Pass profile exponential and asymptotic stability of nonlinear repetitive processes ⋆

Pavel Pakshin * Julia Emelianova, Mikhail Emelianov **
Krzysztof Galkowski *** Eric Rogers ****

* Arzamas Polytechnic Institute of R.E. Alekseev Nizhny Novgorod State Technical University ,19, Kalinina Street, Arzamas, 607227, Russia and Lobachevsky State University of Nizhny Novgorod, Prospekt Gagarina, 23, 603950, Nizhny Novgorod, Russia (e-mail: pakshinpv@gmail.com)
** Arzamas Polytechnic Institute of R.E. Alekseev Nizhny Novgorod State Technical University ,19, Kalinina Street, Arzamas, 607227, Russia (e-mail: EmelianovaJulia@gmail.com)
*** Institute of Control and Computation Engineering, University of Zielona Góra, ul. Szafrana 2, 65-516 Zielona Góra, Poland (e-mail: k.galkowski@issi.uz.zgora.pl)
**** Department of Electronics and Computer Science, University of Southampton, Southampton SO17 1BJ, UK (e-mail: etar@ecs.soton.ac.uk)

Abstract: This paper considers discrete and differential nonlinear repetitive processes using the state-space model setting. These processes are a particular class of 2D systems that have their origins in the modeling of physical processes. Their distinguishing characteristic is that one of the two independent variables needed to describe the dynamics evolves over a finite interval and therefore they are defined over a subset of the upper-right quadrant of the 2D plane. The current stability theory for nonlinear dynamics assumes that they operate over the complete upper-right quadrant and this property may be too strong for physical applications, particulary in terms of control law design. With applications in mind, the contribution of this paper is the use of vector Lyapunov functions to characterize a new property termed pass profile exponential stability.

Keywords: Nonlinear repetitive processes, pass profile exponential stability, vector Lyapunov functions.

1. INTRODUCTION

Repetitive processes belong to the class of systems that complete the same finite duration task over and over again. Each completion is known as a pass and the finite duration is termed the pass length. In this paper the notation for variables is $y_k(p)$, $0 \le p \le \alpha,\ k \ge 0$, where y is the vector or scalar valued variable, the integer k denotes the pass number and $\alpha < \infty$ the pass length.

The distinguishing feature of a repetitive process is that the previous pass profile explicitly contributes to the current pass profile. Let $\{y_k\}$ denote the sequence of pass profiles generated from a given initial pass profile y_0. Then the result can be oscillations in the pass profile that increase from pass-to-pass that cannot be regulated by the application of standard control laws. Early research on repetitive processes was focused on industrial applications, including coal mining. A full treatment of this first research is given in (Rogers et al., 2007) together with the references to the original research.

In coal mining the cutting machine makes repeated passes along the finite length coal face and the major control objective is to maximize the volume of coal extracted without penetrating the coal/stone interface either above the machine or below it, known as the roof and floor respectively. The pass profile in this example is the height of the floor relative to some datum. Once each pass is complete the machine returns to the starting point and cuts the next pass profile resting on the floor produced on the previous pass. The machine weighs at least 5 tonnes and will deform the floor as it passes over and this is the source of the control problem for this industrial repetitive process.

In the case of linear dynamics with a constant pass length, a practically motivated stability theory has been developed together with stability tests and control law design algorithms (Rogers et al., 2007). This theory has been developed for linear dynamics using an abstract model of the dynamics in a Banach space setting with the resulting stability properties given by properties of the associated

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linear operator. Motivated by the unique stability problem, i.e., oscillations that increase in amplitude from pass-to-pass, the stability theory requires that a bounded y_0 produces a pass profile sequence $\{y_k\}$ with bounded entries, where bounded is defined in terms of the norm on the associated Banach space.

The stability theory for linear dynamics imposes the bounded property either over the finite and fixed pass length or uniformly, i.e. for all possible pass lengths. In several particular cases, the abstract model based stability conditions have been converted into tests that can be completed using standard methods, such as Nyquist or Bode plots. This progress has, in turn, led to the application of repetitive process control theory to the design of iterative learning control laws that have been experimentally validated (Hladowski et al., 2010). This validation has used a gantry robot executing a pick and place operation that replicates a common use for such robots in the industrial setting, with further results in, e.g., (Paszke et al., 2013).

Repetitive processes evolve over $\{(k,p) \in [0,\infty] \times [0,\alpha]\}$, i.e., a subset of the upper-right quadrant of the 2D plane. Consequently links exist with 2D systems described by the Roesser (Roesser, 1975) and Fornasini-Marchesini (Fornasini and Marchesini, 1978) state-space models, which evolve of the complete upper-right quadrant of the 2D plane. In previous work, see, e.g, (Rogers et al., 2007) and the relevant cited references, it has been established that some, but not all, systems theoretic properties of repetitive processes can be addressed by application of known methods for Roesser and Fornasini-Marchesini systems. Also there are forms of repetitive process dynamics that cannot be represented by these 2D systems models.

The development of a stability theory for nonlinear 2D systems is underway, e.g., (Kurek, 2012; Yeganefar et al., 2013). Also sufficient conditions guaranteeing Lyapunov stability, asymptotic stability and exponential stability of nonlinear 2D differential-discrete systems were developed in (Knorn and Middleton, 2016), where the conditions for Lyapunov stability and asymptotic stability require that the corresponding 2D Lyapunov function has the negative semi-definite property. Examples were also given to illustrate the results. In (Gałkowski et al., 2016) the first results on a vector Lyapunov function-based approach to the exponential stability analysis of differential repetitive processes were developed.

Exponential stability of repetitive processes imposes a uniform bound on the pass profile dynamics, which is analyzed by letting the pass length $\alpha \to \infty$. Given the finite pass length, a property of the dynamics and not an assumption imposed for analysis, this stability criterion could be too strong for, in particular, control law design. A similar difficulty arose in the application of linear repetitive process theory to, in particular, physical examples. This, in turn, led to the development of strong practical stability for linear repetitive processes, see, e.g., (Paszke et al., 2015). In this paper pass profile exponential stability for nonlinear repetitive processes is defined and characterized using vector Lyapunov functions.

Throughout this paper, the notation $\succ 0$ denotes a symmetric positive definite matrix, the notation $\succeq 0$ denotes a symmetric positive semi-definite matrix and $\preceq 0$ a symmetric negative semi-definite matrix.

2. PASS PROFILE EXPONENTIAL STABILITY OF DISCRETE NONLINEAR REPETITIVE PROCESSES

The discrete nonlinear repetitive processes considered in this paper are described by the following state-space model over $0 \leq p \leq T-1,\ k \geq 0$,

$$\begin{aligned} x_{k+1}(p+1) &= f_1(x_{k+1}(p), y_k(p)), \\ y_{k+1}(p) &= f_2(x_{k+1}(p), y_k(p)), \end{aligned} \tag{1}$$

where the integer $T < \infty$ denotes the number of samples over the pass length (T times the number of samples equals the pass length) and on pass k $x_k(p) \in \mathbb{R}^{n_x}$ is the state vector, $y_k(p) \in \mathbb{R}^{n_y}$ is the pass profile (or output) vector and f_1, and f_2 are nonlinear functions. Also it is assumed that $f_1(0,0) = 0,\ f_2(0,0) = 0.$ and hence an equilibrium at zero. The boundary conditions, i.e., the pass state initial vector sequence and the initial pass profile, are assumed to be of the form

$$x_{k+1}(0) = d_{k+1},\ y_0(p) = f(p),\ k \geq 0,\ 0 \leq p \leq T-1, \tag{2}$$

where the entries in $d_{k+1} \in \mathbb{R}^{n_x}$ are known constants and the entries in $f(p) \in \mathbb{R}^{n_y}$ are known functions of p. It is assumed that d_{k+1} and $f(p)$ have bounded energy, i.e., there exist finite real numbers $M_f > 0$, $\kappa_d > 0$ and $0 < \lambda_d < 1$ such that d_{k+1} and $f(p)$ satisfy

$$|f(p)|^2 \leq M_f,\ |d_{k+1}|^2 \leq \kappa_d \lambda_d^{k+1},\ k \geq 0, \tag{3}$$

where $|q|$ denotes the Euclidian norm of a vector q. From this point onwards, all references to the boundary conditions for the processes considered will assume that they satisfy (3).

In this paper the state initial vector on each pass is independent of the previous pass profile vector but there are applications where this assumption is not applicable. Instead, the more general case where an entry in this sequence must be an explicit function of pass profile samples along the previous pass is required. In the linear case it is known that the structure of this sequence can alone cause instability (Rogers et al., 2007). Hence the entries in this sequence must be adequately modeled. Sufficient progress with the case considered in this paper should prompt further research on this more general case for discrete and differential nonlinear repetitive processes.

In the control and systems theory developed for linear repetitive processes, the stability along the pass property has formed the basis for control law design and experimental verification (Rogers et al., 2007; Hladowski et al., 2010; Paszke et al., 2013, 2016), but there are examples for which a weaker property is all that can be achieved or is required. Stability along the pass demands that a bounded initial pass profile produces a bounded sequence of pass profiles for all possible values of the pass length and is based on linear operator theory in a Banach space setting. Hence it cannot be directly transferred to the nonlinear case.

Stability along the pass requires that the sequence (in k) of pass profiles and state vectors are bounded independent of the pass length and is a strong form of stability for these processes. In the case of discrete and differential nonlinear repetitive processes stability should also enforce boundedness (in k) independent of the pass length of the sequences (in k) of pass profiles and state vectors and one possible approach would be to use Lyapunov functions as in the stability analysis of 1D nonlinear systems.

The Lyapunov approach is based on properties of the function itself and for discrete dynamics of its increments, but the dynamics of discrete repetitive processes are determined by the state vector x and the pass profile vector y, which are functions of the two independent variables p and k. A candidate Lyapunov function for these processes can be chosen as a scalar function, $V(x, y)$, but to construct the discrete equivalent of the gradient along the trajectories of (1) it is required to have $x_{k+1}(p+1) - x_{k+1}(p)$ and $y_k(p) - y_k(p)$ as functions of x and y. These quantities can only be found by solving (1) but then all of the advantages of the Lyapunov approach are lost. This is also true for the differential processes considered the next section.

Previous work developed a stability theory for discrete nonlinear repetitive processes based on the use of vector Lyapunov functions and the discrete counterpart of divergence along the trajectories of the process to characterize the property of exponential stability. The stability property in this previous work was termed exponential stability Emelianova et al. (2014) and for a process described by (1) and (2) requires the existence of real numbers $\kappa > 0$ and $0 < \lambda < 1$ such that

$$|x_k(p)|^2 + |y_k(p)|^2 \leq \kappa\lambda^{k+p}. \quad (4)$$

Exponential stability is imposed on the state and pass profile dynamics over the complete upper-right quadrant of the 2D plane, i.e., $(k, p),\ k \geq 0,\ p \geq 0\}$. In physical applications, however, the pass length will always be finite and only a finite number of passes will ever be completed. Hence, as in the case of linear dynamics (Paszke et al., 2015), it could be the case in some examples that this property is too strong and restrictive in terms of, in particular, control law design. This paper relaxes this condition to so-called pass profile exponential stability defined over the finite and fixed pass length as follows.

Definition 1. A discrete nonlinear repetitive process described by (1) and (2) is said to be pass profile exponentially stable if

$$|y_k(p)|^2 \leq \kappa\lambda^k,\ \kappa > 0,\ 0 < \lambda < 1,\ 0 \leq p \leq T-1. \quad (5)$$

It is easy to see that if (1) is exponentially stable then it also pass profile exponentially stable.

Application of this new stability definition to processes described by (1) and (2) is the subject of the remainder of this section, starting from a vector Lyapunov function of the form

$$V(x_{k+1}(p), y_k(p)) = \begin{bmatrix} V_1(x_{k+1}(p)) \\ V_2(y_k(p)) \end{bmatrix}, \quad (6)$$

where $V_1(x) \geq 0$, $V_2(y) > 0$, $x, y \neq 0$, $V_1(0) = 0$ and $V_2(0) = 0$, i.e., positive semi-definite and positive definite functions respectively. Also the discrete counterpart of the divergence operator of this function along the trajectories of (1) is

$$\mathcal{D}_d V(x_{k+1}(p), y_k(p)) = \Delta_p V_1(x_{k+1}(p)) + \Delta_k V_2(y_k(p)), \quad (7)$$

where

$$\Delta_p V_1(x_{k+1}(p)) = V_1(x_{k+1}(p+1)) - V_1(x_{k+1}(p)),$$
$$\Delta_k V_2(y_k(p)) = V_2(y_{k+1}(p)) - V_2(y_k(p)).$$

Theorem 1. A discrete nonlinear repetitive process described by (1) and (2) is pass profile exponentially stable if there exists a vector function of the form (6) and positive scalars c_0, c_1, c_2 and c_3 such that

$$V_1(x_{k+1}(0)) \leq c_0|x_{k+1}(0)|^2, \quad (8)$$
$$c_1|y_k(p)|^2 \leq V_2(y_k(p)) \leq c_2|y_k(p)|^2, \quad (9)$$
$$\mathcal{D}_d V(x_{k+1}(p), y_k(p)) \leq -c_3|y_k(p)|^2. \quad (10)$$

Proof. It follows from (10) that there exists $\bar{c}_3 > 0$ such that $\bar{c}_3 \leq c_3$ holds and then

$$\mathcal{D}_d V(x_{k+1}(p), y_k(p)) \leq -c_3|y_k(p)|^2 \leq -\bar{c}_3|y_k(p)|^2. \quad (11)$$

Next, using (9) and (10), (11) can be rewritten as

$$V_1(x_{k+1}(p+1)) + V_2(y_{k+1}(p)) \leq V_1(x_{k+1}(p) + \left(1 - \frac{\bar{c}_3}{c_2}\right) V_2(y_k(p)). \quad (12)$$

If $c_3 > c_2(1-\lambda_d^{\frac{1}{2}})$ choose $\bar{c}_3 < c_2(1-\lambda_d^{\frac{1}{2}})$, and if $c_3 \leq c_2(1-\lambda_d^{\frac{1}{2}})$ choose $\bar{c}_3 < c_3 \leq c_2(1-\lambda_d^{\frac{1}{2}})$. Hence in any case it is possible to choose $\bar{c}_3$ such that $\bar{c}_3 \leq c_2(1-\lambda_d^{\frac{1}{2}})$. Define $\lambda = 1 - \frac{\bar{c}_3}{c_2}$ and then it follows from the previous inequality that

$$\lambda_d^{\frac{1}{2}} \leq \lambda < 1. \quad (13)$$

This last inequality guarantees the required convergence properties as shown next.

Rewrite (12) in the form

$$V_1(x_{k+1}(p+1)) \leq V_1(x_{k+1}(p)) + \lambda V_2(y_k(p)) - V_2(y_{k+1}(p)). \quad (14)$$

Solving recursively the inequality (14) with respect to $V_1(x_{k+1}(p))$ gives the following expression in terms of the boundary conditions:

$$V_1(x_{k+1}(p+1)) \leq V_1(x_{k+1}(0)) + \sum_{h=0}^{p}[\lambda V_2(y_k(h)) - V_2(y_{k+1}(h))]$$

and on introducing $H_k(p) := \sum_{h=0}^{p} V_2(y_k(h))$, it follows from the previous inequality that

$$H_{k+1}(p) \leq \lambda H_k(p) + V_1(x_{k+1}(0)) - V_1(x_{k+1}(p+1)). \quad (15)$$

Solving the inequality (15) gives

$$H_k(p) \leq \lambda^k H_0(p) + \sum_{n=0}^{k-1} \lambda^{k-1-n}[V_1(x_{n+1}(0)) - V_1(x_{n+1}(p+1))],$$

or

$$\sum_{n=0}^{k-1} \lambda^{k-1-n} V_1(x_{n+1}(p+1)) + \sum_{h=0}^{p} V_2(y_k(h)) \leq \sum_{n=0}^{k-1} \lambda^{k-1-n} V_1(x_{n+1}(0)) + \lambda^k \sum_{h=0}^{p} V_2(y_0(h)).$$

This last inequality is equivalent to

$$\sum_{n=0}^{k-1} \lambda^{-n} V_1(x_{n+1}(p+1)) + \lambda^{-(k-1)} \sum_{h=0}^{p} V_2(y_k(h)) \leq \sum_{n=0}^{k-1} \lambda^{-n} V_1(x_{n+1}(0)) + \lambda \sum_{h=0}^{p} V_2(y_0(h)). \quad (16)$$

Evaluating the right-hand side of (16) and using (3), (8), (9) and (13) gives

$$\sum_{n=0}^{k-1} \lambda^{-n} V_1(x_{n+1}(0)) + \lambda \sum_{h=0}^{p} V_2(y_0(h)) \leq c_0 \kappa_d \sum_{n=0}^{k-1} \lambda^{-n} \lambda_d^{n+1} + c_2 M_f T$$
$$\leq c_0 \kappa_d \lambda_d \sum_{n=0}^{k-1} \lambda^{n} (\lambda_d^{\frac{1}{2}})^n (\lambda_d^{\frac{1}{2}})^n + c_2 M_f T$$
$$\leq c_0 \kappa_d \lambda_d \sum_{n=0}^{\infty} \lambda^n + c_2 M_f T = \frac{c_0 \kappa_d \lambda_d}{1-\lambda} + c_2 M_f T = C(T)$$

for all k and $0 \leq p \leq T-1$. Also it follows immediately from the left-hand side of (16) that

$$C(T) \geq \lambda^{-(k-1)} \sum_{h=0}^{p} V_2(y_k(h)) \geq (c_1 \lambda) \lambda^{-k} \sum_{h=0}^{p} |y_k(h)|^2 \quad (17)$$

and from (17) that

$$(c_1 \lambda) \lambda^{-k} |y_k(p)|^2 \leq C(T)$$

for all $0 \leq p \leq T-1$. Hence (5) holds with $\kappa = \frac{C(T)}{c_1 \lambda}$ and the proof is complete.

3. PASS PROFILE ASYMPTOTIC STABILITY OF DIFFERENTIAL NONLINEAR REPETITIVE PROCESSES

Consider differential nonlinear repetitive processes with pass length $T < \infty$ described over $0 \leq t \leq T$ by the state-space model

$$\dot{x}_{k+1}(t) = f_1(x_{k+1}(t), y_k(t)),$$
$$y_{k+1}(t) = f_2(x_{k+1}(t), y_k(t)), \quad (18)$$

where on pass k, $x_k(t) \in \mathbb{R}^{n_x}$ is the state vector, $y_k(t) \in \mathbb{R}^{n_y}$ is the pass profile vector, $u_k(t) \in \mathbb{R}^{n_u}$ is the input vector; f_1, f_2 and g are nonlinear functions. Also $f_1(0,0) = 0$, $f_2(0,0) = 0$ is assumed and hence an equilibrium at zero. Also the function f_1 is assumed to satisfy the following Lipschitz condition with respect to variables x and y:

$$|f_1(x', y') - f_1(x'', y'')| \leq L(|x' - x''| + |y' - y''|),$$
$$x', x'' \in \mathbb{R}^{n_x},\ y', y'' \in \mathbb{R}^{n_y}. \quad (19)$$

The boundary conditions are the differential equivalent of those given in (2), i.e.,

$$x_{k+1}(0) = d_{k+1},\ k \geq 0,\ |d_{k+1}|^2 \leq \kappa_d \lambda_d^{k+1},\ k \geq 0,$$
$$y_0(t) = f(t),\ |f(t)|^2 \leq M_f,\ 0 \leq t \leq T, \quad (20)$$

where the entries in $d_{k+1} \in \mathbb{R}^{n_x}$ are known constants, the entries in $f(t) \in \mathbb{R}^{n_y}$ are known functions of t, $M_f > 0$, $\kappa_d > 0$, $0 < \lambda_d < 1$ and it is assumed that the equivalent of (3) holds.

Following (Gałkowski et al., 2016) exponential stability of a differential nonlinear repetitive process described by (18) and (20) requires the existence of real numbers $\kappa > 0$, $\lambda > 0$ and $0 < \zeta < 1$ such that

$$|x_k(t)|^2 + |y_k(t)|^2 \leq \kappa \exp(-\lambda t) \zeta^k. \quad (21)$$

The following is the definition of pass profile exponential stability of the processes considered in this section.

Definition 2. A differential nonlinear repetitive process described by (18) and (20) is said to be pass profile exponentially stable if there exist real numbers $\kappa > 0$, $\lambda > 0$ and $0 < \zeta < 1$

$$|y_k(t)|^2 \leq \kappa \zeta^k,\ \kappa > 0,\ 0 < \zeta < 1. \quad (22)$$

uniformly in $t \in [0, T]$.

It is easy to see that if (18) is exponentially stable it also has the pass profile exponential stability property.

Definition 3. A differential nonlinear repetitive process described by (18) and (20) is said to be pass profile asymptotically stable if $|y_k(t)|$ is bounded and

$$|y_k(t)| \to 0 \quad (23)$$

as $k \to \infty$ uniformly in $t \in [0, T]$.

In this case only conditions for pass profile asymptotic stability can be obtained. To derive these conditions a vector Lyapunov function is again used and, in particular,

$$V(x_{k+1}(t), y_k(t)) = \begin{bmatrix} V_1(x_{k+1}(t)) \\ V_2(y_k(t)) \end{bmatrix} \quad (24)$$

and V_1 and V_2 satisfy the same assumptions as in (6). The divergence operator of the function (24) along the trajectories of (18) is defined as

$$\mathcal{D}_c V(x_{k+1}(t), y_k(t)) = \frac{dV_1(x_{k+1}(t))}{dt} + \Delta_k V_2(y_k(t)), \quad (25)$$

where

$$\Delta_k V_2(y_k(t)) = V_2(y_{k+1}(t)) - V_2(y_k(t)).$$

Theorem 2. A differential nonlinear repetitive process described by (18) and (20) is pass profile exponentially stable if there exists a vector function (24) and positive scalars c_0, c_1, c_2, c_3 and c_4 such that

$$V_1(x) \leq c_0 |x|^2, \quad (26)$$
$$c_1 |y|^2 \leq V_2(y) \leq c_2 |y|^2, \quad (27)$$
$$\mathcal{D}_c V(x_{k+1}(t), y_k(t)) \leq -c_3 |y_k(t)|^2. \quad (28)$$
$$\left| \frac{\partial V_1(x)}{\partial x} \right| \leq c_4 |x|. \quad (29)$$

Proof. By the same arguments as in the proof of the previous theorem, it follows from (27) and (28) that there exists $\bar{c}_3 \le c_3$ such that

$$\lambda_d^{\frac{1}{2}} \le \zeta := 1 - \frac{\bar{c}_3}{c_2} < 1 \tag{30}$$

and

$$\mathcal{D}_c V(x_{k+1}(t), y_k(t)) \le -c_3|y_k(t)|^2 \le -\bar{c}_3|y_k(t)|^2. \tag{31}$$

Also it follows from (28) and (31) that

$$\frac{dV_1(x_{k+1}(t))}{dt} + V_2(y_{k+1}(t)) - \zeta V_2(y_k(t)) \le 0. \tag{32}$$

Solving the inequality (32) with respect to $V_1(x_{k+1}(t))$ gives

$$V_1(x_{k+1}(t)) \le V_1(x_{k+1}(0)) - \int_0^t [V_2(y_{k+1}(s)) - \zeta V_2(y_k(s))]ds. \tag{33}$$

Introducing

$$W_{k+1}(t) := V_1(x_{k+1}(0)) - V_1(x_{k+1}(t)),$$
$$H_k(t) := \int_0^t V_2(y_k(s))ds$$

enables (33) to be rewritten as

$$H_{k+1}(t) \le \zeta H_k(t) + W_{k+1}(t). \tag{34}$$

Solving the inequality (34) gives

$$H_n(t) \le \zeta^n H_0(t) + \sum_{k=1}^{n} W_k(t)\zeta^{n-k} \tag{35}$$

The last inequality is equivalent to

$$\zeta^{-n}\int_0^t V_2(y_n(s))ds + \sum_{k=1}^{n} V_1(x_{k+1}(t))\zeta^{-k} \le \zeta^{-n}\sum_{k=1}^{n} V_1(x_k(0))\zeta^{n-k} + \int_0^t V_2(y_0(s))ds. \tag{36}$$

Evaluating the right-hand side of (36) and using (20), (26), (27) and (30) gives

$$\zeta^{-n}\sum_{k=1}^{n} V_1(x_k(0))\zeta^{n-k} + \int_0^t V_2(y_0(s))ds \le \sum_{k=1}^{n} c_0\kappa_d\lambda_d^k\zeta^{-k} + \int_0^t c_2M_f ds$$
$$\le \sum_{k=1}^{\infty} c_0\kappa_d\zeta^k + c_2M_fT = \frac{c_0\kappa_d}{1-\zeta} + c_2M_fT = C(T) \tag{37}$$

and it follows from (36) and (37) that

$$\int_0^t V_2(y_n(s))ds \le \zeta^n C(T) \to 0 \tag{38}$$

as $n \to \infty$ uniformly in $t \in [0, T]$.

Evaluating $\frac{dV_1(x)}{dt}$ and using (19) and (29) gives

$$\frac{dV_1(x_{k+1}(t))}{dt} = \frac{\partial V_1(x_{k+1}(t))}{\partial x_{k+1}(t)} f_1(x_{k+1}(t), y_k(t))$$
$$\ge -\left|\frac{\partial V_1(x_{k+1}(t))}{\partial x_{k+1}(t)}\right| |f_1(x_{k+1}(t), y_k(t))|$$
$$\ge -c_4L(|x_{k+1}(t)| + \varepsilon|y_k(t)|)(|x_{k+1}(t)| + |y_k(t)|)$$
$$\ge -2c_4L\left(\frac{\varepsilon+1}{2\sqrt{\varepsilon}}|x_{k+1}(t)| + \sqrt{\varepsilon}|y_k(t)|\right)^2$$
$$\ge -2c_4L\left(2\left(\frac{\varepsilon+1}{2\sqrt{\varepsilon}}|x_{k+1}(t)|\right)^2 + 2(\sqrt{\varepsilon}|y_k(t)|)^2\right)$$
$$\ge -\alpha|(x_{k+1}(t)|^2 - \beta\varepsilon V_2(y_k(t)), \tag{39}$$

where $\alpha = \frac{c_4L(\varepsilon+1)^2}{\varepsilon}$, $\beta = \frac{4c_4L}{c_1}$, and ε is arbitrary positive scalar. It then follows from (32) and (39) that

$$V_2(y_{k+1}(t)) - z_0V_2(y_k(t)) \le \alpha|x_{k+1}(t)|^2, \tag{40}$$

where $z_0 = \zeta + \beta\varepsilon$. Choosing ε small enough such that $0 < z_0 < 1$, and solving (40) gives

$$V_2(y_n(t)) \le z_0^nV_2(y_0(t)) + \alpha\sum_{k=1}^{n} z_0^{n-k}|x_{k+1}(t)|^2. \tag{41}$$

Evaluating $\frac{d|x_{k+1}(t)|^2}{dt}$ and using (19) and (29) gives

$$\frac{d|x_{k+1}(t)|^2}{dt} \le \gamma|x_{k+1}(t)|^2 + \delta V_2(y_k(t)), \tag{42}$$

where $\gamma = \frac{2L(\varepsilon+1)^2}{\varepsilon}$, $\delta = \frac{8L\varepsilon}{c_1}$, and ε is an arbitrary positive scalar. It follows from (42) that

$$|x_{k+1}(t)|^2 \le |x_{k+1}(0)|^2e^{\gamma T} + \delta\int_0^T e^{\gamma(T-\tau)}V_2(y_k(\tau))d\tau \tag{43}$$

Given (38) the right-hand side of (43) is bounded uniformly in $t \in [0, T]$. Then it follows from (41) and (27) that $|y_n(t)|$ is also bounded uniformly in $t \in [0, T]$. Moreover, by (38) and (27) $|y_n(t)| \to 0$ as $n \to \infty$ uniformly in $t \in [0, T]$ and the proof is complete.

4. EXAMPLES

For brevity, only discrete dynamics is considered in this section. To highlight the difference between exponential and pass profile exponential stability consider systems described by

$$x_{k+1}(p+1) = A_{11}x_{k+1}(p) + A_{12}y_k(p),$$
$$y_{k+1}(p) = A_{21}x_{k+1}(p) + A_{22}y_k(p) + B_2\varphi(y_{k+1}(p)), \tag{44}$$

where $\varphi(y)$ is nonlinear function satisfying quadratic constraints

$$y^THy + 2y^TS\varphi(y) + \varphi(y)^TR\varphi(y) \ge 0,$$

and H, S and R are matrices of compatible dimensions and in the single-input single-output case for simplicity, $\varphi(y)$ is a scalar sector bounded nonlinearity, i.e., $\varphi(y) \in [k_1y,\ k_2y]$, then $(\varphi(y) - k_1y)(k_2y - \varphi(y)) = -k_1k_2y^2 + (k_1 + k_2)\varphi(y)y - \varphi(y)^2 \ge 0$. If $k_1 = 0$ this sector includes saturation type nonlinearities that arise in many engineering examples and here affect the pass profile.

Choose the entries in the vector Lyapunov function as quadratic forms $V_1(x_{k+1}(p) = x_{k+1}^T(p)P_1x_{k+1}(p)$ and $V_2(y_k(p)) = y_k^T(p)P_2y_k(p)$ with $P_1 \succeq 0,\ P_2 \succ 0$. Then applying Theorem 1 and the S-procedure gives that (44) is pass profile exponentially stable if

$$\bar{x}^T(A^TPA - P + C^THC + G)\bar{x} + 2\bar{x}^T(A^TPB + C^TS)\varphi(y) + \varphi(y)^T(B^TP + R)\varphi(y) \leq 0,$$

where $\bar{x} = [x^T \ y^T]^T$, $C = [0 \ I], B = [0, \ B_2^T]^T, G = \text{diag}[G_1 \ G_2]$ and $G_1 \succeq 0$, $G_2 + H \succ 0$. Moreover, this last inequality holds if the following LMI is feasible

$$\begin{bmatrix} A^TPA - P + C^THC + G & A^TPB + C^TS \\ B^TPA + SC^T & B^TPB + R \end{bmatrix} \preceq 0. \quad (45)$$

Conversely, using the results in (Emelianova et al., 2014) (44) is exponentially stable and hence pass profile exponentially stable if (4) holds with $P_1 \succ 0$, $P_2 \succ 0$ and $G_1 \succ 0$. Hence it is possible for pass profile exponential stability to hold but not exponential stability, due to the less restrictive constraints on the state dynamics.

As a second example suppose that the dynamics are linear and (1) becomes

$$\begin{aligned} x_{k+1}(p+1) &= A_{11}x_{k+1}(p) + A_{12}y_k(p), \\ y_{k+1}(p) &= A_{21}x_{k+1}(p) + A_{22}y_k(p). \end{aligned} \quad (46)$$

Choose the entries in the vector Lyapunov function as the quadratic forms $V_1(x_{k+1}(p)) = x_{k+1}^T(p)P_1x_{k+1}(p)$, $V_2(y_k(p)) = y_k^TP_2y_k(p)$. By Theorem 1 (46) has the pass profile exponential stability property if $P_1 \succeq 0$, $P_2 \succ 0$ and

$$A^TPA - P + Q \preceq 0, \quad (47)$$

where

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \ Q = \text{diag}[Q_1 \ Q_2], \ Q_1 \succeq 0, \ Q_2 \succ 0.$$

Conversely, using the results in (Emelianova et al., 2014) (46) is exponentially stable and hence pass profile exponentially stable if (47) holds with $P_1 \succ 0$ and $Q_1 \succ 0$.

This last result parallels the stability theory in (Rogers et al., 2007) where the property of asymptotic stability demands bounded input bounded output stability of the pass profile sequence over the finite pass length and this property holds if and only if all eigenvalues of the matrix A_{22} have modulus strictly less than unity. Stability along the pass requires this boundedness property independent of α and p and this stronger form of stability requires, as a necessary condition, that both A_{11} and A_{22} have no eigenvalues on or outside the unit circle, which is not guaranteed when pass profile exponential stability is applied to the special case of linear dynamics.

5. CONCLUSIONS AND FUTURE RESEARCH

A new stability property for nonlinear repetitive processes has been developed, motivated by the fact that repetitive process operate over a subset of the upper-right quadrant of the 2D plane where their dynamics are defined but the strongest form of stability previously available imposes conditions over all of the upper-right quadrant of the 2D plane. Moreover, in applications it is the pass profile that must be controlled. Ongoing/future research will aim to use this new stability property in the development of stabilization theory based, e.g., on extension of the dissipativity and passivity theories reported in (Pakshin et al., 2016).

REFERENCES

Emelianova, J., Pakshin, P., Gałkowski, K., and Rogers, E. (2014). Vector Lyapunov function based stability of a class of applications relevant 2D nonlinear systems. *IFAC Proceedings Volumes (IFAC-PapersOnline)*, 47(3), 8247–8252.

Fornasini, E. and Marchesini, G. (1978). Doubly indexed dynamical systems: state models and structural properties. *Mathematical Systems Theory*, 12, 59–72.

Gałkowski, K., Emelianov, M., Pakshin, P., and Rogers, E. (2016). Vector Lyapunov functions for stability and stabilization of differential repetitive processes. *J. Computer Syst. Sci. International*, 55, 503–514.

Hladowski, L., Galkowski, K., Cai, Z., Rogers, E., Freeman, C.T., and Lewin, P.L. (2010). Experimentally supported 2D systems based iterative learning control law design for error convergence and performance. *Control Engineering Practice*, 18, 339– 348.

Knorn, S. and Middleton, R.H. (2016). Asymptotic and exponential stability of nonlinear two-dimensional continuous-discrete Roesser models. *Systems & Control Letters*, 93, 35–42.

Kurek, J. (2012). Stability of nonlinear time-varying digital 2-D Fornasini-Marchesini system. *Multidimensional Systems and Signal Processing*, open access at Springerlink.com, 1 – 10.

Pakshin, P., Emelianova, J., Emelianov, M., Gałkowski, K., and Rogers, E. (2016). Dissipativity and stabilization of nonlinear repetitive processes. *Systems & Control Letters*, 91, 14–20.

Paszke, W., Dabkowski, P., Rogers, E., and Gałkowski, K. (2015). New results on strong practical stability and stabilization of discrete linear repetitive processes. *Systems & Control Letters*, 75, 22–29.

Paszke, W., Rogers, E., and Gałkowski, K. (2016). Experimentally verified generalized KYP lemma based iterative learning control design. *Control Engineering Practice*, 53, 57–67.

Paszke, W., Rogers, E., Galkowski, K., and Cai, Z. (2013). Robust finite frequency range iterative learning control design and experimental verification. *Control Engineering Practice*, 21(10), 1310–1320.

Roesser, R.P. (1975). A discrete state space model for image processing. *IEEE Transactions on Automatic Control*, 20, 1–10.

Rogers, E., Gałkowski, K., and Owens, D. (2007). *Control Systems Theory and Applications for Linear Repetitive Processes. Lecture Notes in Control and Information Sciences, Vol. 349.* Springer-Verlag, Berlin.

Yeganefar, N., Yeganefar, N., Ghamgui, M., and Moulay, E. (2013). Lyapunov theory for 2-D nonlinear Roesser models: Application to asymptotic and exponential stability. *IEEE Transactions on Automatic Control*, 58, 1299–1304.