Higher-order Iterative Learning Control Law Design using Linear Repetitive Process Theory: Convergence and Robustness

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Abstract: Iterative learning control has been developed for processes or systems that complete the same finite duration task over and over again. The mode of operation is that after each execution is complete the system resets to the starting location, the next execution is completed and so on. Each execution is known as a trial and its duration is termed the trial length. Once each trial is complete the information generated is available for use in computing the control input for the next trial. This paper uses the repetitive process setting to develop new results on the design of higher-order ILC control laws for discrete dynamics. The new results include conditions that guarantee error convergence and design in the presence of model uncertainty.

Keywords: iterative learning control, higher-order laws, 2D systems.

1. INTRODUCTION

The first research on Iterative Learning Control (ILC) developed a derivative, or D-type, law for speed control of a voltage-controlled dc-servomotor. Since this first work ILC has been an established area of research and one starting point for the literature is the survey papers (Ahn et al., 2007; Bristow et al., 2006). A large volume of the currently available literature assumes a discrete model of the dynamics is available, by sampling if required and hence direct digital design.

The novel feature of ILC is that all information generated on previous trials is known and can be used in the control law. In higher-order ILC, there are contributions from a finite number $M > 1$ of previous trials. The notation for variables in this paper is of the form $h_k(p)$, $0 \leq p \leq \alpha - 1$, where h is the vector or scalar valued variable of interest, the integer $k \geq 0$ is the trial number and α is the number of samples along the trial (α times the sampling period gives the trial length).

Suppose that $r(p)$, $0 \leq p \leq \alpha - 1$, denotes the supplied reference vector or signal. The error on trial k is $e_k = r(p) - y_k(p)$, where $y_k(p)$ is the output on this trial. Then the ILC design problem is to construct a control sequence $\{u_k\}$ such that the error sequence $\{e_k\}$ converges, i.e., sequentially improve the tracking error from trial-to-trial. Moreover, once a trial is complete all information generated on this trial is available for use in computing the control signal for the next trial and hence non-causal terms are allowed in the current trial input. Higher-order ILC (Bien and Huh, 1989), where information from the previous $M > 1$ previous trials is used in the computation of the current trial control input, offers the possibility of a higher error convergence speed compared to the standard form of an ILC law where only previous trial information is used.

Given the finite trial length, one approach to ILC design for discrete systems is to represent the dynamics by an equivalent standard systems model, where, e.g., the trial output is represented by a column vector formed from the values at the sample instants along the trial. This is often termed lifted ILC design and given the reference trajectory, the trial-to-trial error dynamics can be written as a discrete difference equation in the trial number. The basic task then is to design the ILC law such that trial-to-trial error convergence occurs. In this design setting it is assumed that the system is stable but if not a preliminary feedback control must be applied to ensure stability and acceptable transient dynamics along the trials and ILC designed for the resulting controlled dynamics.

Lifting based ILC design is a two-stage procedure and an alternative is to exploit 2D systems theory where in this setting one indeterminate is the trial number and the other the along the trial variable. Repetitive processes are a distinct class of 2D systems where information propagation in one direction only occurs over a finite duration this is an inherent property of the dynamics and not an assumption introduced for analysis purposes. A detailed treatment of repetitive processes, including industrial examples and how this setting can be used for analysis of other dynamics can be found in (Rogers et al., 2007).

As the trial length is finite, repetitive processes are a closer match to ILC and designs using this setting have seen experimental verification on a gantry robot that replicates the pick and place operation, see, e.g. (Hladowski et al., 2010; Paszke et al., 2016). In the repetitive process setting, it is possible to do control law design for error convergence and transient dynamics along the trials in one step. Moreover, unlike the lifted approach, this setting extends naturally to differential dynamics, i.e., to cases where design by emulation is the preferred or only setting for design.

Robustness, as in other areas, is an important issue in ILC design. In standard linear systems theory one commonly used setting for robustness and control law design is to assume that the uncertainty present lies in a specified model class. Two commonly used classes are termed norm-bounded and polytopic, respectively and in this paper the former is considered in the ILC setting. The result is LMI-based control law design algorithms that extend naturally to the polytopic case.

In recent work (Wang et al., 2016) it was shown that higher-order ILC laws can be developed using linear repetitive process theory. The performance of the resulting design was illustrated using numerical simulations. These previous results show that higher-order ILC law is able to achieve tracking error convergence, but the nature of the convergence was unanswered and, moreover, a rigorous proof of the convergence properties was also missing. In this paper, this previous work is extended by the development of a novel higher-order ILC design with guarantee error convergence, which, in fact, has a form of monotonic (trial-to-trial) convergence. A rigorous convergence proof is developed and it is shown that design in the presence of uncertainty can also be undertaken in the repetitive process setting.

As an essential step to experimental validation, the design algorithms developed in this paper are applied to a model of a physical system. This is a model for one axis of a gantry robot executing a pick and place operation, to which ILC is particularly suited. The model used was obtained by frequency response tests on the robot, which has been used to experimentally verify a number of ILC laws, see, e.g. (Hladowski et al., 2010; Paszke et al., 2016).

Throughout this paper, the null and identity matrices with compatible dimensions are denoted by 0 and I respectively. Also $M \succ 0$ ($\prec 0$) denotes a real symmetric positive (negative) definite matrix and $X \preceq Y$ is used to denote the case when $X-Y$ is a negative semi-definite matrix. Finally, the symbol $\{*\}$ denotes block entries in a symmetric matrix and $\rho(\cdot)$ the spectral radius of its matrix argument, i.e., if a square matrix, say H, has eigenvalues h_i, $1 \leq i \leq j$, $\rho(H) = \max_{1\leq i\leq j} |h_i|$.

2. BACKGROUND

Consider the discrete linear time-invariant state-space model described in the ILC setting by

$$\begin{aligned} x_k(p+1) &= Ax_k(p) + Bu_k(p), \\ y_k(p) &= Cx_k(p),\ p = 0, 1, ..., \alpha - 1, \end{aligned} \tag{1}$$

where $k \geq 0$ is the trial number, $\alpha < \infty$ is the finite number of samples along the trial, i.e., α times the sampling period is equal to the trial length. Also on trial k $x_k(p) \in \mathbb{R}^m$ is the state vector, $y_k(p) \in \mathbb{R}^n$ is the output vector and $u_k(p) \in \mathbb{R}^s$ is the input vector. Let $r(p) \in \mathbb{R}^n$ denote the reference vector and hence the error on kth trial is

$$e_k(p) = r(p) - y_k(p). \tag{2}$$

The ILC design requirement of forcing the error sequence $\{e_k\}$ to converge in k can be formulated mathematically as

$$\lim_{k\to\infty} \|e_k\| = 0, \quad \lim_{k\to\infty} \|u_k - u_\infty\| = 0, \tag{3}$$

where u_∞ is termed the learned control and $\|\cdot\|$ denotes the norm on the underlying function space. One class of widely considered ILC laws computes the current trial input as the sum of that used on the previous trial plus a correction term computed using previous trial data, i.e.,

$$u_{k+1}(p) = u_k(p) + \Delta(u_{k+1}(p)), \tag{4}$$

where $\Delta(u_{k+1}(p))$ is the correction to be designed.

For analysis purposes only, define the following vector from (1)

$$\eta_{k+1}(p+1) = x_{k+1}(p) - x_k(p), \tag{5}$$

i.e., the difference between the state vectors on successive trials. Also consider the case when

$$\Delta(u_{k+1}(p)) = K_1\eta_{k+1}(p+1) + K_2e_k(p+1), \tag{6}$$

where K_1 and K_2 are compatibly dimensioned matrices to be designed. Combing (1), (4) and (6), gives

$$\begin{aligned} \eta_{k+1}(p+1) &= \hat{A}\eta_{k+1}(p) + \hat{B}e_k(p), \\ e_{k+1}(p) &= \hat{C}\eta_{k+1}(p) + \hat{D}e_k(p), \end{aligned} \tag{7}$$

where

$$\begin{aligned} &\hat{A} = A + BK_1,\ \hat{B} = BK_2, \\ &\hat{C} = -C(A + BK_1),\ \hat{D} = I - CBK_2. \end{aligned} \tag{8}$$

Repetitive processes make a series of passes through a set of dynamics defined over a finite duration known as the pass length. Once each pass is complete, the process resets to the starting location and the next pass can begin, either immediately after the resetting operation is complete or after a further period of time has elapsed. On each pass an output, termed the pass profile, is produced, which acts as a forcing function on, and hence contributes to, the dynamics of the next pass. A detailed treatment of the dynamics and control problems for linear repetitive processes can be found in (Rogers et al., 2007). The ILC dynamics (6) are those of a discrete linear repetitive process with previous pass profile $e_k(p)$, current pass state vector $\eta_{k+1}(p)$ and no control input. From this point onwards, pass is replaced by trial to conform with the majority of the ILC literature.

The stability theory for linear repetitive processes (Rogers et al., 2007) requires that a bounded initial trial profile vector produces a bounded sequence of trial profile vectors, where boundedness is defined in terms of the norm on the underlying function space. This theory is based on an abstract model in a Banach space setting that includes all constant trial length linear examples as special cases.

Two forms of stability are defined, termed asymptotic and along the trial, respectively, where the former imposes the boundedness property over the finite and fixed trial length and the latter is stronger since it requires this property for all possible values of the trial length. Moreover, this latter property can be analyzed mathematically by considering $\alpha \to \infty$.

Theorem 1. (Rogers et al., 2007) The state-space model (7) describing the ILC dynamics is stable along the trial if and only if

- $\rho(\hat{D}) < 1$,
- $\rho(\hat{A}) < 1$,
- all eigenvalues of the transfer-function matrix $G(z) = \hat{C}(zI - \hat{A})^{-1}\hat{B} + \hat{D}$ have modulus strictly less than unity for all $|z| = 1$.

In *Theorem* 1 the first entry is the necessary and sufficient condition for asymptotic stability. This is precisely the condition for error convergence of the ILC dynamics under a control law of the form given above with $K_1 = 0$ (Kurek and Zaremba, 1993). Such a control law does not require that the state dynamics is stable. Hence if (1) is unstable a preliminary stabilizing control law would be required to prevent unacceptable transient dynamics along the trials. In the lifting approach this stabilization requirement is met by the design and application of a stabilizing control law, e.g., state feedback and then the ILC law is designed for the resulting controlled dynamics. This requirement is automatically ensured by the repetitive process result.

An alternative to working in the frequency domain for repetitive process control law design is to use a Lyapunov function approach. The following result is the 2D Lyapunov function interpretation of stability along the trial for the systems considered in this paper. Although sufficient but not necessary this result allows the use of LMI based computations in design and was used in the experimental verification results in (Hladowski et al., 2010).

Theorem 2. (Rogers et al., 2007) The state-space model (7) describing the ILC dynamics is stable along the trial if there exist matrices $P_1 \succ 0$ and $P_2 \succ 0$ such that the following LMI holds

$$\begin{bmatrix} -P & (*) \\ \Phi P & -P \end{bmatrix} < 0, \tag{9}$$

where

$$\Phi = \begin{bmatrix} \hat{A} & \hat{B} \\ \hat{C} & \hat{D} \end{bmatrix}, \quad P = \begin{bmatrix} P_1 & 0 \\ 0 & P_2 \end{bmatrix}. \tag{10}$$

3. HIGHER-ORDER ILC DESIGN

One feature of ILC is that data from all previous trials are available for use in the design and application of control laws. A generalization of (4) is to select the control law as

$$u_{k+1}(p) = u_k(p) + K_x \eta_{k+1}(p+1) + \sum_{j=1}^{M} K_{j-1} e_{k-j+1}(p+1), \tag{11}$$

where the integer $M > 1$ denotes the number of previous trial terms used. This is termed a higher-order ILC law and in this section the design to guarantee error convergence in k is developed using linear repetitive process theory.

Substituting (11) into (1) gives the controlled ILC dynamics state-space model

$$\begin{aligned} \eta_{k+1}(p+1) &= \hat{A}\eta_{k+1}(p) + \sum_{j=1}^{M} \hat{B}_{j-1} e_{k-j+1}(p), \\ e_{k+1}(p) &= \hat{C}\eta_{k+1}(p) + \sum_{j=1}^{M} \hat{D}_{j-1} e_{k-j+1}(p), \end{aligned} \tag{12}$$

where

$$\begin{aligned} &\hat{A} = A + BK_x,\ \hat{B}_0 = BK_0,\ \hat{B}_{j-1} = BK_{j-1}, \\ &\hat{C} = -CA - CBK_x,\ \hat{D}_0 = I - CBK_0, \\ &\hat{D}_{j-1} = -CBK_{j-1}, \quad j = 2, ..., M. \end{aligned} \tag{13}$$

In the repetitive process setting this last state-space model is termed non-unit memory , with M denoting the memory length. If $M = 1$ the model of the previous section, termed a unit memory repetitive process, is recovered.

The analysis in this paper requires (12) to be written as a unit memory process, i.e.,

$$\begin{aligned} \eta_{k+1}(p+1) &= \hat{A}\eta_{k+1}(p) + \bar{B}\bar{e}_k(p), \\ \bar{e}_{k+1}(p) &= \bar{C}\eta_{k+1}(p) + \bar{D}\bar{e}_k(p), \end{aligned} \tag{14}$$

where

$$\bar{e}_k(p) = \left[e_{k-M+1}^T(p) \cdots e_{k-1}^T(p)\ e_k^T(p)^T \right],$$

$$\bar{B} = \left[\hat{B}_{M-1}^T \cdots \hat{B}_1^T\ \hat{B}_0^T \right]^T,$$

$$\bar{C} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ \hat{C} \end{bmatrix}, \quad \bar{D} = \begin{bmatrix} 0 & I & 0 & \cdots & 0 \\ 0 & 0 & I & \ddots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & I \\ \hat{D}_{M-1} & \hat{D}_{M-2} & \hat{D}_{M-3} & \cdots & \hat{D}_0 \end{bmatrix}. \tag{15}$$

Applying the z-transform, i.e., $zx_k(p) = x_k(p+1)$, (see (Bristow et al., 2006) for the details of how this transform can be applied despite the finite trial length) to (14) gives

$$\bar{e}_{k+1}(z) = G(z)\bar{e}_k(z), \tag{16}$$

where $G(z) = \bar{C}(zI - \hat{A})^{-1}\bar{B} + \bar{D}$. The terms in transfer-function matrix $G(z)$ that governs trial-to-trial error convergence is the bottom block row, i.e.,

$$e_{k+1}(z) = [G_{M-1}(z), \cdots, G_0(z)]\bar{e}_k(z). \tag{17}$$

The design objective is to select the control law matrices $K_x, K_{i-1}, i = 1, \ldots, M$, such that the norm of the above transfer-function is sufficiently small and thus convergence can be achieved.

Theorem 3. For a given $\gamma > 0$ the discrete linear repetitive process representing the ILC dynamics described by (14) is stable along the trial and satisfies

$$\|[G_{M-1}(z), \cdots, G_0(z)]\|_\infty < \gamma, \tag{18}$$

if there exist matrices $P_1 \succ 0$, N_1 and N_2 with $\mu = \gamma^2$ such that the following LMIs are feasible

$$\begin{bmatrix} -P + Q & * \\ A_1 P + B_1 N & -P \end{bmatrix} \prec 0, \tag{19}$$

$$\begin{bmatrix} -P & * \\ A_2 P + B_2 N & -P \end{bmatrix} \prec 0, \tag{20}$$

where

$$A_1 = \begin{bmatrix} A & 0 \cdots & 0 \\ -CA & 0 \cdots & 1 \end{bmatrix}, \ B_1 = \begin{bmatrix} B & B \\ -CB & -CB \end{bmatrix},$$

$$A_2 = \begin{bmatrix} A & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -CA & 0 & 0 & \cdots & 1 \end{bmatrix}, \ B_2 = \begin{bmatrix} B & B \\ 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \\ -CB & -CB \end{bmatrix}, \tag{21}$$

where

$P = diag\{P_1, I\}, Q = diag\{0, (1-\mu)I\}, N = diag\{N_1, N_2\}$.

If these LMIs are feasible, stabilizing control law matrices are given by

$$K_x = N_1 P_1^{-1},\ [K_{M-1}, ..., \ K_1,\ K_0] = N_2. \tag{22}$$

Proof 1. The LMI (20) can be written as

$$\Phi^T P \Phi - P \prec 0, \tag{23}$$

where

$$\Phi = \begin{bmatrix} \hat{A} & \bar{B} \\ \bar{C} & \bar{D} \end{bmatrix} \tag{24}$$

and the controlled dynamics is stable along the trial (see (Rogers et al., 2007) for details). Also (18) can be written as

$$\begin{bmatrix} (zI-\hat{A})^{-1}\bar{B} \\ I \end{bmatrix}^* \begin{bmatrix} \hat{C} & \widetilde{D} \\ 0 & I \end{bmatrix}^* \begin{bmatrix} I & 0 \\ 0 & -\mu \end{bmatrix} \begin{bmatrix} \hat{C} & \widetilde{D} \\ 0 & I \end{bmatrix} \begin{bmatrix} (zI-\hat{A})^{-1}\bar{B} \\ I \end{bmatrix} < 0. \tag{25}$$

or, on applying the bounded real lemma (Rogers et al. (2007) gives a reference to the original on this lemma),

$$\begin{bmatrix} \hat{A}^T P_1 \hat{A} - P_1 & \hat{A}^T P_1 \bar{B} \\ \bar{B}^T P_1 \hat{A} & \bar{B}^T P_1 \bar{B} \end{bmatrix} + \begin{bmatrix} \hat{C} & \widetilde{D} \\ 0 & I \end{bmatrix}^* \begin{bmatrix} I & 0 \\ 0 & -\mu I \end{bmatrix} \begin{bmatrix} \hat{C} & \widetilde{D} \\ 0 & I \end{bmatrix} < 0. \tag{26}$$

On using the Schur complement formula, the LMI (19) can be obtained from (26) and proof is complete.

The next result shows that when γ is small, the (trial-to-trial) tracking error is guaranteed to converge to zero.

Theorem 4. If for $\gamma \in [0, 1/\sqrt{M})$ the design in Theorem 3 is feasible, the tracking error converges to zero as $k \to \infty$, i.e.

$$\lim_{k \to \infty} e_k = 0.$$

Moreover, the convergence is monotonic in the sense that

$$\max\{\|e_{k+1}\|^2, \ldots, \|e_{k-M+2}\|^2\} < q \max\{\|e_k\|^2, \ldots, \|e_{k-M+1}\|^2\}$$

where $q = \gamma^2 M < 1$.

Proof 2. When the design algorithm of Theorem 3 is feasible, $\|G_{M-1}(z), \ldots, G_0(z)\| < \gamma$ and therefore

$$\|e_{k+1}\|_2^2 < \gamma^2 \sum_{i=0}^{M-1} \|e_{k-i}\|_2^2 \tag{27}$$

Define $q = \gamma^2 M$. Then as $\gamma < 1/\sqrt{M}$, $q < 1$ and hence

$$\begin{aligned} \|e_{k+1}\|_2^2 &< \gamma^2 * (\|e_{k-M+1}\|_2^2 + ... + \|e_k\|_2^2) \\ &< q * max\{\|e_{k-M+1}\|_2^2, ..., \|e_k\|_2^2\}. \end{aligned} \tag{28}$$

Similarly

$$\begin{aligned} \|e_{k+2}\|_2^2 &< q * max\{\|e_{k-M+2}\|_2^2, ..., \|e_{k+1}\|_2^2\} \\ &< q * max\{\|e_{k-M+1}\|_2^2, ..., \|e_k\|_2^2\}, \end{aligned} \tag{29}$$

where the second inequality is a result of (28). By following the same steps as above

$$\|e_{k+i}\|_2^2 < q * max\{\|e_{k-M+1}\|_2^2, ..., \|e_k\|_2^2\}, 1 \le i \le M$$

and hence

$$\begin{aligned} &max\{\|e_{k+1}\|_2^2, ..., \|e_{k-M+2}\|_2^2\} \\ &\quad < q * max\{\|e_{k-M+1}\|_2^2, ..., \|e_k\|_2^2\} \\ &\quad < q^k * max\{\|e_0\|_2^2, ..., \|e_{M-1}\|_2^2\}. \end{aligned} \tag{30}$$

Since $q < 1$, $\lim_{k\to\infty} e_k = 0$, which completes the proof.

From the above theorem it can be seen that for a given M γ characterizes how quickly the tracking error converges to zero under the higher order ILC law. In at least some applications, faster convergence is desirable and this can be achieved by finding the minimum γ as a solution to the following minimisation problem.

$$\min_{s.t. (19),(20), P_1 \succ 0} \gamma. \tag{31}$$

4. ROBUSTNESS

As in the standard linear systems case, the analysis in this section assumes a model structure for the uncertainty. In this paper, norm-bounded uncertainty but the analysis extends in a direct manner to the case of polytopic uncertainty. Moreover, unlike the lifted model approach, the repetitive process setting avoids products of matrices defining the uncertainty and those from the state-space model.

Consider the case of the norm-bounded additive perturbations ΔA, ΔB and ΔC to the state-space model matrices A, B and C respectively. Then the model (1) is replaced by

$$\begin{aligned} x_k(p+1) &= (A + \Delta A)x_k(p) + (B + \Delta B)u_k(p), \\ y_k(p) &= (C + \Delta C)x_k(p), \quad p = 0, 1, ..., \alpha - 1 \end{aligned} \tag{32}$$

and it is assumed that the perturbation matrices can be written in the form

$$\Delta A = H_1 F E_1,\ \Delta B = H_1 F E_2,\ \Delta C = H_2 F E_1, \tag{33}$$

where $H_1 \in \mathbb{R}^{m\times n}$, $H_2 \in \mathbb{R}^{r\times n}$, $E_1 \in \mathbb{R}^{n\times m}$ and $E_2 \in \mathbb{R}^{n\times n}$ are known matrices with real constant entries and $F \in R^{n\times n}$ is an unknown matrix that satisfies $F = F^T$ and $FF^T \preceq I$.

The analysis in this section makes use of the following well known results.

Theorem 5. (Khargonekar et al., 1990) For any $\mathcal{F}^T\mathcal{F} \preceq I$, Σ_1, Σ_2 with proper dimensions and a scalar $\epsilon > 0$ the following holds

$$\Sigma_1 \mathcal{F} \Sigma_2 + \Sigma_2^T \mathcal{F} \Sigma_1^T \preceq \epsilon^{-1} \Sigma_1 \Sigma_1^T + \epsilon \Sigma_2^T \Sigma_2. \tag{34}$$

Theorem 6. (Gahinet and Apkarian, 1994) Given a symmetric matrix $\Psi \in \mathbb{R}^{n\times n}$ and two matrices Λ, Σ of column dimension n, there exists a matrix W such that the following LMI is feasible

$$\Psi + sym\{\Lambda^T W \Sigma\} \prec 0, \tag{35}$$

if and only if the following two inequalities with respect to W are satisfied

$$\Lambda^{\perp^T} \Psi \Lambda^{\perp} \preceq 0,\ \Sigma^{\perp^T} \Psi \Sigma^{\perp} \prec 0. \tag{36}$$

The robust design result can now be established.

Theorem 7. The discrete linear repetitive process representing the ILC dynamics of (14) with additive uncertainty defined by ΔA, ΔB, and ΔC is stable along the trial and satisfies (18) if there exist matrices $P_1 \succ 0$, N_1, N_2, W_{1x}, W_{2x}, W_{1j} for $j = 0, ..., M-1$, W_{20}, W_3, W_{4x}, W_{5x}, W_{4j}, W_{5j} for $j = 0, ..., M-1$, W_6 and real scalars $\epsilon_1 > 0$, $\epsilon_2 > 0$ and $\mu = \gamma^2$ such that the following LMIs are feasible

$$\begin{bmatrix} -P+Q & * & * & * & * \\ A_2P + B_2N & \Delta_{22} & * & * & * \\ \Delta_{31} & \Delta_{32} & \Delta_{33} & * & * \\ \Delta_{41} & \Delta_{42} & \Delta_{43} & \Delta_{44} & * \\ \Delta_{51} & \Delta_{52} & \Delta_{53} & \Delta_{54} & \Delta_{55} \end{bmatrix} \prec 0, \tag{37}$$

$$\begin{bmatrix} -P & * & * & * & * \\ A_3P + B_3N & \Delta_{62} & * & * & * \\ \Delta_{71} & \Delta_{72} & \Delta_{73} & * & * \\ \Delta_{81} & \Delta_{82} & \Delta_{83} & \Delta_{84} & * \\ \Delta_{91} & \Delta_{92} & \Delta_{93} & \Delta_{94} & \Delta_{95} \end{bmatrix} \prec 0, \tag{38}$$

where P, Q, and N are defined in *Theorem* 3 and

$$A_2 = [A\ 0\ 0\ \cdots\ 0]\,,\quad B_2 = [B\ B]\,,$$

$$A_3 = \begin{bmatrix} A & 0 & 0 & \cdots & 0 \\ 0 & 0 & I & \cdots & 0 \\ \vdots & \vdots & 0 & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & I \end{bmatrix} \quad B_3 = \begin{bmatrix} B & B \\ 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \end{bmatrix},$$

$$I_1 = \begin{bmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix},\quad I_2 = [0\ \cdots\ 0\ 1]\,, \tag{39}$$

$$\begin{aligned}
\Delta_{22} &= -P_1 + \epsilon_1 H_1 H_1^T, \\
\Delta_{31} &= [-CW_{1x}^T, -CW_{1M-1}^T, ..., -CW_{11}^T, 1 - CW_{10}^T], \\
\Delta_{32} &= -CW_{2x}^T, \\
\Delta_{33} &= -1 - CW_{20}^T - W_{20}C^T + \epsilon_1 H_2 H_2^T, \\
\Delta_{41} &= A_2\mathbf{P} + B_2\mathbf{N} + [-W_{1x}^T, -W_{1M-1}^T, ..., -W_{10}^T], \\
\Delta_{42} &= -W_{2x}^T, \\
\Delta_{43} &= -W_{20}^T - W_3C^T, \\
\Delta_{44} &= -W_3 - W_3^T + \epsilon_1 H_1 H_1^T, \\
\bar{E}_1 &= [E_1\ 0\ 0, ...,\ 0],\ \bar{E}_2 = [E_2\ E_2],
\end{aligned}$$

$$\Delta_{51} = \begin{bmatrix} & \bar{E}_1\mathbf{P}+\bar{E}_2\mathbf{N} & & \\ -E_1W_{1x}^T & -E_1W_{1M-1}^T & \cdots & -E_1W_{10}^T \\ & \bar{E}_1\mathbf{P}+\bar{E}_2\mathbf{N} & & \end{bmatrix},$$

$$\Delta_{52} = \begin{bmatrix} 0 \\ -E_1W_{2x}^T \\ 0 \end{bmatrix},\quad \Delta_{53} = \begin{bmatrix} 0 \\ -E_1W_{20}^T \\ 0 \end{bmatrix},$$

$$\Delta_{54} = \begin{bmatrix} 0 \\ -E_1W_3^T \\ 0 \end{bmatrix},\quad \Delta_{55} = \begin{bmatrix} -\epsilon_1 & 0 & 0 \\ 0 & -\epsilon_1 & 0 \\ 0 & 0 & -\epsilon_1 \end{bmatrix}, \tag{40}$$

$$\begin{aligned}
\Delta_{62} &= diag\{-P_1 + \epsilon_2 H_1 H_1^T, -I\}, \\
\Delta_{71} &= [-CW_{4x}^T, -CW_{4M-1}^T, ..., -CW_{41}^T, I - CW_{40}^T], \\
\Delta_{72} &= [-CW_{5x}^T, -CW_{5M-1}^T, ..., -CW_{51}^T], \\
\Delta_{73} &= -I - CW_{50}^T - W_{50}C^T + \epsilon_2 H_2 H_2^T, \\
\Delta_{81} &= A_2P + B_2N + [-W_{4x}^T, -W_{4M-1}^T, ..., -W_{40}^T], \\
\Delta_{82} &= [-W_{5x}^T, -W_{5M-1}^T, ..., -W_{51}^T], \\
\Delta_{83} &= -W_{50}^T - W_6C^T, \\
\Delta_{84} &= -W_6 - W_6^T + \epsilon_2 H_1 H_1^T,
\end{aligned}$$

$$\Delta_{91} = \begin{bmatrix} & \bar{E}_1P+\bar{E}_2N & & \\ -E_1W_{4x}^T & -E_1W_{4M-1}^T & \cdots & -E_1W_{40}^T \\ & \bar{E}_1P+\bar{E}_2N & & \end{bmatrix},$$

$$\Delta_{92} = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ -E_1W_{5x}^T & -E_1W_{5M-1}^T & \cdots & -E_1W_{51}^T \\ 0 & 0 & \cdots & 0 \end{bmatrix},$$

$$\Delta_{93} = \begin{bmatrix} 0 \\ -E_1W_{50}^T \\ 0 \end{bmatrix} \quad \Delta_{94} = \begin{bmatrix} 0 \\ -E_1W_6^T \\ 0 \end{bmatrix},$$

$$\Delta_{95} = \begin{bmatrix} -\epsilon_2 & 0 & 0 \\ 0 & -\epsilon_2 & 0 \\ 0 & 0 & -\epsilon_2 \end{bmatrix}. \tag{41}$$

If the LMIs (37) and (38) are feasible, stabilizing control law matrices are given by

$$\begin{aligned} K_x &= N_1 P_1^{-1}, \\ [K_{M-1},\ ...,\ K_1,\ K_0] &= N_2, \end{aligned} \tag{42}$$

with use of the linear objective minimization procedure (31).

Proof 3. Direct substitution of (32) into (19) and (20) introduces nonlinear terms. To avoid these, *Theorem* 5 and *Theorem* 6 are applied and then use of the Schur complement formula gives (37) and (38) and proof is complete.

The following result on error convergence is the counterpart to Theorem 4.

Theorem 8. If Theorem 7 holds for $\gamma \in (0, 1/\sqrt{M})$, the discrete linear repetitive process representing the ILC dynamics of (14) with additive uncertainty defined by ΔA, ΔB, and ΔC is stable along the trial and satisfies $e_k \to 0$ as $k \to \infty$. Moreover, the convergence is monotonic in the sense that

$$\max\{\|e_{k+1}\|^2, \ldots, \|e_{k-M+2}\|^2\} < q \max\{\|e_k\|^2, \ldots, \|e_{k-M+1}\|^2\}.$$

where $q = \gamma^2 M < 1$.

The tracking error norm convergence rate in this last result can also be optimised by following a similar route to that for the nominal systems developed in Section 3. The details are therefore omitted for brevity.

5. SIMULATION BASED CASE STUDY

This section gives a simulation based case study using a model of one axis of gantry robot experimental facility that replicates the pick and place operation that arises in many industrial applications. The the construction of transfer-function approximate models of the dynamics of each axis from frequency response tests is described in (Hladowski et al., 2010) and the relevant cited references. This experimental facility has been used to compare many ILC designs, including those designed in the repetitive process setting (Hladowski et al., 2010).

By construction, the gantry robot is such that each axis can be controlled independently. In this paper, the model of X axis is used (the highest order of the three). Based on the experimentally measured Bode gain plot data a 7th-order continuous-time model has been constructed to model the dynamics, followed by sampling at $T_s = 0.01$ secs to give the minimal state-space model matrices below.

$$A = \begin{bmatrix} 0.3879 & 1.0000 & 0.2138 & 0 & 0.1041 & 0 & 0.0832 \\ -0.3898 & 0.3879 & 0.1744 & 0 & 0.0849 & 0 & 0.0678 \\ 0 & 0 & -0.1575 & 0.2500 & -0.2006 & 0 & -0.1603 \\ 0 & 0 & -0.3103 & -0.1575 & -0.0555 & 0 & -0.0444 \\ 0 & 0 & 0 & 0 & 0.0353 & 0.5000 & 0.2809 \\ 0 & 0 & 0 & 0 & -0.0164 & 0.0353 & -0.2757 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1.0000 \end{bmatrix},$$

$$B = [0\ 0\ 0\ 0\ 0\ 0\ 0.0910]^T\,,$$

$$C = [0.0391\ 0\ 0.0146\ 0\ 0.0071\ 0\ 0.0057]\,.$$

The pass length is 2 sec and the component of the 3D reference trajectory for this axis is shown in Fig. 1.

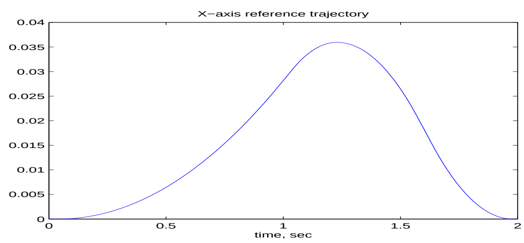

Fig. 1. The reference trajectory for the gantry robot axis.

The first simulation study investigates the effect of the memory length M on the trial-to-trial error convergence

M	γ
1	0.9998
2	0.6212
3	0.5209
4	0.4591
5	0.4158

Table 1. M and γ values for the nominal model design.

performance. Using the design method of Section 3, the results given in Table I were obtained. These confirm that as M increases, i.e., more information from previous trials is used, the value of γ decreases and the tracking error converges faster (see also Fig. 2) as expected. Furthermore, the tracking error converges monotonically to zero, verifying the theoretical predictions in Theorem 4. As one case, the control law matrices when $M = 2$ are $K_x = [-7.7474\ -51.3144\ -10.6598\ -2.7077\ -3.9193\ -2.4118\ -14.7319]$ and $K_i = [62.8982\ 192.7897]$. and the value of γ for different values of M are in Table 1.

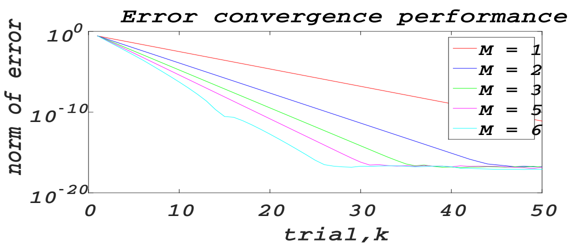

Fig. 2. Error convergence performance along the trial.

To examine the effectiveness of the robust design developed in Section IV, consider the case when the matrices defining the uncertainty model are

$$H_1 = [0.1\ 0\ 0\ 0\ 0\ 0\ 0]^T, H_2 = 0.01, E_2 = 0.01,$$
$$E_1 = [0.1\ -0.1\ 0.1\ -0.1\ 0.05\ -0.05\ 0.1]$$

The results obtained for different M are shown in Fig. 3, which confirm that i) the tracking error decreases monotonically and ii) increasing M improves the convergence speed. Table 2 shows the relationship between M and values of γ.

M	γ
1	0.9999
2	0.7042
3	0.5762
4	0.4980
5	0.4456

Table 2. M and γ values for the robust design example.

In Table 2, γ satisfies $\gamma \in [0, 1)$ and since the design is applied to an uncertain system, the value of γ with model uncertainty is larger than that without, which is the price paid for robust design.

6. CONCLUSIONS

Higher order ILC makes use of the data from a finite number, greater than one, of previous trials in the computation of the input to be applied on the next trial. This paper has used the stability theory for linear repetitive processes to design higher order ILC laws with guaranteed monotonic trial-to-trial error convergence where the computations are LMI based. The analysis and design extends to the case when model uncertainty is present. A model of one axis of a gantry robot obtained from frequency domain tests has been used to illustrate the designs and the results provide evidence that higher order ILC can have advantages. Further work includes design using output rather than state feedback in the ILC law.

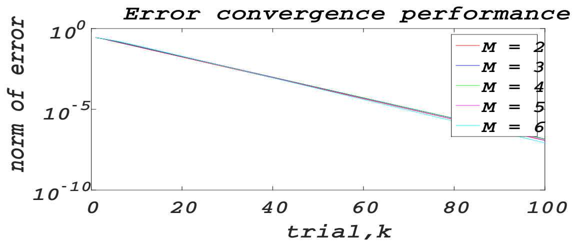

Fig. 3. Error convergence performance along the trial for the robust design.

REFERENCES

Ahn, H.S., Chen, Y.Q., and Moore, K.L. (2007). Iterative learning control: Brief survey and categorization. *IEEE Transactions on Systems, Man and Cybernetics, Part C: Applications and Reviews*, 37(6), 1099–1121.

Bien, Z. and Huh, K.M. (1989). Higher-order iterative learning control algorithm. *IEE Proceedings, Parrt D,. Control Theory and Applications*, 136, 105–112.

Bristow, D., Tharayil, M., and Alleyne, A. (2006). A survey of iterative learning control. *IEEE Control Systems Magazine*, 26(3), 96–114.

Gahinet, P. and Apkarian, P. (1994). A linear matrix inequality approach to H_∞ control. *International Journal of Robust and Nonlinear Control*, 4.

Hladowski, L., Galkowski, K., Cai, Z., Rogers, E., Freeman, C.T., and Lewin, P.L. (2010). Experimentally supported $2D$ systems based iterative learning control law design for error convergence and performance. *Control Engineering Practice*, 18(4), 339–348.

Khargonekar, P.P., Petersen, I.P., and Zhou, K. (1990). Robust stabilization of uncertain linear systems. *IEEE Transactions on Automatic Control*, 35, 356–361.

Kurek, J.E. and Zaremba, M.B. (1993). Iterative learning control synthesis based on $2-D$ system theory. *IEEE Transactions on Automatic Control*, 38(1), 121–125.

Paszke, W., Rogers, E., and Galkowski, K. (2016). Experimentally verified generalized KYP lemma based iterative learning control design. *Control Engineering Practice*, 53, 57–67.

Rogers, E., Galkowski, K., and Owens, D.H. (2007). *Control Systems Theory and Applications for Linear Repetitive Processes*. Springer.

Wang, X., Chu, B., and Rogers, E. (2016). Repetitive process based higher-order iterative learning control law design. In *Proceedings of the American Control Conference*, 378–383.