Acoustic analogy for multiphase or multicomponent flow

Yijun Mao, first author[1]
School of Engineering and the Environment, University of Southampton
University Road, Southampton, SO17 1BJ, United Kingdom

School of Energy and Power Energy, Xi'an Jiaotong University
No.28 Xianning West Road, Xi'an, Shaanxi, 710049, People's Republic of China
e-mail: Y.Mao@soton.ac.uk; maoyijun@mail.xjtu.edu.cn

Zhiwei Hu, second author
School of Engineering and the Environment, University of Southampton
University Road, Southampton, SO17 1BJ, United Kingdom
e-mail: Z.Hu@soton.ac.uk

ABSTRACT

The Ffowcs-Williams and Hawkings (FW-H) equation is widely used to predict sound generated from flow and its interaction with impermeable or permeable surfaces. Owing to the Heaviside function used, this equation assumes that sound only propagates outside the surface. In this paper, we develop a generalized acoustic analogy to account for sound generation and propagation both inside and outside the surface. The developed wave equation provides an efficient mathematical approach to predict sound generated from multiphase or multicomponent flow and its interaction with solid boundaries. The developed wave equation also clearly interprets the physical mechanisms of sound generation, emphasizing that the monopole and dipole sources are dependent on the jump of physical quantities across the interface of multiphase or multicomponent flow rather than the physical quantities on one-side surface expressed in the FW-H equation. Sound generated from gas bubbles in water is analyzed by the newly developed wave equation to investigate parameters affecting the acoustic power output, showing that the acoustic power feature concluded from the Crighton and Ffowcs-Williams equation is only valid in a specific case of all bubbles oscillating in phase.

[1] Corresponding author.

1 Introduction

The acoustic analogy proposed by Lighthill [1] is widely used to analyze sound generated from unbounded turbulence. Further developments have extended its applications in various engineering problems. An extension of the Lighthill's acoustic analogy is to predict sound generated from turbulence and its interaction with solid surfaces or other mediums. Curle [2] and Ffowcs Williams and Hawkings (FW-H) [3] developed the Lighthill's acoustic analogy to consider the effect of solid bodies immersed in turbulent flow on sound generation and propagation. An impermeable data surface was used to describe the fluid-solid interface in the equations, and the effect of the solid body on sound radiation was equivalently expressed by the monopole and dipole sources on the solid surface. After that, the FW-H equation was extended to a more comprehensive version with a permeable data surface [4, 5]. In most cases, the permeable FW-H equation is computationally more efficient than the impermeable FW-H equation to predict the flow induced noise because the sound generated by all the sources inside the permeable data surface, including the volume quadrupole source, can be equivalently represented by the monopole and dipole sources on the permeable data surface. Therefore, sound radiated to the farfield can be calculated without performing any volume integration if the quadrupole sources outside the permeable date surface can be ignored.

Multiphase and multicomponent flows occur in various engineering fields. Crighton and Ffowcs Williams (C-FW) [6] developed the Lighthill's acoustic analogy to analyze the sound generated from multiphase/multicomponent flow (MMF) by using a volume-averaged method to describe the macroscopic properties of the MMF. In the C-FW equation, the effect of the dispersed phase (e.g. gas bubble) on the sound generated from the continuous phase (e.g. water) is also represented by monopole and dipole sources. However, it should be emphasized that the monopole and dipole sources in the C-FW equation are volume sources rather than surface sources as in the FW-H equation. Howe [7] developed an alternative formulation of the Lighthill's acoustic analogy, in which the specific stagnation enthalpy was used as the acoustic variable and acoustic sources were related to the vorticity and entropy gradient. He used this formulation to analyze sound generated from a specific multicomponent flow, i.e., entropy spots in ambient flow, where the density was discontinuous but the pressure was continuous across the surface enclosing the entropy spot. The result showed that the entropy spot was equivalent to a dipole surface source whose strength was related to the density difference across the surface. Note that, in the Howe's equation, the fluid in both the continuous and dispersed phases is restricted to ideal gas because the perfect gas state equation was used to describe the thermodynamic relationship among the enthalpy, pressure and density. Furthermore, Campos [8] extended the Howe's formulation to study sound generated by an ionized inhomogeneity.

The FW-H equation is also applicable for predicting sound generated from MMF, where the monopole and dipole sources in the impermeable FW-H equation are related to mass flow rate and the force on the phase/component interface (PCI), respectively. However, the impermeable FW-H equation is only applicable to analyze sound generated from flow and its interaction with solid surfaces, which implies that the dispersed phase immersed in fluid must be in a solid object. On the other hand, if all the dispersed phase is enclosed by a permeable data surface, the permeable FW-H equation can also be used to predict sound generated from MMF, regardless of the dispersed phase being solid or fluid.

Compared with the impermeable FW-H equation, the C-FW and the permeable FW-H equations have no restriction on the state of the dispersed phase, thus they can be used to predict sound generated from fluid-solid two-phase flow, and also from gas-liquid two-phase flow. However, the volume-averaging method used in the C-FW equation only provides information of the macroscopic flow quantities, therefore cannot describe the detailed flow around the PCI. The permeable FW-H equation does not explicitly account for the dispersed phase in source terms. Therefore, these two equations are not best placed in analyzing the effect of the parameters related to the PCI, such as its shape and velocity, on the sound radiation.

Since some physical parameters, such as density, are discontinuous across the PCI, the PCI is usually modeled via a boundary condition in fluid mechanics. In the FW-H equation, a generalized function, i.e. the Heaviside function $H(f)$, is used to describe the discontinuity across the PCI, where $f=0$ defines the data surface [5, 9]. With the help of the Heaviside function, the generalized continuity and momentum equations, which are valid throughout the entire fluid and solid regions, can be derived. However, the FW-H equation, with either an impermeable or a permeable data surface, only deals with the sound generated from an elastic medium and its interaction with a rigid medium because the Heaviside function forces all parameters inside the data surface to constants. Therefore, the FW-H equation implies that no sound propagates inside the date surface.

For a gas-liquid two-phase flow, the mediums on both sides of the PCI are elastic, and sound synchronously generates and propagates in both mediums. These acoustic phenomena widely exist, such as sound generated from ocean surface, and are of great interest to many engineering applications, such as bubble noise in water. Obviously, the Heaviside function used in the FW-H equation needs to be replaced by another generalized function to describe the discontinuity on the gas-liquid interface to consider the acoustic field inside the data surface.

This paper aims to develop a generalized acoustic analogy for analyzing sound generation and propagation inside and outside the surfaces. The developed wave equation extends the FW-H equation to make it capable of analyzing sound generated from MMF. Compared with the C-FW equation, the developed wave equation provides a more efficient mathematical approach for predicting sound propagation, and also a clear way to interpret the physical mechanisms of sound generation in MMF. The remainder of this paper is organized as follows. In Section 2, some acoustic analogies are reviewed for predicting the sound generated from MMF. Following the derivation of the FW-H equation but without assuming that the medium inside the PCI is rigid, we derive a set of generalized continuity and momentum equations with a new generalized function for the PCI and the corresponding wave equation in Section 3. In Section 4, further extensions of the developed wave equation are conducted to consider effects of solid surfaces and uniform mean flow on sound generated from MMF. Section 5 applies the developed acoustic analogy to calculate sound generated from pulsating gas bubbles immersed in water and to analyze the parameters affecting the acoustic power output. Section 6 presents conclusions from this work.

2 Review of acoustic analogies

2.1 Lighthill’s equation

The continuity and momentum equations describing the motion of viscous compressible fluid are as follows:

$$\frac{\partial \rho}{\partial t}+\frac{\partial(\rho u_i)}{\partial x_i}=0 \tag{1}$$

$$\frac{\partial(\rho u_i)}{\partial t}+\frac{\partial(\rho u_i u_j+p\delta_{ij}-\sigma_{ij})}{\partial x_j}=0 \tag{2}$$

where ρ is the density of the fluid; u_i is the component of flow velocity in the i^{th} coordinate direction x_i; p is the static pressure; σ_{ij} is the viscous stress tensor; δ_{ij} is the Kronecker delta function.

Starting from Eqs. (1) and (2) and assuming that the acoustic medium is at rest, Lighthill [1] derived the following wave equation by performing the temporal derivative over Eq. (1) and the spatial derivative over Eq. (2) and then eliminating terms related to ρu_i

$$\frac{\partial^2 \rho}{\partial t^2}-c_0^2\nabla^2\rho=\frac{\partial^2 T_{ij}}{\partial x_i \partial x_j} \tag{3}$$

with

$$T_{ij}=\rho u_i u_j+(p-\rho c_0^2)\delta_{ij}-\sigma_{ij} \tag{4}$$

where c_0 is the speed of sound in the quiescent medium. Eq. (3) is the well-known Lighthill's acoustic analogy, which describes sound generation and propagation in unbounded flow. The source term on the right-hand side (RHS) of Eq. (3) is known as quadrupole source, representing the acoustic contribution from the fluctuations of fluid.

2.2 C-FW equation

A volume-averaging method was employed by Crighton and Ffowcs Williams [3] to describe a two-phase flow and this method is still widely used for MMF. Assuming that $\hat{\rho}$ and $\breve{\rho}$ are the densities of the continuous phase and dispersed phase, respectively, the macroscopic density of the two-phase fluid is represented by

$$\rho=\beta\breve{\rho}+(1-\beta)\hat{\rho} \tag{5}$$

where β and $1-\beta$ are the bulk concentrations of the dispersed phase and the continuous phase, respectively. With this definition, the continuity and momentum equations of the continuous phase can be expressed as

$$\frac{\partial\hat{\rho}}{\partial t}+\frac{\partial(\hat{\rho}\hat{u}_i)}{\partial x_i}=Q \tag{6}$$

$$\frac{\partial(\hat{\rho}\hat{u}_i)}{\partial t}+\frac{\partial((1-\beta)\hat{\rho}\hat{u}_i\hat{u}_j+\hat{p}\delta_{ij}-\hat{\sigma}_{ij})}{\partial x_j}=L_i \tag{7}$$

with

$$Q=-\hat{\rho}(\frac{\partial}{\partial t}+\hat{u}_j\frac{\partial}{\partial x_j})\ln(1-\beta) \tag{8}$$

$$L_i=F_i+\frac{\partial(\beta\hat{\rho}\hat{u}_i)}{\partial t} \tag{9}$$

where F_i denotes the component of the interphase force in the i^{th} direction. By starting from Eqs. (6) and (7) and following the derivation of the Lighthill's equation, the following C-FW equation can be obtained:

$$\frac{\partial^2 \hat{\rho}}{\partial t^2} - \hat{c}_0^2 \nabla^2 \hat{\rho} = \frac{\partial Q}{\partial t} - \frac{\partial L_i}{\partial x_i} + \frac{\partial^2 T_{ij}}{\partial x_i \partial x_j} \tag{10}$$

with

$$T_{ij} = (1-\beta)\hat{\rho}\hat{u}_i\hat{u}_j + (\hat{p} - \hat{\rho}\hat{c}_0^2)\delta_{ij} - \hat{\sigma}_{ij} \tag{11}$$

In Eq. (10), the quadrupole source T_{ij} given in Eq. (11), which has a subtle difference from the Lighthill's quadrupole source given in Eq. (4), represents the acoustic contribution from the continuous phase. The monopole source Q given in Eq. (8) and the dipole source L_i given in Eq. (9) are the additional sources to represent the effect of the dispersed phase on sound generation. Note that the monopole and dipole sources in Eq. (10) are volume sources, which are computationally more time-consuming than the surface monopole and dipole sources in the FW-H equation. The C-FW equation only employs a parameter, the bulk concentration β, to characterize the bulk fluid parameters on MMF. This macroscopic model does not describe the intrinsic flow of each phase, thus could not account for the effects of variations in the PCI on sound radiation.

2.3 FW-H equation

It is assumed that $f = 0$ defines a data surface, and $f > 0$ and $f < 0$ represent the regions outside and inside the data surface, respectively. It should be noted that the only restriction placed on the data surface $f = 0$ is smoothness, it can move in an arbitrary fashion, and change its shape and orientation [3]. To simplify the derivation of the wave equation, it is reasonable to assume $|\nabla f| = 1$ because all terms associated with these derivatives will cancel exactly in the final result [9]. Under this assumption, we can obtain the following identities:

$$\partial f / \partial x_i = n_i \tag{12}$$

$$\partial f / \partial t = -v_n \tag{13}$$

where n_i is the component of the unit normal vector on the data surface $f = 0$ in the i^{th} direction, v_n is the component of the data surface velocity normal to the surface.

The discontinuity on the data surface can be represented by a Heaviside function defined as

$$H(f) = \begin{cases} 1 & f > 0 \\ 0 & f < 0 \end{cases} \tag{14}$$

With this definition, generalized continuity and momentum equations can be derived as follows:

$$\frac{\partial[H(f)\rho]}{\partial t} + \frac{\partial[H(f)\rho u_i]}{\partial x_i} = Q\delta(f) \tag{15}$$

$$\frac{\partial[H(f)\rho u_i]}{\partial t} + \frac{\partial[H(f)(\rho u_i u_j + p\delta_{ij} - \sigma_{ij})]}{\partial x_j} = L_i \delta(f) \tag{16}$$

with

$$Q = \rho_0 v_n + \rho(u_n - v_n) \tag{17}$$

$$L_i = ((p - p_0)\delta_{ij} - \sigma_{ij})n_j + \rho u_i (u_n - v_n) \tag{18}$$

By performing temporal derivative over Eq. (15) and subtracting the spatial derivative of Eq. (16) to eliminate the terms related to $H(f)\rho u_i$, one can deduce the following FW-H equation:

$$\frac{\partial^2 [H(f)\rho']}{\partial t^2} - c_0^2 \nabla^2 [H(f)\rho'] = \frac{\partial [Q\delta(f)]}{\partial t} - \frac{\partial [L_i \delta(f)]}{\partial x_i} + \frac{\partial^2 [T_{ij} H(f)]}{\partial x_i \partial x_j} \tag{19}$$

with

$$T_{ij} = \rho u_i u_j + (p - \rho' c_0^2)\delta_{ij} - \sigma_{ij} \tag{20}$$

where $\rho' = \rho - \rho_0$ is the density perturbation. Although the quadrupole source expression given in Eq. (20) is slightly different from that in Eq. (4), but they are actually equivalent as the density ρ_0 and the sound speed c_0 for the undisturbed homogeneous fluid are constants. The monopole source Q and the dipole source L_i in the FW-H equation (19) are given in Eqs. (17) and (18), respectively. It is worth pointing out that the Dirac delta function $\delta(f)$ in the monopole and dipole sources in the above FW-H equation limits the sources to the data surface only, which is different from the C-FW equation, where the monopole and dipole sources are volume sources. Moreover, the Heaviside function $H(f)$ combined with the quadrupole source and the wave operator indicates that the quadupole source only exists and sound can only propagate outside the data surface.

When the data surface coincides with the solid surface, the corresponding impermeable FW-H equation is only suitable for predicting sound generated from MMF with a solid dispersed phase. Sound generated from gas-liquid two-phase flow can be predicted with the permeable FW-H equation, however only for propagation outside the data surface. Moreover, effects of the gas-liquid interface on sound generation cannot be considered as the permeable data surface is required to include all interfaces.

Based on the above analysis, both the C-FW and the FW-H equations have pros and cons in predicting sound generated from MMF. The C-FW equation and the permeable

FW-H equation do not limit the state of the dispersed phase, but it is difficult to analyze the effect of the PCI on sound generation. Moreover, the C-FW equation is usually computationally inefficient because volume integrals are required to calculate sound radiated from monopole and dipole sources. The impermeable FW-H equation has an advantage in analyzing the mechanics of sound generation, but it is only suitable for the MMF with a solid dispersed phase. Additionally, the FW-H equation, with either a permeable data surface or an impermeable data surface, assumes that no sound generates and propagates inside the data surface. Therefore, all the wave equations mentioned above are not best placed for analyzing sound generated from MMF, it is meaningful to develop a generalized wave equation which avoids the above disadvantages and combines the merits of the C-FW and FW-H equations.

2.4 Howe's equation

Starting from the continuity and Crocco's equations, Howe [7] deduced the following wave equation to describe sound generated from flow in which vorticity and entropy-gradient vectors are non-vanishing:

$$\left\{\frac{D}{Dt}\left(\frac{1}{c^2}\frac{D}{Dt}\right)+\frac{1}{c^2}\frac{D\mathbf{u}}{Dt}\cdot\nabla-\nabla^2\right\}B=\nabla\cdot\left(\boldsymbol{\omega}\times\mathbf{u}-T\nabla S\right)-\frac{1}{c^2}\frac{D\mathbf{u}}{Dt}\cdot\left(\boldsymbol{\omega}\times\mathbf{u}-T\nabla S\right) \tag{21}$$

where B is the stagnation enthalpy, $\boldsymbol{\omega}$ is the vorticity vector, T is the temperature, c is the local speed of sound and S is the entropy. For mean irrotational flow with low Mach numbers, Eq. (21) can reduce to

$$\left(\frac{1}{c^2}\frac{D^2}{Dt^2}-\nabla^2\right)B=\nabla\cdot\left(\boldsymbol{\omega}\times\mathbf{u}-T\nabla S\right) \tag{22}$$

We consider sound radiated from an entropy spot in ambient fluid, where the density is discontinuous but the pressure is continuous across the PCI. By assuming that the fluid on both sides of the PCI is deal gas, one can deduce the following wave equation to describe sound generated from an entropy spot bounded by a closed surface $f(\mathbf{x},t)=0$:

$$\left(\frac{1}{c_0^2}\frac{D^2}{Dt^2}-\nabla^2\right)B=\nabla\cdot\left(\boldsymbol{\omega}\times\mathbf{u}\right)-\nabla\cdot\left[\frac{\Delta\rho}{\hat{\rho}\breve{\rho}}\frac{\gamma p}{\gamma-1}\mathbf{n}\delta(f)\right] \tag{23}$$

where $\Delta\rho=\hat{\rho}-\breve{\rho}$ is the jump of the density across the surface $f(\mathbf{x},t)=0$, γ is the specific heat ratio. Both terms on the RHS of Eq. (23) are dipole sources. The first is a volume distribution while the second only exists on the data surface. The volume dipole source has the same acoustic contribution as the quadrupole term in Eq. (3) for low-Mach-number flow. The strength of the surface dipole source is related to the jump of the density across the surface $f(\mathbf{x},t)=0$. Compared with the FW-H equation, Howe's equation does not have a monopole source, thus the effect of the moving speed of the entropy spot on the source strength is not explicitly expressed. Moreover, it should be noted that Eq. (23) is only valid for a multicomponent flow of ideal gas.

3 Generalized acoustic analogy for MMF

3.1 Definition and notation

The Heaviside function is used in the FW-H equation to describe the discontinuity on the data surface. Therefore, no sound sources are located and sound does not propagate inside the data surface, which implies that the medium in the region of $f<0$ is rigid and sound only propagates in the region of $f>0$. To consider sound propagation in the entire region, a new generalized/window function $M(\cdot)$ is employed to describe the parameters in the entire space:

$$M(f)\phi = \begin{cases} \hat{\phi} & f > 0 \\ \breve{\phi} & f < 0 \end{cases} \tag{24}$$

where ϕ is a generic flow parameter. Hats $\hat{}$ and $\breve{}$ denote quantities inside and outside the data surface, respectively. An equivalent expression of Eq. (24) is as follows

$$M(f)\phi = \breve{\phi} + H(f)(\hat{\phi} - \breve{\phi}) \tag{25}$$

Especially, if the medium in the region of $f < 0$ is rigid, i.e. $\breve{\phi} = 0$, the generalized function $M(\cdot)$ reduces to the Heaviside function. Actually, similar mathematic treatments have been employed in previous studies related to jet noise [10, 11] where parameters on the interface of the high-speed jet and ambient fluid are discontinuous.

With the above definition, the spatial and temporal derivatives of a generalized parameter are given by

$$\frac{\partial[M(f)\phi]}{\partial x_i} = \frac{\partial \breve{\phi}}{\partial x_i} + H(f)\frac{\partial(\Delta\phi)}{\partial x_i} + \Delta\phi n_i \delta(f) \tag{26}$$

$$\frac{\partial[M(f)\phi]}{\partial t} = \frac{\partial \breve{\phi}}{\partial t} + H(f)\frac{\partial(\Delta\phi)}{\partial t} - \Delta\phi v_n \delta(f) \tag{27}$$

where $\Delta\phi = \hat{\phi} - \breve{\phi}$ is the difference of the parameter ϕ for the two mediums.

3.2 Generalized governing equations

The following two identities can be derived with the newly-defined generalized function $M(f)$ by employing Eqs. (26) and (27):

$$\frac{\partial[M(f)(\rho-\rho_0)]}{\partial t}=\frac{\partial(\breve{\rho}-\breve{\rho}_0)}{\partial t}+H(f)\frac{\partial(\Delta\rho')}{\partial t}-\Delta\rho' v_n\delta(f) \tag{28}$$

$$\frac{\partial[M(f)\rho u_j]}{\partial x_j}=\frac{\partial(\breve{\rho}\breve{u}_j)}{\partial x_j}+H(f)\frac{\partial(\widehat{\rho}\widehat{u}_j-\breve{\rho}\breve{u}_j)}{\partial x_j}+(\widehat{\rho}\widehat{u}_n-\breve{\rho}\breve{u}_n)\delta(f) \tag{29}$$

with $\Delta\rho'=\widehat{\rho}'-\breve{\rho}'$, $\widehat{\rho}'=\widehat{\rho}-\widehat{\rho}_0$ and $\breve{\rho}'=\breve{\rho}-\breve{\rho}_0$. $\widehat{\rho}_0$ and $\breve{\rho}_0$ are the densities of undisturbed mediums outside and inside the data surface, respectively. The sum of Eq. (28) and Eq. (29) gives the following generalized continuity equation:

$$\frac{\partial[M(f)(\rho-\rho_0)]}{\partial t}+\frac{\partial[M(f)\rho u_j]}{\partial x_j}=\Delta Q\delta(f) \tag{30}$$

with

$$\Delta Q=\underbrace{\widehat{\rho}\widehat{u}_n-\widehat{\rho}' v_n}_{\widehat{Q}}-\underbrace{(\breve{\rho}\breve{u}_n-\breve{\rho}' v_n)}_{\breve{Q}} \tag{31}$$

Similarly, by employing the following two identities

$$\frac{\partial[M(f)\rho u_i]}{\partial t}=\frac{\partial(\breve{\rho}\breve{u}_i)}{\partial t}+H(f)\frac{\partial(\widehat{\rho}\widehat{u}_i-\breve{\rho}\breve{u}_i)}{\partial t}-(\widehat{\rho}\widehat{u}_i-\breve{\rho}\breve{u}_i)v_n\delta(f) \tag{32}$$

$$\begin{aligned}\frac{\partial[M(f)(\rho u_i u_j+p\delta_{ij}-\sigma_{ij})]}{\partial x_j}&=\frac{\partial(\breve{\rho}\breve{u}_i\breve{u}_j+\breve{p}\delta_{ij}-\breve{\sigma}_{ij})}{\partial x_j}+H(f)\frac{\partial(\widehat{\rho}\widehat{u}_i\widehat{u}_j+\widehat{p}\delta_{ij}-\widehat{\sigma}_{ij}-\breve{\rho}\breve{u}_i\breve{u}_j-\breve{p}\delta_{ij}+\breve{\sigma}_{ij})}{\partial x_j}\\&+(\widehat{\rho}\widehat{u}_i\widehat{u}_j+\widehat{p}\delta_{ij}-\widehat{\sigma}_{ij}-\breve{\rho}\breve{u}_i\breve{u}_j-\breve{p}\delta_{ij}+\breve{\sigma}_{ij})n_j\delta(f)\end{aligned} \tag{33}$$

the following generalized momentum equation can be deduced

$$\frac{\partial[M(f)\rho u_i]}{\partial t}+\frac{\partial[M(f)(\rho u_i u_j+p\delta_{ij}-\sigma_{ij})]}{\partial x_j}=\Delta L_i\delta(f) \tag{34}$$

with

$$\Delta L_i=\underbrace{(\widehat{p}\delta_{ij}-\widehat{\sigma}_{ij})n_j+\widehat{\rho}\widehat{u}_i(\widehat{u}_n-v_n)}_{\widehat{L}_i}-\underbrace{((\breve{p}\delta_{ij}-\breve{\sigma}_{ij})n_j+\breve{\rho}\breve{u}_i(\breve{u}_n-v_n))}_{\breve{L}_i} \tag{35}$$

By following same steps in deriving the FW-H equation [3], we perform the temporal derivative of Eq. (30) and the spatial derivative of Eq. (34) to obtain the following two equations:

$$\frac{\partial^2[M(f)\rho']}{\partial t^2}+\frac{\partial^2[M(f)\rho u_i]}{\partial t\partial x_i}=\frac{\partial[\Delta Q\delta(f)]}{\partial t} \tag{36}$$

$$\frac{\partial^2[M(f)\rho u_i]}{\partial t\partial x_i}+\frac{\partial^2[M(f)(\rho u_i u_j+p\delta_{ij}-\sigma_{ij})]}{\partial x_i\partial x_j}=\frac{\partial[\Delta L_i\delta(f)]}{\partial x_i} \tag{37}$$

Subtracting Eq. (37) from Eq. (36) yields

$$\frac{\partial^2[M(f)\rho']}{\partial t^2}=\frac{\partial[\Delta Q\delta(f)]}{\partial t}-\frac{\partial[\Delta L_i\delta(f)]}{\partial x_i}+\frac{\partial^2[M(f)(\rho u_i u_j+p\delta_{ij}-\sigma_{ij})]}{\partial x_i\partial x_j} \tag{38}$$

As the speed of sound c_0 usually varies with properties of the medium, the following identity is used for the Laplace operator:

$$\nabla^2[M(f)\rho' c_0^2]=\frac{\partial^2[M(f)\rho' c_0^2]}{\partial x_i\partial x_j}\delta_{ij} \tag{39}$$

where

$$M(f)\rho' c_0^2 = \begin{cases} \hat{\rho}' \hat{c}_0^2 & f > 0 \\ \breve{\rho}' \breve{c}_0^2 & f < 0 \end{cases} \tag{40}$$

Subtracting Eq. (39) from Eq. (38) gives the following generalized wave equation of aeroacoustics:

$$\frac{\partial^2 [M(f)\rho']}{\partial t^2} - \nabla^2 [M(f)\rho' c_0^2] = \frac{\partial [\Delta Q \delta(f)]}{\partial t} - \frac{\partial [\Delta L_i \delta(f)]}{\partial x_i} + \frac{\partial^2 [M(f) T_{ij}]}{\partial x_i \partial x_j} \tag{41}$$

where source terms ΔQ, ΔL_i and T_{ij} are given in Eqs. (31), (35) and (20), respectively. Eq. (41) is the first main result of this paper. Similar to the FW-H equation, Eq. (41) is also an exact rearrangement of the generalized continuity and momentum equations, and three terms on the RHS of Eq. (41) are monopole, dipole and quadrupole sources, respectively. The Dirac delta function indicates that the monopole and dipole sources only appear on the data surface $f = 0$, and this feature is the same as the monopole and dipole sources in the FW-H equation. However, the generalized function $M(f)$ in the quadrupole source indicates that fluctuations of fluid stress on both sides of the data surface contribute to sound generation. Similarly, the generalized function $M(f)$ in the wave operator indicates that sound can propagate both inside and outside the data surface. Therefore, the developed wave equation can account for sound generation and propagation on both sides of the data surface.

3.3 Green's function and integral solution

Eq. (41) can also be expressed as the following equivalent equations:

$$\begin{cases} \dfrac{\partial^2 \hat{\rho}'}{\partial t^2} - \hat{c}_0^2 \nabla^2 \hat{\rho}' = \dfrac{\partial[\Delta Q \delta(f)]}{\partial t} - \dfrac{\partial[\Delta L_i \delta(f)]}{\partial x_i} + \dfrac{\partial^2 [H(f) T_{ij}]}{\partial x_i \partial x_j} & \mathbf{x} \in f > 0 \\ \dfrac{\partial^2 \breve{\rho}'}{\partial t^2} - \breve{c}_0^2 \nabla^2 \breve{\rho}' = \dfrac{\partial[\Delta Q \delta(f)]}{\partial t} - \dfrac{\partial[\Delta L_i \delta(f)]}{\partial x_i} + \dfrac{\partial^2 [H(-f) T_{ij}]}{\partial x_i \partial x_j} & \mathbf{x} \in f < 0 \end{cases} \tag{42}$$

For observers located either in the region of $f > 0$ or $f < 0$, the monopole and dipole sources on the data surface contribute to the sound. However, only the quadrupole source in the same region contributes directly to the sound generated in that region. This feature is similar to the FW-H equation, thus Eq. (42) can also be solved by existing methods, such as time-domain numerical methods [5, 9], frequency-domain numerical methods [12] and spherical harmonic series expansion methods [13, 14]. These methods actually describe the sound propagation once sources are known.

The time-domain Green's function in three-dimensional free space for Eq. (42) can be expressed as

$$g(\mathbf{x},\mathbf{y},t-\tau) = \begin{cases} \dfrac{\delta(t-\tau-|\mathbf{x}-\mathbf{y}|/\hat{c}_0)}{4\pi|\mathbf{x}-\mathbf{y}|} & \mathbf{x} \in f > 0 \\ \dfrac{\delta(t-\tau-|\mathbf{x}-\mathbf{y}|/\breve{c}_0)}{4\pi|\mathbf{x}-\mathbf{y}|} & \mathbf{x} \in f < 0 \end{cases} = \frac{\delta[t-\tau-|\mathbf{x}-\mathbf{y}|/(M(f)c_0)]}{4\pi|\mathbf{x}-\mathbf{y}|} \tag{43}$$

Therefore, the time-domain integral formulation for Eq. (41) is

$$M(f)p'(\mathbf{x},t) = \frac{\partial}{\partial t}\int_{f=0}\int_{-\infty}^{\infty} \Delta Q g dS d\tau - \frac{\partial}{\partial x_i}\int_{f=0}\int_{-\infty}^{\infty} \Delta L_i g dS d\tau + \frac{\partial^2}{\partial x_i \partial x_j}\int_{f\neq 0}\int_{-\infty}^{\infty} M(f) T_{ij} g d\mathbf{y} d\tau \tag{44}$$

with $p'=\rho' c_0^2$. By starting from Eqs. (43) and (44), time-domain and frequency-domain acoustic pressure integral formulations suitable for numerical computation can be derived analytically, which are similar to those obtained by Farassat [9] and Tang [12].

3.4 Discussion

The wave equation (41) is compared with the wave equations reviewed in Section 2. Firstly, we discuss the relationship between existing acoustic analogies and Eq. (41). In the case of a rigid medium in the region of $f<0$, the parameter $\breve{\phi}$ is zero and the function $M(f)$ becomes the Heaviside function $H(f)$. Therefore, sound wave does not propagate in the region of $f<0$, and Eq. (41) reduces to the permeable FW-H equation. Furthermore, if the data surface coincides with the solid surface, there are $u_n=v_n$ and $\breve{\phi}=0$, thus Eq. (41) reduces to the impermeable FW-H equation. If physical parameters are continuous across the PCI, i.e. $\phi=\hat{\phi}=\breve{\phi}$, the monopole and dipole sources disappear and Eq. (41) reduces to the Lighthill's equation. Therefore, the Lighthill's equation and the FW-H equation with either an impermeable or a permeable data surface, are only specific cases of the generalized wave equation (41).

Secondly, we analyze sound generation inside and outside the data surface with the developed wave equation. We assume that the data surface $f=0$ coincides with the PCI, and this data surface can move and change its shape arbitrarily, but it is impermeable because $f>0$ and $f<0$ always represent the regions of different fluids. Therefore, the velocity on the PCI satisfies the following identity:

$$\hat{u}_n=\breve{u}_n=v_n \tag{45}$$

In this situation, the monopole and dipole sources given earlier in Eqs. (31) and (35) can be simplified to

$$\Delta Q=(\hat{\rho}_0-\breve{\rho}_0)v_n=\Delta\rho_0 v_n \tag{46}$$

$$\Delta L_i = (\widehat{p}\delta_{ij} - \widehat{\sigma}_{ij})n_j - (\breve{p}\delta_{ij} - \breve{\sigma}_{ij})n_j \tag{47}$$

Therefore, the velocity of the PCI, the jumps in density and pressure on the PCI are crucial factors affecting sound generated from MMF. Young-Laplace equation denotes that the jump of the static pressure across the PCI is proportional to the interfacial tension, thus the dipole source is mainly from the interfacial tension.

Compared with the impermeable FW-H equation, the developed wave equation enables us to analyze sound generated from gas-liquid two-phase flow. Compared with the C-FW equation and the permeable FW-H equation, the developed wave equation explicitly reveals crucial factors affecting sound generated from MMF, as shown in Eqs. (46) and (47).

For a specific multicomponent flow, i.e., entropy spots in ideal gas, the pressure is continuous but the density is discontinuous across the PCI. In this situation, the Howe's equation (23) shows that the source strength is proportional to the jump of the density. However, the Howe's equation does not account for the effect of the velocity of the PCI on the source strength because the monopole source on the RHS of the continuity equation is not considered in Howe's study [7].

Furthermore, we analyze physical meaning of the sound sources. The FW-H equation denotes that, only the quadruple source is the physical source outputting acoustic energy from turbulence. In the impermeable FW-H equation, the monopole and dipole source terms are equivalent sound sources, representing the effects of motion and scattering of the solid surface on sound propagation. In the permeable FW-H equation, the monopole and dipole terms are still equivalent sources, but they represent the contribution from all sound sources, including the quadrupole source, inside the data surface $f = 0$.

The quadrupole source in Eq. (41) also represents the generation of acoustic energy, but the turbulence in both regions ($f > 0$ and $f < 0$) contributes to sound generation. The monopole and dipole sources in Eqs. (46) and (47) are also equivalent sources, and they represent sound scattering and transmission owing to the discontinuity on the PCI. The sources in the FW-H equation and Eq. (41) are summarized in Table 1.

Thirdly, we analyze sound propagation inside and outside the data surface. All the wave equations can be used to analyze sound outside the data surface. Therefore, we only compare the features of flow and sound inside and through these data surfaces between the FW-H equation and Eq. (41). Obviously, impermeable data surfaces in all the wave equations imply that the fluid cannot go through the surfaces. A subtle difference is that the FW-H equation implies there is no flow inside the impermeable data surface, however, flow exists inside the impermeable data surface of Eq. (41). Moreover, no sound propagates through the impermeable and permeable data surfaces in the FW-H equation, but sound can propagate inside and through the impermeable data surface in Eq. (41). These features are summarized in Table 2.

4 Further extensions to consider the effects of solid surfaces and uniform mean flow

The wave equation developed in Section 3 enables us to analyze sound generation and propagation in MMF, but it does not consider sound scattered by solid surfaces and assumes that the acoustic medium is at rest. In this section, we carry out further extensions to analyze sound generated from MMF and its interaction with solid surfaces. The effect of a uniform mean flow on sound generation and propagation is also considered to develop a convective wave equation. The extended equation can be applied to analyze, e.g. cavitation noise of marine propellers.

4.1 Effect of solid surfaces

We define another data surface $f^s = 0$, which fully includes or coincides with the solid surfaces. Similarly, we have the following identities:

$$\partial f^s / \partial x_i = n_i^s \tag{48}$$

$$\partial f^s / \partial t = -v_n^s \tag{49}$$

where n_i^s is the unit vector normal to the data surface $f^s = 0$, and v_n^s is its normal velocity. The Heaviside function is used to describe the discontinuity on the data surface $f^s = 0$, which is the same as was used in the FW-H equation.

With the preceding definition, we can deduce the following two equations:

$$\begin{aligned}\frac{\partial[M(f)H(f^s)(\rho-\rho_0)]}{\partial t} &= H(f^s)[\frac{\partial\breve{\rho}'}{\partial t} + H(f)\frac{\partial(\hat{\rho}'-\breve{\rho}')}{\partial t} - (\hat{\rho}'-\breve{\rho}')\delta(f)v_n] \\ &\quad - M(f)\delta(f^s)(\rho-\rho_0)v_n^s\end{aligned} \tag{50}$$

$$\begin{aligned}\frac{\partial[M(f)H(f^s)\rho u_j]}{\partial x_j} &= H(f^s)[\frac{\partial(\breve{\rho}\breve{u}_j)}{\partial x_j} + H(f)\frac{\partial(\hat{\rho}\hat{u}_j - \breve{\rho}\breve{u}_j)}{\partial x_j} + (\hat{\rho}\hat{u}_j - \breve{\rho}\breve{u}_j)\delta(f)] \\ &\quad + M(f)\delta(f^s)\rho u_n\end{aligned} \tag{51}$$

The sum of Eq. (50) and Eq. (51) gives the following generalized continuity equation:

$$\frac{\partial[M(f)H(f^s)(\rho-\rho_0)]}{\partial t} + \frac{\partial[M(f)H(f^s)\rho u_j]}{\partial x_j} = H(f^s)\delta(f)\Delta Q + M(f)\delta(f^s)Q^s \tag{52}$$

with

$$Q^s = \rho(u_n - v_n^s) + \rho_0 v_n^s \tag{53}$$

Moreover, we can deduce the following identities with the properties of the generalized function:

$$\begin{aligned}\frac{\partial[M(f)H(f^s)\rho u_i]}{\partial t} &= H(f^s)[\frac{\partial(\breve{\rho}\breve{u}_i)}{\partial t} + H(f)\frac{\partial(\hat{\rho}\hat{u}_i - \breve{\rho}\breve{u}_i)}{\partial t} - (\hat{\rho}\hat{u}_i - \breve{\rho}\breve{u}_i)v_n\delta(f)] \\ &-M(f)\delta(f^s)v_n^s\rho u_i\end{aligned} \tag{54}$$

$$\begin{aligned}\frac{\partial[M(f)H(f^s)(\rho u_i u_j + p\delta_{ij} - \sigma_{ij})]}{\partial x_j} &= H(f^s)\frac{\partial(\breve{\rho}\breve{u}_i\breve{u}_j + \breve{p}\delta_{ij} - \breve{\sigma}_{ij})}{\partial x_j} \\ &+H(f^s)H(f)\frac{\partial(\hat{\rho}\hat{u}_i\hat{u}_j + \hat{p}\delta_{ij} - \hat{\sigma}_{ij} - \breve{\rho}\breve{u}_i\breve{u}_j - \breve{p}\delta_{ij} + \breve{\sigma}_{ij})}{\partial x_j} \\ &+H(f^s)[(\hat{\rho}\hat{u}_i\hat{u}_j + \hat{p}\delta_{ij} - \hat{\sigma}_{ij} - \breve{\rho}\breve{u}_i\breve{u}_j - \breve{p}\delta_{ij} + \breve{\sigma}_{ij})n_j\delta(f)] \\ &+M(f)\delta(f^s)(\rho u_i u_j + p\delta_{ij} - \sigma_{ij})n_j^s\end{aligned} \tag{55}$$

Thus, the generalized momentum equation can be expressed as

$$\frac{\partial[M(f)H(f^s)(\rho u_i)]}{\partial t} + \frac{\partial[M(f)H(f^s)(\rho u_i u_j + p\delta_{ij} - \sigma_{ij})]}{\partial x_j} = \delta(f)H(f^s)\Delta L_i + M(f)\delta(f^s)L_i^s \tag{56}$$

with

$$L_i^s = (p\delta_{ij} - \sigma_{ij})n_j^s + \rho u_i(u_n - v_n^s) \tag{57}$$

Finally, the following wave equation can be deduced from Eqs. (52) and (56):

$$\begin{aligned}\frac{\partial^2[M(f)H(f^s)\rho']}{\partial t^2} - \nabla^2[M(f)H(f^s)\rho' c_0^2] &= \frac{\partial^2[M(f)H(f^s)T_{ij}]}{\partial x_i \partial x_j} \\ &+\frac{\partial[\delta(f)H(f^s)\Delta Q]}{\partial t} - \frac{\partial[\delta(f)H(f^s)\Delta L_i]}{\partial x_i} \\ &+\frac{\partial[M(f)\delta(f^s)Q^s]}{\partial t} - \frac{\partial[M(f)\delta(f^s)L_i^s]}{\partial x_i}\end{aligned} \tag{58}$$

where the quadrupole source T_{ij} is given by Eq. (20); the monopole source ΔQ and dipole source ΔL_i on the data surface $f = 0$ are given by Eqs. (46) and (47), respectively; the monopole source Q^s and dipole source L_i^s on the data surface $f^s = 0$ are given in Eqs. (53) and (57), respectively. Eq. (58) is the second main result of this paper as an extension of Eq. (41). Compared with the FW-H equation, Eq. (58) can also account for sound generated from the interaction between different fluids, apart from the fluid and solid surface interaction. Therefore, Eq. (58) is capable of predicting noise generated from MMF and its interaction with solid surfaces, such as cavitation noise of marine propellers.

We analyze the source terms on the RHS of Eq. (58). The quadrupole source term T_{ij} in the first line accounts for sound generated from turbulence inside and outside the data surface $f = 0$ but outside the data surface $f^s = 0$. The monopole and dipole sources in the second line are the equivalent sources to represent sound scattering and transmission on the data surface $f = 0$. The monopole and dipole sources in the third line are the equivalent sources to represent the sound scattering on the impermeable data surface $f^s = 0$ or all sources inside the permeable data surface $f^s = 0$.

Although the data surface $f = 0$ is fixed on the PCI, the data surface $f^s = 0$ can be defined independently. Fig.1 illustrates four possible relative positions of these two data surfaces. As shown in Fig.1(a), if the data surface $f = 0$ is in the region of $f^s < 0$, the source terms on the second line disappear, and then Eq. (58) reduces to the FW-H equation. As shown in Fig.1(b) and 1(c), if the data surface $f = 0$ is in the region of $f^s > 0$, the monopole and dipole sources in the second line can be equivalently expressed as

$$\frac{\partial[\Delta Q\delta(f)H(f^s)]}{\partial t} - \frac{\partial[\Delta L_i\delta(f)H(f^s)]}{\partial x_i} = \frac{\partial[\Delta Q\delta(f)]}{\partial t} - \frac{\partial[\Delta L_i\delta(f)]}{\partial x_i} \tag{59}$$

Moreover, as shown in Fig.1(c), if the data surface $f^s = 0$ is in the region of $f < 0$, the monopole and dipole sources in the last line can be expressed as

$$\frac{\partial[Q^s M(f)\delta(f^s)]}{\partial t}-\frac{\partial[L_i^s M(f)\delta(f^s)]}{\partial x_i}=\frac{\partial[(\breve{\rho}(\breve{u}_n-v_n^s)+\breve{\rho}_0 v_n^s)\delta(f^s)]}{\partial t} -\frac{\partial[((\breve{p}\delta_{ij}-\breve{\sigma}_{ij})n_j^s+\breve{\rho}\breve{u}_i(\breve{u}_n^s-v_n^s))\delta(f^s)]}{\partial x_i} \tag{60}$$

However, if the position relationship of these two data surfaces is shown in Fig.1(d), the source terms in Eq. (58) cannot be simplified.

4.2 Effect of uniform mean flow

In all the wave equations given in the previous sections, it is assumed that the acoustic medium is at rest. In previous investigations, a convective FW-H equation has been deduced to consider the effect of a uniform mean flow on sound generation and propagation, see for examples [15-17]. Howe [7, 18] proposed a reverse-flow reciprocal theorem to analyze the effect of an irrotational non-uniform mean flow. However, the developed wave equation usually cannot be solved with the method of Green's function because neither the sound speed nor the mean flow velocity is constant. For simplicity, we only consider uniform mean flow in this paper. It is assumed that the acoustic wave is propagating in a uniform mean flow with the velocity $\mathbf{U}$. The local flow velocity is $\mathbf{U}+\mathbf{u}$, where $\mathbf{u}$ is the fluctuation of the local flow velocity. By using this definition and following the derivation of the convective FW-H equation, we can deduce the following convective wave equation:

$$\frac{D^2[M(f)H(f^s)\rho']}{Dt^2}-\nabla^2[M(f)H(f^s)\rho' c_0^2]=\frac{\partial^2[M(f)H(f^s)T_{ij}]}{\partial x_i\partial x_j} +\frac{D[\Delta Q\delta(f)H(f^s)]}{Dt}-\frac{\partial[\Delta L_i\delta(f)H(f^s)]}{\partial x_i} +\frac{D[Q^s M(f)\delta(f^s)]}{Dt}-\frac{\partial[L_i^s M(f)\delta(f^s)]}{\partial x_i} \tag{61}$$

with $\frac{D}{Dt}=\frac{\partial}{\partial t}+U_i\frac{\partial}{\partial x_i}$. The source term T_{ij} is given in Eq. (20), ΔL_i is given in Eq. (47), and the other source terms are expressed by

$$\Delta Q=\Delta\rho_0(v_n-U_n) \tag{62}$$

$$Q^s=\rho_0(v_n^s-U_n)+\rho(u_n-(v_n^s-U_n)) \tag{63}$$

$$L_i^s=((p-p_0)\delta_{ij}-\sigma_{ij})n_j+\rho u_i(u_n-(v_n^s-U_n)) \tag{64}$$

Eq. (61) is the third main result of this paper. The convective wave operator represents the effect of a uniform mean flow on sound propagation. Note that the source terms in Eq. (61) have subtle differences from those in Eq. (58) owing to the effect of the uniform mean flow on sound generation. The quadrupole source term T_{ij} in Eq. (61) has the same expression as that in Eq. (58). However, the parameter $\mathbf{u}$ in Eq. (61) only represents the fluctuating component of the flow velocity whereas $\mathbf{u}$ in Eq. (58) represents the flow velocity. In the monopole sources of the convective wave equation, the partial derivative is replaced by the material derivative. Moreover, the dipole term ΔL_i in Eq. (61) is the same as that in Eq. (58), representing the pressure jump across the PCI, and the dipole term L_i^s given in Eq. (64) has the same physical meaning as that given in Eq. (57). Particularly, if the data surface $f^s=0$ coincides with the solid surface, the second term in Eq. (64) will disappear and L_i^s represents the static pressure on the solid surface.

5 Sound generated from gas bubbles in water

5.1 Definitions and assumptions

Sound generated from gas bubbles immersed in water has been analyzed by Crighton and Ffowcs Williams [6]. This problem is revisited with the developed wave equation (41). We consider gas bubbles in a finite turbulence region. The following four parameters are used to characterize the turbulence: characteristic length of the finite region L, mean flow velocity U, characteristic length of the turbulence eddy l_0 and root-mean-square turbulence velocity u_0. Therefore, $\sigma = u_0/U$ is the turbulence intensity or the relative turbulence level as defined by Crighton and Ffowcs Williams [6], and the angular frequency of turbulent fluctuation ω is in the order of u_0/l_0.

Moreover, we assume this finite turbulence region is acoustically compact, i.e., $L << c_0 l_0/u_0$, and there are N air bubbles in this region and each bubble has a mean radius a in the undisturbed state. Therefore, the bulk concentration of the gas bubbles is

$$\beta = \frac{4\pi N a^3}{3L^3} \tag{65}$$

By following the analysis of Crighton and Ffowcs Williams [6], who only considered the symmetric oscillation mode with a low amplitude, it is reasonable to assume that the gas bubbles always remain a spherical shape. The instantaneous velocity on a single bubble surface is $v_n = V_0 e^{i\omega t}$, where V_0 is the amplitude of the bubble pulsation, which is in the order of u_0.

5.2 Sound generated from a single gas bubble

Based on the above definitions and assumptions and ignoring sound generated from turbulence, the frequency-domain acoustic pressure radiated from a single bubble into water is

$$\tilde{p}'_B(\mathbf{x},\omega) = -i\omega \int_{f=0} \frac{\Delta Q e^{ikr}}{4\pi r} dS + \frac{\partial}{\partial x_i} \int_{f=0} \frac{\Delta L_i e^{ikr}}{4\pi r} dS \tag{66}$$

where $k = \omega/\hat{c}_0$, subscript B represents a single gas bubble. The dipole source strength on the bubble surface varies with the oscillation of the bubble. However, it should be emphasized that the total acoustic contribution from the dipole source is zero for an acoustically compact bubble, because the overall force on the bubble surface is always zero. Crighton and Ffowcs Williams [6] obtained a similar conclusion from the C-FW equation by qualitatively suggesting that the dipole contribution was negligible compared with the monopole contribution. Howe [19] also pointed out that if the radius of the spherical gas bubble is much smaller than the wavelength and the bubble is excited by a uniformly distributed, time harmonic force on the PCI, the sound is mainly contributed from the monopole source and the contribution from the dipole source can be ignored. Therefore, the acoustic pressure radiated from a pulsating gas bubble is computed by

$$\tilde{p}_B'(\mathbf{x},\omega) = -i\omega \int_{f=0} \frac{\Delta Q e^{ikr}}{4\pi r} dS = \frac{-i\omega\Delta\rho_0 V_0 a^2 e^{ikr}}{r} \tag{67}$$

This expression is very similar to Eq. (2.4.1) deduced in reference [19], but there is a subtle difference between these equations. Eq. (2.4.1) of reference [19] is derived from the Lighthill's equation, thus the strength of the monopole source is only related to the fluid density around the bubble. However, Eq. (67) indicates that the strength of the monopole source is actually dependent on the jump of the density across the PCI.

We assume that the sound propagation is lossless, thus the acoustic power output from a single gas bubble is

$$W_B = \lim_{r\to\infty}[4\pi r^2 \frac{\left|\tilde{p}'^2\right|}{\widehat{\rho}_0 \widehat{c}_0}] = \frac{4\pi\omega^2\Delta\rho_0^2 V_0^2 a^4}{\widehat{\rho}_0 \widehat{c}_0} \sim \frac{\Delta\rho_0^2}{\widehat{\rho}_0}\frac{a^4}{l_0^2}U^4 \tag{68}$$

Eq. (68) indicates that the acoustic power output from a single bubble is proportional to the fourth power of the Mach number, which is the fundamental feature of a compact monopole source. Furthermore, we will analyze sound generated from a bubble cloud in a finite region by using this fundamental model.

5.3 Sound generated from bubble cloud

We assume that there are N bubbles with the radius of a in a finite turbulence region with the characteristic length of L. By starting from Eqs. (65) and (67) and employing the assumption of an acoustically compact region, we can calculate the total acoustic pressure radiated from bubbles via a simple summation when the bubbles are oscillating in phase

$$\tilde{p}_C'(\mathbf{x},\omega) = N\frac{-i\omega\Delta\rho_0 V_0 a^2 e^{ikr}}{r} = -\frac{3\beta L^3}{4\pi a}\frac{i\omega\Delta\rho_0 V_0 e^{ikr}}{r} \tag{69}$$

where subscript C represents the bubble cloud. Furthermore, we can deduce that the total acoustic power output is

$$W_C = \lim_{r\to\infty}[4\pi r^2 \frac{\left|\tilde{p}_C'^{\,2}\right|}{\hat{\rho}_0\hat{c}_0}] \sim \beta^2 \frac{\Delta\rho_0^2}{\hat{\rho}_0}\frac{L^6}{a^2 l_0^2}\frac{U^4}{\hat{c}_0} \tag{70}$$

Eq. (70) indicates that the acoustic power is proportional to the square of the bull concentration β, and this conclusion is the same as that obtained by Crighton and Ffowcs Williams [6]. However, we emphasize that Eqs. (69) and (70) are only valid for all bubbles oscillating in phase, otherwise the total acoustic pressure cannot be calculated via a simple summation owing to the phase difference of source pulsation.

For a bubble cloud with many gas bubbles oscillating with a random phase correlation, a more reasonable method for calculating the total acoustic power is the superposition method of energy. Therefore, the acoustic power calculated from Eq. (68) is as follows

$$W_C = N\lim_{r\to\infty}[4\pi r^2 \frac{\left|\tilde{p}_B'^{\,2}\right|}{\hat{\rho}_0\hat{c}_0}] = \frac{3\beta L^3}{4\pi a}\frac{4\pi\omega^2\Delta\rho_0^2 V_0^2 a^4}{\hat{\rho}_0\hat{c}_0} \sim \beta\frac{\Delta\rho_0^2}{\hat{\rho}_0}\frac{aL^3}{l_0^2}\frac{U^4}{\hat{c}_0} \tag{71}$$

Eq. (71) shows that, for gas bubbles oscillating randomly, the acoustic power output is still proportional to the fourth power of the Mach number, but is only proportional to the bull concentration β, which is different from the conclusion obtained from Eq. (70) and the C-FW equation.

Additionally, both Eqs. (70) and (71) indicate that the acoustic power output depends on the jump of the density $\Delta\rho_0$ and the radius of the bubble a. Decreasing the density jump is beneficial for reducing the noise level. The relationship between the acoustic power output and the radius of the bubble depends on the oscillation phases of the bubbles. If all the gas bubbles oscillate in phase, the acoustic power output is inversely proportional to the square of the gas bubble radius. However, if the gas bubbles oscillate with a random phase correlation, the acoustic power output is proportional to the radius of the gas bubble.

6 Conclusion

The FW-H equation deals with sound generated from flow and its interaction with a data surface with an assumption of no sound propagation inside the surface due to the Heaviside function used to describe the discontinuity on the data surface. In this paper, generalized continuity and momentum equations and the corresponding wave equation were derived by replacing the Heaviside function with a newly-defined generalized function to describe the discontinuity on the data surface, which allows for variations of flow parameters both inside and outside the data surface. The developed wave equation consists of three sources: the volume quadrupole source which exists both inside and outside the data surface, the monopole and dipole sources on the data surface. It is worth emphasizing that the surface sources are highly dependent on the jump of the quantities across the surface instead of flow quantities on one side of the surface. The Lighthill's equation and the FW-H equation with impermeable or permeable data surfaces are specific cases of the developed wave equation.

The developed wave equation has the following features in the prediction of sound generated from MMF. Firstly, compared with the impermeable FW-H equation, it does not limit the state of the dispersed phase and it is capable of predicting sound generated from fluid-solid two-phase flow and also from gas-liquid two-phase flow. Secondly, compared with the C-FW equation and the permeable FW-H equation, it explicitly accounts for the effect of the PCI on sound generation. The result indicates that the flow velocity and the jumps of density and pressure on the PCI are the crucial factors affecting sound generated from MMF. Thirdly, it synchronously accounts for sound propagation on both sides of the PCI by introducing a new generalized function, to replace the Heaviside function. This feature provides a direct method for predicting, e.g. ocean surface noise received by observers in air and underwater. In summary, the developed wave equation provides an efficient mathematical approach for predicting sound generated from MMF and also a clear physical explanation for analyzing mechanisms of sound generation.

The developed wave equation is extended to consider the effects of solid surfaces and uniform mean flow on sound generation and propagation. In this situation, the following five sources contribute to sound radiation: quadrupole source in each phase/component of MMF, monopole and dipole sources on the PCI, and monopole and dipole sources on the date surface enclosing all solid surfaces. Note that all the monopole and dipole sources are equivalent sources, which represent the acoustic contribution of sources inside the permeable data surface or the sound scattering and transmission on the impermeable data surface.

Sound generated from gas bubbles in water is analyzed with the developed wave equation. The acoustic power output is proportional to the fourth power of the Mach number, which is consistent with the conclusion drawn from the C-FW equation. Compared with the C-FW equation, the result obtained from the developed equation further shows that decreasing the density jump across the PCI is beneficial for reducing the acoustic power level. Moreover, the oscillation phases of the bubbles have a significant effect on the acoustic power output. If all gas bubbles oscillate in phase, the acoustic power output is proportional to the square of the bulk concentration, which is consistent with the conclusion obtained from the C-FW equation. However, for bubbles oscillating with random phases, the result obtained from a superposition method of energy shows that the total acoustic power output is only proportional to the bulk concentration. This paper focuses on a theoretical analysis of sound generated from gas-liquid two phase flow. It is planned to perform experimental and numerical studies of underwater jet noise to validate the conclusion presented in this paper.

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Nomenclature

a = gas bubble radius, m

B = stagnation enthalpy, m^2/s^2

c_0 = speed of sound, m s^{-1}

f = data surface function

g = time-domain Green's function

H = Heaviside function

L_i = components of local loading intensity in the *i*th direction, Pa

M = new generalized function defined in Eq. (24)

n_i = components of unit vector normal to the data surface

p' = acoustic pressure in time domain, Pa

$\tilde{p}'$ = acoustic pressure in frequency domain, Pa

Q = monopole source strength, $kg \cdot m^{-2} \cdot s^{-1}$

r = distance between source and receiver, $|\mathbf{x}-\mathbf{y}|$, m

S = entropy, kJ/K

T = temperature, K

T_{int} = interval length of time-domain integration, s

t = observer time, s

$\mathbf{U}$ = velocity of uniform mean flow, $m \cdot s^{-1}$

$\mathbf{u}$ = flow velocity, $m \cdot s^{-1}$

v_n = local normal velocity of the data surface, $m \cdot s^{-1}$

W = acoustic power, w

$\mathbf{x}$	=	observer position vector, m
$\mathbf{y}$	=	source position vector, m
ρ	=	local fluid density, $kg \cdot m^{-3}$
ρ'	=	density perturbation, $kg \cdot m^{-3}$
β	=	bulk concentration of the dispersed phase
σ_{ij}	=	components of viscous stress tensor, Pa
$\delta(\cdot)$	=	Dirac delta function
δ_{ij}	=	Kronecker delta function
τ	=	source time, s

Subscripts

0	=	fluid variable in unperturbed medium
B	=	a single gas bubble
C	=	bubble cloud
x	=	observer quantity
y	=	source quantity

References

[1] Lighthill, M. J., 1952, "On sound generated aerodynamically. I. general theory," Proceedings of the Royal Society of London A: Mathematical and Physical Sciences, 211(1107), pp. 564-587.

[2] Curle, N., 1955, "The influence of solid boundaries upon aerodynamic sound," Proceedings of the Royal Society of London A: Mathematical and Physical Sciences, 231(1187), pp. 505-514.

[3] Ffowcs Williams, J., and Hawkings, D., 1969, "Sound generation by turbulence and surfaces in arbitrary motion," Philosophical Transactions of the Royal Society A: Mathematical, Physical and Engineering Sciences, 264(1151), pp. 321-342.

[4] di Francescantonio, P., 1997, "A new boundary integral formulation for the prediction of sound radiation," Journal of Sound and Vibration, 202(4), pp. 491-509.

[5] Brentner, K. S., and Farassat, F., 2003, "Modeling aerodynamically generated sound of helicopter rotors," Progress in Aerospace Science, 39(2-3), pp. 83-120.

[6] Crighton, D. G., and Ffowcs Williams, J. E., 1969, "Sound Generation by Turbulent two-Phase Flow," Journal of Fluid Mechanics, 36, pp. 585-603.

[7] Howe, M. S., 1975, "Contributions to theory of aerodynamic sound, with application to excess jet noise and theory of flute," Journal of Fluid Mechanics, 71(4), pp. 625-673.

[8] Campos, L.M.B.S., 1978, "On the emission of sound by an ionized inhomogeneity," Proceedings of the Royal Society of London A: Mathematical and Physical Sciences, 359, pp. 65-91.

[9] Farassat, F., 2007, "Derivation of Formulations 1 and 1A of Farassat," NASA-TM-2007-214853.

[10] Ffowcs Williams, J. E., 1974, "Sound production at the edge of a steady flow," Journal of Fluid Mechanics., 66(4), pp. 791-816.

[11] Dowling, A. P., Williams, J. E. F., and Goldstein, M. E., 1978, "Sound production in a moving stream," Philosophical Transactions of the Royal Society A: Mathematical, Physical and Engineering Sciences, 288(1353), pp. 321-349.

[12] Tang, H. T., Qi, D. T., and Mao, Y. J., 2013, "Analysis on the frequency-domain numerical method to compute the noise radiated from rotating sources," Journal of Sound and Vibration, 332(23), pp. 6093-6103.

[13] Mao, Y. J., Gu, Y. Y., Qi, D. T., and Tang, H. T., 2012, "An exact frequency-domain solution of the sound radiated from the rotating dipole point source," Journal of the Acoustical Society of America, 132(3), pp. 1294-1302.

[14] Mao, Y. J., Xu, C., Qi, D. T., and Tang, H. T., 2014, "Series expansion solution for sound radiation from rotating quadrupole point source," AIAA Journal, 52(5), pp. 1086-1095.

[15] Wells, V. L., and Han, A. Y., 1995, "Acoustics of a moving source in a moving medium with application to propeller noise," Journal of Sound and Vibration, 184(4), pp. 651-663.

[16] Najafi-Yazdi, A., Bres, G. A., and Mongeau, L., 2011, "An acoustic analogy formulation for moving sources in uniformly moving media," Proceedings of the Royal Society of London A: Mathematical and Physical Sciences, 467(2125), pp. 144-165.

[17] Ghorbaniasl, G., and Lacor, C., 2012, "A moving medium formulation for prediction of propeller noise at incidence," Journal of Sound and Vibration, 331(1), pp. 117-137.

[18] Howe, M. S., 1975, "Generation of sound by aerodynamic sources in an inhomogeneous steady flow," Journal of Fluid Mechanics, 67(3), pp. 597-610.

[19] Howe, M. S., 1998, Acoustics of fluid-structure interaction, Cambridge University Press, Cambridge, UK.

Figure Caption List

Fig. 1 Schematic of different relative positions of the data surfaces

Table Captions List

Table 1

Equation	Quadrupole source	Monopole and dipole sources
C-FW equation	All sources are volume sources and exist in the entire fluid domain	
Impermeable FW-H equation	Only outside the data surface	Equivalent contributions from the motion and scattering of the impermeable data surface (i.e. solid surface)
Permeable FW-H equation	Only outside the data surface	Equivalent contributions from all the sources inside the permeable data surface
Eq. (41)	Inside and outside the data surface	Equivalent contributions from the motion, scattering and transmission of the impermeable data surface (i.e. PCI)

Table 2

	Impermeable FW-H equation	Permeable FW-H equation	Eq. (41)
Flow inside the data surface	No	Yes	Yes
Flow through the data surface	No	Yes	No
Sound inside the data surface	No	No	Yes
Sound through the data surface	No	No	Yes