the study of the second class of gauge groups, namely the gauge groups $\mathcal{G}^{(1,l)}(\mathbb{C}P^2, U(2))$ of bundles with non-trivial first Chern class, we encounter a much more delicate problem. Integral results are particularly tricky and for the most part we make do with p-local statements. We already have the upper bound $\mathcal{G}^{(1,l)}(\mathbb{C}P^2, U(2)) \simeq \mathcal{G}^{(1,l+12)}(\mathbb{C}P^2, U(2))$ of Proposition 4.25. The following provides a lower bound of at least 2 distinct homotopy types. For its proof recall the notation of section 4.5.1; $\lambda_{(k,l)} : U(2) \to Map_*^{(k,l)}(\mathbb{C}P^2, BU(2))$ is the connecting map for the evaluation fibration of the gauge group $\mathcal{G}^{(k,l)}(\mathbb{C}P^2, U(2))$ and $\delta_l : U(2) \to \Omega_0^3 U(2)$ is the connecting map of the evaluation fibration sequence of the gauge group $\mathcal{G}^l(S^4, U(2))$.

Proposition 4.27

$$\pi_1 B\mathcal{G}^{(1,l)}(\mathbb{C}P^2, U(2)) = \begin{cases} 0, & l \equiv 1 \pmod 2 \\ \mathbb{Z}_2, & l \equiv 0 \pmod 2 \end{cases} \tag{4.142}$$

Proof The first step of the proof is to calculate the homomorphisms $\lambda_{(1,0)*} : \pi_1(U(2)) \to \pi_1(Map_*^{(1,0)}(\mathbb{C}P^2, BU(2)))$ and $\delta_l : \pi_1(U(2)) \to \pi_4(U(2))$. Following this we use the decomposition of Theorem 4.11 to obtain the action of $\lambda_{(1,l)*}$ for each l, and thus calculate $\pi_1 B\mathcal{G}^{(1,l)}(\mathbb{C}P^2, U(2)) \cong \mathrm{coker}(\lambda_{(1,l)*})$.

To proceed we examine the evaluation fibration for the gauge group $\mathcal{G}^{(1,0)}(\mathbb{C}P^2, U(2))$, which gives us an exact sequence

$$\pi_1\mathcal{G}^{(1,0)}(\mathbb{C}P^2, U(2)) \xrightarrow{e_*} \pi_1 U(2) \xrightarrow{\lambda_{(1,0)*}} \pi_1 Map_*^{(1,0)}(\mathbb{C}P^2, BU(2)) \to \pi_1 B\mathcal{G}^{(1,0)}(\mathbb{C}P^2, U(2)) \to 0 \tag{4.143}$$

since $BU(2)$ is simply connected. The map $e : \mathcal{G}^{(1,0)}(\mathbb{C}P^2, U(2)) \to U(2)$ is the evaluation. We claim that e_* is surjective and so $\lambda_{(1,0)}$ is trivial. This is seen as follows.

Since its second Chern class vanishes, the bundle $E_{(1,0)} \to \mathbb{C}P^2$ has a reduction of structure to a $U(1)$-bundle $\tilde{E}_1 \to \mathbb{C}P^2$ and there results a bundle-isomorphism $E_{(1,0)} \cong \tilde{E}_1 \times_{U(1)} U(2)$ over $\mathbb{C}P^2$, where $U(1)$ acts on $U(2)$ from the left via its inclusion $i_2 : U(1) \hookrightarrow U(2)$ in the bottom right-hand corner. Define $\alpha : U(1) \to \mathcal{G}(\widetilde{E}_1 \times_{U(1)} U(2)) \cong \mathcal{G}^{(1,0)}(\mathbb{C}P^2, U(2))$ by

$$\alpha(\lambda)(x, A) = (x, i_1(\lambda) \cdot A), \qquad \lambda \in U(1),\ x \in \tilde{E}_1,\ A \in U(2) \tag{4.144}$$

where $i_1 : U(1) \to U(2)$ is the inclusion in the top left-hand corner. It is not difficult to see that this is a well defined homomorphism that satisfies

$$e \circ \alpha = i_1 \tag{4.145}$$

Now recall that i_1 generates $\pi_1 U(n)$. It follows from (4.145) that e_* is surjective and $\lambda_{(1,0)*} = 0$.

On the other hand, $\delta_l : U(2) \to \Omega_0^3 U(2)$ is adjoint to the Samelson product $\langle l \cdot \epsilon_3, 1_{U(2)} \rangle : S^3 \wedge U(2) \to U(2)$ and in degree one homotopy the induced homomorphism is given by $\delta_{l*} : \pi_1 U(2) = \mathbb{Z} \to \pi_4 U(2) = \mathbb{Z}_2$, $\epsilon_1 \mapsto \langle l \cdot \epsilon_3, \epsilon_1 \rangle = l \cdot \epsilon_4$ where ϵ_i generates $\pi_i U(2)$. It is therefore trivial when l is even and a non-trivial surjection when l is odd.

Now we calculate the action of the combined map $\lambda_{(1,l)} = \lambda_{(1,0)} + \delta_l$. Consider the composite $\lambda_{(1,l)*}\epsilon_1 = \lambda_{(1,l)} \circ \epsilon_1$ as a class in $\pi_1 Map_*^{(1,l)}(\mathbb{C}P^2, BU(2))$. With respect to the decomposition of Theorem 4.11 it is represented by the following map

$$\begin{aligned} S^1 &\to Map_*^{(1,l)}(\mathbb{C}P^2, BU(2)) \\ z &\mapsto \left[\mathbb{C}P^2 \xrightarrow{c} \mathbb{C}P^2 \vee S^4 \xrightarrow{(\lambda_{(1,0)*}\epsilon_1(z)) \vee (\delta_{l*}\epsilon_1(z))} BU(2) \vee BU(2) \xrightarrow{\nabla} BU(2) \right]. \end{aligned} \tag{4.146}$$

Since $\lambda_{(1,0)*}\epsilon_1 = 0$, it is homotopic to the constant loop at a chosen basepoint map $f_{(1,0)} \in Map_*^{(1,0)}(\mathbb{C}P^2, BU(2))$. Similarly, $\delta_{l*}\epsilon_1$ is either homotopic to a chosen basepoint loop $\omega_0 \in \Omega_l^4 BU(2)$ when l is even, or, when l is odd, is a generator $\widehat{\epsilon}_1 \in \pi_1 \Omega_l^4 BU(2) = \mathbb{Z}_2$. Thus we have

$$\begin{aligned}\lambda_{(1,l)*}\epsilon_1(z) &= \lambda_{(1,0)*}\epsilon_1(z) + \delta_{l*}\epsilon_1(z) \\ &= f_{(1,0)} + \delta_{l*}\epsilon_1(z) \\ &= \begin{cases} f_{(1,0)} + \omega_0 & , \ l \text{ even} \\ f_{(1,0)} + \widehat{\epsilon}_1(z) & , \ l \text{ odd} \end{cases} \\ &= \begin{cases} f_{(1,l)} & , \ l \text{ even} \\ f_{(1,l)} + (-\omega_0 + \widehat{\epsilon}_1(z)) & , \ l \text{ odd} \end{cases} \\ &= \begin{cases} j(\omega_0) & , \ l \text{ even} \\ j(\widehat{\epsilon}_1(z)) & , \ l \text{ odd} \end{cases}\end{aligned} \tag{4.147}$$

where the last line follows from Proposition 4.10 where the fibre inclusion $j : \Omega^4_0 BU(2) \to Map^{(1,l)}_*(\mathbb{C}P^2, BU(2))$ was identified as $j(\omega) = f_{(1,l)} + \omega$.

Now equation (4.147) shows us that if l is even, then $\lambda_{(1,l)*}\epsilon_1$ is the constant loop at the basepoint map and is thus trivial. On the other hand, if l is odd, then $\lambda_{(1,l)*}\epsilon_1 = j_*\widehat{\epsilon}_1$. Since $j_* : \pi_1\Omega^4_l BU(2) \to \pi_1 Map^{(1,l)}_*(\mathbb{C}P^2, BU(2))$ is an isomorphism by Lemma (4.20), this elemement is non-trivial and we conclude that $\lambda_{(1,l)_*}$ is surjective in this case.

Now (4.143) displays the fact that $\pi_1 B\mathcal{G}^{(1,l)}(\mathbb{C}P^2, U(2)) = \text{coker}\left(\lambda_{(1,l)_*}\right)$. We have just seen that $\lambda_{(1,l)_*}$ is zero when l is even and surjective when l is odd. We know $\pi_1 Map^{(1,l)}_*(\mathbb{C}P^2, BU(2)) = \mathbb{Z}_2$ so therefore

$$\pi_1 B\mathcal{G}^{(1,l)}(\mathbb{C}P^2, U(2)) = \begin{cases} 0, & l \equiv 1 \pmod 2 \\ \mathbb{Z}_2, & l \equiv 0 \pmod 2 \end{cases} \tag{4.148}$$

and we are complete. ∎

The major obstacle to obtaining further integral information on the homotopy types of the gauge groups $\mathcal{G}^{(1,l)}(\mathbb{C}P^2, U(2))$ is actually a lack of 2-local information. Localised at an odd prime p, the homotopy of these objects is much simpler.

Theorem 4.28 *When localised away from 2 there is a product splitting*

$$\mathcal{G}^{(1,l)}(\mathbb{C}P^2, U(2)) \simeq S^1 \times \mathcal{G}^{4l-1}(\mathbb{C}P^2, SU(2)). \tag{4.149}$$

It follows that when localised away from 2 there is a homotopy equivalence $\mathcal{G}^{(1,l)}(\mathbb{C}P^2, U(2)) \simeq \mathcal{G}^{(1,l')}(\mathbb{C}P^2, U(2))$ if and only if gcd$(4l-1, 6)$ = gcd$(4l'-1, 6)$. In particular, when localised at an odd prime $p \geq 5$, the gauge group $\mathcal{G}^{(1,l)}$ has the trivial homotopy type

$$\mathcal{G}^{(1,l)}(\mathbb{C}P^2, U(2)) \simeq U(2) \times Map^*(\mathbb{C}P^2, U(2)). \quad ∎ \tag{4.150}$$

Proof The decomposition statement follows from Corollary 4.16. After this is established it is a simple matter to apply the results of Kono and Tsukuda [127] on the homotopy types of the $\mathcal{G}^{4l-1}(\mathbb{C}P^2, SU(2))$ gauge groups to verify the second statement. The final statement is then an immediate consequence of this. ∎

Before closing we shall return to the integral world to answer one remaining question.

Theorem 4.29 *For any integer values of l, l' it holds that*

$$B\mathcal{G}^{(0,l)}(\mathbb{C}P^2, U(2)) \not\simeq B\mathcal{G}^{(1,l')}(\mathbb{C}P^2, U(2)) \tag{4.151}$$

$$\mathcal{G}^{(0,l)}(\mathbb{C}P^2, U(2)) \not\simeq \mathcal{G}^{(1,l')}(\mathbb{C}P^2, U(2)). \tag{4.152}$$

To demonstrate the statement we shall calculate π_2 of each classifying space. We shall find the result independent of the integers l, l' - up to isomorphism - but different for $B\mathcal{G}^{(0,l)}(\mathbb{C}P^2, U(2))$ and $B\mathcal{G}^{(1,l')}(\mathbb{C}P^2, U(2))$. From this we shall conclude that there are no values of l, l' for which the classifying spaces are homotopy equivalent. Since the information in π_2 is retained after looping we shall be able to conclude the statement for the gauge groups.

Lemma 4.30 *For any integer values of l, l' the following hold.*

$$\pi_2 B\mathcal{G}^{(0,l)}(\mathbb{C}P^2, U(2)) = \mathbb{Z} \oplus \mathbb{Z} \tag{4.153}$$

$$\pi_2 B\mathcal{G}^{(1,l')}(\mathbb{C}P^2, U(2)) = \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}_2. \tag{4.154}$$

Proof The first step comes from an examination of the homotopy exact sequences of the evaluation fibration sequences

$$\ldots U(2) \xrightarrow{\lambda_{(k,l)}} B\mathcal{G}^{(k,l)}_*(\mathbb{C}P^2, U(2)) \to B\mathcal{G}^{(k,l)}(\mathbb{C}P^2, U(2)) \xrightarrow{e} BU(2) \tag{4.155}$$

for $k = 0$, 1. Since $\pi_3 BU(2) = 0$ we have in both cases an exact sequence

$$0 \to \pi_2 B\mathcal{G}^{(k,l)}_*(\mathbb{C}P^2, U(2)) \to \pi_2 B\mathcal{G}^{(k,l)}(\mathbb{C}P^2, U(2)) \xrightarrow{e_*} \pi_2 BU(2) \xrightarrow{\lambda_{(k,l)*}} \pi_1 B\mathcal{G}^{(k,l)}_*(\mathbb{C}P^2, U(2)) \tag{4.156}$$

and we conclude from this that the following is short exact

$$0 \to \pi_2 B\mathcal{G}^{(k,l)}_*(\mathbb{C}P^2, U(2)) \to \pi_2 B\mathcal{G}^{(k,l)}(\mathbb{C}P^2, U(2)) \to \ker(\lambda_{(k,l)*}) \to 0. \tag{4.157}$$

Now Lemma 4.20 and Proposition 4.27 give

$$\ker(\lambda_{(k,l)*}) = \begin{cases} 2\mathbb{Z} & k = 1,\ l \text{ odd} \\ \mathbb{Z} & \text{otherwise} \end{cases} \tag{4.158}$$

so in either case this means that the last group in (4.157) is free abelian so the sequence must split to give

$$\pi_2 B\mathcal{G}^{(k,l)}(\mathbb{C}P^2, U(2)) \cong \pi_2 Map^{(k,l)}_*(\mathbb{C}P^2, BU(2)) \oplus \mathbb{Z}. \tag{4.159}$$

Now the calculation of $\pi_2 Map^{(k,l)}_*(\mathbb{C}P^2, BU(2))$ was completed in Lemma 4.23 and using this information completes the proof. ∎

The hard work is now complete and to prove Theorem 4.3 it only remains to summarise what we have shown.

Proof of Theorem 4.3 The first item 4.3.1 follows from Proposition 4.24 and the comments following it. Item 4.3.2 is Proposition 4.25. Item 4.3.3 follows from Theorem 4.19 and was included in the proof of Proposition 4.26. Item 4.3.4 is the statement of Theorem 4.28. Item 4.3.5. was shown in proposition 4.29 and item 4.3.6 was given in Theorem 4.21. ∎

5 Counting Homotopy Types of Gauge Groups: A History.

In algebraic topology the study of gauge groups has its most primitive roots in the study of groups of self-equivalences. If G is a topological group and $p : E \to B$ is a principal G-bundle then its gauge group is a particular example of a group of self-equivalences and such groups have a rich history in algebraic topology. A good review of the topic is written by Arkowitz [11] where he cites a paper by Barrat and Barcus [23] as the first appearance in the literature of a self-homotopy equivalence group $\mathcal{E}(X) = \{f \in [X, X] \mid f \text{ invertible in } h\mathcal{T}\}$ of a space X, used by the authors in application to the classification of homotopy types. Of more relevance to us is the appearance of $\mathcal{E}(X)$ in a 1964 paper by Kahn [108] which studied the self-homotopy equivalences of Postnikov towers. In particular if $p_n : X_n \to X_{n-1}$ is the nth section then he shows there is an exact sequence $\mathcal{E}_F(X_n) \to \mathcal{E}(X_n) \to \mathcal{E}(X_{n-1})$ where $\mathcal{E}_F(X_n)$ is the group of *fibre homotopy equivalences* of X_n. Since p_n is a fibration this is just the group [134]

$$\mathcal{E}_F(X_n) = \{f \in \mathcal{E}(X_n) \mid p_n \circ f = p_n\}, \tag{5.1}$$

and the similarities with the more refined notion of a gauge group are immediate. In the same paper Kahn went on to show that $\mathcal{E}_F(X_n)$ is naturally a subset of $H^n(X^n; \pi_n X))$.

At around the same time Dold [65] was studying the general notion of fibre homotopy equivalence and produced many interesting results characterising the local and global natures of fibrations. Subsequent applications of fibre homotopy type to the classification of fibre bundles were given by Dold and Lashof [66], Stasheff [175] and Allaud [10]. Later Gottlieb [80], May [135], and Booth, Heath, Morgan and Piccinini [29] used groups of fibre homotopy equivalences to construct classifying spaces for fibrations with fibre F that classified such fibrations up to fibre-homotopy equivalence.

On the other side of the academic globe physicists had introduced the more rigid notion of a gauge group and were studying them in connection with the fundamental forces of nature. Gauge theory arose in physics during the early 20^{th} century as an attempt to provide a unified geometrical framework with which to simultaneously describe the gravitational and electromagnetic interactions [212]. In this setting the principal bundles being studied were often trivial $U(1)$-bundles and the associated gauge groups were not particularly rich in homotopy, with any interest in them being generally of a geometrical nature. It was not until around 1954 when Yang and Mills [143] began to generalise gauge theories to allow for non-abelian structure groups that the importance of the gauge groups to the physical theory began to become apparent.

A word on terminolgy is worthwhile for the mathematician interested in the underlying physics. For a principal G-bundle $p : P \to B$ it is common practice amongst physicists to call the structure group G the *gauge group*, and refer to the object for which we have reserved this term as the *group of gauge transformations*. We shall continue to use our definition (2.27) in the following.

In its modern incarnation gauge theory is used to study field theories associated with Lagrangians that are invariant under some *internal* symmetry group G that is not related to - and is independent of - the spacetime symmetry group. It has been used to great success in modern physics and the Standard Model of the weak, strong and electromagnetic interactions is a gauge theory [211]. To construct a gauge theory one starts with a spacetime manifold B, representing a portion of the universe in which we live, and a set of postulated field equations: the geometrical flavour enters through the use of a principal G-bundle $p : P \to B$ as a model.

A particle propagating through the universe B is described by a field ψ, defined to be a section of a vector bundle $q : P \times_G \mathbb{R}^n \to B$ associated to some n-dimensional representation of G. Specifying a connection ω on P allows one to define a covariant derivative on $P \times_G \mathbb{R}^n$ and so impose the field equations on ψ. It is customary amongst physicists to refer to the connection ω as the gauge connection.

Letting $\mathcal{A}(P)$ denote the space of connections on P, it is an affine space modelled on the vector space $\Lambda^1(B; ad(P))$ of 1-forms with values in the the adjoint bundle $ad(P) = P \times_{ad} \mathfrak{g}$ [15], where $\mathfrak{g}$ is the Lie algebra of G on which it acts through the adjoint representation $(g, X) \mapsto g \cdot X = gXg^{-1}$. A choice of G-invariant inner product on the Lie algebra $\mathfrak{g}$ induces inner products on the spaces of k-forms $\Lambda^k(B; ad(P))$ and one uses this to define the **Yang-Mills functional** $S_{YM} : \mathcal{A}(P) \to \mathbb{R}$, $\omega \mapsto \frac{1}{2}\|F_\omega\|^2$, where $F_\omega \in \Lambda^2(B; ad(P))$ is the curvature form of ω and $\|\cdot\|$ denotes the norm induced by the inner product. The Yang-Mills equations are then obtained as the Euler-Lagrange equations by requiring that S_{YM} remain stationary under variations of the connection.

Now the gauge group $\mathcal{G}(P)$ acts on $\mathcal{A}(P)$ from the right as $(\omega, f) \mapsto \omega \cdot f = f^*\omega$ for $\omega \in \mathcal{A}(P)$, $f \in \mathcal{G}(P)$ and two connections $\omega, \omega' \in \mathcal{A}(P)$ are said to be *gauge equivalent* if there exists a gauge transformation $f \in \mathcal{G}(P)$ such that $\omega \cdot f = \omega'$. In general one must restrict to the based gauge groups $\mathcal{G}_*(P)$ to obtain a free action and the quotient space $\mathcal{O}(P) = \mathcal{A}(P)/\mathcal{G}_*(P)$ is called the **moduli space of gauge potentials**. Its points represent gauge inequivalent connections. The Yang-Mills functional is invariant under the $\mathcal{G}(P)$-action and so factors over the orbits to give a well-defined function on $\mathcal{O}(P)$. It follows that the generally highly non-trivial topology of $\mathcal{O}(P)$ has a profound effect on the solution space of the Yang-Mills equations and so through its relation with the moduli space is the basic manner in which the topology of the gauge group $\mathcal{G}(P)$ becomes physically important.

It it at this stage that we see homotopy theoretic methods begin to enter into physics. In a gauge system the exchange bosons that mediate the interaction being described are represented by the gauge connections and in the Feynman path integral approach to quantisation the need arises to perform a functional integral over the space of physically inequivalent connections. Since the system is to be gauge-invariant each $\mathcal{G}(P)$-orbit in $\mathcal{A}(P)$ should be counted only once. That is, the integral should be performed over the moduli space $\mathcal{O}(P)$. A major difficulty with this approach is that the topology of $\mathcal{O}(P)$ is generally intractable and performing the relevant integral is a mathematically very difficult problem. A usual procedure to circumnavigate the difficulties is to attempt to define a section $\mathcal{O}(P) \to \mathcal{A}(P)$ and perform the integral over its image in $\mathcal{A}(P)$ in a procedure is called *gauge fixing*.

Now it was first observed by Gribov [81] and later studied more rigorously by Singer [173] that the topological non-triviality of the based gauge group $\mathcal{G}_*(P)$ is the primary obstruction to obtaining such a global section. Indeed, a global section section would only be present were there a $\mathcal{G}_*(P)$-bundle equivalence between $\mathcal{A}(P)$ and the trivial bundle $\mathcal{O}(P) \times \mathcal{G}_*(P)$. The easiest way to see that this is an impossibility was first observed by Singer [173]: note that the affine space $\mathcal{A}(P) \simeq *$ is contractible, therefore it only suffices to exhibit a single non-trivial homotopy group of $\mathcal{G}_*(P)$ to disprove the postulated equivalence by homotopy invariance. That $\mathcal{G}_*(P) \not\simeq *$ is certainly true for most examples of physical interest.

Atiyah and Jones subsequently took Singer's remarks to a slightly deeper level, making full use of the contractibility of $\mathcal{A}(P)$ to prove the following.

Theorem 5.1 (Atiyah, Jones [16]) *Let $P \to S^4$ be a principal G-bundle with G a simple, compact Lie group. Then there is a principal $\mathcal{G}_*(P)$-bundle*

$$\mathcal{G}_*(P) \to \mathcal{A}(P) \to \mathcal{O}(P) \tag{5.2}$$

with contractible total space. It follows that there are homotopy equivalences $\mathcal{O}(P) \simeq B\mathcal{G}_*(P) \simeq \Omega_0^3 G$. ∎

Perhaps the first paper to make a concerted application of homotopy theory to gauge theory was Atiyah and Bott's 1983 publication *The Yang-Mills Equations Over Riemann Surfaces* [14]. This ambitious paper ties together algebraic topology, algebraic geometry, differential geometry and even number theory with the unsuspecting thread of the Yang Mills equations. The aim of the paper is to obtain information about the moduli spaces of stable algebraic vector bundles over a given compact Riemann surface M and the authors's tool of choice is infinite dimensional Morse theory, which they use to study the critical points of the Yang-Mills functionals S_{YM} associated to a certain class of principal $U(n)$-bundles $p : P \to M$.

The focal objects of the paper are once again the moduli spaces, but the path that the authors take is to study the the homotopy of $\mathcal{O}(P)$ through the gauge groups $\mathcal{G}(P)$. In fact, the authors reason that the information they require need come this time from the full gauge groups, and not just from the based gauge groups, and the entire second section of the paper is devoted to their study and that of their classifying spaces. An important result that appears in this paper as Proposition 2.4 is the homotopy equivalence $B\mathcal{G}(P) \simeq Map_P(M, BG)$, here stated for G a Lie group and $P \to M$ a principal G-bundle over a smooth manifold M. Although this result can be traced back to the work of Gottlieb [79] many later authors attribute its origin to Atiyah and Bott's paper. The following theorem sumarises the pertinent results of the paper's second section.

Theorem 5.2 (Atiyah, Bott [14]) *Let $p : P \to M$ be a principal $U(n)$-bundle over a compact Riemann surface M with $n \geq 2$. Then $B\mathcal{G}(P)$ is torsion free in homology and has Poincaré series $P_t(B\mathcal{G}(P)) = \prod_{k=1}^{n}(1+t^{2k-1})^{2g} / \left(\prod_{k=1}^{n-1}(1-t^{2k})^2\right)(1-t^{2n})$. Moreover, if $\mathcal{G}' \leq \mathcal{G}(P)$ is a subgroup of finite index then $B\mathcal{G}'$ is torsion free in homology and $P_t(B\mathcal{G}') = P_t(B\mathcal{G}(P))$.* ∎

Atiyah and Bott's paper was incredibly influential and few published results relating to the homotopy theory of gauge groups appear in the wake of [14] whilst the community absorbed the paper's content. During the 1990's, however, there was a surge of interest in the homotopy types of gauge groups. In 1991 the Japanese mathematican Akira Kono published the first in a string of papers on the subject [125]. In this short paper he examined the gauge groups of $SU(2)$-bundles over S^4 and showed the following.

Theorem 5.3 (Kono [125]) *Let $\mathcal{G}_k(S^4, SU(2))$ be the gauge group of the princpal $SU(2)$-bundle over S^4 with second Chern class k. There is a homotopy equivalence $\mathcal{G}_k(S^4, SU(2)) \simeq \mathcal{G}_{k'}(S^4, SU(2))$ if and only if $(12, k) = (12, k')$.* ∎

A large part of the motivation behind Kono's paper may well have come from concurrent interest in the topology of *instanton* moduli spaces $\mathcal{M}_k$ associated with principal $SU(2)$-bundles over S^4 from the physics community. In the same year Boyer, Mann and Waggoner published [35], a paper extending the 1978 results of Atiyah and Jones [16]. Recall the curvature 2-form $F_\omega \in \Lambda^2(B; ad(P))$ of a connection ω on a given principal G-bundle $p : P \to B$. If B is a manifold of dimension 4 then the hodge dual operation $*(-)$ define an automorphism of $\Lambda^2(B; ad(P))$ and F_ω is said to be **G-instanton** if it satisfies $*F_\omega = F_\omega$. Such a curvature form is then a local minima of the Yang-Mills functional and the **instaton moduli space** $\mathcal{M}(P)$ is defined to be the set of $\mathcal{G}(P)$-equivalent G-instantons.

At this stage we pause to introduce some notation. Let B be a compact, connected 4-dimensional manifold and let G be a compact, connected, semisimple Lie group. Then isomorphism classes of principal G-bundles over B are indexed by their second Chern classes, and when G is understood we let $p_k : P_k \to B$ denote the k^{th} such bundle. We use $\mathcal{G}_k(B, G) = \mathcal{G}(P_k)$ to differentiate between the gauge groups of these bundles and we let $\mathcal{O}_k = \mathcal{A}(P_k)/\mathcal{G}_k(B, G)$ denote the moduli space of connections and $\mathcal{M}_k$ the instanton moduli space associated to the appropriate bundle.

In [16] Atiyah and Jones studied the instanton moduli spaces $\mathcal{M}_k$ associated with $SU(2)$-bundles over S^4. In this context $S^4 = \mathbb{R}^4 \cup \infty$ represents the one-point compactification of a real 4-dimensional spacetime. The physical significance of this compactification relates to the boundary conditions placed on solutions to conformally invariant field equations: that charges,and similar quantities should disappear at infinity (S^4 is conformally flat). The relation to the gauge groups is as follows. There is a canonical inclusion $\theta_k : \mathcal{M}_k \hookrightarrow \mathcal{O}_k \simeq \Omega^3_k SU(2) \simeq \Omega^3_0 S^3$ and Atiyah and Jones prove the next result.

Theorem 5.4 (Atiyah Jones [16]) *For $G = SU(2)$ the inclusion induces an epimorphism $H_q\mathcal{M}_k \to H_q\mathcal{O}_k$ for $q \ll k$ which is the projection onto a direct summand.* ∎

Appearing also in [16] is the famous *Atiyah-Jones Conjecture*, which comprised several statements; that 1) the inclusion θ_k induces an isomorphism in homology for a range dependent on k, that 2) the range of the surjection can be calculated explicitly as a function of k, and that 3) similar statements could be made for the induced map on homotopy. Building on their previous papers [33], [35], Boyer, Hurtubise, Mann and Milgram finally settled the long-standing conjecture in 1993 [34] when they in fact proved the slightly stronger result stated below.

In the interim Taubes [183] had related the various moduli spaces by constructing inclusions $\mathcal{M}_k \hookrightarrow \mathcal{M}_{k+1}$ that fit into homotopy commutative diagrams

$$\begin{tikzcd} \mathcal{M}_k \arrow[r, hook] \arrow[d, hook, "\theta_k"] & \mathcal{M}_{k+1} \arrow[d, hook, "\theta_{k+1}"] \\ \Omega_k SU(2) \arrow[r, "\simeq"] & \Omega_{k+1} SU(2). \end{tikzcd} \tag{5.3}$$

In light of the Atiyah-Singer conjecture, it is obviosuly an interesting question to study the space obtained as the limit of these inclusions as k grows large. To this end Taubes provided the following.

Theorem 5.5 (Taubes [183]) *Let $\mathcal{M}_\infty$ be the homotopy direct limit of the moduli spaces $\mathcal{M}_k$. Then the induced map $\mathcal{M}_\infty \to \Omega^3_0 SU(2)$ is a homotopy equivalence.* ∎

His result goes some way towards the Atiyah-Jones conjecture but says nothing about the stucture of $H_*\mathcal{M}_k$ for fixed k, which was after all the object of principal interest. Boyer, Hurtubise, Mann and Milgram used his result as the foundation in a proof of the following theorem.

Theorem 5.6 (Boyer, Hurtubise, Mann, Milgram [34]) *For $k \geq 0$ the inclusion $\mathcal{M}_k \hookrightarrow \mathcal{M}_{k+1}$ is $([k/2]-2)$-connected. In particular, $\mathcal{M}_k \hookrightarrow \Omega^3_0 SU(2)$ is $([k/2]-2)$-connected.* ∎

Now the moduli spaces $\mathcal{M}_k$ have proven to be mathematically important objects outside the realm of theoretical physics. On the one hand they have strong ties to the area of algebraic geometry, as we allueded to a connection in when discussing Atiyah and Bott's paper [14], and this connection has been investigated by Atiyah and Ward [18].

Theorem 5.7 (Atiyah, Ward [18]) *There is a natual correspondence between*

- *Gauge equivalence classes of anti-self-dual connections on principal $SU(2)$-bundles over S^4 with second Chern class $k > 0$.*
- *Isomorphism classes of 2-dimensional algebraic vector bundles E over $\mathbb{C}P^3$ satisfying*
    1. *E has a symplectic structure.*
    2. *The restriction of E to every real line of $\mathbb{C}P^3$ is algebraically trivial.* ∎

On the other hand the moduli spaces $\mathcal{M}_k$ have been proven to have significance in geometry. They were introduced into 4-dimensional geometry by Simon Donaldson in his attempts to answer a previous question of Milnor [145].

Theorem 5.8 (Donaldson [68]) *Let M be a smooth, compact, simply-connected oriented 4-manifold with positive definite intersection form Q_M. Then Q_M is equivalent over the integers to the standard diagonal form.* ∎

Although the theorem appears fairly innocuous its proof is quite complex and marks the introduction of the moduli spaces $\mathcal{M}_k$ into geometry. To prove the theorem Donaldson studies the Moduli space $\mathcal{M}_{-1}$ corresonding to the principal $SU(2)$-bundle over M with second Chern class $-1 \in H^4M$. He first relates $\mathcal{O}_{-1}$ and Q_M by showing that the number of components of the singular subset of $\mathcal{O}_{-1}$ is given by $n(Q_M) = \frac{1}{2}|\{\alpha \in H^2(M) \,|\, Q_M(\alpha) = 1\}|$. In this case the space of irreducible connections $\mathcal{M}_{-1} \subseteq \mathcal{O}_{-1}$ is a 5-dimensional manifold and there is a smooth 5-dimensional open submanifold $\mathcal{U} \subseteq \mathcal{M}_{-1}$ that is diffeomorphic to $M \times (0,1)$ with boundary the disjoint union of M and $n(Q_M)$ copies of $\mathbb{C}P^2$. Donaldson then makes a cobordism invariance argument to show that there must be at least $rank(Q_M)$ copies of $\mathbb{C}P^2$ appearing in the boundary and concludes that $n(Q_M) = rank(Q_M)$ to complete the proof.

Further results were obtained by Donaldson in [69] where he related the moduli spaces $\mathcal{M}_k$ to differential topological invariants of the manifold M that arise as pairings of the fundamental homology class $[\mathcal{M}_k]$ of the $(8k - 3(1 + b^+(M)))$-dimensional manifold with elements in the cohomology of $\mathcal{O}_k$, where $(b^+(M), b^-(M))$ is the signature of Q_M. The kernel of his work on 4-manifolds was later collected in the text [70].

Inspired by these later developments in Donalson's work, Masbaum [133] took up the broader problem of the examination of the cohomology of the classifying spaces of $SU(2)$-gauge groups over simply connected 4-dimensional Poincaré complexes X. The problem is exceptionally complex since the classifying spaces in question have dependence not only on particular bundle over X (classified again by the second Chern class in $H^4X = \mathbb{Z}$) but also on the topology of X, which becomes manifest through the parity of the intersection form on X. As such the paper includes partial results on both the cohomology of the classifying spaces of the based and full gauge groups, with the most complete results relating to the case $X = S^4$. Masbaum is particularly interested in *natural* cohomology classes in $H^*B\mathcal{G}_k(X, SU(2))$ (see the paper for a definition) and describes certain interesting subrings of the cohomology algebra. He gives $H^*(B\mathcal{G}_k(X, SU(2); \mathbb{Z}_2)$ up to extension for favourable spaces X and calculates the divisibility properties of certain elements in the homology module.

Donaldson's work was also a big inspiration for many algebraic topologists interested in dealing directly with the homotopy of the moduli spaces. R. Cohen and R. Milgam, who coauthored [51] in 1994, were one such pair. The paper itself contains few original results but rather compiles the most interseting and important statements from the authors' various surrounding projects into a cohesive survey, making the material accessible to algebraic topologists not specialised in the fields of geometry of physics. Since the authors had uncovered relationships between the moduli spaces and algebraic geometry [34], braid groups [**?**], [54], algebra [182] and combinatorics [33] this survey proved to be very influential.

At the same time other geometers had also taken to further exploring the implications of Donaldson's work and the importance of the homotopy and homology of certain instanton moduli spaces and gauge groups to the classification problem for simply connected 4-manifolds. In 1995 the Turkish mathematician Selman Akbulut published a series of three papers [5], [6], [8] looking closely at the algebraic topology of certain gauge groups. A major theme in these three papers is the study of odd-primary torsion classes in the integral homology $H_*(B\mathcal{G}(P); \mathbb{Z})$ of the classifying spaces of certain $U(2)$- and $SO(3)$-bundles $P \to B$ over a closed, simply connected 4-manifold B. The motivation being that these classes would be able to provide further 4-manifold invariants in a similar manner to the 2-primary classes that had previously been studied in [75], [133]. Another focal point of the papers is the low-dimensional homotopy $\pi_*B\mathcal{G}(P)$ which Akbulut uses to study relations between the various Donaldson invariants.

Although his main interests lie in the algebraic topology of $B\mathcal{G}(P)$, Akbulut's methods carry a very geometric feel. He makes prevalent use of the isomorphisms

$$\pi_r B\mathcal{G}(P) \cong \pi_r Map^P(B, BU(2)) \cong [S^r \times B, BU(2)]^P \tag{5.4}$$

to realise homotopy classes of maps $S^r \to B\mathcal{G}(P)$ geometrically as principal $U(2)$-bundles $E \to S^r \times B$ that restrict to $P \to B$ over any slice $x_0 \times B \subseteq S^r \times B$. When combined with information coming from the low-stages of a Postnikov decomposition of $BU(2)$ that Akbulut also supplies in the papers, this becomes a very powerful technique and he supplies 3-primary information on $\pi_r B\mathcal{G}(P)$ up to degree $r = 7$.

The next interesting paper dealing directly with the homotopy types of gauge groups was written by Sutherland in 1992 [181]. Sutherland had a history in both the topology of function spaces [180] and fibrewise homotopy theory [179], and had succesfully begun to combine the two aspects in papers such as [59], [58]. The paper [181] itself deals primarily with the homotopy types of $U(n)$-bundles over Riemann surfaces and we collect several of its results in the following omnibus theorem which provides a good example of Sutherland's combination of fiberwise homotopy theory with more the more standard homotopy theoretic techniques pioneered by Hansen [93]. In the next statement the bracket notation (k,l) is used to denote the greated common divisor of the integers $k,l \in \mathbb{N}$, and we shall continue to use this notation for the sequel without further mention.

Theorem 5.9 (Sutherland [181]) *Let X_g be a compact Riemann surface of genus g and let $B\mathcal{G}_k(U(n))$ be the classifying space for the gauge group associated with the principal $U(n)$-bundle with first Chern class $k \in H^2X_g = \mathbb{Z}$. Then for all integers k there is a homotopy equivalence $B\mathcal{G}_k(U(n)) \simeq B\mathcal{G}_{k+n}(U(n))$. Moreover if $B\mathcal{G}_k(U(n)) \simeq B\mathcal{G}_l(U(n))$ then $(k,n) = (l,n)$ and conversely, if $(k,n) = (l,n)$ then there is a homotopy equivalence $B\mathcal{G}_k(U(n))^\wedge_p \simeq B\mathcal{G}_l(U(n))^\wedge_p$ after completion at any prime p.*

Proof (Sketch) The first statement is proved using the action of $BU(1)$ on $Map(X_g, BU(n))$ that is induced by the tensor product. The second stament follows from a careful examination of Samelson products in $U(n)$ after applying previous results of both Lang [130] and Bott [31]. The final statement is demonstrated using the action of the unstable Adams operations on the p-completion $BU(n)^\wedge_p$. ∎

After completing the proof of this statement Sutherland goes on to give partial results relating to $SU(n)$- and $Sp(n)$-gauge groups over S^4. These are mainly non-equivalence results, obtained through use of Bott's calculations of Samelson products in $SU(n)$ and $Sp(n)$.

Theorem 5.10 (Sutherland [181]) *If $B\mathcal{G}_k(S^4, SU(n)) \simeq B\mathcal{G}_l(S^4, SU(n))$ then $\left(k, \frac{n(n^2-1)}{(n+1,2)}\right) = \left(l, \frac{n(n^2-1)}{(n+1,2)}\right)$. If $B\mathcal{G}_k(S^4, Sp(n)) \simeq B\mathcal{G}_l(S^4, Sp(n))$ then $(k, n(2n+1)) = (l, n(2n+1))$ if n is even or $(k, 4n(2n+1)) = (l, 4n(2n+1))$ if n is odd.* ∎

Following this the next paper of interest [127] appears in 1996. Written by Kono and his student Tsukuda the paper reprises the earlier work [125] and extends the results of that paper to the gauge groups of $SU(2)$-bundles over simply connected 4-manfolds B. Isomorphism classes of $SU(2)$-bundles over B are again classified by the value of their second Chern class but in this case the signature of the intersection form Q_B on B enters the results. The main result of the paper is as follows.

Theorem 5.11 (Kono, Tsukuda [127]) *There is a homotopy equivalence $\mathcal{G}_k(B, SU(2)) \simeq \mathcal{G}_{k'}(B, SU(2))$ if and only if $(12/d(B), k) = (12/d(b), k')$ where $d(b) = 1$ if Q_B is even and $d(B) = 2$ if Q_B is odd.* ∎

The result is proved using basic homotopy theory to examine the evaluation fibrations for $B\mathcal{G}_k(B, SU(2))$ and $B\mathcal{G}_k(S^4, SU(2))$, comparing the two fibrations with the map induced by the pinch map $B \to S^4$. Kono and Tsukuda also include a section in which they give a geometric interpretation of their theorem. The decision to include this section clearly shows the authors' desire to relate their work to that of their contempory geometers discussed above. This section also displays Tsukuda's influence on the project and around this paper the young mathematician published a string of short papers contain minor related results [202], [204], [205], [207], many of which share this geometrical flavour.

Tsukuda also spent time examining the classifying spaces of the gauge groups [203], [128], [208]. Since homotopy equivalence is a weaker notion than that of an isomorphism groups it is interesting to ponder which, if any, of the homotopy equivalences in Theorems 5.3 and 5.11 can be realised as isomorphisms of (infinite-dimensional Lie) groups. Obviously a homotopy equivalence $B\mathcal{G}_k \simeq B\mathcal{G}_{k'}$ is a required condition for a group isomorphism $\mathcal{G}_k \cong \mathcal{G}_{k'}$ and this is the motivation for Tsukuda's work. The most complete result in this direction appears in the joint paper with Kono [128].

Theorem 5.12 (Kono, Tsukuda [128]) *Let B be an oriented, simply connected, closed 4-manifold. Then $Map^k(B, BSU(2)) \simeq Map^l(B, BSU(2))$ if and only if*

$$\begin{cases} |k| = |l| & \textit{if } B \textit{ admits an orientation reversing homotopy equivalence} \\ k = l & \textit{otherwise} \end{cases} \quad \blacksquare \tag{5.5}$$

The proof depends on intimate knowledge of the cohomology of $Map^k(B, BSU(2))$ and the paper contains corollaries that extend some of Masbaum's results in [133] discussed above. The work in this paper is also related to later work by Crabb and Sutherland, and Tsutaya on A_n-equivalence classes of gauge groups that we will discuss later.

We come now to a turning point in the subject of the algebraic topology of gauge groups and from our particular point of view the next paper to discuss is possibly the single most important paper on the subject. It marks where the homotopy theory of gauge groups becomes of mathematical interest in its own right and ceases to be merely an accessory to deeper geometric and physical applications. The paper in question is *Counting Homotopy Types of Gauge Groups* by Crabb and Sutherland [61] in which they prove the following.

Theorem 5.13 (Crabb, Sutherland [61]) *Let X be a connected, finite complex and let G be a compact, connected Lie group. Then the number of distinct homotopy types amongst the gauge groups of principal G-bundles over X is finite. Moreover the number of H-equivalence classes of these objects is also finite.* $\blacksquare$

The notion of *H-equivalence* introduced here is that of equivalence of H-spaces: two H-spaces G, G' are said to be **H-equivalent** if there is a homotopy equivalence $K \to K'$ that is also an H-map. It is obviously a stronger notion than homotopy equivalence but weaker than group isomorphism if G, G' are topoogical groups.

This theorem puts the previous results, for instance Theorem 5.3, into context and goes some way to explaining why only 6 distinct homotopy types appear amongst the $\mathcal{G}_k(S^4, SU(2))$. Crabb and Sutherland address Kono's 1991 paper and use new fibrewise methods to enumerate the H-equivalence classes of $SU(2)$-gauge groups over S^4, showing that there are precisely 18 distinct H-types amongst the $\mathcal{G}_k(S^4, SU(2))$.

Theorem 5.14 (Crabb, Sutherland [61]) *$\mathcal{G}_k(S^4, SU(2))$ is H-equivalent to $\mathcal{G}_{k'}(S^4, SU(2))$ if and only if $(180, k) = (180, k')$.* $\blacksquare$

Following this we see the publication of several interesting papers. The first of these is a paper is coauthered by Bauer, Crabb and Spreafico, and appears in 2001 [25]. It sees Crabb continue his use of fibrewise techniques to examine the stable homotopy type of the classifying space of $SO(3)$-gauge groups over S^2. Since $\pi_2 BSO(3) = \pi_1 SO(3) = \mathbb{Z}_2$ there are, up to isomorphism, only two distinct $SO(3)$-bundles $P \to S^2$, one of which is the trivial bundle.

Theorem 5.15 (Bauer, Crabb, Spreafico [25]) *There is a stable decomposition*

$$B\mathcal{G}_P(S^2, SO(3)) \simeq_S \begin{cases} BSO(3)_+ \vee \bigvee_{k\geq 1\ odd} BU(1)^{kH}, & \textit{if } P \textit{ is trivial} \\ \bigvee_{k\geq 0\ even} BU(1)^{kH}, & \textit{if } P \textit{ is non-trivial} \end{cases} \tag{5.6}$$

Where H is the complex Hopf line bundle over $BU(1) = \mathbb{C}P^\infty$ and $BU(1)^{kH}$ is the Thom space of the k-fold tensor product of H with itself. ∎

The motivation for the calculation was cohomological. Indeed they cite the previous paper by Tsukuda [206] and extend its results to give a complete description of the integral cohomology ring $H^*B\mathcal{G}_P(S^2, SO(3))$. An auxiliary result of interest in the paper is a fibrewise generalisation of the James-Milnor stable decomposition $\Omega\Sigma F$ of a connected, pointed space F.

Another one-off paper published during this period was written by Terzić [184]. This short paper computes the rational homotopy groups and rational cohomology of $B\mathcal{G}_k(B, G)$ and $\mathcal{G}_k(B, G)$ where B is a compact simply connected 4-manifold and G is a semisimple, compact, simply connected Lie group. A second paper from this period that deals with the subject of the rational homotopy of gauge groups is the paper [218] by Wockel.

Later Felix and Oprea [73] obtained general results for the rational homotopy type of the classifying space $B\mathcal{G}(X, G)$ of the gauge group of any principal G-bundle $P \to X$ over a finite complex X when G is any compact, connected Lie group. Their results generalised those of both Terzić and Wockel and may be considered strong enoguh to have completely settled the subject of the rational homotopy of such gauge groups. An interesting result they obtain is the rational homotopy of the universal gauge group, that is, the gauge group $\mathcal{G}(G) = \mathcal{G}(EG)$ of the universal G-bundle $\pi_G : EG \to BG$.

Theorem 5.16 (Felix, Oprea [73]) $\pi_q\mathcal{G}(G) \otimes \mathbb{Q} = \oplus_{r\geq 0} H^r(BG; \mathbb{Q}) \otimes \pi_{q+r}G.$ ∎

At this point on the timeline we see a flourish of papers from Kono and a new collaborator, a young mathematician by the name of Hamanaka. Their first paper together is [91] in which they examine the homotopy types of $SU(3)$-gauge groups over S^6. These $SU(3)$-bundles are classified by the integer value of their thrird Chern class, and although the motivation for the authors' choice of structure group and base space is not exactly clear, the concise and pleasant answer to their questions makes for good incentive.

Theorem 5.17 (Hamanaka, Kono [91]) *There is a homotopy equivalence* $\mathcal{G}_k(S^6, SU(3)) \simeq \mathcal{G}_{k'}(S^6, SU(3))$ *if and only if* $(120, k) = (120, k')$. ∎

This paper sees the introduction of *unstable K-theory* into the subject. This was an invention of Hamanaka and Kono's that made its first appearance in their joint paper [88] as an extension of techniques previously used by Hamanaka in [83], [84]. Unstable K-theory proves to be very fruitful in applicatin to gauge groups and it is the driving force behind the duo's paper [90] in which they improve Sutherland's lower bounds (c.f. Theorem 5.10) for $SU(n)$-gauge groups over S^4 and completely determine the number of distinct homotopy types of $SU(3)$-gauge groups.

Theorem 5.18 (Hamanaka, Kono [90]) *If* $\mathcal{G}_k(S^4, SU(n)) \simeq \mathcal{G}_{k'}(S^4, SU(n))$ *then* $(n(n^2-1), k) = (n(n^2-1), k')$. *Moreover there is a homotopy equivalence* $\mathcal{G}_k(S^4, SU(3)) \simeq \mathcal{G}_{k'}(S^4, SU(3))$ *if and only if* $(24, k) = (24, k')$. ∎

Continuing the tradition of studying the homology and cohomology of gauge groups and their classifying spaces, a Korean mathematician Choi, who had been a student of Ravenel, published several paper on the subject around the period 2006 to 2008 [45], [45], [47], [48]. Choi had previous studied the homology of instanton moduli spaces [44] and had a keen interest in the homology of iterated loop spaces of compact Lie groups and homogenous spaces.

Choi was also involved in a one-off collaboration with Yoshihiro Hirato and Mamoru Mimura in a more homotopy oriented paper [49]. As its name suggests the primary methods employed in the paper are the compositional techniques for which Mimura became famous for his work with Toda on the homotopy theory of compact Lie groups. The results given in the paper for $SU(3)$-gauge groups were already known to Hamanaka and Kono and the results given for $Sp(2)$-gauge groups are partial. What makes the paper stand out is its attempt to bring new techniques into the study of the homotopy of gauge groups.

Theorem 5.19 (Choi, Hirato, Mimura [49]) *If* $(40, l) = (40, l)$ *then there is a homotopy equivalence* $\mathcal{G}_k(S^4, Sp(2)) \simeq \mathcal{G}_l(S^4, Sp(2))$. *It follows from the bounds in Theorem 5.10 that there are either* 4, 6 *or* 8 *distinct homotopy types amongst these gauge groups.* ∎

Moving back to Europe now we see a major contribution to our subject of choice from the mathematican Stephen Theriault, who at the time was working out of the University of Aberdeen alongside Crabb, Kono and others. Theriault had written his doctoral thesis, 'A Reconstruction of Anick's Spaces', under Paul Selick, studying a certain fibration $S^{2n+1} \to T^{2n-1}_{\infty}(p^r) \to \Omega S^{2n+1}$ originally constructed by David Anick in unrefereed work. The study required intimate knowledge of the homotopical structures of H- and co-H-spaces as well as the properties of various p-localised Samelson and Whitehead products. This was all knowledge that would serve him well as he moved on to examine first the p-primary homotopy types of Lie groups [185], [186], [64], and then to look at the homotopy types of gauge groups.

Theriault's first paper on the subject of gauge groups was [187], published in 2010, in which he extended the p-primary decompositions of compact Lie groups given by Mimura, Nishida and Toda [148] to analogous decompositions for certain gauge groups. He provides decompositions for the gauge groups $\mathcal{G}_k(B; G)$ where B is a simply connected, compact 4-manifold and G is a simple, simply connected compact Lie group, as well as the gauge groups $\mathcal{G}_k(\Sigma_g, U(n))$ where Σ_g is an orientable Riemann surface of genus g and the $U(n)$-bundles are classified by their first Chern classes. The most relevant of the results contained in the paper is the following.

Theorem 5.20 (Theriault [187]) *If $2 \leq n \leq (p-1)(p-2)+1$ then there is a homotopy equivalence*

$$\mathcal{G}_k(S^4, SU(n)) \simeq X_k \times Y_k \times \left(\prod_{j=1,\, j\neq\mu,\nu}^{p-1} B_j \right) \times \left(\prod_{i=1,\, i\neq u,v}^{p-1} \Omega_0^4 B_i \right). \tag{5.7}$$

Here the spaces B_i appear in a mod p homotopy decomposition $SU(n) \simeq \prod_{i=1}^{p-1} B_i$ and B_μ, B_ν are the factors which carry the degree $2n-3$ and $2n-1$ generators in homology, respectively. Moreover X_k, Y_k are the homotopy fibres of certain maps $B_\mu \to \Omega_0^3 B_u$, $B_\nu \to \Omega_0^3 B_v$. ∎

In 2010 Theriault also published a second paper on gauge groups [188], which treated $Sp(2)$-gauge groups over S^4. If we view Kono's influential 1991 paper [125] as having set the stage by enumerating the homotopy types of S^3-gauge groups over S^4, then there are two obvious directions of generalisation to persue. Kono and Hamanaka [90] had already started towards the first by taking a cue from the isomorphism $S^3 \cong SU(2)$ and counting the homotopy types of $SU(3)$-gauge groups over S^4, and Theriault's paper here took the other direction, cueing off of the isomorphism $S^3 \cong Sp(1)$.

The main result of Theriaults paper, given below, should be compared with the previous results of Choi, Hirato and Mimura given in Theorem 5.19.

Theorem 5.21 (Theriault [188]) *The following hold:*

1. *If there is a homotopy equivalence $\mathcal{G}_k(S^4, Sp(2)) \simeq \mathcal{G}_l(S^4, Sp(2))$ then $(40, k) = (40, l)$.*

2. *If $(40, k) = (40, l)$ then there is a local homotopy equivalence $\mathcal{G}_k(S^4, Sp(2) \simeq \mathcal{G}_l(S^4, Sp(2))$ when localised rationally or at any prime.* ∎

It is essentially the appearance of the free factor in $\pi_7 Sp(2) = \mathbb{Z}$ which made extracting an integral homotopy equivalence for the statement much more delicate than, say, the case of $SU(2)$ for which Kono had appealed to a Sullivan arithmetic square in which both spaces on the bottom row were weakly contractible. In an attempt to circumvent the added complications Theriault had provided a certain lemma, which appears in [188] as Lemma 3.1. This result would prove to be beneficial to the future research of several mathematicians.

Lemma 5.22 (Theriault [188]) *Let Y be an H-space with inverse and let $f : X \to Y$ be a map of finite order m. Let F_k be the homotopy fibre of $k \cdot f : Y \to Y$, then if $(m, k) = (m, l)$ there is a local homotopy equivalence $F_k \simeq F_l$ after localisation rationally or at any prime p.* ∎

It is after this that Theriault relocates to the University of Southampton, where he goes on to publish several more papers on the subject of gauge groups. He gives homotopy decomposition of $U(n)$-gauge groups over orientable and non-orientable Rieman surfaces [189], [193], examines the homology of $SU(n)$ and $Sp(n)$-gauge groups over S^4 [191] and extends Hamanaka and Kono's results on $SU(3)$-gauge groups over S^4 to $SU(3)$-gauge groups over simply connected 4-manifolds [190]. He also further extends Kono's original project by studying the homotopy types of $SU(5)$-gauge groups over S^4, a venture made easier by the introduction of his Lemma 5.22.

Theorem 5.23 (Theriault [194]) *There is a homotopy equivalence* $\mathcal{G}_k(S^4, SU(5)) \simeq \mathcal{G}_{k'}(S^4, SU(5))$ *when localised at any prime* p *or rationally if and only if* $(120, k) = (120, k')$. ∎

Other notable works of his from this period include a joint paper [119] with Kono and another Japanese mathematican, Kishimoto, who would go on to produce much important work relating to the homotopy theory of gauge groups. The paper looks at homotopy commutativity in p-local G-gauge groups over S^4, for G a simple, simply connected, compact Lie group.

Theorem 5.24 (Kishimoto, Kono, Theriault [119]) *Let* G *be a simple, simply connected, compact Lie group of type* $(n_1, \ldots, n_l)$ *and let* $p > 2n_l + 1$ *be prime. Localise at* p*. Then* $\mathcal{G}_k(S^4, G)$ *is homotopy commutative.* ∎

The motivation for this study is as follows. Let G be as in the theorem statement, then for $p > 2n_l$ the mutiplication on G becomes homotopy commutative by [137]. Since the gauge groups $\mathcal{G}_k(S^4, G)$ are so intimately connected to the structure group G there are several reasons why it is natural to ask if the feature of homotopy commutativty is transfered from G to $\mathcal{G}_k(S^4, G)$.

Firstly is the appealing nature of abelian gauge theory. Recall that for a G-bundle $P \to B$ there is a representation of the gauge groups $\mathcal{G}(P)$ as the group of functions $\varphi : P \to G$ satisfying $\varphi(e \cdot g) = g^{-1}\varphi(e)g$, for $e \in P$, $g \in G$. Obviously if G is abelian then $\varphi(e \cdot g) = \varphi(e)$, so φ descends to a well-defined function $\bar{\varphi} : B \to G$ and there ensues a group isomorphism $\mathcal{G}(P) \cong Map(B, G)$. Thus in the case of abelian structure group G all the gauge groups of G-bundles over B share a common homotopy type. Moreover, the homotopy type of the mapping space $Map(B, G)$ is generally more easily understood than that of the equivariant mapping space $\mathcal{G}(P) = Map^G(P, G)$.

Now the simple nature of abelian gauge theory motivates a question: *is* G *required to be (homotopy) abelian for* $\mathcal{G}(P)$ *to be (homotopy) abelian?*. This was a question that had previously been raised by Crabb and Sutherland [60] and partially answered in the negative

Theorem 5.25 (Crabb, Sutherland [60]) *Let* $G(n) = SO(n), SU(n)$ *or* $Sp(n)$ *with* $n \geq 4$, $n \neq 7$ *and* $d = 1, 2$ *or* 4 *for each of the respective groups. Let* $P \to X$ *be a principal* $G(n)$*-bundle over a connected complex* X *of dimension* $dimX < d(n+1)/2$*. Then* $\mathcal{G}(P)$ *is not homotopy abelian.* ∎

To this result Kishimoto, Kono and Theriault add the following.

Theorem 5.26 (Kishimoto, Kono and Theriault [119]) *Let* G *be a simply connected, simple, compact Lie group and* $\mathcal{G}_k(G)$ *the gauge groups of the principal* G*-bundle* $P_k \to S^4$ *classified by the degree* k *map* $S^4 \to BG$*. Then the gauge group* $\mathcal{G}_k(G)$ *is not homotopy commutative at* p *in the following cases.*

1. $G = SU(n)$ *and* $p < 2n$*, except possibly for i)* $p = 2$ *and* $n \in \{2, 3, 4\}$*, or ii)* $p = 3$, $n \in \{2, 3\}$ *and* $(3, k) = 1$.

2. $G = SU(n)$ *for* $p = 2n + 1$ *when* $k \not\equiv 0 \pmod{p}$.

3. $G = Sp(n)$, $n \geq 2$ and $p < 4n$ except possibly for $n = 2$ and $p \in \{2, 3, 5\}$.

4. $G = Spin(2n+1)$, $n \geq 3$ and $p < 4n$.

5. $G = Spin(2n)$, $n \geq 4$ and $p < 4(n-1)$. ∎

At this stage we have been introduced to the young Japanese mathematician Kishimoto. His most profound contributions to algebraic topology are perhaps his studies relating to homotopy commutativity in p-local Lie groups [96], [109], [112], [116], [124] and in p-compact groups [110]. Of course much of this work on homotopy commutativity has strong links to understanding the homotopy types of gauge groups and Kishimoto is responsible for many interesting articles on the homotopy theory of gauge groups. In [111], Kamiyama, Kishimoto, Kono and Tsukuda prove the following result for $SO(3)$-gauge groups, providing an interesting complement to the case of $SU(2) \cong Spin(3)$-gauge groups studied previously by Kono.

Theorem 5.27 (Kamiyama, Kishimoto, Kono, Tsukuda [111]) *Let $\mathcal{G}_k(S^4, SO(3))$ be the gauge group of the principal $SO(3)$-bundle $P_k \to S^4$ classified by the degree k map $S^4 \to BSO(3)$. Then $\mathcal{G}_k(S^4, SO(3)) \simeq \mathcal{G}_l(S^4, SO(3))$ if and only if $(12, k) = (12, l)$.* ∎

A similar paper from the same period is [87] which uses unstable KSp-theory [154] to calculate the order of the Samelson product $\langle \epsilon_7, id_{Sp(2)} \rangle : S^7 \wedge Sp(2) \to Sp(2)$, where $\epsilon_7 \in \pi_7 Sp(2) = \mathbb{Z}$ is a generator, and obtain information about the gauge groups of principal $Sp(2)$-bundles over S^8

Theorem 5.28 (Hamanaka, Kaji, Kono [87]) *The order of $\langle \epsilon_7, id_{Sp(2)} \rangle$ is 140. If $\mathcal{G}_k(S^8, Sp(2))$ denotes the gauge group of the principal $Sp(2)$-bundle over S^8 with second symplectic Pontryagin class k, then there is a homotopy equivalence $\mathcal{G}_k(S^8, Sp(2)) \simeq \mathcal{G}_{k'}(S^8, Sp(2))$ if and only if $(140, k) = (140, k')$.* ∎

It is worth introducing Kaji at this point, another young Japanese mathematician with interest in homotopy commutativity and gauge groups. His name appears on many important papers and his contribution to our subject is deep.

Kishimoto has also contributed to the paper [121] written with Theriault and Tsutaya that gives partial results for the number of G_2-gauge groups over S^4.

Theorem 5.29 (Kishimoto, Theriault, Tsutaya [121]) *Let $\mathcal{G}_k(S^4, G_2)$ be the gauge group of the principal G_2-bundle $P_k \to S^4$ classified by the degree k map $S^4 \to BG_2$. Then*

1. *If there is a homotopy equivalence $\mathcal{G}_k(S^4, G_2) \simeq \mathcal{G}_l(S^4, G_2)$, then $(84, k) = (84, l)$.*

2. *If $(168, k) = (168, l)$ then there is a local homotopy equivalence $\mathcal{G}_k(S^4, G_2) \simeq \mathcal{G}_l(S^4, G_2)$ when localised rationally or at any prime.* ∎

Note that this last theorem is not best possible. The paper's methodology is similar to many previous papers, attempting first to determine the order of the Samelson product $\langle \epsilon_3, id_{G_2} \rangle : S^3 \wedge G_2 \to G_2$, where $\epsilon_3 \in \pi_3 G_2$ is a generator, and then apply Theriault's Lemma 5.22 to draw its conclusions. The difficulty in this case is that G_2 is rank 2 but 14-dimensional. There is simply too large a quantity of unstable information to proceed with standard techniques. The authors study the order of $\langle \epsilon_3, id_{G_2} \rangle$ at each prime individually and completely determine its order at all odd primes. At the prime 2, however, they only bound the order to be 4 or 8. Hence the appearance of the number 168 in the second part of the statement versus 84 in the first part.

The difficulty arising from the complexity of the group G_2 is a first example of the difficulties that are faced when studying the gauge groups associated to higher rank and more complex structure groups. An analogous problem is found by increasing the topological complexity of the base space B of the principal G-bundle and this explains why most of the work on the homotopy types of gauge groups has been performed for spheres and simply connected 4-manifolds. It may well be that new methods are required to say much more of interest on the homotopy types of gauge groups of principal G-bundles $P \to B$ with more topologically complex structure groups G and base spaces B.

On the other hand there is also the possibility of extract new types of information from the gauge groups and of delving deeper into their geometric and algebraic structures than just studying their homology or their homotopy types. In this direction Kishimoto has been on the forefront of both problems, the paper [113], written jointly with Kono, being a prime example. Taking inspiration from Crabb and Sutherland [61], who studied gauge groups as H-spaces, and Kono and Tsukuda [128] who studied gauge groups as loop spaces, Kishimoto and Kono propose to study gauge groups as A_n-spaces and, in particular, to study the exact sequence of A_n-spaces

$$1 \to \mathcal{G}_*(P) \to \mathcal{G}(P) \xrightarrow{ev} G \to 1 \tag{5.8}$$

given by evaluation at the basepoint of P. The name of the paper derives from the authors' question: *when does* (5.8) *admit an A_n-splitting?*

Of course their first problem was to formulate what exatly A_n-splitting should mean in this context, and to this end the paper sees the introduction of several new ideas. Firstly the algebraic notion of a fibrewise A_n-space and fibrewise A_n-map is introduced, generalising the well-known notions of fibrewise H-space and fibrewise H-map. Classically the combinatorial definition of an A_n-space is accompanied by a geometric condition: associated to an A_n-space X is a series of projective spaces P_iX, $i = 1, \ldots, n$, whose existence gives a necessary and sufficient condition for the space to be an A_n-space [176]. Kishimoto and Kono briefly comment on fibrewise situation, referring to the fibrewise Milnor construction [57] and subsequently using the fibrewise Dold-Lashof construction to construct fibrewise projective planes for the gauge groups. They go on to give combinatorial and geometric definitions for fibrewise A_n-spaces and show that these definitions share a similar relationship as in the classical case.

The motivation for the introduction of these concepts in the context of gauge theory is the adjoint bundle $Ad(P)$, which is a fibrewise topological group, and the representation of $\mathcal{G}(P)$ as the space of sections $\Gamma(B, Ad(P))$. With their definition of an A_n-splitting, Kishimoto and Kono obtain the following.

Theorem 5.30 (Kono, Kishimoto [113]) *There is an A_n-splitting of* (5.8) *if and only if $Ad(P)$ is fibrewise A_n-equivalent to the trivial bundle $B \times G$.* ∎

The second question the authors ask relates to the geometric interpretation of an A_n-splitting and in their attempt to answer it this they introduce $H(k,l)$-spaces. These spaces are related to Aguadé's T_k-spaces [4], Felix and Tanré's $H(n)$-spaces [74] and Hemmi and Kawamoto's $H_k(n)$-spaces [99].

Definition 5.31 *Let $i_k : P_k\Omega X \to P_\infty\Omega X \simeq X$ be the inclusion of the k^{th} projective plane $P_k\Omega X$ of ΩX. Then X is said to be an $H(k,l)$**-space** if there is an extension $m_{k,l} : P_k\Omega X \times P_l\Omega X \to X$ making the following diagram commute up to homotopy*

$$\begin{tikzcd} P_k\Omega X \vee P_l\Omega X \arrow[r, "(i_k,i_l)"] \arrow[d, hook] & X \arrow[d, equal] \\ P_k\Omega X \times P_l\Omega X \arrow[r, "m_{k,l}"] & X \end{tikzcd} \tag{5.9}$$

where the left-hand vertical map is the canonical inclusion. □

Note that an H-space is an $H(\infty,\infty)$-space and the loop space of an $H(1,1)$-space is a homotopy commutative H-space. Note also that an $H(k,l)$-space is an $H(k',l')$-space for $k' \leq k$, $l' \leq l$. It follows that the loop space of an $H(k,l)$-spaces is an intermediate object between an H-space satisfying some higher homotopy commutativity conditions and the loopspace of an H-space (which is a Sugawara C_∞-space).

The $H(k,l)$-spaces give useful geometric criteria with which to study the A_n-splitting problem. Using their definition Kishimoto and Kono prove the following.

Theorem 5.32 (Kishimoto, Kono [113]) *There is an A_n-splitting of* (5.8) *if BG is an $H(k,n)$-space and $cat(X) \leq k$.* ▮

The notion of $H(1,n)$-space is also sufficiently powerful as to let them prove a much deeper result. Let $\pi_G : EG \to BG$ be the universal G-bundle with contractible total space and let $\mathcal{G}(G) = \mathcal{G}(EG)$ denote its gauge group. This bundle is classified by the identity $id_{BG} : BG \to BG$, so $B\mathcal{G}(G) \simeq Map^{id_G}(BG,BG)$, $B\mathcal{G}_*(G) \simeq Map_*^{id_G}(BG,BG)$ and therefore the connecting map in the 'universal' evaluation fibration is a map $\delta_G : G \to Map_*^{id_G}(BG,BG)$.

Theorem 5.33 (Kishimoto, Kono [113]) *The connecting map δ_G is given for $g \in G$ by $\delta_G(g) = Bad(g)$, where $Bad(g) : BG \to BG$ is the map that classifies the homomorphism $ad(g) : G \to G$, $h \mapsto g^{-1}hg$.* ▮

Let $P \xrightarrow{p} X$ be a principal G-bundle classified by a map $\alpha : X \to BG$. Then the universal property of δ_G comes from the fact that we can write $\alpha = id_{BG} \circ \alpha$ and use this to factor $\delta_P \simeq \alpha^* \circ \delta_G$, where $\delta_P : G \to Map_*^\alpha(X,BG) \simeq B\mathcal{G}_*(P)$ is the connecting map for the evaluation fibration sequence of the gauge group $\mathcal{G}(P)$ of P and $\alpha^* : Map^{id_{BG}}(BG,BG) \to Map_*^\alpha(X,BG)$ is the induced map of mapping spaces.

Although it will generally be difficult to work with the map δ_G and the gauge group $\mathcal{G}(G)$ due to the infinte dimensional constuction of the universal bundle π_G, the geometric approach bestows the theorem a certain attraction. In particular it serves as a primary example of the way in which their geometric techniques may be applied to study the more general problem of the connecting map $\delta_f : \Omega Y \to Map_*^f(X,Y)$ of the evaluation fibration associated to any map $f : X \to Y$ between spaces X, Y.

We have already mentioned the relationship between $H(k,n)$-spaces and Hemmi and Kawamoto's $H_k(n)$-spaces. In [99] Hemmi and Kawamoto define a class of topological monoids satisfying certain higher homotopy commutativity conditions whose members they call $C_k(n)$-spaces, and they show that the loopspace of an $H_k(n)$ is a $C_k(n)$-space. It stands to reason that the loops on an $H(k,l)$-space will also satisfy some higher homotopy commutativity conditions and this idea is explored by Kishimoto and Kono in the final section of [113]. Following Hemmi and Kawamoto they define $C(k,l)$-spaces and prove the following relation with the $H(k,l)$-spaces.

Theorem 5.34 (Kishimoto, Kono [113]) *A connected topological monoid G is a $C(k,l)$-space if and only if its classifying space is an $H(k,l)$-space.* ∎

It follows from this that if G is a $C(k,l)$-space and $P \xrightarrow{p} X$ is a principal G-bundle, then there is an A_n-splitting of (5.8).

The idea of studying the higher homotopy associativity and higher homotopy commutativity of gauge groups was something that Kishimoto would go on to develop further together with his student Tsutaya. Writing under Kishimoto's tutelage, Tsutaya took to the study of A_n-equivalences of gauge groups in [209]. Building on the previous work of Crabb and Sutherland [61], the main result of Tsutaya's paper was a direct generalisation of Theorem 5.13.

Theorem 5.35 (Tsutaya [209]) *Let X be a finite complex, G a compact, connected Lie group and $n \geq 1$ a finite positive integer. Then as P ranges over all principal G-bundles with base X, the number of A_n-equivalence types amongst the gauge groups $\mathcal{G}(P)$ is finite.* ∎

In this statement an A_n**-equivalence** between A_n-spaces G, H is a homotopy equivalence $\varphi : G \xrightarrow{\simeq} H$ that is also an A_n-map, that is, admits an A_n-form [102]. Note that an A_1-space is a space and and A_2-space is an H-space, so the theorem is indeed a direct generalisation of Theorem 5.13.

In theory Tsutaya's work here prompts analogous generalisations of Theorem 5.14 to count, in the simplest of cases, the number of A_n-types of gauge groups of $SU(2)$-bundles over S^4. For $n \geq 3$, however, the calculations soon become exceedingly complex - even for the low rank Lie group $SU(2)$. Nevertheless, Tsutaya does obtain various equivalence and non-equivalence statements for the number of A_n-types of these gauge groups and they are included in the paper. The most promient of these results is the following.

Theorem 5.36 (Tsutaya [209]) *The number of A_n-types of gauge groups of principal $SU(2)$-bundles over S^4 is at least $2^{\pi(2n+1)-\pi(n+1)}3^{\pi(n+1)}$, where $\pi(m)$ is the number of primes less than or equal to m.* ∎

This statement gives lower bounds of 6 distinct homotopy types, for $n = 1$, and 18 distinct H-types, for $n = 2$. Recall that these are in fact the exact number of distinct types of each respective class. For $n = 3$ the statement yields the lower bound of 36 distinct A_3-types amongst the $SU(2)$-gauge groups. The exact number of A_3-types remains an open question and it is interesting to ponder whether Tsutaya's result may be exact in this case as well.

Tsutaya continued his study of A_n-classes of gauge groups in [210] and presented there further results on the number of A_n-types of $SU(2)$-gauge groups over S^4. The main body of the paper develops certain aspects of the theory of A_n-structure and much of the paper is comprised of technical details. Section 4 of the paper examines the interaction between A_n-structure and homotopy pullback squares and although the following result was not new [102], it was the first time that a detailed proof had appeared in print.

Theorem 5.37 (Tsutaya [210]) *Let $f_1 : X_1 \to X_3$, $f_2 : X_2 \to X_3$ be A_n-maps between A_n-spaces. Then the homotopy pullback X of $X_1 \xrightarrow{f_1} X_3 \xleftarrow{f_2} X_2$ admits an A_n-form such that the canonical projections $q_1 : X \to X_1$, $q_2 : X \to X_2$ are A_n-homomorphisms and there is a homotopy of A_n-maps $f_1 \circ q_1 \simeq f_2 \circ q_2 : X \to X_3$.* ∎

Tsutaya's constructions are complicated somewhat due to their intended applications to the fibrewise A_n-spaces of the gauge groups $\mathcal{G}(P)$, and the second set of technical details included in the paper relates to this structure. For an A_n-space G and a connected, pointed space X let $\mathcal{E}^{A_n}G(X)$ denote the set of equivalence classes of fiberwise A_n-spaces over X with fibers A_n-equivalent to G.

Theorem 5.38 (Tsutaya [210]) *Let G be a well-pointed A_n-space of the homotopy type of a CW complex for n a finite positive integer. Then there exists a fibrewise A_n-space $E_n(G) \to M_n(G)$ with fibres A_n-equivalent to G such that for any well-pointed, connected CW complex X, the map $[X, M_n(G)] \to \mathcal{E}^{A_n}G(X)$, $f \mapsto f^*E_n(G)$ defines a bijective correspondence.* ∎

When applied to the study of A_n-equivalence classes of $SU(2)$-bundles over S^4 Tsutaya's theory yields the following result.

Theorem 5.39 (Tsutaya [210]) *For each positive integer n, the gauge groups $\mathcal{G}_k(SU(2))$ and $\mathcal{G}_{k'}(SU(2))$ are A_n-equivalent if $\min\{2n, \nu_2(k)\} = \min\{2n, \nu_2(k')\}$ and $\min\{[2n/(p-1)], \nu_p(k)\} = \min\{[2n/(p-1)], \nu_p(k')\}$ for any odd prime p, where $\nu_p(k)$ denotes the p-exponent of k and $[m]$ is the maximum integer not greater than m. Moreover if $\nu_2(k) \leq 1$, then the converse is also true.* ∎

An important point to bear in mind is that Theorems 5.35 and 5.35 are only applicable for finite n. We have already discussed the paper [128] and its Theorem 5.12 which provides a counterexample when $n = \infty$. In [123] Kishimoto and Tsutaya showed that A_∞-equivalence of G-gauge groups over S^4 is always a much stronger notion than A_n-equivalence for finite n.

Theorem 5.40 (Kishimoto, Tsutaya [123]) *Let G be a compact, connected simple Lie group. As P ranges over all principal G-bundles over S^d, if there are infinitely many P, then there are infinitely many distinct A_∞-types amongst the gauge groups $\mathcal{G}(P)$.* ∎

The proof relies on the obstruction-theoretic approach to A_n-triviality through geometric methods developed in [113] and makes prominent use of the theory of unstable modules over the Steenrod algebra. For G as in the statement, let $P_\alpha \to S^d$ be the principal G-bundle classified by $\alpha : S^d \to BG$. Working under the assumption that α is of infinite order, Kishimoto and Tsutaya work at a large prime p and study the A_n-triviality of the p-localised gauge groups $\mathcal{G}(P_\alpha)$ for n a function of p. They derive conditions on the p-divisibility of α for $\mathcal{G}(P_\alpha)$ to be A_n-trivial and explicitly produce non-A_n-equivalent gauge groups. It is the interplay between the theory of unstable modules of the Steenrod algebra and their geometric techniques that make the result noteworthy, and it serves as a demonstration of the flexibiliy and power of their new methods for potential application in other areas of unstable homotopy theory.

Kishimoto's work with Tsutaya here extended the examination of A_n-equivalence classes of gauge groups that essentialyl began in [113]. That paper also touched upon the higher homotopy commutativity of gauge groups with the study of $C(k,l)$-spaces and this is a theme to which Kishimoto would also return. In 2016 Hasui, Kishimoto and Tsutaya jointly published [97].

The main result in [97] relates to the higher homotopy commutativity of Lie groups and they extend the previous work of McGibbon [137], [138] and Saumell [169]. Following [137] they say that an A_n-space G is a **Sugawara C_n-space** if the multiplication $m_2 : G \times G \to G$ admits an A_n-form which respects the product A_n-form on $G \times G$.

Theorem 5.41 (Hasui, Kishimoto, Tsutaya [97]) *Let G be a compact, connected, simple Lie group of type $(n_1, \ldots, n_l)$. Let p be a prime and k a positive integer. If $p > kn_l$ then the p-localisation $G_{(p)}$ is a Sugawara C_k-space.* ∎

They base their methods once again around a geometric approach, avoiding the arduous cominatorial calculations often inherent in such reasoning by resorting to stuctural arguments using higher projective planes. If G is an A_n-space then there is a series of quasifibrations $\pi_n : EG_n \to BG_n$ that fit together into a ladder

$$\begin{tikzcd} G = E_0G \arrow[r, hook] \arrow[d, "\pi_0^G"'] & E_1G \arrow[r, hook] \arrow[d, "\pi_1^G"'] & \ldots \arrow[r, hook] & E_nG \arrow[d, "\pi_n^G"'] \\ * = B_0G \arrow[r, hook] & B_1G \arrow[r, hook] & \ldots \arrow[r, hook] & B_nG \end{tikzcd} \tag{5.10}$$

with $E_nG \simeq *^{n+1}G$ and $E_iG \xrightarrow{\pi_i} B_iG \to B_{i+1}G$ a cofiber sequence [175]. If G, H are A_n-spaces and $\varphi : G \to H$ is a map, then φ is an A_n-map if and only if the composite $B_1G = \Sigma G \xrightarrow{\Sigma\varphi} \Sigma H = B_1H \hookrightarrow B_nH$ extends to B_nG. The spaces B_iG are the projective planes associated with the A_n-space G and if G is an A_∞-space then the induced map $\pi^G : EG \to BG$ between the homotopy colimits of the top and bottom rows of (5.10) is equivalent to Milnor's constuction of the universal G-bundle using its loop structure. For G as in Theorem 5.41 Hasui, Kishimoto and Tsutaya show that the p-regular decomposition of G as a product of spheres is compatible with both the A_∞-stuctures and the theorem follows directly from this using an obstuction argument.

With respect to gauge groups the authors apply their machinery first to study the higher homotopy commutativity of the localised gauge groups $\mathcal{G}(E_nG)$, obtaining the first part of Theorem 5.42 below. With this established the final parts of the theorem follow easily using the arguments in [113].

Theorem 5.42 (Hasui, Kishimoto, Tsutaya [97]) *Let G be a compact, connected, simple Lie group of type $(n_1, \ldots, n_l)$. Let p be prime and let n, k be positive integers, then the following assertions hold.*

1. *If $p > (n+k)n_l$, then $\mathcal{G}(E_nG)_{(p)}$ is a Sugawara C_k-space.*

2. If $(n+1)n_l < p < (n+k)n_l$, then $\mathcal{G}(E_nG)_{(p)}$ is not a Williams C_k-space.

Let $P \to X$ be a principal G-bundle, p a prime and k a positive integer. If $p > (cat(X)+k)n_l$, then $\mathcal{G}(P)_{(p)}$ is a Sugawara C_k-space. If $X = S^{2n_i}$ and $p \geq kn_l+n_i$ then $\mathcal{G}(P)_{(p)}$ is a Sugawara C_k-space. ∎

Together with the present author's own modest contribution to the theory there are just a few further papers we wish to briefly discuss before bringing this section of the article to a close. The first is KishimotoKonoTsutaya:2013 in which Kishimoto, Kono and Tsutaya extend previous results on mod p decompositions of gauge groups due to Kishimoto and Kono [114].

Theorem 5.43 (Kishimoto, Kono, Tsutaya KishimotoKonoTsutaya:2013) *Let G be a compact, simply connected, simple Lie group such that $H_*(G;\mathbb{Z})$ is p-torsion free and $G \neq Spin(2n)$, and let $p : P \to S^{2d+2}$ be a principal G-bundle with d an integer in the type $t(G)$ of G. Then there are spaces $\mathcal{B}_i^P$, a homotopy equivalence $\mathcal{G}(P)_{(p)} \simeq \mathcal{B}_1^P \times \cdots \times \mathcal{B}_{p-1}^P$ and a homotopy fibre sequence $\Omega(\Omega_0^{2d+1}B_i \to \mathcal{B}_i^P \to B_{i-d-1}$, where $\mathcal{G}(P) \simeq B_1 \times \cdots \times B_{p-1}$ [148].* ∎

The paper presents an interesting amalgam of techniques, with newer fibrewise p-local splitting arguments used alongside more classical arguments that draw from [148].

The second paper [122] appeared soon after, furthering again the authors' numerous results of mod p decompositions of gauge groups

Theorem 5.44 (Kishimoto, Kono, Tsutaya [122]) *Let G be a compact, simply connected, simple Lie group such that $H_*(G;\mathbb{Z})$ is p-torsion free and $G \neq Spin(2n)$. Let $\epsilon \in \pi_{2d-1}G$, $k \in \mathbb{Z}$ and let $P_k \to S^{2d}$ denote the principal G-bundle classified by $k\epsilon$. Then $\mathcal{G}(P_k)_{(p)} \simeq \mathcal{G}(P_l)_{(p)}$ if and only if $min\{\nu_p(k), \lambda(\epsilon)\} = min\{\nu_p(l), \lambda(\epsilon)\}$, where $\nu_p(m)$ is the p-exponent of m, $\lambda(\epsilon)$ is the exponent of $\langle \epsilon, \alpha \rangle$ and α is the inclusion of a certain generating subcomplex $A(G)$ of G* ∎

The final paper to which we should like to bring attention appeared only recently [95]. The paper counts the number of homotopy types of $PU(3)$- and $PSp(3)$-gauge groups over S^4 and the reader is invited to compare its results with Theorems 5.18 and 5.21. The principal bundles in this care may be indexed by the degree of the classifying map in $\pi_4 BG \cong \pi_3 G \cong \mathbb{Z}$ for $G = PU(3), PSp(3)$.

Theorem 5.45 (Hasui, Kishimoto, Kono, Sato) *The following hold*

1. *$\mathcal{G}_k(S^4, PU(3)) \simeq \mathcal{G}_k(S^4, PU(3))$ if and only if $(24,k) = (24,l)$.*

2. *$\mathcal{G}_k(S^4, PSp(3))_{(p)} \simeq \mathcal{G}_k(S^4, PSp(3))_{(p)}$ for any prime if and only if $(40,k) = (40,l)$.*

It is interesting to see gauge groups associated with non-simply connected Lie groups discussed, and one wonders if these objects may soon become the focus of further research.

5.1 Relations and Interactions With Other Areas of Mathematics.

5.1.1 Homotopy Commutativity of Compact Lie Groups.

Lang's Theorem [130] has been a prominent feature in the work of many authors studying the homotopy types of gauge groups, mainly for the following reason. Consider the evaluation fibration for a given map $f : \Sigma A \to B$ out of a suspension space, and its boundary map $\Delta_f : \Omega B \to Map_*^f(\Sigma A, B)$. Then Lang's Theorem says that the adjoint of Δ_f is the generalised Whitehead product $[f, ev] : \Sigma(A \wedge \Omega B) \to B$, where $ev : \Sigma\Omega B \to B$ is adjoint to the identity on ΩB. In turn the generalised Whitehead product is adjoint to the generalised Samelson product $\langle ad(f), id_{\Omega B}\rangle : A \wedge \Omega B \to \Omega B$, where $ad(f) : A \to \Omega B$ is the adjoint to f.

In the context of gauge theory, if we have a principal G-bundle $p : P \to \Sigma A$ classified by a map $f : \Sigma A \to BG$ then the gauge group $\mathcal{G}(P)$ is the homotopy fibre of $\Delta_f \simeq ad(\langle ad(f), id_G\rangle)$. If G is a compact, connected, simple Lie group then $\langle ad(f), id_G\rangle$, and thus Δ_f also, are maps of finite order [103]. It follows that the orders of all such Samelson products for all maps $\Sigma A \to BG$ serves as a rough upper bound for the number of distinct homotopy types amongst the G-gauge groups over ΣA.

In this manner emphasis is shifted towards the commutator $c : G \wedge G \to G$ during the study of the homotopy types of G-gauge groups and several papers have made this basic relationship explicit. The best example is perhaps [126] in which the authors make our previous comments explicit.

Theorem 5.46 (Kono, Theriault [126]) *Let G be a compact, connected Lie group and let $f : \Sigma A \to BG$ be a map from a suspension space ΣA. For an integer k let $P_k \to \Sigma A$ be the principal G-bundle classified by $kf \in [\Sigma A, BG]$. If the commutator $c : G \wedge G \to G$ has order m, then the number of distinct p-local homotopy types for the gauge groups ${}^{(}P_k)$ is at most $\nu_p(m) + 1$, where $\nu_p(m)$ is the largest integer r such that p^r divides m but p^r does not divide m.* ∎

To prove this theorem their tool of choice was *unstable K-theory*, which we shall now briefly describe. The premise for unstable K-theory is to study the (loopspace of the) infinite Stiefel manifold $W_n = U(\infty)/U(n)$. There is a principal fibration sequence $\Omega W_n \xrightarrow{\Omega\pi} U(n) \to U(\infty) \xrightarrow{\pi} W_n$ where the infinite dimensional unitary group $U(\infty)$ is well-known to be a homotopy commutative H-space [152]. Naturality then implies that the commutator $c : U(n) \wedge U(n) \to U(n)$ lifts through $\Omega\pi$ to a map $\lambda : U(n) \wedge U(n) \to U(n)$. In particular all Samelson products into $U(n)$ factor through the maps λ and $\Omega\pi$. This basic setup is made more appealing by the results contained in [89] where Hamanaka and Kono explicitly describe a particular choice of lift λ with many appealing properties.

Application of unstable K-theory to the study of homotopy commutativity in compact Lie groups has proved extremely fruitful and we have already discussed the three papers [87], [90], [91] that use it in this connection with later application to the homotopy types of gauge groups in mind. It has also been extensively used by the Japanese schools to calculate Samelson products in compact Lie group [85], [155], [116] and self-homotopy sets of compact Lie groups [92], [115], [86]. [122].

5.1.2 Groups of Self-Equivalences.

We noted in the open paragraphs of this article that the gauge group $\mathcal{G}(P)$ is a special type of self-equivalence group and likened it to the more homotopy theoretic fibre equivalence group $\mathcal{E}_F(X_n)$ first studied by Kahn [108]. In passing we mention that this group $\mathcal{E}_F(X_n)$ was also later studied by Nomura [159], Rutter [167], Scheerer [170], Sasao [168] and Tsukiyama [200]. The gauge group $\mathcal{G}(P)$ is also an example of an equivariant equivalence group and they have been studied in this connection by Tsukiyama [201] and Tsukiyama and Oshima [162].

Something that we would like to draw attention to for its connection with gauge groups is the automorphism group $Aut^G(P)$ of a given principal G-bundle $p : P \to B$. This is the group of G-equivariant homeomorphisms $P \to P$, with no restriction placed on the maps which they cover. To see the relation with gauge groups assume that $\alpha : B \to BG$ classifies the bundle P. Then there is a Serre fibration [104]

$$\mathcal{G}(P) \to G\text{-}Map(P, P) \xrightarrow{\hat{p}} Map^{[\alpha]}(B, B) \tag{5.11}$$

where $Map^{[\alpha]}(B, B)$ is the space of all maps $f : B \to B$ satisfying $\alpha \circ f \simeq \alpha$ and $\hat{p}$ assigns to an equivariant self-map $\theta : P \to P$ the unique map $f = \theta/G : B \to B$ such that $f \circ p = p \circ \theta$.

Restricting to the space of homeomorphisms then produces an exact sequence of topological groups [63]

$$1 \to \mathcal{G}(E) \to Aut^G(P) \xrightarrow{\hat{p}} Aut(B) \tag{5.12}$$

where $Aut(B)$ is the group of self-homeomorphisms $B \to B$ and $\hat{p}$ assigns to a G-equivariant homeomorphism $\theta : P \to P$ the unique homeomorphism $f = \theta/G : B \to B$ such that $f \circ p = p \circ \theta$. Note that $\hat{p}$ is not surjective in general. An example of this is given when $B = S^n$ and P is classified by an element $\alpha \in \pi_n BG$ of order different from 2. In this case the antipodal map $(-1) : S^n \to S^n$ is a self-homeomorphism but α and $\alpha \circ (-1) = -\alpha$ classify different G-bundles so there can be no G-equivariant self-map of P that covers it. To rectify this we define $Aut^P(B) = \{f \in Aut(B) \mid f^*P \cong P\}$. Then the following is indeed an exact sequence of topological groups.

$$1 \to \mathcal{G}(E) \to Aut^G(P) \xrightarrow{\hat{p}} Aut^P(B) \to 1. \tag{5.13}$$

The groups $Aut^G(P)$ have not been much studied in the literature. Various attempts have been made in [63], [79], [104], [162], but few concrete results exist. This is mainly due to the difficult nature of the groups $Aut^G(P)$, and they are particularly unwieldly objects. With modern techniques, however, their homotopy should be accessible - at least in special cases - and this becomes more promising given their relation with the much better understood gauge groups through the exact sequence (5.12). It would be an interesting problem to persue the homotopy theoretic propreties of the groups $Aut^G(P)$ further.

5.1.3 The Gauge Group As An Infinite Dimensional Lie Group.

When gauge groups arise in physics there is generally the added stipulation of smooth structure on everything in sight: the base and total space of the focal G-principal bundle $P \xrightarrow{p} X$ are both smooth manifolds, p is a smooth map and G itself is a smooth Lie group. In this context it is natural (and equivalent in our sense [216]) to require that the gauge transformations $P \to P$ be smooth maps, and under this condtion the gauge group $\mathcal{G}(P)$ inherits a natural Lie structure.

Now in general the gauge groups that arise in application cannot be given a local Banach structure in the sense of Birkhoff [26] and must be treated with other techniques of infinite dimensional differential geometry. One setting that is applicable is that of infinite dimensional Lie groups modelled on a locally convex space [2], and in his 1983 lecture at Les Houches [146], Milnor built on the previous work ot Omori [161] showing in an example that these gauge groups become smooth, *locally exponentiable* Lie groups. Milnor's example became a folklore results and only recently has a rigourous and modern proof of this statement been provided by Wockel [218], and we cover his result below after setting up the required notation.

Fix a smooth principal G-bundle $P \xrightarrow{p} X$ over a compact manifold X with G a smooth, compact Lie group with Lie algebra $\mathfrak{g}$. Define the **smooth gauge group** $\mathcal{G}_s(P) = G\text{-}C^\infty(P, G)$ to be the space of all G-equivariant smooth maps $P \to G$, where G acts on itself by conjugation, and define also the **gauge algebra** $\mathfrak{G}(P) = G\text{-}C^\infty(P, \mathfrak{g})$ of P to be the space of smooth G-equivariant maps $P \to \mathfrak{g}$, where G acts on its Lie algebra through the adjoint representation $g \cdot X = ad_g(X) = g^{-1}Xg$. As the name suggests there is a natural locally convex Lie algebra structure on this space and it becomes the Lie algebra of $\mathcal{G}_s(P)$.

Theorem 5.47 (Wockel [218]) *Let $P \xrightarrow{p} X$ be as above. Then $\mathcal{G}_s(P)$ carries a Lie Group structure modelled on $\mathfrak{G}(P)$.* ∎

Wockel's statement in [218] is actually slightly more general, allowing for infinite dimensional structure groups, based around a property of the trivialising open cover of X called *property SUB*.

A second result covered by Wockel in the same work that is more important to the current article connects the world of smooth structures with the looser topological structures used throughout the rest of this exposition.

Theorem 5.48 (Wockel [218]) *Let $P \to X$ be a smooth principal G-bundle over a compact manifold X with locally exponentiable structure group G. Then the natural incusion $\mathcal{G}_s(P) = C^\infty(P, G)^G \hookrightarrow Map(P, G)^G = \mathcal{G}(P)$ is a weak homotopy equivalence.* ∎

A source of motivation for placing a smooth structure on the gauge group relates to our earlier comments surrounding Singer's work [173] on the Gribov Ambiguity. In physical applications one needs to make sense of an integral over the infinite dimensional moduli space $\mathcal{O}(P) = \mathcal{A}(P)/\mathcal{G}_{s*}(P)$. There is also the need in geometry to make sense of the manifold structures on the compact yet singular instanton moduli spaces $\mathcal{M}(P)$. See [3] for some further comments in this direction.

There is also the weaker idea of the convenient setting [129] that is applicable to gauge groups. In this wider context the ideas of an infinite dimensional Lie group structure on the gauge group $\mathcal{G}(P)$ have been discussed in [1], [50], [140].

The introduction of the gauge algebra $\mathfrak{G}(P)$ opened up a new area of study, namely that of its Lie algebra cohomology. This turned out to have application in physics and it was first considered in [27] where it was used to study anomalies arising during the quantisation of gauge theories. In attempting to surmount the lack of local sections of the $\mathcal{G}_*(P)$-bundle $\mathcal{A} \to \mathcal{O}$ physicists introduced *ghost* fields into the gauge theory Lagrangian, resulting in an effective Lagrangian that was not gauge-invariant and possesed an extra, nilpotent symmetry operation s called the BRST operater which relates bosonic and fermionic fields [211]. An interesting result that emphasises the importance of the topology of the gauge group in quantum field theory is the following.

Theorem 5.49 (Schmid [171]) *Let $\Gamma^*_{loc}(P, sg) = \sum^r \Gamma^r_{loc}(P, sg)$ be the complex of Ad-equivariant local r-forms on P with values in the Lie algebra $\mathfrak{g}$, and let $\mathcal{G}(P)$ act on this complex by pullback. Form the Chevalley-Eilenberg complex $C^{(p,q)}(\mathfrak{G}(P), \Gamma_{loc}(P, \mathfrak{g}))$ and let $H^{(1,0)}(\mathfrak{G}(P), \Gamma_{loc}(P, \mathfrak{g}))$ denote its cohomology. Then the chiral anomaly $\partial^\mu J_\mu$ is represented as a cohomology class $\omega \in H^{(1,0)}(\mathfrak{G}(P), \Gamma_{loc}(P, \mathfrak{g}))$.* ∎

In classical field theory it is Noether's theorem which exhibits the reltion between symmetries and conservation laws: a local one-parament symmetry of a Lagrangian system $\mathcal{L}$ gives rise to a conserved current j^μ. After the system is quantised it is not necessarily true that the quantised current j^μ will remain a conserved quantity and such a loss of conservation upon quantisation is called an anomaly. If the symmetry in question is a gauge symmetry then the anomaly manifests itself as a modificaction of the commutation relations on $\mathfrak{G}$ [19] and this gives rise to a central extension of Lie algebra $1 \to \mathfrak{a} \to \widehat{\mathfrak{G}} \to \mathfrak{G} \to 1$.

Another way that central extensions arise in the study of anomalies is as in [139]. One has a projective $\mathcal{G}$-equivarant vector bundle over the space of connections $\mathcal{A}$ which descends to a projective bundle over the moduli space $\mathcal{O} = \mathcal{A}/\mathcal{G}$. One is interested in lifting this projective bundle to a Hilbert space bundle over $\mathcal{O}$. In general it will not be possible to lift the $\mathcal{G}$-action to the Hilbert bundle, and the gauge group will need to be replaced with a central extension $\hat{\mathcal{G}}$ by $U(1)$ to define the lifted $\hat{\mathcal{G}}$-equivariant bundle. This is discussed in [41].

Such central extensions of infinite dimensional Lie algebras have been studied [156], and topologial considerations enter into the question of their integrability. Results dealing directly with central extensions of gauge groups and gauge algebras may be found in [158], [157], [106].

5.2 Future Directions.

In this section we hope to tie up any loose ends and present a brief outlook on some of the exciting open avenues that have recently presented themselves for further research with regards to the homtopy theory of gauge groups.

5.2.1 The Extended Programme.

Although we have discussed many studies of the homotopy types of gauge groups, each having its own unique approach, there has been one fundamental principle underlying the entire scheme: to count the number of G-gauge groups over S^4 for G a simply connected, compact, simple Lie group. Put in this perspective one should realise that so far there have only appeared four studies boasting complete results: those of $SU(2) \cong Sp(1) \cong Spin(3)$, $SU(3)$, $SU(5)$ and $Sp(2) \cong Spin(5)$ - and even two of these ($SU(5)$ and $Sp(2)$) give only complete local information. A first place to look for further research would be to simply continue in a linear, case-by-case examination as G ranges over increasingly higher rank $SU(n)$, $Sp(n)$ and $Spin(n)$ groups.

Now we have already discussed the inherent difficulty in this task for increasing rankm but the next case, $SU(4)$, should be approachable. Moreover there are good theoretical motivations for doing so. Firstly its dimension, 15, is still fairly low and as a rank 4 Lie group its cell structure is not particularly dense. Furthermore the special unitary groups are among the most topologically well-understood Lie groups and with the advent of the stable and unstable K-theories there is a wealth of topological information already accessible. Finally the accidental isomorphisms give $SU(4) \cong Spin(6)$ so there is more homotopy theoretic information available through the subgroup inclusions $Sp(2) \cong Spin(5) \hookrightarrow SU(4)$ and $SU(4) \hookrightarrow Spin(7)$. With all this it is perhaps surprising that no complete results have yet been announced. Further motivation for this case may be found by considering the lower bounds on the number of homotopy types amongst the $SU(4)$-gauge groups given by Theorem 5.18. Plugging in the numbers reaveals an in interesting pattern, either in the number of 2-local homotopy types or a potential improvement for the bounds.

Now the case-by-case plan as set out is obviously doomed to fail at some point but there may yet be rewards for the knowledgeable homotopy theorist willing to work sufficiently hard. Anyone following this linear direction would almost certainly be forced to invent new techniques and methods with which to accomplish the goal. Viewed from this angle these new methods may result in being of more theoretical interest than their modest application to gauge groups and they may even be able to say new things about the homoptopy types of the simply connected compact Lie groups themselves. Unstable K-theory is the primary example of a past occurance of this.

There may also be further scope to make universal statements rather than attempt a case-by-case study each each and every Lie group. For $SU(n)$-gauge groups, the bounds on the number of homtopy types obtained in 5.18 remain the best published, and for $Sp(n)$-gauge groups the bounds in [181] are the best that so far appear in print. As of yet there are has been no effort made to obtain bounds on the number of homotopy types of $Spin(n)$-gauge groups. Looking outside the integral realm there may also be statements similar to those in [195] that may be made for $Sp(n)$- and $Spin(n)$-gauge groups that could characterise the p-local homotopy types of these gauge groups completely.

Finally there are also the exceptional Lie groups to consider and these gauge groups remain almost untouched. So far there has been the single study on the homotopy types of G_2-gauge groups [121]. The large sizes of the exceptional groups makes the prospect of studying the topology of their gauge groups daunting but there is physical motivation for doing so. It has been long known that the standard model group $SU(3) \times SU(2) \times U(1)$ fits natually inside E_6 [82] and this group has been proposed as a potential candidate for a grand unified theory. There are also applications of E_7-gauge theory to $N = 8$ supergravity [62], E_8-gauge theory to 11-dimensional M-theory [215] and there have been models for 11-dimensional supergravity proposed with F_4-gauge structures [164].

Finally there is also the extension of all of these results to the gauge groups of principal $SU(n)$-, $Sp(n)$- and $Spin(n)$-bundles over more complex base spaces - the prime example being that of simply connected 4-manifolds. In this case only the homtopy types of $SU(2)$- and $SU(3)$-gauge groups have been successfully counted, even the case of $Sp(2)$ being problematic. Whilst there are a wealth of odd-primary local statements available for these gauge groups [120] very little information about the 2-local or integral types has emerged.

An example in which such an extension has already begun to take shape is [56]. In the paper Claudio and Spreafico combine composition methods with information on the homotopy groups of spheres gleamed from Toda's masterful book [197] to completely classify the homotopy types of S^3-gauge groups over S^n for $n = 5, \ldots, 25$. Whilst it is doubtful that such a complete result could be reproduced for higher rank Lie groups, the information presented in the paper would be invaluable for anyone wishing to study S^3-gauge groups over higher dimensional manifolds.

Finally we should mention the possibility of extending all of the previous results to manifolds of higher dimension. The study of the homotopy types of gauge groups over simply connected 4-manifolds has been largely motivated by geometry and the work of Donaldson, partly because of its relevance to the marked behaviour of 4-dimensional objects. Recently there have been observances of objects of complex dimension 4 with properties that may be equally amenable to study with gauge theoretic methods [72], [71]. This has led to interest from geometers in $SU(4)$-gauge groups over 4-(complex) dimensional Calabi Yau manifolds, $Spin(7)$-gauge groups over 8-manifolds, and also G_2-gauge groups over 7-manifolds [7], [9].

5.2.2 Higher Gauge Theory.

Higher gauge theory arises naturally during the process of *categorification* - the procedure of replacing set theoretic notions with categorical generalisations. To start the process spaces are replaced by topological categories that carry with them a set of morphisms. Continuous maps become continuous functors, and equality of maps becomes natural isomorphism between functors. The process need not stop there, and theoretically we could use n-categorical notions throughout. One can also allow for various degrees of strictness in any coherence conditions required at any stage of the construction, replacing equality of n-morphisms by the looser condition of natural $(n+1)$-equivalence.

When the basic process is complete it remains to step back and attempt to make sense of what has been assembled. Categorification is a best a rough guideline, and there is no guarantee that the ensuing theory will be useful or even interesting. We shall not persue the ideas past their basic stages here since we will look to theoretical physics for motivation, where the subject is currently still in its infancy.

Recall that the starting point of a gauge theory is a principal G-bundle $P \xrightarrow{p} X$ over a space X for some topological group G: the gauge group $\mathcal{G}(P)$ is the group, under composition, of G-equivariant maps $P \to P$ that cover the identity on X. The categorification process, at the level of generality with which we wish to work, is a follows. There is first to define the notion of a 2-space and replace X with such an object $\mathcal{X}$. Following this we shall also need the concept of a 2-group $\mathbb{G}$ to replace G. Once these basic notions are in place we can introduce a functor $\mathcal{P} \xrightarrow{\pi} \mathcal{X}$ and decide conditions for it to become a locally trivial 2-bundle. To complete the picture one needs further to define the notion of a 2-action of a 2-group on the 2-space $\mathcal{P}$ and decide upon what extra structure should be apparent in a locally trivial G-2-bundle. The higher gauge 2-group $\mathcal{G}(\mathcal{P})$ may then defined as the automorphism group of the 2-projection π in a suitable sense.

A great deal of effort has already been put in to making these definitions. The theory of topological categories has a rich history [30] and there are even definitions of smooth categories [42], [174] that may be suitable for 2-spaces. 2-groups are particularly well understood [21], mainly due to the fact that there is a one-to-one correspondence between 2-groups and *crossed modules*, which arose in Whitehead's work on homotopy theory and have been rigorously studied [37]. The difficult ideas of locally trivial principal 2-bundles have been tackled in [24], [219] and their definitions are similar to those of bundle gerbes, which have been studied in geometry [38] and algebraic geometry [78].

As should be apparent by now, the process is somewhat involved, and at this stage its benefits are not exactly clear. To help motivate all of this as a natural process it is worth tracing the history of gauge theory back to its roots. Gauge theory originally arose in the study of electromagnetism in the context of abelian structural group $U(1)$ [212]. It was only after Yang and Mills introduced the study of non-abelian gauge theory [143] that much of the beauty - and many of the geometric and topological subtleties - of the theory became apparent. In particular it was the generalisation to non-abelian structure groups allowed for the self-interacting gauge fields needed to describe the weak and strong interactions.

Now string theory is currently being exhaustively studied by the physical community. Its basic premise is to replace the point-like particles of classical mechanics with higher objects extending into some higher-dimensional space. Classical gauge theory uses the theory of connections on a principal bundle to describe the parallel transport of classical point-like particles, and Baez [20] reasons that a natural way to describe the parallel transport of strings through spacetime is vía some form of higher connection whose higher holonomy can track the extra information required of it.

Indeed applications of bundle gerbes have been made to M-theory [13] for this very reasoning. These applications are related to twisted K-theory [17] and the use of twisted k-theory to classify D-brane charges in string theory [32]. Further applications of gerbes has been made to the study of Wess-Zumino-Witten models [77] and Chern-Simons theory [76].

Outside of string theory there have also been physical applications of gerbes to topological quantum field theory [166] and its variant homotopy quantum field theory [39]. Further applications in regular quantum field theory can be found in [40]. The full generality of principal 2-bundles has also been exploited, and later applications have been given to $\mathcal{N} = (2,0)$ superconformal field theory in six-dimensions [107] and even gravity [22].

The previous paragraphs have given the reader a brief taster of a promising new area of study for algebraic topologists. Evidently the scope of potential application of algebraic topology to higher gauge theory is very broad and we shall reserve our comments to where it may touch upon the study of *higher gauge 2-groups*. Although we have seen two subtly different approaches to higher gauge theory appearing in the literature, there do exist careful definitions for what the gauge 2-group of a principal 2-bundle should be [21]. In fact there has already been at least one study of higher gauge 2-groups [219] using this definition, although its questioning is directed towards the categorical, generalised analytic and generalised Lie aspects rather than any homotopy theoretic aspects.

Now before setting down a programme to study the homotopy types of physically relevant higher gauge 2-groups several comments are in order. Firstly it is not exactly clear what rôle the gauge 2-groups and their topologies will play in the physical theories currently being studied: the homotopy of these objects may not have the same relevance to the new theories as that of regular gauge groups did to Yang-Mills theory. On a more fundamental level it is not even clear what the future holds for the physical theories themselves, and it remains to be seen whether higher gauge theory will make a truly deep contribution to theoretical physics.

Another issue that we should like to point out is the inherent difficulty of the proposed task. The objects in question are no longer topological spaces or groups, but rather categorical generalisations thereof. The homotopy theory of these objects is still under development and it is not even clear how current methods are best suited to the task. Without proper context it may well be that a gauge 2-group is simply too general of a concept to be useful, and as we have already remarked, such a context has yet to emerge from the physics community. In any case, most of the applications so far have been through bundle gerbles, and these objects abound in geometric and algebraic structure; it may be that geometry and algebra are better tools with which to study the gauge 2-groups that arise, and that they may not even be particularly rich in homotopy.

References

[1] M. Abbati, R. Cirelli, A. Maniá, P. Michor, *The Lie Group of Automorphisms of a Principle Bundle*, J. Geom. and Phys., **2** (1989), 215-235.

[2] R. Abraham, J. Marsden, *Hamiltonian Mechanics on Lie Groups and Hydrodynamics* Proc. Pure Math XVI, Amer. Math. Soc. (1970), 237243.

[3] M.Adams, T. Ratiu, R. Schmid, *The Lie Group Structure of Diffeomorphism Groups and Invertible Fourier Integral Operators with Applications* Infinite Dimensional Groups With Applications, Springer, 1985, 1-70.

[4] J. Aguadé, *Decomposable Free Loop Spaces*, Canad. J. Math., **39** (1987), 938-955.

[5] S. Akbulut, *Torsion in Gauge Theory*, Doga J. Math, **19** (1995), 167-177.

[6] S. Akbulut, *On Universal Relations In Gauge Theory*, TAMS, **4** (1996), 1329-1355.

[7] S. Akbulut, M. Kalafat, *Algebraic Topology of G_2 Manifolds*, Expo. Math., **34** (2016), 106-129.

[8] S. Akbulut, T. Mrowka, Y. Ruan, *Torsion Classes and a Universal Constraint on Donalson Invariants for Odd Manifolds*, TAMS, **1** (1995), 63-76.

[9] S. Akbulut, S. Salur, *Calibrated Manifolds and Gauge Theory* https://arxiv.org/abs/math/0402368

[10] G. Allaud, *On the Classification of Fiber Spaces*, Math. Z., **92** (1966), 110-125.

[11] M. Arkowitz, *The Group of Self-Homotopy Equivalences - A Survey*, Groups of Self-Equivalences and Related Topics., Springer, New York (1990), 170-203.

[12] M. Arkowitz, *Introduction to Homotopy Theory*, Springer New York, 2010.

[13] P. Aschieri, B. Jurčo, *Gerbes, M5-Brane Anomalies and E8 Gauge Theory* J. High. En. Phys., **0410** (2004).

[14] M. Atiyah, R. Bott, *The Yang-Mills Equations Over Riemann Surfaces*, Phil. Trans. R. Soc. Lond., **308** (1983), 523-615.

[15] M. Atiyah, N. Hitchin, W. Singer, *Self-Duality in Four-Dimensional Riemannian Geometry*, Proc. Roy. Soc. London Ser., **362** (1978), 425461.

[16] M. Atiyah, J. Jones, *Topological Aspects of Yang-Mills Theory*, Comm. Math. Phys., **2** (1978), 97-118.

[17] M. Atiyah, G. Segal, *Twisted K-Theory*, https://arxiv.org/abs/math/0407054

[18] M. Atiyah, R. Ward *Instantons and Algebraic Geometry*, Comm. Math. Phys., **2** (1977), 117-124.

[19] J. Azcárraga, J. Izquierdo, *Lie Groups, Lie Algebras, Cohomology and Some Applications in Physics*, Cambridge University Press, Second Edition, 1998.

[20] J. Baez, J. Huerta, *An Invitation to Higher Gauge Theory*, Gen. Rel. and Grav., **9** (2011), 23352392.

[21] J. Baez, A. Lauda, *Higher-Dimensional Algebra. V. 2-Groups*, Theory Appl. Categ. **12** (2004), 423491.

[22] J. Baez, D. Wise, *Teleparallel Gravity as a Higher Gauge Theory*, Comm. Math. Phys., **333** (2015), 153186.

[23] W. Barcus, M. Barratt, *On the Homotopy Classification of the Extensions of a Fixed Map*, Trans. Am. Math. Soc., **1** (1958), 57-74.

[24] T. Bartels, *Higher Gauge Theory I: 2-Bundles* https://arxiv.org/abs/math/0410328.

[25] S. Bauer, M. Crabb, M. Spreafico, *The Classifying Space of the Gauge Group of an* $SO(3)$*-Bundle Over* S^2, Proc. R. Soc. Edinburgh, **131** (2001), 767783.

[26] G. Birkhoff, *Analytic Groups*, Trans. Am. Math. Soc. **43** (1938), 61-101.

[27] L. Bonora, P. Cotta-Ramusino, *Some Remarks on BRS Transformations, Anomalies and the Cohomology of the Lie Algebra of the Group of Gauge Transformations*, Comm. Math. Phys., **4** (1982), 589-603.

[28] L. Bonora, M. Schnabl, M.M. Sheikh-Jabbari, A. Tomasiello, *Noncommutative SO(n) and Sp(n) Gauge Theories*, Nucl. Phys. B **589**, 461 (2000).

[29] P. Booth, P Heath, C. Morgan, R. Piccinini, *H-Spaces of Self-Equivalences of Fibrations and Bundles*, Proc. London Math. Soc., **1** (1984), 111-127

[30] F. Borceux, *Handbook of Categorical Algebra 2. Categories and Structures*, Cambridge University Press, 1994.

[31] R. Bott, *A Note on the Samelson Product in the Classical Groups*, Comment. Math. Helv. **34** (1960), 245-256.

[32] P. Bouwknegt, A. Carey, V. Mathai, M. Murray, D. Stevenson, *Twisted K-Theory and K-Theory of Bundle Gerbes*, Commun. Math. Phys., **228** (2002), 17-49.

[33] P. Boyer, B. Mann, *Homology Operations on Instantons*, J. Differential Geom., **3** (1988), 423-465.

[34] C. Boyer, J. Hurtubise, B. Mann, R. Milgram, *The Topology of Instanton Moduli Spaces I: The Atiyah-Jones Conjecture*, Ann. Math., **137** (1993), 561-609.

[35] C. Boyer, B. Mann, D. Waggoner, *On the Homology of $SU(n)$ Instantons*, Trans. Am. Math. Soc., **2** (1991), 529-561.

[36] T. Bröcker, T. Tom Dieck, *Representations of Compact Lie Groups*, Springer-Verlag, 1985.

[37] R. Brown, P. Higgins, R. Sierva, *Nonabelian Algebraic Topology*, European Mathematical Society, 2011.

[38] J. Brylinski, *Loop Spaces, Characteristic Classes and Geometric Quantization*, Birkhäuser Boston, 2008.

[39] U. Bunke, P. Turner, S. Willerton, *Gerbes and Homotopy Quantum Field Theories*, Algebr. Geom. Topol., **4** (2004), 407-437.

[40] A. Carey, J. Mickelsson, M. Murray, *Bundle Gerbes Applied to Quantum Field Theory*, Rev. Math. Phys., **12** (2000), 65-90.

[41] A. Carey, M. Murray, *Mathematical Remarks on the Cohomology of Gauge Groups and Anomalies*, Confronting the Infinite, World Scientific, 1995.

[42] K. Chen, *On Differentiable Spaces*, Categories in Continuum Physics, Springer, **1174** (1986), 38-42.

[43] T. Cheng, L. Li, *Gauge Theory of Elementary Particle Physics*, Claredon Press, Oxford, (1982).

[44] Y. Choi, *The Mod 2 Homology of $Sp(n)$ Instantons and the Classifying Space of the Gauge Group*, Bull. Austral. Math. Soc. **3** (1997), 503512.

[45] Y. Choi, *Homology of the Classifying Space of $Sp(n)$ Gauge Groups* Israel J. Math., **151** (2006), 167177.

[46] Y. Choi, *Homology of Based Gauge Groups Associated With Spinor Group*, Bull. Korean Math. Soc. **1** (2006), 6375.

[47] Y. Choi, *Homology of Gauge Groups Associated with Special Unitary Groups*, Top. Appl., **12** (2008), 13401349.

[48] Y. Choi, *Homology of the Gauge Group of Exceptional Lie Group G2*, J. Korean Math., **3** (2008), 699709.

[49] Y. Choi, Y. Hirato, M. Mimura, *Composition Methods and Homotopy Types of the Gauge Groups of* $Sp(2)$ *and* $SU(3)$, Bull. Belg. Math. Soc., **15** (2008), 409417.

[50] R. Cirelli, A. Maniá, *The Group of Gauge Transformations as a Schwartz-Lie Group*, J. Math. Phys., **26** (1985), 3036-3041.

[51] R. Cohen, R. Milgram *"The Homotopy Type of Gauge Theoretic Moduli Spaces*, Alg. Top. App., **27** (1994), 15-55.

[52] R. Cohen, *The Topology of Fiber Bundles Lecture Notes*, Lecture Notes. Available at http://math.stanford.edu/∼ralph/fiber.pdf.

[53] F. Cohen, R. Cohen, B. Mann, R. Milgram, *The Topology of Rational Functons and Divisors of Surfaces*, Acta Math., **166** (1991), 163-221.

[54] R. Cohen, J. Jones, *Representations of Braid Groups and Operators Coupled to Monopoles*, Geometry of Low-Dimensional Manifolds 1: Gauge Theory and Algebraic Surfaces, Cambridge University Press (1991), 191-206.

[55] G. Cooke, L. Smith, *Mod p Decompositions of co-H-Spaces and Applications*, Math. Zeit **157**, (1977), 155-177.

[56] M. Claudio, M. Spreafico, *Homotopy Types of Quaternionic Line Bundles Over Spheres*, Top. and App., **156** (2009), 643-651.

[57] M. Crabb, I. James, *Fibrewise Homotopy Theory* Springer, London, 1998.

[58] M. Crabb, W. Sutherland, *Function Spaces and Hurwitz-Radon Numbers*, Math. Scand. **55** (1984), 67-90.

[59] M. Crabb, W. Sutherland, *The Space of Sections of a Sphere-Bundle I* Proc. Edinburgh Math. Soc., **29** (1986), 383-403.

[60] M. Crabb, W. Sutherland, *How Non-Abelian is Non-Abelian Gauge Theory?*, Q. J. Math., **46** (1995), 279290.

[61] M. Crabb, W. Sutherland, *Counting Homotopy Types of Gauge Groups*, Proc. London Math. Soc. **83**, (2000), 747-768.

[62] E. Cremmer, B. Julia, *The* $N = 8$ *Supergravity Theory. 1. The Lagrangian*, Phys. Lett., **80** (1978), 49-51.

[63] W. Curtis, *The Automorphism Group of a Compact Group Action*, Trans. Am. Math. Soc. **203** (1975), 45-54.

[64] D. Davis, S, Theriault, *Odd-Primary Homotopy Exponents of Compact Simple Lie Groups*, Geom. Top., **13** (2008).

[65] A, Dold, *Partitions of Unity in the Theory of Fibrations*, Ann. Math. **78**, (1962).

[66] A. Dold, R. Lashof, *Principal Quasifibrations and Fibre Homotopy Equivalence of Bundles*, Ill. J. Math., **3** (1959), 285-305.

[67] S. Donaldson, *An Application of Gauge Theory to Four Dimensional Topology*, J. Differential Geom. **18**, (1983), 279-315.

[68] S. Donaldson, *Self-Dual Connections and the Topology of Smooth 4-Manifolds*, Bull. Amer. Math. Soc., **1** (1983), 81-83.

[69] S. Donaldson, *Polynomial Invariants for Smooth Four-Manifolds* Top., **3** (1990), 257-315.

[70] S. Donaldson, P. Kronheimer, *The Geometry of Four-Manifolds*, Oxford University Press, (1990).

[71] S. Donaldson, E. Segal, *Gauge Theory in Higher Dimensions, II* https://arxiv.org/abs/0902.3239.

[72] S. Donaldson, R. Thomas, *Gauge Theory in Higher Dimensions* The Geometric Universe, Huggett et al. Eds. Oxford U.P., 1998.

[73] Y. Felix, J. Oprea, *Rational Homotopy of Gauge Groups*, Proc. Am. Math. Soc., **4** (2009), 1519-1527.

[74] Y. Felix, D. Tanré, *H-Space Structure on Pointed Mapping Spaces*, Algebr. Geom. Topol., **5** (2005), 713724

[75] R. Fintushel, R. Stern, *2-Torsion Instanton Invariants*, J. Am. Math. Soc., **2** (1993), 299-339.

[76] D. Fiorenza, H. Sati, U. Schreiber, *Multiple M5-Branes, String 2-Connections, and 7d Nonabelian Chern-Simons Theory*, Adv. Theor. Math. Phys., **18** (2014), 229-321.

[77] K. Gawedzki, N. Reis *WZW Branes and Gerbes*, Rev. Math. Phys., **14**, (2002), 1281-1334.

[78] J. Giraud, *Cohomologie non abélienne*, Springer-Verlag, 1971.

[79] D. Gottlieb, *Applications of Bundle Map Theory*, Trans. Am. Math. Soc., **171** (1972), 23-50.

[80] D. Gottlieb, *On Fibre Spaces and the Evaluation Map*, Ann. of Math. **87**, (1968), 42-55.

[81] V. Gribov, *Instability of Nonabelian Gauge Theories and Impossibility of Choice of Coulomb Gauge*, SLAC Translation, **176** (1977)., 24-38.

[82] F. Gursey, P. Ramond, P. Sikivi *A Universal Gauge Theory Model Based on E6* Phys. Lett., **60** (1976), 177-180.

[83] H. Hamanaka, *Homotopy-Commutativity in Rotation Groups*, J. Math. Kyoto Univ., **3** (1996), 519537

[84] H. Hamanaka, *Homotopy-Commutativity in Spinor Groups*, J. Math. Kyoto Univ., **2** (2000), 389405.

[85] H. Hamanaka, *On Samelson Products in p-Localized Unitary Groups*, Top. and its App., **154** (2007), 573583.

[86] H. Hamanaka, *Nilpotency of Unstable K-Theory*, Top. and its App., **154** (2007), 13681376.

[87] H. Hamanaka, S. Kaji, A. Kono, *Samelson Products in* $Sp(2)$, Top. and App., **11** (2008), 1207-1212.

[88] H. Hamanka, A. Kono, *On* $[X, U(n)]$ *When dimX is* $2n$, J. Math. Kyoto Univ., **2** (2003), 333348

[89] H. Hamanaka, A. Kono, *On* $[X, U(n)]$ *When dimX* $= 2n$, J. Math. Kyoto Univ., **43** (2003), 333348.

[90] H. Hamanaka, A. Kono, *Unstable* K^1*-Group* $[\Sigma^{2n-2}\mathbb{C}P^2, U(n)]$ *and Homotopy Type of Certain Gauge Groups*, Proc. Roy. Soc. Edinburgh Sect. A **136**, (2006), 149-155.

[91] H. Hamanka, A. Kono, *Homotopy Type of Gauge Groups of* $SU(3)$*-Bundles Over* S^6, Top. Appl., **7** (2007), 13771380.

[92] H. Hamanaka, A. Kono, *On the Commutativity of the Localized Self Homotopy Groups of* $SU(n)$, Top. and App. **157** (2010) 11821187.

[93] V. Hansen, *Contributions to the Homotopy Theory of Mapping Spaces*, PhD thesis, University of Warwick (1972).

[94] J. Harper, *Secondary Cohomology Operations* Graduate Studies in Mathematics **49**. Amer. Math. Soc., Providence, RI, 2002.

[95] S. Hasui, D. Kishimoto, A. Kono, T. Sato, *The Homotopy Types of* $PU(3)$ *and* $PSp(3)$*-Gauge Groups*, Alg. Geom. Top., **16** (2016), 18131825.

[96] S. Hasui, D. Kishimoto, A. Ohsita, *Samelson Products in p-Regular Exceptional Lie groups*, Top. Appl., **178** (2014), 1729.

[97] S. Hasui, D. Kishimoto, M. Tsutaya, *Higher Homotopy Commutativity in Localized Lie Groups and Gauge Groups*, https://arxiv.org/abs/1612.08816.

[98] A. Hatcher, *Algebraic Topology*, Cambridge University Press, (2002).

[99] Y. Hemmi, Y. Kawamoto, *Higher homotopy commutativity and the resultohedra*, J. Math. Soc. Jap., $\widehat{\text{bf}}$2 (2011), 443-471.

[100] D. Husemoller, *Fibre Bundles*, Third Edition. Springer-Verlag New York, 1994.

[101] K. Ishiguro, *A p-Local Splitting of BU(n)* Proc. Am. Math. Soc. **2**, (1985), 307-311.

[102] N. Iwase, M. Mimura, *Higher Homotopy Associativity*, Springer Lecture Notes in Math., **1370** (1986), 193-220.

[103] I. James, *On Lie Groups and Their Homotopy Groups*, Proc. Cambridge Phil. Soc., **55** (1959), 244-247.

[104] I. James, *The Space of Bundle Maps*, Top., **2** (1963), 45-59.

[105] I. James, J. Whitehead, *The Homotopy Theory of Sphere-Bundles Over Spheres I*, Proc. London Math. Soc. **4** (1954), 196-218.

[106] B. Janssens, K. Neeb, *Covariant Central Extensions of Gauge Lie Algebras*, https://arxiv.org/abs/1510.07938.

[107] B. Jurčo, C. Sämann, M. Wolf, *Higher Groupoid Bundles, Higher Spaces, and Self-Dual Tensor Field Equations*, Fortschritte der Physik, **64** (2016), 674-717.

[108] D. Kahn, *The Group of Fibre Homotopy Equivalences*, Math. Z., **84** (1964), 1-8.

[109] S. Kaji, D. Kishimoto, *Homotopy Nilpotency in p-Regular Loop Spaces*, Math. Z. **1** (2010), 209224.

[110] S. Kaji, D. Kishimoto, *Homotopy Nilpotency in p-Compact Groups*, https://arxiv.org/abs/0710.3975.

[111] Y. Kamiyama, D. Kishimoto, A. Kono, S. Tsukuda, *Samelson Products of SO(3) and Applications*, Glasg. Math. J. **49** (2007), 405-409.

[112] D. Kishimoto, *Homotopy Nilpotency in localized* $SU(n)$, Hom. Hom. Appl., **1** (2009), 6179

[113] D. Kishimoto, A. Kono, *Splitting of Gauge Groups*, Trans. Amer. Math. Soc., **12** (2010), 67156731.

[114] D. Kishimoto, A. Kono, *Mod p Decompositions of Gauge Groups*, Proc. Jap. Acad. Ser. A Math. Sci., **86** (2010), 15-17.

[115] D. Kishimoto, A. Kono, T. Nagao, *Commutativity of Localized Self-Homotopy Groups of Symplectic Groups*, Top. and its App., **158** (2011), 10251032.

[116] D. Kishimoto, T. Nagao, *Commutativity in Special Unitary Groups at Odd Primes*, Top. Appl., **12** (2010), 19491954.

[117] D. Kishimoto, A. Kono, M Tsutaya, *Mod p Decompositions of Gauge Groups*, Alg. Geom. Top., **13** (2013), 17571778 .

[118] D. Kishimoto, A. Kono, M Tsutaya, *On Localized Unstable K^1-Groups and Applications to Self-Homotopy Groups*, Canad. Math. Bull., **57** (2014), 344356.

[119] D. Kishimoto, A. Kono, S. Theriault, *Homotopy Commutativity in p-Localized Gauge Groups*, Proc. Roy. Soc. Edinburgh, **4** (2013), 851870.

[120] D. Kishimoto, A. Kono, S. Theriault, *Refined Gauge Group Decompositions*, Kyoto J. Math., **3** (2014), 679-691.

[121] D. Kishimoto, S. Theriault, M. Tsutaya, *The Homotopy Types of G_2-Gauge Groups*, Preprint.

[122] D. Kishimoto, A. Kono, M. Tsutaya, *On p-Local Homotopy Types of Gauge Groups*, Proc. Roy. Soc. Edin., **1** (2014), 149-160.

[123] D. Kishimoto, M. Tsutaya, *Infiniteness of A_∞-Types of Gauge Groups*, J. Top., **1** (2016), 181-191.

[124] D. Kishimoto, M. Tsutaya, *Samelson Products in p-Regular SO(2n) and its Homotopy Normality*, Glasgow Math. J. **1** (2017), 1-10.

[125] A. Kono, *A Note on the Homotopy Type of Certain Gauge Groups*, Proc. Roy. Soc. Edinburgh Sect. A, **117** (1991), 295-297.

[126] A. Kono, S. Theriault, *The Order of the Commutator on $SU(3)$ and an Application to Gauge Groups*, Bull. Belg. Math. Soc., **2** (2013), 359-370.

[127] A. Kono, S. Tsukuda, *Remark on the Homotopy Type of Certain Gauge Groups*, J. Math. Kyoto Univ. **36**, (1996), 115-121.

[128] A. Kono, S. Tsukuda, *4-Manifolds X Over $BSU(2)$ and the Corresponding Homotopy Types $Map(X, BSU(2))$*, J. Pure Appl. Algebra, **3** (2000), 227237.

[129] A. Kriegl, P. Michor, *The Convenient Setting of Global Analysis*, Am. Math. Soc. **54**, 1997.

[130] G. Lang, *The Evaluation Map and EHP Sequences*, Pacific J. Math. **44** (1973), 201-210.

[131] J. Lin, *H-Spaces With Finiteness Conditions*, North Holland, 1995.

[132] G. Lupton, S. Smith, *Criteria for Components of a Function Space to be Homotopy Equivalent*, Math. Proc. Camb. Philos. **145**, (2008), 95-106.

[133] G. Masbaum, *On the cohomology of the classifying space of the gauge group over some 4-complexes*, Bull. Soc. Math. France, **1** (1991), 1-31.

[134] P. May, *A Concise Course in Algebraic Topology*, University of Chicago Press, 1999.

[135] P. May, *Classifying Spaces and Fibrations*, Mem. Amer. Math. Soc., No. 155, 1975.

[136] P. May, K. Ponto, *More Concise Algebraic Topology*, University of Chicago Press, 2012.

[137] C. McGibbon, *Homotopy Commutativity in Localized Groups*, Amer. J. Math., (3), (1984), 665-687.

[138] C. McGibbon, *Higher Forms of Homotopy Commutativity and Finite Loop Spaces*, Math. Z., **3** (1989),363374.

[139] J. Mickelsson, *Current Algebras and Groups*, Springer, 1989.

[140] P. Michor, *Gauge Theory for Fiber Bundles*, Mon. and Text. in Phys. Sci, **19**, 1991.

[141] R. Milgram, *The Bar Construction and Abelian H-Spaces*, Ill. J. Math., **11** (1967), 242-250.

[142] H. Miller, *Stable Splittings of Stiefel Manifolds*, Topology **24** (1985), 411-419.

[143] R. Mills, C. Yang, *Conservation of Isotopic Spin and Isotopic Gauge Invariance*, Phys.Rev., **96** (1954), 191-195.

[144] J. Milnor, *Construction of Universal Bundles, I*, Ann. Math. **2** (1956), 272-284.

[145] J. Milnor, *On Simply-Connected 4-Manifolds*, Proc. Int. Symp. Algebraic Topology, Mexico (1958), 122-128.

[146] J. Milnor, *Remarks On Infinite-Dimensional Lie Groups* Relativity, groups and topology: 2, North-Holland, 1984, 1007-1057.

[147] M. Mimura, *On the Number of Multiplications on SU(3) and Sp(2)*, Trans. Amer. Math. Soc. **146** (1969), 473-492.

[148] M. Mimura, G. Nishida, H. Toda, *Mod p Decomposition of Compact Lie Groups*, Publ. Res. Inst. Math. Sci., **13** (1977), 627680.

[149] M. Mimura, H. Toda, *Homotopy Groups of SU(3), SU(4) and Sp(2)*, J. Math. Kyoto Univ. Vol. **3** (1963), 217-250.

[150] M. Mimura, H. Toda, *Homotopy Groups of Symplectic Groups*, J. Math. Kyoto Univ. **3** (1963), 251-273.

[151] M. Mimura, H. Toda, *Cohomology Operations and the Homotopy of Compact Lie Groups I*, Topology **9** (1970), 317-336 .

[152] M. Mimura, H. Toda, *Topology of Lie Groups, I and II*, Trans. Am. Math. Soc., **91** (1991).

[153] J. Munkres, *Topology*, Second Edition. Prentice Hall, 2000.

[154] T. Nagao, *On the Groups* $[X, Sp(n)]$ *With* $dim X \leq 4n+2$, J. Math. Kyoto Univ. **48** (2008), 149165 .

[155] T. Nagao, *On Samelson products in* $Sp(n)$, J. Math. Kyoto Univ., **49** (2009), 225234.

[156] K. Neeb, *Central Extensions of Infinite-Dimensional Lie Groups*, Annales de linstitut Fourier, **5** (2002) , 1365-1442.

[157] K. Neeb, *Lie Groups of Bundle Automorphisms and Their Extensions*, Developments and Trends in Infinite-Dimensional Lie Theory, **288** (2010), 281-338.

[158] K. Neeb, C. Wockel, *Central Extensions of Groups of Sections*, Ann. Global Anal. Geom. **36** (2009), 381418.

[159] Y. Nomura, *A Note on Fibre Homotopy Equivalences*, Bull. Nagoya Inst. Tech. **17** (1965), 66-71.

[160] K. Oguchi, *Generators of 2-Primary Components of Homotopy Groups of Spheres, Unitary Groups and Symplectic Groups*, J. Fac. Sci., Univ. of Tokyo, 1 A, **11** (1964). 65-111.

[161] H. Omori, *Infinite Dimensional Lie Transformations Groups*, Springer-Verlag, 1974.

[162] H. Oshima, K. Tsukiyama, *On the Group of Equivariant Self Equivalences of Free Actions*, Pub. Res. Inst. Math. Sci. Kyoto Univ. **22** (1986), 905-923.

[163] R. Palais, *On the Existence of Slices for Actions of Non-Compact Lie Groups*, Ann. Math. **2** (1961), 295-323.

[164] T. Pengpan, P. Ramond, *M(ysterious) Patterns in* $SO(9)$, Phys. Rept., **315** (1999), 137-152.

[165] R. Piccinini, M. Spreafico, *The Milgram-Steenrod Construction of Classifying Spaces for Topological Groups*, Expo. Math., **2** (1998), 97-130.

[166] R. Picken, *TQFT's and Gerbes*, Algebr. Geom. Topol., **4** (2004), 243-272.

[167] J. Rutter, *Groups of Self-Homotopy Equivalences of Induced Spaces* Comm. Math. Helv. **45** (1970), 236-255.

[168] S. Sasao, *Fibre Homotopy Self-Equivalences*, Kodai Math. J. **5** (1982), 446-453.

[169] L. Saumell, *Higher Homotopy Commutativity in Localized Groups*, Math. Z., **2** (1995), 203213.

[170] H. Scheerer, *Arithmeticity of Groups of Fibre Homotopy Equivalence Classes*, Manuscripta Math. **31** (1980), 413-424.

[171] *Local Cohomology in Gauge Theories, BRST Transformations and Anomalies*, Diff. Geom. and App., **2** (1994,) 107-116.

[172] G. Segal, *Classifying Spaces and Spectral Sequences*, Pub. Math. de l'IHÉS, **34** (1968), 105-112.

[173] I. Singer, *Some Remarks on the Gribov Ambiguity*, Comm. Mat. Phys. **60**, (1978), 7-12.

[174] J. Souriau, *Groupes Differentiels*, Differential Geometrical Methods in Mathematical Physics, Springer **836** (1980), 91128.

[175] J. Stasheff, *A Classification Theorem for Fibre Spaces*, Top., **2** (1963), 239-246.

[176] J. Stasheff, *Homotopy Associativity of H-Spaces. I*, Trans. Am. Math. Soc., **2** (1963), 275-292.

[177] N. Steenrod, *Topology of Fibre Bundles*, Third Edition. Princeton University Press, 1960.

[178] N. Steenrod, *Milgram's Classifying Space of a Topological Group*, Top. **7** (1968), 349-368.

[179] W. Sutherland *Fibre Homotopy Equivalence and Vector Fields*, Proc. Lon. Math. Soc., **1** (1965), 543-556.

[180] W. Sutherland, *Path-Components of Function Spaces*, Quart. J. Math. Oxford **2** (1983), 223-233.

[181] W. Sutherland, *Function Spaces Related to Gauge Groups*, Proc. Royal Soc. Edinburgh. A **121**, (1992), 185-190.

[182] G. t'Hooft, *On the Phase Transisition Towards Permanent Quark Confinement*, Nucl. Phys. B, **1** (1978), 1-25.

[183] C. Taubes, *The Stable Topology of Self-Dual Moduli Spaces*, J. Diff. Geom., **1** (1989), 163-230.

[184] S. Terzić, *The Rational Topology of Gauge Groups and of Spaces of Connections*, Compos. Math., **1** (2005), 262270.

[185] S. Theriault, *2-Primary Exponent Bounds for Lie Groups of Low Rank*, Canad. Math. Bull., **1** (2004), 119132.

[186] S. Theriault, *The Odd Primary H-Structure of Low Rank Lie Groups and its Application to Exponents*, Trans. Amer. Math. Soc., **9** (2007), 45114535.

[187] S. Theriault, *Odd Primary Homotopy Decompositions of Gauge Groups*, Alg. Geom. Top., **1** (2010), 535564.

[188] S. Theriault, *The Homotopy Types of Sp(2)-Gauge Groups*, Kyoto J. Math. **50** (2010), 591-605.

[189] S. Theriault, *Homotopy Decompositions of Gauge Groups Over Riemann Surfaces and Applications to Moduli Spaces*, Int. J. Math. **22**, (2011), 1711-1719.

[190] S. Theriault, *The Homotopy Types of SU(3)-gauge Groups over Simply Connected 4-manifolds*, Publ. Res. Inst. Math. Sci. **48**, (2012), 543-563.

[191] S. Theriault, *The Odd Primary Homology of $SU(n)$, $Sp(n)$ and $Spin(n)$-Gauge Groups*, J. Pure Appl. Alg., **3** (2012), 679687.

[192] S. Theriault, *The Homotopy Types of SU(5)-Gauge Groups*, Osaka J. Math. **52** (2015), 15-31.

[193] S. Theriault, *The Homotopy Types of Gauge Groups of Nonorientable Surfaces and Applications to Moduli Spaces*, Illinois J. of Math. **57**, (2013), 59-85.

[194] S. Theriault, *The Homotopy Types of $SU(5)$-Gauge Groups*, Osaka J. Math., **1** (2015), 1529.

[195] S. Theriault, *Odd Primary Homotopy Types of $SU(n)$ Gauge Groups* Alg. Geom. Top., 1-18.

[196] R. Thom, *L'homologie des Espaces Fonctionnels*, Colloq. Topol. Algébrique, Louvain, (1956), 29-39.

[197] H. Toda, *Composition Methods in Homotopy Groups of Spheres*, Annals of Mathematics Studies **49**, Princeton University Press, Princeton N.J., 1962.

[198] T. tom Dieck, *Algebraic Topology*, European Mathematical Society, 2008.

[199] T. tom Dieck, *Transformation Groups*, Walter de Gruyter, Berlin, 1987.

[200] K. Tsukiyama, *On the Group of Fibre Homotopy Equivalences*, Hiroshima Math. J. **12** (1982), 349-376.

[201] K. Tsukiyama, *Equivariant Self-Equivalences of Principal Fibre Bundles*, Math. Proc. Camb. Phil. Soc. **98** (1985), 87-92.

[202] S. Tsukuda, *Notes On the Group of S^1 equivariant homeomorphisms*, Publ. Res. Inst. Math. Sci., **6** (1995), 10971105.

[203] S. Tsukuda, *A Remark on the Homotopy Type of the Classifying Space of Certain Gauge Groups*, J. Math. Kyoto Univ. **1** (1996), 123128.

[204] S. Tsukuda, *On Maps from BS^1 to Classifying Spaces of Certain Gauge Groups*, J. Math. Kyoto Univ., **2** (1997), 327342.

[205] S. Tsukuda, *On Maps from BS^1 to Classifying Spaces of Certain Gauge Groups II*, J. Math. Kyoto Univ., **3** (1997), 547-552.

[206] S. Tsukuda, *On the Cohomology of the Classifying Space of a Certain Gauge Group*, Proc. Roy. Soc. Edinburgh, **2** (1997), 407409.

[207] S. Tsukuda, *On Isomorphism Classes of Gauge Groups*, Top., **3** (1998), 173187

[208] S. Tsukuda, *Comparing the Homotopy Types of the Components of $Map(S4, BSU(2))$*, J. Pure Appl. Algebra, **161** (2001), 235243.

[209] M. Tsutaya, *Finiteness of An-Equivalence Types of Gauge Groups*, J. London Math. Soc., **1** (2012), 142-164.

[210] M. Tsutaya, *Homotopy Pullback of A_n-Spaces and its Applications to A_n-Types of Gauge Groups*, Top. and App., **187** (2015), 1-25.

[211] S. Weinberg, *The Quantum Theory of Fields Vol. II: Modern Applications*, Cambridge University Press, 1995.

[212] H. Weyl, *Gravitation und Elektrizität*, Sitzungsber. Deutsch. Akad. Wiss. Berlin, Klossefü. 465-480.

[213] G. Whitehead, *On Products in Homotopy Groups*, Ann. Math. **47** (1946), 460475.

[214] E. Witten, *Current Algebra, Baryons, and Quark Confinement*, Nucl. Phys. B **223**, 2, (1983), 433-444.

[215] E. Witten, *On Flux Quantization in M Theory and the Effective Action*, J. Geom. Phys., **22** (1997), 1-13.

[216] C. Wockel, *Infinite-Dimensional Lie Theory for Gauge Groups*, Dissertation, TU Darmstadt, 2006.

[217] C. Wockel, *The Samelson Product and Rational Homotopy for Gauge Groups*, Abh. Math. Sem. Univ. Hamburg, **77** (2007), 219228.

[218] C. Wockel, *Lie Group Structure On Symmetry Groups of Principal Bundles*, J. Funct. Anal. **1** (2007), 254-288.

[219] C. Wockel, *Principal 2-Bundles and Their Gauge 2-Groups*, Forum Math. 23 (3) (2011) 565-610.

[220] L. Woodward, *The Classification of Principal PU_n-Bundles Over a 4-Complex*, J. London Math. Soc. **25**, (1982), 513-524.

[221] A. Yamaguchi, *On the p-Regularity of Stiefel Manifolds*, Publ RIMS, Kyoto Univ. **25** (1989), 355-380.

[222] A. Zabrodsky, *Hopf Spaces*, North-Holland Publishing Company, 1976.