Stability analysis of 2D Roesser systems via vector Lyapunov functions [★]

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Abstract: The paper gives new results that contribute to the development of a stability theory for 2D nonlinear discrete and differential systems described by a state-space model of the Roesser form using an extension of Lyapunov's method. One of the main difficulties in using such an approach is that the full derivative or its discrete counterpart along the trajectories cannot be obtained without explicitly finding the solution of the system under consideration. This has led to the use of a vector Lyapunov function and its divergence or its discrete counterpart along the system trajectories. Using this approach, new conditions for asymptotic stability are derived in terms of the properties of two vector Lyapunov functions. The properties of asymptotic stability in the horizontal and vertical dynamics, respectively, are introduced and analyzed. This new properties arise naturally for repetitive processes where one of the two independent variables is defined over a finite interval. Sufficient conditions for exponential stability in terms of the properties of one vector Lyapunov function are also given as a natural follow on from the asymptotic stability analysis.

Keywords: 2D systems, vector Lyapunov functions, asymptotic stability, exponential stability.

1. INTRODUCTION

This paper considers 2D systems with particular emphasis on stability theory. Representations for the dynamics of these systems include the Roesser (Roesser, 1975) and Fornasini-Marchesini state-space model (Fornasini and Marchesini, 1978). The Roesser model, originally proposed for image processing applications, partitions the state vector into two sub-vectors, commonly termed horizontal and vertical respectively. In Fornasini-Marchesini state-space model (Fornasini and Marchesini, 1978)) one state vector is used.

The most commonly analyzed Roesser models have discrete dynamics but there are cases where differential 2D systems will arise, either differential in both sub-vectors or differential in one and discrete in the other, where the latter are termed differential-discrete in this paper. One

example of the former case is Dymkov et al. (2011) where a 2D Roesser model was developed for a sorption process, which arises in waste water and sewage treatment. Conditions for a unique solution was also developed together with a representation formula.

Recent years have seen research reported on the development of a stability theory for 2D nonlinear systems. For example, the stability of nonlinear deterministic systems described by a Fornasini-Marchesini model was analyzed in (Kurek, 2014). Discrete nonlinear systems described by the Roesser model were considered in Yeganefar et al. (2013) where Lyapunov theorems to check for asymptotic and exponential stability were established and a converse Lyapunov theorem developed for exponential stability.

In Emelianova et al. (2014) a stability theory was developed for discrete nonlinear repetitive processes where vector Lyapunov functions were used to characterize the exponential stability property and Pakshin et al. (2015a) extended these results to nD ($n \geq 2$ discrete nonlinear systems described a Roesser model. Sufficient conditions

for exponential stability of nonlinear 2D systems described by Fornasini-Marchesini and Roesser state-space models were developed in Pakshin et al. (2015b) using vector Lyapunov functions and a converse stability theorem was established. Sufficient conditions guaranteeing Lyapunov stability, asymptotic stability and exponential stability of nonlinear 2D continuous-discrete systems were developed in Knorn and Middleton (2016) and require the corresponding two-dimensional Lyapunov function to have the negative semi-definite property.

The underlying motivation for this paper is that the existing literature considers exponential stability for nonlinear Roesser systems. Using results in (Pakshin et al., 2015a) it follows that the conditions for asymptotic stability obtained in (Yeganefar et al., 2013) actually guarantee exponential stability and the conditions for asymptotic stability were derived as a consequence of exponential stability. Exponential stability is a natural property of linear dynamics and it is well known that for linear standard, termed $1D$ in this paper, systems asymptotic stability and exponential stability are equivalent. For nonlinear systems this is not true and an example may be stable or asymptotically stable but never exponentially stable, see (Knorn and Middleton, 2016) and example therein. Moreover, there are no conditions that guarantee asymptotic, but not exponential, stability, where the former property may be less restrictive in terms of stabilization.

In this paper, new properties termed asymptotic stability in horizontal and vertical dynamics, respectively, are defined and the main new results are the derivation of conditions for both of them and comparison to the commonly used asymptotic stability of 2D systems, i.e., in both the vertical and horizontal dynamics, in terms of vector Lyapunov functions for discrete and differential 2D nonlinear systems described by a Roesser model. Sufficient conditions for exponential stability are also developed in terms of the properties of a vector Lyapunov function as natural development of the asymptotic stability analysis. Asymptotic stability in horizontal or vertical dynamics arises naturally for cases when one of two independent variables is defined over finite interval. Physical examples include the sorption process considered in Dymkov et al. (2011).

2. 2D DISCRETE NONLINEAR ROESSER MODEL ANALYSIS

The systems considered in this paper are described by a discrete nonlinear Roesser state-space model of the form

$$x_1(i+1,j) = f_1(x_1(i,j), x_2(i,j)),$$
$$x_2(i,j+1) = f_2(x_1(i,j), x_2(i,j)), \quad (1)$$

where $x_k \in \mathbb{R}^{n_k}, k = 1,2$, are the horizontal and vertical state sub-vectors, $f_k(x_1, x_2), k = 1,2$, are nonlinear functions, where it is assumed that $f_k(0,0) = 0, k = 1,2$ and hence an equilibrium at the origin. The boundary conditions are given by

$$x_1(0,j) = g_1(j), \ x_2(i,0) = g_2(i), \quad (2)$$

where $g_k, \ k = 1,2$, are known vector-valued functions. Also it is assumed that

$$|g_k(n)|^2 \leq \kappa_k \rho^n, \ 0 < \rho < 1, \ n = i,j, \quad (3)$$

for $\kappa_k \ (k = 1,2) > 0$ and $| \cdot |$ denotes the norm on the underlying function space. From this point onwards all references to the boundary conditions will assume that they satisfy (3). Also the notation $x(i,j) = [x_1(i,j)^T \ x_2(i,j)^T]^T$ is used.

The stability properties considered in this paper are defined as follows.

Definition 1. A 2D nonlinear system described by (1) and (2) is said to be asymptotically stable in the horizontal dynamics if $|x_1(i,j)|$ is bounded and

$$|x_1(i,j)| \to 0 \ \text{as} \ i + j \to \infty. \quad (4)$$

Definition 2. A 2D nonlinear system described by (1) and (2) is said to be asymptotically stable in the vertical dynamics if $|x_2(i,j)|$ is bounded and

$$|x_2(i,j)| \to 0 \ \text{as} \ i + j \to \infty. \quad (5)$$

Definition 3. A 2D nonlinear system described by (1) and (2) is said to be asymptotically stable if $|x(i,j)|$ is bounded and

$$|x(i,j)| \to 0 \ \text{as} \ i + j \to \infty. \quad (6)$$

Definition 4. A 2D nonlinear system described by (1) and (2) is said to be exponentially stable if there exist scalars $\kappa > 0$ and $0 < \zeta < 1$ such that

$$|x_1(i,j)|^2 + |x_2(i,j)|^2 \leq \kappa \zeta^{i+j}. \quad (7)$$

2.1 Asymptotic Stability Analysis

The asymptotic stability property of a system described by (1) and (2) is analyzed using a vector Lyapunov function of the form

$$V_h(x(i,j)) = \begin{bmatrix} V_{h1}(x_1(i,j)) \\ V_{h2}(x_2(i,j)) \end{bmatrix}, \quad (8)$$

where $V_{h1}(x_1(i,j)) > 0$ and $V_{h2}(x_2(i,j)) \geq 0$ and $V_{h1}(0) = V_{h2}(0) = 0$.

$$V_v(x(i,j)) = \begin{bmatrix} V_{v1}(x_1(i,j)) \\ V_{v2}(x_2(i,j)) \end{bmatrix}, \quad (9)$$

where $V_{v1}(x_1(i,j)) \geq 0$ and $V_{h2}(x_2(i,j)) > 0$ and $V_{v1}(0) = V_{v2}(0) = 0$ The discrete counterpart of the divergence operators for these functions along the trajectories of (1) are

$$\mathcal{D}V_h(x(i,j)) = V_{h1}(x_1(i+1,j)) - V_{h1}(x_1(i,j))$$
$$+ V_{h2}(x_2(i,j+1)) - V_{h2}(x_2(i,j)). \quad (10)$$

$$\mathcal{D}V_v(x(i,j)) = V_{v1}(x_1(i+1,j)) - V_{v1}(x_1(i,j))$$
$$+ V_{v2}(x_2(i,j+1)) - V_{v2}(x_2(i,j)). \quad (11)$$

The following theorem gives sufficient conditions for asymptotic stability in the horizontal dynamics of a system described by (1) and (2) in terms of a vector Lyapunov function of the form (8).

Theorem 1. A system described by (1) and (2) is asymptotically stable in the horizontal dynamics if there exists a function V_h of the form (8) and positive constants $c_{hi}, i = 0,1,2,3$, satisfying the following inequalities along the trajectories of (1):

$$c_{h1}|x_1|^2 \leq V_{h1}(x_1) \leq c_{h2}|x_1|^2, \quad (12)$$
$$V_{h2}(x_2(i,0)) \leq c_{h0}|x_2(i,0)|^2, \quad (13)$$
$$\mathcal{D}V_h(x) \leq -c_{h3}|x_1|^2. \quad (14)$$

Moreover, there exist positive constants M_h and $0 < \zeta_h < 1$, such that

$$|x_1(n,j)|^2 \leq M_h \zeta_h^n, \quad \text{for all } j. \quad (15)$$

Proof. It follows from (14) that there exists $\bar{c}_{h3}$ such that $\bar{c}_{h3} \leq c_{h3}$ holds and then

$$\mathcal{D}V_h(x(i,j)) \leq -c_{h3}|x_1(i,j)|^2 \leq -\bar{c}_{h3}|x_1(i,j)|^2. \quad (16)$$

Then using (12), (16) can be rewritten as

$$V_{h2}(x_2(i,j+1)) - V_{h2}(x_2(i,j))$$
$$+ V_{h1}(x_1(i+1,j)) - \left(1 - \frac{\bar{c}_{h3}}{c_{h2}}\right)V_{h_1}(x_1(i,j)) \leq 0. \quad (17)$$

If $c_{h3} > c_{h2}(1 - \rho^{\frac{1}{2}})$ choose $\bar{c}_{h3} < c_{h2}(1 - \rho^{\frac{1}{2}})$, and if $c_{h3} \leq c_{h2}(1 - \rho^{\frac{1}{2}})$ choose $\bar{c}_{h3} < c_{h3} \leq c_{h2}(1 - \rho^{\frac{1}{2}})$. Hence in any case it is possible to choose $\bar{c}_{h3}$ such that $\bar{c}_{h3} \leq c_{h2}(1 - \rho^{\frac{1}{2}})$. Define $\zeta_1 = 1 - \frac{\bar{c}_{h3}}{c_{h2}}$ and then it follows from the previous inequality that

$$\rho^{\frac{1}{2}} \leq \zeta_1 < 1. \quad (18)$$

This last inequality guarantees the required convergence properties as shown next.

Rewrite (17) in the form

$$V_{h2}(x_2(i,j+1)) - V_{h2}(x_2(i,j))$$
$$+ V_{h1}(x_1(i+1,j)) - \zeta_1 V_{h_1}(x_1(i,j)) \leq 0. \quad (19)$$

Solving (19) with respect to $V_{h2}(i,m)$ gives

$$V_{h2}(x_2(i,m)) \leq V_{h2}(x_2(i,0))$$
$$+ \sum_{j=0}^{m-1}[V_{h1}(x_1(i+1,j)) - \zeta_1 V_{h1}(x_1(i,j))]. \quad (20)$$

Introduce

$$W_i(m) = \sum_{j=0}^{m-1} V_{h1}(x_1(i+1,j)), \quad (21)$$

and it follows from (20) that

$$W_{i+1}(m) - \zeta_1 W_i(m) \leq V_{h2}(x_2(i,0))$$
$$- V_{h2}(x_2(i,m)) \leq V_{h2}(x_2(i,0)). \quad (22)$$

Solving (22) with respect to $W_n(m)$ gives

$$W_n(m) \leq \zeta_1^n W_0(m) + \sum_{i=0}^{n-1} \zeta_1^{n-1-i} V_{h2}(x_2(i,0))$$
$$= \zeta_1^n \sum_{j=0}^{m-1} V_{h1}(x_1(0,j)) + \sum_{i=0}^{n-1} \zeta_1^{n-1-i} V_{h2}(x_2(i,0)). \quad (23)$$

Evaluating the last two terms in (23) and using (12), (13), (3) and (18) yields

$$\sum_{j=0}^{m-1} V_{h1}(x_1(n+1,j)) = \zeta_1^n \sum_{j=0}^{m-1} V_{h1}(x_1(0,j))$$
$$+ \sum_{i=0}^{n-1} \zeta_1^{n-1-i} V_{h2}(x_2(i,0))$$
$$\leq \left(\frac{c_{h2}\kappa_1}{(1-\rho)} + \frac{c_{h0}\kappa_2}{\zeta_1(1-\rho^{\frac{1}{2}})}\right)\zeta_1^n = C_1 \zeta_1^n. \quad (24)$$

Hence

$$|x_1(n,j)|^2 \leq \frac{C_1}{c_{h1}}\zeta_1^n, \quad \text{for all } j, \quad (25)$$

and also

$$\lim_{m \to \infty} \sum_{j=0}^{m-1} |x_1(n+1,j)|^2 \leq \frac{C_1}{c_{h1}}\zeta_1^n < \infty. \quad (26)$$

It follows from (26) that

$$|x_1(n+1,j)|^2 \to 0 \quad \text{as } j \to \infty \quad \text{for all } n. \quad (27)$$

Also (25) and (27) imply that $|x_1(i,j)|$ is bounded and $|x_1(i,j)| \to 0$ as $i + j \to \infty$. Moreover (15) is valid with $M_h = \frac{C_1}{c_{h1}}$ and $\zeta_h = \zeta_1$. This completes the proof.

The following theorem gives sufficient conditions for asymptotic stability in vertical dynamics of a system described by (1) and (2) in terms of a vector Lyapunov function of the form (9).

Theorem 2. A system described by (1) and (2) is asymptotically stable in the vertical dynamics if there exists a function V_v of the form (9) and positive constants $c_{vi}, i = 0, 1, 2, 3$, satisfying the following inequalities along the trajectories of (1):

$$V_{v1}(x_1(0,j)) \leq c_{v0}|x_1(0,j)|^2, \quad (28)$$
$$c_{v1}|x_2|^2 \leq V_{v2}(x_2) \leq c_{v2}|x_1|^2, \quad (29)$$
$$\mathcal{D}V_v(x) \leq -c_{v3}|x_2|^2. \quad (30)$$

Moreover, there exist positive constants M_v and $0 < \zeta_v < 1$, such that

$$|x_2(i,m)|^2 \leq M_v \zeta_v^m, \quad \text{for all } i. \quad (31)$$

Proof. The proof follows the outline of that for Theorem 1 with obvious modifications. The details are therefore omitted.

The following theorem gives sufficient conditions for asymptotic stability of a system described by (1) and (2) in terms of two vector Lyapunov functions of the form (8) and (9).

Theorem 3. A system described by (1) and (2) is asymptotically stable if there exists functions V_h and V_v of the form (8) and (9) and positive constants c_{hi} and $c_{vi}, i = 0, 1, 2, 3$, satisfying the inequalities (12)–(14) and (28)–(30), respectively, along the trajectories of (1).

Proof. The proof follows from an obvious combination of those for Theorems 1 and 2.

2.2 Exponential Stability Analysis

The theorem below gives sufficient conditions for exponential stability of systems described by (1) and (2) using a vector Lyapunov function of the form

$$V(x(i,j)) = \begin{bmatrix} V_1(x_1(i,j)) \\ V_2(x_2(i,j)) \end{bmatrix}, \quad (32)$$

where $V_1(x_1(i,j)) > o$ and $V_2(x_2(i,j)) > 0$. The discrete counterpart of the divergence operator for this function along the trajectories of (1) is

$$\mathcal{D}V(x(i,j)) = V_1(x_1(i+1,j)) - V_1(x_1(i,j))$$
$$+ V_2(x_2(i,j+1)) - V_2(x_2(i,j)). \quad (33)$$

Theorem 4. A system described by (1) and (2) is exponentially stable if there exist a function V of the form (32) and positive constants c_1, c_2 and c_3 satisfying the following inequalities along the trajectories of (1):

$$c_1|x_i|^2 \leq V_i(x_i) \leq c_2|x_i|^2, \ i = 1, 2, \qquad (34)$$

$$\mathcal{D}V(x) \leq -c_3|x|^2. \qquad (35)$$

Proof. The proof follows the outline of that for Theorem 3 with routine modifications. It also follows from Pakshin et al. (2015a) as a particular case when $n=2$.

Remark 2. It follows from the results above that Theorem 3.1 in (Yeganefar et al., 2013) gives sufficient conditions for exponential stability but (Yeganefar et al., 2013) claim that their result gives sufficient conditions for asymptotic stability only.

3. STABILITY OF NONLINEAR CONTINUOUS-TIME ROESSER TYPE SYSTEMS

The continuous-time case is more complicated because of the need to deal with integrals instead of sums and boundedness of integrals does not imply, in general, boundedness of the corresponding integrands. By Barbalat's lemma, the integrand tends to zero if it is uniformly continuous and the integral is bounded and progress is possible by imposing additional constraints on the gradients of the entries in the vector Lyapunov function.

The state-space model for the 2D systems considered in this section is of the form

$$\frac{\partial}{\partial t_i} x_i(t_1, t_2) = f_i(x_1(t_1, t_2), x_2(t_1, t_2)), \ i = 1, 2, \quad (36)$$

where $x_i \in \mathbb{R}^{n_i}, i = 1, 2$, are the horizontal and vertical state vectors, respectively, $f_i(x_1, x_2), i = 1, 2$, are vector-valued nonlinear functions satisfying the Lipschitz conditions

$$|f_i(x_1', x_2') - f_i(x_1'', x_2'')| \leq L(|x_1' - x_1''| + |x_2' - x_2''|),$$
$$x_1', x_1'' \in \mathbb{R}^{n_1} \ x_2', x_2'' \in \mathbb{R}^{n_2}, \ u \in \mathbb{R}^{n_u}, \quad (37)$$

and $f_i(0,0) = 0, i = 1, 2$, to ensure an equilibrium at the origin. The boundary conditions are assumed to be of the form

$$x_1(0, \tau) = g_1(\tau), \ x_2(\tau, 0) = g_2(\tau), \qquad (38)$$

and $g_i, i = 1, 2$, are known vector-valued functions such that

$$|g_i(\tau)|^2 \leq \kappa_i \mathrm{e}^{-\alpha \tau}, \qquad (39)$$

where $\alpha, \kappa_i, i = 1, 2$, are positive constants. From this point onwards all references to the functions $f_i, i = 1, 2$, will assume that they satisfy (38). Similarly, all references to the boundary conditions will assume that they satisfy (39). Also the notation

$$x(t_1, t_2) = [x_1(t_1, t_2)^T \ x_2(t_1, t_2)^T]^T$$

is used.

In some application similar to sorption processes considered in Dymkov et al. (2011) one or both independent variables are given on finite interval. If $t_2 \leq T_2$ and t_1 is unbounded or $t_1 \leq T_1$ and t_2 is unbounded it is reasonable to define stability by the following way

Definition 3. A 2D nonlinear system described by (36) and (38) is said to be asymptotically stable in the horizontal dynamics if $|x(t_1, t_2)|$ is bounded and

$$|x(t_1, t_2)| \to 0 \ \text{as} \ t_1 \to \infty \qquad (40)$$

uniformly in $t_2 \in [0, T_2]$.

Definition 4. A 2D nonlinear system described by (36) and (38) is said to be asymptotically stable in the vertical dynamics if $|x(t_1, t_2)|$ is bounded and

$$|x(t_1, t_2)| \to 0 \ \text{as} \ t_2 \to \infty \qquad (41)$$

uniformly in $t_1 \in [0, T_1]$.

Definition 5. A 2D nonlinear system described by (36) and (38) is said to be exponentially stable if there exist scalars $\kappa > 0$ and $\lambda_0 > 0$ such that

$$|x_1(t_1, \ t_2)|^2 + |x_2(t_1, \ t_2)|^2 \leq \kappa e^{-\lambda_0(t_1+t_2)}. \qquad (42)$$

3.1 Asymptotic Stability Analysis

The asymptotic stability property of a system described by (36) and (38) is analyzed using vector Lyapunov functions of the form

$$V_h(x(t_1, t_2)) = \begin{bmatrix} V_{h1}(x_1(t_1, t_2)) \\ V_{h2}(x_2(t_1, t_2)) \end{bmatrix}, \qquad (43)$$

where $V_{hi}(0) = 0, i = 1, 2, V_{h1}(x_1) > 0, V_{h2}(x_2) \geq 0$,

$$V_v(x(t_1, t_2)) = \begin{bmatrix} V_{v1}(x_1(t_1, t_2)) \\ V_{v2}(x_2(t_1, t_2)) \end{bmatrix}, \qquad (44)$$

with $V_i(0) = 0, i = 1, 2$. $V_{v1}(x_1) \geq 0, \ V_{v2}(x_2) > 0$ The divergence operators of theses function along the trajectories of (36) are

$$\mathrm{div}V_h(x(t_1, t_2)) = \frac{\partial V_{h1}(x_1(t_1, t_2))}{\partial t_1} + \frac{\partial V_{h2}(x(t_1, t_2))}{\partial t_2}. \qquad (45)$$

$$\mathrm{div}V_v(x(t_1, t_2)) = \frac{\partial V_{v1}(x_1(t_1, t_2))}{\partial t_1} + \frac{\partial V_{v2}(x(t_1, t_2))}{\partial t_2}. \qquad (46)$$

Theorem 5. A system described by (36) and (38) is asymptotically stable in the horizontal dynamics if there exists a function V_h of the form (43) such that $V_{h1}(x_1) > 0$, $V_{h2}(x_2) \geq 0$, and positive constants $c_{hi}, i = 0, 1, 2, 3$, satisfying the following inequalities along the trajectories of (36):

$$c_{h1}|x_1|^2 \leq V_{h1}(x_1) \leq c_{h2}|x_1|^2, \qquad (47)$$

$$V_{h2}(x_2(t_1, 0)) \leq c_{h0}|x_2(t_1, 0)|^2, \qquad (48)$$

$$\mathrm{div}V_h(x) \leq -c_{h3}|x_1|^2, \qquad (49)$$

$$\frac{\partial V_2(x_2)}{\partial x_2} \leq c_{h4}|x_2|. \qquad (50)$$

Proof. It follows from (47) that there exists $\bar{c}_3 < c_3$ such that $\lambda = \frac{\bar{c}_3}{c_2}$ and select $\alpha > \lambda_1$. Moreover

$$\mathrm{div}V_h(x(t_1, t_2)) \leq -c_{h3}|x_1(t_1, t_2)|^2$$
$$\leq -\bar{c}_{h3}|x_1(t_1, t_2)|^2. \qquad (51)$$

and using (48), (51) can be rewritten as

$$\frac{\partial V_{h2}(x_2(t_1, t_2))}{\partial t_2} \leq -\frac{\partial V_{h1}(x_1(t_1, t_2))}{\partial t_1}$$
$$- \lambda_1 V_{h1}(x_1(t_1, t_2)). \quad (52)$$

Solving the inequality (52) with respect to $V_{h2}(x_2(t_1, t_2))$ gives

$$V_{h2}(x_2(t_1, t_2)) \leq V_{h2}(x_2(t_1, 0))$$
$$- \int_0^{t_2} \left[\frac{\partial V_{h1}(x_1(t_1, \tau_2))}{\partial t_1} + \lambda_1 V_{h1}(x_1(t_1, \tau_2)) \right] d\tau_2. \quad (53)$$

and it follows from (53) that

$$\frac{\partial}{\partial t_1} \int_0^{t_2} V_{h1}(x_1(t_1,\tau_2))d\tau_2 + \lambda_1 \int_0^{t_2} V_{h1}(x_1(t_1,\tau_2))d\tau_2$$
$$\leq V_{h2}(x_2(t_1,0)) - V_{h2}(x_2(t_1,t_2)) \quad (54)$$

Also solving (54) with respect to

$$\int_0^{t_2} V_{h1}(x_1(t_1,\tau_2))d\tau_2$$

gives

$$\int_0^{t_2} V_{h1}(x_1(t_1,\tau_2))d\tau_2 \leq e^{-\lambda_1 t_1} \int_0^{t_1} V_{h1}(x_1(0,\tau_2))d\tau_2$$
$$+ \int_0^{t_1} e^{-\lambda_1(t_1-\tau_1)} V_{h2}(x_2(\tau_1,0))d\tau_1$$
$$- \int_0^{t_1} e^{-\lambda_1(t_1-\tau_1)} V_{h2}(x_2(\tau_1,t_2))d\tau_1. \quad (55)$$

Evaluating (55) and using (47), (48) and (39) yields

$$\int_0^{t_2} V_{h1}(x_1(t_1,\tau_2))d\tau_2$$
$$\leq \frac{c_{h2}\kappa_1}{\alpha} + \frac{c_{h0}\kappa_2}{(\alpha - \lambda_1)} = C_1 e^{-\lambda_1 t_1}. \quad (56)$$

and

$$\int_0^{t_2} |x_1(t_1,\tau_2)|^2 d\tau_2 \leq \frac{C_1}{c_{h1}} e^{-\lambda_1 t_1}. \quad (57)$$

Evaluating $\frac{\partial V_{h2}(x_2(t_1,t_2))}{\partial t_2}$ and using(37) and (50) gives

$$\frac{\partial V_{h2}(x_2(t_1,t_2))}{\partial t_2}$$
$$\geq -\left|\frac{\partial V_{h2}(x_2(t_1,t_2))}{\partial x_2}\right| |f_2(x_1(t_1,t_2),x_2(t_1,t_2)|$$
$$\geq -c_{h4}L(|x_2(t_1,t_2)| + \varepsilon|x_1(t_1,t_2)|)(|x_1(t_1,t_2)|$$
$$+ |x_2(t_1,t_2)|) \geq -2c_{h4}L[(\varepsilon+1)(2\sqrt{\varepsilon})^{-1}|x_2(t_1,t_2)|$$
$$+ \sqrt{\varepsilon}|x_1(t_1,t_2)|)^2 \geq -2c_{h4}L(2((\varepsilon$$
$$+1)(2\sqrt{\varepsilon})^{-1}|x_2(t_1,t_2)|)^2 + 2(\sqrt{\varepsilon}|x_1(t_1,t_2)|)^2)$$
$$\geq -c_{h4}\varepsilon^{-1}L(\varepsilon+1)^2|x_2(t_1,t_2)|^2$$
$$- 4c_{h4}c_{h1}^{-1}L\varepsilon V_{h1}(x_1(t_1,t_2)). \quad (58)$$

Substituting (58) into (52) gives

$$\frac{\partial V_{h1}(x_1(t_1,t_2))}{\partial t_1} + \lambda_0 V_{h1}(x_1(t_1,t_2)) - \beta|x_2(t_1,t_2)|^2 \leq 0,$$
$$\quad (59)$$

where $\lambda_0 = \lambda_1 - \frac{4c_{h4}L\varepsilon}{c_{h1}}$, $\beta = \frac{c_{h4}L(\varepsilon+1)^2}{\varepsilon}$. Choose ε sufficiently small such that $\lambda_0 > 0$ and solving (59) with respect to $V_{h1}(x_1(t_1,t_2))$ gives

$$V_{h1}(x_1(t_1,t_2)) \leq V_{h1}(x_1(0,t_2))e^{-\lambda_0 t_1}$$
$$+ \beta \int_0^{t_1} |x_2(\tau_1,t_2)|^2 e^{-\lambda_0(t_1-\tau_1)}d\tau_1. \quad (60)$$

Evaluating $\frac{\partial|x_2(t_1,t_2)|^2}{\partial t_2}$ in the same way as in (58) gives

$$\frac{\partial|x_2(t_1,t_2)|^2}{\partial t_2} \leq \gamma|x_2(t_1,t_2)|^2 + \alpha V_{h1}(x_1(t_1,t_2)), \quad (61)$$

where $\gamma = \frac{2L(\varepsilon+1)^2}{\varepsilon}$, $\alpha = \frac{8L\varepsilon}{c_{h1}}$. Solving (61) and evaluating of the right-hand side and using (56) and (39) gives

$$|x_2(t_1,t_2)|^2 \leq |x_2(0,t_2)|^2 e^{\gamma t_2}$$
$$+ \delta \int_0^{t_2} V_{h1}(x_1(t_1,t_2))e^{\gamma(t_2-\tau_2)}d\tau_2$$
$$\leq e^{\gamma t_2}(\kappa_2 e^{-\alpha t_1} + C_1\delta) \leq e^{\gamma T_2}(\kappa_2 + C_1\delta). \quad (62)$$

Then by (60), (62) and (47)

$$|x_1(t_1,t_2)|^2 \leq \frac{1}{c_{h1}} V_{h1}(x_1(t_1,t_2))$$
$$\leq \frac{\kappa_1 c_{h2}}{c_{h1}} + \frac{\beta e^{\gamma T_2}(\kappa_2 + C_1\delta)}{c_{h1}\lambda_0} < \infty.$$

Hence $|x_1(t_1,t_2)|^2$ is bounded. Also by (57)

$$\lim_{t_1 \to \infty} \int_0^{t_2} |x_1(t_1,\tau_2)|^2 d\tau_2 = 0. \quad (63)$$

and hence $|x_1(t_1,t)| \to 0$ as $t_1 \to \infty$ uniformly in $t \in [0,T_2]$ and the proof is complete.

The following theorem, whose proof is omitted as it follows the same steps as that for the previous result, gives sufficient conditions of asymptotic stability in the vertical dynamics.

Theorem 6. A system described by (36) and (38) is asymptotically stable in vertical dynamics if there exists a function V_v of the form (44), such that $V_{v1}(x_1) \geq 0$, $V_{h2}(x_2) > 0$ and positive constants $c_{vi}, i = 0,1,2,3$, satisfying the following inequalities along the trajectories of (36):

$$V_{v1}(x_1(0,t_2)) \leq c_{v0}|x_1(0,t_2)|^2,$$
$$c_{v1}|x_2|^2 \leq V_{v2}(x_2) \leq c_{v2}|x_2|^2,$$
$$\text{div}V_v(x) \leq -c_{h3}|x_2|^2,$$
$$\frac{\partial V_1(x_1)}{\partial x_1} \leq c_{v4}|x_1|.$$

3.2 Exponential Stability Analysis

The following theorem gives sufficient conditions for exponential stability of systems described by (36) and (38) in terms of properties of the vector Lyapunov function.

$$V(x(t_1,t_2)) = \begin{bmatrix} V_1(x_1(t_1,t_2)) \\ V_2(x_2(t_1,t_2)) \end{bmatrix}, \quad (64)$$

where $V_i(x_i) > 0, i = 1,2$. The divergence operator of this function along the trajectories of (36) is

$$\text{div}V(x(t_1,t_2)) = \frac{\partial V_1(x_1(t_1,t_2))}{\partial t_1} + \frac{\partial V_2(x_n(t_1,t_2))}{\partial t_2}. \quad (65)$$

Theorem 7. A system described by (36) and (38) is exponentially stable if there exists a function V of the form (64) and positive constants $c_i, i = 1,2,3,4$ satisfying the following inequalities along the trajectories of (36):

$$c_1|x_i|^2 \leq V_i(x_i) \leq c_2|x_i|^2, \ i=1,2, \quad (66)$$
$$\text{div}V(x) \leq -c_3|x|^2, \quad (67)$$
$$\frac{\partial V_i(x_i)}{\partial x_i} \leq c_4|x_i|, \ i=1,2. \quad (68)$$

Proof. Following the steps of the proof of Theorem 5 in this case gives

$$\frac{\partial V_1(x_1(t_1,t_2))}{\partial t_1} + \lambda V_1(x_1(t_1,t_2)) \leq -\frac{\partial V_2(x_2(t_1,t_2))}{\partial t_2}$$
$$- \lambda V_2(x_2(t_1,t_2)), \quad (69)$$

where $\lambda = \frac{\bar{c}_3}{c_2}$ and select $\lambda < \alpha$. Solving the inequality (69) with respect to $V_1(x_1(t_1, t_2))$ gives

$$\frac{\partial}{\partial t_2} \int_0^{t_1} V_2(x_2(\tau_1, t_2)) \mathrm{e}^{-\lambda(t_1 - \tau_1)} d\tau_1$$

$$+ \lambda \int_0^{t_1} \mathrm{e}^{-\lambda(t_1 - \tau_1)} V_2(x_2(\tau_1, t_2)) d\tau_1$$

$$\leq V_1(x_1(0, t_2)) \mathrm{e}^{-\lambda t_1} - V_1(x_1(t_1, t_2)) \quad (70)$$

and solving (70) with respect to

$$\int_0^{t_1} \mathrm{e}^{-\lambda(t_1 - \tau_1)} V_2(x_2(\tau_1, t_2)) d\tau_1$$

yields

$$\mathrm{e}^{\lambda t_2} \int_0^{t_1} \mathrm{e}^{\lambda \tau_1} V_2(x_2(\tau_1, t_2)) d\tau_1$$

$$+ \mathrm{e}^{\lambda t_1} \int_0^{t_2} \mathrm{e}^{\lambda \tau_2} V_1(x_1(t_1, \tau_2)) d\tau_2$$

$$\leq \int_0^{t_1} \mathrm{e}^{\lambda \tau_1} V_2(x_2(\tau_1, 0)) d\tau_1 + \int_0^{t_2} \mathrm{e}^{\lambda \tau_2} V_1(x_1(0, \tau_2)) d\tau_2. \quad (71)$$

Evaluating the right-hand side of this last inequality and using (66) and (67):

$$\int_0^{t_1} \mathrm{e}^{\lambda \tau_1} V_2(x_2(\tau_1, 0)) d\tau_1 + \int_0^{t_2} \mathrm{e}^{\lambda \tau_2} V_1(x_1(0, \tau_2)) d\tau_2$$

$$\leq \int_0^\infty c_2 \kappa_2 \mathrm{e}^{-\gamma \tau_1} d\tau_1 + \int_0^\infty c_2 \kappa_1 \mathrm{e}^{-\gamma \tau_2} d\tau_2$$

$$= \frac{c_2(\kappa_1 + \kappa_2)}{\gamma}, \quad (72)$$

where $\gamma = \alpha - \lambda > 0$. The right-hand side of (72) is bounded for all $t_i \in [0, \infty], i = 1, 2$ and hence both the integrals on the left-hand side of (71) are bounded for all $t_i \in [0, \infty], i = 1, 2$. Evaluating $\frac{\partial V_1(x_1)}{\partial t_1}$ and using (37) and (68) gives

$$\frac{\partial V_2(x_2(t_1, t_2))}{\partial t_2} + \lambda_0 V_2(x_2(t_1, t_2)) - \beta V_1(x_1(t_1, t_2)) \leq 0, \quad (73)$$

where ε is sufficiently small such that $0 < \lambda_0 = \lambda - \frac{4 L c_4 \varepsilon}{c_1}$ and $\beta = \frac{L c_4 (\varepsilon + 1)^2}{c_1 \varepsilon}$. Similarly

$$\frac{\partial V_1(x_1(t_1, t_2))}{\partial t_1} + \lambda_0 V_1(x_1(t_1, t_2)) - \beta V_2(x_2(t_1, t_2)) \leq 0. \quad (74)$$

Solving (73) and (74) gives, on multiplying both sides in each case by $\mathrm{e}^{\lambda_0(t_1 + t_2)}$,

$$V_2(x_2(t_1, t_2)) \mathrm{e}^{\lambda_0(t_1 + t_2)}$$

$$\leq \mathrm{e}^{\lambda_0 t_1} V_2(x_2(t_1, 0)) + \beta \mathrm{e}^{\lambda t_1} \int_0^{t_2} \mathrm{e}^{\lambda \tau_2} V_1(x_1(t_1, \tau_2)) d\tau_2,$$

$$V_1(x_1(t_1, t_2)) \mathrm{e}^{\lambda_0(t_1 + t_2)}$$

$$\leq \mathrm{e}^{\lambda_0 t_2} V_1(x_1(0, t_2)) + \beta \mathrm{e}^{\lambda t_2} \int_0^{t_1} \mathrm{e}^{\lambda \tau_1} V_2(x_1(\tau_1, t_2)) d\tau_1.$$

By (71) and (72), the right-hand sides of the last two inequalities are bounded for $t_i \in [0, \infty], i = 1, 2$ and it follows from these inequalities on using (66) that (42) holds and the proof is complete.

4. CONCLUSIONS AND FUTURE RESEARCH

This paper has established sufficient conditions of asymptotic stability of both discrete and continuous-time 2D systems described by state-space models of the Roesser form in terms of properties of two vector Lyapunov functions. The novelty of these new results is that their conditions guarantee asymptotic stability as opposed to the stronger property of exponential stability and hence the requirements imposed on the Lyapunov functions. Ongoing research aims to develop a stabilization theory for these systems using the new stability results in this paper as a basis. The less stringent conditions on the vector Lyapunov functions should be of significant benefit in this respect. Also systems described by 2D Fornasini-Marchesini models should be considered.

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