UNIVERSITY OF SOUTHAMPTON

FACULTY OF SOCIAL, HUMAN AND MATHEMATICAL SCIENCES

Mathematical Sciences

AdS/CFT: Dictionary and Applications

by

Maria Ioanna Stylianidi Christodoulou

Thesis for the degree of Doctor of Philosophy

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ABSTRACT

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This thesis discusses two separate questions within the field of gauge/gravity dualities. The first is *what is the gravity dual of a CFT state* and it is based on work done in collaboration with my advisor, Prof. Skenderis, and which was presented in [1]. In particular, we develop a systematic perturbative construction of bulk solutions that are dual to CFT excited states. The second question concerns four dimensional theories that admit charged planar AdS black hole solutions carrying axionic charge and which can support additional scalar hair. More specifically, we discuss a number of analytic solutions in which the axions have a linear profile in the boundary directions and the additional scalar field satisfies mixed boundary conditions. We focus on the effect of these features on the thermodynamics and dynamic stability of the solutions and we use our results to study phase transitions between solutions with the same asymptotic charges and which satisfy the same boundary conditions. These results are based on work done in collaboration with M. Caldarelli, I. Papadimitriou and K. Skenderis and which was presented in [2].

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Declaration of Authorship

I, Maria Ioanna Stylianidi Christodoulou, declare that the thesis entitled *AdS/CFT: Dictionary and Applications* and the work presented in the thesis are both my own, and have been generated by me as the result of my own original research. I confirm that:

- this work was done wholly or mainly while in candidature for a research degree at this University;
- where any part of this thesis has previously been submitted for a degree or any other qualification at this University or any other institution, this has been clearly stated;
- where I have consulted the published work of others, this is always clearly attributed;
- where I have quoted form the work of others, the source is always given. With the exception of such quotations, this thesis is entirely my own work;
- I have acknowledged all main sources of help;
- where the thesis is based on work done by myself jointly with others, I have made clear exactly what was done by others and what I have contributed myself;
- parts of this work have been published as:
  - A. Christodoulou, K. Skenderis, *Holographic Construction of Excited CFT States*, JHEP **04** (2016) 096, `arXiv:1602.02039` [1]
  - M. Caldarelli, A. Christodoulou, I. Papadimitriou, K. Skenderis, *Phases of planar AdS black holes with axionic charge*, JHEP **04** (2017) 001, `arXiv:1612.07214` [2]

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To my mother

Chapter 1

Introduction

1.1 From the Holographic Principle to the AdS/CFT Correspondence

The holographic principle states that the physics governing quantum gravity in a $d+1$ dimensional volume of spacetime is encoded in a quantum field theory without gravity defined on the d dimensional boundary. Each degree of freedom of the gravitational theory can be holographic projected to a degree of freedom on the boundary in such a way that the two theories are in fact describing the same physics.

The holographic principle is a deep and fundamental property of quantum gravity that emerged from observations about black holes dating back to the early 70s. In 1972, Bekenstein introduced the notion of black hole entropy [3] as a measure of inaccessibility of information about the interior of a black hole, in analogy to thermodynamic entropy which is a measure of our ignorance about the microscopic configurations of a system, when our knowledge is restricted to its macroscopic properties. Since the entropy of any system must be non–decreasing, he asserted that the black hole entropy is proportional to the area of its event horizon which had already been shown to be non–decreasing by Christodoulou and Hawking [4, 5, 6].

About twenty years after Bekenstein's area law for black hole entropy, 't Hooft proposed a radical interpretation for it. Combining black hole thermodynamics with ideas from quantum mechanics he postulated that at Planckian length scales where quantum gravity takes over, the world is not 3+1 dimensional but instead the observable degrees of freedom live on a 2 dimensional surface that evolves in time [7]. Said differently, given a closed surface in spacetime enclosing a quantum gravitational system, all information contained in the interior of the surface can be holographically projected onto the surface. Moreover, the theory of quantum gravity governing the interior or bulk physics can be described by a gauge field theory on the boundary surface. This was not the first time a gauge field theory description was proposed for a theory of quantum gravity and vice versa. Klebanov and Susskind [8], and Thorn [9] discovered that string theory can be described by a 2+1 dimensional gauge theory. However, 't Hooft's result is much stronger as it states that *any* theory of quantum gravity must be holographic.

This idea was further refined by Susskind [10] and many others (see for example [8, 9]), especially in the context of string theory, and in 1997 the first concrete example of a holographic theory of quantum gravity was proposed by Juan Maldacena [11]. By studying the low energy limit of brane configurations in string theory and M–theory, Maldacena was led to the conjecture that string theory and M–theory on anti–de Sitter (AdS) spacetimes times compact manifolds are dual to lower dimensional gauge field theories. The gauge theories are defined on spacetimes conformal to the asymptotic boundary of AdS and, for this reason, we say that they "live on the boundary" and refer to them as the "boundary theory" and to string theory as the "bulk theory". The duality between the two theories implies that their degrees of freedom and dynamics are in one–to–one correspondence. For each field on the gravity side of the duality there is a boundary operator of the dual field theory and one can study the equations of motion of the bulk field to learn about the boundary operator dynamics and vice versa.

The first and most famous example of the duality emerged from the study of N coincident D3 branes in string theory and it states that type IIB string theory on $\text{AdS}_5 \times S^5$ is dual to 4 dimensional $\mathcal{N} = 4$ super–Yang–Mills (SYM) with gauge group $SU(N)$ and coupling constant g_{SYM}. The SYM is a conformal field theory (CFT) which led to the term *AdS/CFT correspondence*. This duality holds for all values of N, g_{SYM} and g_s, the string coupling constant. As such, it provides a definition for the full quantum IIB string theory as a lower dimensional non–gravitational theory. Although this is a very powerful result, its utilitarian power is limited because of our limited understanding of string theory at strong coupling. More commonly, one studies the duality in its weak form, obtained when N and $g_{\text{SYM}}^2 N$ are taken to infinity while g_s is kept finite and small. In this limit the gravity side reduces to classical type IIB supergravity which can be treated perturbatively. On the other hand, the SYM becomes strongly coupled and can not be studied using conventional methods such as perturbation theory. Hence, this form of the AdS/CFT correspondence is a strong/weak coupling duality and it offers a powerful tool for the study of strongly coupled field theories.

In the prototypical example of the correspondence the field theory is highly symmetric, making it unrealistic for real world applications. This, however, is not an issue as the duality can be extended to less symmetric more realistic setups such as field theories with some or all supersymmetries and/or the conformal symmetry broken. For example, perturbing the field theory Lagrangian by a relevant operator can cause the theory flow to less symmetric theories. Moreover, the bulk spacetime need not be AdS but only asymptotically (locally) AdS (AAdS or AlAdS). Such generalisations of AdS/CFT are referred to as gauge/gravity dualities, although the term AdS/CFT is also used to refer to them, and they allow for a wide range of applications of the duality.

There are two possible routes to obtaining the dual theories. The first is known as the top down approach and it involves starting from string theory and M–theory and studying the low energy dynamics of brane configurations, in analogy to what Maldacena did. This method is quite involved and in principle it provides the dual field theory but there is no control over what this theory is. In other words, this method will provide the field theory Lagrangian which is fixed by the string theory configuration one considers. The alternative, known as the bottom up approach, bypasses the high energy physics and it involves postulating a gravitational theory on an asymptotically AdS spacetime which contains supergravity fields dual to a desire set of field theory operators. The choice of "desired" operators depends on field theory system being modelled which could be for example a strongly coupled condensed matter system. The field theory is not known in this case and one only knows of the elements they placed in the theory by hand. This approach makes use of the *AdS/CFT dictionary*, the map that relates objects and features of the bulk theory to objects and features of the boundary field theory. For example, the bulk theory necessarily contains the gravitational field which, according to the AdS/CFT dictionary, sources the field theory stress energy tensor. In addition, one may want to have symmetry currents and operators of various dimensions in the field theory which requires turning on gauge fields and matter fields in the bulk. Moreover, one may want to study the field theory at finite temperature. In the bulk this translates to considering black hole solutions in AdS. Once the building blocks of the field theory under consideration have been placed in the bulk, one can compute its observables and study its properties by performing the corresponding bulk computations. In fact, in the bottom up approach there is no specific Lagrangian for the field theory and the only way to study it is through the bulk. The work presented in this thesis is an application of the bottom–up method and we begin by presenting the elements of the AdS/CFT dictionary needed to understand the main part of the thesis.

1.2 The AdS/CFT dictionary

According to the AdS/CFT correspondence, the two sides of the duality describe the same physics. This implies that the symmetries, degrees of freedom and dynamics of the two theories must be in one–to–one correspondence. In this section we will make this statement explicit.

1.2.1 Matching Global Symmetries

We begin by considering the global symmetries of the two theories. According to the AdS/CFT dictionary, gauge symmetries of the bulk theory are mapped to global symmetries of the boundary theory. In particular the isometries of the bulk are mapped to the global symmetry group of the boundary theory. For example, in the example of $\text{AdS}_5 \times S^5$ the isometry group of AdS_5 is SO(4,2) and of the S^5 SO(6). In the $\mathcal{N} = 4$ SYM we encounter the same symmetry groups. The SO(4,2) is the conformal group in four dimensions and the SO(6)$\simeq$SU(4) is the group associated to the R–symmetry of the theory. Moreover, the two theories have the same number of supersymmetry generators.

Scale/Radius Correspondence

The identification of the diffeomorphisms of AdS with the conformal symmetry of the boundary suggests that the extra dimension of the bulk, namely the holographic or radial direction, is related to the energy scale of the field theory. In particular, studying the radial evolution of the bulk field equations tells us something about the renormalisation group (RG) flow of the dual operators in the field theory. To illustrate the relation between the bulk holographic direction and the field theory energy scale we consider AdS_{d+1} in Poincaré coordinates in which the metric takes the form

$$\mathrm{d}s^2 = \frac{\ell^2}{z^2}\left(\eta_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu + \mathrm{d}z^2\right). \tag{1.2.1}$$

z is the holographic direction and ℓ is the AdS radius. Constant z hypersurfaces are parametrised by x^μ, $\mu = 0, \dots d$, and their topology is $\mathbb{R}^{1,d-1}$. According to the AdS/CFT dictionary, the field theory metric is given by the asymptotic limit of the bulk metric, up to conformal transformations. Hence, in this case, it can be taken to be the Minkowski metric parametrised by the same coordinates x^μ.

The field theory is invariant under rigid scale transformations $x^\mu \to \alpha x^\mu$ which rescale the energies of particles according to $E \to E/\alpha$. In the bulk, this transformation corresponds to the diffeomorphism $x^\mu \to \alpha x^\mu$, $z \to \alpha z$. This leads to the identifications of the extra bulk dimension with the inverse energy scale of the gauge theory, $z \sim 1/E$, giving rise to a scale/radius or UV/IR duality. High energies or equivalently short distances on the field theory side translate in the bulk to large radii, that is, to moving closer to the boundary. Another way of understanding this duality is to say that, as we move a bulk excitation closer to the boundary of AdS, it localises in the field theory, i.e. the wavelength of the field theory excitation becomes smaller and its energy larger. Conversely, moving the excitation towards the interior of AdS smears the boundary excitation over a larger area.

Understanding the relation between the UV of the field theory and the bulk IR is very important for the understanding and treatment of divergences. Quantum field theories suffer from short distance UV divergences. In accordance with the UV/IR relation of AdS/CFT these divergences correspond to infinite volume IR divergences of the bulk gravitational theory that arise when one integrates over the holographic direction. In section 1.2.4 we explain how to deal with such divergences in the bulk using holographic renormalisation.

1.2.2 Matching Bulk Fields to Boundary Operators

One of the most important ingredients of the AdS/CFT dictionary is the map between bulk fields and boundary operators. The map between the spectrum of type IIB supergravity on $\text{AdS}_5 \times S^5$ and 4 dimensional, $\mathcal{N} = 4$ super–Yang–Mills was derived shortly after the publication of Maldacena's paper in [12] and it provided an important check of the conjecture. We do not delve into the details of this specific mapping as it requires knowledge of the two theories which is beyond the scope of this thesis. Instead, we provide general rules for mapping supergravity fields to field theory operators. The results and relations we discuss here are employed in the main part of the thesis to build bottom–up models.

In general, for every field $\Phi(x,z)$ that propagates in the bulk, there is a local, gauge invariant operator $\mathcal{O}(x)$. The boundary operator couples to the restriction of the bulk field on the boundary $\phi_{(0)}(x)$ via a term of the form $\int_{\mathcal{B}_d} \phi_{(0)}\mathcal{O}$ where $\mathcal{B}_d$ is the boundary manifold. Subleading terms in the asymptotic expansion of the bulk field are related to the expectation value of the field theory operator. Accordingly, the bulk field and operator must have the same Lorentz structure and quantum numbers. In particular, a scalar boundary operator is dual to a bulk scalar field, a conserved current associated with a boundary global symmetry couples to a bulk dynamical gauge field, the boundary stress energy tensor couples to the bulk metric, a boundary p–form operator couples to a bulk p–form and fermionic fields in the bulk are dual to fermionic operators on the boundary. Moreover, the mass of the bulk fields is related to the conformal dimension of the dual boundary operators. The relations between the masses of various bulk fields and the conformal dimension of the dual operators are

$$\begin{aligned}
&\text{scalar or massive spin 2 field :} && m^2\ell^2 = \Delta(\Delta-d).\\
&\text{massless spin 2 field:} && m^2\ell^2 = 0,\ \Delta = d,\\
&\text{spin } \tfrac{1}{2} \text{ or } \tfrac{3}{2} \text{ field:} && |m|\ell = \Delta - \tfrac{d}{2},\\
&p\text{–form field:} && m^2\ell^2 = (\Delta-p)(\Delta+p-d),\\
&\text{rank } s \text{ symmetric traceless tensor:} && m^2\ell^2 = (\Delta+s-2)(\Delta-s+2-d)\,. \qquad (1.2.2)
\end{aligned}$$

Below we briefly explain how these relations arise by considering the simplest case of a massive scalar field.

Mass/Conformal Dimension Relation

Consider a bulk free scalar field $\Phi(x,z)$ with mass m propagating in the Poincaré patch of AdS_{d+1}. The dynamics of the field is governed by the massive Klein–Gordon equation which admits two linearly independent solutions. Near the asymptotic boundary ($z \to 0$) the two solutions behave like $z^{d-\Delta}$ and z^{Δ} where Δ is the largest root of $\Delta(\Delta-d) = m^2\ell^2$. Generically the mode that asymptotes to $\phi_{(0)}(x)z^{d-\Delta}$ for some function $\phi_{(0)}(x)$ of the transverse coordinates, is non–normalisable whereas the mode that asymptotes to $\phi_{(2\Delta-d)}(x)z^{\Delta}$ is always normalisable. $\phi_{(2\Delta-d)}(x)$ is again some function of the transverse coordinates which can be determined by solving the equations of motion in the bulk but not from the asymptotic analysis. We discuss the significance of these two functions with regards to the dual theory in depth in section 1.2.4. For now we simply state that the coefficient of the non–normalisable mode, namely $\phi_{(0)}(x)$, is related to sources that couple to the field theory operator $\mathcal{O}(x)$ dual to $\Phi(x,z)$ whereas the coefficient of the normalisable mode, namely $\phi_{(2\Delta-d)}(x)$, is related to its expectation value $\langle\mathcal{O}(x)\rangle$. A generic solution to the field equation is a combination of the two modes but for now we focus on a purely normalisable solution $\Phi(x,z)$ which asymptotes to

$$\Phi(x,z) \sim z^{\Delta}\phi_{(2\Delta-d)}(x) + \ldots \tag{1.2.3}$$

In chapters 3 and 4 we demonstrate that such purely normalisable field configurations in the bulk are dual to excited field theory states obtained by acting with $\mathcal{O}(x)$ on the field theory vacuum. Moreover, they only exist in Lorentzian AdS; in Euclidean AdS they are no normalisable modes.

Returning to our discussion about $\Phi(x,z)$, up to an overall factor, the expectation value of the dual operator is related to the $\Phi(x,z)$ according to

$$\langle\mathcal{O}(x)\rangle \sim \lim_{z\to 0} z^{-\Delta}\Phi(x,z). \tag{1.2.4}$$

Under the bulk diffeomorphism $(x,z)\to(\mu x,\mu z)$ the expectation value of the dual operator transforms as follows

$$\langle\mathcal{O}(x)\rangle \to \lim_{z\to 0} z^{-\Delta}\Phi(\mu x,\mu z) = \lim_{z\to 0}(z/\mu)^{-\Delta}\ \Phi(\mu x,z) \sim \mu^{\Delta}\langle\mathcal{O}(\mu x)\rangle\,. \tag{1.2.5}$$

Hence the conformal dimension of $\mathcal{O}(x)$ is Δ where Δ is given in (1.2.2). Although this calculation relies on the behaviour of a bulk scalar field under diffeomorphisms, the same method applies for tensor fields if we use tangent indices rather than coordinate indices.

1.2.3 Matching Observables

One of the most important statements of the AdS/CFT correspondence is identification of the partition functions of the two theories,

$$\mathcal{Z}_{\text{string}} = \mathcal{Z}_{\text{CFT}}. \tag{1.2.6}$$

For example, the partition function of type IIB string theory defined on $\text{AdS}_5\times S^5$ is equivalent to the partition function of $\mathcal{N}=4$ SYM theory, defined on the conformal boundary of AdS_5 [13, 12]. This statement implies that one can compute field theory observables by performing string theory calculations and vice versa. Below we will review how such calculations are performed. For this we must restrict to the weak form of the correspondence in which string theory reduces to semiclassical supergravity. Moreover, at the present we will focus on the Euclidean case. The Lorentzian signature requires additional care and it is treated in section 1.2.5.

For the gauge theory, the weak limit of the correspondence amounts to taking the large N large g^2_{YM} limit. In this limit the n-point functions of the theory are dominated by the planar contributions. Equation (1.2.6) becomes

$$\mathcal{Z}_{\text{string}} \simeq \mathrm{e}^{-S_{\text{SUGRA}}} = \mathcal{Z}_{\text{CFT}} = \mathrm{e}^{W} \tag{1.2.7}$$

where W is the generating functional for connected Green's functions in the gauge theory and S_{SUGRA} is the supergravity (SUGRA) on-shell action. This relation is a general result that holds for all gauge/gravity dualities and it plays a central role in the computation of observables.

Equation (1.2.7) is only schematic and additional information is needed to make it meaningful. Focusing first on the field theory side, the generating functional of connected Green's functions for a field theory is obtained by modifying the action to include source terms $f_{(0)}(x)$ for the operators $\mathcal{O}(x)$,

$$S'_{\text{CFT}}[f_{(0)};\mathcal{O}] = S_{\text{CFT}}[\mathcal{O}] - \int \mathrm{d}^d x\; f_{(0)}(x)\mathcal{O}(x). \tag{1.2.8}$$

Then,

$$\mathcal{Z}_{\text{CFT}}\left[\mathcal{O}; f_{(0)}\right] = \mathrm{e}^{W[\mathcal{O};f_{(0)}]} = \left\langle \exp\left(\int \mathrm{d}^d x\; f_{(0)}(x)\mathcal{O}(x)\right)\right\rangle_{\text{CFT}}. \tag{1.2.9}$$

On the gravity side both $\langle\mathcal{O}(x)\rangle$ and $f_{(0)}(x)$ are related to the dual field $\mathcal{F}(x,z)$. In particular, the general solution to the bulk equations of motion for $\mathcal{F}(x,z)$ has the following asymptotic form

$$\mathcal{F}(x,z) \sim z^{d-\Delta} f_{(0)}(x) + \dots + z^{\Delta} f_{(2\Delta-d)}(x) + \dots \tag{1.2.10}$$

where the ... represent subleading terms in the expansion that are determined in terms of $f_{(0)}$ and $f_{(2\Delta-d)}$. $f_{(0)}$ and $f_{(2\Delta-d)}$ are the integration functions obtained when solving the differential equations that govern the z-behaviour of $\mathcal{F}(x,z)$. In the simple case of a free theory where the equations of motion are linear, these are the coefficients of the two linearly independent modes in the asymptotic expansion of the general solution. Δ is the conformal dimension of the field theory operator and it is related to the mass of the bulk field according to equations (1.2.2). Usually one imposes Dirichlet boundary conditions on the field which fix $f_{(0)}$,

$$\lim_{z\to 0}\left(z^{-(d-\Delta)}\mathcal{F}(x,z)\right) = f_{(0)}(x). \tag{1.2.11}$$

The boundary condition $f_{(0)}$ is associated to the source that couples to $\mathcal{O}$ and the coefficient of z^{Δ} in expansion (1.2.10), i.e. $f_{(2\Delta-d)}$, is proportional to the expectation value of the dual operator $\langle\mathcal{O}\rangle$. Putting all these concepts together, the supergravity generating function is related to the field theory generating function according to the relation

$$\mathcal{Z}_{\text{SUGRA}}[\mathcal{F}(x,z); f_{(0)}(x)] = \exp\left(-S_{\text{on-shell}}[\mathcal{F}(z,x)]\Big|_{\lim_{z\to 0}\left(z^{\Delta-d}\mathcal{F}(z,x)\right)=f_{(0)}(x)}\right) \tag{1.2.12}$$

and the supergravity on–shell action and field theory generating function of connected n–point functions according to

$$S_{\text{on-shell}}[\mathcal{F}(z,x) \sim f_{(0)}(x)] = -W_{\text{QFT}}[f_{(0)}] \tag{1.2.13}$$

where the supergravity action is evaluated on the solutions to the bulk equations that satisfy the Dirichlet boundary condition (1.2.11).

Equation (1.2.13) allows us to compute field theory correlation functions for $\mathcal{O}$ through the functional differentiation of the on–shell supergravity action with respect to the boundary condition $f_{(0)}(x)$,

$$\langle \mathcal{O}(x_1) \dots \mathcal{O}(x_n) \rangle = (-1)^{n+1} \frac{\delta^n S_{\text{on-shell}}}{\delta f_{(0)}(x_1) \dots \delta f_{(0)}(x_n)} \bigg|_{f_{(0)}=0} . \tag{1.2.14}$$

There are however a number of issues with this prescription. The field theory side suffers from UV divergences. In accordance with the UV/IR relation between the two theories, these correspond to infinite volume IR divergences on the bulk side. Hence, the quantities on either side of equation (1.2.13) are infinite and need to be renormalised. Moreover, as was mentioned at the start of this section, this prescription applies to the Euclidean version of the correspondence. For the Lorentzian case, the dictionary between the two theories needs to be modified. All these issues are addressed in the following sections by looking at a single massive scalar field propagating in empty AdS$_{d+1}$ in Poincaré coordinates. We first study the Euclidean case and demonstrate the renormalisation prescription for the bulk theory. Once this is done, we study the Lorentzian analogue, emphasising the differences between the Lorentzian and Euclidean cases and providing a prescription for dealing with the issues that arise in Lorentzian signature.

1.2.4 Euclidean AdS$_{d+1}$/CFT$_d$: Computing Expectation Values

Consider a massive scalar field $\Phi(x,z)$ propagating in empty AdS$_{d+1}$. Recall that in Poincaré coordinates the metric for AdS$_{d+1}$ is

$$\mathrm{d}s^2 = G_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu = \frac{\ell^2}{z^2}\left(\mathrm{d}z^2 + \delta_{ij}\mathrm{d}x^i\mathrm{d}x^j\right). \tag{1.2.15}$$

The action for the scalar field is

$$S_0 = \frac{1}{2}\int_{\text{AdS}} \mathrm{d}z\, \mathrm{d}^d x\, \sqrt{G}\left(G^{\mu\nu}\partial_\mu\Phi\partial_\nu\Phi + m^2\Phi^2\right) \tag{1.2.16}$$

and the corresponding bulk equation of motion is

$$\left(-\Box_G + m^2\right)\Phi = -\frac{1}{\sqrt{G}}\partial_\mu\left(\sqrt{G}G^{\mu\nu}\partial_\nu\Phi\right) + m^2\Phi = 0. \tag{1.2.17}$$

We use latin indices $i, j, \dots = 0, \dots d$ to label the transverse coordinates x^i and they are contracted using the Euclidean metric δ_{ij}. Stability of AdS requires the mass of the field is bounded from below by the Breitenlohner–Freedman bound, $m^2 \geq -d^2/4$ [14, 15]. Here we are treating the scalar field as a perturbation on a fix AdS background.

The general solution to (1.2.17) is [12, 16]

$$\Phi(x,z) = \int_{\partial\text{AdS}} \mathrm{d}^d x' \frac{\alpha(x) z^{\Delta_+}}{\left(z^2 + |x-x'|^2\right)^{\Delta_+}} + \int_{\partial\text{AdS}} \mathrm{d}^d x' \frac{\beta(x) z^{\Delta_-}}{\left(z^2 + |x-x'|^2\right)^{\Delta_-}} \tag{1.2.18}$$

where α and β are arbitrary functions of the boundary coordinates x. $\Delta_\pm$ are the roots of equation $m^2\ell^2 - \Delta(\Delta - d) = 0$ with $\Delta_+ = d/2 + \sqrt{d^2/4 + m^2\ell^2}$ the larger and $\Delta_- = d - \Delta_+ = d/2 - \sqrt{d^2/4 + m^2\ell^2}$ the smaller. We only consider cases where $\Delta_\pm$ are equal to $d/2 + n_\pm$ with $n_\pm$ two positive integers. This choice is motivated by supergravity and the spectrum of field theories that admit a holographic dual. Near the asymptotic boundary at $z = 0$ the first term behaves as z^{Δ_+} and the second as z^{Δ_-}. In particular, using the relation

$$\lim_{z\to 0} \frac{z^{2\Delta_+ - d}}{\left(z^2 + |x-x'|^2\right)^{\Delta_+}} = \pi^{d/2} \frac{\Gamma\left(\Delta_+ - \frac{d}{2}\right)}{\Gamma\left(\Delta_+\right)} \delta^{(d)}(x - x') \tag{1.2.19}$$

one finds that near the asymptotic boundary the bulk field $\Phi(x,z)$ has the form

$$\begin{aligned}\Phi(x,z) &= z^{\Delta_-}\left(\phi_{(0)}(x) + z^2\phi_{(2)}(x) + \dots\right) + z^{\Delta_+}\left(\phi_{(2\Delta_+ - d)}(x) + z^2\phi_{(2\Delta_+ - d + 2)}(x) + \dots\right) \\ &= z^{\Delta_-}\left(\phi_{(0)}(x) + z^2\phi_{(2)}(x) + \dots + z^{2\Delta_+ - d}\phi_{(2\Delta_+ - d)}(x) + z^{2\Delta_+ - d + 2}\phi_{(2\Delta_+ - d + 2)} + \dots\right)\end{aligned} \tag{1.2.20}$$

where $\phi_{(0)}(x)$ and $\phi_{(2\Delta_+ - d)}(x)$ are linear functionals of $\alpha(x)$ and $\beta(x)$. All subleading terms to order $z^{\Delta_+ - 1}$ can be determined in terms of $\phi_{(0)}(x)$ alone and all terms of order $z^{\Delta_+ + 1}$ and higher can be determined in terms of $\phi_{(0)}(x)$ and $\phi_{(2\Delta_+ - d)}$, using the equations of motion. In the special case $\Delta_+ = \Delta_- = \frac{d}{2}$ an additional logarithmic term appears at order $z^{2\Delta_+ - d}$.

Usually $\phi_{(0)}(x)$ and $\phi_{(2\Delta_+ - d)}(x)$ are interpreted as the source that couples to the dual operator and its expectation value, respectively. In this case conformal dimension of the dual operator is Δ_+. This configuration corresponds to imposing Dirichlet boundary conditions at the asymptotic boundary. However, depending on the mass of the field, this is not the only choice of bulk boundary conditions. In the next section we discuss the different choices of boundary conditions and how these relate to the mass of the scalar field and the conformal dimension of the dual operator. We then proceed to renormalise the scalar on-shell action and compute the expectation value of the dual operator which concludes our discussion of Euclidean AdS/CFT.

Boundary Conditions

In this section we discuss the possible boundary conditions for the bulk field and their field theory interpretation. According to the AdS/CFT correspondence the mass of the bulk field is related to the conformal dimension of the dual operator, Δ, which is given by the solution to the equation $\Delta(\Delta - d) = m^2\ell^2$. As we have seen already this equation has two roots, namely Δ_+ and Δ_-. The choice of either root determines the asymptotic behaviour of the bulk field, as well as the dimension of the dual operator. Modifying equation (1.2.20) slightly to allow either choice for Δ one requires that near the boundary

$$\Phi(x,z) \to z^{d-\Delta}\left(\phi_{(0)}(x) + \mathcal{O}(z^2)\right) + z^{\Delta}\left(\phi_{(2\Delta-d)}(x) + \mathcal{O}(z^2)\right) \tag{1.2.21}$$

where Δ can be either root. Usually one assumes that $\Delta = \Delta_+$ corresponding to imposing Dirichlet boundary conditions on the bulk field. However, under certain conditions $\Delta = \Delta_-$ is also allowed, corresponding to imposing Neumann boundary conditions on the bulk field. The two different choices of boundary conditions give rise to two distinct AdS invariant quantisations each of which is dual to a distinct field theory, one containing an operator of conformal dimension Δ_+ and one containing an operator of conformal dimension Δ_-.

The conditions under which each boundary behaviour of the bulk field is allowed were first studied by Breitenlohner and Freedman [15] who demonstrated that positivity of the classical energy and the existence of a well-defined quantum field theory imply that the mass of the scalar field must satisfy

$$-\frac{d^2}{4} \le m^2\ell^2 \tag{1.2.22}$$

for Dirichlet boundary conditions to be admissible and

$$-\frac{d^2}{4} \le m^2\ell^2 \le -\frac{d^2}{4} + 1 \tag{1.2.23}$$

for Neumann boundary conditions. Using the relations between $\Delta_\pm$ and $m^2\ell^2$ we see that the dimension of the scalar operator in the field theory has to be greater than or equal to $d/2$ when Dirichlet boundary conditions are imposed in the bulk and greater than $d/2 - 1$ and less than $d/2$ in the case of Neumann. It follows that the minimum dimension the dual operator can have is $d/2 - 1$ which is the unitarity bound for scalar operators in a d–dimensional field theory.

From the perspective of the dual field theory, the former choice corresponds to a theory with a scalar operator of dimension Δ_+ and the latter corresponds to a different theory with a scalar operator of conformal dimension Δ_-. We proceed to make these statements explicit, starting with the case where $\Delta = \Delta_+$. Given a boundary condition $\phi_{(0)}(x)$, the bulk field can be constructed using the bulk–to–boundary propagator

$K_{\Delta_+}(z, x, x')$ [16],

$$K_{\Delta_+}(z,x,x') = \pi^{-d/2}\frac{\Gamma(\Delta_+)}{\Gamma\left(\Delta_+ - d/2\right)}\frac{z^{\Delta_+}}{\left(z^2+|x-x'|^2\right)^{\Delta_+}} \xrightarrow[z\to 0]{} z^{d-\Delta_+}\delta^{(d)}(x-x') \quad (1.2.24)$$

and

$$\Phi(x,z) = \int \mathrm{d}^d x' K_{\Delta_+}(z,x,x')\,\phi_{(0)}(x'). \quad (1.2.25)$$

It follows that

$$\phi_{(2\Delta_+-d)}(x) = \pi^{-d/2}\frac{\Gamma(\Delta_+)}{\Gamma\left(\Delta_+ - d/2\right)}\int \mathrm{d}^d x' \phi_{(0)}(x')|x-x'|^{-2\Delta_+} \quad (1.2.26)$$

Modulo subtleties associated with divergences, this expression suggests that $\phi_{(2\Delta_+-d)}(x)$ is the expectation value of an operator $\mathcal{O}(x)$ of conformal dimension Δ_+ in the presence of another operator $\mathcal{O}$ located at x'. To demonstrate this we proceed to compute the on–shell action which will also allow us to make a direct connection with the discussion in section 1.2.3. Introducing for convenience a new field $\phi(x,z)$ through $\Phi(x,z) = z^{d-\Delta_+}\phi(x,z)$ and substituting into the action (1.2.16) we obtain

$$S_{\text{on-shell}} = -\frac{1}{2}\int_{z=\epsilon} \mathrm{d}^d x\, z^{d+1-2\Delta_+}\phi(x,z)\partial_z\phi(x,z) + S_{\text{ct}} \quad (1.2.27)$$

where we have introduced a cut–off surface $\mathcal{B}_\epsilon = \{z = \epsilon\}$ to regularise the integral over z. In the limit $\epsilon \to 0$ this surface is pushed to the AdS boundary, $\mathcal{B}_\epsilon \to \partial\text{AdS}$. S_{ct} consists of a set of counter–terms necessary to subtract divergences associated with infinite volumes in the bulk and short distances from the field theory perspective. Here we bypass these issues by restricting to $d/2 < \Delta_+ < d/2+1$ for which there are no such divergences and we postpone talking about the treatment of divergences until section 1.2.4. Using the expression for $\Phi(x,z)$ in terms of $\phi_{(0)}(x)$ as well as the series expansion of K_{Δ_+} around $z = 0$ one eventually finds

$$S_{\text{on-shell}}[\phi_{(0)}] = -\left(\Delta_+ - \frac{d}{2}\right)\pi^{-d/2}\frac{\Gamma\left(\Delta_+\right)}{\Gamma\left(\Delta_+ - d/2\right)}\int \mathrm{d}^d x' \mathrm{d}^d x \frac{\phi_{(0)}(x)\phi_{(0)}(x')}{|x-x'|^{2\Delta_+ +}} \quad (1.2.28)$$

Then, according to the discussion in section 1.2.3 and, in particular, from equation (1.2.14), it follows that the one–point function of the operator dual to $\Phi(x,z)$ in the CFT is

$$\begin{aligned}\langle\mathcal{O}(x)\rangle = -\frac{\delta S_{\text{on-shell}}}{\delta\phi_{(0)}(x)}\Big|_{\phi_{(0)}=0} &= (2\Delta_+ - d)\left[\pi^{-d/2}\frac{\Gamma(\Delta_+)}{\Gamma(\Delta_+ - d/2)}\int \mathrm{d}^d x' \frac{\phi_{(0)}(x')}{|x-x'|^{2\Delta_+}}\right] \\ &= (2\Delta_+ - d)\phi_{(2\Delta_+-d)}(x). \end{aligned} \quad (1.2.29)$$

Hence, we have proven that given a bulk theory with a scalar operator whose asymptotic behaviour is

$$\Phi(x,z) = z^{d-\Delta_+}\left(\phi_{(0)}(x) + \cdots + z^{2\Delta_+-d}\phi_{(2\Delta_+-d)}(x) + \dots\right) \tag{1.2.30}$$

the dual field theory contains a scalar operator of conformal dimension Δ_+ for which there is a source deformation turned on given by $\int \mathrm{d}^d x \phi_{(0)}(x)\mathcal{O}$. Moreover, the expectation value of $\mathcal{O}$ is proportional to $\phi_{(2\Delta_+-d)}$.

We now turn our attention to the case where the dual operator has dimension less than $d/2$, i.e. where $\Delta = \Delta_-$. Although this is a different field theory, it is not independent from the theory with Δ_+. The two are related via a canonical transformation that interchanged the roles of $\phi_{(0)}$ and $\phi_{(2\Delta_+-d)}$ as the "source" and the "fluctuating field" [16]. In particular, from the point of view of the Δ_- theory, $\phi_{(0)}$ is the field whose conformal dimension is Δ_+ and the source that couples to it is $-(2\Delta_+ - d)\phi_{(2\Delta_+-d)}$. Accordingly, the generating functional for the Δ_- theory is given by the Legendre transform of the generating functional of the Δ_+ theory,

$$\Gamma_{\text{QFT}}[\phi_{(2\Delta_+-d)}(x)] = W_{\text{QFT}}[\phi_{(0)}] - \int \mathrm{d}^d x\, \phi_{(0)}(x)\Big((2\Delta_+ - d)\phi_{(2\Delta_+-d)}(x)\Big) \tag{1.2.31}$$

and $\phi_{(0)}(x)$ and $(2\Delta_+ - d)\phi_{(2\Delta_+-d)}(x)$ are related to them via

$$\phi_{(0)}(x) = -\frac{\delta\Gamma[\phi_{(2\Delta_+-d)}]}{\delta\left((2\Delta_+ - d)\,\phi_{(2\Delta_+-d)}\right)}, \qquad (2\Delta_+ - d)\phi_{(2\Delta_+-d)} = \frac{\delta W[\phi_{(0)}]}{\delta\phi_{(0)}}. \tag{1.2.32}$$

From the perspective of the bulk, the Δ_- theory corresponds to fixing $\phi_{(2\Delta-d)}(x)$ instead of $\phi_{(0)}$. In particular, introducing a more general notation for the boundary deformation, $\exp\left(-\int d^d x\, J\mathcal{O}\right)$, where J is the source that couples to $\mathcal{O}$, the usual Dirichlet boundary conditions can be written as

$$J_{\text{D}} = \phi_{(0)}(x) \tag{1.2.33}$$

with the usual relation between the bulk on-shell action and the generating functional of the dual theory

$$\mathrm{e}^{-S_{\text{on-shell}}[\phi_{(0)}\sim J_{\text{D}}]} = \left\langle \exp\int \mathrm{d}^d x\, J\mathcal{O}\right\rangle = \mathrm{e}^{W_{\text{QFT}}[J]}. \tag{1.2.34}$$

For Neumann boundary conditions one fixes $(2\Delta_+ - d)\phi_{(2\Delta_+-d)}$,

$$J_{\text{N}} = -(2\Delta_+ - d)\phi_{(2\Delta_+-d)} \tag{1.2.35}$$

which requires an additional boundary term to be added to the bulk action, in accordance with equation (1.2.31),

$$S_{\text{on-shell}} \to S_{\text{on-shell}}\left[\phi_{(2\Delta_+-d)}\left(\phi_{(0)}\right) \sim J_{\text{N}}\right] + \int \mathrm{d}^d x\, \phi_{(0)} J_{\text{N}}. \tag{1.2.36}$$

Boundary condition	Mass range of bulk field	Conformal dimension of the dual operator	Range of conformal dimension
Dirichlet	$m^2\ell^2 \geq -\frac{d^2}{4}$	$\Delta_+ = \frac{d}{2} + \sqrt{\frac{d^2}{4} + m^2\ell^2}$	$\Delta \geq \frac{d}{2}$
Neumann/Mixed	$-\frac{d^2}{4} \leq m^2\ell^2 \leq -\frac{d^2}{4} + 1$	$\Delta_- = \frac{d}{2} - \sqrt{\frac{d^2}{4} + m^2\ell^2}$	$\frac{d}{2} - 1 \leq \Delta \leq \frac{d}{2}$

Table 1.2.1: Boundary conditions for a free scalar field of mass m in AdS and corresponding conformal dimensions for the dual operator.

Note that one must first solve $\phi_{(2\Delta_+-d)}$ in terms of $\phi_{(0)}$ and then use the resulting expression to evaluate the on-shell action.

A final possibility not discussed thus far is to impose mixed boundary conditions on bulk field in which case one fixes a particular combination of $\phi_{(0)}$ and $\phi_{(2\Delta_+-d)}$. From the perspective of the field theory this choice introduces a multi-trace deformation. In particular, suppose we want to introduce a deformation $\int \mathrm{d}^d x\, f(\mathcal{O})$ to the field theory action $S_{f,\mathrm{QFT}}[\mathcal{O}] = S_{\mathrm{QFT}} + \int \mathrm{d}^d x\, f(\mathcal{O})$ where $f(0) = 0$. This modifies the generating functional according to

$$W_{f,\mathrm{QFT}} = W_{\mathrm{QFT}}[J] + \int \mathrm{d}^d x\, \left(f(\sigma) - \sigma\, f'(\sigma)\right)\Big|_{\sigma = \frac{\delta W[J]}{\delta J}} \tag{1.2.37}$$

where σ is the vev of $\mathcal{O}$. In theory the vev of $\mathcal{O}$ can be chosen to be proportional to either $\phi_{(0)}$ or $\phi_{(2\Delta-d)}$ and have dimension Δ_- or Δ_+ respectively. However, the deformation is relevant only if we choose $\phi_{(0)}$ and Δ_-, i.e. if we are deforming the Neumann theory. The boundary condition in this case reads

$$J_{\mathrm{f}} = -(2\Delta_+ - d)\phi_{(2\Delta_+-d)} - f'(\phi_{(0)}). \tag{1.2.38}$$

and order to impose it we must modify the bulk action according to

$$S_{\mathrm{bulk}} \to S_{\mathrm{bulk}} + \int \mathrm{d}^d x\, \left[-\phi_{(0)}(2\Delta_+ - d)\phi_{(2\Delta_+-d)} + f(\phi_{(0)}) - \phi_{(0)} f'(\phi_{(0)})\right]. \tag{1.2.39}$$

The additional boundary terms in the bulk action lead to a modification of the field theory stress-energy tensor. This topic is discussed further in chapter 6 as well as chapters 7 and 8.

Boundary condition	Source	Vev	W[J]
Dirichlet	$J_{\mathrm{D}} = \phi_{(0)}$	$(2\Delta_+ - d)\phi_{(2\Delta_+-d)}$	$-S_{\text{on-shell}}[J_{\mathrm{D}}]$
Neumann	$J_{\mathrm{N}} = -(2\Delta_+ - d)\phi_{(2\Delta_+-d)}$	$\phi_{(0)}$	$-S_{\text{on-shell}}[J_{\mathrm{N}}] - \int \mathrm{d}^d x\, \phi_{(0)} J_{\mathrm{N}}$
Mixed	$J_{\mathrm{f}} = J_{\mathrm{N}} - f'(\phi_{(0)})$	$\phi_{(0)}$	$-S_{\text{on-shell}}[J_{\mathrm{N}}] - \int \mathrm{d}^d x\, \phi_{(0)} J_{\mathrm{N}} - \int \mathrm{d}^d x\, f(\phi_{(0)}) + \phi_{(0)} f'(\phi_{(0)})$

Table 1.2.2: Source, vev and generating functionals for a field theory dual to AdS_{d+1} with a single free scalar field.

Holographic Renormalisation

Next we shall compute the on-shell action and address the issue of divergences associated with the infinite volume of AdS. To do this we must first solve the asymptotic expansion of the field, given by equation (1.2.20),

$$\Phi(x,z) = z^{\Delta_-}\left(\phi_{(0)}(x) + z^2\phi_{(2)}(x) + \cdots + z^{2\Delta_+-d}\phi_{(2\Delta_+-d)}(x) + z^{2\Delta_+-d+2}\phi_{(2\Delta_+-d+2)} + \dots\right)$$

and determine its coefficients. Plugging this expression into the equation of motion (1.2.17) and solving order by order in z one finds that $\phi_{(k)} = 0$ for k odd and

$$\phi_{(2n)}(x) = \frac{1}{2n\,(2\Delta - d - 2n)}\Box_0\phi_{(2n-2)}(x) \tag{1.2.40}$$

where $\Box_0 = \delta^{ij}\partial_i\partial_j$ and $n < \Delta - d/2$. If $2\Delta = 2m + d$ for some positive integer m then there is an additional logarithmic term at order z^{Δ},

$$\Phi(x,z) = z^{d-\Delta}\left(\phi_{(0)}(x) + z^2\phi_{(2)}(x) + \cdots + z^{2\Delta-d}\left(\phi_{(2\Delta-d)}(x) + \tilde{\phi}_{(2\Delta-d)}(x)\log z\right) + \dots\right) \tag{1.2.41}$$

with

$$\tilde{\phi}_{(2\Delta-d)}(x) = -\frac{1}{2^{2m}\Gamma(m)\,\Gamma(m+1)}\left(\Box_0\right)^m\phi_{(0)}(x). \tag{1.2.42}$$

All terms up to $\phi_{(2\Delta-d)}(x)$, including $\tilde{\phi}_{(2\Delta-d)}(x)$ are determined in terms of $\phi_{(0)}(x)$ by solving algebraic equations. $\phi_{(2\Delta-d)}(x)$ appears at order z^{Δ} which is the order of the second linearly independent solution, explaining why it can not be determined in terms of $\phi_{(0)}(x)$. An attempt to apply the same procedure outlined above to this order leads to a trivial equation[1]. In order to obtain $\phi_{(2\Delta-d)}(x)$ one must solve the equation of motion

[1] In the cases where the bulk field is the metric of a bulk gauge field, the bulk equations of motion determine the trace and divergence of the fields. These equations then lead to Ward identities for the dual

in the bulk and not only asymptotically. Then, regularity in the interior combined with the boundary condition $\phi_{(0)}(x)$ uniquely determines $\phi_{(2\Delta-d)}(x)$ as well.

We already saw that $\phi_{(0)}(x)$ is the source term in the field theory and $\phi_{(2\Delta-d)}(x)$ is proportional to the vacuum expectation value of the dual operator. The new term, $\tilde{\phi}_{(2\Delta-d)}(x)$, has a field theory interpretation as well, it is related to conformal anomalies.

Having determined all possible coefficients in the asymptotic expansion of $\Phi(x,z)$, the next step is to compute the on-shell action. The on-shell action suffers from infinite volume IR divergences related to field theory UV divergences. To deal with them we introduce a cut-off at $z=\epsilon$ and integrate over $z \geq \epsilon$ where ϵ is a small positive parameter. The on-shell action will have a finite number of terms that diverge in the limit $\epsilon \to 0$. These terms give the counter terms that have to be subtracted to remove the IR divergences. This method of dealing with divergences in AdS/CFT is known as *holographic renormalisation*. It was initiated in [17, 18] and developed in [19]. Detailed discussions can be found in [20, 19, 21, 22]. A radial Hamiltonian version of the method was introduced in [23, 24]. Here we follow the exposition in [25].

The determination of the divergent terms requires that, in addition to the asymptotic expansion of the field, we also have an asymptotic expansion for the metric. The appropriate asymptotic form for the metric is given by the Fefferman-Graham gauge (FG),

$$\mathrm{d}s^2 = \frac{\ell^2}{z^2}\left(\mathrm{d}z^2 + g_{ij}(x,z)\mathrm{d}x^i\mathrm{d}x^j\right) \tag{1.2.43}$$

with

$$g_{ij}(x,z) = g_{(0)ij}(x) + z g_{(1)ij}(x) + z^2 g_{(2)ij}(x) + \ldots \tag{1.2.44}$$

In general, when studying as asymptotically AdS spacetime, one must also determine the coefficients $g_{(k)ij}(x)$, either by computing the asymptotic expansion of the bulk metric or, if the bulk metric is not known, by solving the corresponding equations of motion order by order in z, as we did for $\Phi(x,z)$.

Using these results, the regularised action is

$$\begin{aligned} S_{\text{reg}} &= \frac{1}{2}\int_{z\geq\epsilon} \mathrm{d}z\,\mathrm{d}^d x\,\sqrt{G}\left(G^{\mu\nu}\partial_\mu\Phi\partial_\nu\Phi + m^2\phi^2\right) \\ &= \frac{1}{2}\int_{z\geq\epsilon} \mathrm{d}z\,\mathrm{d}^d x\,\sqrt{G}\,\Phi\left(-\Box_G + m^2\right)\Phi - \frac{1}{2}\int_{z=\epsilon}\mathrm{d}^d x\left(\sqrt{G}G^{zz}\,\Phi\partial_z\Phi\right)_{z=\epsilon} \end{aligned}$$

Evaluating S_{reg} on-shell eliminates the bulk term and we are left only with boundary terms,

$$\begin{aligned} S_{\text{reg, on-shell}} &= -\frac{\ell^{d-1}}{2}\int_{z=\epsilon}\mathrm{d}^d x\,\epsilon^{d-2\Delta}\left((d-\Delta)\phi^2(x,\epsilon) + \epsilon\phi(x,\epsilon)\partial_\epsilon\phi(x,\epsilon)\right) \\ &= \ell^{d-1}\int_{z=\epsilon}\mathrm{d}^d x\,\epsilon^{d-2\Delta}\left(a_{(0)} + \epsilon^2 a_{(2)} + \epsilon^4 a_{(4)} + \ldots + \log\epsilon\; a_{(2\Delta-d)}\right) \end{aligned} \tag{1.2.45}$$

operators.

where the coefficients $a_{(2k)}$ are local functions of $\phi_{(0)}$ that are determined using the series expansion of $\phi(x,\epsilon)$. We present here four of them to demonstrate their structure.

$$a_{(0)} = -\frac{1}{2}(d-\Delta)\phi_{(0)}^2, \qquad a_{(2)} = -(d-\Delta+1)\phi_{(0)}\phi_{(2)} = -\frac{d-\Delta+1}{2\,(2\Delta-d-2)}\phi_{(0)}\Box_0\phi_{(0)},$$
$$a_{(4)} = -\frac{1}{2}\left((d-\Delta)\phi_{(2)}^2 + \phi_{(0)}\phi_{(4)}\right) = -\frac{1}{2}\left(\frac{(d-\Delta)\left(\Box_0\phi_{(0)}\right)^2}{4\,(2+d-2\Delta)^2} + \frac{\phi_{(0)}\left(\Box_0\right)^2\phi_{(0)}}{8\,(4+d-2\Delta)\,(2+d-2\Delta)}\right)$$
$$a_{(2\Delta-d)} = -\frac{d}{2^{2m+1}\Gamma(m)\Gamma(m+1)}\phi_{(0)}\left(\Box_0\right)^m\phi_{(0)}.$$

All terms in (1.2.45) that appear at negative order in ϵ diverge in the limit $\epsilon \to 0$ and we need to add counter terms to the on-shell action to eliminate them. The counterterms are obtained by re-expressing (1.2.45) in terms of bulk fields. In particular, the derived expression for $S_{\text{reg, on-shell}}$ is a function of $\phi_{(0)}(x)$ and $g_{ij} = \delta_{ij}$. We need to expressed these fields as functions of $\Phi(x,\epsilon)$ and γ_{ij}, the restrictions of the bulk field and metric on the hypersurface $z = \epsilon$. This ensures that the action transforms correctly under bulk diffeomorphisms.

In general $\gamma_{ij} = g_{ij}\,\ell^2/\epsilon^2$ which, for empty AdS in Poincaré coordinates becomes $\gamma_{ij} = \delta_{ij}\ell^2/\epsilon^2$. To obtain an expression for $\phi_{(0)}(x)$ in terms of $\Phi(x,\epsilon)$ one must invert the expansion (1.2.41) evaluated at $z = \epsilon$ order by order in ϵ. One must perform the inversion to high enough order in ϵ such that we are able to rewrite all divergent terms in S_{reg} in terms of $\Phi(x,\epsilon)$. The first terms of the covariant counterterm action are

$$S_{\text{ct}} = -S_{\text{reg}} = \frac{1}{L}\int_{z=\epsilon} \mathrm{d}^d x\sqrt{\gamma}\left(\frac{1}{2}(d-\Delta)\Phi^2(x,\epsilon) + \frac{\ell^2}{2(2\Delta-d-2)}\Phi(x,\epsilon)\Box_\gamma\Phi(x,\epsilon) + \dots\right.$$
$$\left. -L^4\frac{(d-\Delta+2)}{8(2\Delta-2-d)^2}\left(\Box_\gamma\Phi(x,\epsilon)\right)^2 + \dots\right).$$

The counterterm action computed this way contains the minimal set of terms necessary for the theory to be renormalisable. We are free to add further finite terms which will give rise to scheme dependence in the field theory. In either case, the renormalised on-shell action is computed by adding the counterterm action to the regularised on-shell action and taking the limit $\epsilon \to 0$,

$$S_{\text{sub}} = S_{\text{reg}} + S_{\text{ct}} = \frac{1}{2}\int_{z\geq\epsilon} \mathrm{d}z\,\mathrm{d}^d x\,\sqrt{G}\left(G^{\mu\nu}\partial_\mu\Phi(x,z)\partial_\nu\Phi(x,z) + m^2\Phi^2(x,z)\right)$$
$$+\frac{1}{\ell}\int_{z=\epsilon} \mathrm{d}^d x\sqrt{\gamma}\left(\frac{1}{2}(d-\Delta)\Phi^2(x,\epsilon) + \frac{\ell^2}{2(2\Delta-d-2)}\Phi(x,\epsilon)\Box_\gamma\Phi(x,\epsilon)\right.$$
$$\left. -\ell^4\frac{(d-\Delta+2)}{8(2\Delta-2-d)^2}\left(\Box_\gamma\Phi(x,\epsilon)\right)^2 + \dots\right)$$

and

$$S_{\text{ren}} = \lim_{\epsilon\to 0} S_{\text{sub}}. \tag{1.2.46}$$

The final part of our discussion is the computation of the vacuum expectation value for the dual operator. The expectation value is computed by varying the subtracted bulk action, S_{sub}, with respect to $\Phi(x,\epsilon)$ and then taking the limit $\epsilon \to 0$,

$$\langle \mathcal{O} \rangle_{\text{s}} = \lim_{\epsilon\to 0}\left(\frac{\ell^d}{\epsilon^{\Delta}\sqrt{\gamma}}\frac{\delta S_{\text{sub}}}{\delta\Phi(x,\epsilon)}\right) - (2\Delta - d)\phi_{(2\Delta-d)}(x) + C(\phi_{(0)}). \tag{1.2.47}$$

The subscript s is to stress that this is the one point function of $\mathcal{O}$ in the presence of a source and $C(\phi_{(0)})$ is a scheme dependent local function of $\phi_{(0)}$ [19].

This prescription is straight forwardly generalised to other fields. In section 1.3.2 we apply it to Einstein–Maxwell theories and in chapter 6 to a class of theories containing, in addition to the Maxwell field, massive and massless scalars.

Most of the prescription described in this section did not rely on the signature of the spacetime and can therefore be applied to both Lorentzian and Euclidean AdS/CFT. However, there is a crucial difference between the two signatures related to the determination of $\phi_{(2\Delta-d)}$. This is the subject of the next section in which we describe the process for doing Lorentzian AdS/CFT.

1.2.5 Real Time Holography

In the previous section the field equation for $\Phi(x,z)$ was solved only asymptotically and boundary conditions were imposed which fixed the leading coefficient in the asymptotic expansion, namely $\phi_{(0)}(x)$. $\phi_{(2\Delta-d)}(x)$, the coefficient of the normalisable mode, appears at a subleading order in z and as we saw, it can not be determined from the boundary conditions. Moreover, despite it being an integration function, we do not have the freedom to select $\phi_{(2\Delta-d)}(x)$, it has to be determined dynamically by the theory. This is reflected in the fact that choosing a generic pair $\big(\phi_{(0)}(x), \phi_{(2\Delta-d)}(x)\big)$ in general will result in singular fields. The analogue of this on the field theory side is that the vacuum structure is a dynamical question and we can not tune $\langle\mathcal{O}\rangle$. The first step towards obtaining a solution that is regular and unique is to solve the field equations in the interior of AdS and impose regularity of the field. In Euclidean AdS this is sufficient; given $\phi_{(0)}(x)$, imposing bulk regularity uniquely selects $\phi_{(2\Delta-d)}$, resulting in a unique, regular $\Phi(x,z)$. However, in Lorentzian AdS there are normalisable modes which are regular at the interior and decay at the boundary and hence they can be added to any regular solution, modifying $\phi_{(2\Delta-d)}$ but not $\phi_{(0)}$. This issue is discussed further is chapter 2, section 2.1.

On the field theory side this issue is related to the existence of multiple types of correlators in Lorentzian signature. Moreover, one can compute correlation functions in non-trivial states, something which is not possible in Euclidean signature. One way to deal with these issues in quantum field theory is to specify a time contour in the complex time plane. Consider for example the time ordered vacuum correlator $\langle\Omega|\,\mathcal{T}\left(\mathcal{O}(x_1)\mathcal{O}(x_2)\right)|\Omega\rangle$. This can be obtained from $\langle\phi_-,-T|\,\mathcal{T}\left(\mathcal{O}(x_1)\mathcal{O}(x_2)\right)|\phi_+,T\rangle$ by extending the time T along the imaginary axis. Here $|\phi_-,-T\rangle$ and $|\phi_+,T\rangle$ correspond to initial and final states $|\phi_-\rangle$ and $|\phi_+\rangle$ at times $-T$ and $+T$ respectively. These are time dependent states and they

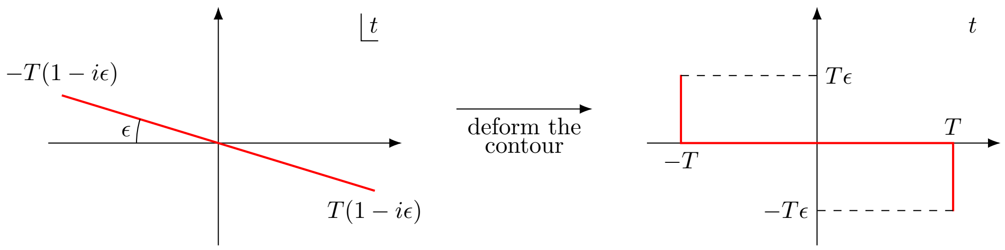


Figure 1.2.3: **(Left)** Complex time contour used in the example to obtain $\langle\Omega|\mathcal{T}(\ldots)|\Omega\rangle$ starting from $\langle\Phi_-,-T|\mathcal{T}(\ldots)|\Phi_+,T\rangle$ and taking the limit $T\to\infty$. The contour can be deformed in the complex t plane, as long as it does not cross any poles.
(Right) Deformation of the original contour shown on the left. Here the contour runs from $-T(1-i\epsilon)$ to $-T$ to $+T$ and finally to $T(1-i\epsilon)$. Both contours project onto the vacuum in the limit $T\to\infty$.

can be written in a time-independent form using the time-evolution operator $U(t) = \exp(-iHt)$. Then $|\phi_+,T\rangle = \mathrm{e}^{-iHT}|\phi_+\rangle$, $\langle\phi_-,-T| = \langle\phi_-|\,\mathrm{e}^{-iHT}$ and

$$\langle\phi_-,-T|\,\mathcal{T}\left(\mathcal{O}(x_1)\mathcal{O}(x_2)\right)|\phi_+,T\rangle = \langle\phi_-|\,\mathrm{e}^{-iHT}\mathcal{T}\left(\mathcal{O}(x_1)\mathcal{O}(x_2)\right)\mathrm{e}^{-iHT}\,|\phi_+\rangle\,. \tag{1.2.48}$$

Complexifying time according to $T\to T(1-i\epsilon)$ and taking the limit $T\to\infty$ projects the desired vacuum correlator since

$$\lim_{T\to\infty}\mathrm{e}^{-iHT(1-i\epsilon)}\,|\phi_+\rangle = \lim_{T\to\infty}\int\mathcal{D}\psi\,\mathrm{e}^{-iHT(1-i\epsilon)}\,|\psi\rangle\,\langle\psi|\phi_+\rangle = A\,|\Omega\rangle \tag{1.2.49}$$

where $A=\lim_{T\to\infty}\mathrm{e}^{-iE_{\text{vac}}T(1-i\epsilon)}\,\langle\Omega|\phi_+\rangle$ is a constant. A similar expression holds for $\langle\phi_-|$ which leads to

$$\lim_{T\to\infty}\langle\phi_-,-T(1-i\epsilon)|\,\mathcal{T}\left(\mathcal{O}(x_1)\mathcal{O}(x_2)\right)|\phi_+,T\rangle = C\,\langle\Omega|\,\mathcal{T}\left(\mathcal{O}(x_1)\mathcal{O}(x_2)\right)|\Omega\rangle \tag{1.2.50}$$

where C is another constant. The $(1-i\epsilon)$ factor corresponds to tilting the time line in the complex time plane as shown in the left panel of figure 1.2.3. The right panel of the figure shows a deformed version of this path consisting of a real segment and two imaginary ones. Such path deformations are allowed as long as they do not cross any singularities and the end points remain fixed. Upon taking the limit $T\to\infty$, the vertical segments project the theory onto some initial and final states; in the example above the vacuum. These states are evolved respectively forward and backwards in time and become the initial and final states for the real field theory computations, associated with the real time segment $-T$ to T.

More generally, one may wish to compute $\langle\Psi|\,\mathcal{T}\left(\mathcal{O}(x_1)\ldots\mathcal{O}(x_n)\right)|\Phi\rangle$ for initial and final states $|\Phi\rangle$ and $|\Psi\rangle$. In this case one can use time contour shown in figure 1.2.4, where the vertical segments extend to complex infinity. The crosses along the contour represent operator insertions at different times. The initial and final states $|\Phi\rangle$ and $|\Psi\rangle$ are created

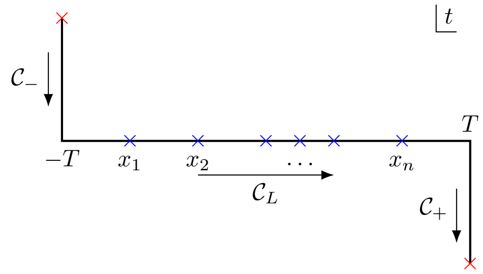


Figure 1.2.4: Complex time contour used to evaluate $\langle\Psi|\,\mathcal{T}\left(\mathcal{O}(x_1)\ldots\mathcal{O}(x_n)\right)|\Phi\rangle$. The initial and final states $|\Phi\rangle$ and $\langle\Psi|$ are generated by operator insertions along the imaginary segments of the contour. These insertions are indicated by the red crosses along $\mathcal{C}_-$ and $\mathcal{C}_+$. The operators $\mathcal{O}(x_1)$, $\mathcal{O}(x_2)$, ..., $\mathcal{O}(x_n)$ whose real–time expectation value is to be computed are inserted along $\mathcal{C}_L$, the real segment of the time contour. These represented here by the blue crosses. The choice of contour automatically enforces the time ordering.

by operator insertions along the semi–infinite vertical lines $\mathcal{C}_-$ and $\mathcal{C}_+$, respectively. The operators $\mathcal{O}(x_i)$ whose correlator we wish to compute are inserted along the real time segment $\mathcal{C}_L$. The path integral for this setup is given by

$$\mathcal{Z}_{\text{QFT}}\left[\phi_{(0)};\mathcal{C}\right] = \int [\mathcal{D}\mathcal{O}]\,\exp\left(-i\int_{\mathcal{C}}\mathrm{d}t\int \mathrm{d}^{d-1}x\sqrt{g}\left(\mathcal{L}_{\text{QFT}}[\mathcal{O}] + \phi_{(0)}\mathcal{O}\right)\right) \tag{1.2.51}$$

where $\mathcal{O}$ represents the collective set of operators in the theory and $\phi_{(0)}$ are sources that couple to these operators. The time integration of the Lagrangian is along the complex time contour. More specifically, t is a complex variable which runs from $-T-i\infty$ to $-T$ on $\mathcal{C}_-$, the from $-T$ to T along $\mathcal{C}_L$ and then from T to $T+i\infty$ on $\mathcal{C}_+$. Alternatively, one can parametrise the contour using real variables. For the vertical segments we can write $t=-T+i\tau_-$ with $\tau_-\leq 0$ for $\mathcal{C}_-$, and $t=-T+i\tau_+$ with $\tau_+\geq 0$ for $\mathcal{C}_+$. For the horizontal, real–time segment we can simply use t with $-T\leq t\leq T$. The time–ordered real time correlator is computed simply by functional differentiation of $\mathcal{Z}_{\text{QFT}}$ with respect to the sources. The time ordering is naturally implemented by the choice of time contour.

Other types of correlators and different states can be computed by appropriate choices of contours and operator insertions. The left panel of figure 1.2.5 shows examples of some field theory complex time contours. Starting from the top, the first is the in–out contour used to compute time order correlators, the second is the in–in time contour used to compute path ordered correlators and the third is the thermal contour used to compute thermal correlators. This formalism for specifying initial and final states and computing correlators using the path integral is the Schwinger–Keldysh formalism [26, 27] and it has been extended to holography.

More precisely, to obtain unique fields in Lorentzian AdS, in addition to the boundary conditions at the conformal boundary, one must provide initial and/or final data for the fields. The initial and final data can be imposed by complexifying the AdS manifold to include both Euclidean and Lorentzian segments, in analogy to how the time contour in quantum field theory is complexified. Then, the complex segments of the manifold, i.e. the Euclidean ones, provide the initial and final data. In particular, they can be thought of as providing a Hartle–Hawking wave function which is matched to the Lorentzian theory on spacelike interfaces between Euclidean and Lorentzian segments.

The holographic analogue of the Schwinger–Keldysh formalism is known as *real-time gauge/gravity duality* [28, 29] and it dictates that one should start with a complex field theory time contour and "fill it in" with AdS spatial directions to obtain an AdS manifold consisting of Euclidean and Lorentzian AdS segments, dual to the complex and real time segments in the field theory. That is to say, given a field theory spacetime consisting of a line in the complex time plane times a real space $\mathbb{R} \times \Sigma^{d-1}$, the bulk spacetime should be an asymptotically AdS spacetime whose boundary is conformal to $\mathbb{R} \times \Sigma^{d-1}$. Under this prescription, imaginary field theory time segments become Euclidean (A)AdS manifolds and real time segments become Lorentzian (A)AdS manifolds. Figure 1.2.5 shows examples of field theory time contours $\mathcal{C}$ on the left and the corresponding bulk manifolds $M_{\mathcal{C}}$ on the right.

For the construction of unique bulk fields one begins by solving the field equations, including the Einstein equations, for each segment of the manifold and for the metric signature dictated by the field theory contour. The solutions are then glued together and matching conditions are imposed at the spacelike hypersurfaces between the submanifolds. For example, consider two adjacent manifolds M_- and M_+ and a spatial surface $\partial_t M$ connecting them. To obtain a field Φ in $M_- \cup M_+$ we first solve the field equations for Φ in M_- and M_+ separately and then impose the following matching conditions on the resulting solutions

$$\begin{aligned} \Phi_-\big|_{\partial_t M} &= \Phi_+\big|_{\partial_t M} \\ \partial_t \Phi_-\big|_{\partial_t M} &= \partial_t \Phi_+\big|_{\partial_t M}. \end{aligned} \tag{1.2.52}$$

The first condition implies that the field must be continuous on $\partial_t M$ and the second that its conjugate momentum is also continuous. The derivative in the second line is with respect to the complex time t introduced above in the context of the field theory. Solving these matching conditions provides the initial and final data for the Lorentzian fields.

Under this construction, the relation between the field theory partition function and the supergravity action becomes

$$\mathcal{Z}_{\text{QFT}}[\mathcal{O}; \phi_{(0)}, \mathcal{C}] = \exp\left(i \int_{M_{\mathcal{C}}} \mathrm{d}^{d+1}x \sqrt{-G}\, \mathcal{L}^{\text{on-shell}}_{\text{AdS}} \left[\Phi \sim \phi_{(0)}\right] \right) \tag{1.2.53}$$

where $M_{\mathcal{C}}$ is bulk manifold corresponding to $\mathcal{C}$. For example, let $\mathcal{C}$ be the complex time contour shown in figure 1.2.4 and discussed above. A version of this contour along with

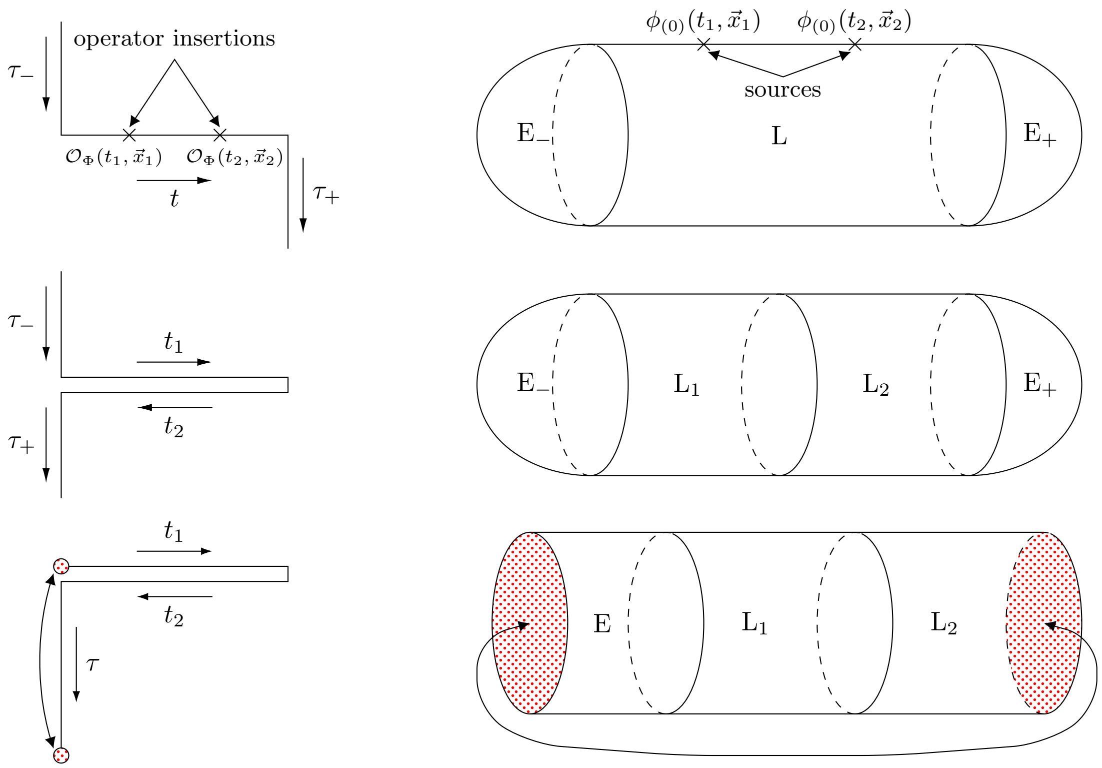


Figure 1.2.5: Field theory complex time contours and corresponding AdS manifolds. (**Top**) In–out time contour with two operator insertions along the real time segment used to compute time ordered correlators (left) and corresponding AdS manifold (right). The two field theory semi–infinite Euclidean time segments τ_- and τ_+ become the half Euclidean AdS manifold E_- and E_+. These provide the initial and final conditions for all the Lorentzian fields, including the metric. The insertion of operators in the field theory corresponds to the boundary conditions $\phi_{(0)}$.
(Middle) In–In time contour used to compute path ordered correlators (left) and corresponding AdS manifold (right). Again, the semi–infinite Euclidean time segments of the field theory contour become half Euclidean AdS manifolds, providing initial and final data for Lorentzian fields. The field theory time contour consists of two real time segments with opposite time evolution directions. On the AdS side this translates to having two Lorentzian AdS manifolds with the time also evolving in an analogous manner.
(Bottom) Complex time contour for a thermal field theory (left) and corresponding AdS manifold (right). For both theories, the Euclidean time is periodically identified, as indicated by the shaded circles/surfaces. In the field theory this results in a periodic Euclidean time with the period identified with the inverse temperature β. On the AdS side, this corresponds to imposing appropriate boundary conditions on the two shaded spacelike surfaces which are identified.

the corresponding AdS manifold $M_{\mathcal{C}}$ are shown in the top panel of figure 1.2.5. In this case, the right hand side of (1.2.53) is given schematically by

$$\exp\left(-\int_{M_{\mathcal{C}_-}} \mathrm{d}\tau_- \mathrm{d}^d x \sqrt{G_E}\, \mathcal{L}^E_{\text{on-shell}}\left[\Phi \sim \phi^-_{(0)}\right] + i\int_{M_{\mathcal{C}_L}} \mathrm{d}t\, \mathrm{d}^d x \sqrt{-G_L}\, \mathcal{L}^L_{\text{on-shell}}\left[\Phi \sim \phi^L_{(0)}; \phi^-_{(0)}, \phi^+_{(0)}\right]\right.$$
$$\left. - \int_{M_{\mathcal{C}_+}} \mathrm{d}\tau_+\, \mathrm{d}^d x \sqrt{G_E}\, \mathcal{L}^E_{\text{on-shell}}\left[\Phi \sim \phi^+_{(0)}\right]\right)$$

where $\mathcal{L}^L_{\text{on-shell}}$ is the Lorentzian Lagrangian and $\mathcal{L}^E_{\text{on-shell}}$ is its Wick rotated version. $\phi^-_{(0)}$, $\phi^L_{(0)}$ and $\phi^+_{(0)}$ are the boundary conditions imposed at the asymptotic boundaries of $M_{\mathcal{C}_-}$, $M_{\mathcal{C}_L}$ and $M_{\mathcal{C}_+}$ respectively. Note that the Lorentzian action depends on both the boundary condition $\phi^L_{(0)}$ as well as $\phi^\pm_{(0)}$. From the perspective of the Lorentzian fields, the latter are related to the initial and final conditions imposed.

The above expression suffers from IR divergences due to the non–compactness of the radial direction, just as was the case for Euclidean AdS/CFT. Moreover, there are additional divergences associated with the non–compactness of the temporal direction of Lorentzian AdS. However, these new divergences cancel out and, thus, one can apply the holographic renormalisation prescription introduced in section 1.2.4 "piecewisely" to each segment of the manifold, without any modifications.

For a more detailed discussion of real time holography and examples of applications can be found in [28, 29]. In the first part of the thesis we study a scalar field in Lorentzian AdS and we use the formalism of real time gauge/gravity to derive the bulk setup dual to an excited field theory state. In particular, we confirm that turning on source in the Euclidean submanifolds corresponds to having the field theory in an excited state.

1.3 Applied AdS/CFT: Statistical Field Theories

In this section we will apply many of the concepts introduced and extend what was already said to study a gauge/gravity dual system where the field theory is at finite temperature and chemical potential. Such a system can be thought of as a toy model for the study of condensed matter phenomena. Conventionally, condensed matter systems are studied using Landau's theory and the Fermi liquid theory. The former relies on the symmetries of the system and the latter makes use of perturbation theory. However, many interesting phenomena such as high critical temperature superconductivity, the fractional quantum Hall effect, heavy fermion compounds and spin liquids, fall outside the spectrum of applicability of either of these two theories. There is, thus, a need for new theoretical tools for the study of systems that are beyond the scope of the traditional tools.

Gauge/gravity dualities, being strong/weak coupling dualities, provide an excellent framework for studying strongly correlated systems. By considering bulk theories with a $U(1)$ gauge field and by looking at asymptotically AdS (AAdS) solutions with black holes, we obtain gauge theories at finite chemical potential and temperature. It is then possible to study the phase diagram and thermodynamics of the gauge theories by studying the black hole thermodynamics and the asymptotic behaviour of bulk fields. Moreover, by perturbing bulk fields and studying the response of the bulk system one can compute the transport coefficients of the gauge theory such as the conductivity.

The construction of dual theories with the desired features often relies on the bottom–up approach in which one chooses the bulk setup phenomenologically as opposed to deriving it through a consistent truncation of supergravity. In this case the field theory Lagrangian is not known. This, however, does not pose an issue since the properties of the theory are inferred from gravity computations. Moreover, condensed matter systems often display universality, implying that one can obtain useful information even when studying toy models whose microscopic feature are unknown or are not realised in a physical system. Such considerations motivate the study of bulk setups with different features.

In part II we study the thermodynamics of a three field theories which have special features such scalar fields with multitrace deformations and topological axionic charges. The bulk solutions dual to these theories are known analytically and they are all solutions of the same class of theories whose action has the form

$$S_{\text{bulk}} = \int_{\mathcal{M}} \mathrm{d}^{d+1}x\sqrt{-G}\left(R - \frac{1}{2}\left(\partial\phi\right)^2 - V(\phi) - \frac{1}{2}W(\phi)\sum_{I=1}^{d-1}\left(\partial\psi_I\right)^2 - \frac{1}{4}Z(\phi)F^2\right) \quad (1.3.1)$$

where we have set $G_{\text{N}} = 1/16\pi$. The fields ϕ and ψ are a massive scalar and $d-1$ massless scalars respectively, $W(\phi)$ is the coupling between ϕ and ψ, $F_{\mu\nu}$ is the field strength associated with a $U(1)$ symmetry, $Z(\phi)$ is couplings between the corresponding gauge field and the massive scalar and $V(\phi)$ is the potential for ϕ which, at zero order in ϕ, gives the negative cosmological constant associated with asymptotically AdS spacetimes. A full analysis of this action is given in chapter 6. This includes the asymptotic solution of the action, the renormalisation, the computation of the one point functions and the thermodynamic properties of the dual theory. However, before we study the theories defined by (1.3.1) it is useful to look at the much simpler setup of Einstein–Maxwell theories with a negative cosmological constant which provide the minimal setup for the study of field theories at finite temperature and density.

1.3.1 Field Theories at Finite Temperature and Charge Density

In this section we review briefly the notions of the canonical and grand canonical ensemble in statistical physics and demonstrate how the physical quantities of the system in thermal equilibrium can be derived from a thermodynamic potential. This will motivate then next section in which we will discuss how the thermodynamic potential and expectation values of various operators can be obtained holographically.

Consider a field theory with a time independent Hamiltonian $\hat{H}$ and a global U(1) symmetry associated with a conserved current J^i and a conserved charge $\hat{Q}$. The expectation value of this conserved charge is associated to the number of charged particles N. At finite temperature the theory will experience thermal and quantum fluctuations and to study it we need to use statistical physics. If the charge density or, equivalently, the number of charged particles, is kept fixed the statistical ensemble representing the possible configurations of the theory is the canonical ensemble in which, in addition to the charge density, the temperature and volume are kept fixed. The expectation values of operators are obtained as a statistical average over the ensemble. More precisely, one defines the partition function Z and thermal density matrix $\hat{\rho}$ of the theory,

$$Z = \text{Tr}\left(\mathrm{e}^{-\beta \hat{H}}\right) \tag{1.3.2}$$

and

$$\hat{\rho} = \frac{\mathrm{e}^{-\beta \hat{H}}}{Z} \tag{1.3.3}$$

where the trace is over the Hilbert space of the field theory and $\beta = 1/T$[2]. Then, the expectation values of various operators are given by

$$\langle \hat{\mathcal{O}} \rangle = \text{Tr}\left(\hat{\mathcal{O}}\hat{\rho}\right). \tag{1.3.4}$$

The partition function also provides a definition for the thermodynamic potential of the theory which, for the canonical ensemble, is the Helmholtz free energy $\mathcal{F}$,

$$\mathcal{F} = \mathcal{F}(T, \mathcal{V}, N) = -T \ln Z. \tag{1.3.5}$$

Moreover, defining the entropy of the system as

$$S = -\langle \ln \hat{\rho} \rangle \tag{1.3.6}$$

and identifying $\mathcal{E} \equiv \langle \hat{H} \rangle$ as its energy, we can write

$$\mathcal{F}(T, \mathcal{V}, N) = \mathcal{E} - TS. \tag{1.3.7}$$

Partial variations of the free energy with respect to T, $\mathcal{V}$ and N give the entropy S, pressure $\mathcal{P}$ and chemical potential μ of the theory,

$$S = -\frac{\partial \mathcal{F}}{\partial T}\Big|_{V,N=\text{const.}}, \qquad \mathcal{P} = -\frac{\partial \mathcal{F}}{\partial \mathcal{V}}\Big|_{T,N=\text{const.}}, \qquad \mu = \frac{\partial \mathcal{F}}{\partial N}\Big|_{\mathcal{V},T=\text{const.}}. \tag{1.3.8}$$

The exact differential of the free energy is thus given by

$$\mathrm{d}\mathcal{F} = -S\mathrm{d}T - \mathcal{P}\mathrm{d}\mathcal{V} + \mu \mathrm{d}N. \tag{1.3.9}$$

[2] We have set the Boltzmann constant equal to 1

Alternatively one may wish to consider setups where the number of particles N in the system is not fixed in which case the appropriate ensemble describing the Hilbert space of the theory is the grand canonical ensemble where T and $\mathcal{V}$ are kept fixed but N can fluctuate. The corresponding partition function and thermal density matrix are given by

$$Z = \text{Tr}\left(\mathrm{e}^{-\beta(\hat{H}-\mu\hat{Q})}\right) \quad \text{and} \quad \hat{\rho} = \frac{\mathrm{e}^{-\beta(\hat{H}-\mu\hat{Q})}}{Z} \tag{1.3.10}$$

respectively. The thermodynamic potential associated with the grand canonical ensemble is $\mathcal{W} = \mathcal{W}(T, \mathcal{V}, \mu)$ given by

$$\mathcal{W} = \mathcal{E} - TS - \mu N = -T \ln Z \tag{1.3.11}$$

where $\mathcal{E}$ and N and S are obtained by averaging over the grand canonical ensemble. The potential is related to the free energy of the system through a Legendre transform,

$$\mathcal{F}(T, \mathcal{V}, N) = \mathcal{W}(T, \mathcal{V}, \mu) + \mu N. \tag{1.3.12}$$

As can be seen from the above relations, knowing the thermodynamic potential of a system allows us to compute its physical properties. However, within statistical physics and thermal field theory, computing the potential is often a formidable task. The AdS/CFT correspondence provides an alternative method for computing both the thermodynamic potential and the expectation values of the operators of the field theory. In the next section we introduce the necessary bulk ingredients to describe thermal field theories with finite charge densities. Using these ingredients we build a toy model and use it to develop the elements of the AdS/CFT dictionary that are used in the main part of the thesis.

1.3.2 Gauge/Gravity Duality for Einstein–Maxwell Theories

We begin this section by examining the bulk ingredients that are needed to describe a field theory at finite temperature and charge density. Any field theory has an energy momentum tensor which is sourced in the bulk by a metric, thus we must introduce gravity. Moreover, the bulk spacetime must be asymptotically AdS and therefore we require a negative cosmological constant. The finite temperature of the field theory is achieved by considering black hole and black brane gravity solutions. Then the Hawking temperature and entropy of the black hole or black brane are identified with the temperature and entropy of the field theory. Finally, in the field theory the finite charge density is realised by a conserved current J^i. According to the AdS/CFT dictionary, such currents are sourced by bulk gauge fields. It follows that the minimal bulk action which has solutions dual to a thermal field theory with finite charge density is the Einstein–Maxwell action with negative cosmological constant. The thermodynamic potential of the field theory is associated with the renormalised on-shell action. The expectation values of the field theory operators such as the energy momentum tensor T^{ij}, the current J^i and the charge Q are obtained by functional differentiation of the on–shell bulk action.

In addition to the above ingredients, one may wish to include other field theory operators or symmetry currents by turning on the corresponding dual bulk fields. The fields can be charged under the $U(1)$ symmetry. For example, in applications of the gauge/gravity duality for the study of superconductors one must introduce a charged field in the bulk and search for solutions with both vanishing and non–vanishing values for the field. When the field vanishes the U(1) symmetry is present but for solutions with non–vanishing field the symmetry is spontaneously broken. In the dual theory the charged field corresponds to a charged operator that acquires a vacuum expectation value and spontaneously breaks the U(1) symmetry, in analogy with what one observes in superconductors.

Here we consider the simplest bulk setup dual to to a thermal field theory with finite charge density, namely the Einstein–Maxwell theory with negative cosmological constant. For concreteness we work in 4 dimensions. The action is given by

$$S = \int_{\mathcal{M}} \mathrm{d}^4x \, \sqrt{-G} \left(R - 2\Lambda - \frac{1}{4} F^{\mu\nu} F_{\mu\nu} \right) \tag{1.3.13}$$

where we use units in which $16\pi G_{\mathrm{N}} = 1$. $\mathcal{M}$ is the bulk manifold and $\Lambda = -6/\ell^2$ is the cosmological constant. The Greek indices $\mu, \nu, \ldots$ label bulk coordinates and range from 0 to 4 and the Latin indices $i, j, \ldots$ label the transverse coordinates, including the boundary coordinates, and range from 0 to 3.

Asymptotic Solutions to the Equations of Motion

The equations of motion are

$$\begin{aligned} &R_{\mu\nu} - \frac{R}{2} G_{\mu\nu} - \frac{6}{\ell^2} G_{\mu\nu} = F_{\mu\rho} F^{\rho}{}_{\nu} - \frac{1}{4} G_{\mu\nu} F^2 \\ &D_\mu F^{\mu\nu} = 0 \end{aligned} \tag{1.3.14}$$

where D_μ is the covariant derivative with respect to $G_{\mu\nu}$. To solve the equations of motion we gauge fix both fields. The appropriate gauge for the metric is the Fefferman–Graham gauge given by

$$\begin{aligned} \mathrm{d}s^2 &= \frac{\ell^2}{z^2} \left(\mathrm{d}z^2 + g_{ij}(x,z) \mathrm{d}x^i \mathrm{d}x^j \right) \\ g_{ij}(x,z) &= g_{(0)ij}(x) + z g_{(1)ij}(x) + z^2 g_{(2)ij}(x) + \ldots \end{aligned} \tag{1.3.15}$$

For the gauge field we choose the radial gauge in which $A_z = 0$. The remaining components admit the following asymptotic expansion

$$A_i(x,z) = A_i^{(0)}(x) + z A_i^{(1)}(x) + z^2 A_i^{(2)}(x) + \ldots \tag{1.3.16}$$

To facilitate our analysis we split the Einstein equations into three sets which we treat separately. The first set consists of the transverse components, i.e. $\mu = i, \nu = j$ and we refer to these as the tensorial Einstein equations. The second, which we refer to as the vector Einstein equations, corresponds to setting $\mu = z$, $\nu = i$. This leaves only the $\mu = z, \nu = z$ component of the equations and we refer to this as the scalar Einstein equation.

In terms of g, the scalar Einstein equation is given by

$$-\frac{1}{2}\,\mathrm{Tr}\,\left(g^{-1}g''\right) + \frac{1}{4}\,\mathrm{Tr}\,\left(g^{-1}g'g^{-1}g'\right) + \frac{1}{2z}\,\mathrm{Tr}\,\left(g^{-1}g'\right) = \frac{z^2}{4\ell^2}\left(g^{ij}A'_iA'_j - \frac{1}{2}\widetilde{F}^2\right) \quad (1.3.17)$$

where $'$ implies differentiation with respect to the holographic direction z and $\widetilde{F}^2 = g^{ij}g^{kl}F_{ik}F_{jl} = 4g^{ij}_{(0)}g^{kl}_{(0)}\partial_{[i}A^{(0)}_{k]}\,\partial_{[j}A^{(0)}_{l]} + \ldots$. Similarly, the tensorial Einstein equations are

$$\begin{aligned} {}^{(g)}R_{ij} - \frac{1}{2}g''_{ij} + \frac{1}{2z}g_{ij}\,\mathrm{Tr}\,\left(g^{-1}g'\right) + \frac{1}{2}\left(g'g^{-1}g'\right)_{ij} + \frac{1}{z}g'_{ij} - \frac{1}{4}g'_{ij}\,\mathrm{Tr}\,\left(g^{-1}g'\right) = \\ = \frac{z^2}{2\ell^2}\left(A'_iA'_j - \frac{1}{4}g_{ij}\left(2A'^2 + \tilde{F}^2\right) + g^{kl}F_{ik}F_{jl}\right) \end{aligned} \quad (1.3.18)$$

where the Ricci tensor admits the asymptotic expansion

$${}^{(g)}R_{ij} = {}^{(g)}R^{(0)}_{ij} + z\,{}^{(g)}R^{(1)}_{ij} + \ldots \quad (1.3.19)$$

Finally, the vector Einstein equations are

$$\frac{1}{2}g^{jk}\left(D_kg'_{ki} - D_ig'_{jk}\right) = \frac{z^2}{2\ell^2}g^{jk}F_{ij}A'_k. \quad (1.3.20)$$

These equations give the Ward identities associated with the invariance up to anomalies of the renormalised bulk action under boundary diffeomorphisms.

The equations of motion for the gauge field expressed in terms of g are

$$D^i\left(A'_i\right) = 0, \quad (1.3.21)$$

for the z component, and

$$A''_i + \frac{1}{2}\,\mathrm{Tr}\,\left(g^{-1}g'\right)A'_i - \left(g^{-1}g'\right)^j{}_i\,A'_j + D^jF_{ji} = 0 \quad (1.3.22)$$

for the remaining components. Equation (1.3.21) gives the Ward identity associated with the U(1) symmetry.

Solving the equations of motion for the components of the asymptotic expansions of the metric and gauge field, we find

$$
\begin{aligned}
g_{(1)ij} &= 0, \\
g_{(2)ij} &= -\left({}^{(g)}R^{(0)}_{ij} - \frac{1}{4} g_{(0)ij}\, {}^{(g)}R_{(0)}\right) \\
\mathrm{Tr}\left(g^{-1}_{(0)} g_{(2)}\right) &= -\frac{1}{4}\, {}^{(g)}R_{(0)} \\
\mathrm{Tr}\left(g^{-1}_{(0)} g_{(3)}\right) &= 0 \\
A^{(1)}_i &= \frac{1}{2} D^j_{(0)} F^{(0)}_{ij} \\
A^{(2)}_i &= \frac{1}{6} D^j_{(0)} F^{(1)}_{ij} + \frac{1}{6}\left(g^{-1}_{(0)} g_{(2)}\right)^j{}_i A^{(0)}_j - \frac{1}{6} \mathrm{Tr}\left(g^{-1}_{(0)} g_{(2)}\right) A^{(0)}_i
\end{aligned}
$$

as well as the divergence identities,

$$D^j_{(0)} A^{(0)}_j = 0, \qquad D^j_{(0)} A^{(1)}_j = 0 \tag{1.3.23}$$

and

$$D^i_{(0)}\left({}^{(g)}R_{(0)ij} - \frac{1}{2} g_{(0)ij}\, {}^{(g)}R_{(0)}\right) = 0, \qquad D^j_{(0)} g_{(3)ij} = 0 \tag{1.3.24}$$

$D^i_{(0)}$ is the covariant derivative associated with $g_{(0)ij}$.

We have obtained the asymptotic solutions for the bulk fields which are needed to compute the on-shell action. However, this is not sufficient. If one proceeds to compute the on-shell action, performing first the necessary holographic regularisation, varying the result with respect to the sources, as was described in the previous sections, the process will not produce the correct stress-energy tensor. The reason for this is that we must first add additional surface terms which impose the correct boundary conditions for the metric. It is then this augmented version of the action that one renormalises and subsequently perturbs to obtain the expectation values of the dual operators.

In particular, when we perform the functional differentiation with respect to the metric, we obtain a surface term proportional to $\partial_z \delta G_{\mu\nu}$. Imposing Dirichlet boundary conditions at fixed z is not sufficient to eliminate this term, one must add a boundary term to the action. This is the well-known Gibbons-Hawking (or Gibbons-Hawking-York) term. We proceed by reviewing this term and demonstrate why it is necessary.

Extrinsic curvature and Gibbons-Hawking term

Consider the timelike hypersurface $\mathcal{B}_\epsilon = \{z = \epsilon\}$. This will be the cutoff surface we will use to regularise the action, and we will take the $\epsilon \to 0$ limit after adding the counterterms needed to cancel the divergences. Let n be the unit normal to $\mathcal{B}_\epsilon$, pointing outwards. For the metric in the Fefferman-Graham gauge we have

$$n^\mu = -\frac{z}{\ell}(\partial_z)^\mu, \qquad \text{or equivalently} \quad n_\mu = -\frac{\ell}{z}(\mathrm{d}z)_\mu. \tag{1.3.25}$$

The first fundamental form of the embedding of $\mathcal{B}_\epsilon$ in the spacetime is defined as the projector $h_{\mu\nu}$ on $\mathcal{B}_\epsilon$,

$$h_{\mu\nu} = G_{\mu\nu} - n_\mu n_\nu, \qquad \Rightarrow \qquad h_{ij} = \frac{\ell^2}{\epsilon^2} g_{ij}, \quad h_{zz} = 0, \quad h_{zi} = 0, \tag{1.3.26}$$

and the induced metric γ_{ij} on $\mathcal{B}_\epsilon$ is[3]

$$\gamma_{ij} = \frac{\ell^2}{\epsilon^2} g_{ij}. \tag{1.3.27}$$

To avoid confusion, we will still use greek indices μ, ν,... for raising/lowering indices using the induced metric γ, and reserve latin indices i, j,... for raising/lowering indices using the metric g_{ij}. We define the extrinsic curvature of the embedding of $\mathcal{B}_\epsilon$ in the spacetime as[4]

$$K_{\mu\nu} = -h_\mu{}^\rho h_\nu{}^\sigma D_{(\rho} n_{\sigma)}. \tag{1.3.28}$$

For our geometry, we find that $D_z n_\mu = D_\mu n_z = 0$, and the only non-vanishing terms of the covariant derivative of the normal vector are

$$D_i n_j = -\frac{\ell}{2\epsilon} g'_{ij} + \frac{\ell}{\epsilon^2} g_{ij} \tag{1.3.29}$$

yielding the extrinsic curvature tensor

$$K_{ij} = \frac{\ell}{2\epsilon} g'_{ij} - \frac{\ell}{\epsilon^2} g_{ij}. \tag{1.3.30}$$

Its trace $K = \gamma^{ij} K_{ij}$, or mean curvature, is then given by

$$K = \frac{\epsilon}{2\ell} \operatorname{Tr}\left(g^{-1} g'\right) - \frac{3}{\ell}. \tag{1.3.31}$$

Let us see now why we need the Gibbons-Hawking term. When varying the action with respect to the metric, we can use the relation

$$G^{\mu\nu} \delta_G R_{\mu\nu} = D^\mu \left(G^{\rho\sigma} D_\rho \delta G_{\mu\sigma} - G^{\rho\sigma} D_\mu \delta G_{\rho\sigma} \right). \tag{1.3.32}$$

[3]Using the coordinates $\{x^i\}$ on $\mathcal{B}_\epsilon$, the embedding $X^\mu(x^i)$ of $\mathcal{B}_\epsilon$ in the spacetime is given by $X^z = \epsilon$, $X^i = x^i$. Then the induced metric is given by $\gamma_{ij} = \partial_i X^\mu \partial_j X^\nu G_{\mu\nu}$, and the first fundamental form is obtained as $h^{\mu\nu} = \partial_i X^\mu \partial_j X^\nu \gamma^{ij}$.

[4]Notice that we have conventionally a minus sign in our definition. Equivalently, it can be calculated as $K_{\mu\nu} = -\frac{1}{2}\mathcal{L}_n h_{\mu\nu} = \frac{\epsilon}{2\ell}\partial_\epsilon h_{\mu\nu}$. For codimension higher than one, the extrinsic curvature tensor is defined as $K_{\mu\nu}{}^\rho = -h_\mu{}^\Lambda h_\nu{}^\sigma D_\sigma h_\lambda{}^\rho$, and the mean curvature vector by $K^\rho = h^{\mu\nu} K_{\mu\nu}{}^\rho$.

This is a total derivative; after an integration by parts the variation of S_{bulk} gives,

$$\delta_G S_{\text{bulk}} = \int_{z>\epsilon} d^{d+1}x\sqrt{-G}\ [\text{bulk e.o.m.}]_{\mu\nu}\, \delta G^{\mu\nu} + \int_{z=\epsilon} d^d x\sqrt{-\gamma}\, n^\mu \left(G^{\rho\sigma} D_\rho \delta G_{\mu\sigma} - G^{\rho\sigma} D_\mu \delta G_{\rho\sigma}\right). \tag{1.3.33}$$

Substituting $G^{\rho\sigma} = h^{\rho\sigma} + n^\rho n^\sigma$, the boundary term becomes

$$\int_{z=\epsilon} d^d x\sqrt{-\gamma}\, n^\mu \left(h^{\rho\sigma} D_\rho \delta G_{\mu\sigma} - h^{\rho\sigma} D_\mu \delta G_{\rho\sigma}\right). \tag{1.3.34}$$

If we impose Dirichlet boundary conditions, $\delta G_{\mu\nu}|_{\mathcal{B}_\epsilon} = 0$, the first term in the integrand vanishes, since it is the derivative of $\delta G_{\mu\nu}$, along some boundary direction. On the other hand, the second term does not vanish, as it is the change of $\delta G_{\mu\nu}$ moving away from the boundary. The variational principle with Dirichlet boundary conditions $\delta G_{\mu\nu} = 0$ on $\mathcal{B}_\epsilon$ is thus not well-posed. To make the action functionally differentiable, we need to add the Gibbons-Hawking term

$$S_{\text{GH}} = -2\int_{z=\epsilon} d^d x\sqrt{-\gamma}\, K. \tag{1.3.35}$$

Indeed, using the expression

$$\delta_G \Gamma^\rho_{\mu\nu} = \frac{1}{2} G^{\rho\sigma}\left(D_\mu \delta G_{\sigma\nu} + D_\nu \delta G_{\sigma\mu} - D_\sigma \delta G_{\mu\nu}\right), \tag{1.3.36}$$

we find that

$$G^{\mu\nu}\delta_G K_{\mu\nu} = \frac{1}{2} h^{\rho\sigma} n^\mu \left(D_\rho \delta G_{\mu\sigma} + D_\sigma \delta G_{\mu\rho} - D_\mu \delta G_{\rho\sigma}\right) \tag{1.3.37}$$

and the variation of the Gibbons-Hawking term gives

$$\delta_G S_{\text{GH}} = \int_{z=\epsilon} d^d x\sqrt{-\gamma}\, \left[(K\gamma_{ij} - 2K_{ij})\,\delta G^{ij} - 2n^\mu h^{\rho\sigma}\nabla_\rho \delta G_{\mu\sigma} + n^\mu h^{\rho\sigma}\nabla_\mu \delta G_{\rho\sigma}\right]. \tag{1.3.38}$$

Combining all contributions

$$\delta_G (S_{\text{bulk}} + S_{\text{GH}}) = \int_{z=\epsilon} d^d x\sqrt{-\gamma}\, \left[(K\gamma_{ij} - 2K_{ij})\,\delta G^{ij} - n^\mu h^{\rho\sigma} D_\rho \delta G_{\mu\sigma}\right]. \tag{1.3.39}$$

To deal with the last term, we write it as

$$-n^\mu h^{\rho\sigma} D_\rho \delta G_{\mu\sigma} = -h^{\rho\sigma} D_\rho \left(n^\mu \delta G_{\mu\sigma}\right) - h^{\rho\sigma} D_\rho n^\mu \delta_{\mu\sigma} \tag{1.3.40}$$

$$= \mathcal{D}_i\left(\gamma^{ij} n^\mu \delta G_{\mu j}\right) - h^{\rho\sigma} G^{\mu\nu}\left(D_\rho n_\nu\right)\delta G_{\mu\sigma} \tag{1.3.41}$$

with $\mathcal{D}_i$ the covariant derivative associated to the induced metric γ_{ij}. The total derivative yields zero when integrated, and the remaining term becomes $K_{\mu\nu}\delta G^{\mu\nu}$. Finally, the metric variation of S_0 is

$$\delta_G(S_{\mathsf{bulk}} + S_{\mathsf{GH}}) = \int_{z>\epsilon} d^{d+1}x\sqrt{-G}\,E_{\mu\nu}\delta G^{\mu\nu} - \int_{z=\epsilon} d^d x\sqrt{-\gamma}\,(K_{ij} - K\gamma_{ij})\,\delta\gamma^{ij}, \quad (1.3.42)$$

where we took $\delta\gamma^{ij} = \delta G^{ij}$. Hence, the addition of the Gibbons-Hawking term renders the action functional differentiable when imposing Dirichlet boundary conditions on $\mathcal{B}_\epsilon$, leading to a well-defined variational problem. The resulting equations of motion are $E_{\mu\nu} = 0$.

As a final remark we point out that the above discussion is not quite correct since the bulk fields, including the metric, do not induce boundary fields on the conformal boundary but instead a conformal class of boundary fields. As a consequence, although one can impose Dirichlet boundary conditions on the cut-off surface $\mathcal{B}_\epsilon$, in the limit where $\mathcal{B}_\epsilon \to \partial$ such boundary conditions need to be replaced by a weaker set of boundary conditions where the boundary fields are kept fixed up to Weyl transformations [30]. For the metric this implies the relationship

$$\delta\gamma_{ij} = 2\gamma_{ij}\delta\sigma(x) \quad (1.3.43)$$

where $\delta\sigma(x)$ an arbitrary infinitesimal function of the boundary coordinates. Once the covariant counterterm necessary for the finiteness of the on-shell action are also taken into account, the variational problem of the resulting action is well-posed. However, there is an important subtlety associated with a non-zero conformal anomaly, since one can no longer impose (1.3.43). In this case one has to choose a specific representative of the boundary conformal structure to impose the boundary conditions which means that the bulk diffeomorphisms are partly broken. However, when both the Gibbons-Hawking and the counterterms are added, the on-shell action has a well defined transformation under the broken diffeomorphisms. In particular, in this case the violation of the variational problem depends on the conformal class γ_{ij} only and therefore, imposing Dirichlet boundary conditions leads to a well-posed variational problem. For a detailed discussion of boundary terms, counterterms and the well-posedness of the variational problem in the presence of conformal anomalies see [30].

Holographic Renormalisation

The next step in our analysis is the computation of the regularised on-shell action. Starting with the action (1.3.13) and using the equations of motion we find

$$S_{\text{on-shell}} = \int_{\mathcal{M}} \mathrm{d}^4x\sqrt{-G}\left(2\Lambda - \frac{1}{4}F^2\right) + S_{\text{GH}} \quad (1.3.44)$$

where we have included the Gibbons-Hawking term,

$$S_{\text{GH}} = -2\int_{z=\epsilon} \mathrm{d}^3x\sqrt{-\gamma}\,K. \quad (1.3.45)$$

The on-shell action suffers from infinite volume IR divergences and has to be renormalised. As we saw in section 1.2.4, we must first regulate the action by introducing a cut-off at $z = \epsilon$,

$$S_{\text{reg}} = \int_{z>\epsilon} \mathrm{d}^4x \sqrt{-G} \left(2\Lambda - \frac{1}{4}F^2\right) - 2\int_{z=\epsilon} \mathrm{d}^3x \sqrt{-\gamma}\, K. \tag{1.3.46}$$

We wish to identify terms which, after we integrate the on-shell action with respect to z for $z > \epsilon$, are divergent in the limit $\epsilon \to 0$. These will provide the counter-term action necessary to make the on-shell action finite. Using the asymptotic form of the fields, (1.3.15) and (1.3.16), and the corresponding solutions, we find that the divergent terms are

$$S_{\text{reg}} = \int_{z=\epsilon} \mathrm{d}^3x \sqrt{-g_{(0)}} \left(\frac{4\ell^2}{\epsilon^3} + \frac{\ell^2}{2\epsilon} {}^{(g)}R_{(0)}\right) + O\big(\epsilon^0\big)\,. \tag{1.3.47}$$

More details on the derivation of this result can be extracted from the discussion in section 6.3.4 by turning off any fields that are not present here. The expression (1.3.47) provides the counterterm action that one must add to (1.3.44) to render the on-shell action finite. However, the above expression is not covariant so it does not respect bulk diffeomorphisms. To obtain a covariant expression we must invert the field expansions to obtain the expressions for the boundary values of the fields in terms of covariant bulk fields that live on the cut-off surface $\mathcal{B}_\epsilon$. In terms of these fields, the covariant counterterm action is

$$S_{\text{ct}} = -\int_{z=\epsilon} \mathrm{d}^3x \sqrt{-\gamma} \left(\frac{4}{\ell^2} + \ell\, {}^{(\gamma)}R\right), \tag{1.3.48}$$

and the renormalised on-shell action is

$$S_{\text{ren}} = \lim_{\epsilon\to 0} \big(S_{\text{reg}} + S_{\text{ct}}\big)\,. \tag{1.3.49}$$

Varying this expression with respect to the sources of the fields, i.e. $g_{(0)ij}$ and $A_i^{(0)}$, we obtain the one point functions of the dual operators,

$$\delta S_{\text{ren}} = \int \mathrm{d}^3x \sqrt{-g_{(0)}} \left(\frac{1}{2}\left\langle T^{ij}\right\rangle \delta g_{(0)ij} + \left\langle J^i\right\rangle \delta A_i^{(0)}\right) \tag{1.3.50}$$

where

$$\left\langle T^{ij}\right\rangle = 3\ell^2 g_{(3)}^{ij}, \qquad \left\langle J^i\right\rangle = A^i_{(1)}. \tag{1.3.51}$$

Moreover, using the divergence equations (1.3.23) and (1.3.24) we obtain the Ward identities associated with boundary Weyl and U(1) transformations

$$\begin{aligned} -D^j_{(0)} \left\langle T_{ij}\right\rangle + \left\langle J^i\right\rangle F^{(0)}_{ij} &= 0, \\ D_{(0)i} \left\langle J^i\right\rangle &= 0. \end{aligned} \tag{1.3.52}$$

So far we have only studied the asymptotic behaviour of the bulk fields but the vacuum expectation values of the operators remain undetermined. In order to find the explicit expressions for the vevs we must solve the field equations everywhere in the bulk, providing initial and final data for the fields to obtain unique solutions. We do not carry out this analysis here and instead we study a known solutions of the action (1.3.13), namely the 4 dimensional AdS Reissner–Nordström black hole or black brane.

1.3.3 AdS Reissner–Nordström Black Brane

The simplest solution to the equations of motion (1.3.14) is obtained by setting the gauge field equal to zero. In this case the spacetime is empty AdS and its metric is

$$\mathrm{d}s^2 = \frac{\ell^2}{z^2}\left(-\mathrm{d}t^2 + \mathrm{d}z^2 + \mathrm{d}\vec{x}^2\right). \tag{1.3.53}$$

This is already in the Fefferman–Graham gauge with $g_{ij}(x,z) = \eta_{ij}$. Moreover, $A_i(x,z) = 0$. It follows that the dual field theory has vanishing stress–energy tensor and U(1) current, as one would expect.

A more interesting solution is the static planar AdS Reissner–Nordström black hole or black brane for which the metric and gauge field are given by

$$\begin{aligned}
\mathrm{d}s^2 &= \frac{\ell^2}{z^2}\left(-f(z)\mathrm{d}t^2 + \frac{\mathrm{d}z^2}{f(z)} + \mathrm{d}\vec{x}^2\right)\\
f(z) &= 1 - M\frac{z^3}{z_h^3} + Q^2\frac{z^4}{z_h^4}, \qquad M = 1 + Q^2, \qquad Q^2 = \frac{z_h^2\mu^2}{4\ell^2},\\
A_t(z) &= \mu\left(1 - \frac{z}{z_h}\right).
\end{aligned} \tag{1.3.54}$$

The parameters M and Q are related to the ADM mass and the charge of the black brane and z_h is the position of the horizon. For the dual field theory to be at finite charge density, the time component of the gauge field must have a non-trivial profile. Here we are interested in dual theories that preserve rotational symmetry in the spatial directions and therefore $A_x = A_y = 0$ and $A_t = A_t(z)$. Moreover, A_t vanishes at the horizon in order for $A_t\mathrm{d}t$ to have finite norm. As we will see below, the parameter μ corresponds to the chemical potential of the dual theory and μ/z_h, the coefficient of term proportional to z, to its density.

We proceed with the computation of the temperature and entropy density of the black brane which are identified with the corresponding quantities of the field theory. We then study the asymptotic behaviour of the bulk solution by expressing the fields in the Fefferman–Graham gauge and identify the one point functions of the dual operators. Finally, we review the thermodynamics of the dual field theory.

Temperature and Entropy

The temperature of the black brane can be computed by Wick rotating the metric and then studying the near horizon limit of the resulting Euclidean metric. The Euclidean metric has a conical singularity at the horizon unless the Euclidean metric is period. Its period is identified with the inverse temperature.

The four dimensional Euclidean AdS Reissner–Nordström black brane, obtained by Wick rotating the metric (1.3.54), is

$$\mathrm{d}s^2 = f(z)\mathrm{d}\tau^2 + \frac{\mathrm{d}z^2}{f(z)} + z^2\mathrm{d}\vec{x}^2. \tag{1.3.55}$$

To study the near horizon region we introduce the radial coordinate ρ given by

$$z = z_h\left(1 - \frac{\rho^2}{4\ell^2}(3-Q^2)\right), \qquad \rho(z_h) = \rho_h = 0. \tag{1.3.56}$$

To lowest order in ρ the metric is given by

$$\mathrm{d}s^2 = \frac{(3-Q^2)^2}{4z_h^2}\rho^2\mathrm{d}\tau^2 + \mathrm{d}\rho^2 + \frac{\ell^2}{z_h^2}\mathrm{d}\vec{x}^2. \tag{1.3.57}$$

This has a conical singularity at $\rho = 0$ unless τ is periodic. Its period is identified with the inverse temperature $\beta = 1/T$. To find the period we rescale τ to $\phi = \tau(3-Q^2)/2z_h$. Then the (τ, ρ) surface becomes

$$\mathrm{d}s^2 = \mathrm{d}\rho^2 + \rho^2\mathrm{d}\phi^2. \tag{1.3.58}$$

Upon imposing periodicity, $\phi \sim \phi + 2\pi$, this is a plane in polar coordinates. In terms of τ this implies that $\Delta\tau = 2\pi\left(2z_h/(3-Q^2)\right) = \beta$ and hence

$$T = \frac{1}{4\pi z_h}\left(3-Q^2\right). \tag{1.3.59}$$

Having found the temperature of the black brane we proceed with the calculation of its entropy which is given by the Bekenstein–Hawking formula,

$$S = 4\pi A_h \tag{1.3.60}$$

where A_h is the area of the horizon. Recall that we are using units in which $16\pi G = 1$. For a planar black holes the area of the horizon is infinite and therefore we compute the entropy density instead, given by $S/\mathcal{V}_{d-1}$, where $\mathcal{V}_{d-1}$ is the area of the transverse space. Then, from (1.3.54) it follows that

$$s = 4\pi\frac{\ell^2}{z_h^2}. \tag{1.3.61}$$

According to the AdS/CFT dictionary the temperature (1.3.59) and entropy density (1.3.61) are identified with the temperature and entropy density of the dual field theory.

Field Theory Expectation Values

Next we holographically compute the expectation values of the stress–energy tensor and U(1) current of the dual field theory. We do this by making use of the general expressions derived previously relating the one point functions of the field theory operators to asymptotic expansions of the bulk fields. In particular, we found that

$$\langle T^{ij} \rangle = 3\ell^2 g^{ij}_{(3)}, \qquad \langle J^i \rangle = A^i_{(1)} \tag{1.3.62}$$

where $g^{ij}_{(3)}$ and $A^i_{(1)}$ are obtained from the asymptotic expansion of the bulk fields written in the Fefferman–Graham gauge. To express the metric in the Fefferman–Graham gauge we need to perform a coordinate transformation $z = z(r)$ satisfying

$$g_{zz}\mathrm{d}z^2 = \frac{\ell^2}{r^2}\mathrm{d}r^2 \tag{1.3.63}$$

under which the original metric takes the form

$$\mathrm{d}s^2 = \frac{\ell^2}{r^2}\left(\mathrm{d}r^2 + g_{ij}(x,r)\mathrm{d}x^i\mathrm{d}x^j\right). \tag{1.3.64}$$

To derive this transformation we consider a series expansion for $z(r)$[5],

$$z(r) = r\left(1 + a_1 r + a_2 r^2 + a_3 r^3 + a_4 r^4 + O(r^5)\right), \tag{1.3.65}$$

and solve order by order in r. Note that the boundary is located at $r = 0$. From the $z(r)$ expansion we compute $\mathrm{d}z^2$ and $g_{zz}\mathrm{d}z^2$ in terms of r and $\mathrm{d}r$,

$$\mathrm{d}z^2 = \left(1 + 4ra_1 + 2r^2\left(2a_1^2 + 3a_2\right) + 4r^3\left(3a_1a_2 + 2a_3\right) + O(r^4)\right)\mathrm{d}r^2, \tag{1.3.66}$$

$$g_{zz}\mathrm{d}z^2 = \frac{\ell^2}{r^2}\left(1 + 2ra_1 + r^2\left(-a_1^2 + 4a_2\right) + r^3\left(\frac{M}{z_h^3} - 2a_1a_2 + 6a_3\right) + O(r^4)\right). \tag{1.3.67}$$

Requiring that the right hand side of this expression agrees with the g_{rr} component of the Fefferman–Graham metric (1.3.64) $a_1 = 0$, $a_2 = 0$ and $a_3 = -M/6z_h^3$. To determine more coefficients in the $z(r)$ expansion we need to go to higher orders in r. In particular, to order $O(r^5)$ we obtain

$$z(r) = r\left(1 - r^3\frac{M}{6z_h^3} + r^4\frac{Q^2}{8z_h^4} + O(r^5)\right), \tag{1.3.68}$$

[5] $z(r)$ is independent of the transverse coordinates x since the original metric has no x dependence.

$$g_{ij} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + r^3 \begin{pmatrix} \frac{2M}{3z_h^3} & 0 & 0 \\ 0 & \frac{M}{3z_h^3} & 0 \\ 0 & 0 & \frac{M}{3z_h^3} \end{pmatrix} + r^4 \begin{pmatrix} -\frac{3Q^2}{4z_h^4} & 0 & 0 \\ 0 & -\frac{Q^2}{4z_h^4} & 0 \\ 0 & 0 & -\frac{Q^2}{4z_h^4} \end{pmatrix} + O(r^5), \tag{1.3.69}$$

and

$$A_t = \mu \left(1 - \frac{r}{z_h} + \frac{r^4}{z_h^4} \frac{M}{6} + O(r^5) \right). \tag{1.3.70}$$

Comparing these expressions to the definitions of the one point functions of the dual operators (1.3.62) we obtain $\langle T^{ij} \rangle$ and $\langle J^i \rangle$,

$$\langle T^{ij} \rangle = 3\ell^2 \begin{pmatrix} \frac{2M}{3z_h^3} & 0 & 0 \\ 0 & \frac{M}{3z_h^3} & 0 \\ 0 & 0 & \frac{M}{3z_h^3} \end{pmatrix}$$

$$\langle J^i \rangle = \left(\frac{\mu}{z_h}, 0, 0, 0 \right). \tag{1.3.71}$$

The $\langle T^{00} \rangle$ component of the stress–energy tensor is related to the energy density of the field theory,

$$\varepsilon = 2\ell^2 \frac{M}{z_h^3} \tag{1.3.72}$$

and the time component of the U(1) current is the charge density of the field theory,

$$\rho = \frac{\mu}{z_h}. \tag{1.3.73}$$

From the discussion of the grand canonical ensemble and in particular from equation (1.3.10) we know that in the field theory Lagrangian the charge operator is coupled with the chemical potential. Moreover, from the AdS/CFT dictionary we know that the source that couples to J^i and ρ is given by the leading asymptotic term of the gauge field, namely $A_i^{(0)}$. Hence, we conclude that $A_t^{(0)}$ is indeed the chemical potential of the dual field theory.

Another way of computing the charge density is to vary the thermodynamic potential of the theory with respect to μ. In terms of bulk quantities, the potential is given by the Euclidean renormalised on–shell action. Using the general results obtain above for the counterterms and including the Gibbons–Hawking term, the thermodynamic potential of the dual theory is

$$\mathcal{W} = -S_{\text{ren}} = -\left(\frac{\ell^2}{z_h^3} + \frac{\mu^2}{4z_h} \right) \mathcal{V}_2 \tag{1.3.74}$$

where the inverse temperature β comes from integrating over one Euclidean time cycle and $\mathcal{V}_2 = \int \mathrm{d}x\mathrm{d}y$ which is formally divergent. This can be resolved by compactifying these directions or by working with local densities obtained by dividing by $\mathcal{V}_2$. Accordingly, the corresponding density w is defined as $\mathcal{W}/\mathcal{V}_2$ and the charge density is

$$\rho = -\frac{\partial w}{\partial \mu}\Big|_{T=\text{const.}} = \frac{\mu}{z_h} \tag{1.3.75}$$

in agreement with our previous result. Similarly, the entropy density can be computed by varying w with respect to the temperature while keeping the chemical potential fixed. The result obtained using this method is identical to the expression for the Bekenstein–Hawking entropy computed in the previous section. Using all the above expressions it is easy to verify that

$$w = \varepsilon - Ts - \mu\rho. \tag{1.3.76}$$

Moreover, we can compute the pressure of the system by varying the total energy $\mathcal{E} = \varepsilon\mathcal{V}_2$ with respect to the volume $\mathcal{V}_2$, keeping the total entropy $S = s\mathcal{V}_2$ and total charge $Q = \rho\mathcal{V}_2$ fixed. From the expression from ε we have

$$\mathcal{E} = \frac{1}{4\ell}\frac{S^{3/2}}{\pi^{3/2}\sqrt{\mathcal{V}_2}} + \frac{\ell Q^2\sqrt{\pi}}{\sqrt{S}\sqrt{\mathcal{V}_2}} \tag{1.3.77}$$

where we inverted the expressions for S and Q to obtain $\mu(S,Q)$ and $z_h(S,Q)$. Then the pressure is

$$\mathcal{P} = -\frac{\partial\mathcal{E}}{\partial\mathcal{V}_2}\Big|_{S,Q} = \frac{1}{8\ell}\frac{S^{3/2}}{\pi^{3/2}\mathcal{V}_2\sqrt{\mathcal{V}_2}} + \frac{\ell Q^2\sqrt{\pi}}{2\sqrt{S}\mathcal{V}_2\sqrt{\mathcal{V}_2}} = \frac{\ell^2}{z_h^3} + \frac{\mu^2}{4z_h}. \tag{1.3.78}$$

Note that $w = -\mathcal{P}$ and, furthermore,

$$\varepsilon + \mathcal{P} = Ts + \mu\rho. \tag{1.3.79}$$

This last expression is the Gibbs–Duhem relation.

This concludes our introduction of the AdS/condensed matter theory (CMT) dictionary. To summarise, we studied the Einstein–Maxwell action with negative cosmological constant which provides the necessary building blocks for the dual field theory to be at finite temperature and charge density. In terms of statistical field theory, this dual theory can be studied using the grand–canonical potential. We demonstrated how this can be obtained from a bulk calculation; it is simply the renormalised Euclidean on–shell action. Moreover, we computed the expectations values of the stress–energy tensor and U(1) current of the quantum field theory and deduced the charge and energy densities of the theory. Finally, we studied some of the thermodynamic properties and relations of the dual theory by combining all the results from the bulk calculations.

This example is a warm–up exercise for chapter 6 where we repeat this analysis for a more complicated setup that includes, in addition to gravity and the gauge field, massive and massless scalar fields with non–trivial couplings to the gauge field. In analogy to what was done here, we first study the theory in full generality by expressing the fields as asymptotic expansions with undetermined coefficients. The results obtained are then applicable to any specific solution. In chapters 7 and 8 we apply the general results to three analytic solutions of the general theory.

1.4 Concluding Remarks

In this chapter we reviewed the basic elements of the AdS/CFT dictionary and then applied them to the holographic study of a simple condensed matter system. More specifically, in section 1.2 we established the relation between fields in AdS and operators in the dual field theory and demonstrated how one finds the expectation values of the field theory operators from semiclassical supergravity calculations in the bulk (1.2.3). This involves holographically renormalising the bulk on–shell action in order to treat infinite volume IR divergences which are related to UV divergences of the field theory (1.2.4). The one–point functions of the dual operators are then obtained through functional differentiation of the renormalised on–shell action with respect the respective sources. This calculation was carried out explicitly for a scalar field in the Poincaré patch of AdS_{d+1} which is dual to a scalar operator in a flat background. Although the calculation did not rely explicitly on the signature of the metric, we saw that Lorentzian AdS requires special treatment due to its causal structure. In particular, the unique determination of bulk fields requires that, in addition to the usual boundary conditions, we specify initial and /or final conditions for the Lorentzian fields. The procedure for obtaining these initial and final conditions was the subject of section 1.2.5 which concluded our introduction to the AdS/CFT dictionary.

The remaining of the chapter was dedicated to the application of AdS/CFT to condensed matter systems with particular focus on the derivation of the thermodynamics of the field theory from bulk calculations. In section 1.3.1 we briefly reviewed the thermodynamics of field theories at finite temperature and charge density. This review revealed the power of the thermodynamic potential and motivated the use of gauge/gravity dualities for the study thermal field theories since they provide a straightforward way to compute the thermodynamic potential. In section 1.3.2 we studied the Einstein–Maxwell theory in 3+1 dimensions which admits solutions dual to field theories at finite charge density and temperature. We first solve the equations of motion for the gauge field and metric near the conformal boundary and computed the renormalised on–shell action and one–point functions of the dual operators, namely the stress energy tensor and the conserved U(1) current. This calculation was performed in full generality. In particular, we did not specify the boundary and initial and final conditions for the fields and therefore the solutions obtained were in terms of arbitrary coefficients that appear in the asymptotic expansion of the fields. These coefficients are determined in the next section, 1.3.3, where we considered a known solution to this theory, namely the AdS Reissner–Nordström black brane. Having an analytic solution allowed us to compute the temperature, entropy density, energy density and charge density of the dual theory as well as the expectation values of its stress energy tensor and conserved U(1) current. Moreover, the renormalised on–shell action evaluated on this solution provided us with the grand canonical potential for the field theory. As a final exercise we used all the field theory quantities we computed to study the thermodynamics of the dual field theory.

The rest of the thesis is dedicated to the further development and application these concepts. It consists of two distinct parts. The first part addresses the question of how one can holographically construct excited field theory states and it follows [1]. The construction of such states is only possible in Lorentzian AdS/CFT and it requires the use of the real time gauge/gravity dictionary introduced in section 1.2.5. In this thesis we focus on a simple example: a field theory state that to leading order in the large N limit can be described by a scalar field in a fixed AdS background. For our construction we make use of the in–in field theory contour which corresponds to an AdS manifold consisting of two half Euclidean balls and two Lorentzian cylinders sandwiched between them. Our analysis demonstrates that the excited field theory state $\mathcal{O}\,|0\rangle$, where $\mathcal{O}$ is the scalar operator dual to the bulk scalar field, is created by turning on sources for the bulk scalar in the two Euclidean caps. We show this explicitly for global AdS_3 in chapter 3 and for the Poincaré patch of AdS_3 in chapter 4.

The second part of the thesis is an application of the correspondence to a class of theories that admit planar black brane solutions that can carry electric and/or magnetic charges and can be supported by two types of scalar fields. The scalars are a set of axion fields and a scalar with a running profile. Both types of scalars are neutral in the theories we study. The axion fields admit a linear profile in the spatial boundary directions and are constant along the radial direction. As we will see in chapter 6 these are 0–forms that carry magnetic–like charges. As such, they are primary hair and their charge enters in the first law of the dual field theory. Moreover, it was shown in [31, 32] that fields with this profile break translation invariance in the dual theory and lead to momentum dissipation and finite DC conductivity when the scalar fields are charged. In contrast, the scalar with the running profile corresponds to secondary hair and there are no conserved charges associated with it. However, as we will see, its mass is in the window that allows for mixed and Neumann boundary conditions (see section 1.2.4). In chapters 7 and 8 we study solutions for which the scalar satisfies mixed boundary conditions and we find that they modify bulk the on–shell action and therefore the holographic stress tensor, conserved charges and free energy of the field theory. By correctly implementing these modifications in the first law we confirm that there are no charges associated to this scalar field, as should be the case for secondary hair. From the field theory perspective, mixed boundary conditions are associate to multi–trace deformations. This type of deformations introduce a new parameter in the field theory that can be tuned to control the condensate of the corresponding scalar operator. We will see that the phases of the theory are governed by this operator and therefore, this new tuning parameter can be dialled to move across different phases and change the theory. For this reason we will refer to this field as the *dialton*. The main objective and outcome of this part of the thesis is the development of the holographic dictionary for this class of theories and the correct identification and derivation of the conserved charges and thermodynamic properties and it follows [2]. The theory is first solved in full generality in chapter 6 and the results are then applied to specific solutions whose analytic form is known in chapters 7 and 8.

The remaining of the thesis is organised as follows. We begin by discussing the holographic construction of excited field theory states in part I. In chapter 2 we formally introduce the question that we set out to answer, namely how can we holographically construct CFT states of the form $\mathcal{O}\,|0\rangle$ where $\mathcal{O}$ is a scalar operator. We then proceed to answer this question for global AdS_3 in chapter 3 and for the Poincaré patch of AdS_3 in chapter 4. We conclude this part of the thesis with the summary of the results in chapter 5. The second part of the thesis focuses on the a class of four dimensional Einstein–Maxwell theories which carry axionic and magnetic charges and can support an additional running scalar field satisfying mixed boundary conditions. In chapter 6 we introduce this class of theories and discuss the features of the solutions we are interested in. We then proceed to perform the holographic analysis of a generic solution possessing these features and develop the holographic dictionary as well as the thermodynamic properties of the dual theory. These results are used to study solutions to two theories that explicit realisations of the class of theories we are interested in. In particular, we discuss the thermodynamics properties and dynamical stability of both bold and hairy solutions to these theories, as well as phase transitions between them. This is done is chapters 7 and 8.

Part I

Holographic Construction of Excited CFT States

Chapter 2

General Discussion

2.1 Real time holography

A central question in holography is how the bulk is reconstructed from QFT data. The first part of this thesis will address a simpler question:"what is the bulk dual of a CFT state?" While it has been clear since the early days of AdS/CFT that normalisable bulk solutions are related to states [33], a precise construction of a bulk solution given a state has not been available prior to [1] on which this part of the thesis is based[1].

The construction is an application of the real-time gauge/gravity dictionary [28, 29] discussed in section 1.2.5 and it can be applied to any state that has a (super)gravity description. We will however focus on a simple example: a state that to leading order in a large N limit can be described by a scalar field in a fixed AdS background. An additional motivation for studying this example is that the bulk solution appeared also in related work [40]. That paper was part of a bigger program where an attempt is made to reconstruct bulk operators from boundary data [41, 42, 43, 44, 45, 46] (see also related work [47, 48, 49]). We will return to this in the next section where we give a brief overview the general formalism used in these works. Once we have performed the reconstruction for specific cases, we shall return again to this matter and discuss similarities and differences with that work.

In chapter 1 section 1.2.4 we studied the asymptotic behaviour of a massive scalar in AdS. Let us briefly revisit the subject and use it as a stepping stone to review what is known about bulk reconstruction starting with a scalar field in a fixed background and in Euclidean signature. It is well known that a field Φ of mass $m^2 = \Delta(\Delta - d)$ in AdS$_{d+1}$ is dual to an operator $\mathcal{O}_\Delta$ of dimension Δ. The bulk field has an asymptotic expansion

[1] A related question that received more attention over the years is the converse: given a bulk solution with normalisable asymptotics what is the dual state? For such solutions, the leading order asymptotic behaviour of the solution is related with the 1-point function of the gauge invariant operators in a state and from the 1-point functions one may extract information about the dual states. Examples of such computations include the computation of 1-point functions for the solutions corresponding to the Coulomb branch of $\mathcal{N} = 4$ SYM [34], the 1-point functions for the LLM solutions [35] in [36] and 1-point functions for fuzzball solutions [37, 38, 39].

of the form [19]

$$\Phi(r,x) = r^{d-\Delta}\phi_{(0)}(x) + \cdots + r^{\Delta}\log r^2\psi_{(2\Delta-d)}(x) + r^{\Delta}\phi_{(2\Delta-d)}(x) + \dots \tag{2.1.1}$$

where r is the holographic (radial) direction and x denotes the collective set of boundary coordinates. $\phi_{(0)}(x)$ is the source for the dual operator and $\phi_{(2\Delta-d)}(x)$ is related to the 1-point function,

$$\langle\mathcal{O}_\Delta\rangle = (2\Delta - d)\phi_{(2\Delta-d)}(x) + X(\phi_{(0)}), \tag{2.1.2}$$

where $X(\phi_{(0)})$ is a local function of the source $\phi_{(0)}$ (whose exact form depends on the bulk theory under discussion). $\phi_{(0)}(x)$ and $\phi_{(2\Delta-d)}(x)$ are the only two arbitrary coefficient functions in the above expansion. All subleading terms down to r^Δ (including $\psi_{(2\Delta-d)}$ but not $\phi_{(2\Delta-d)}(x)$) are locally related to $\phi_{(0)}(x)$ and similarly all terms that appear at higher orders can be determined in terms of $\phi_{(0)}$ and $\phi_{(2\Delta-d)}(x)$. Thus, given the pair ($\phi_{(0)}(x)$, $\phi_{(2\Delta-d)}(x)$) one can iteratively construct a unique bulk solution. A different (non-perturbative) argument for uniqueness is to note that the 1-point function is the canonical momentum π_Δ in a radial Hamiltonian formalism [23] and by a standard Hamiltonian argument, specifying a conjugate pair $(\phi_{(0)}, \pi_\Delta)$ uniquely picks a solution of the theory. This argument however does not tell us whether the solution is regular in the interior. Indeed in quantum field theory, the vacuum structure is a dynamical question: in general one cannot tune the value of $\langle\mathcal{O}_\Delta\rangle$. The counterpart of this statement is that a generic pair $(\phi_{(0)}, \pi_\Delta)$ leads to a singular solution[2] and it is regularity in the interior that selects $\langle\mathcal{O}_\Delta\rangle$.

In Lorentzian signature new complications arise as we discussed in section 1.2.5. As we saw, in the bulk boundary conditions alone do not determine a unique solution: Lorentzian AdS is a non–hyperbolic manifold. Indeed, there exist normalisable modes which are regular in the interior and vanish at the boundary, leaving the boundary data unaffected.

On the QFT side, there are related issues. While in Euclidean signature there is only one type of correlator, in Lorentzian signature, there are multiple types of correlators (time-ordered, Wightman functions, advanced, retarded, etc.). In addition, one may wish to consider these correlators on non–trivial states (such as thermal states, states that spontaneously break some symmetries, general non-equilibrium states). All of this data may be nicely encoded by providing a contour in the complex time plane and considering the path integral defined along this contour. Different types of correlators and different initial/final states are encoded by operator insertions along this contour. This is known as the Schwinger-Keldysh formalism [26, 50, 51, 27].

In section 1.2.5 we saw how this formalism can be extended to AdS/CFT. In particular, the bulk version of this formalism dictates the gauge/gravity duality acts in a piece-wise

[2] Some of these pairs do not correspond to QFT data at all while others are singular in supergravity but they would be regular in string theory. It is not currently known how to distinguish between the two cases.

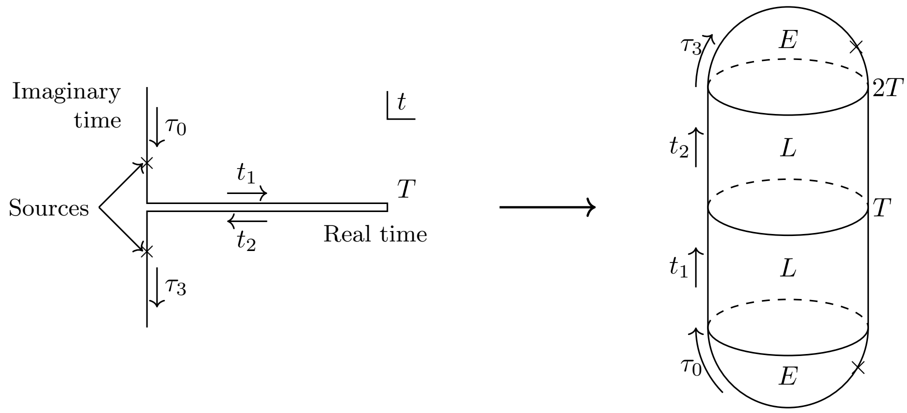


Figure 2.1.1: In–in time contour (left) and corresponding AdS manifold (right). The manifolds labelled by L are empty Lorentzian AdS and those labelled by E are empty, Euclidean AdS.

fashion on the various parts of the time contour and appropriate matching condition are imposed at the corners. More specifically, real time pieces of the contour are associated with Lorentzian AdS manifolds, imaginary time pieces with Euclidean AdS manifolds and the matching conditions require that the fields and their conjugate momenta are continuous across the different manifolds. In this way, the initial conditions are traded for boundary condition in the Euclidean parts of the spacetime. In this formalism, imposing boundary conditions on the entire bulk manifold, uniquely specifies the bulk solution, as in the Euclidean case.

This is a general method that may be used to study correlation functions in general non-equilibrium states. In this paper we will use it to construct a bulk solution that corresponds to an excited CFT state. By the operator-state correspondence any such state may be obtained by acting with scalar primary operators $\mathcal{O}_\Delta$ on the CFT vacuum,

$$|\Delta\rangle = \mathcal{O}_\Delta|0\rangle. \tag{2.1.3}$$

In the Schwinger-Keldysh formalism, in–in correlators in this state may be obtained by considering the in-in contour $\mathcal{C}$ on the left panel of Fig. 2.1.1. On the gravity side we consider the manifold corresponding to the in–in field theory time contour shown in the right panel of figure 2.1.1. The operator $\mathcal{O}_\Delta$ corresponds to a massive bulk scalar field and we will solve the scalar field equation in all four parts of the bulk spacetime. The boundary conditions we use are sources turned on in the two Euclidean manifolds , *i.e.* $\phi_{(0)}(x) \neq 0$ for $x \in \partial E$ where ∂E the boundary of the Euclidean manifolds. In the Lorentzian manifolds we want purely normalisable solutions so we set the sources equal to zero, *i.e.* $\phi_{(0)}(x) = 0$ for $x \in \partial L$ where ∂L is the boundary of the Lorentzian manifolds.

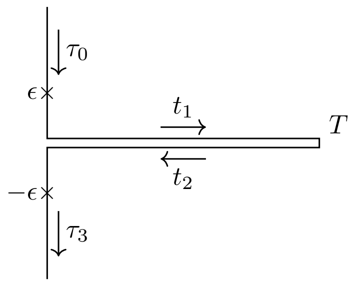


Figure 2.2.1: In–in complex time contour with operator insertions at $t = 0 \pm i\epsilon$.

2.2 Quantum field theory considerations

In this section we setup the problem using the Schwinger-Keldysh formalism. Let us denote by $\phi_{(0)}$ the source that couples to $\mathcal{O}_\Delta$. We would like to compute expectation values in the state $|\Delta\rangle = \mathcal{O}_\Delta|0\rangle$, inserted at $\vec{x} = t = 0$. To realise this set up we consider the contour shown in Fig. 2.2.1. We insert the operator $\mathcal{O}_\Delta$ at small imaginary distance $\tau_0 = -\epsilon$ at $t = 0$ and at $\tau_3 = \epsilon$ at $t_2 = 2T$, where τ_0, t_1, t_2 and τ_3 are contour times in the four segments. In complexified time the insertions are at $t = 0 + i\epsilon$ and $t = 0 - i\epsilon$. Performing the Euclidean path integral over the imaginary part of the contour provides the initial and final conditions for the Lorentzian path integral. Altogether the path integral under consideration is

$$Z\left[\phi_{(0)};\mathcal{C}\right] = \int [\mathcal{D}\phi] \exp\left[-i\int_{\mathcal{C}} \mathrm{d}t\mathrm{d}^{d-1}x\sqrt{-g_{(0)}}\left(\mathcal{L}_{QFT} + \phi_{(0)}(x)\mathcal{O}_\Delta(x)\right)\right] \tag{2.2.1}$$

If we compute this path integral for general $\phi_{(0)}(x)$ and then differentiate w.r.t. $\phi^+_{(0)}$ and $\phi^-_{(0)}$, where $\phi^\pm_{(0)} = \phi_{(0)}(0_\pm, \vec{0})$ and $0_\pm = 0 \pm i\epsilon$, and then set to zero the sources in the imaginary part of the contour, the resulting expression will be the desired generating functional of in-in correlators in the state $|\Delta\rangle$.

In later sections we will construct the gauge/gravity analogue of (2.2.1). Corresponding to $\phi_{(0)}$ there is bulk scalar field Φ and the best we can currently do holographically is to construct (2.2.1) perturbatively in the bulk fields (or perturbatively in a large N limit, see below). Correspondingly we will consider the source $\phi_{(0)}(x)$ in the imaginary part as being infinitesimal, with the product of the two sources at the same point set to zero, $(\phi_{(0)}(x))^2 = 0$, so that we generate a single insertion. If we relax this condition we will generate states that are superpositions of the states associated with "single trace" and "multi-trace" operators. The path integral (2.2.1) with $\phi_{(0)}(x)$ infinitesimal also contains terms linear in the sources which would not contribute if we were to differentiate w.r.t. both $\phi^+_{(0)}$ and $\phi^-_{(0)}$. However, these linear terms still provide a non-trivial check that we are constructing holographically the correct path integral and as such we will consider them in detail.

Let $\mathcal{O}_i$ be gauge invariant operators. Their 1-point function is given by

$$\langle \mathcal{O}_i(t,\vec{x})\rangle = \int [\mathcal{D}\phi]\, \mathcal{O}_i(t,\vec{x}) \exp\Big[-i \int_{\mathcal{C}} \mathrm{d}t' \mathrm{d}^{d-1}\vec{x}' \sqrt{-g_{(0)}}\, (\mathcal{L}_{QFT} + \phi_{(0)}(x')\mathcal{O}_\Delta(x')\Big)\Big]. \tag{2.2.2}$$

Expanding in the sources we obtain

$$\begin{aligned}\langle \mathcal{O}_i(t,\vec{x})\rangle =&\phi^+_{(0)}\langle 0|\mathcal{O}_\Delta(0_+,\vec{0})\mathcal{O}_i(t,\vec{x})|0\rangle + \phi^-_{(0)}\langle 0|\mathcal{O}_i(t,\vec{x})\mathcal{O}_\Delta(0_-,\vec{0})|0\rangle \\ &+ \phi^+_{(0)}\phi^-_{(0)}\langle 0|\mathcal{O}_\Delta(0_+,\vec{0})\mathcal{O}_i(t,\vec{x})\mathcal{O}_\Delta(0_-,\vec{0})|0\rangle. \\ =&\phi^+_{(0)}\langle \Delta|\mathcal{O}_i(t,\vec{x})|0\rangle + \phi^-_{(0)}\langle 0|\mathcal{O}_i(t,\vec{x})|\Delta\rangle + \phi^+_{(0)}\phi^-_{(0)}\langle \Delta|\mathcal{O}_i(t,\vec{x})|\Delta\rangle\end{aligned} \tag{2.2.3}$$

Note that the correlators that appear here are all Wightman functions, as can be seen from the time contour. The expectation value of $\mathcal{O}_i$ in the state $|\Delta\rangle$ appears in the terms quadratic in the sources. As mentioned above, we kept the terms linear in the sources because these terms may be used as a non-trivial check that we construct the correct path integral.

If we linearise in the sources then only the contribution of the first line of (2.2.3) survives. This corresponds in gauge/gravity duality to linearising the bulk field equations. In this case the 1-point function is related to the 2-point function at the conformal point. Since 2-point functions in CFT are diagonal then the only operator that has a non-zero 1-point function is precisely the operator associated with the excited state

$$\langle \mathcal{O}_\Delta\rangle \neq 0, \qquad \langle \mathcal{O}_i\rangle = 0 \qquad (\textit{linear approximation}). \tag{2.2.4}$$

This implies that if we want to work out the linearised bulk solution dual to the state $|\Delta\rangle$, it suffices to only consider the bulk field that is dual to the operator $\mathcal{O}_\Delta$ in a fixed AdS background.[3]

This is no longer the case if we consider the full field equations, as now the second line in (2.2.3) is also relevant and

$$\langle \mathcal{O}_\Delta\rangle \neq 0, \qquad \langle \mathcal{O}_i\rangle \neq 0, \tag{2.2.5}$$

for all operators $\mathcal{O}_i$ that appear in the OPE of $\mathcal{O}_\Delta$ with itself (so that the 3-point function in (2.2.3) is non-zero). This implies that the bulk solution will now include all bulk fields that are dual to these operators. In particular, the energy momentum tensor T_{ij} appears

[3]Note that if we set $\phi^+_{(0)} = \phi^-_{(0)} \equiv \phi_{(0)}$ (with $\phi_{(0)}$ infinitesimal) and the bulk action is quadratic in Φ so that the linear approximation is exact, the bulk solution would have the interpretation as being dual to the state $|0\rangle + \phi_{(0)}|\Delta\rangle$. In this paper we are taking the view that the bulk action contains interaction terms and the linear approximation is the first step towards constructing the full solution perturbatively. From the full solution one may extract the in-in correlators in the state $|\Delta\rangle$ by computing the renormalised on-shell action and keeping the terms proportional to $\phi^+_{(0)}\phi^-_{(0)}$.

in the OPE so one can no longer ignore the back-reaction to the metric.

The CFTs that appear in gauge/gravity duality admit a 't Hooft large N limit and one may also use the large N limit to organise the bulk reconstruction. In particular, if we normalise the operators such that their 2-point function is independent of N, then 3- and higher-point functions go to zero as $N \to \infty$. With this normalisation, the first line in (2.2.3) is the leading order term in the large N limit. We would like to emphasise however that with this normalisation not all $1/N^2$ terms correspond to non-planar corrections (quantum corrections in the bulk).

An alternative normalisation is to normalise the operators such that all connected n-point functions scale as N^2 to leading order (i.e. computed using planar diagrams). With this normalisation all $1/N^2$ corrections are associated with non-planar diagrams. In AdS/CFT this normalisation is known as the "supergravity normalisation": all leading order factors of N come from Newton's constant and $1/N^2$ corrections are due to quantum corrections (loop diagrams).

Either way the leading order construction of the bulk solution dual to a state is universal while the higher order terms depend on the CFT under consideration. In this paper we will discuss in detail the universal part of the construction. The method can be readily extended to higher order once the CFT input is given.

To keep the technicalities at the minimum we will discuss the case of $2d$ CFT either on $R \times S^1$ (with coordinates (t, ϕ)) or on $R^{1,1}$ (with coordinates (t, x)) and we set the source equal to one, $\phi^{\pm}_{(0)} = 1$. For a CFT on $R \times S^1$ the 1-point function in the first line in (2.2.3) then gives,

$$\langle \mathcal{O}_\Delta(t,\phi)\rangle_{\text{exc}} = \frac{C}{(\cos(t-i\epsilon)-\cos\phi)^\Delta} + \frac{C}{(\cos(t+i\epsilon)-\cos\phi)^\Delta}, \tag{2.2.6}$$

while for a CFT on $R^{1,1}$ we obtain

$$\langle \mathcal{O}_\Delta(t,\phi)\rangle_{\text{exc}} = \frac{\tilde{C}}{(-(t-i\epsilon)^2+x^2)^\Delta} + \frac{\tilde{C}}{(-(t+i\epsilon)^2+x^2)^\Delta}, \tag{2.2.7}$$

where we have used the subscript "exc" to emphasise that these are the 1–point functions in the excited state. C and $\tilde{C}$ are the normalisations of the 2-point functions in the two cases[4]. The bulk solution dual to this state in global AdS should reproduce (2.2.6) while the bulk solution in Poincaré AdS should yield (2.2.7).

2.3 Bulk Reconstruction from Boundary Data: A Brief Overview of the Existing Work

As was mentioned already, there exists a series of papers on the reconstruction of bulk fields in Lorentzian AdS from their boundary values. These papers have a different fo-

[4] Actually, since $R \times S^1$ and $R^{1,1}$ are conformally related one may relate (2.2.6) and (2.2.7) and then $\tilde{C} = 2^\Delta C$ [29].

cus from the work presented here. Their results and claims are, however, tangentially related to our work, and therefore, we dedicate this section to reviewing the formalism and language used there. Once again we will consider the simplest case of the free massive scalar field $\Phi(r,x)$, propagating in AdS_{d+1} without backreaction. The field is now in Lorentzian AdS and there are no Euclidean segments in the manifold. Immediately one should worry that this does not lead to a well posed problem since, as we have mentioned, Lorentzian AdS is a non-hyperbolic manifold and one needs to provide additional boundary conditions to uniquely determine any field.

Near the asymptotic boundary the field admits the series expansion given in equation (2.1.1)

$$\Phi(r,x) = r^{d-\Delta}\phi_{(0)}(x) + \dots + r^{\Delta}\log r^2 \psi_{(2\Delta-d)}(x) + r^{\Delta}\phi_{(2\Delta-d)}(x) + \dots \tag{2.3.1}$$

We already mentioned that $\phi_{(0)}(x)$, the coefficient of $r^{d-\Delta}$, is the source for the dual operator $\mathcal{O}_\Delta$. If it is non-zero, the bulk field has a non-normalisable drop-off near the boundary. However, the authors of the *bulk reconstruction* papers want to interpret the bulk field as a bulk excitation. As such, it must be normalisable and, therefore, $\phi_{(0)}$ is set to zero. Then, the reconstruction camp postulates a relation between the operator in the field theory and a local bulk operator $\hat{\Phi}(r,x)$ in the bulk,

$$\hat{\Phi}(r,x) \sim r^{\Delta}\hat{\mathcal{O}}_\Delta + \dots \tag{2.3.2}$$

This resembles the relation (2.1.2) which relates the bulk field with the same boundary conditions imposed to the one point function of the dual operator. However, this is a fundamentally very different statement. Note that the proportionality coefficient here is not necessarily the one given in equation (2.1.2) as now we are not imposing the standard normalisation used when deriving it. Equation (2.3.2) can be inverted to get an expression for $\hat{\mathcal{O}}_\Delta(x)$ in terms of $\hat{\Phi}(r,x)$,

$$\hat{\mathcal{O}}_\Delta = \lim_{r\to 0} r^{-\Delta}\hat{\Phi}(r,x). \tag{2.3.3}$$

The main objective of the *bulk reconstruction* papers is to obtain an expression for $\hat{\Phi}(r,x)$ in terms of $\hat{\mathcal{O}}_\Delta(x)$, of the form

$$\hat{\Phi}(r,x) = \int \mathrm{d}x'\; K(r,x|x')\hat{\mathcal{O}}_\Delta(x'). \tag{2.3.4}$$

$K(r,x|x')$ is referred to as the smearing function. According to their claim this is to be interpreted as a one-to-one correspondence between local bulk operators, encoded in $\hat{\Phi}(r,x)$, and non-local boundary operators, obtained by smearing the dual operators, here $\hat{\mathcal{O}}_\Delta$, over a subregion of the boundary. We postpone discussing this claim and focus for now on the procedure by which one obtains a relation of the form of (2.3.4). The first step is to solve the equation of motion for the bulk field. For a free scalar field the relevant equation is the massive Klein–Gordon equation,

$$\left(\Box_{(r,x)} - m^2\right)\Phi(r,x) = \frac{1}{\sqrt{|G|}}\partial_\mu\left(\sqrt{|G|}G^{\mu\nu}\partial_\nu\Phi(r,x)\right) - m^2\Phi(r,x) = 0, \tag{2.3.5}$$

where $G_{\mu\nu}$ the AdS$_{d+1}$ metric, $G = \det(G_{\mu\nu})$. This can be solved in terms of a complete set of orthonormal modes $F_k(r,x)$, each of which satisfies equation (2.3.5),

$$\hat{\Phi}(r,x) = \int \mathrm{d}k\, \hat{a}_k F_k(r,x) + \text{c.c.} \tag{2.3.6}$$

k is the set of eigenvalues labelling each mode and $\hat{a}_k$ are constant coefficients[5]. Equation (2.3.3) implies a corresponding expansion for $\hat{\mathcal{O}}_\Delta$,

$$\hat{\mathcal{O}}_\Delta = \int \mathrm{d}k\; \hat{a}_k f_k(x) + \text{c.c.} \tag{2.3.7}$$

where $f_k(x) = \lim_{r\to\infty} r^\Delta F_k(r,x)$. If the boundary modes $f_k(x)$ are orthogonal, it is possible to extract the coefficients a_k from expansion for $\hat{\mathcal{O}}_\Delta(x)$,

$$\hat{a}_k g(k) = \int \mathrm{d}x f_k^*(x)\hat{\mathcal{O}}_\Delta(x), \tag{2.3.8}$$

where $g(k)$ accounts for the fact that the boundary modes are not, in general, normalised. Substituting this expression for a_k into the $\hat{\Phi}(r,x)$ expansion one finds,

$$\begin{aligned}\hat{\Phi}(r,x) &= \int \mathrm{d}k \left[\int \mathrm{d}x'\, \frac{f_k^*(x')}{g(k)}\langle\mathcal{O}_\Delta(x')\rangle\right] F_k(r,x)\\ &= \int \mathrm{d}x' \left[\int \mathrm{d}k\, \frac{f_k^*(x')F_k(r,x)}{g(k)}\right]\hat{\mathcal{O}}_\Delta(x').\end{aligned} \tag{2.3.9}$$

The first line implies the second only if the integral over x' is convergent and one must be careful in swapping the order of integration. Assuming the exchange is allowed and, by comparing with equation (2.3.4), we read off the smearing function,

$$K(r,x|x') = \int \mathrm{d}k\, \frac{1}{g(k)} f_k^*(x')F_k(r,x). \tag{2.3.10}$$

The integral in equation (2.3.10) is not always convergent and it requires appropriate $i\epsilon$ insertions or an analytic continuation in the boundary coordinates in order to converge.

In the first part of this thesis we shall perform a similar construction of bulk fields. Our main objective in doing this is to find the bulk dual to an excited field theory state. However, as a bi-product, we obtain a method for performing the reconstruction outlined here. The main difference is that, in addition to the boundary data encoded in the one point function of the dual operator, namely $\langle\mathcal{O}_\Delta\rangle$, we also provide initial and final conditions, as was explained above. The latter are necessary for the uniqueness of the bulk

[5] If the eigenvalues k are discrete, as is the case for global AdS, the integral is replaced by a sum.

solutions, something that is not addressed in by the *bulk reconstruction* camp. Furthermore, since the initial and final conditions are imposed by complexifying the manifold, one expects that the $i\epsilon$ insertions and the analytic continuations necessary to make the integral in equation (2.3.10) convergent, should arise in a natural way, rather than being added by hand. We return to this when we perform explicit reconstructions for global AdS_{2+1} in chapter 3, and for the Poincaré patch of AdS_{2+1} in chapter 4.

This first part of the thesis is organised as follows. We begin by solving the bulk equations of motion for the scalar field in AdS, both for the Lorentzian and Euclidean signature, and impose the boundary conditions discussed above. We then use the result to confirm that the bulk solution obtained represents the field theory state discussed in section 2.2. In particular, the asymptotic behaviour of the bulk field is related to the one-point function of the dual operator, as we discussed above. By obtaining this expression and comparing it with the expected field theory expression, we conclude that the bulk solution indeed represents the state it should. In chapter 3 we discuss the construction of the solution dual to a state of a two dimensional CFT on $R \times S^1$, while in chapter 4 we solve the same problem for a CFT on $R^{1,1}$. We conclude in chapter 5, where we also discuss further the relation with the work of the *bulk reconstruction* camp.

Chapter 3

Global AdS

This chapter is dedicated to the construction of bulk scalar field solutions that are dual to the state $|\Delta\rangle = \mathcal{O}_\Delta|0\rangle$ of a CFT on $R \times S^1$ (see chapter 2). In order to perform the construction to linear order in the sources, it suffices to consider a free scalar Φ of mass $m^2 = \Delta(\Delta-2)$ in global AdS – this field is dual to the operator $\mathcal{O}_\Delta$. We will take $\Delta = 1+l$ with $l = 0, 1, 2, \ldots$, as this is the case in most models embedded in string theory, though the results hold for any $\Delta \geq 1$ with minimal changes. We will also set $1/16\pi G_N = 1, \ell = 1$, where G_N is the three dimensional Newton constant and ℓ is the AdS radius.

The appropriate spacetime shown in the right panel of Fig. 3.0.1, with the Lorentzian pieces, labelled by L, being global Lorenzian AdS spacetimes and the Euclidean ones, labelled by E, their Wick rotated version. The left panel shows the corresponding quantum field theory time contour. It is an in–in time contour with operator insertions along the imaginary time axis.

The real-time gauge/gravity prescription instructs us to solve the field equations of the scalar Φ in the four different parts of the spacetime and then match them. Since we are only aiming at constructing the leading order universal part, it suffices to solve the free field equations.

3.1 Lorentzian Solution

The metric for global AdS_{2+1} and for Lorentzian signature can be written as

$$\mathrm{d}s^2 = -(1+r^2)\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{1+r^2} + r^2\mathrm{d}\phi^2. \tag{3.1.1}$$

In these coordinates the conformal boundary of AdS is at $r \to \infty$. The field equation describing a massive scalar field propagating in this background without back–reaction is given by

$$\left((1+r^2)\partial_r^2 + \frac{1+3r^2}{r}\partial_r - \frac{1}{1+r^2}\partial_t^2 + \frac{1}{r^2}\partial_\phi^2 - m^2\right)\Phi(t,r,\phi) = 0. \tag{3.1.2}$$

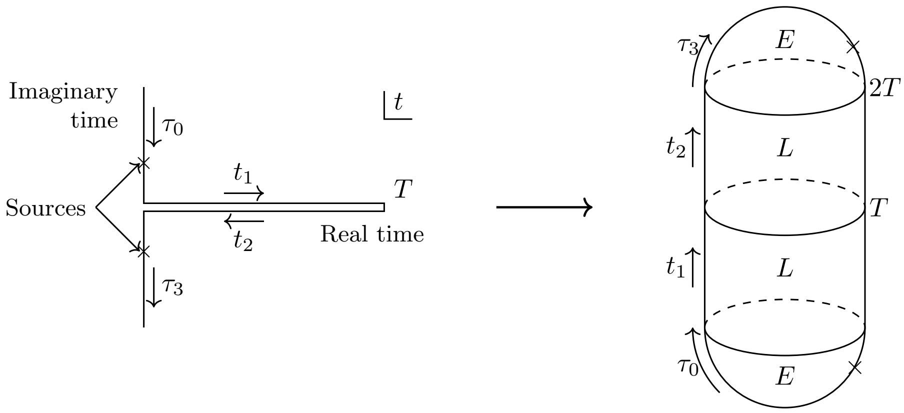


Figure 3.0.1: In–in time contour (left) and corresponding AdS manifold (right). The manifolds labelled by L are empty Lorentzian AdS and those labelled by E are empty, Euclidean AdS.

Substituting the solution ansatz

$$\mathrm{e}^{-i\omega t+ik\phi}f(\omega,k,r) \tag{3.1.3}$$

one finds that $f(\omega,k,r)$ satisfies

$$0=(1+r^2)f''+\frac{3r^2+1}{r}f'-\left(\frac{k^2}{r^2}-\frac{\omega^2}{r^2+1}+m^2\right)f. \tag{3.1.4}$$

where the prime denotes a derivative w.r.t. r. The solution of this ODE is given in terms of a hypergeometric function,

$$f(\omega,k,r)=C_{\omega kl}(1+r^2)^{\omega/2}r^{|k|}\,{}_2F_1(\hat{\omega}_{kl},\hat{\omega}_{kl}-l;|k|+1;-r^2) \tag{3.1.5}$$

where $l=\Delta-1=\{0,1,2,\dots\}$, $\Delta=1+\sqrt{1+m^2}$, $\hat{\omega}_{kl}=(\omega+|k|+l+1)/2$, $k\in\mathbb{Z}$, $\omega\in\mathbb{R}$ and $C_{\omega kl}=(\Gamma(\hat{\omega}_{kl})\Gamma(\hat{\omega}_{kl}-\omega))/((l-1)!\,|k|!)$. The normalisation constant has been chosen to make the coefficient of the leading order term in the near boundary expansion of $f(\omega,k,r)$ equal to 1. Note that $f(\omega,k,r)=f(\omega,-k,r)=f(\omega,|k|r)$ and $f(\omega,k,r)=f(-\omega,k,r)$.

Near the conformal boundary the solution admits the following series expansion in r,

$$f(\omega,|k|,r)=r^{l-1}+\dots+r^{-l-1}\alpha(\omega,|k|,l)\left[\ln(r^2)+\beta(\omega,|k|,l)\right]+\dots \tag{3.1.6}$$

where

$$\alpha(\omega,|k|,l)=\frac{(\hat{\omega}_{kl}-l)_l(\hat{\omega}_{kl}-|k|-l)_l}{l!\,(l-1)!} \tag{3.1.7a}$$

$$\beta(\omega,|k|,l)=-\psi(\hat{\omega}_{kl})-\psi(\hat{\omega}_{kl}-l-\omega). \tag{3.1.7b}$$

From this expression we see that the modes have simple poles in the ω plane which appear at normalisable order, i.e. at $r^{-l-1} = r^{-\Delta}$. Thus, by integrating over ω, in the absence of sources, we obtain the normalisable modes.

The poles of $f(\omega, k, r)$ are at $\omega = \omega_{nk}^{\pm} = \pm(2n+|k|+l+1)$, $n \in \mathbb{N}$. This is not immediately obvious. Based on the digamma functions alone one expects the poles to occur at $\omega = -(2n+|k|+l+1)$ (poles of $\psi(\hat{\omega}_{kl})$) and $\omega = 2n+|k|+1-l$ (poles of $\psi(\hat{\omega}_{kl}-l-\omega)$) for $n = 0, 1, 2, \ldots$. However, $\alpha(\omega, |k|, l)$ has simple zeros at $\omega = 2n+|k|+1-l, n = 0, 1, \ldots, l-1$ which cancel some of the simple poles of $\psi(\hat{\omega}_{kl}-l-\omega)$ leaving only the ones at $\omega_{nk}^{\pm}$. It follows that near the conformal boundary the normalisable modes are given by

$$\begin{aligned} g(\omega_{nk}, |k|, r) =& \frac{1}{4\pi^2 i} \oint_{\omega_{nk}} d\omega\, f(\omega, |k|, r) \\ =& \frac{r^{-l-1}}{4\pi^2 i} \oint_{\omega_{nk}} d\omega \Big[\text{non-norm. term} + \frac{(\hat{\omega}_{kl}-l)_l\, (\hat{\omega}_{kl}-|k|-l)_l}{l!(l-1)!} \big(\ln(r^2) \\ & - \psi(\hat{\omega}_{kl}) - \psi(\hat{\omega}_{kl}-\omega-l) \big) + \ldots \Big] \\ =& \frac{1}{\pi} r^{-l-1} \frac{(n+|k|+1)_l (n+l)!}{n! l! (l-1)!} + \ldots \end{aligned} \tag{3.1.8}$$

where the contours are defined clockwise for the poles at ω_{nk}^{+} and counterclockwise for poles at ω_{nk}^{-} such that $g(\omega_{nk}^{+}, |k|, r) = g(\omega_{nk}^{-}, |k|, r)$. Combining this result with equation (3.1.5) allows us to extend the normalisable modes to finite r,

$$\begin{aligned} g(\omega_{nk}, |k|, r) =& \frac{1}{\pi} r^{|k|} (1+r^2)^{-\frac{|k|+l+1}{2}} \frac{(n+1)_l (n+|k|+1)_l}{l!(l-1)!} \\ & {}_2F_1\left(n+|k|+l+1, -n; l+1; \frac{1}{1+r^2} \right). \end{aligned} \tag{3.1.9}$$

Details of how this result is obtained are presented in appendix 3.A

Then, a normalisable Lorentzian solution has the form

$$\Phi_L(t, r, \phi) = \sum_{k\in\mathbb{Z}} \sum_{n=0}^{\infty} \left(b_{nk}\, \mathrm{e}^{-i\omega_{nk}^{+} t + ik\phi} + b_{nk}^{\dagger}\, \mathrm{e}^{-i\omega_{nk}^{-} t - ik\phi} \right) g(\omega_{nk}, |k|, r), \tag{3.1.10}$$

where b_{nk} and $b_{nk}^{\dagger}$ are arbitrary coefficients, to be determined from the matching conditions.

3.1.1 Euclidean Solution

The metric for global AdS_{2+1} and for Euclidean signature can be obtained from the Lorentzian one, (3.1.1), by Wick rotation, $t = -i\tau$. Similarly, one may obtain the Euclidean solutions by analytically continuing the Lorentzian modes,

$$\begin{aligned} \mathrm{e}^{-\omega\tau + ik\phi} f(\omega, k, r) =& C_{\omega k l}\, \mathrm{e}^{-\omega\tau + ik\phi} (1+r^2)^{\omega/2} r^{|k|} \\ & {}_2F_1(\hat{\omega}_{kl}, \hat{\omega}_{kl}-l; |k|+1; -r^2). \end{aligned} \tag{3.1.11}$$

In accordance with our choice of boundary conditions, the general solution in the Euclidean caps requires that we turn on a source $\phi_{(0)}(\tau,\phi)$ on the boundary. Since we are working with momentum modes, we need to express the source in momentum space. For a general source $\phi^-_{(0)}(\tau,\phi)$ with support on the boundary of the past Euclidean cap and away from the matching surface at $\tau=0$ we have

$$\phi^-_{(0)}(\omega,k) = \int_0^{2\pi} \mathrm{d}\phi \int_{-\infty}^{0} \mathrm{d}\tau\, \mathrm{e}^{\omega\tau - ik\phi}\phi^-_{(0)}(\tau,\phi) \tag{3.1.12}$$

Since the range of τ is over the half real line only, it is natural to use Laplace rather than Fourier transforms. For the past Euclidean cap, τ runs from $-\infty$ to zero. When we treat the future Euclidean cap, the range of integration for τ is over the positive half line, from zero to $+\infty$.

Using the momentum space expression for the source, the most general solution in the past Euclidean cap is

$$\begin{aligned}\Phi^-_E(\tau,r,\phi) =& \frac{1}{4\pi^2 i}\sum_{k\in\mathbb{Z}}\int_{-i\infty}^{i\infty} d\omega\, \mathrm{e}^{-\omega\tau+ik\phi}\phi^-_{(0)}(\omega,k) f(\omega,|k|,r) \\ &+ \sum_{k\in\mathbb{Z}}\sum_{n=0}^{\infty} d^-_{nk}\mathrm{e}^{-\omega^-_{nk}\tau + ik\phi} g(\omega_{nk},k,r)\end{aligned} \tag{3.1.13}$$

where $g(\omega_{nk},|k|,r)$ is defined in (3.1.9). Notice that integration over ω is along the imaginary axis, as it should be for the inverse Laplace transform. The $1/4\pi^2 i$ factor is required for agreement with the definition of $g(\omega_{nk},|k|,r)$.

The second term in equation (3.1.13) is included to make the solution as general as possible. It behaves as r^{-l-1} near the boundary and it decays exponentially as $\tau\to-\infty$ so it does not affect the asymptotic behaviour of the solution and, therefore, it can not be excluded.

To explicitly see that the solution has a source term, recall that for large r, f has the expansion in (3.1.6) and thus the Euclidean solution asymptotes to[1]

$$\begin{aligned}\Phi^-_E(\tau,r,\phi) =& r^{l-1}\frac{1}{4\pi^2 i}\sum_{k\in\mathbb{Z}}\int_{-i\infty}^{i\infty} d\omega\, \mathrm{e}^{-\omega\tau+ik\phi}\phi^-_{(0)}(\omega,k) + O(r^{l-2}) \\ =& r^{l-1}\phi_{(0)}(\tau,\phi) + O(r^{l-2})\end{aligned} \tag{3.1.14}$$

In our analysis we choose the source profile to be a δ–function localised at $(\tau,\phi) = (-\epsilon,0)$, $\epsilon>0$, i.e.

$$\phi^-_{(0)}(\tau,\phi) = \delta(\tau+\epsilon)\delta(x), \tag{3.1.15}$$

which implies $\phi^-_{(0)}(\omega,k) = \exp(-\omega\epsilon)$.

[1] Here we assume that the source admits a Laplace transform. This is true in particular if $\phi_{(0)}(\omega,k)$ can be extended to a meromorphic function with no singularities for $\mathrm{Re}(\omega) > c$, for some finite c. Here for simplicity we take $c=0$.

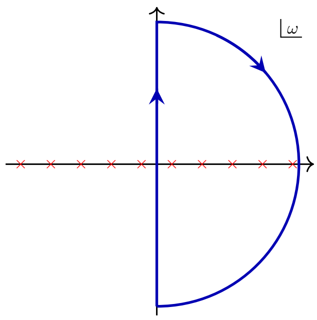


Figure 3.1.1: The integration contour for ω for the past Euclidean cap.

The integral over ω can be done explicitly close to the matching surface using contour integration. Let the time in the past Euclidean cap be denoted by τ_0 and let $-\epsilon < \tau_0 \leq 0$. The latter ensures that we are close to the matching surface, located at $\tau_0 = 0$. The integrant contains the exponential factor $\mathrm{e}^{-\omega(\tau_0+\epsilon)}$. Since $\tau_0 + \epsilon < 0$, we close the ω–contour to the right, as shown in figure 3.1.1, such that $\mathrm{Re}(\omega) > 0$. This ensures that the integral around the semi-circular part of the contour vanishes when its radius is send to infinity. According to the Cauchy residue theorem, we then pick up the contributions from the poles at $\omega = \omega^+_{nk}$ of the subleading terms of the expansion of f, obtaining

$$\Phi^-_E(\tau_0, r, \phi) = \sum_{k\in\mathbb{Z}} \sum_{n=0}^{\infty} \Big(\phi^-_{(0)}(\omega^+_{nk}, k) \mathrm{e}^{-\omega^+_{nk}\tau_0 + ik\phi} + d^-_{nk} \mathrm{e}^{-\omega^-_{nk}\tau_0 + ik\phi} \Big) \, g(\omega_{nk}, |k|, r), \tag{3.1.16}$$

where recall that the contour of integration around the ω^+_{nk} poles was defined clockwise which explains the sign of Φ_E.

The analysis for the future Euclidean cap follows along the same lines. In particular, denoting Euclidean time in the future Euclidean cap by τ_3, $0 \leq \tau_3 < \infty$, and using a δ–function source localised at $(\tau_3, \phi) = (\epsilon, 0)$ where ϵ is the same as for the past Euclidean cap, $\phi^+_{(0)}(\tau_3, \phi) = \delta(\tau_3 - \epsilon)\delta(\phi)$, $\phi^+_{(0)}(\omega, k) = \exp(\omega\epsilon)$ and considering the solution close to the matching surface, $0 \leq \tau_3 < \epsilon$, we obtain

$$\Phi^+_E(\tau_3, r, \phi) = \sum_{k\in\mathbb{Z}} \sum_{n=0}^{\infty} \Big(\phi^+_{(0)}(\omega^-_{nk}, k) \mathrm{e}^{-\omega^-_{nk}\tau_3 + ik\phi} + \tilde{d}^+_{nk} \mathrm{e}^{-\omega^+_{nk}\tau_3 + ik\phi} \Big) \, g(\omega_{nk}, |k|, r). \tag{3.1.17}$$

3.1.2 Matching Conditions

The time contour considered here is the in–in contour shown on the left of figure 3.1.2, with the corresponding AdS manifold shown on the right. It runs from $i\infty$ to 0, then to

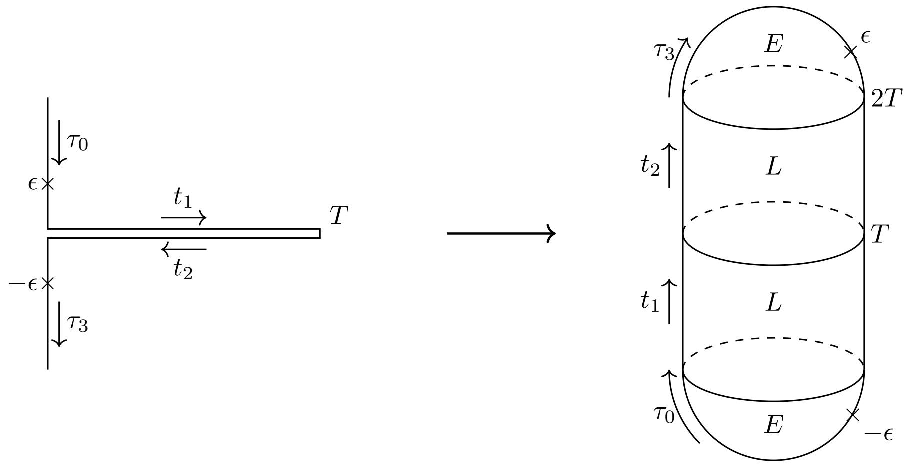


Figure 3.1.2: In–in time contour (left) and corresponding AdS manifold (right).

T, then back to 0 and then to $-i\infty$. Accordingly, the contour–integrated action is

$$\begin{aligned} S = &- \int_{-\infty}^{0} \mathrm{d}\tau_0 \; L_E(\Phi_E^-) + i \int_0^T \mathrm{d}t_1 \; L_L(\Phi_L^1) - i \int_T^{2T} \mathrm{d}t_2 \; L_L(\Phi_L^2) \\ &- \int_0^{\infty} \mathrm{d}\tau_3 \; L_E(\Phi_E^+). \end{aligned} \tag{3.1.18}$$

where

$$L_E = \tfrac{1}{2} \int \mathrm{d}^3x \sqrt{g} \left(g^{\mu\nu} \partial_\mu \Phi_E \partial_\nu \Phi_E + m^2 \Phi_E^2 \right) \tag{3.1.19}$$

and

$$L_L = -\tfrac{1}{2} \int \mathrm{d}^3x \sqrt{-g} \left(g^{\mu\nu} \partial_\mu \Phi_L \partial_\nu \Phi_L - m^2 \Phi_L^2 \right). \tag{3.1.20}$$

The matching conditions are

$$\begin{aligned} \Phi_E^-\big|_{\tau_0=0} &= \Phi_L^1\big|_{t_1=0}, & \partial_{\tau_0} \Phi_E^-\big|_{\tau_0=0} &= -i\partial_{t_1} \Phi_L^1\big|_{t_1=0} \\ \Phi_L^1\big|_{t_1=T} &= \Phi_L^2\big|_{t_2=T}, & \partial_{t_1} \Phi_L^1\big|_{t_1=T} &= -\partial_{t_2} \Phi_L^2\big|_{t_2=T} \\ \Phi_L^2\big|_{t_2=2T} &= \Phi_E^+\big|_{\tau_3=0}, & \partial_{t_2} \Phi_L^2\big|_{t_2=2T} &= -i\partial_{\tau_3} \Phi_E^+\big|_{\tau_3=0}. \end{aligned} \tag{3.1.21}$$

From the previous section we have that the solutions in the four manifolds are

$$-\epsilon < \tau_0 \le 0 :$$

$$\Phi_E^-(\tau_0, r, \phi) = \sum_{k\in\mathbb{Z}} \sum_{n=0}^{\infty} \left(\phi_{(0)}^-(\omega_{nk}^+, k) \mathrm{e}^{-\omega_{nk}^+ \tau_0 + ik\phi} + d_{nk}^- \mathrm{e}^{-\omega_{nk}^- \tau_0 + ik\phi} \right) g(\omega_{nk}, |k|, r) \tag{3.1.22a}$$

$$0 \le \tau_3 < \epsilon :$$

$$\Phi_E^+(\tau_3, r, \phi) = \sum_{k\in\mathbb{Z}} \sum_{n=0}^{\infty} \left(\phi_{(0)}^+(\omega_{nk}^-, k) \mathrm{e}^{-\omega_{nk}^- \tau_3 + ik\phi} + \tilde{d}_{nk}^+ \mathrm{e}^{-\omega_{nk}^+ \tau_3 + ik\phi} \right) g(\omega_{nk}, |k|, r) \tag{3.1.22b}$$

$$0 \le t_1 \le T :$$

$$\Phi_L^1(t_1, r, \phi) = \sum_{k\in\mathbb{Z}} \sum_{n=0}^{\infty} \left(b_{nk}\, \mathrm{e}^{-i\omega_{nk}^+ t_1 + ik\phi} + b_{nk}^\dagger\, \mathrm{e}^{-i\omega_{nk}^- t_1 - ik\phi} \right) g(\omega_{nk}, |k|, r), \tag{3.1.22c}$$

$$T \le t_2 \le 2T :$$

$$\Phi_L^2(t_2, r, \phi) = \sum_{k\in\mathbb{Z}} \sum_{n=0}^{\infty} \left(\tilde{b}_{nk}\, \mathrm{e}^{-i\omega_{nk}^+ t_2 + ik\phi} + \tilde{b}_{nk}^\dagger\, \mathrm{e}^{-i\omega_{nk}^- t_2 - ik\phi} \right) g(\omega_{nk}, |k|, r). \tag{3.1.22d}$$

Applying to these the matching conditions we obtain the following relations: from the matching conditions at $\tau_0 = 0, t_1 = 0$

$$b_{nk} = \phi_{(0)}^-(\omega_{nk}^+, k) = \mathrm{e}^{-\omega_{nk}^+ \epsilon}, \tag{3.1.23a}$$

$$b_{nk}^\dagger = d_{nk}^-. \tag{3.1.23b}$$

From the matching conditions at $t_1 = T, t_2 = T$

$$b_{nk}^\dagger = \tilde{b}_{nk} \mathrm{e}^{-2i\omega_{nk}^+ T}, \tag{3.1.24a}$$

$$b_{nk} = \tilde{b}_{nk}^\dagger \mathrm{e}^{-2i\omega_{nk}^- T}. \tag{3.1.24b}$$

Finally, from the matching conditions at $t_2 = 2T, \tau_3 = 0$

$$\tilde{b}_{nk} = \phi_{(0)}^+(\omega_{nk}^-, k) \mathrm{e}^{-2i\omega_{nk}^- T} = \mathrm{e}^{-i\omega_{nk}^- (2T + i\epsilon)}, \tag{3.1.25a}$$

$$\tilde{b}_{nk}^\dagger = \tilde{d}_{nk}^+ \mathrm{e}^{-2i\omega_{nk}^+ T}. \tag{3.1.25b}$$

Note that had we chosen the position in complex time where we insert the sources to be different for the two caps, say $\tau_{0,\text{source}} = -\epsilon$ and $\tau_{3,\text{source}} = \tilde{\epsilon}$, where $\tilde{\epsilon} > 0$, then the relationships $b_{nk} = \left(b_{nk}^\dagger\right)^*$ and $\tilde{b}_{nk} = \left(\tilde{b}_{nk}^\dagger\right)^*$ would have implied that $\epsilon = \tilde{\epsilon}$.

In what follows we refer to terms proportional to $\mathrm{e}^{-i\omega^+_{nk}t}$ ($\mathrm{e}^{\omega^+_{nk}\tau}$ for Euclidean) as the positive frequency modes and $\mathrm{e}^{-i\omega^-_{nk}t}$ ($\mathrm{e}^{-\omega^-_{nk}\tau}$ for Euclidean) as the negative frequency modes. From the matching conditions we observe that the positive frequency exponential source modes from the past Euclidean cap source the positive frequency oscillatory normalisable modes in the first Lorentzian manifold. As these modes evolve into the second Lorentzian manifold they give rise to the negative frequency oscillatory normalisable modes. Finally, they become positive frequency normalisable modes in the future Euclidean cap. The negative frequency source modes from the past Euclidean manifold decay and do not enter the Lorentzian manifolds. In addition to source modes, there are negative frequency normalisable modes in the past Euclidean manifold. These modes come from negative frequency source modes in the future Euclidean cap which become positive frequency normalisable modes in the second Lorentzian manifold, then evolve into negative frequency normalisable modes in the first Lorentzian manifold and finally they give rise to negative normalisable modes in the past Euclidean cap. The absence of positive frequency normalisable modes in the past Euclidean manifold is due to the fact that these grow exponentially as $\tau_0 \to -\infty$. Schematically, the different modes evolved as shown below: Starting from the past Euclidean modes,

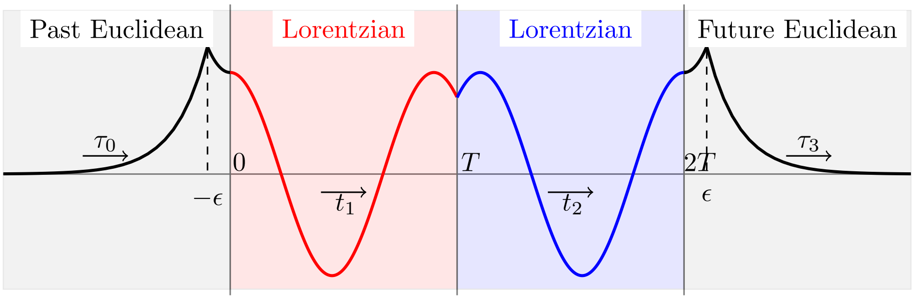
$$\begin{aligned}
&\phi_0^-(\omega^+_{nk},k) \longrightarrow b_{nk} \longrightarrow \tilde{b}^\dagger_{nk}\mathrm{e}^{-2i\omega^-_{nk}T} \longrightarrow \tilde{d}^+_{nk} \\
&\phi^-_{(0)}(\omega^-_{nk},k) \longrightarrow \text{decay} \\
&d^-_{nk} \longrightarrow b^\dagger_{nk} \longrightarrow \tilde{b}_{nk}\mathrm{e}^{-2i\omega^+_{nk}T} \longrightarrow \phi^+_{(0)}(\omega^-_{nk},k),
\end{aligned} \tag{3.1.26}$$

and, similarly, starting from the future Euclidean cap,

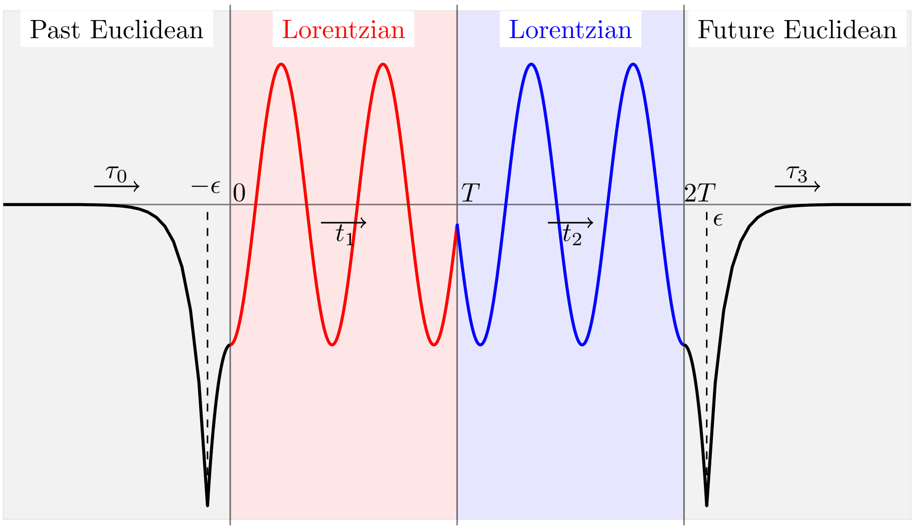
$$\begin{aligned}
&\phi^+_{(0)}(\omega^-_{nk},k) \longrightarrow \tilde{b}_{nk}\mathrm{e}^{-2i\omega^+_{nk}T} \longrightarrow b^\dagger_{nk} \longrightarrow d^-_{nk} \\
&\phi^+_{(0)}(\omega^+_{nk},k) \longrightarrow \text{decay} \\
&\tilde{d}^+_{nk} \longrightarrow \tilde{b}^\dagger_{nk}\mathrm{e}^{-2i\omega^-_{nk}T} \longrightarrow b_{nk} \longrightarrow \phi^-_{(0)}(\omega^+_{nk},k).
\end{aligned} \tag{3.1.27}$$

Figure 3.1.3 shows plots of the time evolution of individual modes from exponentially decaying source modes in the Euclidean manifolds to oscillatory, normalisable modes in the Lorentzian manifolds. These plots were obtained by fixing r and ϕ to be 1 and 0 respectively, and with the source insertions located at $\epsilon = 0.1$. The vertical axis corresponds to the amplitude of the scalar mode and the horizontal axis to contour time. Then these plots show two individual modes as they evolve from imaginary time in the past Euclidean manifold, to real time in the two Lorentzian manifolds and then back to imaginary time in the future Euclidean manifolds.

Combining all three sets of relationships between the coefficients of the different modes



(a) Mode parameters: $n = 2,\ k = 1,\ l = 1,\ r = 1,\ \phi = 0,\ \epsilon = 0.1$



(b) Mode parameters: $n = 4,\ k = 2,\ l = 3,\ r = 1,\ \phi = 0,\ \epsilon = 0.1$

Figure 3.1.3: Tracing individual modes through the four segments of the manifold.

we find

$$b^{\dagger}_{nk} = \phi^{+}_{(0)}(\omega^{-}_{nk}, k), \tag{3.1.28a}$$

$$\tilde{b}^{\dagger}_{nk} = \phi^{-}_{(0)}(\omega^{+}_{nk}, k)\mathrm{e}^{-2i\omega^{+}_{nk}T}. \tag{3.1.28b}$$

Returning to the Lorentzian fields, we can now replace the original, arbitrary coefficients $b^{\pm}_{nk}$ and $\tilde{b}^{\pm}_{nk}$ with the above results to obtain expressions in terms of the Euclidean source modes.

$$\Phi^1_L(t,r,\phi)=\sum_{n=0}^{\infty}\sum_{k\in\mathbb{Z}}\Big[\phi^-_{(0)}(\omega^+_{nk},k)\mathrm{e}^{-i\omega^+_{nk}t+ik\phi} +\phi^+_{(0)}(\omega^-_{nk},k)\mathrm{e}^{-i\omega^-_{nk}t-ik\phi}\Big]g(\omega_{nk},|k|,r) \tag{3.1.29a}$$

$$\Phi^2_L(t,r,\phi)=\sum_{n=0}^{\infty}\sum_{k\in\mathbb{Z}}\Big[\phi^+_{(0)}(\omega^-_{nk},k)\mathrm{e}^{i\omega^+_{nk}t+ik\phi} +\phi^-_{(0)}(\omega^+_{nk},k)\mathrm{e}^{i\omega^-_{nk}t-ik\phi}\Big]g(\omega_{nk},|k|,r). \tag{3.1.29b}$$

where we used the relation between physical and contour time, $t_1 = t$ and $2T - t_2 = t$.

3.1.3 1-point function

Having constructed normalisable Lorentzian solutions, we will now extract the 1-point function to verify that this solution is indeed dual to the state $|\Delta\rangle$. For this we need to obtain the asymptotic expansion of the bulk field near the conformal infinity as in (2.1.1),

$$\Phi(r,x) = r^{d-\Delta}\phi_{(0)}(x) + \cdots + r^{\Delta}\log r^2\psi_{(2\Delta-d)}(x) + r^{\Delta}\phi_{(2\Delta-d)}(x) + \ldots \tag{3.1.30}$$

and use [19],

$$\langle\mathcal{O}_\Delta(t,\phi)\rangle = -(2\Delta-2)\phi_{(2\Delta-d)}(t,\phi). \tag{3.1.31}$$

We can choose to consider the insertion either in the upper part of the contour or in the lower. In the former case the 1-point function can be extracted from the asymptotic expansion of Φ^1_L while in the latter case from the asymptotic expansion of Φ^2_L. In both cases, the answer should be the same.

For concreteness, we consider the case the operator is in the upper part of the contour so the relevant field is Φ^1_L. Since this a normalisable mode, $\phi_{(2\Delta-2)}$ is the coefficient of the leading order term as $r\to\infty$,

$$\phi_{(2\Delta-2)} = \frac{1}{\pi}\sum_{n=0}^{\infty}\sum_{k\in\mathbb{Z}}\mathrm{e}^{-\omega^+_{nk}\epsilon}\left(\mathrm{e}^{-i\omega^+_{nk}t+ik\phi}+\mathrm{e}^{-i\omega^-_{nk}t-ik\phi}\right)\alpha(\omega_{nk},|k|,l), \tag{3.1.32}$$

where we have used

$$g(\omega_{nk},|k|,r) = \frac{1}{\pi}r^{-\Delta}\alpha(\omega_{nk},|k|,l) + \mathrm{O}\big(r^{-\Delta-1}\big) \tag{3.1.33}$$

Performing the sums over n and k and inserting in (3.1.31) we finally get

$$\langle\mathcal{O}_\Delta(t,\phi)\rangle_{\text{exc}} = \frac{l^2}{2^l\pi}\left(\frac{1}{(\cos(t-i\epsilon)-\cos\phi)^\Delta}+\frac{1}{(\cos(t+i\epsilon)-\cos\phi)^\Delta}\right) \tag{3.1.34}$$

where we have used the subscript "exc" to emphasise that this is the 1–point function of $\mathcal{O}_\Delta$ in the excited state. This is indeed equal to value we got via a QFT computation in

(2.2.6). In our case, $C = l^2/(2^l\pi)$, which is the standard supergravity normalisation of the 2-point function.

3.2 Reconstruction of bulk fields from boundary data

In this section we demonstrate how one can reconstruct the Lorentzian fields from their asymptotic value which is given by

$$\Psi(t,\phi) = \lim_{r\to\infty} r^{l+1}\Phi(t,r,\phi). \tag{3.2.1}$$

The motivation for this construction comes from the work of several groups which try to extend the holographic dictionary to the interior of AdS (see for example [40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 52, 53, 54, 55, 56, 57, 58, 59]), a work motivated by the "extrapolate" dictionary of Banks, Douglas, Horowitz and Martinec [60]. Here we follow the formalism of Hamilton et. al. and express the normalisable, Lorentzian solutions, in terms of their asymptotic value,

$$\Phi(t,r,\phi) = \int_{\partial\text{AdS}} \mathrm{d}\hat{t}\,\mathrm{d}\hat{\phi}\; K(t,r,\phi|\hat{t},\hat{\phi})\Psi(\hat{t},\hat{\phi}). \tag{3.2.2}$$

$K(t,r,\phi|t',\phi')$ which we refer to as the smearing function, is a type of Green's function satisfying special boundary conditions. When such an expression exists the claim is that one can use the standard AdS/CFT dictionary (often referred to as the "differentiate" dictionary), which relates the boundary field, $\Psi(t,\phi)$, to the one point function of the dual operator,

$$\Psi(t,\phi) \propto \langle \mathcal{O}_\Delta(t,\phi)\rangle, \tag{3.2.3}$$

to write a relation between local bulk operators and field theory operators. This requires quantisation of $\Phi(t,r,\phi)$ and the replacement of the expectation value in (3.2.3) by the operator itself. Then, the claim is that equation (3.2.2) gives a representation of a bulk operator $\hat{\Phi}(t,r,\phi)$ in terms of a field theory operator $\mathcal{O}_\Delta(t,\phi)$, smeared over some region of spacetime. However, there are some subtleties and issues that arise in doing this which will be address in chapter 5.

Here we construct the smearing function and obtain relation (3.2.2) for the bulk Lorentzian fields for global AdS.

We begin by extending the past and future Euclidean fields to the past and future of the source insertions, respectively. Recall that when the integral over ω was perform in section 3.1.1, we restricted to a neighbourhood near the matching surface, in the future of the past Euclidean source and in the past of the future one. The solutions can be extended to any point in Euclidean time, except the exact location of the source insertion, as follows,

$$\Phi_E^-(\tau_0,r,\phi) = \sum_{k\in\mathbb{Z}}\sum_{n=0}^{\infty}\Big(\theta(-\tau_0-\epsilon)\phi_{(0)}^-(\omega_{nk}^-,k)\mathrm{e}^{-\omega_{nk}^-\tau_0+i\,k\phi} + \theta(\tau_0+\epsilon)\phi_{(0)}^-(\omega_{nk}^+,k)\mathrm{e}^{-\omega_{nk}^+\tau_0+ik\phi} + d_{nk}^-\mathrm{e}^{-\omega_{nk}^-\tau_0+ik\phi}\Big)g(\omega_{nk},|k|,r) \quad (3.2.4a)$$

$$\Phi_E^+(\tau_3,r,\phi) = \sum_{k\in\mathbb{Z}}\sum_{n=0}^{\infty}\Big(\theta(\tau_3-\epsilon)\phi_{(0)}^+(\omega_{nk}^-,k)\mathrm{e}^{-\omega_{nk}^-\tau_3+ik\phi} + \theta(-\tau_3+\epsilon)\phi_{(0)}^+(\omega_{nk}^-,k)\mathrm{e}^{-\omega_{nk}^-\tau_3+ik\phi} + \tilde{d}_{nk}^+\mathrm{e}^{-\omega_{nk}^+\tau_3+ik\phi}\Big)g(\omega_{nk},|k|,r) \quad (3.2.4b)$$

Combining these expressions, as well as the corresponding ones for the bulk Lorentzian fields given by equations (3.1.29a) and (3.1.29b), with the definition of the boundary field, (3.2.1), we find the expressions for the boundary fields,

$$\Psi_E^-(\tau_0,\phi) = \sum_{k\in\mathbb{Z}}\sum_{n=0}^{\infty}A_{nkl}\Big(\theta(-\tau_0-\epsilon)\phi_{(0)}^-(\omega_{nk}^-,k)\mathrm{e}^{-\omega_{nk}^-\tau_0+i\,k\phi} + \theta(\tau_0+\epsilon)\phi_{(0)}^-(\omega_{nk}^+,k)\mathrm{e}^{-\omega_{nk}^+\tau_0+ik\phi} + d_{nk}^-\mathrm{e}^{-\omega_{nk}^-\tau_0+ik\phi}\Big) \quad (3.2.5a)$$

$$\Psi_E^+(\tau_3,\phi) = \sum_{k\in\mathbb{Z}}\sum_{n=0}^{\infty}A_{nkl}\Big(\theta(\tau_3-\epsilon)\phi_{(0)}^+(\omega_{nk}^-,k)\mathrm{e}^{-\omega_{nk}^-\tau_3+ik\phi} + \theta(-\tau_3+\epsilon)\phi_{(0)}^+(\omega_{nk}^-,k)\mathrm{e}^{-\omega_{nk}^-\tau_3+ik\phi} + \tilde{d}_{nk}^+\mathrm{e}^{-\omega_{nk}^+\tau_3+ik\phi}\Big) \quad (3.2.5b)$$

$$\Psi_L^1(t_1,\phi) = \sum_{n=0}^{\infty}\sum_{k\in\mathbb{Z}}A_{nkl}\Big(\phi_{(0)}^-(\omega_{nk}^+,k)\mathrm{e}^{-i\omega_{nk}^+t_1+ik\phi} + \phi_{(0)}^+(\omega_{nk}^-,k)\mathrm{e}^{-i\omega_{nk}^-t_1-ik\phi}\Big) \quad (3.2.5c)$$

$$\Psi_L^2(t_2,\phi) = \sum_{n=0}^{\infty}\sum_{k\in\mathbb{Z}}A_{nkl}\Big(\phi_{(0)}^+(\omega_{nk}^-,k)\mathrm{e}^{i\omega_{nk}^+(t_2-2T)+ik\phi} + \phi_{(0)}^-(\omega_{nk}^+,k)\mathrm{e}^{i\omega_{nk}^-(t_2-2T)-ik\phi}\Big). \quad (3.2.5d)$$

where we have defined for compactness

$$A_{nkl} = \frac{1}{\pi}\frac{(n+1)_l(n+|k|+1)_l}{l!(l-1)!}. \quad (3.2.6)$$

We proceed by using the expressions for the Euclidean boundary fields to obtain expressions for $\phi^-_{(0)}(\omega^+_{nk},k)$ and $\phi^+_{(0)}(\omega^-_{nk},k)$,

$$A_{nkl}\phi^-_{(0)}(\omega^+_{nk},k) = \frac{1}{4\pi^2 i}\int_0^{2\pi}\mathrm{d}\phi\int_{-\delta-i\infty}^{-\delta+i\infty}\mathrm{d}\tau_0\ \mathrm{e}^{\omega^+_{nk}\tau_0-ik\phi}\ \Psi^-_E(\tau_0,\phi) \tag{3.2.7a}$$

$$A_{nkl}\phi^+_{(0)}(\omega^-_{nk},k) = \frac{1}{4\pi^2 i}\int_0^{2\pi}\mathrm{d}\phi\int_{\tilde{\delta}-i\infty}^{\tilde{\delta}+i\infty}\mathrm{d}\tau_3\ \mathrm{e}^{\omega^-_{nk}\tau_3-ik\phi}\ \Psi^+_E(\tau_3,\phi) \tag{3.2.7b}$$

where $\delta,\tilde{\delta}>0$, are needed to make the step functions appearing in the expressions for the Euclidean fields well defined.

By comparing the expressions for the Lorentzian and Euclidean boundary fields and performing the appropriate coordinate transformations we obtain expressions for the source modes in terms of the Lorentzian boundary fields,

$$A_{nkl}\phi^-_{(0)}(\omega^+_{nk},k) = \frac{1}{4\pi^2}\int_0^{2\pi}d\phi\int_{-\infty}^{+\infty}dt\,\mathrm{e}^{i\omega^+_{nk}(t+i\delta)-ik\phi}\Psi^1_L(t+i\delta,\phi) \tag{3.2.8a}$$

$$A_{nkl}\phi^+_{(0)}(\omega^-_{nk},k) = \frac{1}{4\pi^2}\int_0^{2\pi}d\phi\int_{-\infty}^{+\infty}dt\,\mathrm{e}^{i\omega^-_{nk}(t-i\tilde{\delta})-ik\phi}\Psi^1_L(t-i\tilde{\delta},\phi). \tag{3.2.8b}$$

The final step is to substitute these expressions for the source modes in the expressions for bulk Lorentzian fields, (3.1.29a) and (3.1.29b), combined with the expression for $\tilde{g}(\omega_{nk},|k|,r)$ which can be found in appendix 3.A, and perform the summation over modes.

For $\Phi^1_L(t_1,r,\phi)$ we find

$$\Phi^1_L(t_1,r,\phi) = \sum_{n=0}^{\infty}\sum_{k\in\mathbb{Z}}A_{nkl}(1+r^2)^{-\frac{|k|+l+1}{2}}\Big[\phi^-_{(0)}(\omega^+_{nk},k)\mathrm{e}^{-i\omega^+_{nk}t_1+ik\phi} + \phi^+_{(0)}(\omega^-_{nk},k)\mathrm{e}^{-i\omega^-_{nk}t_1-ik\phi}\Big]\,{}_2F_1\left(n+|k|+l+1,-n;l+1;\frac{1}{1+r^2}\right). \tag{3.2.9}$$

Substituting the expressions for the source modes, (3.2.8a) and (3.2.8b),

$$\Phi^1_L(t_1,r,\phi) = \int_0^{2\pi}\mathrm{d}\hat{\phi}\int_{-\infty}^{+\infty}\mathrm{d}\hat{t}\Bigg[\frac{1}{4\pi^2}\lim_{\delta,\tilde{\delta}\to 0}\sum_{n=0}^{\infty}\sum_{k\in\mathbb{Z}}\left(\mathrm{e}^{-i\omega^+_{nk}(t_1-\hat{t}-i\delta)} + \mathrm{e}^{-i\omega^-_{nk}(t_1-\hat{t}+i\tilde{\delta})}\right)\mathrm{e}^{ik(\phi-\hat{\phi})}\,r^{|k|}(1+r^2)^{-\frac{|k|+l+1}{2}}\ {}_2F_1\left(n+|k|+l+1,-n;l+1;\frac{1}{1+r^2}\right)\Bigg]\Psi^1_L(\hat{t},\hat{\phi}) \tag{3.2.10}$$

where we set $\delta,\tilde{\delta}=0$ in Ψ^1_L. The expression in the square brackets is the smearing function, as defined by equation (3.2.2),

$$K(t_1, r, \phi|\hat{t}, \hat{\phi}) = \frac{1}{4\pi^2} \lim_{\delta,\tilde{\delta}\to 0} \sum_{n=0}^{\infty} \sum_{k\in\mathbb{Z}} \left[\left(\mathrm{e}^{-i\omega^+_{nk}(t_1-\hat{t}-i\delta)} + \mathrm{e}^{-i\omega^-_{nk}(t_1-\hat{t}+i\tilde{\delta})} \right) \mathrm{e}^{ik(\phi-\hat{\phi})} \right.$$
$$\left. r^{|k|}(1+r^2)^{-\frac{|k|+l+1}{2}} {}_2F_1\left(n+|k|+l+1, -n; l+1; \frac{1}{1+r^2} \right) \right] \tag{3.2.11}$$

The isometries of AdS enable us to compute the smearing function at $r = 0$ since we can always translate any point in the bulk to the origin, perform the reconstruction, and then map back to the original point. By setting $r = 0$ only the s–wave remains non zero, eliminating the sum over k. Thus, (using the $\omega^{\pm}_{nk} = \pm(2n + |k| + l + 1)$)

$$\begin{aligned} K(t_1, 0, \phi|\hat{t}, \hat{\phi}) &= \frac{1}{4\pi^2} \lim_{\delta,\tilde{\delta}\to 0^+} \sum_{n=0}^{\infty} \left[\left(\mathrm{e}^{-i(2n+l+1)(t_1-\hat{t}-i\delta)} + \mathrm{e}^{i(2n+l+1)(t_1-\hat{t}+i\tilde{\delta})} \right) \right. \\ &\qquad \left. {}_2F_1(n+l+1, -n; l+1; 1) \right] \\ &= \frac{1}{4\pi^2} \lim_{\delta,\tilde{\delta}\to 0^+} \sum_{n=0}^{\infty} \left[\left(\mathrm{e}^{-i(2n+l+1)(t_1-\hat{t}-i\delta)} + \mathrm{e}^{i(2n+l+1)(t_1-\hat{t}+i\tilde{\delta})} \right) \right. \\ &\qquad \left. \frac{\Gamma(0)\Gamma(l+1)}{\Gamma(-n)\Gamma(n+l+1)} \right] \end{aligned} \tag{3.2.12}$$

Using the properties of the gamma function and writing l as $\Delta - 1$ we find

$$K(t_1, 0, \phi|\hat{t}, \hat{\phi}) = \frac{1}{2\pi^2} \lim_{\delta\to 0^+} \mathrm{Re} \left[\mathrm{e}^{i\Delta(\hat{t}-t_1)} {}_2F_1\left(1, 1; \Delta; -\mathrm{e}^{2i(\hat{t}-t_1+i\delta)}\right) \right] \tag{3.2.13}$$

In [40] the authors give the following expression for the positive frequency part of K, for $t_1 = r = 0$ and for AdS_d,

$$K_+ = \frac{1}{\pi\,\mathrm{vol}\,(S^{d-1})} \mathrm{e}^{i\Delta\tau} {}_2F_1\left(1, \frac{d}{2}; \Delta - \frac{d}{2} + 1; -\mathrm{e}^{2i\tau}\right). \tag{3.2.14}$$

To make the connection with our result we use $d = 2$ and identify τ with $\hat{t}$ in our expression. This gives

$$K_+ = \frac{1}{2\pi^2} \mathrm{e}^{i\Delta\hat{t}} {}_2F_1\left(1, 1; \Delta; -\mathrm{e}^{2i\hat{t}}\right) \tag{3.2.15}$$

in agreement with our expression, up to a factor of 2. This factor comes about because the above authors are working with a compact time direction, $-\pi/2 \le \tau \le \pi/2$. Consequently, there is a factor of two difference in the orthogonality relations of our frequency modes. We have

$$\int_{-\infty}^{\infty} \mathrm{d}t\, \mathrm{e}^{-i(\omega_{nk}-\omega_{mk})t} = 2\pi\delta_{n,m} \tag{3.2.16}$$

whereas, in the aforementioned paper they have

$$\int_{-\pi/2}^{\pi/2} \mathrm{d}\tau \, \mathrm{e}^{-i(\omega_{nk}-\omega_{mk})\tau} = \pi\delta_{n,m} \tag{3.2.17}$$

which explains why our result is twice theirs.

3.A Global AdS: Full Bulk Solutions

In section 3.1 we saw that the equation of motion that governs the radial behaviour of a massive scalar field in AdS is solved in terms of a hypergeometric function,

$$f(\omega, |k|, r) = C_{\omega kl}(1+r^2)^{\omega/2} r^{|k|} {}_2F_1(\hat{\omega}_{kl},\, \hat{\omega}_{kl} - l; |k|+1; -r^2). \tag{3.A.1}$$

The bulk scalar field is then the Fourier transform in ω and k of this function. In section 3.1 we performed the ω integral near the asymptotic boundary, by first obtaining the series expansion of $f(\omega, |k|, r)$ for large r, and then performing a contour integration. The result obtained,

$$g(\omega_{nk}, |k|, r) = \frac{1}{\pi} r^{-\ell-1} \frac{(n+|k|+1)_l (n+l)!}{n! l! (l-1)!} + \dots, \tag{3.A.2}$$

holds only for $r > 1$. To obtain an expression that holds for $r < 0$ one must determine the subleading terms in this expression. The final answer is once this is done is

$$\begin{aligned} g(\omega_{nk}, |k|, r) = \frac{1}{\pi} r^{|k|} (1+r^2)^{-\frac{|k|+l+1}{2}} \frac{(n+1)_l (n+|k|+1)_l}{l!(l-1)!} \\ {}_2F_1\left(n+|k|+l+1, -n; l+1; \frac{1}{1+r^2}\right). \end{aligned} \tag{3.A.3}$$

Here, we will derive this expression using two independent methods.

Extending to the Bulk: Method A

We begin by transforming the original modes, given by equation (3.A.1), to obtain an expression for $f(\omega, |k|, r)$ which is valid for $0 \leq r < \infty$. This requires using the properties of the hypergeometric function [61]. Then, since we know from the asymptotic analysis that the frequency of the modes in the final result is quantised, we set $\omega = \omega_{nk}^{\pm}$. Finally, we determine the normalisation of $f(\omega_{nk}, |k|, r)$ by requiring that its asymptotic value agrees with (3.A.2).

Using the transformation properties of the hypergeometric function, the radial modes of the scalar field (3.A.1) can be rewritten as

$$\widetilde{f}(\omega,|k|,r) = \widetilde{C}_{\omega kl} r^{|k|}\Bigg[(-1)^n \frac{(1+r^2)^{-\frac{|k|+l+1}{2}} k!\Gamma(-l)}{\Gamma(\hat{\omega}_{kl}-l)\Gamma(\hat{\omega}_{kl}-\omega-l)} \\ {}_2F_1\left(\hat{\omega}_{kl},\hat{\omega}_{kl}-\omega;l+1;\frac{1}{1+r^2}\right) + \frac{(1+r^2)^{-\frac{|k|-l+1}{2}} k!\Gamma(l)}{\Gamma(\hat{\omega}_{kl})\Gamma(\hat{\omega}_{kl}-\omega)} \\ {}_2F_1\left(\hat{\omega}_{kl}-l,\hat{\omega}_{kl}-\omega-l;1-l;\frac{1}{1+r^2}\right)\Bigg], \tag{3.A.4}$$

where the normalisation constant $C_{\omega kl}$ has been replaced by $\widetilde{C}_{\omega kl}$, allowing for the possibility of a different value than the one found through the asymptotic analysis. Setting $\omega = \omega^{\pm}_{nk}$ gives

$$\widetilde{f}(\omega^+_{nk},|k|,r) = \widetilde{C}_{\omega kl} r^{|k|}\Bigg[\frac{(-1)^n(1+r^2)^{-\frac{|k|+1+l}{2}} k!(l+n)!}{\Gamma(n+|k|+1)l!} \\ {}_2F_1\left(n+|k|+l+1,-n;l+1;\frac{1}{1+r^2}\right) + \underbrace{\frac{(1+r^2)^{-\frac{|k|-l+1}{2}} k!\Gamma(l)}{\Gamma(n+|k|+l+1)\Gamma(-n)}}_{\substack{\text{Unless } \widetilde{C}_{\omega kl} \text{ introduces infinities} \\ \text{this is zero due to } \Gamma(-n)}} \\ {}_2F_1\left(-n-l,n+|k|+1;1-l;\frac{1}{1+r^2}\right)\Bigg] = \widetilde{f}(\omega^-_{nk},|k|,r). \tag{3.A.5}$$

In the limit $r \to \infty$ this reduces to

$$\widetilde{f}(\omega_n k,|k|,r) \to \widetilde{C}_{\omega kl}\Big[r^{-l-1}\frac{(-1)^n(l+n)!k!}{l!\Gamma(n+|k|+1)} + r^{l-1}\frac{k!\,\Gamma(l)}{\Gamma(n+|k|+l+1)\Gamma(-n)}\Big]. \tag{3.A.6}$$

Comparing to the asymptotic expression for $g(\omega_{nk},|k|,r)$, we obtain $\widetilde{C}_{\omega kl}$, and, thus, $\widetilde{f}(\omega_{nk},|k|,r)$,

$$\widetilde{f}(\omega_{nk},|k|,r) = \frac{1}{\pi} r^{|k|}\Bigg[\frac{(1+r^2)^{-\frac{|k|+l+1}{2}}(l+n)!(n+|k|+l)_l}{n!l!(l-1)!} \\ {}_2F_1\left(n+|k|+l+1,-n;l+1;\frac{1}{1+r^2}\right) + (-1)^n \underbrace{\frac{(1+r^2)^{-\frac{|k|-l+1}{2}}\Gamma(l)}{\Gamma(-n)n!(l-1)!}}_{=0 \text{ due to } \Gamma(-n)} \\ {}_2F_1\left(n+|k|+1,-n-l;1-l;\frac{1}{1+r^2}\right)\Bigg]. \tag{3.A.7}$$

Finally, this yields the desired expression for $g(\omega_{nk}, |k|, r)$, valid in the bulk,

$$\begin{aligned}\widetilde{f}(\omega_{nk}, |k|, r) =& g(\omega_{nk}, |k|, r)\\ =& \frac{1}{\pi} r^{|k|}(1+r^2)^{-\frac{|k|+l+1}{2}} \frac{(n+1)_l (n+|k|+1)_l}{l!(l-1)!}\\ & {}_2F_1\left(n+|k|+l+1, -n; l+1; \frac{1}{1+r^2}\right)\end{aligned} \tag{3.A.8}$$

Extending to the Bulk: Method B

The second method for deriving the full bulk solution involves starting with the same expression for $f(\omega, |k|, r)$ as the one used in method A, namely (3.A.4). This is normalised such that the leading term as $r \to \infty$ is $1 \cdot r^{l-1}$. The new solution, labelled by $\hat{f}(\omega, |k|, r)$, is integrated over ω, to yield the full bulk solution.

As $r \to \infty$, the transformed solution, given by equation (3.A.1), behaves like

$$\begin{aligned}\hat{f}(\omega, |k|, r) \xrightarrow{r\to\infty} & \widehat{C}_{\omega kl} r^{-l-1} \frac{k!\Gamma(-l)}{\Gamma(\hat{\omega}_{kl}-l)\Gamma(\hat{\omega}_{kl}-\omega-l)}\\ & + \widehat{C}_{\omega kl} r^{l-1} \frac{k!\Gamma(l)}{\Gamma(\hat{\omega}_{kl})\Gamma(\hat{\omega}_{kl}-\omega)}.\end{aligned} \tag{3.A.9}$$

Requiring that, in this limit, $\hat{f}(\omega, |k|, r) \sim 1 \cdot r^{l-1}$, allows us to determine the normalisation constant $\widehat{C}_{\omega kl}$,

$$\widehat{C}_{\omega kl} = \frac{\Gamma(\hat{\omega}_{kl}-l)\Gamma(\hat{\omega}_{kl}-\omega-l)}{k!\Gamma(-l)}.$$

Substituting into $\hat{f}(\omega, |k|, r)$, we find

$$\begin{aligned}&\hat{f}(\omega, |k|, l) =\\ &= \overbrace{r^{|k|}(1+r^2)^{-\frac{|k|+l+1}{2}} \frac{\Gamma(-l)\Gamma(\hat{\omega}_{kl})\Gamma(\hat{\omega}_{kl}-\omega)}{\Gamma(l)\Gamma(\hat{\omega}_{kl}-l)\Gamma(\hat{\omega}_{kl}-\omega-l)}\, {}_2F_1\left(\hat{\omega}_{kl}, \hat{\omega}_{kl}-\omega; l+1; \frac{1}{1+r^2}\right)}^{\text{Normalisable fall-off}}\\ &+ \underbrace{r^{|k|}(1+r^2)^{-\frac{|k|+1-l}{2}}\, {}_2F_1\left(\hat{\omega}_{kl}-l, \hat{\omega}_{kl}-\omega-l; 1-l; \frac{1}{1+r^2}\right)}_{\text{Non-normalisable fall-off}}.\end{aligned} \tag{3.A.10}$$

These modes, once integrated over ω, yield $g(\omega_{nk}, |k|, r)$ (see also (3.1.8),

$$g(\omega_{nk}, |k|, r) = \frac{1}{4\pi^2 i} \oint_{\omega_{nk}} d\omega\ \hat{f}(\omega, |k|, r). \tag{3.A.11}$$

To perform the integral we must identify the singularities of this function in the ω plane and use Cauchy's residue theorem. The term which has normalisable fall-off has simple poles at $\hat{\omega}_{kl} = -n$ and at $\hat{\omega}_{kl} - \omega = -n$, $n \in \mathbb{N}_0$. Notice that the poles of this term coincide with the poles of the asymptotic expansion of the solution. Furthermore, notice that after setting $\hat{\omega}_{kl} = -n$ or $\hat{\omega}_{kl} - \omega = -n$, the singular term $\Gamma(-l)$ cancels with the zeros in the denominator coming from $\Gamma(\hat{\omega}_{kl} - l)$ or $\Gamma(\hat{\omega}_{kl} - \omega - l)$, respectively.

The term with non–normalisable fall–off is analytic in ω and by Cauchy's integral theorem, it vanishes when integrated in the complex ω plane.

Performing the integration, we find

$$\begin{aligned}\widetilde{f}(\omega_{nk}, |k|, r) &= g(\omega_{nk}, |k|, r) \\ &= \frac{1}{\pi} r^{|k|} (1+r^2)^{-\frac{|k|+l+1}{2}} \frac{(n+1)_l (n+|k|+1)_l}{l!(l-1)!} \\ &\quad {}_2F_1\left(n+|k|+l+1, -n; l+1; \frac{1}{1+r^2}\right),\end{aligned} \tag{3.A.12}$$

where we used that $\mathrm{Res}\left\{\Gamma(z); z=-n\right\} = \frac{(-1)^n}{n!}$. This is precisely the answer obtained using method A, (3.A.8), as expected.

Chapter 4

Poincaré AdS

In this chapter we will address the same problem, namely the construction of bulk scalar field solutions that are dual to the state $|\Delta\rangle = \mathcal{O}_\Delta|0\rangle$, but for a CFT on $R^{1,1}$. Then the relevant bulk problem is to solve the free field equation for a massive scalar field in Poincaré AdS.

4.1 Lorentzian Solutions

The metric for the Poincaré patch of Lorentzian AdS_{2+1} is given by

$$\mathrm{d}s^2 = \frac{1}{z^2}\left(-\mathrm{d}t^2 + \mathrm{d}z^2 + \mathrm{d}x^2\right) \tag{4.1.1}$$

with the asymptotic boundary at $z = 0$. In this background the Klein-Gordon equation is given by

$$\left(\partial_z^2 - \frac{1}{z}\partial_z - \partial_t^2 + \partial_x^2 - \frac{m^2}{z^2}\right)\Phi(t,z,x) = 0. \tag{4.1.2}$$

Substituting the ansatz

$$\Phi\left(t,z,x\right) = \mathrm{e}^{-i\omega t + ikx} f_{\omega k}(z) \tag{4.1.3}$$

we get

$$f''_{\omega k}(z) - \frac{1}{z} f'_{\omega k}(z) + \left(\omega^2 - k^2 - \frac{m^2}{z^2}\right) f_{\omega k}(z) = 0. \tag{4.1.4}$$

To solve this ODE we need to consider the cases $-\omega^2 + k^2 > 0$ (spacelike modes) and $-\omega^2 + k^2 \leq 0$ (timelike modes).

4.1.1 Timelike Modes

For timelike modes

$$-\omega^2 + k^2 = -q^2 \leq 0. \tag{4.1.5}$$

The two linearly independent solutions to the z–ODE are

$$f_1(z) = zJ_l(qz) \tag{4.1.6a}$$

$$f_2(z) = zY_l(qz) \tag{4.1.6b}$$

where $l = \sqrt{1+m^2} \in \{0,1,2,\dots\}$, $q^2 = \omega^2 - k^2$. The boundary behaviour of these solutions is

$$zJ_l(qz) \xrightarrow[z\to 0]{} z^{1+l}\left(\frac{q^l}{2^l\Gamma(l)} - \dots\right) \qquad \text{normalisable} \tag{4.1.7a}$$

$$zY_l(qz) \xrightarrow[z\to 0]{} z^{1-l}\left(\frac{-2^l\Gamma(l)}{q^l\pi} + \dots + z^{2l}\frac{(-1)^l q^l\Gamma(-l)}{2^l\pi} + \dots\right) \qquad \text{non-normalisable.} \tag{4.1.7b}$$

As $z \to \infty$,

$$zJ_l(qz) \xrightarrow[z\to\infty]{} z^{1/2}\sin\left(\frac{\pi}{4} - \frac{l\pi}{2} + qz\right)\sqrt{\frac{2}{\pi q}} + z^{-1/2}\sin\left(\frac{\pi}{4} + \frac{l\pi}{2} - qz\right)\frac{(4l^2-1)}{4\sqrt{2\pi q^3}} + \dots \tag{4.1.8a}$$

$$zY_l(qz) \xrightarrow[z\to\infty]{} -z^{1/2}\sin\left(\frac{\pi}{4} + \frac{l\pi}{2} - qz\right)\sqrt{\frac{2}{\pi q}} - z^{-1/2}\cos\left(\frac{\pi}{4} + \frac{l\pi}{2} - qz\right)\frac{(4l^2-1)}{4\sqrt{2\pi q^3}} + \dots \tag{4.1.8b}$$

From these expressions we observe that there are no individual timelike modes that remain finite in the bulk. Therefore, any solution that is finite must be constructed by integrating over infinitely many such modes.

4.1.2 Spacelike Modes

For spacelike modes

$$-\omega^2 + k^2 = q^2 \geq 0. \tag{4.1.9}$$

The two linearly independent solutions to the z–ODE become

$$f_1(z) = zI_l\left(q_\delta z\right) \tag{4.1.10a}$$

$$f_2(z) = zK_l\left(q_\delta z\right) \tag{4.1.10b}$$

where l is as defined above and $q_\delta = \left(-\omega^2 + k^2 - \mathrm{i}\delta\right)^{1/2}$, with $\delta > 0$ an infinitesimal parameter. Looking again at the near boundary behaviour of the solutions we find

$$zI_l(qz) \xrightarrow[z\to 0]{} z^{1+l}\left(\frac{q^l}{2^l\Gamma(l)} + \frac{q^{2+l}z^2}{2^{2+l}(1+l)\Gamma(1+l)} + O\left(z^3\right)\right) \qquad \text{normalisable} \tag{4.1.11a}$$

$$zK_l(qz) \xrightarrow[z\to 0]{} z^{1-l}\left(\frac{2^{l-1}\Gamma(l)}{q^l} - \frac{2^{l-3}\Gamma(l)z^2}{q^{l-2}(l-1)} + O\left(z^3\right)\right)$$
$$+z^{1+l}\left(\frac{q^l\Gamma(-l)}{2^{l+1}} + \frac{q^{l+2}z^2\Gamma(-l)}{2^{l+3}(1+l)} + O\left(z^3\right)\right)$$
$$\text{non-normalisable.} \qquad (4.1.12a)$$

As $z \to \infty$,

$$zI_l(qz) \xrightarrow[z\to\infty]{} \frac{z^{1/2}}{\sqrt{2\pi q}}\left[\mathrm{e}^{qz}\left(1 + \mathrm{O}\left(z^{-1}\right)\right) + \mathrm{e}^{-qz}\left(i(-1)^l + \mathrm{O}\left(z^{-1}\right)\right)\right] \qquad (4.1.13)$$

$$zK_l(qz) \xrightarrow[z\to\infty]{} z^{1/2}\mathrm{e}^{-qz}\left[\sqrt{\frac{\pi}{2q}} + \frac{4l^2-1}{8z}\sqrt{\frac{\pi}{2q^3}} + O\left(z^{-2}\right)\right]. \qquad (4.1.14)$$

Here one set of modes, namely the non-normalisable $zK_l(q\,z)$ modes, remain finite at the interior whereas the normalisable ones diverge. Consequently, the only physical spacelike modes are the non-normalisable ones.

We are now in position to construct the Lorentzian solutions using the physical modes we have found. Our choice of boundary conditions for the Lorentzian manifolds dictates that there are no sources present. Accordingly, we construct Lorentzian solutions using only normalisable modes,

$$\Phi_L\left(t,z,x\right) = \int_{-\infty}^{\infty}\frac{\mathrm{d}k}{2\pi}\int_{-\infty}^{\infty}\frac{\mathrm{d}\omega}{2\pi}\left[a_{\omega k}\,\mathrm{e}^{-i\omega t+ikx}z\,\theta\left(\omega^2-k^2\right)J_l\left(\sqrt{\omega^2-k^2}\,z\right) + \text{c.c.}\right]. \qquad (4.1.15)$$

4.2 Euclidean Solutions

The metric for Euclidean Poincaré patch of AdS_{2+1} can be obtained from the Lorentzian one, (4.1.1), by Wick rotating $t = -i\tau$,

$$\mathrm{d}s^2 = \frac{1}{z^2}\left(\mathrm{d}\tau^2 + \mathrm{d}z^2 + \mathrm{d}x^2\right). \qquad (4.2.1)$$

The massive Klein-Gordon in this background is given by

$$\left(\partial_z^2 - \frac{1}{z}\partial_z + \partial_\tau^2 + \partial_x^2 - \frac{m^2}{z^2}\right)\Phi(\tau,z,x) = 0 \qquad (4.2.2)$$

Since we are solving this on half Euclidean manifolds for which $\tau \geq 0$ or $\tau \leq 0$, it is possible to have both oscillatory and exponentially decaying modes in our solutions. We consider both cases separately.

4.2.1 Exponentially Decaying Modes

Substituting the ansatz[1]

$$\Phi\left(\tau, z, x\right) = \mathrm{e}^{\pm|\omega|\tau + ikx} f_{\omega k}(z) \tag{4.2.3}$$

into equation (4.2.2), we obtain the radial ODE,

$$f''_{\omega k}(z) - \frac{1}{z} f'_{\omega k}(z) - \left(\frac{m^2}{z^2} - \omega^2 + k^2\right) f_{\omega k}(z) = 0. \tag{4.2.4}$$

As was the case in Lorentzian signature, we can have both timelike, $\omega^2 - k^2 \geq 0$, and spacelike, $\omega^2 - k^2 \leq 0$, modes. The analysis of the Euclidean case follows along the same lines of the Lorentzian and therefore we will directly state and use the results from above.

In particular, from the Lorentzian case we have that the only physical, exponentially decaying modes are the timelike normalisable modes. These are used to construct the Euclidean normalisable solution. For $\tau \leq 0$ this has the general form

$$\Phi_E^-\left(\tau, z, x\right) = \int_{-\infty}^{\infty} \frac{\mathrm{d}k}{2\pi} \int_0^{\infty} \frac{\mathrm{d}\omega}{2\pi i} \left[d_{\omega k}\, \mathrm{e}^{\omega\tau + ikx} z\, \theta\left(\omega^2 - k^2\right) J_l\left(\sqrt{\omega^2 - k^2}\, z\right)\right], \tag{4.2.5a}$$

and for $\tau \geq 0$

$$\Phi_E^+\left(\tau, z, x\right) = \int_{-\infty}^{\infty} \frac{\mathrm{d}k}{2\pi} \int_0^{\infty} \frac{\mathrm{d}\omega}{2\pi i} \left[\tilde{d}_{\omega k}\, \mathrm{e}^{-\omega\tau + ikx} z\, \theta\left(\omega^2 - k^2\right) J_l\left(\sqrt{\omega^2 - k^2}\, z\right)\right]. \tag{4.2.5b}$$

No physical solutions can be constructed using exponentially decaying non-normalisable modes.

4.2.2 Oscillatory Modes

To study the oscillatory modes of the Euclidean fields we substitute the ansatz

$$\Phi(\tau, z, x) = \mathrm{e}^{i\omega\tau + ikx} f_{\omega k}(z) \tag{4.2.6}$$

into equation (4.2.2). The resulting radial ODE is

$$f''_{\omega k}(z) - \frac{1}{z} f'_{\omega k}(z) - \left(\frac{m^2}{z^2} + \omega^2 + k^2\right) f_{\omega k}(z) = 0. \tag{4.2.7}$$

[1] The choice of sign of the exponential modes depends on whether τ is positive or negative, i.e. whether we are considering the solution in the past or future Euclidean cap.

Defining $p^2 = \omega^2 + k^2 \geq 0$, the solutions to this ODE are

$$f_1(z) = zI_l(p\,z) \tag{4.2.8a}$$

$$f_2(z) = zK_l(p\,z) \tag{4.2.8b}$$

where $p = \left(\omega^2 + k^2\right)^{1/2}$.

$K_l(z)$ has a branch cut along the negative real axis. However, by restricting to $p \geq 0$ we need not worry about this, although there might be subtleties due to $p = 0$. From the asymptotic analysis of $I_l(pz)$and $K_l(pz)$ discussed in the previous section we have that the only physical solution in this case is the one constructed by integrating the modes proportional to $zK_l(p\,z)$. These are non–normalisable, source modes which we normalise such that, as $z \to 0$,

$$C_{\omega k} z K_l(p\,z) = 1 \cdot z^{1-l} + \dots \tag{4.2.9}$$

The resulting modes are convoluted with the modes of a source with a δ–function profile, localised in spacetime on the boundary. We consider a delta function source localised at $\tau = -\epsilon, x = 0$, where $\epsilon > 0$. Then the corresponding bulk solution is given by

$$\Phi^-_E(\tau, z, x) = \frac{z}{\Gamma(l)2^{l-1}} \int_{-\infty}^{\infty} \frac{\mathrm{d}k}{2\pi} \int_{-\infty}^{\infty} \frac{\mathrm{d}\omega}{2\pi} \left[\mathrm{e}^{i\omega\tau + ikx} \phi^-_{(0)}(\omega, k) \left(\omega^2 + k^2\right)^{l/2} K_l\Big(\sqrt{\omega^2 + k^2}\, z\Big)\right]$$

$$\phi^-_{(0)}(\omega, k) = \mathrm{e}^{i\omega\epsilon}. \tag{4.2.10}$$

Indeed, it is easy to see that in the limit $z \to 0$ this is δ–function source localised at $(\tau, x) = (-\epsilon, 0)$. Similarly, for $\tau \geq 0$ and for a source localised at $(\tau, x) = (\epsilon, 0)$, the solution takes the form

$$\Phi^+_E(\tau, z, x) = \frac{z}{\Gamma(l)2^{l-1}} \int_{-\infty}^{\infty} \frac{\mathrm{d}k}{2\pi} \int_{-\infty}^{\infty} \frac{\mathrm{d}\omega}{2\pi} \left[\phi^+_{(0)}(\omega, k)\, \mathrm{e}^{i\omega\tau + ikx} \left(\omega^2 + k^2\right)^{l/2} K_l\Big(\sqrt{\omega^2 + k^2}\, z\Big)\right]$$

$$\phi^+_{(0)}(\omega, k) = \mathrm{e}^{-i\omega\epsilon}. \tag{4.2.11}$$

The final solutions for the two Euclidean caps are linear combinations of normalisable and non–normalisable pieces.

4.3 Matching Conditions

The field theory time contour and bulk manifold that we will considered here are the in–in contour and the corresponding manifold used in the case of global AdS, discussed in the previous chapter and shown in figure 4.3.1. For the sake of completeness we repeat the results for the matching conditions but, for more details, we refer the reader to

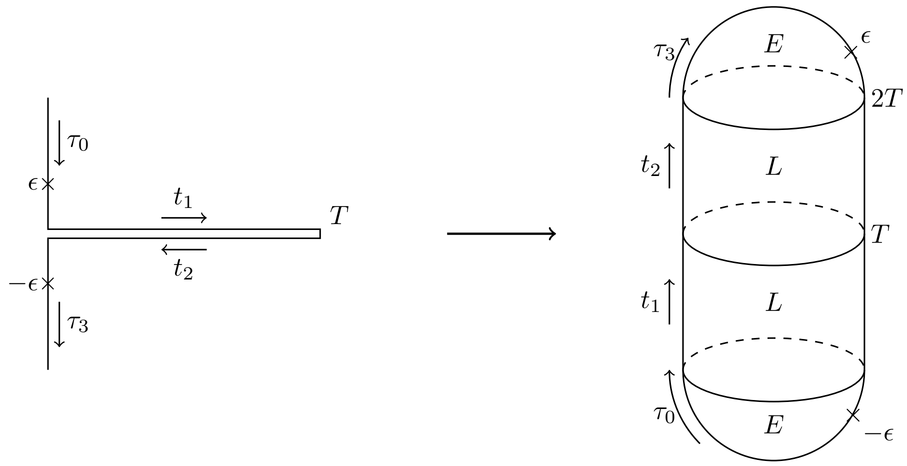


Figure 4.3.1: In–in time contour (left) and corresponding AdS manifold (right).

section 3.1.2. The matching conditions are

$$\begin{aligned} \Phi_E^-\big|_{\tau_0=0} &= \Phi_L^1\big|_{t_1=0}, & \partial_{\tau_0}\Phi_E^-\big|_{\tau_0=0} &= -i\partial_{t_1}\Phi_L^1\big|_{t_1=0} \\ \Phi_L^1\big|_{t_1=T} &= \Phi_L^2\big|_{t_2=T}, & \partial_{t_1}\Phi_L^1\big|_{t_1=T} &= -\partial_{t_2}\Phi_L^2\big|_{t_2=T} \\ \Phi_L^2\big|_{t_2=2T} &= \Phi_E^+\big|_{\tau_3=0}, & \partial_{t_2}\Phi_L^2\big|_{t_2=2T} &= -i\partial_{\tau_3}\Phi_E^+\big|_{\tau_3=0}. \end{aligned} \tag{4.3.1}$$

The solutions in each manifold, which are constructed by appropriate modifications of the general solutions obtained above, are

$$\begin{aligned} &0 \le t_1 \le T: \\ &\Phi_L^1(t_1,z,x) = \int_{-\infty}^{\infty}\frac{\mathrm{d}k}{2\pi}\int_{-\infty}^{\infty}\frac{\mathrm{d}\omega}{2\pi}\Big[a_{\omega k}\,\mathrm{e}^{-i\omega t_1+ikx}z\,\theta\left(\omega^2-k^2\right)J_l\left(\sqrt{\omega^2-k^2}\,z\right) \\ &\qquad\qquad +\text{c.c.}\Big], \end{aligned} \tag{4.3.2a}$$

$$\begin{aligned} &T \le t_2 \le 2T: \\ &\Phi_L^2\left(t_2,z,x\right) = \int_{-\infty}^{\infty}\frac{\mathrm{d}k}{2\pi}\int_{-\infty}^{\infty}\frac{\mathrm{d}\omega}{2\pi}\Big[\tilde{a}_{\omega k}\,\mathrm{e}^{-i\omega t_2+ikx}z\,\theta\left(\omega^2-k^2\right)J_l\left(\sqrt{\omega^2-k^2}\,z\right) \\ &\qquad\qquad +\text{c.c.}\Big], \end{aligned} \tag{4.3.2b}$$

for the two Lorentzian segments, and

$$\begin{aligned} &-\infty < \tau_0 \le 0: \\ &\Phi_E^-\left(\tau_0,z,x\right) = \\ &= \frac{z}{\Gamma(l)2^{l-1}}\int_{-\infty}^{\infty}\frac{\mathrm{d}k}{2\pi}\int_{-\infty}^{\infty}\frac{\mathrm{d}\omega}{2\pi}\Big[\mathrm{e}^{i\omega(\tau_0+\epsilon)+ikx}\left(\omega^2+k^2\right)^{l/2}K_l\Big(\sqrt{\omega^2+k^2}\,z\Big) \\ &\quad+\int_{-\infty}^{\infty}\frac{\mathrm{d}k}{2\pi}\int_{0}^{\infty}\frac{\mathrm{d}\omega}{2\pi i}\,d_{\omega k}\,\mathrm{e}^{\omega\tau_0+ikx}z\,\theta\left(\omega^2-k^2\right)J_l\left(\sqrt{\omega^2-k^2}\,z\right)\Big], \end{aligned} \tag{4.3.3a}$$

$$
\begin{aligned}
&0 \leq \tau_3 < \infty : \\
&\Phi_E^+(\tau_3, z, x) = \\
&= \frac{z}{\Gamma(l) 2^{l-1}} \int_{-\infty}^{\infty} \frac{\mathrm{d}k}{2\pi} \int_{-\infty}^{\infty} \frac{\mathrm{d}\omega}{2\pi} \Big[\mathrm{e}^{i\omega(\tau_3 - \epsilon) + ikx} \left(\omega^2 + k^2\right)^{l/2} K_l\Big(\sqrt{\omega^2 + k^2}\, z\Big) \\
&\quad + \int_{-\infty}^{\infty} \frac{dk}{2\pi} \int_0^{\infty} \frac{d\omega}{2\pi i} \tilde{d}_{\omega k}\, \mathrm{e}^{-\omega\tau_3 + ikx} z\, \theta\left(\omega^2 - k^2\right) J_l\left(\sqrt{\omega^2 - k^2}\, z\right)\Big], \qquad (4.3.3b)
\end{aligned}
$$

for the two Euclidean segments. The Lorentzian solutions are purely normalisable whereas the Euclidean solutions are linear combinations of a non–normalisable piece and a normalisable piece. In momentum space we saw that the individual modes are either Bessel functions of the first kind, J_l, or modified Bessel functions of the second kind, K_l. These functions are not orthogonal to each other. We circumvent this complication by making use of the following two integrals of Bessel functions [62]

$$\int_0^{\infty} dz\; z J_n(za) J_n(zb) = \frac{1}{a}\delta(b-a), \qquad a, b \in \mathbb{R} \qquad (4.3.4a)$$

$$\int_0^{\infty} dz\; z K_\nu(za) J_\nu(zb) = \frac{b^\nu}{a^\nu (a^2 + b^2)}, \qquad \mathrm{Re}(a) > 0,\; b > 0. \qquad (4.3.4b)$$

To extract individual modes from our solutions we perform the following steps. Given a field $\Phi(t, z, x)$ or its time derivative $\partial_t \Phi(t, z, x)$, where t here can be either real or imaginary time, we multiply by $\theta\left(\omega^2 - k^2\right) J_l\left(\sqrt{\omega^2 - k^2}\, z\right) \mathrm{e}^{-ikx}$ and integrate first over x from $-\infty$ to $+\infty$ and then over z from zero to $+\infty$,

$$\int_0^{\infty} \mathrm{d}z\, \theta(\omega^2 - k^2) J_l\left(\sqrt{\omega^2 - k^2}\, z\right) \int_{-\infty}^{\infty} dx\, \mathrm{e}^{-ikx} \Phi(t, z, x)\big|_{\text{on matching surface}}. \qquad (4.3.5)$$

To perform the z integral one needs to use either equation (4.3.4a) or (4.3.4b). The Heaviside step function is to ensure that the conditions associated with these two equations are satisfied. Some of the details of this calculation are given in appendix 4.A.

Applying the matching conditions to these solutions and using the above prescription to extract individual modes we finally obtain the following relations which hold for $\omega^2 > k^2$. Note that normalisable modes exist only for $\omega^2 > k^2$ so the above matching conditions are sufficient for our purposes.

From the matching conditions at $\tau_0 = 0,\ t_1 = 0$, between the past Euclidean cap and the first Lorentzian manifold, we obtain

$$
\begin{aligned}
a_{|\omega| k} + a^\dagger_{-|\omega|\, -k} &= \frac{\left(\omega^2 - k^2\right)^{l/2} \pi}{\Gamma(l) 2^{l-1}} \mathrm{e}^{-|\omega|\epsilon} \\
&= \frac{\left(\omega^2 - k^2\right)^{l/2} \pi}{\Gamma(l) 2^{l-1}} \phi^-_{(0)}(i|\omega|, k) \qquad (4.3.6a) \\
a_{-|\omega| k} + a^\dagger_{|\omega|\, -k} &= -i d_{|\omega| k}. \qquad (4.3.6b)
\end{aligned}
$$

From the matching conditions at $t_1 = T$, $t_2 = T$, between the two Lorentzian manifolds,

$$a_{|\omega|k} + a^{\dagger}_{-|\omega|\,-k} = \left(\tilde{a}_{-|\omega|k} + \tilde{a}^{\dagger}_{|\omega|-k}\right) \mathrm{e}^{2i|\omega|T} \tag{4.3.7a}$$

$$a_{-|\omega|k} + a^{\dagger}_{|\omega|-k} = \left(\tilde{a}_{|\omega|k} + \tilde{a}^{\dagger}_{-|\omega|-k}\right) \mathrm{e}^{-2i|\omega|T} \tag{4.3.7b}$$

Finally, the matching conditions at $t_2 = 2T$, $\tau_3 = 0$, between the second Lorentzian manifold and the future Euclidean cap give

$$\begin{aligned} \tilde{a}_{|\omega|k} + \tilde{a}^{\dagger}_{-|\omega|\,-k} &= \frac{\left(\omega^2 - k^2\right)^{l/2} \pi}{\Gamma(l) 2^{l-1}} \mathrm{e}^{-|\omega|(\epsilon - 2iT)} \\ &= \frac{\left(\omega^2 - k^2\right)^{l/2} \pi}{\Gamma(l) 2^{l-1}} \mathrm{e}^{2i|\omega|T} \phi^{+}_{(0)}(-i|\omega|, k) \end{aligned} \tag{4.3.8a}$$

$$\tilde{a}_{-|\omega|k} + \tilde{a}^{\dagger}_{|\omega|\,-k} = -i\tilde{d}_{|\omega|k} \mathrm{e}^{-2i|\omega|T}. \tag{4.3.8b}$$

Given the matching relations it is easier to redefine the Lorentzian coefficients by introducing $b_{\omega k} = a_{|\omega|k} + a^{\dagger}_{-|\omega|-k}$ and $b^{\dagger}_{\omega -k} = a_{-|\omega|k} + a^{\dagger}_{|\omega|-k}$ for the first Lorentzian manifold and $\tilde{b}_{\omega k} = \tilde{a}_{|\omega|k} + \tilde{a}^{\dagger}_{-|\omega|-k}$ and $\tilde{b}^{\dagger}_{\omega -k} = \tilde{a}_{-|\omega|k} + \tilde{a}^{\dagger}_{|\omega|-k}$ for the second Lorentzian manifold. In terms of these new coefficients the solutions become

$$\begin{aligned} \Phi^1_L(t_1, z, x) = \int_{-\infty}^{\infty} \frac{\mathrm{d}k}{2\pi} \int_0^{\infty} \frac{\mathrm{d}\omega}{2\pi} \Bigg[& \left(b_{\omega k}\, \mathrm{e}^{-i\omega t_1 + ikx} + b^{\dagger}_{\omega -k} \mathrm{e}^{i\omega t_1 + ikx}\right) \\ & z\, \theta\left(\omega^2 - k^2\right) J_l\left(\sqrt{\omega^2 - k^2}\, z\right)\Bigg], \end{aligned} \tag{4.3.9}$$

with an analogous expression for $\Phi^2_L(t_2, z, x)$.

Re–expressing the matching conditions in terms of b's and $\tilde{b}$'s,

$$b_{\omega k} = \frac{\left(\omega^2 - k^2\right)^{l/2} \pi}{\Gamma(l) 2^{l-1}} \phi^{-}_{(0)}(i\omega, k) = \tilde{b}^{\dagger}_{\omega -k} \mathrm{e}^{2i\omega T} = -i\tilde{d}_{\omega k} \tag{4.3.10a}$$

$$b^{\dagger}_{\omega -k} = -id_{\omega k} = \tilde{b}_{\omega k} \mathrm{e}^{-2i\omega T} = \frac{\left(\omega^2 - k^2\right)^{l/2} \pi}{\Gamma(l) 2^{l-1}} \phi^{+}_{(0)}(-i\omega, k) \tag{4.3.10b}$$

where the frequency ω is greater or equal to zero. Note that had we not chosen the source insertion points in the past and future Euclidean caps to be the same, reality conditions for the Lorentzian solutions would dictate that they have to be the same.

Identifying the coefficients of $\mathrm{e}^{-i\omega t}$ ($\mathrm{e}^{-\omega\tau}$) as the positive frequency oscillatory (exponential) modes and the coefficients of $\mathrm{e}^{+i\omega t}$ as the negative ones, we see that our modes evolve in an analogous way as we saw in the global case. In particular, the positive frequency normalisable modes in the first Lorentzian manifold are sourced by exponentially decaying positive frequency source modes in the past Euclidean manifold whereas the positive frequency source modes decay. The positive frequency Lorentzian modes from the first manifold then evolve across the matching surface at $t_1 = T = t_2$ to become negative frequency modes in the second Lorentzian manifold and finally they become negative frequency normalisable modes in the future Euclidean manifold. There are no positive frequency normalisable modes in the future manifold as these grow exponentially as $\tau_3 \to \infty$.

The negative frequency normalisable modes in the first Lorentzian manifold are the evolution of positive frequency normalisable modes which we have included in the past Euclidean manifold. As they evolve across the matching surface into the second Lorentzian manifold they become the positive frequency normalisable modes which are associated to negative frequency source modes turned on in the future Euclidean manifold.

Returning to the Lorentzian fields, we can now replace the arbitrary coefficients $b_{\omega k}$ and $\tilde{b}_{\omega k}$ with the above results to obtain

$$\Phi^1_L(t_1, z, x) = \frac{z}{\Gamma(l)2^l}\int_{-\infty}^{\infty}\frac{\mathrm{d}k}{2\pi}\int_0^{\infty}\mathrm{d}\omega\Big[\Big(\phi^-_{(0)}(i\omega, k)\mathrm{e}^{-i\omega t_1} + \phi^+_{(0)}(-i\omega, k)\mathrm{e}^{i\omega t_1}\Big) \\ \mathrm{e}^{ikx}\left(\omega^2 - k^2\right)^{l/2}\theta\left(\omega^2 - k^2\right)J_l\left(\sqrt{\omega^2 - k^2}\, z\right)\Big], \tag{4.3.11a}$$

$$\Phi^2_L(t_2, z, x) = \frac{z}{\Gamma(l)2^l}\int_{-\infty}^{\infty}\frac{\mathrm{d}k}{2\pi}\int_0^{\infty}\mathrm{d}\omega\Big[\Big(\phi^+_{(0)}(-i\omega, k)\mathrm{e}^{-i\omega t_2} + \phi^-_{(0)}(i\omega, k)\mathrm{e}^{i\omega t_2}\Big) \\ \mathrm{e}^{ikx}\left(\omega^2 - k^2\right)^{l/2}\theta\left(\omega^2 - k^2\right)J_l\left(\sqrt{\omega^2 - k^2}\, z\right)\Big]. \tag{4.3.11b}$$

Equations (4.3.11a) and (4.3.11b) demonstrate explicitly how the Euclidean source modes generate the purely normalisable solutions in the Lorentzian bulk.

4.3.1 1-point function

We will now extract the 1-point function to verify that the solution indeed describes an excited state. For this we need to extract the coefficient $\phi_{(2\Delta-2)}$, which in our case is the leading order coefficient of the bulk solution. As in the case of global AdS, we consider the case where the operator is in the upper part of the contour so the relevant field is Φ^1_L. Then

$$\phi_{(2\Delta-2)}(t, x) = \lim_{z\to 0} z^{\Delta}\Phi^1_L(z, t, x) = \frac{1}{2^{2l-1}\Gamma(l)\Gamma(l+1)} \\ \int_{-\infty}^{\infty}\frac{\mathrm{d}k}{2\pi}\int_0^{\infty}\mathrm{d}\omega\left[\theta\big(\omega^2 - k^2\big)\left(\omega^2 - k^2\right)^l \mathrm{e}^{-\omega\epsilon + ikx}\cos(\omega t)\right] \tag{4.3.12}$$

Eliminating first the Heaviside step function and setting $\omega = rk$, we obtain

$$\phi_{(2\Delta-2)}(t, x) = \frac{1}{2^{2l-1}\Gamma(l)\Gamma(l+1)}\int_0^{\infty}\frac{\mathrm{d}k}{2\pi}\int_1^{\infty}\mathrm{d}r\Big[k^{2l+1}\left(r^2 - 1\right)^l \mathrm{e}^{-kr\epsilon} \\ \big(\cos\big(k(rt + x)\big) + \cos\big(k(rt - x)\big)\big)\Big]. \tag{4.3.13}$$

Then we perform the k integral,

$$\phi_{(2\Delta-2)}(t,x) = \frac{(-1)^{\Delta}\Gamma(2\Delta)}{2^{2\Delta-1}\pi\Gamma(\Delta-1)\Gamma(\Delta)}\int_1^{\infty}\mathrm{d}r\left[(r(t+i\epsilon)-x)^{-2\Delta}+\right. \tag{4.3.14}$$
$$\left.(r(t+i\epsilon)+x)^{-2\Delta}+(r(t_1-i\epsilon)+x)^{-2\Delta}+(r(t-i\epsilon)-x)^{-2\Delta}\right]\left(r^2-1\right)^{\Delta-1},$$

and finally, we compute the r integral,

$$\phi_{(2\Delta-2)}(t,x) = -\frac{l}{\pi}\left(\frac{1}{(-(t-i\epsilon)^2+x^2)^{\Delta}}+\frac{1}{(-(t+i\epsilon)^2+x^2)^{\Delta}}\right) \tag{4.3.15}$$

and thus,

$$\langle\mathcal{O}_{\Delta}(t,x)\rangle_{\text{exc}} = \frac{2l^2}{\pi}\left(\frac{1}{(-(t-i\epsilon)^2+x^2)^{\Delta}}+\frac{1}{(-(t+i\epsilon)^2+x^2)^{\Delta}}\right) \tag{4.3.16}$$

where we used the subscript "exc" to emphasise that this is the 1–point function in the excited state. This is indeed equal to value we got via a QFT computation in (2.2.7). In our case, $\tilde{C} = 2l^2/\pi$, which is the standard supergravity normalisation of the 2-point function. Note also that the normalisations in (3.1.34) and (4.3.16) are related as in the footnote 4, as they should.

4.3.2 Reconstruction of Bulk Fields From Boundary Data

In this section we perform the reconstruction of the Lorentzian bulk fields living in the Poincaré patch of AdS, from their asymptotic value. The procedure follows closely what was done for global AdS in section 3.2. However, there are some additional complications and subtleties related to the Poincaré patch which are addressed below.

We begin by extracting the source modes from the boundary Euclidean fields. For Poincaré coordinates, the definition of the boundary fields used for global AdS in section 3.2, becomes

$$\Psi(t,x) = \lim_{z\to 0} z^{-l-1}\Phi(t,z,x). \tag{4.3.17}$$

However, this definition can not be used for the Euclidean fields in the Poincaré patch because the z dependence of the source term is z^{1-l} and, hence, with this definition it would blow up. To avoid this complication we focus only on the normalisable part of the Euclidean solution, ignoring the source piece. This is justified by the fact that, near the boundary, the source term is a delta function, localised in time. Thus, $\tau_0 \neq -\epsilon$ and $\tau_3 \neq \epsilon$, the source terms are zero. Applying this to the Euclidean fields, given by equations (4.3.3a) and (4.3.3b) we find,

$$\begin{aligned}\Psi_E^-(\tau_0,x) &= \frac{1}{2^{2l}\Gamma(l)\Gamma(l+1)}\int_{-\infty}^{\infty}\frac{\mathrm{d}k}{2\pi}\int_0^{\infty}\mathrm{d}\omega\,\Big[\mathrm{e}^{\omega\tau_0+ikx}\phi^+_{(0)}(-i\omega,k)\\ &\quad \left(\omega^2-k^2\right)^l\theta\left(\omega^2-k^2\right)\Big], \qquad (4.3.18a)\\ \Psi_E^+(\tau_3,x) &= \frac{1}{2^{2l}\Gamma(l)\Gamma(l+1)}\int_{-\infty}^{\infty}\frac{\mathrm{d}k}{2\pi}\int_0^{\infty}\mathrm{d}\omega\,\Big[\mathrm{e}^{-\omega\tau_3+ikx}\phi^-_{(0)}(i\omega,k)\\ &\quad \left(\omega^2-k^2\right)^l\theta\left(\omega^2-k^2\right)J_l\left(\sqrt{\omega^2-k^2}\,z\right)\Big] \qquad (4.3.18b)\end{aligned}$$

where $\phi^+_{(0)}(-i\omega,k)=\mathrm{e}^{-\omega\epsilon}=\phi^-_{(0)}(i\omega,k)$. In deriving these we have used that

$$J_l(qz)\xrightarrow[z\to 0]{} z^l\frac{q^l}{2^l\Gamma(l+1)}. \qquad (4.3.19)$$

Similarly, the boundary Lorentzian fields are

$$\begin{aligned}\Psi_L^1(t_1,x) &= \frac{1}{2^{2l}\Gamma(l)\Gamma(l+1)}\int_{-\infty}^{\infty}\frac{\mathrm{d}k}{2\pi}\int_0^{\infty}\mathrm{d}\omega\bigg[\Big(\phi^-_{(0)}(i\omega,k)\mathrm{e}^{-i\omega t_1}\\ &\quad +\phi^+_{(0)}(-i\omega,k)\mathrm{e}^{i\omega t_1}\Big)\mathrm{e}^{ikx}\left(\omega^2-k^2\right)^l\theta\left(\omega^2-k^2\right)\bigg],\\ \Psi_L^2(t_2,x) &= \frac{1}{2^{2l}\Gamma(l)\Gamma(l+1)}\int_{-\infty}^{\infty}\frac{\mathrm{d}k}{2\pi}\int_0^{\infty}\mathrm{d}\omega\bigg[\Big(\phi^+_{(0)}(-i\omega,k)\mathrm{e}^{-i\omega t_2}\\ &\quad +\phi^-_{(0)}(i\omega,k)\mathrm{e}^{i\omega t_2}\Big)\mathrm{e}^{ikx}\left(\omega^2-k^2\right)^l\theta\left(\omega^2-k^2\right)\bigg]. \qquad (4.3.20a)\end{aligned}$$

where $0\le t_1\le T$ and $T\le t_2\le 2T$.

Next we extract the source modes, $\phi^{\pm}_{(0)}(\omega,k)$, from the Euclidean boundary fields. We will demonstrate in detail how to extract $\phi^+_{(0)}$ from $\Psi^-_E(\tau_0,x)$. This is sufficient for the reconstruction as $\phi^-_{(0)}(\omega,k)=\phi^+_{(0)}(-\omega,k)$. The first step is to multiply the expression for $\Psi^-_E(\tau_0,x)$ by e^{-ikx} and integrate over x, from $-\infty$ to $+\infty$. This yields,

$$\begin{aligned}&\int_{-\infty}^{\infty}dx\,\mathrm{e}^{-ikx}\Psi^-_E(\tau_0,x)=\\ &=\frac{1}{2^{2l}\Gamma(l)\Gamma(l+1)}\int_0^{\infty}d\omega\,\left(\omega^2-k^2\right)^l\phi^+_{(0)}(-i\omega,k)\,\mathrm{e}^{\omega\tau_0}\,\theta\left(\omega^2-k^2\right).\end{aligned}$$

This expression is multiplied by $\mathrm{e}^{-\omega\tau_0}$ and integrated over τ_0, from $-\delta-i\infty$ to $-\delta+i\infty$, where $0\le\delta<\epsilon$. Some explanation is needed with regards to the range of integration of τ_0. This comes from the theory of Laplace transforms. In particular, the Laplace transform, $F(\tau)$, of a function $f(\omega)$ is defined by the integral

$$F(\tau)=\int_0^{\infty}d\omega\,f(\omega)\mathrm{e}^{\omega\tau}, \qquad \mathrm{Re}(\tau)<0$$

If f is summable over all finite intervals, and there is a constant c for which

$$\int_0^\infty d\omega\, |f(\omega)| \mathrm{e}^{-c\omega} < \infty, \tag{4.3.21}$$

then the Laplace transform exists when $\tau = \sigma + it$ is such that $\sigma \geq c$. For the inversion, $F(\tau)$ must be of $\mathcal{O}(\tau^{-k}),\ k > 1$. Then

$$f(\omega) = \frac{1}{2\pi i}\int_{\gamma - i\infty}^{\gamma + i\infty} d\tau\ F(\tau)\mathrm{e}^{-\omega\tau} \tag{4.3.21}$$

where γ must be to the right of the singularities of $F(\tau)$.

Returning to the case we are considering,

$$f(\omega) = \frac{\left(\omega^2 - k^2\right)^l \theta\left(\omega^2 - k^2\right)}{2^{2l}\Gamma(l)\Gamma(l+1)} \phi^{+}_{(0)}(-i\omega, k) \tag{4.3.22}$$

and

$$F(\tau_0) = \int_0^\infty d\omega \left\{ \frac{\left(\omega^2 - k^2\right)^l \theta\left(\omega^2 - k^2\right)}{2^{2l}\Gamma(l)\Gamma(l+1)} \phi^{+}_{(0)}(-i\omega, k) \right\} \mathrm{e}^{\omega\tau_0} \tag{4.3.22}$$

which is well defined for $\mathrm{Re}(\tau_0) < 0$.

Accordingly,

$$\begin{aligned}\int_{-\delta - i\infty}^{-\delta + i\infty} d\tau_0 \int_{-\infty}^{\infty} dx\, \mathrm{e}^{-\omega\tau_0 - ikx}\Psi^{-}_{E}(\tau_0, x) &= \frac{1}{2^{2l}\Gamma(l)\Gamma(l+1)}\int_{-\delta - i\infty}^{-\delta + i\infty} d\tau_0 \int_0^\infty d\omega' \\ &\quad \left(\omega'^2 - k^2\right)^l \phi^{+}_{(0)}(-i\omega', k)\, \mathrm{e}^{(\omega' - \omega)\tau_0}\, \theta\big(\omega'^2 - k^2\big) \\ &= \frac{i\pi \left(\omega^2 - k^2\right)^l \phi^{+}_{(0)}(-i\omega, k)\, \theta\big(\omega^2 - k^2\big)}{2^{2l-1}\Gamma(l)\Gamma(l+1)}.\end{aligned} \tag{4.3.23}$$

This is relation gives $\phi^{+}_{(0)}(-i\omega, k)$ in terms of the past Euclidean boundary field $\Psi^{-}_{E}(\tau_0, x)$. Similarly, we can extract $\phi^{-}_{(0)}(i\omega, k)$ from $\Psi^{+}_{E}(\tau_3, x)$. The final expressions are

$$\begin{aligned}\frac{\left(\omega^2 - k^2\right)^l \theta\big(\omega^2 - k^2\big)}{2^{2l}\Gamma(l)\Gamma(l+1)}\phi^{+}_{(0)}(-i\omega, k) &= \\ &= \frac{1}{2\pi i}\int_{-\delta - i\infty}^{-\delta + i\infty} \mathrm{d}\tau_0 \int_{-\infty}^{\infty} dx\, \mathrm{e}^{-\omega\tau_0 - ikx}\Psi^{-}_{E}(\tau_0, x),\end{aligned} \tag{4.3.24a}$$

$$\begin{aligned}\frac{\left(\omega^2 - k^2\right)^l \theta\big(\omega^2 - k^2\big)}{2^{2l}\Gamma(l)\Gamma(l+1)}\phi^{-}_{(0)}(i\omega, k) &= \\ &= \frac{1}{2\pi i}\int_{\delta - i\infty}^{\delta + i\infty} d\tau_3 \int_{-\infty}^{\infty} dx\, \mathrm{e}^{\omega\tau_3 - ikx}\Psi^{+}_{E}(\tau_3, x).\end{aligned} \tag{4.3.24b}$$

The above equations give the source modes in terms of the Euclidean boundary fields. The matching conditions derived in section 4.3, allow us to re-write these relations in terms of Lorentzian boundary fields. More specifically, starting from equation (4.3.24a) we can write

$$\frac{\left(\omega^2-k^2\right)^l\,\theta\left(\omega^2-k^2\right)}{2^{2l}\Gamma(l)\Gamma(l+1)}\phi^+_{(0)}(-i\omega,k)=\frac{1}{2^{2l}\Gamma(l)\Gamma(l+1)}\int_{-\delta-i\infty}^{-\delta+i\infty}\mathrm{d}\tau_0$$
$$\int_{-\infty}^{\infty}\mathrm{d}x\,\mathrm{e}^{-\omega\tau_0-ikx}\int_{-\infty}^{\infty}\frac{dk'}{2\pi}\int_0^{\infty}\frac{\mathrm{d}\omega'}{2\pi i}\left(\omega'^2-k'^2\right)^l\theta\left(\omega'^2-k'^2\right)$$
$$\left[\phi^+_{(0)}(-i\omega',k')\mathrm{e}^{\omega'\tau_0+ik'x}\underbrace{+\phi^-_{(0)}(i\omega',k')\mathrm{e}^{-\omega'\tau_0+ik'x}}_{\substack{\text{integrating this over }\tau_0\\ \text{gives }\delta(\omega'+\omega)\text{ which vanishes}\\ \text{for }\omega',\omega>0\Rightarrow\text{ can add them freely}}}\right]$$

Next perform the following changes of variables:

1. $\tau_0\to s=-i\tau_0$
2. $s\to t=s-i\epsilon$.

This give,

$$\frac{\left(\omega^2-k^2\right)^l\,\theta\left(\omega^2-k^2\right)}{2^{2l}\Gamma(l)\Gamma(l+1)}\phi^+_{(0)}(-i\omega,k)=\int_{-\infty}^{+\infty}\mathrm{d}t\int_{-\infty}^{\infty}\mathrm{d}x\,\mathrm{e}^{-i\omega(t+i\delta)-ikx}$$
$$\left\{\frac{1}{2^{2l}\Gamma(l)\Gamma(l+1)}\int_{-\infty}^{\infty}\frac{dk'}{2\pi}\int_0^{\infty}\frac{d\omega'}{2\pi}\left(\omega'^2-k'^2\right)^l\theta\left(\omega'^2-k'^2\right)\mathrm{e}^{ik'x}\right.$$
$$\left.\left[\phi^+_{(0)}(-i\omega',k')\mathrm{e}^{i\omega'(t+i\delta)}+\phi^-_{(0)}(i\omega',k')\mathrm{e}^{-i\omega'(t+i\delta)}\right]\right\}$$
$$=\frac{1}{2\pi}\int_{-\infty}^{+\infty}dt\int_{-\infty}^{\infty}dx\,\mathrm{e}^{-i\omega(t+i\delta)-ikx}\Psi^1_L(t+i\delta,x)$$

Hence, the final result is

$$\frac{\left(\omega^2-k^2\right)^l\,\theta\left(\omega^2-k^2\right)}{2^{2l}\Gamma(l)\Gamma(l+1)}\phi^+_{(0)}(-i\omega,k)=$$
$$=\frac{1}{2\pi}\int_{-\infty}^{+\infty}dt\int_{-\infty}^{\infty}dx\,\mathrm{e}^{-i\omega(t+i\delta)-ikx}\Psi^1_L(t+i\delta,x). \tag{4.3.25}$$

Similarly, one finds that

$$\frac{\left(\omega^2-k^2\right)^l\,\theta\left(\omega^2-k^2\right)}{2^{2l}\Gamma(l)\Gamma(l+1)}\phi^-_{(0)}(i\omega,k)=$$
$$=\frac{1}{2\pi}\int_{-\infty}^{+\infty}dt\int_{-\infty}^{\infty}dx\,\mathrm{e}^{i\omega(t-i\delta)-ikx}\Psi^1_L(t-i\delta,x). \tag{4.3.26}$$

Having obtained these expressions, we now have all the necessary tools to express the bulk Lorentzian fields in terms of their boundary values. Starting from equation (4.3.11a) for the Lorentzian bulk field and using the above results, we have

$$\begin{aligned}\Phi^1_L(t_1,z,x) =& \, 2^l z\Gamma(l+1)\int_{-\infty}^{\infty}\frac{dk}{2\pi}\int_0^{\infty}\frac{d\omega}{2\pi}\,\left(\omega^2-k^2\right)^{-l/2}\theta\left(\omega^2-k^2\right)\mathrm{e}^{ikx}\\ & J_l\left(\sqrt{\omega^2-k^2}\,z\right)\left\{\mathrm{e}^{-i\omega t_1}\int_{-\infty}^{+\infty}dt\int_{-\infty}^{\infty}dy\mathrm{e}^{-i\omega(t+i\delta)-iky}\Psi^1_L(t+i\delta,y)\right.\\ & \left.+\mathrm{e}^{i\omega t_1}\int_{-\infty}^{+\infty}dt\int_{-\infty}^{\infty}dy\,\mathrm{e}^{i\omega(t-i\delta)-iky}\Psi^1_L(t-i\delta,y)\right\}\\ =& \int_{-\infty}^{+\infty}dt\int_{-\infty}^{\infty}dy\left\{2^l z\Gamma(l+1)\int_{-\infty}^{\infty}\frac{dk}{2\pi}\int_0^{\infty}\frac{d\omega}{2\pi}\,\theta\big(\omega^2-k^2\big)\right.\\ & \left(\omega^2-k^2\right)^{-l/2}J_l\left(\sqrt{\omega^2-k^2}\,z\right)\Big(\mathrm{e}^{-i\omega(t+t_1+i\delta)+ik(x-y)}\Psi^1_L(t+i\delta,y)\\ & \left.+\mathrm{e}^{i\omega(t-t_1-i\delta)+ik(x-y)}\Psi^1_L(t-i\delta,y)\Big)\right\}.\end{aligned}$$

Assuming now that we can take the limit δ and δ goes to zero in Ψ^1_L without any complications[2], we can write

$$\begin{aligned}\Phi^1_L(t_1,z,x) =& \int_{-\infty}^{+\infty}dt\int_{-\infty}^{\infty}dy\left\{2^l z\Gamma(l+1)\int_{-\infty}^{\infty}\frac{dk}{2\pi}\int_0^{\infty}\frac{d\omega}{2\pi}\,\theta\big(\omega^2-k^2\big)\right.\\ & \left(\omega^2-k^2\right)^{-l/2}J_l\left(\sqrt{\omega^2-k^2}\,z\right)\\ & \left.\left(\mathrm{e}^{-i\omega(t+t_1+i\epsilon)+ik(x-y)}+\mathrm{e}^{i\omega(t-t_1-i\epsilon)+ik(x-y)}\right)\right\}\Psi^1_L(t,y).\end{aligned}$$

Then, using the fact that $\Psi^1_L(t,y)=\Psi^1_L(-t,y)$,

$$\begin{aligned}\Phi^1_L(t_1,z,x) =& \int_{-\infty}^{+\infty}dt\int_{-\infty}^{\infty}dy\left\{2^{l+1}z\Gamma(l+1)\int_{-\infty}^{\infty}\frac{dk}{2\pi}\int_0^{\infty}\frac{d\omega}{2\pi}\,\theta\big(\omega^2-k^2\big)\right.\\ & \left.\left(\omega^2-k^2\right)^{-l/2}J_l\left(\sqrt{\omega^2-k^2}\,z\right)\mathrm{e}^{-i\omega(t_1-t+i\epsilon)+ik(x-y)}\right\}\Psi^1_L(t,y).\end{aligned}$$

Comparing this expression to equation (3.2.2) which defines the smearing function we have

$$K(t_1,z,x|t,y)=\frac{2^{l-1}\Gamma(l+1)z}{\pi^2}\int_{-\infty}^{\infty}dk\int_0^{\infty}d\omega\,\theta\big(\omega^2-k^2\big)\big(\omega^2-k^2\big)^{-l/2}J_l\left(\sqrt{\omega^2-k^2}\,z\right)\times \\ \times\mathrm{e}^{-i\omega(t_1-t+i\epsilon)-ik(y-x)}. \tag{4.3.27}$$

To perform the integration we follow [40] and set $t_1=x=y=0$ and define $\omega^+=\frac{1}{2}(\omega+k)=r\mathrm{e}^{\xi}$ and $\omega^-=\frac{1}{2}(\omega-k)=r\mathrm{e}^{-\xi}$.

[2]The $i\delta$ insertions are needed for convergence of the integral over frequency. Ψ^1_L is a position space expression which is indeed finite.

$$\begin{aligned} K(0,0,z|t,0) &= \frac{2^{l+1}\Gamma(l+1)z}{\pi^2}\int_0^\infty rdr\int_{-\infty}^\infty d\xi\,(2r)^{-l}J_l(2rz)\,\mathrm{e}^{2ir\cosh\xi\,(t-i\epsilon)} \\ &= \frac{2^{l-1}\Gamma(l+1)z}{\pi^2}\int_0^\infty dr\,r^{1-l}K_0\big(-ir(t-i\epsilon)\big)J_l(rz) \\ &= -\frac{z^{l+1}}{\pi^2(t-i\epsilon)^2}\,{}_2F_1\left(1,1;l+1;\frac{z^2}{(t-i\epsilon)^2}\right) \end{aligned}$$

where we used the definition of the modified Bessel function of the second kind,

$$K_\nu(z) = \frac{z^\nu\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\nu+\frac{1}{2}\right)}\int_0^\infty d\xi\,\mathrm{e}^{-z\cosh\xi}\sinh^{2\nu}\xi \tag{4.3.28}$$

and the identity

$$\begin{aligned} \int_0^\infty dx\,x^{-\lambda}K_\mu(ax)J_\nu(bx) = &\frac{b^\nu\Gamma\left(\frac{\nu-\lambda+\mu+1}{2}\right)\Gamma\left(\frac{\nu-\lambda-\mu+1}{2}\right)}{2^{\lambda+1}a^{\nu-\lambda+1}\Gamma\left(1+\nu\right)}\times \\ &\times{}_2F_1\left(\frac{\nu-\lambda+\mu+1}{2},\frac{\nu-\lambda-\mu+1}{2};\nu+1;-\frac{b^2}{a^2}\right). \end{aligned}$$

Using the isometries of the boundary, the Lorentz invariant generalisation of $K(0,0,z|t,0)$ is

$$K(0,0,z|t,y) = -\frac{1}{\pi^2}\frac{z^{l+1}}{(t-i\epsilon)^2-y^2}\,{}_2F_1\left(1,1;l+1;\frac{z^2}{(t-i\epsilon)^2-y^2}\right) \tag{4.3.29}$$

which is in perfect agreement with the result obtained in [40], including the $i\epsilon$ insertions.

4.A Matching conditions for the Poincaré AdS

Here we demonstrate how individual modes can be extracted from the solutions obtained for the Poincaré patch of AdS. We only present the calculations for the matching surface at $\tau_0 = 0, t_1 = 0$ but the same method can be applied straightforwardly to the other matching surfaces.

Our analysis makes use of the following two identities of the Bessel functions

$$\int_0^\infty dz\;zJ_n(za)J_n(zb) = \frac{1}{a}\delta(b-a) \tag{4.A.1}$$

$$\int_0^\infty dz\;zK_\nu(za)J_\nu(zb) = \frac{b^\nu}{a^\nu(a^2+b^2)}. \tag{4.A.2}$$

Focusing first on the Lorentzian solution, on the hypersurface located at $t_1 = 0$ the field and its derivative are given by

$$\Phi_L^1\left(t_1, z, x\right)\big|_{t_1=0} = \int_{-\infty}^{\infty}\frac{\mathrm{d}k}{2\pi}\int_{-\infty}^{\infty}\frac{\mathrm{d}\omega}{2\pi}\Bigg[\left(a_{\omega k}\mathrm{e}^{ikx} + a^*_{\omega k}\mathrm{e}^{-ikx}\right) z \\ \theta\left(\omega^2 - k^2\right) J_l\left(\sqrt{\omega^2 - k^2}\, z\right)\Bigg] \tag{4.A.3a}$$

$$-i\partial_{t_1}\Phi_L^1\left(t_1, z, x\right)\big|_{t_1=0} = \int_{-\infty}^{\infty}\frac{\mathrm{d}k}{2\pi}\int_{-\infty}^{\infty}\frac{\mathrm{d}\omega}{2\pi}\Bigg[\left(-a_{\omega k}\mathrm{e}^{ikx} + a^*_{\omega k}\mathrm{e}^{-ikx}\right)\omega z \\ \theta\left(\omega^2 - k^2\right) J_l\left(\sqrt{\omega^2 - k^2}\, z\right)\Bigg]. \tag{4.A.3b}$$

Multiplying the above expressions by $\theta\left(\omega^2 - k^2\right) J_l\left(\sqrt{\omega^2 - k^2}\, z\right) \mathrm{e}^{-ikx}$ and integrating first over x from $-\infty$ to $+\infty$ and then over z from zero to $+\infty$, we find

$$\int_0^{\infty}\mathrm{d}z\,\theta\left(\omega^2 - k^2\right) J_l(\sqrt{\omega^2 - k^2}\, z)\int_{-\infty}^{\infty}\mathrm{d}x\,\mathrm{e}^{-ikx}\Phi_L^1\left(t_1, z, x\right)\big|_{t_1=0} = \\ = \frac{\theta(\omega^2 - k^2)}{2\pi|\omega|}\left(a_{|\omega|k} + a_{-|\omega|,k} + a^*_{|\omega|\,-k} + a^*_{-|\omega|\,-k}\right) \tag{4.A.4a}$$

$$\int_0^{\infty}\mathrm{d}z\,\theta\left(\omega^2 - k^2\right) J_l(\sqrt{\omega^2 - k^2}\, z)\int_{-\infty}^{\infty}\mathrm{d}x\,\mathrm{e}^{-ikx}\left(-i\partial_{t_1}\Phi_L^1\left(t_1, z, x\right)\big|_{t_1=0}\right) = \\ = \frac{\theta(\omega^2 - k^2)}{2\pi}\left(-a_{|\omega|k} + a_{-|\omega|k} - a^*_{-|\omega|\,-k} + a^*_{|\omega|\,-k}\right) \tag{4.A.4b}$$

In more details:

$$\int_0^{\infty}\mathrm{d}z\,\theta\left(\omega^2 - k^2\right) J_l(\sqrt{\omega^2 - k^2}\, z)\int_{-\infty}^{\infty}\mathrm{d}x\,\mathrm{e}^{-ikx}\Phi_L^1\left(t_1, z, x\right)\big|_{t_1=0} = \\ = \int_0^{\infty}\mathrm{d}z\int_{-\infty}^{\infty}\mathrm{d}x\int_{-\infty}^{\infty}\frac{\mathrm{d}k'}{2\pi}\int_{-\infty}^{\infty}\frac{\mathrm{d}\omega'}{2\pi}\theta\left(\omega^2 - k^2\right)\theta\left(\omega'^2 - k'^2\right)\Big[a_{\omega'k'}\mathrm{e}^{i(k'-k)x} \\ + a^*_{\omega'k'}\mathrm{e}^{-i(k'+k)x}\Big]\, z\, J_l\left(\sqrt{\omega^2 - k^2}\, z\right) J_l\left(\sqrt{\omega'^2 - k'^2}\, z\right) \\ = \int_{-\infty}^{\infty}\frac{\mathrm{d}\omega'}{2\pi}\int_0^{\infty}\mathrm{d}z\,\theta\left(\omega'^2 - k^2\right)\theta\left(\omega^2 - k^2\right)\left(a_{\omega'k} + a^*_{\omega'\,-k}\right) z \\ J_l\left(\sqrt{\omega^2 - k^2}\, z\right) J_l\left(\sqrt{\omega'^2 - k^2}\, z\right) \\ = \int_{-\infty}^{\infty}\frac{\mathrm{d}\omega'}{2\pi}\theta\left(\omega'^2 - k^2\right)\theta\left(\omega^2 - k^2\right)\left(a_{\omega'k} + a^*_{\omega'\,-k}\right)\frac{\delta\left(\sqrt{\omega'^2 - k^2} - \sqrt{\omega^2 - k^2}\right)}{\sqrt{\omega^2 - k^2}} \tag{4.A.5}$$

where in the last line we used (4.A.1) to perform the z integral.

To proceed we make use of the relation

$$\delta\left(\sqrt{\omega'^2 - k^2} - \sqrt{\omega^2 - k^2}\right) = \frac{\sqrt{\omega^2 - k^2}}{|\omega|}\left[\delta\left(\omega' + |\omega|\right) + \delta\left(\omega' - |\omega|\right)\right] \tag{4.A.6}$$

to obtain

$$\begin{aligned}&\int_0^\infty \mathrm{d}z\,\theta\left(\omega^2-k^2\right) J_l(\sqrt{\omega^2-k^2}\,z)\int_{-\infty}^\infty \mathrm{d}x\,\mathrm{e}^{-ikx}\Phi_L^1\left(t_1,z,x\right)\big|_{t_1=0} =\\ &=\int_{-\infty}^\infty \frac{\mathrm{d}\omega'}{2\pi|\omega|}\left(a_{\omega' k}+a^*_{\omega'\,-k}\right)\theta\big(\omega^2-k^2\big)\,\theta\big(\omega'^2-k^2\big)\Big[\delta\left(\omega'+|\omega|\right)\\ &\quad+\delta\left(\omega'-|\omega|\right)\Big]\\ &=\frac{\theta(\omega^2-k^2)}{2\pi|\omega|}\Big(a_{|\omega|k}+a_{-|\omega|k}+a^*_{|\omega|\,-k}+a^*_{-|\omega|\,-k}\Big)\,. \qquad \square\end{aligned} \tag{4.A.7}$$

The computation for the derivative is very similar.

Focusing now on the Euclidean solution, on the hypersurface located at $\tau_0=0$, the field and its derivative are given by

$$\begin{aligned}\Phi_E^-\left(\tau_0,z,x\right)\big|_{\tau_0=0}=&\frac{z}{\Gamma(l)2^{l-1}}\int_{-\infty}^\infty\frac{\mathrm{d}k}{2\pi}\int_{-\infty}^\infty\frac{\mathrm{d}\omega}{2\pi}\Big[\mathrm{e}^{i\omega\epsilon+ikx}\,\left(\omega^2+k^2\right)^{l/2}\\ &K_l\Big(\sqrt{\omega^2+k^2}\,z\Big)\Big]+\int_{-\infty}^\infty\frac{\mathrm{d}k}{2\pi}\int_0^\infty\frac{\mathrm{d}\omega}{2\pi i}\Big[b_{\omega k}\,\mathrm{e}^{ikx}z\\ &\theta\left(\omega^2-k^2\right)J_l\left(\sqrt{\omega^2-k^2}\,z\right)\Big]\end{aligned} \tag{4.A.8a}$$

$$\begin{aligned}\partial_{\tau_0}\Phi_E^-\left(t_0,z,x\right)\big|_{\tau_0=0}=&\frac{z}{\Gamma(l)2^{l-1}}\int_{-\infty}^\infty\frac{\mathrm{d}k}{2\pi}\int_{-\infty}^\infty\frac{\mathrm{d}\omega}{2\pi}\Big[i\,\omega\mathrm{e}^{i\omega\epsilon+ikx}\,\left(\omega^2+k^2\right)^{l/2}\\ &K_l\Big(\sqrt{\omega^2+k^2}\,z\Big)\Big]+\int_{-\infty}^\infty\frac{\mathrm{d}k}{2\pi}\int_0^\infty\frac{\mathrm{d}\omega}{2\pi i}\Big[\omega\,b_{\omega k}\,\mathrm{e}^{ikx}z\\ &\theta\left(\omega^2-k^2\right)J_l\left(\sqrt{\omega^2-k^2}\,z\right)\Big]\end{aligned} \tag{4.A.8b}$$

By using the same method we find

$$\begin{aligned}&\int_0^\infty \mathrm{d}z\,\theta\left(\omega^2-k^2\right)J_l(\sqrt{\omega^2-k^2}\,z)\int_{-\infty}^\infty\mathrm{d}x\,\mathrm{e}^{-ikx}\Phi_E^-\left(\tau_0,z,x\right)\big|_{\tau_0=0}=\\ &=\theta\big(\omega^2-k^2\big)\left(\frac{\left(\omega^2-k^2\right)^{l/2}}{2^l\Gamma(l)|\omega|}\mathrm{e}^{-|\omega|\epsilon}+\frac{d_{|\omega|k}}{2\pi i|\omega|}\right)\end{aligned} \tag{4.A.9a}$$

$$\begin{aligned}&\int_0^\infty \mathrm{d}z\,\theta\big(\omega^2-k^2\big)\,J_l\left(\sqrt{\omega^2-k^2}\,z\right)\int_{-\infty}^\infty\mathrm{d}x\,\mathrm{e}^{-ikx}\left(\partial_{\tau_0}\Phi_E^-\left(\tau_0,z,x\right)\big|_{\tau_0=0}\right)=\\ &=\theta\big(\omega^2-k^2\big)\left(-\frac{\left(\omega^2-k^2\right)^{l/2}}{2^l\Gamma(l)}\mathrm{e}^{-|\omega|\epsilon}+\frac{d_{|\omega|k}}{2\pi i}\right).\end{aligned} \tag{4.A.9b}$$

Obtaining these results requires a bit of extra work because our Euclidean solutions consists of two terms, one of which is in terms of the modified Bessel function of the second kind and therefore we need to use (4.A.2) and perform a contour integration in the ω plane.

In more detail, this is done as follows,

$$\begin{aligned}
&\int_0^\infty \mathrm{d}z\, \theta\left(\omega^2-k^2\right) J_l(\sqrt{\omega^2-k^2}\, z) \int_{-\infty}^{\infty} \mathrm{d}x\, \mathrm{e}^{-ikx} \Phi_E^-(\tau_0,z,x)\,\big|_{\tau_0=0} = \\
&= \int_0^\infty \mathrm{d}z \int_{-\infty}^{\infty} \mathrm{d}x \int_{-\infty}^{\infty} \frac{\mathrm{d}k'}{2\pi} \int_{-\infty}^{\infty} \frac{\mathrm{d}\omega'}{2\pi} \Bigg[\frac{z\,\theta\big(\omega^2-k^2\big)\left(\omega'^2+k'^2\right)^{l/2} \mathrm{e}^{i\omega'\epsilon - i(k-k')x}}{2^{l-1}\Gamma(l)} \\
&\quad J_l\left(\sqrt{\omega^2-k^2}\, z\right) K_l\left(\sqrt{\omega'^2+k'^2}\, z\right) \Bigg] + \int_0^\infty \mathrm{d}z \int_{-\infty}^{\infty} \mathrm{d}x \int_{-\infty}^{\infty} \frac{\mathrm{d}k'}{2\pi} \int_0^\infty \frac{\mathrm{d}\omega'}{2\pi i} \Bigg[\\
&\quad d_{\omega' k'} \mathrm{e}^{i(k-k')x} \theta\big(\omega^2-k^2\big)\, \theta\big(\omega'^2-k'^2\big)\, z J_l\left(\sqrt{\omega^2-k^2}\, z\right) J_l\left(\sqrt{\omega'^2-k'^2}\, z\right) \Bigg] \\
&= I_1 + I_2
\end{aligned} \tag{4.A.10}$$

where

$$\begin{aligned}
I_1 = &\int_0^\infty \mathrm{d}z \int_{-\infty}^{\infty} \mathrm{d}x \int_{-\infty}^{\infty} \frac{\mathrm{d}k'}{2\pi} \int_{-\infty}^{\infty} \frac{\mathrm{d}\omega'}{2\pi} \Bigg[\frac{z\,\theta\big(\omega^2-k^2\big)\left(\omega'^2+k'^2\right)^{l/2} \mathrm{e}^{i\omega'\epsilon - i(k-k')x}}{2^{l-1}\Gamma(l)} \\
&J_l\left(\sqrt{\omega^2-k^2}\, z\right) K_l\left(\sqrt{\omega'^2+k'^2}\, z\right) \Bigg],
\end{aligned} \tag{4.A.11a}$$

$$\begin{aligned}
I_2 = &\int_0^\infty \mathrm{d}z \int_{-\infty}^{\infty} \mathrm{d}x \int_{-\infty}^{\infty} \frac{\mathrm{d}k'}{2\pi} \int_0^\infty \frac{\mathrm{d}\omega'}{2\pi i} \Bigg[d_{\omega' k'} \mathrm{e}^{i(k-k')x} \theta\big(\omega^2-k^2\big)\, \theta\big(\omega'^2-k'^2\big) \\
&z J_l\left(\sqrt{\omega^2-k^2}\, z\right) J_l\left(\sqrt{\omega'^2-k'^2}\, z\right) \Bigg].
\end{aligned} \tag{4.A.11b}$$

The computation of I_2 is identical to what we did for the Lorentzian field above,

$$\begin{aligned}
I_2 &= \int_0^\infty \mathrm{d}z \int_{-\infty}^{\infty} \mathrm{d}x \int_{-\infty}^{\infty} \frac{\mathrm{d}k'}{2\pi} \int_0^\infty \frac{\mathrm{d}\omega'}{2\pi i} \Bigg[z\, d_{\omega' k'} \mathrm{e}^{i(k-k')x} \theta\big(\omega^2-k^2\big)\, \theta\big(\omega'^2-k'^2\big) \\
&\quad J_l\left(\sqrt{\omega^2-k^2}\, z\right) J_l\left(\sqrt{\omega'^2-k'^2}\, z\right) \Bigg] \\
&= \int_0^\infty \mathrm{d}z \int_0^\infty \frac{\mathrm{d}\omega'}{2\pi i} \Bigg[z d_{\omega' k} \theta\big(\omega^2-k^2\big)\, \theta\big(\omega'^2-k^2\big) J_l\left(\sqrt{\omega^2-k^2}\, z\right) \\
&\quad J_l\left(\sqrt{\omega'^2-k^2}\, z\right) \Bigg] \\
&= \int_0^\infty \frac{\mathrm{d}\omega'}{2\pi i}\, d_{\omega' k} \theta\big(\omega^2-k^2\big)\, \theta\big(\omega'^2-k^2\big) \frac{\delta\Big(\sqrt{\omega'^2-k^2} - \sqrt{\omega^2-k^2}\Big)}{\sqrt{\omega^2-k^2}} \\
&= \int_0^\infty \frac{\mathrm{d}\omega'}{2\pi|\omega| i}\, d_{\omega' k} \theta\big(\omega^2-k^2\big)\, \theta\big(\omega'^2-k^2\big) \big(\delta\big(\omega-|\omega'|\big) + \delta(\omega+|\omega|)\big) \\
&= \theta\big(\omega^2-k^2\big) \frac{d_{|\omega| k}}{2\pi i|\omega|}
\end{aligned} \tag{4.A.12}$$

where we used equations (4.A.1) and (4.A.6).

The computation of I_1 goes as follows,

$$I_1 = \int_0^\infty \mathrm{d}z \int_{-\infty}^\infty \mathrm{d}x \int_{-\infty}^\infty \frac{\mathrm{d}k'}{2\pi} \int_{-\infty}^\infty \frac{\mathrm{d}\omega'}{2\pi} \left[z\,\theta\big(\omega^2 - k^2\big) \frac{\big(\omega'^2 + k'^2\big)^{l/2}}{2^{l-1}\Gamma(l)} \mathrm{e}^{i\omega'\epsilon - i(k-k')x} J_l\left(\sqrt{\omega^2 - k^2}\,z\right) K_l\left(\sqrt{\omega'^2 + k'^2}\,z\right)\right]$$
$$= \int_0^\infty \mathrm{d}z \int_{-\infty}^\infty \frac{\mathrm{d}\omega'}{2\pi} \left[z\,\theta\big(\omega^2 - k^2\big) \frac{\big(\omega'^2 + k^2\big)^{l/2}}{2^{l-1}\Gamma(l)} \mathrm{e}^{i\omega'\epsilon} J_l\left(\sqrt{\omega^2 - k^2}\,z\right) K_l\left(\sqrt{\omega'^2 + k^2}\,z\right)\right]$$
$$= \int_{-\infty}^\infty \frac{\mathrm{d}\omega'}{2\pi} \frac{\theta\big(\omega^2 - k^2\big)}{2^{l-1}\Gamma(l)} \mathrm{e}^{i\omega'\epsilon} \frac{\big(\omega^2 - k^2\big)^{l/2}}{\omega'^2 + \omega^2} \tag{4.A.13}$$

where for the last line we used equation (4.A.2). The integral over ω' is performed using contour integration. Closing the contour in the upper half plane and picking up the contribution from the pole at $i|\omega|$ we obtain,

$$I_1 = \frac{\theta\big(\omega^2 - k^2\big)\,\big(\omega^2 - k^2\big)^{l/2}}{2^l \pi \Gamma(l)} 2\pi i \,\mathrm{Res}\left[\frac{\mathrm{e}^{i\omega'\epsilon}}{\omega'^2 + \omega^2}; \omega' = i|\omega|\right]$$
$$= i \frac{\theta\big(\omega^2 - k^2\big)\,\big(\omega^2 - k^2\big)^{l/2}}{2^{l-1}\Gamma(l)} \left[\frac{\mathrm{e}^{-|\omega|\epsilon}}{2i|\omega|}\right] = \frac{\theta\big(\omega^2 - k^2\big)\,\big(\omega^2 - k^2\big)^{l/2}}{2^l\Gamma(l)|\omega|} \mathrm{e}^{-|\omega|\epsilon}. \tag{4.A.14}$$

Combining the results for I_1 and I_2,

$$\int_0^\infty \mathrm{d}z\,\theta\big(\omega^2 - k^2\big)\, J_l\big(\sqrt{\omega^2 - k^2}\,z\big) \int_{-\infty}^\infty \mathrm{d}x\, \mathrm{e}^{-ikx} \left(\partial_{\tau_0} \Phi_E^- (\tau_0, z, x)\,\big|_{\tau_0=0}\right) =$$
$$= \theta\big(\omega^2 - k^2\big) \left(-\frac{\big(\omega^2 - k^2\big)^{l/2}}{2^l\Gamma(l)} \mathrm{e}^{-|\omega|\epsilon} + \frac{d_{|\omega|k}}{2\pi i}\right). \tag{4.A.15}$$

The computations for the derivative follow along the same lines.

Chapter 5

Discussion

In this first part of this thesis we presented the construction of a bulk solution dual to a general excited CFT state, $|\Delta\rangle$, where Δ is the scaling dimension. By the operator-state correspondence, the state is generated by an operator $\mathcal{O}_\Delta$ acting on the vacuum. The corresponding bulk solution at linearised level involves only the bulk scalar Φ which is dual to the operator $\mathcal{O}_\Delta$. This part is universal: it is the same for all CFTs whose spectrum contains an operator with such dimension. To construct the full bulk solution we need more information about the CFT. In particular, we need to know the OPE of $\mathcal{O}_\Delta$ with itself. All bulk fields that are dual to operators that appear in this OPE are necessarily turned on in the bulk.

In this thesis we presented in detail the construction of the universal part, for states of two dimensional CFTs either on $R\times S^1$ or $R^{1,1}$. From the bulk perspective this leads to the construction of solutions of free scalar field equations either in global AdS_3 or Poincaré AdS_3. The solutions describe normalisable modes and their coefficients are directly related to the dual state. In more detail, the CFT state is generated by a Euclidean path integral which contains a source for $\mathcal{O}_\Delta$ and the coefficients of the bulk normalisable modes are given in terms of the source. Normalisable modes describe bulk local excitations and thus our results give a direct relation between CFT states and bulk excitations. To substantiate the claim that these solutions are dual to the state $|\Delta\rangle$, we computed the 1-point function of local operators both in the CFT and in the bulk and found perfect agreement[1]. Our discussion generalizes straightforwardly to higher dimensions.

To go beyond this leading order computation, one needs to be more specific about the CFT (as mentioned above). In particular, one would need to take into account the back-reaction to the metric. Given appropriate CFT data (for a CFT with a known bulk dual), the construction of the bulk solution dual to any given state can proceed along the same lines. It would be interesting to explicitly carry this out in detail in concrete examples.

[1]As emphasised in section 2.2, this agreement is a non-trivial check that we are constructing the correct path integral. To holographically compute expectation values in the state $|\Delta\rangle$ we would need the solution to quadratic order in the bulk fields.

In our discussion we explicitly demonstrated how a solution of the bulk field equations is reconstructed from QFT data: given a Schwinger-Keldysh contour and insertions we constructed a unique bulk solution. We also demonstrated how one can rewrite the bulk solution in the Lorentzian part in terms of its boundary value, resulting in expressions of the following form,

$$\Phi(t,r,\phi) = \int_{\partial \text{AdS}} \mathrm{d}t' \, \mathrm{d}\phi' \; K(t,r,\phi|t',\phi')\langle O(t',\phi')\rangle \tag{5.0.1}$$

where $K(t,r,\phi|\hat{t},\hat{\phi})$, is the smearing function. One must use caution when write such an expression. For us (5.0.1) is a map between expectation values of the boundary theory and classical fields in the bulk. In [40] the idea was different. The main point was to look for CFT operators that behave like bulk local operators. The initial ansatz in [40] was

$$\hat{\Phi}(t,r,\phi) = \int_{\partial \text{AdS}} \mathrm{d}t' \, \mathrm{d}\phi' \; K(t,r,\phi|t',\phi')\hat{O}(t',\phi'), \tag{5.0.2}$$

and the smearing function $K(t,r,\phi|t',\phi')$ was fixed by rewriting the bulk normalisable modes in this form. The hat on the left hand side indicates that this is a quantum operator. If we quantize canonically the bulk scalar field then the coefficients b_{nk} and $b^\dagger_{nk}$ of the normalisable modes (see (3.1.10)) are promoted to creation and annihilation operators. However, the matching condition relates these coefficients to a CFT source and the latter is not a quantum operator. One may still reconcile the two pictures if one considers the bulk solutions as being associated with a coherent state, as was recently argued in [63]. Then the eigenvalue of the annihilation operator acting on the coherent state would be equal to the value of the source. This would give a map from states $|\Delta\rangle$ of the CFT to coherent states in the bulk and it would be interesting to understand this map in more detail.

As emphasised, (5.0.1) and (5.0.2) hold at the linearised level in the bulk (free fields)[2]. While (5.0.1) and (5.0.2) may be related at this order, it is not clear this will continue to be the case at non-linear level. There has been work in extending (5.0.2) to higher orders, see for example [43, 44, 45, 46, 64]. In these papers, the map is modified by including additional terms on the RHS of (5.0.2), which are double-trace operators. The coefficients are then fixed by requiring bulk locality. In our case, the full bulk solution will instead involve many additional bulk fields, which are dual to single-trace operators. It would be interesting to clarify the relation between the two reconstruction formulae at non-linear order.

Another application of our construction is in the context of the fuzzball program [65, 66, 67, 68]. As was argued in [69, 37, 38, 67], the fuzzball solutions for black holes with AdS

[2] This is also the leading term in the 't Hooft large N limit, if we normalise the CFT operators such that their 2-point function has coefficient 1 in the large N limit. One should keep in mind however that with this normalisation the subleading terms in N do not necessarily correspond to quantum loops, see the discussion in section 2.2.

throats are the bulk solutions dual to the states that account for black hole entropy. In all previous works, fuzzball solutions were constructed by solving supergravity equations and the relation to CFT states was only studied afterwards (for a class of fuzzballs). The construction here allows one to pursue a direct (iterative) construction of bulk solutions dual to individual states. It would be interesting to carry out such computations. One may also use the results here to sharpen an old argument [70] that the number of *supergravity* solutions dual to the 3-charge BPS black holes cannot exceed that of the 2-charge ones.

Part II

Einstein–Maxwell–Dialton–Axion Theories

Chapter 6

The Model:

Asymptotic Analysis, Thermodynamics and Stability

6.1 Introduction

A big challenge of modern theoretical physics is the demystification of the underlying physics governing strongly coupled systems such as the quark gluon plasma (QGP) created during heavy ion collisions, cold atom systems and strongly correlated electrons. The reason why these systems remained for a long time a mystery is because they lie outside the regime of validity of the conventional tools usually employed by theoretical physicists. In particular, since they are strongly coupled, perturbation theory does not apply and, although lattice regularisation of quantum chromodynamics (QCD) is sufficient for the study of static observables, using it for time dependent problems such as the thermalisation of the QGP poses great difficulties. Gauge/gravity dualities, being strong/weak coupling dualities, offer a promising candidate for a new framework that will ultimately fill the gap in the toolset of theoretical physics. Through the AdS/CFT dictionary one can map questions about the macroscopic properties of strongly coupled gauge theories to tractable problems in supergravity on asymptotically AdS backgrounds. For example, by performing supergravity calculations, string theorists can derive real-time correlation functions of field theory operators and study them to learn about non-equilibrium processes such as diffusion and sound wave propagation in strongly coupled non-conformal plasmas. A notable result obtained through this program and relating primarily to the QGP is the calculation of the ratio of the shear viscosity to entropy density in strongly coupled plasmas [71, 72]. Equally interesting are the results relating to the phase structure of strongly coupled condensed matter systems which emerged from the study of the of supergravity solutions with charged asymptotically AdS black holes supporting additional matter fields. For example, it was shown in [73] that charged black holes in AdS can develop charged scalar hair which spontaneously breaks the gauge symmetry, thus providing a holographic description of superconductivity. Moreover, probing further the phase space of such bulk theories revealed that, in addition to superconducting phase transitions, the dual theories have phases exhibiting emergent quantum criticality, non–relativistic scaling properties as well as hyperscaling violation.

The second part of this thesis is based on [2] and it focuses on the study of a class of 3+1 dimensional planar AdS black holes that carry electric and/or magnetic charges, axionic hair and can support additional scalar hair associated with a running scalar. There are many motivations for studying supergravity solutions with these characteristics. Firstly, running scalar fields in planar asymptotically AdS backgrounds describe holographic Renormalisation Group (RG) flows between conformal fixed points and between UV fixed points and IR theories exhibiting more interesting scaling behaviours such as hyperscale violation and/or Lifshitz scaling [74, 75, 76, 77, 78, 79]. These RG flows play a key role in understanding the physics of the QGP in heavy ion collisions and of condensed matter system which have been observed to display hyperscaling violation. Secondly, as mentioned above, when coupled to the U(1) gauge field, the scalar field provides a mechanism for the spontaneous breaking of the U(1) symmetry giving rise to holographic description of superconductors [73, 80]. In the theories we study the scalar field is not charged under the U(1) gauge symmetry and therefore the ordered phase of these theories is not superconducting. Nonetheless, we find that they do exhibit phase transitions which are controlled by the condensate of the operator dual to the scalar. Inspired by this connection of the scalar to the tuning or *dialling* of the theory between different phases we have dubbed it the *dialton*. Another appealing feature of the theories we study is that the mass of the dialton lies in the range which allows for mixed and Neumann boundary conditions, in addition to the conventional Dirichlet. By choosing to impose mixed boundary conditions we introduce in the theory an additional tuning parameter that can be used tuned to control the condensation of the operator dual to the scalar and induce new phase transitions in the dual theory. In particular, from the perspective of the field theory, imposing mixed boundary conditions corresponds to turning on a multitrace deformation which introduces a new coupling ϑ and which can be tuned to destabilise the theory [81]. However, this is not the only significance of the mixed boundary conditions. As we will see in section 6.2.2, in order to impose mixed boundary conditions on the dialton one must add extra boundary terms to the bulk action. These terms ultimately modify the holographic stress tensor and the on—shell action [82], and, hence, the associated conserved charges and free energy [30] but they leave the thermodynamic relations unaffected. In particular, we will see that by correctly accounting for the modifications of the charges we arrive at the standard thermodynamics relations, including the first law, without any new "charges" associated with the dialton. This is exactly what one expects since running scalars in AdS black holes are *secondary hair* and not primary [83].

Returning to discussion of the motivation for studying this class of theories, the second set of scalar hair, namely the axion fields, provide a mechanism for momentum relaxation in the dual theory, a highly desired featured for holographic condensed matter systems. More specifically, the axions admit solutions with a linear profile in the boundary spatial directions. From the field theory perspective, such bulk solutions correspond to turning on spatially dependent sources for the dual operators which break translation invariance and lead to the momentum relaxation. Linear axion backgrounds were first considered in [84] where the authors obtained supergravity solutions that are dual to theories exhibiting Lifshitz–like fixed points in the IR with anisotropic scale invariance. Their results were later generalised to finite temperatures [85, 86] providing a description for strongly coupled homogeneous and anisotropic plasmas which can play a key role in understanding the physics of the QGP. More generally, linear axion backgrounds are a special case of the holographic Q–lattices first discovered in [31] as solutions to 3+1 dimensional Einstein–Maxwell theory coupled to a gauge neutral complex massive scalar that enjoys a global U(1) symmetry. This global symmetry combined with a spatially periodic ansatz for the scalar field lead to dual systems that explicitly break translational invariance leading to states with finite DC conductivity. In a similar spirit, the authors of [32] studied supergravity solutions with scalar fields that have linear profiles along the boundary directions which explicitly break translational invariance but preserve isotropy and they obtained a dual metallic system exhibiting momentum dissipation and finite DC conductivity.

As was mentioned above, the dialton is secondary hair and therefore there are no conserved charges associated with it. The axionic hair is fundamentally different in this sense; the axions with a linear profile along the spatial boundary directions are *primary hair*, carrying magnetic–like charges and satisfying global Ward identities. On the boundary, they correspond to deforming the QFT action by terms of the form $\int_{\partial\mathcal{M}} x^I \mathcal{O}_{\psi_I}$ where $\mathcal{O}_{\psi_I}$ are the operators dual to the axions. As we will see, the global Ward identities imply that $\mathcal{O}_{\psi_I}$ are locally exact and thus, one might be tempted to integrate by parts the deformation term and eliminate explicit x dependence of the field theory action. However, there is an important subtlety preventing us from doing so. The boundary terms associated with the integration by parts diverge as $x^I \to \pm\infty$ and can not be ignored. It follows that solutions with linear axion profiles in boundary spatial directions correspond to turning on topological axion charges analogous to turning on magnetic charges. These charges modify the first law of thermodynamics and they should be treated on the same footing as conventional magnetic fields turned on on the boundary.

The main focus of this work is the analysis of the thermodynamic properties and phase structure of theories dual to the bulk configurations described above, namely electric and/or magnetically charged planar black branes carrying primary axionic charges and which can support additional secondary scalar hair satisfying mixed boundary conditions. In this chapter we introduce the model and perform the general analysis of the action. We begin in section 6.2 where we present the action and revisit the properties of the axions and the dialton in more detail. In section 6.3 we derive the equations which we solve asymptotically in section 6.3.2. We then proceed to compute the boundary terms necessary to impose our choice of boundary conditions and for the variation of the on–shell action with respect to the sources to be well defined 6.3.3. These terms, combined with the asymptotic solutions of the equations of motion, allow us to derive the renormalised on–shell action, 6.3.4, and the one–point functions of the dual operators 6.3.5. We also obtain the local Ward identities associated with diffeomorphism invariance and the U(1) gauge symmetry, 6.3.6, as well as global Ward identities for the axions, 6.3.7. We then go on to derive the field theory thermodynamics in section 6.4 and discuss the results obtained in [2] for dynamical stability of hairy solutions in section 6.5.

In chapters 7 and 8 we apply the general results of the current chapter to certain analytic solutions of the theory. In particular we will revisit the exact axionic black holes found in [87], which do not have a running profile for the dialton, as well as those obtained in [88] and have a running dialton in addition to the non trivial axion background. A running dialton is also a feature present in the electrically charged black brane solutions found analytically in [89], which we also discuss. Finally, in chapter 8, we obtain a new family of exact magnetically charged axionic black holes, which may be viewed as the magnetic version of those presented in [89]. Using the results we obtain in this chapter, we derive the thermodynamic properties of the dual theories and study their phase structure and dynamical stability.

6.2 The Action

The theories we are interested in are $d+1$ dimensional Einstein–Maxwell theories with $d-1$ massless scalar fields ψ_I and a single massive scalar field ϕ, described by the action

$$S_{\mathsf{bulk}} = \int_{\mathcal{M}} \mathrm{d}^{d+1}x\sqrt{-G}\left(R - \frac{1}{2}(\partial\phi)^2 - V(\phi) - \frac{1}{2}W(\phi)\sum_{I=1}^{d-1}(\partial\psi_I)^2 - \frac{1}{4}Z(\phi)F^2\right), \tag{6.2.1}$$

where we are using units in which $16\pi G_{\mathsf{N}} = 1$. The $d-1$ massless scalar fields ψ_I are the axions, the field ϕ is the dialton, F is the field strength associated to the Maxwell field A, $F = \mathrm{d}A$ and $G_{\mu\nu}$ is the spacetime metric used to raise and lower Greek indices $\mu, \nu, \ldots$. The capital indices $I, J, \ldots$ denote the flavour of the axions and they run from 1 to $d-1$. The functions $V(\phi)$, $W(\phi)$ and $Z(\phi)$ define the dialton potential, the coupling of the dialton to the scalar fields ψ_I and the coupling of the dialton to the Maxwell field, respectively. The dialton potential is chosen such that it generates an effective cosmological constant Λ when the dialton vanishes, $V(0) = 2\Lambda$. Expressed in terms of the radius of AdS ℓ, the cosmological constant is defined by

$$\Lambda = -\frac{d(d-1)}{2\ell^2} \tag{6.2.2}$$

The coupling of the dialton to the axions has been included for generality but for the theories we will study $W(\phi) = 1$. Finally, the coupling between the dialton field and the

Maxwell field is normalised such that, for vanishing dialton, $Z(0) = 1$.

Before proceeding with the study of the equations of motion it is instructive to examine more closely the properties of the $(d-1)$ axions and of the dialton.

6.2.1 The $(d-1)$ Axions

The $(d-1)$ scalar fields ψ_I are massless and only enter the action through their derivatives. This implies that they enjoy a global shift symmetry,

$$\psi_I \mapsto \psi_I + c_I, \tag{6.2.3}$$

where c_I is a constant translation vector, as well as a global SO$(d-1)$ flavour rotation symmetry,

$$\psi_I \mapsto \Lambda_I^J \psi_J, \tag{6.2.4}$$

where Λ_I^J is an SO$(d-1)$ constant rotation. These scalars are in fact 0-form fields with field strength $F_{[1]}^I = \mathrm{d}\psi_I$, which, as can be seen from (6.2.1) with $W(\phi) = 1$, only enter the action through their field strength. Consequently, it is possible to construct homogeneous solutions whose stress-energy tensor does not depend on the boundary coordinates. These solutions correspond to turning on sources for ψ_I which are linear in the boundary coordinates. Furthermore, by having $d-1$ axions, i.e. the same number as the number of the boundary spatial directions, allows for the possibility to arrange their sources is such a way so as to preserve the $\mathrm{SO}(d-1)$ rotational symmetry. This means that their contribution to the stress-energy tensor does not break isotropy. The axion fields associated with such homogeneous and isotropic bulk solutions have the form

$$\psi_I = p x^I \tag{6.2.5}$$

where p is a constant.

Solutions of the form of (6.2.5) have a number of important features. Firstly, they act as vacuum energy for the horizon giving rise to topological AdS black holes with a flat horizon but with a lapse function that resembles the one usually associated with hyperbolic AdS black holes [88]. This means that they enjoy both a flat boundary as well as the additional length scale associated to the hyperbolic radius.

Secondly, they break translation invariance in the transverse x^I directions, providing a mechanism for momentum dissipation in the dual field theory. This can be seen from the Ward identity associated with boundary diffeomorphisms, namely

$$D_{(0)}^j \langle T_{ij} \rangle + \langle \mathcal{O}_\phi \rangle \partial_i \varphi_{(0)} + \sum_{I=1}^{d-1} \langle \mathcal{O}_{\psi_I} \rangle \partial_i \psi_I^{(0)} + \left\langle J^j \right\rangle F_{ij}^{(0)} = 0 \tag{6.2.6}$$

where i, j label boundary coordinates $x^i = (t, x^a)$, $\langle \mathcal{O}_\phi \rangle$ and $\langle \mathcal{O}_\psi \rangle$ are the vacuum expectation values (*vevs*) of the operators dual to ϕ and ψ_I, $\left\langle J^i \right\rangle$ is the $U(1)$ conserved current dual to the bulk gauge field and J_f, $\psi_I^{(0)}$ and $F_{ij}^{(0)}$ are the sources for $\mathcal{O}_\phi$, $\mathcal{O}_{\psi_I}$ and J_i respectively. A version of this identity is derived in section 6.3.6. Note that this expression assumes Dirichlet boundary conditions for the dialton although in our subsequent analysis this will not be the case. This does not affect the current analysis. This Ward identity demonstrates that turning on spatially dependent sources for the axions gives rise to the non–conservation of $\langle P^i \rangle = \langle T^{0i} \rangle$ in the presence of non–zero axion *vevs*. This mechanism for momentum dissipation was first introduced in [32].

Finally, solutions of the form (6.2.5) have non–zero flux, i.e. $F_I = p \neq 0$, and they correspond to turning on topological "magnetic" axion charge densities in the background dual field theory. The effect of these charges with respect to the thermodynamics of the dual field theory are completely analogous to that of conventional magnetic backgrounds. In particular, these charges contribute to the free energy of the theory and enter the first law, just as a magnetic field. We derive these expressions, including the axionic "magnetisation" conjugate to the axionic charges in section 6.4 were we discuss the thermodynamic properties of the theory.

6.2.2 The Dialton Field

In this section we focus on some of the features of the dialton field. Our choice of potential $V(\phi)$ allows us to impose mixed boundary conditions for the dialton where we keep fixed a particular combination of the two modes of the dialton. The combination we will choose to keep fixed is motivated by string theory and has the interpretation of a triple trace deformation of the dual field theory. This in turn has important implications for the phase diagram of the theories.

To understand the properties of the dialton we first need to study more closely its potential. For our theories it possesses a negative extremum at $\phi = 0$. More precisely, near $\phi = 0$ our potentials have the form

$$V(\phi) = -\frac{d(d-1)}{\ell^2} + \frac{1}{2}\left(-\frac{d^2}{4\ell^2} + \frac{1}{4\ell^2}\right)\phi^2 + \mathrm{O}(\phi^4). \tag{6.2.7}$$

Note that $V(\phi)$ is manifestly an even function and therefore are no odd powers of ϕ (to all orders). The first term in this expansion corresponds to an effective cosmological constant whereas the second generates a mass for the dialton,

$$m_\phi^2 = \frac{1}{2}V''(0) = -\frac{d^2}{4\ell^2} + \frac{1}{4\ell^2}. \tag{6.2.8}$$

This mass lies in the window

$$-\frac{d^2}{4} \leq m_\phi^2 \ell^2 \leq -\frac{d^2}{4} + \frac{1}{4}, \tag{6.2.9}$$

which, as we saw in section 1.2.4 implies that in addition to Dirichlet boundary condition the scalar can admit Neumann and mixed boundary conditions as well. In particular, recall that near the asymptotic boundary, located at $z = 0$, the scalar field admits an asymptotic expansion of the form

$$\phi(x,z) \to z^{d-\Delta}\left(\varphi_{(0)}(x) + \mathcal{O}(z^2)\right) + z^{\Delta}\left(\varphi_{(2\Delta-d)}(x) + \mathcal{O}(z^2)\right) \qquad (6.2.10)$$

where for the mass range (6.2.8) Δ can be either root of $\Delta(\Delta - d) = m^2\ell^2$, the equation that defines the relation between the conformal dimension of the dual operator and the mass of the scalar field. If one chooses to fix $\varphi_{(0)}$ on the bound at the boundary[1], i.e. impose Dirichlet boundary conditions, then $\varphi_{(0)}$ is interpreted in the field theory as the source that couples to the dual operator via a term in the QFT action of the form $\int \mathrm{d}^d x\, J_{\mathrm{D}}\mathcal{O}$. In this case $\varphi_{(2\Delta-d)}$ becomes the "fluctuating field" and it is found to be proportional to the *vev* of $\mathcal{O}$ in the presence of the source. Moreover, the conformal dimension of $\mathcal{O}$ is given by the larger root of the mass–conformal dimension equation, namely $\Delta_+ = d/2 + \sqrt{d^2/4 + m^2\ell^2}$.

Alternative, for fields satisfying (6.2.9), one may choose to keep $\varphi_{(2\Delta-d)}$ fixed at the boundary and allow $\varphi_{(0)}$ to fluctuate. In this case the source that couples to the dual operator is $J_{\mathrm{N}} = -\ell^2(2\Delta - d)\phi_{(2\Delta-d)}$ and $\varphi_{(0)}$ is its *vev*. This choice corresponds to imposing Neumann boundary conditions and it requires that one adds an additional boundary term to the bulk action to ensure that the source is indeed kept fixed under variations,

$$S_{\text{bulk}} \to S_{\text{bulk}} + \int d^d x\, J_{\mathrm{N}}\varphi_{(0)}. \qquad (6.2.11)$$

With this choice of boundary conditions the conformal dimension of the dual operator is given by $\Delta_- = d/2 - \sqrt{d^2/4 + m^2\ell^2}$. Note that the constraint on the mass of the bulk field implies that Δ_- satisfies the unitarity bound for a scalar operator, $\Delta_- \geq d/2 - 1$.

The final choice, which is the one of interest to us, is to impose mixed boundary conditions on the bulk field in which case the source kept fixed at the boundary is a function of both $\varphi_{(0)}$ and $\varphi_{(2\Delta-d)}$ of the form

$$J_{\mathcal{F}} = -\ell^2(2\Delta_+ - d)\varphi_{(2\Delta_+-d)} - \mathcal{F}'(\varphi_{(0)}) \qquad (6.2.12)$$

$\mathcal{F}(\varphi_{(0)})$ is a polynomial of degree n with $2 \leq n \leq d/\Delta_-$ satisfying $\mathcal{F}(0) = 0$. From the perspective of the field theory this choice introduces a multi–trace deformation, $\int \mathrm{d}^d x\, \mathcal{F}(\mathcal{O})$. For the deformation to be relevant or marginal, the conformal dimension of $\mathcal{O}$ must be chosen to be Δ_- and its *vev* $\varphi_{(0)}$, i.e. we deform the Neumann theory [90, 91, 92, 82]. Moreover, to ensure that the source kept fixed under variations with respect to the scalar field is indeed $J_{\mathcal{F}}$, we must once again add boundary terms to the bulk

[1]Strictly speaking, since the boundary is actually a conformal boundary, one can only keep $\varphi_{(0)}$ fixed up to conformal transformation. That is to say we can demand that $\delta\varphi_{(0)} = -(d - \Delta_+)\delta\sigma(x)\varphi_{(0)}$, where $\delta\sigma(x)$ is an arbitrary infinitesimal scalar function on the boundary. The variation of the action will still vanish under these generalised Dirichlet boundary conditions by virtue of the trace Ward identity, as long as there is no conformal anomaly.

action. These are now given by

$$S_{\text{bulk}} \to S_{\text{bulk}} + \int \mathrm{d}^d x \left(J_{\mathcal{F}} \varphi_{(0)} + \mathcal{F}(\varphi_{(0)}) \right) \tag{6.2.13}$$

The theories we will be studying have a triple trace deformation. The function $f(\varphi_{(0)})$ in this case is

$$f\left(\varphi_{(0)}\right) = \frac{1}{3} \vartheta \varphi_{(0)}^3 \tag{6.2.14}$$

where ϑ is a coupling that depends on the particular solution we are considering. This is discussed in section 6.3.3.

In [81] the authors studied the effect of multitrace deformations in the context of holographic superconductors and found that they destabilise the vacuum, giving rise to a new mechanism for constructing holographic superconductors. This new source of instability is present even if the scalar field is neutral under the U(1) gauge field, as is the case with our theories. However, in this case the ordered phase is no longer superconducting since it does not break the $U(1)$ symmetry but instead a $\mathbb{Z}_2$ symmetry. In chapter 8 we study a family of solutions in which ϑ is a parameter that we can tune. In this case we also find that varying ϑ allows us to tune the theory between different phases.

6.3 Asymptotic Analysis of the Bulk

In this section we compute the bulk renormalised on–shell action by performing a full asymptotic analysis of the bulk fields. The analysis is done in full generality, i.e. without specifying the form of the dialton potential and of the coupling between the dialton and the gauge field. This allows us to apply our results immediately to the specific theories we will study without needing to do any additional calculations. The theories we will study are four dimensional and therefore, to simplify the analysis, we set $d = 3$ from now on.

The first step in our analysis is to derive the equations of motion from the action (6.2.1) and solve them near the asymptotic boundary, subject to appropriate boundary conditions. We begin by expanding the bulk fields in the holographic direction. In accordance with the AdS/CFT dictionary, we write the bulk metric in the Fefferman–Graham gauge which allows us to directly relate the coefficients in the boundary expansion of the fields to field theory observables. The equations of motion are solved order by order in the holographic direction and the result is then substituted back into the bulk action, augmented by boundary terms necessary to enforce our choice of boundary conditions. The resulting on–shell action is divergent and we need to perform holographic renormalisation. Once this is done, the renormalised on–shell action can varied with respect to the sources to obtain the expectations values of the dual operators.

6.3.1 The Equations of Motion

Variation of the action (6.2.1) leads to the following equations of motion

$$\begin{aligned}
&\nabla^\mu \left(Z(\phi)F_{\mu\nu}\right) = 0 \\
&\Box\psi_I = 0 \qquad I = 1,2 \\
&\Box\phi - V'(\phi) = \frac{1}{4}Z'(\phi)F^2 \\
&R_{\mu\nu} = \frac{1}{2}G_{\mu\nu}R + \frac{1}{2}Z(\phi)\left(F_\mu{}^\lambda F_{\nu\lambda} - \frac{1}{4}F^2G_{\mu\nu}\right) + \frac{1}{2}\sum_{i=1}^{2}\left(\partial_\mu\psi_I\partial_\nu\psi_I - \frac{1}{2}\left(\partial\psi_I\right)^2\right) \\
&\qquad + \frac{1}{2}\left(\partial_\mu\phi\partial_\nu\phi - \frac{1}{2}G_{\mu\nu}\left(\partial\phi\right)^2 - G_{\mu\nu}V(\phi)\right)
\end{aligned} \tag{6.3.1}$$

Tracing the Einstein equation we obtain an expression for the Ricci scalar in terms of the other fields,

$$R[G] = \frac{1}{2}\sum_{i=1}^{2}\left(\partial\psi_I\right)^2 + \frac{1}{2}\left(\partial\phi\right)^2 + 2V(\phi) \tag{6.3.2}$$

which we use to simplify the Einstein equation,

$$R_{\mu\nu} - \frac{1}{2}Z(\phi)\left(F_\mu{}^\lambda F_{\nu\lambda} - \frac{1}{4}F^2G_{\mu\nu}\right) - \frac{1}{2}\sum_{i=1}^{2}\partial_\mu\psi_I\partial_\nu\psi_I - \frac{1}{2}\left(\partial_\mu\phi\partial_\nu\phi + V(\phi)G_{\mu\nu}\right) = 0 \tag{6.3.3}$$

We now proceed to solve these equations asymptotically, in the vicinity of the conformal boundary.

6.3.2 Asymptotic Solutions

Asymptotic Expansions of the Fields

In this section we obtain an asymptotic solution to the equations of motion. We begin by considering the following expansions for our fields,

$$\begin{aligned}
\mathrm{d}s^2 &= G_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu = \frac{\ell^2}{z^2}\mathrm{d}z^2 + \frac{\ell^2}{z^2}g_{ij}(x,z)\mathrm{d}x^i\mathrm{d}x^j \\
g_{ij}(x,z) &= g_{(0)ij}(x) + zg_{(1)ij}(x) + z^2g_{(2)ij}(x) + \ldots
\end{aligned} \tag{6.3.4}$$

where the boundary is located at $z = 0$ and g_{ij} is a 3 dimensional metric thus $i, j = 1, 2, 3$. This is the Fefferman–Graham gauge for the metric. From the point of view of g_{ij}, z is just a parameter. The coefficients in the expansion, $g_{(0)ij}$, $g_{(1)ij}$, . . . , are to be determined, or at least constrained, using the equations of motion.

For the other fields the asymptotic expansion has the form

$$A_i(x,z) = A_i^{(0)}(x) + zA_i^{(1)}(x) + z^2 A_i^{(2)}(x) + \dots \tag{6.3.5}$$

$$\psi_I(x,z) = \psi_I^{(0)}(x) + z\psi_I^{(1)}(x) + z^2\psi_I^{(2)}(x) + \dots \tag{6.3.6}$$

$$\phi(x,z) = z^{\Delta_-}\left(\varphi_{(0)}(x) + z\varphi_{(1)}(x) + z^2\varphi_{(2)}(x) + \dots\right), \tag{6.3.7}$$

We are interested in theories for which the potential satisfies

$$\frac{1}{2}\frac{\mathrm{d}^2V(\phi)}{\mathrm{d}\phi^2}\Big|_{\phi=0} = -\frac{2}{\ell^2} = m_\phi^2. \tag{6.3.8}$$

Consequently, $\Delta_- = 1$ and

$$\phi(x,z) = z\left(\varphi_{(0)}(x) + z\varphi_{(1)}(x) + z^2\varphi_{(2)}(x) + \dots\right). \tag{6.3.9}$$

Next we have to substitute these expressions in the equations on motion, (6.3.1) and (6.3.3), and solve order by order in z. To facilitate this process we rewrite the equations of motion in terms of g_{ij} and A_i. Some useful results associated with the metric and Christoffel symbols that have been used in the analysis of the equations of motion are given in appendix 6.A. In what follows we use D to refer to the covariant derivative with respect to the 3–dimensional metric g and $' \equiv \mathrm{d}/\mathrm{d}z$.

Axion

In terms of g, the axion equations of motion are

$$\psi_I'' + g^{ij}\partial_i\partial_j\psi_I + \left(\frac{1}{2}\,\mathrm{Tr}\,\left(g^{-1}g'\right) - \frac{2}{z}\right)\psi_I' + \left(\partial_i g^{ij} + \frac{1}{2}g^{ij}\,\mathrm{Tr}\,\left(g^{-1}\partial_i g\right)\right)\partial_j\psi_I = 0. \tag{6.3.10}$$

Solving these order by order in z we obtain the following results,

$$\psi_I^{(1)}(x) = 0 \tag{6.3.11}$$

$$\psi_I^{(2)}(x) = \frac{1}{2}\,{}^{(g)}\Box_{(0)}\psi_I^{(0)}(x) = \frac{1}{2\sqrt{-g_{(0)}}}\partial_i\left(\sqrt{-g_{(0)}}g_{(0)}^{ij}\partial_j\psi_I^{(0)}(x)\right) \tag{6.3.12}$$

where ${}^{(g)}\Box_{(0)}$ the D'Alembertian with respect to $g_{(0)ij}$ and $g_{(0)}$ the determinant of $g_{(0)ij}$. As we shall see, the next order term in the expansion, $\psi_I^{(3)}$, is the *vev* of the axion. This term is unconstrained by the asymptotic equations of motion and thus our analysis of the axion terminates here.

Dialton

In terms of g, the dialton equation of motion are

$$\phi'' + g^{ij}\partial_i\partial_j\phi + \left(\frac{1}{2}\,\mathrm{Tr}\,\left(g^{-1}g'\right) - \frac{2}{z}\right)\phi' + \left(\partial_i g^{ij} + \frac{1}{2}g^{ij}\,\mathrm{Tr}\,\left(g^{-1}\partial_i g\right)\right)\partial_j\phi = \\ = \frac{\ell^2}{z^2}\frac{\mathrm{d}V(\phi)}{\mathrm{d}\phi} + \frac{z^2}{4\ell^2}\frac{\mathrm{d}Z(\phi)}{\mathrm{d}\phi}\left(2\tilde{A}'^2 + \tilde{F}^2\right) \tag{6.3.13}$$

where

$$\tilde{A}'^2 = g^{ij}A'_iA'_j = g^{ij}_{(0)}A^{(1)}_iA^{(1)}_j + \dots \tag{6.3.14}$$

$$\tilde{F}^2 = g^{ij}g^{kl}F_{ik}F_{jl} = 4g^{ij}_{(0)}g^{kl}_{(0)}\partial_{[i}A^{(0)}_{k]}\,\partial_{[j}A^{(0)}_{l]} + \dots \tag{6.3.15}$$

We also need the expansions for $\mathrm{d}Z(\phi)/\mathrm{d}\phi$ and $\mathrm{d}V(\phi)/\mathrm{d}\phi$. First we define

$$Z_n = \lim_{z\to 0}\frac{\mathrm{d}^{(n)}Z(\phi)}{\mathrm{d}\phi^{(n)}} \tag{6.3.16}$$

and

$$V_n = \lim_{z\to 0}\frac{\mathrm{d}^{(n)}V(\phi)}{\mathrm{d}\phi^{(n)}}. \tag{6.3.17}$$

Using these we obtain the desired expansions,

$$\frac{\mathrm{d}Z(\phi)}{\mathrm{d}\phi} = = Z_1 + z\varphi_{(0)}Z_2 + z^2\left(Z_3\varphi^2_{(0)} + 2Z_2\varphi_{(1)}\right) + \dots \tag{6.3.18}$$

$$\frac{\mathrm{d}V(\phi)}{\mathrm{d}\phi} = V_1 + z\varphi_{(0)}V_2 + \frac{z^2}{2}\left(V_3\varphi^2_{(0)} + 2V_2\varphi_{(1)}\right) \\ + \frac{z^3}{6}\left(V_4\varphi^3_{(0)} + 6V_3\varphi_{(0)}\varphi_{(1)} + 6V_2\varphi_{(2)}\right) + \dots \tag{6.3.19}$$

Turning back to the dialton equation of motion and making use of these expansions we find the following results,

$$V_1 = 0 \tag{6.3.20}$$

$$V_2 = -\frac{2}{l^2} \tag{6.3.21}$$

$$V_3\varphi_{(0)} = \frac{1}{\ell^2}\,\mathrm{Tr}\,\left(g^{-1}_{(0)}g_{(1)}\right) \tag{6.3.22}$$

$$\varphi_{(2)} = -\frac{1}{2}\,{}^{(g)}\Box_{(0)}\varphi_{(0)} - \frac{1}{2}\left(\frac{1}{2}\,\mathrm{Tr}\,\left(g^{-1}_{(1)}\,g_{(1)}\right) + \,\mathrm{Tr}\,\left(g^{-1}_{(0)}g_{(2)}\right) - \frac{\ell^2}{6}V_4\varphi^2_{(0)}\right)\varphi_{(0)} \tag{6.3.23}$$

Note that the asymptotic analysis did not constraint $\varphi_{(1)}$. This reflects the fact that $\varphi_{(1)}$ is the *vev* of the dual operator when the theory is subject to Dirichlet boundary conditions in which case $\varphi_{(0)}$ is the corresponding source, or, if Neumann or mixed boundary conditions are imposed, it is related to the source for the dual operator and $\varphi_{(0)}$ is the *vev*.

Scalar Einstein Equation

The rr component of the Einstein equations or scalar Einstein equation as we refer to it, given in terms of g is

$$-\frac{1}{2}\,\mathrm{Tr}\,\left(g^{-1}g''\right)+\frac{1}{4}\,\mathrm{Tr}\,\left(g^{-1}g'g^{-1}g'\right)+\frac{1}{2z}\,\mathrm{Tr}\,\left(g^{-1}g'\right)=\frac{z^2}{2\ell^2}\frac{Z(\phi)}{2}\left(g^{ij}A_i'A_j'-\frac{1}{2}\tilde{F}^2\right)+ \\ +\frac{1}{2}\sum_{I=1}^{2}\psi_I'^2+\frac{1}{2}\left(\phi'^2+\frac{\ell^2}{z^2}V(\phi)-2\Lambda\right) \tag{6.3.24}$$

where $\tilde{F}^2$ is given by equation (6.3.15). The expansions of $Z(\phi)$ and $V(\phi)$ are given by

$$Z(\phi)=Z_0+z\varphi_{(0)}Z_1+\dots \tag{6.3.25}$$

$$V(\phi)=V_0+z\varphi_{(0)}V_1+\frac{z^2}{2}\left(V_2\varphi_{(0)}^2+2V_1\varphi_{(1)}\right)+\frac{z^3}{6}\left(V_3\varphi_{(0)}^3+6V_2\varphi_{(0)}\varphi_{(1)}+6V_1\varphi_{(2)}\right)+\dots \\ =V_0-\frac{z^2}{\ell^2}\varphi_{(0)}^2+\frac{z^3}{6}\left(V_3\varphi_{(0)}^3-\frac{12}{\ell^2}\varphi_{(0)}\varphi_{(1)}\right)+\dots \tag{6.3.26}$$

where we have used the results obtained previously for V_1 and V_2. By solving the scalar Einstein equation order by order in z we obtain the following results.

$$V_0=2\Lambda \tag{6.3.27}$$

$$\mathrm{Tr}\,\left(g_{(0)}^{-1}g_{(1)}\right)=0 \tag{6.3.28}$$

$$\mathrm{Tr}\,\left(g_{(0)}^{-1}g_{(1)}g_{(0)}^{-1}g_{(1)}\right)=0 \tag{6.3.29}$$

$$\mathrm{Tr}\,\left(g_{(0)}^{-1}g_{(1)}g_{(0)}^{-1}g_{(2)}\right)-3\,\mathrm{Tr}\,\left(g_{(0)}^{-1}g_{(3)}\right)=2\phi_{(0)}\varphi_{(1)}+\frac{1}{6}\,\mathrm{Tr}\,\left(g_{(0)}^{-1}g_{(1)}\right)\varphi_{(0)}^2 \tag{6.3.30}$$

where we have used that $\psi_I^{(1)}=0$ to get the last two relations.

At this point one may be tempted to combine relations (6.3.28) and (6.3.22) and erroneously conclude that V_3 is zero. The reason this would be wrong is that the asymptotic expansions, and in particular the expansion for ϕ (6.3.7), is not the most general allowed and one should in general include a logarithmic term $z^2\log z\,\widetilde{\varphi}_{(1)}$, where we have set $\Delta_-=1$. Then, one find that V_3 is not zero but related to $\widetilde{\varphi}_{(1)}$ (see e.g. equations (2.18) and (2.19) in [93]). A logarithmic term in the ϕ expansion or, equivalently, a non–vanishing V_3, signal the presence of a conformal anomaly. However, for both of the theories we study in the next two chapters, the potential is such that $V_3=0$. As a consequence, in these cases, the coefficient of the logarithmic term vanishes and it is in anticipation of this result that we have omitted it, and the additional complications arising from it. Thus, although this analysis is valid and sufficient for the theories we are interested in, it should not be applied to V_3 since the result we find in this case, namely $V_3=0$, has, essentially, been put in as an assumption when choosing the ϕ expansion.

Tensorial Einstein Equations

In terms of g the ij components of the Einstein equations or, as we will refer to them, the tensorial Einstein equations, read

$$
\begin{aligned}
{}^{(g)}R_{ij} - \frac{1}{2}g_{ij} + \frac{1}{2z}g_{ij}\,\mathrm{Tr}\,\left(g^{-1}g'\right) + \frac{1}{2}\left(g'g^{-1}g'\right)_{ij} + \frac{1}{z}g'_{ij} = \frac{1}{2}\sum_{I=1}^{2}\partial_i\psi_I\partial_j\psi_I + \frac{1}{2}\partial_i\phi\partial_j\phi + \\
+\frac{\ell^2}{2z^2}(V-2\Lambda) + \frac{z^2}{\ell^2}\frac{Z(\phi)}{2}\left(A'_iA'_j - \frac{1}{4}g_{ij}\left(2\left(A'\right)^2 + \tilde{F}^2\right) + g^{kl}F_{ik}F_{jl}\right) \qquad (6.3.31)
\end{aligned}
$$

where the Ricci tensor admits the asymptotic expansion

$$
{}^{(g)}R_{ij} = {}^{(g)}R^{(0)}_{ij} + z\,{}^{(g)}R^{(1)}_{ij} + \dots \qquad (6.3.32)
$$

Using the asymptotic expansions and solving order by order in z we find the following results,

$$
g_{(1)ij} = 0 \qquad (6.3.33)
$$

$$
{}^{(g)}R^{(0)}_{ij} + g_{(2)ij} + g_{(0)ij}\,\mathrm{Tr}\,\left(g^{-1}_{(0)}g_{(2)}\right) = \frac{1}{2}\left(\sum_{I=1}^{2}\partial_i\psi^{(0)}_I\partial_j\psi^{(0)}_I - \varphi^2_{(0)}g_{(0)ij}\right) \qquad (6.3.34)
$$

$$
\mathrm{Tr}\,\left(g^{-1}_{(0)}g_{(3)}\right) = -\frac{2}{3}\varphi_{(0)}\varphi_{(1)}. \qquad (6.3.35)
$$

Furthermore, by tracing equation (6.3.34) we obtain

$$
4\,\mathrm{Tr}\,\left(g^{-1}_{(0)}g_{(2)}\right) = -\,{}^{(g)}R_{(0)} + \frac{1}{2}g^{ij}_{(0)}\sum_{I=1}^{2}\partial_i\psi^{(0)}_I\partial_j\psi^{(0)}_I - 3\varphi^2_{(0)}. \qquad (6.3.36)
$$

Finally, substituting this expression back into equation (6.3.34) we obtain an expression for $g_{(2)ij}$,

$$
\begin{aligned}
g_{(2)ij} = -\left({}^{(g)}R^{(0)}_{ij} - \frac{1}{4}g_{(0)ij}\,{}^{(g)}R_{(0)}\right) + \frac{1}{2}\sum_{I=1}^{2}\left(\partial_i\psi^{(0)}_I\partial_j\psi^{(0)}_I - \frac{1}{4}g_{(0)ij}g^{kl}_{(0)}\partial_k\psi^{(0)}_I\partial_l\psi^{(0)}_I\right) \\
-\frac{1}{8}g_{(0)ij}\varphi^2_{(0)} \qquad (6.3.37)
\end{aligned}
$$

Maxwell Equations

The equations of motion for the gauge field expressed in terms of g are

$$
D^i\left(Z(\phi)A'_i\right) = 0, \qquad (6.3.38)
$$

for the r component, and

$$
\left(Z(\phi)A'_i\right)' + \frac{1}{2}\,\mathrm{Tr}\,\left(g^{-1}g'\right)Z(\phi)A'_i - \left(g^{-1}g'\right)^j{}_i\,Z(\phi)A'_j + D^j\left(Z(\phi)F_{ji}\right) = 0 \qquad (6.3.39)
$$

for the remaining components. ${}^{(g)}D_i$ is the covariant derivative associated with g_{ij}.

Focusing first on (6.3.38) we find the following result,

$$D^i_{(0)}A^{(1)}_i = 0 \tag{6.3.40}$$

$$D^i_{(0)}A^{(2)}_i = -\frac{1}{2}g^{ij}_{(0)}\frac{Z_1}{Z_0}\partial_i\varphi_{(0)}A^{(1)}_j. \tag{6.3.41}$$

Note that the *vev* of the dual field theory current that couples to the gauge field is proportional to $A^{(1)}_i$ and hence, the coefficients $A^{(n)}_i$ for $n \geq 2$ do not contain additional information and it should be possible to obtain expressions for them in terms of the source coefficient $A^{(0)}_i$, and the *vev* coefficient $A^{(1)}_i$. We have included the expression for $A^{(2)}_i$ as a confirmation.

Repeating for equation (6.3.39) we find the following result,

$$D^j_{(0)}F^{(0)}_{ji} = -\left(\frac{Z_1}{Z_0}\varphi_{(0)}A^{(1)}_i + 2A^{(2)}_i\right). \tag{6.3.42}$$

Vector Einstein Equations

The ri components of the Einstein equations, or vector Einstein equations, are constraints which give the Ward identities associated with diffeomorphism. In terms of g these equations are given by

$$\frac{1}{2}g^{jk}\left(D_j g'_{ik} - D_i g'_{jk}\right) = \frac{z^2}{2\ell^2}Z(\phi)g^{jk}A'_jF_{ik} + \frac{1}{2}\sum_{I=1}^{2}\psi'_I\partial_i\psi_I + \frac{1}{2}\phi'\partial_i\phi. \tag{6.3.43}$$

Solving again order by order in z we obtain the following relation,

$$D^j_{(0)}g_{(3)ij} = \frac{2}{3\ell^2}Z_0 g^{jk}_{(0)}A^{(1)}_j\partial_{[i}A^{(0)}_{k]} + \sum_{I=1}^{2}\psi^{(3)}_I\partial_i\psi^{(0)}_I - \frac{1}{3}\varphi_{(0)}\partial_i\varphi_{(1)} \tag{6.3.44}$$

Once we derive the one point functions for the dual operators, we will return to this expression and re-express it in terms of field theory quantities.

This concludes the asymptotic construction of solutions to the equations of motion. Below we provide a summary of the results, keeping mostly terms up to the *vev*.

Metric:

$$\begin{aligned} g_{(1)ij} &= 0, \\ g_{(2)ij} &= -\left({}^{(g)}R^{(0)}_{ij} - \frac{1}{4}g_{(0)ij}\,{}^{(g)}R_{(0)}\right) - \frac{1}{8}g_{(0)ij}\varphi^2_{(0)} \\ &\quad + \frac{1}{2}\sum_{I=1}^{2}\left(\partial_i\psi^{(0)}_I\partial_j\psi^{(0)}_I - \frac{1}{4}g_{(0)ij}g^{kl}_{(0)}\partial_k\psi^{(0)}_I\partial_l\psi^{(0)}_I\right) \end{aligned} \tag{6.3.45}$$

$$\mathrm{Tr}\left(g_{(0)}^{-1}g_{(2)}\right) = -\frac{1}{4}\,{}^{(g)}R_{(0)} + \frac{1}{8}\sum_{I=1}^{2}\left(\partial\psi_I^{(0)}\right)^2 - \frac{3}{8}\varphi_{(0)}^2,$$
$$\mathrm{Tr}\left(g_{(0)}^{-1}g_{(3)}\right) = -\frac{2}{3}\varphi_{(0)}\varphi_{(1)} \tag{6.3.46}$$

Dialton potential:

$$V_0 = 2\Lambda = -\frac{6}{\ell^2}, \qquad V_1 = 0, \qquad V_2 = -\frac{2}{\ell^2}. \tag{6.3.47}$$

Dialton:

$$\varphi_{(2)} = -\frac{1}{2}{}^{(g)}\Box_{(0)}\varphi_{(0)} - \frac{1}{2}\,\mathrm{Tr}\left(g_{(0)}^{-1}g_{(2)}\right) + \frac{\ell^2}{12}V_4\varphi_{(0)}^3 \tag{6.3.48}$$

Note that $\varphi_{(2)}$ appears at higher order than the source and *vev* of the dual operator and, therefore, it does not contain any useful information for our analysis. We have nevertheless included it as a confirmation that it can be determined in terms of $\left(\varphi_{(0)}, \varphi_{(1)}\right)$.

Axions: The coefficients associated with the source and *vev* of the operator dual to the axions are $\psi_I^{(0)}$ and $\psi_I^{(3)}$, respectively, and hence they are not determined by the asymptotic analysis. Here, we present the results for $\psi_I^{(1)}$ and $\psi_I^{(2)}$ but in fact only $\psi_I^{(1)}$ would be needed in our subsequent calculation of the renormalised on-shell action.

$$\psi_I^{(1)}(x) = 0, \qquad \psi_I^{(2)}(x) = \frac{1}{2}{}^{(g)}\Box_{(0)}\psi_I^{(0)}(x) \tag{6.3.49}$$

Divergence identities - Gauge field: The coefficients associated with the source and the *vev* of the current dual to the gauge field are $A_i^{(0)}$ and $A_i^{(1)}$, respectively. Hence, the only useful relation we obtain for the gauge field from the asymptotic analysis is the divergence identity for $A_j^{(1)}$ which gives the Ward identity associated with the U(1) symmetry.

$$D_{(0)}^j A_j^{(1)} = 0. \tag{6.3.50}$$

Divergence identities - Metric: For the metric we find the following two divergence identities

$$D_{(0)}^i\left({}^{(g)}R_{(0)ij} - \frac{1}{2}g_{(0)ij}\,{}^{(g)}R_{(0)}\right) = -\varphi_{(0)}\partial_i\varphi_{(0)}$$
$$D_{(0)}^j g_{(3)ij} = \frac{2}{3\ell^2}Z_0 g_{(0)}^{jk}A_j^{(1)}\partial_{[i}A_{k]}^{(0)} + \sum_{I=1}^{2}\psi_I^{(3)}\partial_i\psi_i^{(0)} - \frac{1}{3}\varphi_{(0)}\partial_i\varphi_{(1)}. \tag{6.3.51}$$

The first one is trivially satisfied and it can be used to confirm that our results are correct. The second one gives rise to the Ward identity associated with diffeomorphism invariance.

6.3.3 Boundary Terms

In the previous section we derived the asymptotic solutions to the bulk equations of motions. In this section we discuss the boundary terms needed to impose our choice of boundary conditions and to make the variational problem well defined. In particular, if one proceeds to compute the renormalised on-shell action and attempts to vary the result with respect to the restrictions of the bulk fields on the conformal boundary, the process will not produce the correct stress tensor or 1-point function for the dialton. This is because varying the on-shell action with respect to the metric on the conformal boundary gives rise to a surface term proportional to $\partial_z \delta G_{\mu\nu}$ that does not vanish when imposing Dirichlet boundary conditions at fixed z. The boundary term necessary to remedy this issue is the well-known Gibbons-Hawking-York term discussed in section 1.3.2.

In the case of the dialton, the issue arises because we are imposing mixed boundary conditions as opposed to the standard Dirichlet conditions. As we briefly mentioned above, in order to keep an arbitrary function of $J_{\mathcal{F}}$ of $\varphi_{(0)}$ and $\varphi_{(2\Delta-d)} = \varphi_{(1)}$ we need to augment the bulk action by boundary terms that ensure that, upon varying the on-shell action with respect to the boundary value of the scalar, the resulting integrant is proportional $\delta J_{\mathcal{F}} \langle \mathcal{O}_\phi \rangle$.

Before we proceed to compute the on-shell action we discuss these two sets of boundary terms in more detail and derive the terms that we must add to the bulk action in order for obtain the correct one-point functions for the dual operators.

Extrinsic curvature and Gibbons-Hawking term

As we have already mentioned, the term needed to render the variation of the bulk action with respect to the bulk metric $G_{\mu\nu}$, subject to Dirichlet boundary conditions $\delta G_{\mu\nu} = 0$ well-posed is the Gibbons-Hawking term,

$$S_{\mathrm{GH}} = -2 \int_{z=\epsilon} \mathrm{d}^d x \, \sqrt{\gamma} K \tag{6.3.52}$$

where we have re-instated d for this discussion. $z = \epsilon$ defines the timelike cutoff surface $\mathcal{B}_\epsilon$ used to regularise the action, γ_{ij} is the induced metric on $\mathcal{B}_\epsilon$,

$$\gamma_{ij} = G_{ij}\big|_{z=\epsilon} = \frac{\ell^2}{\epsilon^2} g_{ij}, \tag{6.3.53}$$

and K is the trace of the extrinsic curvature tensor, $K_{\mu\nu}$, of the embedding of $\mathcal{B}_\epsilon$ in the spacetime. In particular, let n be the outward-pointing, unit normal to $\mathcal{B}_\epsilon$ which, for the metric (6.3.4), reads $n^\mu = -\frac{z}{\ell}(\partial_z)^\mu$. Using this we define the first fundamental form of the embedding as

$$h_{\mu\nu} = G_{\mu\nu} - n_\mu n_\nu. \tag{6.3.54}$$

Then, the extrinsic curvature tensor is

$$K_{\mu\nu} = -h_\mu{}^\rho h_\nu{}^\sigma \nabla_{(\rho} n_{\sigma)} \tag{6.3.55}$$

and the mean curvature, K is

$$K = G^{\mu\nu} K_{\mu\nu}. \tag{6.3.56}$$

The only non-vanishing components of $h_{\mu\nu}$ and $K_{\mu\nu}$ are

$$h_{ij} = \gamma_{ij} = \frac{\ell^2}{\epsilon^2} g_{ij} \quad \text{and} \quad K_{ij} = \frac{\epsilon}{2\ell}\partial_\epsilon \gamma_{ij} = \frac{\ell}{2\epsilon} g'_{ij} - \frac{\ell}{\epsilon^2} g_{ij}. \tag{6.3.57}$$

Using these expressions we finally obtain the expression for the Gibbons–Hawking term, both in terms of γ_{ij} and g_{ij}. The latter will be useful when we calculate the counter-terms needed to remove divergences of the on-shell action.

$$\begin{aligned} S_{\mathsf{GH}} &= -2 \int_{z=\epsilon} \mathrm{d}^d x \, \sqrt{-\gamma} \left(\frac{\epsilon}{2\ell} \gamma^{ij} \partial_\epsilon \gamma_{ij} \right) \\ &= -2 \int_{z=\epsilon} \mathrm{d}^d x \, \frac{\ell^d}{\epsilon^d} \sqrt{-g} \left(\frac{\epsilon}{2\ell} \operatorname{Tr}\left(g^{-1} g'\right) - \frac{d}{\ell} \right) \end{aligned} \tag{6.3.58}$$

Let us see now why we need the Gibbons-Hawking term. Varying the bulk action with respect to the metric, ignoring the Gibbons–Hawking term for now, we find upon integrating by parts

$$\begin{aligned} \delta_G S_{\mathsf{bulk}} = \int_{z>\epsilon} & d^{d+1}x \sqrt{-G} \, [\, \text{bulk e.o.m.} \,]_{\mu\nu} \, \delta G^{\mu\nu} \\ & + \int_{z=\epsilon} d^d x \sqrt{-\gamma} \, n^\mu \left(G^{\rho\sigma} \nabla_\rho \delta G_{\mu\sigma} - G^{\rho\sigma} \nabla_\mu \delta G_{\rho\sigma} \right). \end{aligned} \tag{6.3.59}$$

Substituting $G^{\rho\sigma} = h^{\rho\sigma} + n^\rho n^\sigma$, the boundary term becomes

$$\int_{z=\epsilon} d^d x \sqrt{-\gamma} \, n^\mu \left(h^{\rho\sigma} \nabla_\rho \delta G_{\mu\sigma} - h^{\rho\sigma} \nabla_\mu \delta G_{\rho\sigma} \right). \tag{6.3.60}$$

If we impose Dirichlet boundary conditions, $\delta G_{\mu\nu}|_{\mathcal{B}_\epsilon} = 0$, the first term in the integrant vanishes, since it is the derivative of $\delta G_{\mu\nu}$, along some boundary direction. On the other hand, the second term does not vanish, as it is the change of $\delta G_{\mu\nu}$ moving away from the boundary. The variational principle with Dirichlet boundary conditions $\delta G_{\mu\nu} = 0$ on $\mathcal{B}_\epsilon$ is thus not well-posed.

Turning our attention now to the Gibbons–Hawking term, varying it with respect to the metric we find

$$\delta_G S_{\mathsf{GH}} = \int_{z=\epsilon} d^d x \sqrt{-\gamma} \, \left[\left(K\gamma_{ij} - 2K_{ij} \right) \delta G^{ij} - 2 n^\mu h^{\rho\sigma} \nabla_\rho \delta G_{\mu\sigma} + n^\mu h^{\rho\sigma} \nabla_\mu \delta G_{\rho\sigma} \right]. \tag{6.3.61}$$

Combining all contributions

$$\delta_G(S_{\mathsf{bulk}} + S_{\mathsf{GH}}) = \int_{z=\epsilon} d^d x \sqrt{-\gamma} \left[(K\gamma_{ij} - 2K_{ij}) \, \delta G^{ij} - n^\mu h^{\rho\sigma} \nabla_\rho \delta G_{\mu\sigma} \right]. \tag{6.3.62}$$

To deal with the last term, we write it as

$$-n^\mu h^{\rho\sigma} \nabla_\rho \delta G_{\mu\sigma} = -h^{\rho\sigma} \nabla_\rho \left(n^\mu \delta G_{\mu\sigma} \right) - h^{\rho\sigma} \nabla_\rho n^\mu \delta_{\mu\sigma} \tag{6.3.63}$$

$$= \mathcal{D}_i \left(\gamma^{ij} n^\mu \delta G_{\mu j} \right) - h^{\rho\sigma} G^{\mu\nu} \left(\nabla_\rho n_\nu \right) \delta G_{\mu\sigma} \tag{6.3.64}$$

with $\mathcal{D}_i$ the covariant derivative associated to the induced metric γ_{ij}. The total derivative yields zero when integrated, and the remaining term becomes $K_{\mu\nu} \delta G^{\mu\nu}$. Finally, the metric variation of S_0 is

$$\delta_G(S_{\mathsf{bulk}} + S_{\mathsf{GH}}) = \int_{z>\epsilon} d^{d+1} x \sqrt{-G} \, E_{\mu\nu} \delta G^{\mu\nu} - \int_{z=\epsilon} d^d x \sqrt{-\gamma} \left(K_{ij} - K\gamma_{ij} \right) \delta\gamma^{ij}, \tag{6.3.65}$$

where we took $\delta\gamma^{ij} = \delta G^{ij}$. Hence, the addition of the Gibbons-Hawking term renders the action functional differentiable when imposing Dirichlet boundary conditions on $\mathcal{B}_\epsilon$, leading to a well-defined variational problem. The resulting equations of motion are $E_{\mu\nu} = 0$.

Mixed Boundary Conditions for the Dialton

As we have already mentioned, we are interested in theories where the dual field theory contains a triple trace deformation of the operator dual to the dialton. In the bulk, a deformation of this form is related to mixed boundary conditions for the dialton. More precisely, near the asymptotic boundary at $z = 0$ we saw that the dialton admits the series expansion

$$\phi(x, z) = z^{\Delta_-} \varphi_{(0)}(x) + z^{\Delta_+} \varphi_{(1)}(x) + \dots \tag{6.3.66}$$

where $\Delta_- = 1$ and $\Delta_+ = 2$ and the ellipses correspond to higher order terms in z. Imposing Dirichlet boundary conditions corresponds to fixing $\varphi_{(0)}$ at the boundary which is then interpreted as the source that couples to the dual operator. The expectation value of the dual operator is then proportional to $\varphi_{(1)}$ and its conformal dimension is $\Delta_+ = 2$. Alternatively, when the mass of the dialton lies in the range $-d/4 \leq m^2 \ell^2 \leq -d^2/4 + \frac{1}{4}$ one can consider a setup where $\varphi_{(0)}$ is the expectation value of the operator whose dimension is now Δ_- and $\varphi_{(1)}$, or more precisely a term proportional to $\varphi_{(1)}$, is the source that couples to it. This of course corresponds to imposing Neumann boundary conditions on the bulk field, as we have already discussed. On the field theory side, the Δ_- theory is obtained from the Δ_+ theory via a Legendre transform of its generating functional that exchanges the roles of the source and expectation value of the operator.

Finally, the last option which is the one relevant to our analysis, is to impose mixed boundary conditions in the bulk, thus introducing a multitrace deformation in the dual theory of the form $\int \mathrm{d}^d x \, \mathcal{F}\left(\mathcal{O}\right)$ where $\mathcal{F}$ is a polynomial of degree n, $2 \leq n \leq d/\Delta_- = 3$.

The source we keep fix in this case is given by

$$J_{\mathcal{F}} = -\ell^2 \varphi_{(1)} - \mathcal{F}'\left(\varphi_{(0)}\right). \tag{6.3.67}$$

In order to ensure that the variation with respect to the dialton of the regularised on-shell bulk action (plus boundary terms) is proportional to $\delta J_{\mathcal{F}}$, i.e.

$$\delta_\phi\left(S_{\text{reg}} + S_{\text{ct}} + S_{\text{boundary}}\right) = \int_{z=\epsilon} \mathrm{d}^d x \sqrt{-g_{(0)}}\, \langle \mathcal{O}_\phi \rangle\, \delta J_{\mathcal{F}}, \tag{6.3.68}$$

we have to augment the action by adding a boundary term, S_f, given by

$$S_f = \int_{z=\epsilon} \mathrm{d}^d x \sqrt{-g_{(0)}}\, \left(J_{\mathcal{F}} \varphi_{(0)} + \mathcal{F}\left(\varphi_{(0)}\right)\right). \tag{6.3.69}$$

S_{reg} is the regulated on-shell action and S_{ct} is the counter-term action necessary to cancel the divergences of the on-shell action. For the derivation and complete expressions of both these terms we refer the reader to section 6.3.4. For now it is sufficient to note that

$$\delta_\phi\left(S_{\text{reg}} + S_{\text{ct}}\right) = \ell^2 \varphi_{(1)} \delta\varphi_{(0)} + O(\epsilon). \tag{6.3.70}$$

Using this result and combining it with the expression for S_f we confirm (6.3.68) and find the expectation value of $\mathcal{O}_\phi$,

$$\begin{aligned}\delta_\phi\left(S_{\text{reg}} + S_{\text{ct}} + S_f\right) =& \int_{z=\epsilon} \mathrm{d}^d x \sqrt{-g_{(0)}}\, \Big(\ell^2 \varphi_{(1)} \delta\varphi_{(0)} + \left(\delta J_f\right) \varphi_{(0)} \\ &+ J_f \delta\varphi_{(0)} + \mathcal{F}'\left(\varphi_{(0)}\right) \delta\varphi_{(0)} + O(\epsilon)\Big) \\ =& \int_{z=\epsilon} \mathrm{d}^d x \sqrt{-g_{(0)}}\, \Big(\varphi_{(0)} \delta J_f + O(\epsilon)\Big).\end{aligned}$$

Hence, we have confirmed that the addition of S_f to the bulk action ensures we obtain the correct variational problem, subject to the desired boundary condition $\delta J_f = 0$. Furthermore, the 1-point function of the dual operator is

$$\langle \mathcal{O}_\phi \rangle = \varphi_{(0)}. \tag{6.3.71}$$

This is the 1-point function one obtains when imposing Neumann boundary condition and, hence, the mixed boundary conditions lead to a deformation of the Neumann theory. The dimension of the dual operator in this case is Δ_-.

Next we turn our attention to the function $\mathcal{F}\left(\varphi_{(0)}\right)$. We are interested in triple trace deformations of the boundary theory and therefore we choose $\mathcal{F}$ to be of the form

$$\mathcal{F}\left(\varphi_{(0)}\right) = \frac{1}{3} \vartheta \varphi_{(0)}^3 \tag{6.3.72}$$

where the coupling constant ϑ can be a free parameter or it could be constrained by the theory we are considering. The boundary term we must add to the bulk action is given by

$$S_f = \int_{z=\epsilon} \mathrm{d}^3x \sqrt{-g_{(0)}} \left(-\ell^2 \varphi_{(0)}\varphi_{(1)} - \frac{2}{3}\vartheta\varphi_{(0)}^3 \right). \tag{6.3.73}$$

Finally, note that since $d = 3$ and $\Delta_- = 1$ we have $d/\Delta_- = 3$ and thus this is a marginal deformation.

6.3.4 Renormalised On–Shell Action

Holographic Renormalisation: A Brief Introduction

We have now identified two surface terms that need to be added to the bulk action to make the theory described by (6.2.1) a well defined variational problem. Hence, the full action is

$$S = S_{\text{bulk}} + S_{\text{GH}} + S_f. \tag{6.3.74}$$

Next we shall compute the renormalised on–shell action. Let us begin by briefly reviewing the program of holographic renormalisation [17, 19, 22, 25].

According to the AdS/CFT correspondence, the on–shell bulk action evaluated as a function of the boundary values of fields, collectively denoted by $f_{(0)}$ here, is identified with the generating functional of the field theory correlation functions,

$$-W_{\text{QFT}}[f_{(0)}] = S_{\text{on-shell}}[f_{(0)}]. \tag{6.3.75}$$

However, both objects suffer from divergences; for the field theory these are short distance UV divergences which are identified as IR divergences in the bulk. Furthermore, the IR in the bulk corresponds to the near boundary region. Since in quantum field theory UV divergences do not depend on the IR physics, it follows that dealing with the corresponding bulk divergences requires only the near boundary behaviour of the fields. Therefore, we only need the asymptotic solutions of the fields in order to renormalise the on–shell bulk action.

The program of holographic renormalisation involves first regularising the on–shell action by introducing a cutoff spacetime surface $\mathcal{B}_\epsilon$ near the asymptotic boundary. This allows us to integrate over the holographic direction and find the terms which diverge when $\mathcal{B}_\epsilon$ is send to the boundary. There will be a finite number of divergent terms and to deal with them we introduce counterterms on $\mathcal{B}_\epsilon$. The counterterms must be covariant and this entails inverting the asymptotic expansions of the fields to the appropriate order in ϵ. More specifically, let $\mathcal{F}(x, z)$ be a bulk field. On $\mathcal{B}_\epsilon$ is admits a series expansion of the form

$$\mathcal{F}(x, \epsilon) = \epsilon^m f_{(0)}(x) + \epsilon^{m+1} f_{(1)}(x) + \ldots + \epsilon^n \left(f_{(n)}(x) + \log(\epsilon)\tilde{f}_{(n)}(x) \right) + \ldots \tag{6.3.76}$$

where the logarithmic term is only present if d is even. The on–shell action and counterterms are evaluated as a function of the "source" $f_{(0)}$ which is not covariant in the bulk.

To make the expressions covariant we must invert this expansion to obtain $f_{(0)}(x) = f_{(0)}\left(\mathcal{F}(x,\epsilon),\epsilon\right)$ and this is done order by order in ϵ as we will see later.

Once we have obtained the subtracted on-shell action, $S_{\text{sub}}\left[\mathcal{F}(x,\epsilon);\epsilon\right]$, the 1-point function of the dual, field theory operator is given by

$$\langle \mathcal{O}_{\mathcal{F}} \rangle = \lim_{\epsilon\to 0}\left(\frac{1}{\epsilon^{d-m}}\frac{1}{\sqrt{\gamma}}\frac{\delta S_{\text{sub}}}{\delta \mathcal{F}(x,\epsilon)}\right). \tag{6.3.77}$$

We proceed with the renormalisation of the action (6.2.1) using the asymptotic solutions obtained in section 6.3.2. This will allow us to compute the 1-point functions of the operators in the dual field theory.

Regularised On-Shell Action

We start with the action (6.2.1) and use the equations of motion to obtain

$$S_{\text{on-shell}} = \int_{\mathcal{M}} \mathrm{d}^4x\sqrt{-G}\left(V(\phi) - \frac{1}{4}Z(\phi)F^2\right) + S_{\text{GH}} + S_f \tag{6.3.78}$$

where

$$S_{\text{GH}} = -2\int_{z=\epsilon}\mathrm{d}^3x\sqrt{-\gamma}\,K \tag{6.3.79}$$

and

$$S_f = \int_{z=\epsilon}\mathrm{d}^3x\sqrt{-g_{(0)}}\left(-\ell^2\varphi_{(0)}\varphi_{(1)} - \frac{2}{3}\vartheta\varphi_{(0)}^3\right) \tag{6.3.80}$$

Using the asymptotic form of the fields, (6.3.4) – (6.3.7), and the corresponding solutions (6.3.45)–(6.3.49), we rewrite the fields as

$$\mathrm{d}s^2 = G_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu = \frac{\ell^2}{z^2}\mathrm{d}z^2 + \frac{\ell^2}{z^2}g_{ij}(x,z)\mathrm{d}x^i\mathrm{d}x^j \tag{6.3.81}$$

where

$$g_{ij}(x,z) = g_{(0)ij}(x) + z^2 g_{(2)ij} + z^3 g_{(3)ij} + O(z^4), \tag{6.3.82}$$

$$g^{ij}(x,z) = g_{(0)}^{ij}(x) - z^2\left(g_{(0)}^{-1}g_{(2)}g_{(0)}^{-1}\right)^{ij} + O(z^3), \tag{6.3.83}$$

$$g_{(2)ij} = -\left({}^{(g)}R_{ij}^{(0)} - \frac{1}{4}g_{(0)ij}\,{}^{(g)}R_{(0)}\right) + \frac{1}{2}\sum_{I=1}^{2}\left(\partial_i\psi_I^{(0)}\partial_j\psi_I^{(0)} - \frac{1}{4}g_{(0)ij}g_{(0)}^{kl}\partial_k\psi_I^{(0)}\partial_l\psi_I^{(0)}\right) - \frac{1}{8}g_{(0)ij}\varphi_{(0)}^2, \tag{6.3.84}$$

and

$$A_i(x,z) = A_i^{(0)}(x) + zA_i^{(1)}(x) + \frac{z^2}{2}\left({}^{(g)}D_{(0)}^jF_{ij}^{(0)} - \frac{Z_1}{Z_0}\varphi_{(0)}A_i^{(1)}\right) + O(z^3) \tag{6.3.85}$$

$$\psi_I(x,z) = \psi_I^{(0)}(x) + \frac{z^2}{2}\,{}^{(g)}\Box_{(0)}\psi_I^{(0)}(x) + O(z^3) \tag{6.3.86}$$

$$\phi(x,z) = z\varphi_{(0)}(x) + z^2\varphi_{(1)}(x) + O(z^3) \tag{6.3.87}$$

$g_{(3)ij}$, $\varphi_{(1)}(x)$ and $A_i^{(1)}(x)$ are related to the *vevs* of the dual operators and to determine them we need the full bulk solutions.

In addition to these expansions we have that

$$V(\phi) = -\frac{6}{\ell^2} - \frac{z^2}{\ell^2}\varphi_{(0)} - \frac{2z^3}{\ell^2}\varphi_{(0)}\varphi_{(1)} + O(z^4) \tag{6.3.88}$$

$$Z(\phi) = Z_0 + O(z) \tag{6.3.89}$$

$$\sqrt{-G} = \frac{\ell^4}{z^4}\sqrt{-g_{(0)}}\left[1 - \frac{z^2}{8}\left({}^{(g)}R_{(0)} - \frac{1}{2}\sum_{I=1}^{2}\left(\partial\psi_I^{(0)}\right)^2 + \frac{3}{2}\varphi_{(0)}^2\right) - \frac{z^3}{3}\varphi_{(0)}\varphi_{(1)}\right] \tag{6.3.90}$$

Focusing first on the bulk action given by the first term in (6.3.78), we are interested in terms which, after we integrate the on–shell action with respect to z for $z > \epsilon$, are divergent in the limit $\epsilon \to 0$. To obtain these, we only need to keep terms in the integrant that are of order $O(z^{-1})$. $\sqrt{-G}\, Z(\phi)F^2$ is of order $O(z^0)$ and it does not lead to divergences. Hence, we only need to worry about the contribution due to

$$\int_{z>\epsilon} \mathrm{d}^4x\sqrt{-G}\, V(\phi).$$

Using the above expressions,

$$\begin{aligned}
S_{\text{bulk}} =& \int_{z>\epsilon} \mathrm{d}^4x\, \frac{\ell^4}{z^4}\sqrt{-g_{(0)}}\left[1 - \frac{z^2}{8}\left({}^{(g)}R_{(0)} - \frac{1}{2}\sum_{I=1}^{2}\left(\partial\psi_I^{(0)}\right)^2 + \frac{3}{2}\varphi_{(0)}^2\right)\right. \\
&\left. - \frac{z^3}{3}\varphi_{(0)}\varphi_{(1)} + O(z^4)\right]\left[-\frac{6}{\ell^2} - \frac{z^2}{\ell^2}\varphi_{(0)}^2 - 2z^3\frac{1}{\ell^2}\varphi_{(0)}\varphi_{(1)} + O(z^4)\right] \\
=& \int_{z>\epsilon} \mathrm{d}^4x\sqrt{-g_{(0)}}\,\frac{\ell^4}{z^4}\left[-\frac{6}{\ell^2} + \frac{3z^2}{4\ell^2}\left({}^{(g)}R_{(0)} - \frac{1}{2}\sum_{I=1}^{2}\left(\partial\psi_I^{(0)}\right)^2 + \frac{1}{6}\varphi_{(0)}^2\right) + O(z^4)\right] \\
=& \int_{z=\epsilon} \mathrm{d}^3x\sqrt{-g_{(0)}}\left[-\frac{2\ell^2}{\epsilon^3} + \frac{3\ell^2}{4\epsilon}\left({}^{(g)}R_{(0)} - \frac{1}{2}\sum_{I=1}^{2}\left(\partial\psi_I^{(0)}\right)^2 + \frac{1}{6}\varphi_{(0)}^2\right)\right] + \mathrm{O}(\epsilon)
\end{aligned} \tag{6.3.91}$$

To this we must add the contribution from S_{GH} and S_f. To compute S_{GH} we need to compute the trace of the extrinsic curvature of the embedding of $\mathcal{B}_\epsilon$ in the spacetime. From equation (6.3.58) we have

$$K = \frac{\epsilon}{2\ell}\,\mathrm{Tr}\,\left(g^{-1}g'\right)\Big|_{z=\epsilon} - \frac{3}{\ell}. \tag{6.3.92}$$

Using the expansions for g^{-1} and g, as well as the expressions for the traces $\mathrm{Tr}\left(g_{(0)}^{-1}g_{(2)}\right)$ and $\mathrm{Tr}\left(g_{(0)}^{-1}g_{(3)}\right)$ from (6.3.46),

$$K = -\frac{3}{\ell} - \frac{\epsilon^2}{4\ell}\left({}^{(g)}R_{(0)} - \frac{1}{2}\sum_{I=1}^{2}\left(\partial\psi_I^{(0)}\right)^2 + \frac{3}{2}\varphi_{(0)}^2\right) - \frac{\epsilon^3}{\ell}\varphi_{(0)}\varphi_{(1)} + O(\epsilon^4) \tag{6.3.93}$$

Furthermore, using the relation $\gamma_{ij} = \frac{\ell^2}{\epsilon^2} g_{ij}\Big|_{z=\epsilon}$, the determinant of γ_{ij} is

$$\sqrt{-\gamma} = \frac{\ell^3}{\epsilon^3}\sqrt{-g_{(0)}}\left[1 - \frac{\epsilon^2}{8}\left({}^{(g)}R_{(0)} - \frac{1}{2}\sum_{I=1}^{2}\left(\partial\psi_I^{(0)}\right)^2 + \frac{3}{2}\varphi_{(0)}^2\right) - \frac{\epsilon^3}{3}\varphi_{(0)}\varphi_{(1)} + O(\epsilon^4)\right]. \tag{6.3.94}$$

Combining these expressions and keeping only terms that do not vanish in the limit $\epsilon \to 0$, we find the following Gibbons–Hawkings term,

$$S_{\text{GH}} = \int_{z=\epsilon} \mathrm{d}^3x\,\sqrt{-g_{(0)}}\left[\frac{6\ell^2}{\epsilon^3} - \frac{\ell^2}{4\epsilon}\left({}^{(g)}R_{(0)} - \frac{1}{2}\sum_{I=1}^{2}\left(\partial\psi_I^{(0)}\right)^2 + \frac{3}{2}\varphi_{(0)}^2\right) + O(\epsilon)\right]. \tag{6.3.95}$$

The final term we need to evaluate is the boundary term associated with the dialton, S_f. Using equations (6.3.67) and (6.3.69) with $d = 3$ we have

$$S_f = \int_{z=\epsilon} \mathrm{d}^3x\sqrt{-g_{(0)}}\left(-\ell^2\varphi_{(0)}\varphi_{(1)} - \frac{2}{3}\vartheta\,\varphi_{(0)}^3\right). \tag{6.3.96}$$

Combining all three terms, S_{bulk}, S_{GH} and S_f, we find the regularised on-shell action,

$$\begin{aligned} S_{\text{reg}} =& S_{\text{Bulk}} + S_{\text{GH}} + S_f \\ =& \int_{z=\epsilon} \mathrm{d}^3x\sqrt{-g_{(0)}}\left[\frac{4\ell^2}{\epsilon^3} + \frac{\ell^2}{2\epsilon}\left({}^{(g)}R_{(0)} - \frac{1}{2}\sum_{I=1}^{2}\left(\partial\psi_I^{(0)}\right)^2 - \frac{1}{2}\varphi_{(0)}^2\right)\right. \\ & \left. -\ell^2\varphi_{(0)}\varphi_{(1)} - \frac{2}{3}\vartheta\varphi_{(0)}^3\right] \end{aligned} \tag{6.3.97}$$

Counterterm Action

From (6.3.97) one immediately identifies the divergent terms which need to be subtracted to render the action finite,

$$S_{\text{ct}} = -\int_{z=\epsilon} \mathrm{d}^3x\sqrt{-g_{(0)}}\left[\frac{4\ell^2}{\epsilon^3} + \frac{\ell^2}{2\epsilon}\left({}^{(g)}R_{(0)} - \frac{1}{2}\sum_{I=1}^{2} g_{(0)}^{ij}\partial_i\psi_I^{(0)}\partial_j\psi_I^{(0)} - \frac{1}{2}\varphi_{(0)}^2\right)\right]. \tag{6.3.98}$$

This action is not covariant so it does not respect the symmetries of the bulk. To obtain a covariant expression we must invert the expressions for the boundary values of the fields and expressed them in terms of the covariant fields that live on the the cutoff surface $\mathcal{B}_\epsilon$.

We begin with the "forward" expansions for the fields, given by equations (6.3.82)–(6.3.87). We will also need to invert $\sqrt{-g_{(0)}}$. This admits the expansion

$$\sqrt{-g_{(0)}} = \frac{\epsilon^3}{\ell^3}\sqrt{-\gamma} = \sqrt{-g_{(0)}}\left(1 + \frac{\epsilon^2}{2}\,\mathrm{Tr}\,\left(g_{(0)}^{-1}g_{(2)}\right) + O\left(\epsilon^3\right)\right). \tag{6.3.99}$$

These expressions can be inverted order by order in ϵ to obtain expressions for $g_{(0)ij}(x)$, $\varphi_{(0)}(x)$, $\psi_I^{(0)}(x)$, $A_i^{(0)}(x)$ and $\sqrt{-g_{(0)}}$.

Leading order in ϵ:

To lowest order, the inverted expressions for the fields are simply

$$g_{(0)ij}(x) = \frac{\epsilon^2}{\ell^2}\gamma_{ij}(x,\epsilon), \quad \varphi_{(0)}(x) = \frac{1}{\epsilon}\phi(x,\epsilon), \quad \psi_I^{(0)}(x) = \psi_I(x,\epsilon), \quad A_i^{(0)}(x) = A_i(x,\epsilon). \tag{6.3.100}$$

Furthermore, it is easy to see that ${}^{(g)}\Box_{(0)} \equiv \frac{1}{\sqrt{-g_{(0)}}}\partial_i\left(\sqrt{-g_{(0)}}g_{(0)}^{ij}\partial_j\right)$ can be rewritten in terms of ${}^{(\gamma)}\Box$ as

$${}^{(g)}\Box_{(0)} = \frac{\ell^2}{\epsilon^2}\,{}^{(\gamma)}\Box \tag{6.3.101}$$

and similarly for all contracted quantities. For example ${}^{(g)}R_{(0)} = \frac{\ell^2}{\epsilon^2}\,{}^{(\gamma)}R$.
Finally, inverting the expression for $\sqrt{-g_{(0)}}$ to leading order we find

$$\sqrt{-g_{(0)}} = \frac{\epsilon^3}{\ell^3}\sqrt{-\gamma}. \tag{6.3.102}$$

Next to leading order in ϵ

To obtain the inversions to the next order we substitute the leading order expressions (6.3.100) into the forward expansions keeping the next order terms. That is to say, we substitute the expressions (6.3.100) into the expressions for $g_{(2)ij}$ and $\psi_I^{(2)}$. $\varphi_{(0)}(x)$ and $A_i^{(0)}(x)$ can not be computed to higher order in ϵ since the next order terms correspond to the *vevs* of the dual operators and these can not be determined from the asymptotics of the fields. Starting from the expression for $\psi_I^{(2)}(x)$, $\psi_I^{(2)}(x) = \frac{1}{2}{}^{(g)}\Box_{(0)}\psi_I^{(0)}(x)$, we obtain

$$\psi_I^{(2)}(x) = \frac{\ell^2}{2\epsilon^2}\,{}^{(\gamma)}\Box\psi_I(x,\epsilon), \tag{6.3.103}$$

and,

$$\psi_I^{(0)}(x) = \psi_I(x,\epsilon) - \frac{\ell^2}{2}{}^{(\gamma)}\Box\psi_I(x,\epsilon) + O(\epsilon^3). \tag{6.3.104}$$

Rewriting $g_{(2)ij}$ in terms of the inverted expressions is straightforward, albeit more tedious. Using equations (6.3.100) and the fact that ${}^{(g)}R_{(0)ij} = {}^{(\gamma)}R_{ij} + O(\epsilon^2)$ we obtain

$$g_{(2)ij} = -\left({}^{(\gamma)}R_{ij} - \frac{1}{4}\gamma_{ij}\,{}^{(\gamma)}R\right) + \frac{1}{2}\sum_{I=1}^{2}\left(\partial_i\psi_I\partial_j\psi_I - \frac{1}{4}\gamma_{ij}\gamma^{kl}\partial_k\psi_I\partial_l\psi_I\right) - \frac{1}{8\ell^2}\gamma_{ij}\phi^2 \tag{6.3.105}$$

and

$$g_{(0)ij} = \epsilon^2 \Bigg[\frac{1}{\ell^2}\gamma_{ij} + \left({}^{(\gamma)}R_{ij} - \frac{1}{4}\gamma_{ij}\, {}^{(\gamma)}R \right) - \frac{1}{2}\sum_{I=1}^{2} \left(\partial_i\psi_I \partial_j \psi_I - \frac{1}{4}\gamma_{ij} \left(\partial\psi_I\right)^2 \right) + \frac{1}{8\ell^2}\gamma_{ij}\phi^2 \Bigg] + O\left(\epsilon^4\right). \tag{6.3.106}$$

To compute $\sqrt{-g_{(0)}}$ to the next order in ϵ we make use of the relation

$$\mathrm{Tr}\left(g_{(0)}^{-1} g_{(2)}\right) = -\frac{1}{4}\, {}^{(g)}R_{(0)} + \frac{1}{8}\sum_{I=1}^{2}\left(\partial\psi_I^{(0)}\right)^2 - \frac{3}{8}\varphi_{(0)}^2. \tag{6.3.107}$$

In terms of the covariant fields,

$$\mathrm{Tr}\left(g_{(0)}^{-1} g_{(2)}\right) = -\frac{\ell^2}{4\epsilon^2}\, {}^{(\gamma)}R + \frac{\ell^2}{8\epsilon^2}\sum_{I=1}^{2}\left(\partial\psi_I\right)^2 - \frac{3}{8\epsilon^2}\phi^2 \tag{6.3.108}$$

and

$$\sqrt{-g_{(0)}} = \frac{\epsilon^3}{\ell^3}\left[1 + \frac{\ell^2}{8}\left({}^{(\gamma)}R - \frac{1}{2}\sum_{I=1}^{2}\left(\partial\psi_I\right)^2 - \frac{3}{2\ell^2}\phi^2 \right)\right] \tag{6.3.109}$$

It is not possible to compute the boundary fields to higher order as the next terms in both the metric and axion expansions correspond to *vevs* of their respective dual operators. However, the expressions obtained are sufficient to rewrite all terms in the counterterm action in a covariant form.

Summarising, we have the following covariant expressions

$$g_{(0)ij}(x) = \epsilon^2 \Bigg[\frac{1}{\ell^2}\gamma_{ij} + \left({}^{(\gamma)}R_{ij} - \frac{1}{4}\gamma_{ij}\, {}^{(\gamma)}R \right) - \frac{1}{2}\sum_{I=1}^{2} \left(\partial_i\psi_I \partial_j \psi_I - \frac{1}{4}\gamma_{ij} \left(\partial\psi_I\right)^2 \right) + \frac{1}{8\ell^2}\gamma_{ij}\phi^2 \Bigg] + O\left(\epsilon^4\right). \tag{6.3.110}$$

$$\varphi_{(0)}(x) = \frac{1}{\epsilon}\phi(x,\epsilon) + O(\epsilon) \tag{6.3.111}$$

$$\psi_I^{(0)}(x) = \psi_I(x,\epsilon) - \frac{\ell^2}{2}{}^{(\gamma)}\Box\psi_I(x,\epsilon) + O(\epsilon^3) \tag{6.3.112}$$

$$A_i^{(0)}(x) = A_i(x,\epsilon) + O(\epsilon) \tag{6.3.113}$$

$$\sqrt{-g_{(0)}} = \frac{\epsilon^3}{\ell^3}\sqrt{-\gamma}\left[1 + \frac{\ell^2}{8}\left({}^{(\gamma)}R - \frac{1}{2}\sum_{I=1}^{2}\left(\partial\psi_I\right)^2 - \frac{3}{2\ell^2}\phi^2 \right)\right] \tag{6.3.114}$$

The final step is to substitute these expressions in the counterterm action (6.3.98), ensuring that we keep all terms that do not vanish in the limit $\epsilon \to 0$. After some algebra we obtain

$$S_{\rm ct} = -\int_{z=\epsilon} \mathrm{d}^3x\sqrt{-\gamma}\left(\frac{4}{\ell^2} + \ell^{(\gamma)}R + \frac{1}{2\ell}\phi^2 - \frac{\ell^2}{2}\gamma^{ij}\partial_i\psi_I\partial_j\psi_I\right), \tag{6.3.115}$$

and,

$$S_{\rm ren} = \lim_{\epsilon\to 0}\left(S_{\rm reg} + +S_{\rm ct}\right), \tag{6.3.116}$$

where $S_{\rm reg} = \left(S_{\rm Bulk} + S_{\rm GH} + S_f\right)_{z=\epsilon}$

6.3.5 1–Point Functions of the Dual Operators

In this section we proceed to compute the 1–point functions of the operators dual to the bulk fields. This involved varying the renormalised action with respect to the boundary values of the bulk fields, i.e. with respect to $g_{(0)ij}$, $A_i^{(0)}$, $\psi_I^{(0)}$ and $\varphi_{(0)}$. When Dirichlet boundary conditions are imposed, these are identified as the sources that couple to the dual operator. For the theory at hand, this is still true for all fields except the dialton. As we have already explained, we are imposing mixed boundary fields on the dialton and $\varphi_{(0)}$ is no longer the source. This is taken care of by the boundary term we have added to the action S_f. Thus, varying the renormalised action–including S_f–with respect to the boundary values of the fields, we obtain

$$\delta S_{\rm ren} = [\text{bulk e.o.m}] + \int \mathrm{d}^3x\sqrt{-g_{(0)}}\Bigg(-\frac{1}{2}\left\langle T^{ij}\right\rangle\delta g_{(0)ij} + \left\langle J^i\right\rangle\delta A_i^{(0)} + \sum_{I=1}^{2}\left\langle \mathcal{O}_{\psi_I}\right\rangle\delta\psi_I^{(0)} + \left\langle\mathcal{O}_\phi\right\rangle\delta J_f\Bigg) \tag{6.3.117}$$

Here T^{ij} is the energy–momentum tensor of the dual field theory, J_i is a conserved $U(1)$ current associated to a global $U(1)$ symmetry on the boundary and $\mathcal{O}_{\psi_I}$ and $\mathcal{O}_\phi$ are field theory scalar operators dual to the axions and dialton, respectively. Clearly the boundary term vanishes when $g_{(0)ij}$, $A_i^{(0)}$, $\psi_I^{(0)}$ and J_f are kept fixed on the boundary and so, the variation of the total action implies the bulk equations of motion. Next we will perform the variation of the renormalised action and from the result we will read off the 1–point functions, according to equation (6.3.117).

Using the unintegrated form of the bulk action, the renormalised action can be written as

$$\begin{aligned} S_{\rm ren} = \lim_{\epsilon\to 0}\Bigg[&\int_{z\geq\epsilon}\mathrm{d}^4x\sqrt{-G}\Bigg(R - \frac{Z(\phi)}{4}F^2 - \frac{1}{2}\sum_{I=1}^{2}\left(\partial\psi_I\right)^2 - \frac{1}{2}\left(\partial\phi\right)^2 - V(\phi)\Bigg)\\ &-2\int_{z=\epsilon}\mathrm{d}^3x\sqrt{-\gamma}\,K + \int_{z=\epsilon}\mathrm{d}^3x\sqrt{-g_{(0)}}\left(-\ell^2\varphi_{(0)}\varphi_{(1)} - \frac{2}{3}\vartheta\,\varphi_{(0)}^3\right)\\ &-\int_{z=\epsilon}\mathrm{d}^3x\sqrt{-\gamma}\left(\frac{4}{\ell} + \ell\,^{(\gamma)}R + \frac{1}{2\ell}\phi^2 - \frac{\ell}{2}\sum_{I=1}^{2}\left(\partial\psi_I\right)^2\right)\Bigg] \end{aligned} \tag{6.3.118}$$

where the first term is the bulk action, S_{bulk}, the second is the Gibbons–Hawking term S_{GH}, the third is the boundary term necessary for the dialton boundary condition, S_f, and the fourth is the counterterm action, S_{ct}. Note that all terms in the action have been expressed in terms of covariant fields except for S_f.

Perturbing each term in S_{ren} we obtain the following results,

$$\begin{aligned}
\delta S_{\text{bulk}} =& \int_{z\geq\epsilon} \mathrm{d}^4x\; E_{\mu\nu}\delta G^{\mu\nu} - \int_{z\geq\epsilon} \mathrm{d}^4x\sqrt{-G}\; G^{\mu\nu}G^{\rho\sigma}F_{\nu\sigma}\delta\left(\partial_\mu A_\rho\right) \\
&- \int_{z\geq\epsilon} \mathrm{d}^4x\sqrt{-G}\; G^{\mu\nu}\partial_\nu\psi_I\partial_\mu\left(\delta\psi_I\right) - \int_{z\geq\epsilon} \mathrm{d}^4x\sqrt{-G}\Bigg(G^{\mu\nu}\partial_\mu\phi\partial_\nu\left(\delta\phi\right) \\
&+ \left(\frac{\mathrm{d}V}{\mathrm{d}\phi} + \frac{F^2}{4}\frac{\mathrm{d}Z}{\mathrm{d}\phi}\right)\delta\phi\Bigg) \\
=& \int_{z=\epsilon} \mathrm{d}^3x\sqrt{-\gamma}\left(\gamma^{ij}F_{zi}\delta A_j + \frac{\ell^2}{\epsilon^2}\partial_\epsilon\phi_I(x,\epsilon)\,\delta\phi_I(x,\epsilon) + \frac{\ell^2}{\epsilon^2}\partial_\epsilon\phi(x,\epsilon)\,\delta\phi(x,\epsilon)\right),
\end{aligned} \tag{6.3.119}$$

$$\delta S_{\text{GH}} = -\int_{z=\epsilon} \mathrm{d}^3x\sqrt{-\gamma}\left(K_{ij} - K\gamma_{ij}\right), \tag{6.3.120}$$

$$\begin{aligned}
\delta S_f =& \int_{z=\epsilon} \mathrm{d}^3x\sqrt{-g_{(0)}}\Bigg(\left(\frac{\ell^2}{2}\varphi_{(0)}\varphi_{(1)} + \frac{1}{3}\vartheta\varphi_{(0)}^3\right)g_{(0)ij}\delta g_{(0)}^{ij} \\
&- \left(\ell^2\varphi_{(1)}\delta\varphi_{(0)} - \varphi_{(0)}\delta J_f\right)
\end{aligned} \tag{6.3.121}$$

$$\begin{aligned}
\delta S_{\text{ct}} =& -\int_{z=\epsilon} \mathrm{d}^3x\sqrt{-\gamma}\Bigg(\ell\,{}^{(\gamma)}R_{ij} - \frac{\ell}{2}\partial_i\psi_I\partial_j\psi_I - \gamma_{ij}\left(\frac{2}{\ell} + \frac{\ell}{2}{}^{(\gamma)}R + \frac{1}{4\ell}\phi^2 - \frac{\ell}{4}\left(\partial\psi_I\right)^2\right)\Bigg)\delta\gamma^{ij} \\
&- \int_{z=\epsilon} \mathrm{d}^3x\sqrt{-\gamma}\,\frac{\phi}{\ell}\,\delta\phi + \int_{z=\epsilon} \mathrm{d}^3x\sqrt{-\gamma}\,\ell\,\gamma^{ij}\,\partial_i\psi_I\,\partial_j\delta\psi_I
\end{aligned} \tag{6.3.122}$$

where $\delta J_f = -\ell^2\delta\varphi_{(1)} - 2\vartheta\varphi_{(0)}\delta\varphi_{(0)}$. Combining all terms and using the expansions for the fields, the renormalised action is

$$\begin{aligned}
\delta S_{\text{ren}} =& \lim_{\epsilon\to0}\int_{z=\epsilon} \mathrm{d}^3x\sqrt{-\gamma}\Bigg[\frac{\epsilon}{\ell}\gamma^{ij}Z\,\partial_\epsilon A_i\,\delta A_j + \left(\frac{\epsilon}{\ell}\partial_\epsilon\psi_I - \ell\,\Box_\gamma\psi_I\right)\delta\psi_I + \left(\frac{\epsilon}{\ell}\partial_\epsilon\phi - \frac{\phi}{\ell}\right)\delta\phi \\
&- \Bigg(K_{ij} - K\gamma_{ij} + \ell\,{}^{(\gamma)}R_{ij} - \frac{\ell}{2}\partial_i\psi_I\partial_j\psi_I - \gamma_{ij}\Big(\frac{2}{\ell} + \frac{\ell}{2}{}^{(\gamma)}R + \frac{1}{4\ell}\phi^2 - \frac{\ell}{4}\left(\partial\psi_I\right)^2\Big)\Bigg)\delta\gamma^{ij}\Bigg] \\
&+ \delta S_f.
\end{aligned} \tag{6.3.123}$$

As was already explained above, comparing this expression with equation (6.3.117) allows us to read off the expectations values of the dual operators. In particular, let $\mathcal{F}(x,z)$ one of the bulk fields and let its near boundary behaviour be $\mathcal{F}(x,z) \sim z^m f_{(0)}(x) + \ldots$. Then, the 1-point function of the operator dual to $\mathcal{F}(x,z)$ is given by

$$\langle\mathcal{O}_{\mathcal{F}}\rangle = \lim_{\epsilon\to0}\left[\left(\frac{\ell}{\epsilon}\right)^{d-m}\frac{1}{\sqrt{-\gamma}}\frac{\delta S_{\text{ren}}[\mathcal{F}(x,\epsilon);\epsilon]}{\delta\mathcal{F}(x,\epsilon)}\right] \tag{6.3.124}$$

or, in terms of the sources,

$$\langle \mathcal{O}_{\mathcal{F}} \rangle = \frac{1}{\sqrt{-g_{(0)}}} \frac{\delta S_{\text{ren}}\left[f_{(0)}(x)\right]}{\delta f_{(0)}(x)}. \tag{6.3.125}$$

Upon applying this prescription we obtain the following results,

$$\begin{aligned} \langle \mathcal{O}_\phi \rangle &= \varphi_{(0)}, & \langle \mathcal{O}_{\psi_I} \rangle &= 3\ell^2 \psi_I^{(3)} \\ \left\langle J^i \right\rangle &= Z_0 A^i_{(1)}, & \left\langle T^{ij} \right\rangle &= 3\ell^2 g^{ij}_{(3)} - \frac{2}{3}\vartheta \varphi^3_{(0)}\, g^{ij}_{(0)}. \end{aligned} \tag{6.3.126}$$

6.3.6 Local Ward Identities

The 1–point functions combined with the divergence relations for $g_{(3)ij}$ and $A_i^{(1)}$, given by equations (6.3.50) and (6.3.51) respectively, give the local Ward identities,

$$D^j_{(0)} \left\langle T_{ij} \right\rangle = \langle \mathcal{O}_\phi \rangle \partial_i J_f + \sum_{I=1}^{2} \langle \mathcal{O}_{\psi_I} \rangle \partial_i \psi_I^{(0)} + \left\langle J^j \right\rangle F^{(0)}_{ij}, \tag{6.3.127}$$

$$D_{(0)i} \left\langle J^i \right\rangle = 0. \tag{6.3.128}$$

The former is the result of diffeomorphism invariance and the latter of gauge invariance of the theory. Moreover, starting from (6.3.46) and using the expression for $\langle T^{ij} \rangle$ we find

$$\left\langle T^i{}_i \right\rangle - (d - \Delta_-)\varphi_{(0)} J_f - d\, \mathcal{F}(\varphi_{(0)}) + \Delta_- \varphi_{(0)} \mathcal{F}'(\varphi_{(0)}) = 0. \tag{6.3.129}$$

This is the trace Ward identity associated with the invariance of the theory under Weyl transformations. One immediately notices that, in addition to the usual $(d - \Delta_-)\varphi_{(0)} J_f$ term one expects in the presence of a scalar field, there are two extra terms. These are associated with the multi–trace deformation $\mathcal{F}(\mathcal{O}_\phi)$. Another important feature of the trace Ward identity is the vanishing of the right hand side. If this symmetry is broken then the trace is no longer equal to zero but instead it is equal to the conformal anomaly $\mathcal{A}(\varphi_{(0)})$. In $d = 3$ the only source for the conformal anomally is the scalar operator dual to the dialton. From the prespective of the bulk there are two possibilities that can give rise to a non–zero conformal anomaly. The first, which was already mentioned in section 6.3.2 on page 108, is the presence of an effective ϕ^3 term in the bulk action due to a non–zero V_3 in the expansion for the dialton potential [93]. In this case the dialton expansion has an additional logarithmic term whose coefficient contributes to the conformal anomaly. However, for the theories we are interested in, the potentials have $V_3 = 0$ and therefore there is no such logarithmic term for the dialton. The second possibility has to do with the mixed boundary conditions or, equivalently, the multi–trace deformation of the field theory action. This deformation introduces additional terms in $\langle T^{ij} \rangle$ which may still break scale invariance, even in the absence of the cubic term in the bulk action. However, if the multi–trace deformation is marginal i.e. if $\mathcal{F}(\mathcal{O}_\phi) \propto \mathcal{O}_\phi^{d/\Delta_-}$, as is the case here, then it does not break conformal invariance.

Although these relations have been derived using the explicit expressions for the 1-point functions, it is also possible to derive them using a Noether procedure, using the invariance of the renormalised bulk action under boundary boundary diffeorphisms and U(1) gauge transformations.

6.3.7 Global Ward Identities

In addition to the local Ward identities derived above, there are global Ward identites associated with the invariance of the bulk action, including the counterterms, under the global ISO($d-1$) symmetry of the axions, $\psi_I \to \Lambda_I^J \psi_J + c_I$, and hence, so is the renormalised action S_{ren}. In particular, substituting the infinitesimal form of this transformation in equation (6.3.117), we obtain the integral constraints,

$$\begin{aligned} &\int_{\partial\mathcal{M}} \mathrm{d}^d x \sqrt{-g_{(0)}}\, \langle \mathcal{O}_{\psi_I} \rangle = 0, \\ &\int_{\partial\mathcal{M}} \mathrm{d}^d x \sqrt{-g_{(0)}}\, \psi_{[I}^{(0)} \langle \mathcal{O}_{\psi_{J]}} \rangle = 0 \end{aligned} \tag{6.3.130}$$

where the axion indices are antisymmetrised in the second identity. The first identity is derived using the invariance of the theory under global shifts of the axions. From the perspective of 0-forms, global shifts correspond to gauge transformations and this identity is thus the analogue of the local current conservation for 0-forms. The second identity is a special feature of the theories we are studying and it associated with the flavour rotation symmetry of axions.

6.4 Field Theory Thermodynamics

We now have the tools needed to study the thermodynamic properties of our model. In particular, the renormalised Euclidean on-shell action gives the Gibbs free energy, or grand canonical potential, of the field theory [94],

$$\beta \mathcal{W}(T, \mathcal{V}, \mu, \mathcal{B}, \Pi) = S_{\text{ren}}^{\text{E}} = -S_{\text{ren}}. \tag{6.4.1}$$

Since we are interested in solutions satisfying mixed boundary conditions, the appropriate action to use in the above expression is the one augmented by the appropriate boundary actions, $S_{\text{ren}} = S_{\text{bulk}} + S_{\text{GH}} + S_f$ given in equation (6.3.118). In the grand canonical ensemble, described by the Gibbs free energy, the control variables are the temperature $T = 1/\beta$, the spatial volume $\mathcal{V}$, the chemical potential μ, the magnetic field $\mathcal{B}$ (if present) and the axionic strength Π which, for the isotropic configuration of axionic fields that we are interested is defined as

$$\Pi = \frac{1}{\ell} \sqrt{\frac{1}{(d-1)} \sum_{I=1}^{d-1} (\partial \psi_I)^2} = \frac{1}{\ell} \sqrt{\frac{1}{2} \sum_{I=1}^{2} (\partial \psi_I)^2}. \tag{6.4.2}$$

Varying the free energy with respect to these control variables provides the expressions for their conjugate variables, namely the entropy $\mathcal{S}$, the pressure of the system $\mathcal{P}$, the electric charge Q_{e}, the magnetisation $\mathcal{M}$ and the axionic magnetisation ϖ which are conjugate to the temperature, volume, chemical potential, magnetic field and axion strength respectively. These calculations simplify significantly by noting that for spatially homogeneous systems, such as the solutions we are interested in, the volume $\mathcal{V}$ factorises in the expression for the on-shell action allowing us to define the free energy density w which is independent of the volume

$$w(T,\mu,\mathcal{B},\Pi) = \mathcal{W}(T,\mathcal{V},\mu,\mathcal{B},\Pi)/\mathcal{V}. \tag{6.4.3}$$

Using this we obtain the following expressions

$$\left(\frac{\partial w}{\partial T}\right)_{\mu,\mathcal{B},\Pi} = -s, \quad \left(\frac{\partial w}{\partial \mu}\right)_{T,\mathcal{B},\Pi} = -\rho, \quad \left(\frac{\partial w}{\partial \mathcal{B}}\right)_{T,\mu,\Pi} = -\mathcal{M}, \quad \left(\frac{\partial w}{\partial \Pi}\right)_{T,\mu,\mathcal{B}} = -\varpi. \tag{6.4.4}$$

where $s = \mathcal{S}/\mathcal{V}$ is the entropy density of the system and $\rho = Q_{\mathsf{e}}$ its charge density. Note that the spatial volume of the system is formally infinite and therefore it is more appropriate to talk densities. The above quantities can be independently derived from the holographic analysis of the conserved black hole charges. This is a straight forward task when it comes to the chemical potential, charge density, temperature, entropy density and magnetic and axionic charge densities for which we have, in addition to the definition (6.4.2),

$$T = T_{\mathsf{BH}}, \qquad s = s_{\mathsf{BH}}, \qquad \mu = \lim_{z\to 0} A_t, \qquad \rho = \ell^2 \left\langle J^t \right\rangle, \qquad \mathcal{B} = \frac{1}{\ell^3}\lim_{z\to 0} F_{xy}, \tag{6.4.5}$$

where we have assumed that the gauge potential vanishes at the horizon. However, this task is a lot more involve when it comes to the thermodynamic potentials conjugate to the magnetic and axionic charges, i.e. the magnetization and axionic magnetization. We circumvent this complication by using general thermodynamic relations such as the ones given in equation (6.4.4) to define these quantities. This however, implies that our analysis does not provide an independent confirmation of thermodynamic relations such as the first law. A first principles definition of all thermodynamic variables for planar black holes with axionic charge, and correspondingly a general derivation of the first law, will appear in [95].

Returning to the discussion of the free energy, we see from the above expressions that the exact differential of the free energy density is

$$dw = -s\,\mathrm{d}T - \rho\,\mathrm{d}\mu - \mathcal{M}\,\mathrm{d}\mathcal{B} - \varpi\,\mathrm{d}\Pi, \tag{6.4.6}$$

Another important quantities characterising a system is its internal energy–or in this case energy density ε. It can be derived from the free energy density by performing a

Legendre transformation with respect to the temperature and the chemical potential,

$$\varepsilon(s, \rho, \mathcal{B}, \Pi) = w + Ts + \mu\rho. \tag{6.4.7}$$

Alternatively, it can be read off from the holographic stress energy tensor,

$$\varepsilon = \ell^2 \left\langle T^{00} \right\rangle . \tag{6.4.8}$$

The exact differential of the internal energy density gives the first law of thermodynamics for an infinitesimal volume,

$$\mathrm{d}\varepsilon = T\,\mathrm{d}s + \mu\,\mathrm{d}\rho - \varpi\,\mathrm{d}\Pi - \mathcal{M}\,\mathrm{d}\mathcal{B}. \tag{6.4.9}$$

Moreover, re-instating the volume dependence, one obtains the total energy $\mathcal{E}$ which depends on the total entropy $S = s\mathcal{V}$, charge $Q_{\mathsf{e}} = \rho\mathcal{V}$, and volume $\mathcal{V}$ of the system,

$$\mathcal{E}(S, \mathcal{V}, Q_{\mathsf{e}}, \mathcal{B}, \Pi) = \mathcal{V}\,\varepsilon\left(\frac{S}{\mathcal{V}}, \frac{Q_{\mathsf{e}}}{\mathcal{V}}, \mathcal{B}, \Pi\right). \tag{6.4.10}$$

The exact differential of this expression gives the first law of thermodynamics in its usual form, i.e.

$$\mathrm{d}\mathcal{E} = T\,\mathrm{d}S - \mathcal{P}\,\mathrm{d}\mathcal{V} + \mu\,\mathrm{d}Q_{\mathsf{e}} - (\varpi\mathcal{V})\mathrm{d}\Pi - (\mathcal{M}\mathcal{V})\,\mathrm{d}\mathcal{B}, \tag{6.4.11}$$

where the pressure of the system is given by

$$\mathcal{P} = -\left(\frac{\partial\mathcal{E}}{\partial\mathcal{V}}\right)_{S, Q_{\mathsf{e}}, \mathcal{B}, \Pi}. \tag{6.4.12}$$

Combining (6.4.11) with (6.4.6) and (6.4.7), we find that the pressure is related to the free energy as

$$w = -\mathcal{P}, \tag{6.4.13}$$

and satisfies the Gibbs-Duhem relation

$$\varepsilon + \mathcal{P} = Ts + \mu\rho. \tag{6.4.14}$$

Finally, since the Gibbs free energy density w is a function of the variable T, μ, $\mathcal{B}$ and Π, in order to compare solutions with the same charge densities we need to use the Helmholtz free energy density $\mathfrak{f}$, which is a function of $T, \rho, \mathcal{B}, \Pi$. This is related to the Gibbs free energy via a Legendre transform with respect to the chemical potential μ,

$$\mathfrak{f} = w + \mu\rho = \varepsilon - Ts. \tag{6.4.15}$$

The thermodynamic identity (6.4.6) implies that

$$d\mathfrak{f} = -s\,\mathrm{d}T + \mu\,\mathrm{d}\rho - \mathcal{M}\,\mathrm{d}\mathcal{B} - \varpi\,\mathrm{d}\Pi, \tag{6.4.16}$$

and so $\mathfrak{f}$ is indeed a function of the variables $T, \rho, \mathcal{B}, \Pi$, as desired.

The analysis up to now has been independent of any particular features of the theory under consideration and it is therefore valid for any black hole. To apply it to specific systems one needs in addition an equation of state, relating the thermodynamic variables. This is strongly constrained by symmetries, such as conformal invariance. For the theories we are interested in, we saw in section 6.3.6 that the dialton does not explicitly break conformal invariance which is only broken explicitly by the background magnetic field and axion charge. A simple scaling argument allows us to generalise the equation of state for conformal theories to theories where conformal symmetry is explicitly broken by magnetic and axionic charges.

6.4.1 Equation of State in the Presence of Magnetic and Axionic Charges

In a d-dimensional theory with no explicit breaking of conformal symmetry, the tracelessness of the stress tensor the vacuum implies that in a state of thermal equilibrium the system is governed by the equation of state $\mathcal{P} = \varepsilon/(d-1)$. To see how this relation is modified for the theories we are interested in, which (can) have a magnetic field, a chemical potential and an isotropic distribution of axionic charges, we first notice that the Gibbs free energy $\mathcal{W}$ can be re-written as

$$\mathcal{W}(T, \mathcal{V}, \mu, \mathcal{B}, \Pi) = (\varepsilon - Ts - \mu\rho)\mathcal{V}. \tag{6.4.17}$$

where we have used (6.4.7). Conformal invariance and extensivity restrict thus the form of the state function $\mathcal{W}$ to

$$\mathcal{W}(T, \mathcal{V}, \mu, \mathcal{B}, \Pi) = -\mathcal{V}T^d\, h\left(\frac{\mu}{T}, \frac{\mathcal{B}}{T^2}, \frac{\Pi}{T}\right), \tag{6.4.18}$$

where the function h depends only on the dimensionless ratios μ/T, $\mathcal{B}/T^2$ and Π/T. We should stress that in writing this relation we assume that there are no dimensionfull couplings, either single- or multi-trace, for the scalar operator dual to ϕ. This assumption is justified for the planar black holes we consider here, but in general dimensionfull scalar couplings must be included in the scaling argument (see e.g. [96]). As a consequence of (6.4.18), $\mathcal{W}$ possesses the scaling property

$$\mathcal{W}(\lambda T, \lambda^{1-d}\mathcal{V}, \lambda\mu, \lambda^2\mathcal{B}, \lambda\Pi) = \lambda\mathcal{W}(T, \mathcal{V}, \mu, \mathcal{B}, \Pi). \tag{6.4.19}$$

Differentiating this relation with respect to λ and setting $\lambda = 1$ we obtain

$$-sT - (1-d)\mathcal{P} - \rho\mu - 2\mathcal{M}\mathcal{B} - \varpi\Pi = \mathcal{W}/\mathcal{V}, \tag{6.4.20}$$

where we have used the conjugate variables introduced in (6.4.4) and (6.4.12). Equivalently,

$$\frac{\partial \mathcal{W}}{\partial T} = -s\mathcal{V}, \quad \frac{\partial \mathcal{W}}{\partial \mathcal{V}} = -\mathcal{P}, \quad \frac{\partial \mathcal{W}}{\partial \mu} = -\rho\mathcal{V}, \quad \frac{\partial \mathcal{W}}{\partial \mathcal{B}} = -\mathcal{M}\mathcal{V}, \quad \frac{\partial \mathcal{W}}{\partial \Pi} = -\varpi\mathcal{V}. \tag{6.4.21}$$

Combining equation (6.4.20) with the defining relation (6.4.17) finally gives the equation of state

$$\varepsilon = (d-1)\mathcal{P} - 2\mathcal{M}\mathcal{B} - \varpi\Pi. \tag{6.4.22}$$

In chapters 7 and 8 we explicitly confirm this relation for the explicit solutions we study.

6.5 Dynamical Stability and the Energy Density

In this section we discuss the holographic effective potential for the *vev* of the scalar operator dual to the dialton, which tell us whether a particular solution corresponds to stable (thermal) vacuum of the dual theory. The quantum effective action for the scalar *vev* $\sigma = \varphi_{(0)}$ is given by the Legendre transform of the generating function of the theory, given by the renormalised on-shell action S_{ren} (6.3.116), with respect to the scalar source, keeping all other sources to their values in the solutions,

$$\Gamma[\sigma] = S_{\text{ren}} - \ell^2 \int \mathrm{d}^3x\, \sigma J_f = \ell^2 \int \mathrm{d}^3x\, \Big(\mathcal{V}_{\text{QFT}}(\sigma) + \text{derivatives}\Big), \tag{6.5.1}$$

where we used the fact that the QFT is on Minkowski (with metric $g_{(0)} = \text{diag}(-1, \ell^2, \ell^2)$) and $\mathcal{V}_{\text{QFT}}(\sigma)$ is the quantum effective potential for σ. Since we are only interested in homogeneous solutions, we can neglect the derivative terms. From (6.3.117) and (6.5.1) it follows that the source of $\mathcal{O}_{\Delta_-}$ is then given by

$$J_f = -\frac{\delta\Gamma[\sigma]}{\delta\sigma}, \tag{6.5.2}$$

and, hence, vacua of the theory are extrema of the effective action:

$$\left.\frac{\delta\Gamma[\sigma]}{\delta\sigma}\right|_{\sigma=\sigma_*} = 0. \tag{6.5.3}$$

To compute the effective action we observe that from (6.3.69) and (6.3.116) it follows that [82]

$$\Gamma[\sigma] = S_{\text{D,ren}} + \ell^2 \int d^3x \mathcal{F}(\sigma), \tag{6.5.4}$$

where $S_{\text{D,ren}}$ is the generating function of the Dirichlet theory or equivalently the effective action of the Neumann theory, given by

$$S_{\text{D,ren}} = \lim_{\epsilon\to 0} \left(S_{\text{Bulk}} + S_{\text{GH}} + S_{\text{ct}}\right), \tag{6.5.5}$$

where S_{Bulk} is the regularised on-shell bulk action. As for Poincaré domain walls [97, 21, 98, 24], for the homogeneous solutions we are interested in here S'_{ren} can be expressed in terms of a fake superpotential that governs non-relativistic flows [99].

We do not present the details of the calculation here and we refer the reader to appendix B of [2]. We would like to mention however, that the result of this calculation was shown to take a rather universal form, that applies to all hairy black holes solutions we will be studying. In particular, the effective potential for the scalar *vev* σ in all cases takes the form

$$\begin{aligned}\mathcal{V}_{\text{QFT}}(\sigma) &= \mathcal{V}_0 + \frac{\mu q_{\text{e}}}{\ell^2} + \frac{\varepsilon}{2\ell^2\sigma_*^3}\left(\sigma^3 - 3\sigma_*^2\sigma\right) \\ &= \mathcal{V}_0 + \frac{\mu q_{\text{e}}}{\ell^2} + \frac{\varepsilon}{2\ell^2\sigma_*^3}\left(-2\sigma_*^3 + 3\sigma_*(\sigma-\sigma_*)^2 + (\sigma-\sigma_*)^3\right), \end{aligned} \tag{6.5.6}$$

where $\mathcal{V}_0$ is a constant and σ_* is the value of the *vev* at the extremum, i.e. the value corresponding to the specific background solution, and ε is the corresponding energy density. Dynamical stability is now determined by the sign of the effective mass term, *i.e.* the coefficient of the quadratic term, and we see that it is equivalent to the positivity of the energy density, as one may have expected. This result also provides an alternative method for computing the energy density. Using the specific expressions for the energy density in each of the solutions, therefore, determines the range of parameters for which they are dynamically stable.

6.6 Concluding Remarks

In this chapter we performed a detailed analysis of a class of supergravity theories that admit planar black brane solutions carrying electromagnetic and axionic charges and can support an additional running scalar which we named *the dialton*. We focused on solutions satisfying mixed boundary conditions for the dialton field and which have axions that are linear in the boundary spatial directions. We discussed the implications of these features in length, stressing their implications for the dual field theory. We then went on to solve the bulk equations of motion near the conformal boundary and used the results to compute the renormalised on-shell action and, subsequently, the one-point functions of the dual operators. Combining our results of the holographic analysis with general thermodynamic relations, we then derived the thermodynamic properties of the dual theory. Finally, we briefly discussed the results obtained in [2] regarding the dynamic stability of the solutions which have non-trivial dialtons by deriving holographic effective potential for the *vev* of the scalar operator dual to the dialton.

In the next two chapters we will employ the results of this chapter to analyse two theories that are explicit realisations of the model discussed here. The first theory, studied in chapter 7, corresponds to the choice $Z(\phi) = 1$ and

$$V(\phi) = 2\Lambda\left[\cosh^4\left(\frac{\phi}{2\sqrt{3}}\right) - \alpha\sinh^4\left(\frac{\phi}{2\sqrt{3}}\right)\right], \qquad 0 \leq \alpha \leq 1. \tag{6.6.1}$$

This theory was discussed in [87] where the authors found exact axionic black hole solutions which do not have a running profile for the dialton, as well as in [88] where they found solutions that have both a non trivial axion background and also a running dialton. The second theory we will analyse is defined by

$$V(\phi) = \sigma_1 e^{\frac{(d-2)(d-1)\delta^2-2}{2(d-1)\delta}\phi} + \sigma_2 e^{\frac{2\phi}{\delta(1-d)}} + \sigma_3 e^{(d-2)\delta\phi}, \qquad Z(\phi) = e^{-(d-2)\delta\phi}, \tag{6.6.2}$$

where

$$\sigma_0 = -\frac{d(d-1)}{\ell^2} = 2\Lambda, \qquad \sigma_1 = \sigma_0 \frac{8(d-2)(d-1)^2\delta^2}{d(2+(d-2)(d-1)\delta^2)^2},$$
$$\sigma_2 = \sigma_0 \frac{(d-2)^2(d-1)\delta^2(d(d-1)\delta^2-2)}{d(2+(d-2)(d-1)\delta^2)^2}, \quad \sigma_3 = -2\sigma_0 \frac{(d-2)^2(d-1)\delta^2-2d}{d(2+(d-2)(d-1)\delta^2)^2}, \tag{6.6.3}$$

and δ is a free parameter in the potential. This action, and a corresponding family of analytic electrically charged black brane solutions with axion charges and a running dialton, were presented in appendix C of [89]. In chapter 8 we examine this family of solutions and we also derive and study a new family of exact magnetically charged axionic black holes. Finally, using our results for the thermodynamics of these exact black branes, we study their thermodynamic and dynamic stability, and the corresponding phase structure. Moreover, since we have analytical expressions for our solutions, we are also able to compute analytically the off-shell holographic quantum effective potential of the scalar operator dual to the dialton field by employing the formalism developed in section 6.5. This computation allows us to show that dynamical stability of hairy planar AdS black holes with respect to scalar fluctuations is equivalent to positivity of the energy density.

6.A Fefferman–Graham Gauge: Useful Identities

In the Fefferman–Graham gauge the metric has the form

$$\begin{aligned} \mathrm{d}s^2 &= G_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu = \frac{\ell^2}{z^2}\mathrm{d}z^2 + \frac{\ell^2}{z^2} g_{ij}(x,z)\mathrm{d}x^i \mathrm{d}x^j \\ g_{ij}(x,z) &= g_{(0)ij}(x) + z g_{(1)ij}(x) + z^2 g_{(2)ij}(x) + \dots \end{aligned} \tag{6.A.1}$$

where the boundary is located at $z=0$. We consider a 4 dimensional spacetime thus g_{ij} is a 3 dimensional and $i,j = 1,2,3$.

Below we give expressions for the coefficients of the inverse metric asymptotic expansion and for the Christoffel symbols.

1. Inverse metric:

   Using the relationship

$$g^{ij}(x,z) g_{jk}(x,z) = \delta^i_k \tag{6.A.2}$$

where,

$$g^{ij}(x,z) = g^{ij}_{(0)}(x) + z g^{ij}_{(1)}(x) + z^2 g^{ij}_{(2)}(x) + \dots \tag{6.A.3}$$

we find

$$g^{ij}_{(1)}(x) = -\left(g^{-1}_{(0)} g_{(1)} g^{-1}_{(0)}\right)^{ij} \tag{6.A.4}$$

$$g^{ij}_{(2)}(x) = \left(g^{-1}_{(0)} g_{(1)} g^{-1}_{(0)} g_{(1)} g^{-1}_{(0)}\right)^{ij} - \left(g^{-1}_{(0)} g_{(2)} g^{-1}_{(0)}\right)^{ij}. \tag{6.A.5}$$

2. Christoffel Symbols:

$$\begin{aligned} &\Gamma^z_{zz} = -\frac{1}{z} && \Gamma^i_{jz} = \Gamma^i_{zj} = \frac{1}{2}\left(g^{-1}g'\right)^i{}_j - \frac{1}{z}\delta^i_j \\ &\Gamma^z_{ij} = \frac{1}{z} g_{ij} - \frac{1}{2} g'_{ij} && \Gamma^i_{ij} = {}^{(g)}\Gamma^i_{jk} \\ &\Gamma^z_{zi} = \Gamma^z_{iz} = 0 && \Gamma^i_{zz} = 0 \end{aligned} \tag{6.A.6}$$

Useful identities for the Christoffel symbols:

$$\begin{aligned} \Gamma^\mu_{\mu z} &= -\frac{4}{z} + \frac{1}{2}\,\mathrm{Tr}\,\left(g^{-1}g'\right) & \Gamma^\mu_{\mu i} &= \frac{1}{2}\,\mathrm{Tr}\,\left(g^{-1}\partial_i g\right) \\ G^{\mu\nu}\Gamma^z_{\mu\nu} &= \frac{2}{\ell^2} z - \frac{z^2}{2\ell^2}\,\mathrm{Tr}\,\left(g^{-1}g'\right) & G^{\mu\nu}\Gamma^i_{\mu\nu} &= -\partial_j g^{ij} - \frac{1}{2} g^{ij}\,\mathrm{Tr}\,\left(g^{-1}\partial_j g\right) \end{aligned} \tag{6.A.7}$$

Chapter 7

Theory I

The first model we study is obtained from the general action (6.2.1) by setting $Z(\phi) = 1$ and choosing $V(\phi)$ to be

$$V(\phi) = 2\Lambda\left[\cosh^4\left(\frac{\phi}{2\sqrt{3}}\right) - \alpha\sinh^4\left(\frac{\phi}{2\sqrt{3}}\right)\right], \qquad 0 \leq \alpha \leq 1. \tag{7.0.1}$$

This model was originally studied in [87] where the authors found analytic expessions for bald charged black branes carrying axionic charges but which did not have without the additional dialton scalar and subsequently in [88] where they obtained hairy solutions which have a running scalar field in addition to the axions.

The above potential can be obtained through field redefinition from AdS gravity with cosmological constant $\Lambda = -3/\ell^2$, conformally coupled to a scalar field with a conformal self interaction coupling α [100, 82]. For $\alpha = 1$ this potential was discussed earlier in [101], where a black hole solution with a horizon of constant negative curvature was obtained, and in [102], where this potential was embedded in the $U(1)^4$ truncation of maximally supersymmetric gauge supergravity in four dimensions. In fact, for $\alpha = 1$, taking $Z(\phi)$ to be a specific exponential function of the scalar ϕ and setting the axions ψ_I to zero, the full action (6.2.1) can be embedded in the $U(1)^4$ truncation of gauged supergravity. No embedding is known for $\alpha \neq 1$, or for $Z(\phi) = 1$. Our analysis will focus on solutions with non-trivial axion profiles, and so the corresponding action should be treated as a bottom up model.

In this chapter we revisit the above solutions in light of the analysis performed in the previous chapter. We begin with the study of the bald solutions in section 7.1. We review the properties of these black branes in section 7.1.3. In particularly we find the location of their horizon and compute their temperature and their entropy density. These solutions admit an extremal limit which we obtain in section 7.1.4. We obtain these extremal solutions and derive their near horizon geometry. Next, by rewriting the metric in the Fefferman–Graham gauge and using the results of the previous chapter we obtain the one-point functions of the dual operators, 7.1.5. Finally, we conclude our discussion of the bald solutions with the discussion of their thermodynamic properties in section 7.1.6, which also relies on the results of the previous chapter. In section 7.2 we turn our attention to the hairy solutions of the theory and repeat the analysis. In particular, we discuss their horizon properties, temperature and entropy density is section 7.2.1. Unlike their bald version, these solutions do not have a regular extremal limit. The one-point functions of the dual operators are discussed in section 7.2.2 and the thermodynamic properties in section 7.2.3. We conclude the chapter in section 7.3 were we discuss phase transitions of the theory between the bald and hairy phases, as well as the stability of these phases.

7.1 Bald Solution

7.1.1 Electric Solution

When the dialton field vanishes the potential (7.0.1) reduces to a constant $V(0) = -6/\ell^2 = 2\Lambda$, and the theory to AdS gravity coupled to a gauge field and two free scalars ψ_I. The bald solutions to this theory, endowed with the properties discussed in the previous chapter, namely, electrically charged planar black holes solutions with axions that are linear in the boundary directions are given by

$$\begin{aligned} \mathrm{d}s^2 &= -f(r)\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{f(r)} + r^2\left(\mathrm{d}x^2 + \mathrm{d}y^2\right) \\ f(r) &= -\frac{p^2}{2} - \frac{m}{r} + \frac{q^2}{4r^2} + \frac{r^2}{\ell^2} \\ \psi_1 &= px, \qquad \psi_2 = py \\ A &= \mu\left(1 - \frac{r_0}{r}\right)\mathrm{d}t, \qquad F = \mathrm{d}A = -\frac{q}{r^2}\mathrm{d}t\wedge\mathrm{d}r, \end{aligned} \tag{7.1.1}$$

where r_0 is the position of the horizon and m, p and q are related to the mass, axion charge and $U(1)$ charge densities respectively. Since the gauge potential A vanishes at the horizon it follows that μ is the chemical potential in the dual field theory and $q = \mu r_0$ is the U(1) charge density.

We would like to draw the reader's attention to the form of the metric. In particular, note that the form of $f(r)$ is the one commonly associated with a hyperbolic black hole. Nevertheless, this black hole has a flat horizon. As was explained in chapter 6, section 6.2.1, this particular feature of the metric is attributed to the axions which modify the vacuum energy at the horizon, making it flat.

7.1.2 Dyonic Solution

The electric solution presented above can easily be extended to include the presence of a magnetic field by letting $q^2 = q_{\mathsf{e}}^2 + q_{\mathsf{m}}^2$ in the expression for the metric. The gauge field is also modified to

$$A = \mu_{\mathsf{e}}\left(1 - \frac{r_0}{r}\right)\mathrm{d}t + q_{\mathsf{m}}x\mathrm{d}y, \qquad F = \mathrm{d}A = -\frac{q_{\mathsf{e}}}{r^2}\mathrm{d}t\wedge\mathrm{d}r + q_{\mathsf{m}}\mathrm{d}x\wedge\mathrm{d}y, \tag{7.1.2}$$

where the previous relation $q = \mu r_0$ is replaced by $q_\mathsf{e} = \mu_\mathsf{e} r_0$.

7.1.3 Black Brane Properties: Horizon and Extremality

In this section we investigate the properties associated with the horizon of these bald solutions. We focus on the dyonic solution only. The corresponding results for the purely electric solution can be retrieved by taking the $q_\mathsf{m} \to 0$ limit.

The horizon, r_0, of the black brane is located at the largest root of $f(r)$. By solving the equation $f(r_0) = 0$ for m, we obtain an expressions for the mass parameter in terms of the horizon r_0, axion charge p and chemical potential μ_e,

$$m = \frac{r_0^3}{\ell^2}\left(1 + \frac{\ell^2}{4r_0^2}\left(\mu_\mathsf{e}^2 - 2p^2\right) + \frac{\ell^2}{4r_0^4} q_\mathsf{m}^2\right). \tag{7.1.3}$$

This expression will be used to eliminate m in subsequent results.

Temperature

The temperature of the black branes can derived by requiring that the Euclidean metric does not have a conical singularity,

$$T = \frac{f'(r_0)}{4\pi} = \frac{3r_0}{4\pi\ell^2}\left(1 - \frac{\ell^2}{12r_0^2}\left(\mu_\mathsf{e}^2 + 2p^2\right) - \frac{\ell^2}{12r_0^4} q_\mathsf{m}^2\right). \tag{7.1.4}$$

Entropy

The entropy density of the black hole is

$$s = \frac{a_h}{4G_N} = 4\pi r_0^2 \tag{7.1.5}$$

where a_h is the area of the horizon and we have used our convention, $16\pi G_N = 1$.

Symmetry enhancement

As shown by Davison and Goutéraux [103], the bald solution becomes conformal to $\mathrm{AdS}_2 \times \mathbb{R}^2$ for particular values of the parameters, and enjoys in that case an enhanced $SL(2,\mathbb{R}) \times SL(2,\mathbb{R})$ symmetry. In this case, one can solve the linearized perturbation equations exactly in terms of hypergeometric functions.[1] This happens precisely when the form of the lapse function simplifies to

$$f(r) = -\frac{p^2}{2} + \frac{r^2}{\ell^2}, \tag{7.1.6}$$

[1] Notice that such a symmetry is also enjoyed by the scalar wave equation in the nonextremal Kerr black hole background, in the low frequency limit [104] (see [105] for the Schwarschild black hole case). This *hidden conformal symmetry* is not derived from an underlying symmetry of the spacetime itself, but is rather related to the fact that black hole scattering amplitudes are given in terms of hypergeometric functions, which are well-known to form representations of the conformal group $SL(2,\mathbb{R})$. What is notable in the bald black hole case is that this symmetry becomes an exact symmetry of the linearized gravitational perturbation equations for those values of the parameters.

i.e. when m and q both vanish. The crucial point is that the metric becomes *conformal* to a patch of $\text{AdS}_2 \times \mathbb{R}^2$. In section 7.2 we study the hairy solutions of these theory where we find that, for a particular range of the parameters, these solutions exhibit the same symmetry enchancement. It is however unlikely that the coupled linearised perturbation equations for the fields $g_{\mu\nu}$, A_μ, ϕ, and ψ_I remain exactly solvable as a consequence of the enhanced symmetry. This is left for further exploration.

7.1.4 Extremal Black Branes

The solution becomes extremal when r_0 is a double root of $f(r)$ or equivalently when its temperature vanishes. Setting T, given in equation (7.1.4), equal to zero, and solving for r_0 we find

$$r_{0,\text{ext}}^2 = \frac{\ell^2}{24}\left(2p^2 + \mu_{\text{e}}^2\right)\left[1 + \sqrt{1 + \frac{48 q_{\text{m}}^2}{\ell^2\left(2p^2 + \mu_{\text{e}}^2\right)^2}}\right]. \tag{7.1.7}$$

The entropy of these black holes is

$$s = \frac{\ell^2}{96 G_N}\left(2p^2 + \mu_{\text{e}}^2\right)\left[1 + \sqrt{1 + \frac{48\, q_{\text{m}}^2}{\ell^2\left(2p^2 + \mu_{\text{e}}^2\right)^2}}\right]. \tag{7.1.8}$$

As is usually the case with charged AdS black holes, the entropy does not vanish at $T = 0$, signalling that the ground state of the dual theory is degenerate.

Note that for black holes which carry only electric charge,

$$r_{0,\text{ext}}^2 = \frac{\ell^2}{12}\left(2p^2 + \mu_{\text{e}}^2\right), \tag{7.1.9}$$

$$m = \ell^3\left(\frac{\mu^2 + 2p^2}{12}\right)^{3/2}\left(1 + 3\,\frac{\mu^2 - 2p^2}{\mu^2 + 2p^2}\right), \tag{7.1.10}$$

and

$$s = \frac{r_0^2}{4G_{\text{N}}} = \frac{\ell^2}{48 G_{\text{N}}}\left(\mu^2 + 2p^2\right). \tag{7.1.11}$$

Near Horizon Geometry

Next we look at the near horizon geometry of the extremal black branes which teaches us about the IR of the dual theory. We begin by expanding the lapse function given in (7.1.1) near $r = r_{0,\text{ext}}$ given by equation (7.1.7). Using the fact that $f(r_{0,\text{ext}}) = f'(r_{0,\text{ext}})$ we find

$$f(r) = \frac{1}{2 r_{0,\text{ext}}^4}\Big(\left(\mu_{\text{e}}^2 + p^2\right) r_{0,\text{ext}}^2 + q_{\text{m}}^2\Big)(r - r_{0,\text{ext}})^2 + \text{O}\big((r - r_{0,\text{ext}})^3\big). \tag{7.1.12}$$

Substituting this result into the metric and performing the following change of coordinates $r = r_{0,\text{ext}} + \epsilon\rho$, $t = \tau/\epsilon$ we find

$$\mathrm{d}s^2 = -\frac{\left(\mu_{\mathsf{e}}^2 + p^2\right) r_{0,\text{ext}}^2 + q_{\mathsf{m}}^2}{2r_{0,\text{ext}}^4}\rho^2\mathrm{d}\tau^2 + \frac{2r_0^2}{\left(\mu_{\mathsf{e}}^2 + p^2\right) r_{0,\text{ext}}^2 + q_{\mathsf{m}}^2}\frac{\mathrm{d}\rho^2}{\rho^2} + r_{0,\text{ext}}^2\left(\mathrm{d}x^2 + \mathrm{d}y^2\right). \quad (7.1.13)$$

This metric describes an $\mathrm{AdS}_2 \times \mathbb{R}^2$ geometry in the IR with the radius of AdS_2 given by

$$\ell_{\text{IR}}^2 = \frac{2r_{0,\text{ext}}^4}{\left(\mu_{\mathsf{e}}^2 + p^2\right) r_{0,\text{ext}}^2 + q_{\mathsf{m}}^2}. \quad (7.1.14)$$

For purely electric black branes with $q_{\mathsf{m}} = 0$ this expression reduces to

$$\ell_{\text{IR}}^2 = \frac{\ell^2}{6}\frac{\mu^2 + 2\pi^2}{\mu^2 + p^2}. \quad (7.1.15)$$

Thus, we find that the full geometry interpolates between AdS_4 in Poincaré coordinates in the UV and a near horizon $\mathrm{AdS}_2 \times \mathbb{R}^2$ geometry in the IR.

This near horizon geometry introduces a source of instability for massive fields. In particular, consider a scalar field in an extremal (or near extremal) charged AdS black hole background whose mass is above the BF bound for the 4 dimensional spacetime but below the bound for the 2 dimensional space, i.e. $-9/4\ell^2 \leq m^2 < -1/\ell^2$. Although this scalar does not introduce an instability far from the near horizon region, in the vicinity of the horizon where the geometry is $\mathrm{AdS}_2 \times \mathbb{R}^2$, its mass violates the corresponding BF bound, making it unstable. Note that this instability is not associated with superconductivity or superfluidity a priori since it does not rely on the field being charged under a $U(1)$ symmetry.

7.1.5 Fefferman–Graham Gauge and 1–point functions

Fefferman–Graham Expansion

In chapter 6 we derived the 1–point functions for any theory whose action has the form (6.2.1). This was done by rewriting the metric in the Fefferman–Graham gauge and expressing all the fields in this gauge (see section 6.3.5). We will now use those results to read off the 1–point functions for the bald solution 7.1.1 by first obtain the asymptotic expansion for the metric in the Fefferman–Graham gauge. In particular, we want to find the transformation $r = r(z)$ such that

$$\frac{\mathrm{d}r^2}{f(r)} = \frac{\ell^2}{z^2}\mathrm{d}z^2 \quad (7.1.16)$$

where r and $f(r)$ are the holographic coordinate and lapse function in (7.1.1) and z is the Fefferman–Graham holographic coordinate.

To derive this transformation we consider a series expansion for $r(z)$ and solve order by order in z,

$$r(z) = \frac{\ell^2}{z}\left(1 + a_0 z + a_1 z^2 + a_2 z^3 + O(z^4)\right). \tag{7.1.17}$$

The largest power of z in the expansion is found by considering the largest power of r in $f(r)$, i.e. r^2/ℓ^2. If $f(r) = r^2/\ell^2$ then

$$\frac{\ell^2}{r^2}\mathrm{d}r^2 = \frac{\ell^2}{z^2}\mathrm{d}z^2 \Rightarrow \ln r = \ln z^{-1} \Rightarrow r = \frac{\ell^2}{z}, \tag{7.1.18}$$

where we used the fact that $r = \infty$ corresponds to $z = 0$. From the $r(z)$ expansion we compute $\mathrm{d}r^2$,

$$\begin{aligned}\mathrm{d}r^2 &= \ell^4\left(-\frac{1}{z^2} + a_1 + 2a_2 z + 3a_3 z^2 + \dots\right)^2 \mathrm{d}z^2 \\ &= \frac{\ell^4}{z^4}\left(1 - 2a_1 z^2 - 4a_2 z^3 + \dots\right)\mathrm{d}z^2.\end{aligned} \tag{7.1.19}$$

We also need the series expansion for $\mathrm{d}r^2/f(r)$ in terms of z. First we compute $1/f(r)$ by substituting $r(z)$ in $f(r)$,

$$\frac{1}{f(r)} = \frac{z^2}{\ell^2}\left(1 - 2a_0 z - 2a_1 z^2 + \frac{p^2}{2\ell^2}z^2 + \dots\right) \tag{7.1.20}$$

Combining this result with the expression for $\mathrm{d}r^2$, (7.1.19), and after some algebra, we find

$$\frac{\mathrm{d}r^2}{f(r)} = \frac{\ell^2}{z^2}\mathrm{d}z^2 + \frac{\ell^2}{z^2}\left(-2a_0 z - \left(4a_1 - \frac{p^2}{2\ell^2}\right)z^2 + \dots\right)\mathrm{d}z^2. \tag{7.1.21}$$

Comparing this to equation (7.1.16) we see that for (7.1.16) to hold we need the second term on the right hand side of (7.1.21) to be identically zero, i.e.

$$-2a_0 z - \left(4a_1 - \frac{p^2}{2\ell^2}\right)z^2 + \dots = 0. \tag{7.1.22}$$

This is solved order by order in z to obtain the expansion coefficients a_i. We immediately observe that

$$a_0 = 0 \qquad \text{and} \qquad a_1 = \frac{p^2}{8\ell^2}, \tag{7.1.23}$$

implying that to $\mathrm{O}\left(z^2\right)$ the coordinate transformation is

$$r(z) = \frac{\ell^2}{z}\left(1 + \frac{p^2}{8\ell^2}z^2 + \dots\right). \tag{7.1.24}$$

Repeating the procedure, one can obtain the transformation to any order. However, as we go to higher orders, the calculations are much longer and tedious and it is best to be done using a computer program. Once this relationship has been obtained to the desired order we find the asymptotic form of the fields by simply substituting $r = r(z)$ in our solutions. This was also done using a computer program. The resulting expressions are

$$r(z)=\frac{\ell^2}{z}\left(1+\frac{p^2}{8\ell^2}z^2+\frac{m}{6\ell^4}z^3-\frac{q_{\mathsf{e}}^2+q_{\mathsf{m}}^2}{32\ell^6}z^4+\dots\right) \tag{7.1.25}$$

for the holographic coordinate,

$$g_{tt}=-1+z^2\frac{p^2}{4\ell^2}+z^3\frac{2m}{3\ell^4}-z^4\left(\frac{p^4}{64\ell^4}+\frac{3\left(q_{\mathsf{e}}^2+q_{\mathsf{m}}^2\right)}{16\ell^6}\right)+\mathrm{O}(z^5) \tag{7.1.26}$$

$$g_{zz}=1 \tag{7.1.27}$$

$$g_{xx}=g_{yy}=\ell^2+z^2\frac{p^2}{4}+z^3\frac{m}{3\ell^2}+z^4\left(\frac{p^4}{64\ell^2}-\frac{q_{\mathsf{e}}^2+q_{\mathsf{m}}^2}{16\ell^4}\right)+\mathrm{O}(z^5), \tag{7.1.28}$$

for the metric components and

$$A=\left(\frac{q_{\mathsf{e}}}{r_0}-\frac{q_{\mathsf{e}}}{\ell^2}z+\frac{p^2q_{\mathsf{e}}}{8\ell^4}z^3+\frac{mq_{\mathsf{e}}}{6\ell^6}z^4+\mathrm{O}(z^5)\right)\mathrm{d}t+q_{\mathsf{m}}x\mathrm{d}x \tag{7.1.29}$$

for the gauge field. The axions are unchanged since they depend only on the boundary spatial directions. Moreover, $Z_0=1$ and $V_0=-6/\ell^2$ with all other coefficients for $Z(\phi)$ and $V(\phi)$ vanishing.

1–point functions

To compute the 1–point functions for the dual theory, we simply substitute the expressions for the coefficients of the asymptotic expansions obtained here in the general expressions in chapter 6, section 6.3.5, equation (6.3.126). In particular, we find

$$\begin{gathered}\langle\mathcal{O}_\phi\rangle=0,\qquad\langle\mathcal{O}_{\psi_I}\rangle=0,\qquad\left\langle J^i\right\rangle=\left(\frac{r_0\mu_{\mathsf{e}}}{\ell^2},0,0\right)=\left(\frac{q_{\mathsf{e}}}{\ell^2},0,0\right),\\ \left\langle T^{ij}\right\rangle=\begin{pmatrix}2m/\ell^2&0&0\\0&m/\ell^4&0\\0&0&m/\ell^4\end{pmatrix}\end{gathered} \tag{7.1.30}$$

7.1.6 Thermodynamics

The final step of the analysis for the bald solution is to study the thermodynamics of the dual field theory using the definitions and relations discussed in chapter 6, section 6.4. The temperature and entropy density are equal to the corresponding quantities for the black hole, given by equations (7.1.4) and (7.1.5), respectively. Namely,

$$T=\frac{3r_0}{4\pi\ell^2}-\frac{1}{16\pi r_0}\left(2p^2+\frac{q_{\mathsf{e}}^2+q_{\mathsf{m}}^2}{r_0^2}\right)\qquad\text{and}\qquad s=4\pi r_0^2. \tag{7.1.31}$$

The energy density, ε, chemical potential μ, charge density ρ, magnetic charge density $\mathcal{B}$

and axionic charge density Π are

$$\begin{aligned}
&\varepsilon = \ell^2 \left\langle T^{tt} \right\rangle = 2m, \qquad \mu = \lim_{r\to\infty} A_t = \frac{q_{\mathsf{e}}}{r_0}, \qquad \rho = \ell^2 \left\langle J^t \right\rangle = q_e, \\
&\mathcal{B} = \frac{1}{\ell^3} F_{(0)xy} = \frac{q_{\mathsf{m}}}{\ell^3}, \quad \Pi = \frac{|p|}{\ell}
\end{aligned} \tag{7.1.32}$$

where we have used the relations between bulk fields and boundary charges and potentials, discussed in section 6.4. We have not derived the magnetisation $\mathcal{M}$ conjugate to the magnatic charge $\mathcal{B}$ and the axionic magnetisation ϖ conjugate to the axionic charge Π holographically. Instead we will obtain these quantities through the standard thermodynamic relations, from the Gibbs free energy or grand canonical potential. As we saw in section 6.4, this is equal to the renormalised Euclidean on-shell action. To compute this we must Wick rotate the Lorentzian renormalised on-shell action given by equation (6.3.118). Note that this action takes into account the contribution from the boundary action associated with the dialton, despite the fact that it vanishes in this case. The reason we must include it is because we want to consider the bald black branes as solutions to the same theory that admits the hairy solutions and hence we should impose the same boundary conditions on ϕ as those satisfied by the hairy solutions. These boundary conditions are trivially satisfied for a vanishing dialton.

Returning to the calculation of the renormalised on-shell action, we must first evaluate S_{bulk} by integrating over the radial coordinate r, from the horizon r_0, to a UV cutoff $\tilde{r} = \tilde{r}(\epsilon)$ in the r coordinate. At the end of this calculation we express $\tilde{r}(\epsilon)$ in terms of the cutoff ϵ in the z coordinate using the $r(z)$ expansion (7.1.25). The resulting expression is

$$S_{\mathsf{bulk}} = \int_{r_0}^{r(\epsilon)} \mathrm{d}^4x\sqrt{-G}\left(R - \frac{1}{4}F^2\right) = \int_{z=\epsilon} \mathrm{d}^3x \left[-\frac{2\ell^4}{\epsilon^3} - \frac{3\ell^2p^2}{4\epsilon} + \left(\frac{2r_0^3}{\ell^2} - m + \frac{q_{\mathsf{e}}^2 - q_{\mathsf{m}}^2}{2r_0}\right) + \mathrm{O}\left(\epsilon\right)\right]. \tag{7.1.33}$$

To this we must add the Gibbons–Hawking term S_{GH}, the boundary action associated with the dialton S_f and the counterterm action, evaluated on the bald solution. Using the expressions derived in section 6.3.4 we have

$$\begin{aligned}
S_{\mathsf{GH}} &= \int_{z=\epsilon} \mathrm{d}^3x \left[\frac{6\ell^4}{\epsilon^3} + \frac{\ell^2p^2}{4\epsilon} + \mathrm{O}(\epsilon)\right], \\
S_{\mathsf{ct}} &= -\int_{z=\epsilon} \mathrm{d}^3x \left[\frac{4\ell^4}{\epsilon^3} - \frac{\ell^2p^2}{2\epsilon} + \mathrm{O}(\epsilon)\right], \\
S_f &= 0,
\end{aligned} \tag{7.1.34}$$

and

$$S_{\mathsf{ren}} = \lim_{\epsilon\to 0}\left(S_{\mathsf{Bulk}} + S_{\mathsf{GH}} + S_{\mathsf{ct}} + S_f\right) = \int \mathrm{d}t \left(-m + \frac{2r_0^3}{\ell^2} + \frac{q_{\mathsf{e}}^2 - q_{\mathsf{m}}^2}{2r_0}\right)\mathcal{V}_2 \tag{7.1.35}$$

where $\mathcal{V}_2 = \mathrm{Vol}(\mathbb{R}^2)$. This is of course infinite but we circumvent this issue by using densities. Finally, by Wick rotating we can perform the integral over Euclidean time to obtain the Gibbs free energy, $\mathcal{W}(T,\mu)$, and the corresponding free energy density $w(T,\mu)$,

$$\mathcal{W}(T,\mu) = -S_{\mathsf{ren}}/\beta = w\mathcal{V}_2 = \left(m - \frac{2r_0^3}{\ell^2} + \frac{q_{\mathsf{m}}^2 - q_{\mathsf{e}}^2}{2r_0}\right)\mathcal{V}_2. \tag{7.1.36}$$

Using the expressions for ε, T, s, μ and ρ one can verify the relation

$$w = \varepsilon - sT - \mu\rho, \tag{7.1.37}$$

as required. As a consistency check, one can also check that

$$s = -\left(\frac{\partial w}{\partial T}\right)_{\mu,\mathcal{B},\Pi}, \qquad \rho = -\left(\frac{\partial w}{\partial \mu}\right)_{T,\mathcal{B},\Pi}. \tag{7.1.38}$$

This is done by parametrising $w = w(r_0, q_{\mathsf{e}}, T)$, $T = T(r_0, q_{\mathsf{e}}, p)$, and $\mu = \mu(r_0, q_{\mathsf{e}}, p)$, in terms of r_0, q_{e} and p, and using the implicit function theorem, leading for example to

$$\left(\frac{\partial w}{\partial T}\right)_{\mu,\pi,\mathcal{B}} = \frac{\left.\frac{\partial w}{\partial r_0}\right|_{q_{\mathsf{e}},p,q_{\mathsf{m}}} - \left.\frac{\partial w}{\partial q_{\mathsf{e}}}\right|_{r_0,p,q_{\mathsf{m}}} \frac{\left.\partial\mu\middle/\partial r_0\right|_{q_{\mathsf{e}},p,q_{\mathsf{m}}}}{\left.\partial\mu\middle/\partial q_{\mathsf{e}}\right|_{r_0,p,q_{\mathsf{m}}}}}{\left.\frac{\partial T}{\partial r_0}\right|_{q_{\mathsf{e}},p,q_{\mathsf{m}}} - \left.\frac{\partial T}{\partial q_{\mathsf{e}}}\right|_{r_0,p,q_{\mathsf{m}}} \frac{\left.\partial\mu\middle/\partial r_0\right|_{q_{\mathsf{e}},p,q_{\mathsf{m}}}}{\left.\partial\mu\middle/\partial q_{\mathsf{e}}\right|_{r_0,p,q_{\mathsf{m}}}}}. \tag{7.1.39}$$

Moreover, given that the total energy, electric charge and entropy are obtained from the corresponding densities by multiplying by the spatial volume $\mathcal{V}_2$, i.e. $\mathcal{E} = \varepsilon\mathcal{V}_2$, $Q_{\mathsf{e}} = \rho\mathcal{V}_2$, and $S = s\mathcal{V}_2$, the thermodynamic identity (6.4.12) determines the pressure to be

$$\mathcal{P} = -\left(\frac{\partial \mathcal{E}}{\partial \mathcal{V}_2}\right)_{S,Q_{\mathsf{e}},\mathcal{B},\Pi} = \langle T_{xx}\rangle - \frac{q_{\mathsf{m}}^2}{r_0} + p^2 r_0. \tag{7.1.40}$$

From (7.1.36) and the one point functions (7.1.30) then follows that $\mathcal{P} = -w$, in agreement with the general result (6.4.13). Finally, (7.1.36) and (7.1.37) give the Gibbs-Duhem relation (6.4.14),

$$\varepsilon + \mathcal{P} = Ts + \mu\rho. \tag{7.1.41}$$

Next we use the definitions (6.4.4) to compute the conjugate potentials to the magnetic and axionic charge densities,

$$\mathcal{M} = -\left(\frac{\partial w}{\partial \mathcal{B}}\right)_{T,\mu,\Pi} = -\ell^3\frac{q_{\mathsf{m}}}{r_0}, \qquad \varpi = -\left(\frac{\partial w}{\partial \Pi}\right)_{T,\mu,\mathcal{B}} = 2\ell\pi r_0. \tag{7.1.42}$$

Using these results it is straightforward to confirm that both the equation of state (6.4.22),

$$\varepsilon = (d-1)\mathcal{P} - 2\mathcal{M}\mathcal{B} - \varpi\Pi, \tag{7.1.43}$$

and the first law (6.4.9),

$$\mathrm{d}\varepsilon = T\,\mathrm{d}s + \mu\,\mathrm{d}\rho - \varpi\,\mathrm{d}\Pi - \mathcal{M}\,\mathrm{d}\mathcal{B}, \tag{7.1.44}$$

are satisfied. The extra contribution to the pressure (7.1.40) is thus due to the pressure $\mathcal{B}\mathcal{M} + \frac{1}{2}\Pi\varpi$ exerted by the magnetisation. Finally, in order to compare this solution to the hairy solution with the same charge densities, we need to use the Helmholtz free energy density $\mathfrak{f}$, which is a function of $T, \rho, \mathcal{B}, \Pi$. This is obtained from the Gibbs free energy by Legendre transforming with respect to the chemical potential μ,

$$\mathfrak{f} = w + \mu\rho = \varepsilon - Ts = m - \frac{2r_0^3}{\ell^2} + \frac{q_{\mathsf{e}}^2 + q_{\mathsf{m}}^2}{2r_0} \tag{7.1.45}$$

This concludes our current discussion of the bald solution of the theory. We return to it in section 7.3 were we discuss phase transitions of the theory between the bald and hairy phases, as well as the stability of these phases.

7.2 Hairy Solution

In addition to the bald solution, we have an analytical electrically charged hairy black brane solution with a non-trivial dialton. This solution is given by

$$\begin{aligned}
\mathrm{d}s^2 &= \Omega(r)\left[-f(r)\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{f(r)} + r^2\left(\mathrm{d}x^2 + \mathrm{d}y^2\right)\right] \\
f(r) &= \frac{r^2}{\ell^2} - \frac{p^2}{2}\left(1 + \frac{\sqrt{\alpha}v}{r}\right)^2, \qquad \Omega(r) = 1 - \frac{v^2}{(r + \sqrt{\alpha}\,v)^2} \\
\psi_1 &= px, \qquad \psi_2 = py \\
\phi(r) &= 2\sqrt{3}\tanh^{-1}\left(\frac{v}{r + \sqrt{\alpha}v}\right) \\
A &= \mu\left(1 - \frac{r_0}{r}\right)\mathrm{d}t, \qquad F = \mathrm{d}A = -\frac{q}{r^2}\mathrm{d}t \wedge \mathrm{d}r,
\end{aligned} \tag{7.2.1}$$

where again r_0 is the position of the horizon and p and q are related to the axion and $U(1)$ charge densities respectively. The parameter v is related to the vacuum expectation value (vev) of the scalar operator dual to ϕ, as we shall see later. Furthermore, the gauge potential vanishes at the horizon and thus μ is the chemical potential in the dual field theory and $q = \mu r_0$. This solution can be obtained from the corresponding solution found in [88], by rewriting it in the Einstein frame. An important point to note is that for these fields to be solutions to the equations of motion, given by (6.3.1) with $V(\phi)$ given by (7.0.1) and $Z(\phi) = 1$, we need to impose the constraint

$$q^2 = 2p^2v^2(1 - \alpha). \tag{7.2.2}$$

This constraint requires we restrict the range of α to $0 \leq \alpha \leq 1$.

Dyonic Solution

In analogy to the bald solution, the hairy solution (7.2.1) can be easily extended to include a magnetic field strength, that is constant everywhere since we are in four dimensions. The main difference lies in the fact that the constraint relating the parameters now fixes the electromagnetic duality invariant quantity $q_{\mathsf{e}}^2+q_{\mathsf{m}}^2$ instead of the electric charge alone,

$$q_{\mathsf{e}}^2 + q_{\mathsf{m}}^2 = 2p^2v^2(1-\alpha). \tag{7.2.3}$$

The metric, dialton, and axion fields are the same as in the purely electric solution (7.2.1), but the electromagnetic field picks up an extra magnetic contribution, as given in equation (7.1.2),

$$A = \mu_{\mathsf{e}}\left(1-\frac{r_0}{r}\right)\mathrm{d}t + q_{\mathsf{m}}x\,\mathrm{d}y, \qquad F = \mathrm{d}A = -\frac{q_{\mathsf{e}}}{r^2}\,\mathrm{d}t\wedge\mathrm{d}r + q_{\mathsf{m}}\,\mathrm{d}x\wedge\mathrm{d}y, \tag{7.2.4}$$

with $q_{\mathsf{e}} = \mu_{\mathsf{e}}r_0$. The resulting planar black hole is dyonic, and carries both an electric charge density q_{e} and a magnetic charge density q_{m}. When the magnetic field is turned on, the parameter space changes slightly, however the blackening function $f(r)$, the conformal factor $\Omega(r)$, and the dialton profile remain unchanged and, therefore, the geometric and thermodynamic properties of these dyonic branes can be related in a straightforward way to those of the electrically charged branes. The analysis here will be performed for the dyonic case. By setting q_{m} equal to zero, one retrieves the electric case.

7.2.1 Black Brane Properties: Horizon and Extremality

Horizon analysis

As usual, the event horizon of the black hole is located at the largest root of the lapse function, $f(r_0)=0$. However, the presence of the conformal factor $\Omega(r)$ in our metric introduces an additional constraint on our solutions. For a regular geometry, free of naked singularities, we require that $\Omega(r)$ does not vanish outside the horizon. Solving $\Omega(r_\Omega)=0$ for r_Ω we find

$$r_\Omega = |v| - \sqrt{\alpha}v. \tag{7.2.5}$$

For any genuine black hole solutions, $r_0 > r_\Omega$. Without loss of generality let $p \geq 0$. Then the largest root of $f(r_0)=0$ is

$$r_0 = \begin{cases} \dfrac{\ell p}{2\sqrt{2}}\left(1+\sqrt{1+4\sqrt{2\alpha}\dfrac{v}{\ell p}}\right) & \text{for } v \geq -\dfrac{\ell p}{4\sqrt{2\alpha}} \quad \text{(Case A)} \\ -\dfrac{\ell p}{2\sqrt{2}}\left(1-\sqrt{1-4\sqrt{2\alpha}\dfrac{v}{\ell p}}\right) & \text{for } v < -\dfrac{\ell p}{4\sqrt{2\alpha}} \quad \text{(Case B).} \end{cases} \tag{7.2.6}$$

Imposing the constraint $r_0 > r_\Omega$, we see that all Case B geometries suffer from naked singularities and are therefore rejected. The only regular solutions correspond to a subset of Case A for which the mass parameter satisfies

$$-\frac{\ell p}{\sqrt{2}\,(1+\sqrt{\alpha})^2} < v < \frac{\ell p}{\sqrt{2}\,(1-\sqrt{\alpha})^2}. \tag{7.2.7}$$

From this we can obtain the condition for r_0 by first solving $f(r_0) = 0$ for v,

$$v = \sqrt{\frac{2}{\alpha}}\frac{r_0^2}{\ell p} - \frac{r_0}{\sqrt{\alpha}}, \tag{7.2.8}$$

and substituting into equation (7.2.7),

$$\frac{\ell p}{\sqrt{2}\,(1+\sqrt{\alpha})} < r_0 < \frac{\ell p}{\sqrt{2}\,(1-\sqrt{\alpha})}. \tag{7.2.9}$$

The lower bound is in fact more restricted. We will see in section 7.3 that, for stability, $v > 0$ and hence

$$\frac{\ell p}{\sqrt{2}} < r_0 < \frac{\ell p}{\sqrt{2}(1-\sqrt{\alpha})}. \tag{7.2.10}$$

Temperature and absence of extremal solutions

The temperature of the black brane is

$$T = \frac{f'(r_0)}{4\pi} = \frac{1}{\pi \ell^2}\left(r_0 - \frac{p\ell}{2\sqrt{2}}\right). \tag{7.2.11}$$

Solving for r_0 and substituting in (7.2.10) we find that the temperature of the hairy black brane solutions is bounded by

$$\frac{p}{2\sqrt{2}\,\pi\ell} < T < \frac{1+\sqrt{\alpha}}{1-\sqrt{\alpha}}\frac{p}{2\sqrt{2}\,\pi\ell}. \tag{7.2.12}$$

An important thing to observe is that the temperature can never vanish for regular hairy solutions, implying there are no regular extremal hairy black brane solutions for this theory. In particular, the largest zero of $f(r)$ becomes a double zero located at $r_0 = \ell p/2\sqrt{2}$ when $v = -\ell p/4\sqrt{2\alpha}$, and so extremal solutions would be dynamically unstable, if they existed. However, when $v = -\ell p/4\sqrt{2\alpha}$ the conformal factor $\Omega(r)$ vanishes at

$$r_\Omega = \frac{1+\sqrt{\alpha}}{4\sqrt{2\alpha}}\ell p > r_0, \tag{7.2.13}$$

and, hence, extremal solutions are singular.[2]

[2]When $\alpha = 1$ we have $r_\Omega = r_0$ and so Ω vanishes on the horizon. However, the entropy vanishes too in that case.

Entropy density

The entropy density of these solutions is given by

$$s = 4\pi\,\Omega(r_0) r_0^2 = 4\pi \left((1+\alpha) - \frac{\ell p}{r_0\sqrt{2}} \right) \frac{r_0^2}{\alpha} \tag{7.2.14}$$

where we made use of the fact that we are using units in which $16\pi G_N = 1$.

Symmetry enhancement

We saw in the previous section that the bald solution to this theory becomes conformal to $\mathrm{AdS}_2 \times \mathbb{R}^2$ and its symmetry is enhanced to $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$. Furthermore, we said that this symmetry enhancement should also be a feature of the hairy solution. Indeed, when its lapse function takes the form shown in equation (7.1.6), namely,

$$f(r) = -\frac{p^2}{2} + \frac{r^2}{\ell^2}, \tag{7.2.15}$$

the hairy solution enjoys the same symmetry enhancement, irrespective of the presence of the conformal factor $\Omega(r)$. There are two instances where this occurs,

1. $\alpha = 0$ and $q_{\mathsf{e}}^2 + q_{\mathsf{m}}^2 = 2p^2 v^2$,
2. $v = 0$ and $q_{\mathsf{e}} = q_{\mathsf{m}} = 0$.

The second case coincides with the bald solution since $\phi = 0$ when $v = 0$ and therefore only the first is of relevance here. However, it is unlikely that the enhanced symmetry will lead to exactly solvable coupled linearised perturbation equations for the fields $g_{\mu\nu}$, A_μ, ϕ, and ψ_I, as is the case for the bald solutions. This is left for further exploration.

7.2.2 Fefferman–Graham gauge and 1–point functions

Fefferman–Graham Expansion

In this section we give the asymptotic expansions of the fields. These are obtained by applying the procedure outlined in section 7.1.5 to the hairy solution (7.2.1). The transformation of the holographic coordinate is,

$$\begin{aligned} r(z) = \frac{\ell^2}{z} + \frac{1}{8}\left(p^2 - \frac{2v^2}{\ell^2}\right) z + \frac{v\sqrt{\alpha}}{6\ell^2}\left(p^2 + \frac{2v^2}{\ell^2}\right) z^2 \\ + \frac{v^2}{16\ell^4}(1+\alpha)\left(p^2 - \frac{1+3\alpha}{1+\alpha}\frac{2v^2}{\ell^2}\right) z^3 + \mathrm{O}\left(z^4\right). \end{aligned} \tag{7.2.16}$$

Using this, the metric components are

$$g_{tt} = -1 + z^2 \frac{1}{\ell^2}\left(\frac{p^2}{4} + \frac{3v^2}{2\ell^2}\right) + z^3 \frac{8v\sqrt{\alpha}}{3\ell^4}\left(\frac{p^2}{4} - \frac{v^2}{\ell^2}\right) + O\left(z^4\right), \tag{7.2.17}$$

$$g_{zz} = 1 \tag{7.2.18}$$

$$g_{xx} = g_{yy} = \ell^2 + z^2\left(\frac{p^2}{4} - \frac{3v^2}{2\ell^2}\right) + z^3 \frac{4v\sqrt{\alpha}}{3\ell^2}\left(\frac{2v^2}{\ell^2} + \frac{p^2}{4}\right) + O\left(z^4\right) \tag{7.2.19}$$

the gauge field is

$$A(z) = \left(\frac{q_{\mathrm{e}}}{r_0} - \frac{q_{\mathrm{e}}}{\ell^2} z + \frac{p^2 q_{\mathrm{e}}}{8\ell^4} z^3 + \frac{v q_{\mathrm{e}}}{6\ell^6} z^4 + \mathrm{O}\left(z^5\right)\right) \mathrm{d}t + q_{\mathrm{m}} x \mathrm{d}x, \tag{7.2.20}$$

the dialton is

$$\phi(z) = z\frac{2v\sqrt{3}}{\ell^2} - z^2 \frac{2v^2\sqrt{3\alpha}}{\ell^4} + \mathrm{O}\left(z^3\right), \tag{7.2.21}$$

and the dialton potential is,

$$V(\phi) = -\frac{6}{\ell^2} - z^2\frac{12v^2}{\ell^6} + z^3 \frac{24v^3\sqrt{\alpha}}{\ell^8} + \mathrm{O}\left(z^4\right). \tag{7.2.22}$$

The axion solutions remain unchanged since they do not depend on the holographic direction and, similarly, $Z_0 = 1$ with all other coefficients for $Z(\phi)$ vanishing.

Dialton boundary conditions

As we discussed in 6, section 6.2.2, the theories we study satisfy mixed boundary conditions for the dialton, imposed by requiring that the function

$$J_f = -\ell^2 \varphi_{(1)} - f'\left(\varphi_{(0)}\right) = -\ell^2 \varphi_{(1)} - \vartheta \varphi_{(0)}^2 \tag{7.2.23}$$

is kept fixed. Accordingly, $\delta J_f = -\ell^2 \delta\varphi_{(1)} - 2\vartheta\varphi_{(0)}\delta\varphi_{(0)} = 0$. Here we confirm that the dialton field for this solution satisfy a boundary condition of this form and we derive the coupling ϑ.

From equation (7.2.21) we have

$$\varphi_{(0)} = \frac{2\sqrt{3}v}{\ell^2} \qquad \text{and} \qquad \varphi_{(1)} = -\frac{2\sqrt{3\alpha}\, v^2}{\ell^4}, \tag{7.2.24}$$

from which it follows that $\varphi_{(1)} = -\left(\sqrt{\alpha}/2\sqrt{3}\right)\varphi_{(0)}^2$. Hence, $\delta\varphi_{(1)} = -\sqrt{\alpha/3}\,\varphi_{(0)}\delta\varphi_{(0)}$ and
$\delta J_f = \left(\ell^2\sqrt{\alpha/3} - 2\vartheta\right)\varphi_{(0)}\delta\varphi_{(0)}$. It follows that the solution (7.2.1) satisfies mixed boundary conditions for the dialton field with

$$\vartheta_{\mathrm{I}} = \frac{\ell^2\sqrt{3\alpha}}{6}. \tag{7.2.25}$$

Note that when $\alpha = 0$, $\vartheta = 0$ and we have Neumann boundary conditions. In all other cases however, $\vartheta \neq 0$ and we are in the presence of a triple–trace deformation of the Neumann theory. Moreover, we have already seen that the vev of the dual operator is $\langle \mathcal{O}_\phi \rangle = \varphi_{(0)}$ and its conformal dimension is 1.

1–point functions

The 1–point functions for the dual theory can be read–off from the asymptotic expansions of the bulk fields, using the holographic relations in equation (6.3.126). In particular, we find

$$\langle \mathcal{O}_\phi \rangle = 2\sqrt{3}\,\frac{v}{\ell^2}, \qquad \langle \mathcal{O}_{\psi_I} \rangle = 0, \qquad \left\langle J^i \right\rangle = \left(\frac{r_0 \mu_{\mathsf{e}}}{\ell^2}, 0, 0\right) = \left(\frac{q_{\mathsf{e}}}{\ell^2}, 0, 0\right),$$
$$\left\langle T^{ij} \right\rangle = \begin{pmatrix} 2p^2 v\sqrt{\alpha}/\ell^2 & 0 & 0 \\ 0 & p^2 v\sqrt{\alpha}/\ell^4 & 0 \\ 0 & 0 & p^2 v\sqrt{\alpha}/\ell^4 \end{pmatrix}$$

7.2.3 Thermodynamics

In this section we use the definitions and relations discussed in section 6.4 to study the thermodynamic properties of the field theory dual to the hairy solution (7.2.1) of theory I (7.0.1). The analysis is identical to the analysis performed for the bald solutions in 7.1.6 and therefore we omit some of the details and refer the reader to the afformationed section if additional information is required.

The temperature and entropy density are equal to the corresponding quantities for the black hole, given by equations (7.2.11) and (7.2.14), respectively.

$$T = \frac{1}{\pi \ell^2}\left(r_0 - \frac{p\ell}{2\sqrt{2}}\right), \qquad s = 4\pi\left((1+\alpha) - \frac{\ell p}{r_0\sqrt{2}}\right)\frac{r_0^2}{\alpha}. \tag{7.2.26}$$

The energy density, ε, chemical potential μ, charge density ρ, magnetic charge density $\mathcal{B}$ and axionic charge density Π are

$$\varepsilon = \ell^2 \left\langle T^{tt} \right\rangle = 2p^2 v\sqrt{\alpha}, \qquad \mu = \lim_{r\to\infty} A_t = \frac{q_{\mathsf{e}}}{r_0}, \qquad \rho = \ell^2 \left\langle J^t \right\rangle = q_e,$$
$$\mathcal{B} = \frac{1}{\ell^3} F_{(0)xy} = \frac{q_{\mathsf{m}}}{\ell^3}, \qquad \Pi = \frac{|p|}{\ell}. \tag{7.2.27}$$

An important property of the hairy solutions of Theory I is that, as a direct consequence of the condition (7.2.3), the variables $(T, \mu, \mathcal{B}, \Pi)$ are *not* independent and satisfy the constraint

$$\mathcal{B}^2 + \left(\pi T + \frac{\Pi}{2\sqrt{2}}\right)^2 \left[\frac{\mu^2}{\ell^2} - \frac{4}{\alpha}(1-\alpha)\left(\pi T - \frac{\Pi}{2\sqrt{2}}\right)^2\right] = 0. \tag{7.2.28}$$

Since all the variables $T, \mu, \mathcal{B}, \Pi$ are a priori external tunable parameters, we conclude that these black holes exist only when these external parameters lie on the constraint submanifold defined by (7.2.28).

Next we compute the Gibbs free energy and free energy density of the field theory. For this we need the Euclidean renormalised on–shell action which is computed by following the same steps as those performed for the bald solutions in section 7.1.6. After some algebra, the Euclidean renormalised on–shell action for these black branes is

$$S^{\mathsf{E}}_{\mathsf{ren}} = -\beta\left(\frac{2r_0^3}{\ell^2} + \frac{q_{\mathsf{e}}^2 - q_{\mathsf{m}}^2}{2r_0} - p^2 v\sqrt{\alpha} - \frac{2v^3}{\ell^2}\frac{r_0^3\sqrt{\alpha}}{(r_0 + v\sqrt{\alpha})^3}\right)\mathcal{V}_2. \tag{7.2.29}$$

Note that the integral over the holographic direction has a contribution both from the asymptotic boundary and also from the horizon. This expression is in terms of the parameters of the solution whereas the Gibbs free energy is a function of $(T, \mu, \mathcal{B}, \Pi)$. Thus, to obtain the Gibbs free energy, one must first invert the expressions 7.2.27 and then substitute the result in the expression for $S^{\mathsf{E}}_{\mathsf{ren}}$ (7.2.29). This gives the following expression for the Gibbs free energy and free energy density,

$$w(T, \mu, \mathcal{B}, \Pi) = -\sqrt{2}\,\ell^4\Pi\left[\left(\pi T + \frac{\Pi}{2\sqrt{2}}\right)^2 + \frac{\mu^2}{4\ell^2(1-\alpha)}\right] + \ell^4\mathcal{B}^2\,\frac{\pi T - \alpha\left(\pi T + \frac{\Pi}{2\sqrt{2}}\right)}{(1-\alpha)\left(\pi T + \frac{\Pi}{2\sqrt{2}}\right)^2}, \tag{7.2.30}$$

where again all the variables $T, \mu, \mathcal{B}, \Pi$ lie on the constraint submanifold (7.2.28). Using the above expressions for the energy density, temperature, entropy, electric charge density and chemical potential, one can verify that the free energy density satisfies the thermodynamic relation (7.1.37), provided the constraint (7.2.28) is taken into account. For zero magnetic field (7.2.30) simplifies to

$$w = -\frac{\ell}{2}\left[(\Xi + 1)^2 + \frac{1}{\alpha}(\Xi - 1)^2\right]\left(\frac{p}{\sqrt{2}}\right)^3, \tag{7.2.31}$$

where $\Xi = 2\sqrt{2}\,\pi\ell T/p$, in complete agreement with the free energy obtained in eqn. (5.6) of [88] using a (real time) Hamiltonian approach to the thermodynamics. This means that we can use the thermodynamic analysis of [88] and, in particular, the results on the phase structure of the system obtained there.

If the variables $T, \mu, \mathcal{B}, \Pi$ were all independent, the expression (7.2.30) for the free energy density could be used to determine the thermodynamic potentials conjugate to the magnetic and axion charge densities through the relations (6.4.4). However, these variables are not independent, due to the constraint (7.2.28). Nevertheless, since we know already the values for the entropy and electric charge densities, we can use a Lagrange multiplier for the constraint (7.2.28) to obtain the potentials conjugate to the magnetic and axion charge densities. Considering the variation of $w + \lambda\mathcal{C}$, where λ is a Lagrange multiplier and $\mathcal{C}$ is the constraint (7.2.28), and identifying the coefficient of $\mathrm{d}T$ with $-s$

fixes the value of the Lagrange multiplier to

$$\lambda = \frac{4\ell^2(\sqrt{2}\,\ell^2\mu\Pi - 2(1-\alpha)q_{\mathsf{e}})}{(1-\alpha)\mu(4\pi T + \sqrt{2}\,\Pi)^2}. \tag{7.2.32}$$

This allows us to read off the potentials

$$\mathcal{M} = -\ell^3\frac{q_{\mathsf{m}}}{r_0}, \qquad \varpi = 2p\ell\left(r_0 + \frac{v^2}{r_0 + v\sqrt{\alpha}}\right). \tag{7.2.33}$$

Using these results one can verify that the first law (6.4.9) and the equation of state (6.4.22) hold.

Finally, in preparation for the discussion of the phase structure of the theory, we compute the Helmholz free energy density by Legendre transforming the Gibbs free energy with respect to the chemical potential μ,

$$\mathfrak{f} = w + \mu\rho = p^2 v\sqrt{\alpha} + \frac{2v^3}{\ell^2}\frac{r_0^3\sqrt{\alpha}}{(r_0 + v\sqrt{\alpha})^3} - \frac{2r_0^3}{\ell^2} + \frac{q_{\mathsf{e}}^2 + q_{\mathsf{m}}^2}{2r_0}. \tag{7.2.34}$$

The Helmholz free energy density is a function of $T, \rho, \mathcal{B}$ and Π which means that it allows us to compare solutions with the same charge densities.

BPS-like structure

Intriguingly, all thermodynamic variables of these solutions, including the temperature, are completely fixed by the charges. In particular, the energy density is given by

$$\varepsilon(\rho, \mathcal{B}, \Pi) = \left(\frac{2\alpha}{1-\alpha}\right)^{\frac{1}{2}} \ell\Pi\sqrt{\rho^2 + \mathcal{B}^2\ell^6}\,, \tag{7.2.35}$$

Similarly the entropy density is also completely determined in terms of $\rho, \mathcal{B}$ and Π, but the corresponding expression is too complicated to usefully reproduce it here.

This evokes the analogous property of extremal black holes, and despite having a non-vanishing temperature, the hairy black holes of Theory I behave along the constraint (7.2.3) as extremal black holes. The energy density itself is linear in the charges, and we can think of the black hole as composed of elementary blocks carrying unit axionic and electric charges, $(\Pi, \rho, \mathcal{B}) = (1, 1, 0)$, and magnetic elementary blocks with charges $(1, 0, 1)$ (in suitable units). We can thus investigate the stability of such black holes towards fragmentation of the charges by comparing the entropies of the system before and after fragmentation. We find

$$\begin{aligned}
s(\Pi, \rho_1, 0) + s(\Pi, \rho_2, 0) &\geq s(\Pi, \rho_1 + \rho_2, 0),\\
s(\Pi, 0, \mathcal{B}_1) + s(\Pi, 0, \mathcal{B}_2) &\geq s(\Pi, 0, \mathcal{B}_1 + \mathcal{B}_2),\\
s(\Pi_1, \rho, 0) + s(\Pi_2, \rho, 0) &\leq s(\Pi_1 + \Pi_2, \rho, 0),\\
s(\Pi_1, 0, \mathcal{B}) + s(\Pi_2, 0, \mathcal{B}) &\leq s(\Pi_1 + \Pi_2, 0, \mathcal{B}).
\end{aligned} \tag{7.2.36}$$

It is thus entropically favorable for these black holes to decay to a bound state of smaller black holes carrying a smaller electric charge/magnetic field. On the other hand, the axionic charge is stable against fragmentation.

Quantum Effective Potential

In this section we present the quantum effective potential for these solutions, derived in [2],

$$\mathcal{V}_{\mathsf{QFT}}(\sigma)= \mathcal{V}_0 + \frac{\mu q_{\mathsf{e}}}{\ell^2} + \frac{\varepsilon}{2\ell^2\sigma_*^3}\left(-2\sigma_*^3 + 3\sigma_*(\sigma-\sigma_*)^2 + (\sigma-\sigma_*)^3\right), \tag{7.2.37}$$

where $\mathcal{V}_0$ is a constant, ε is the energy density given in (7.2.27), σ is the *vev* and σ_* is its value at the extrema of the effective action of the theory, i.e. at the vacua. It follows that $\sigma = \sigma_*$ is a stable local extremum of the effective potential provided $\varepsilon > 0$, i.e. $v > 0$.

7.3 Phase Transitions and Dynamical Stability

The final part of our analysis focuses on the phase structure of the theory. For the solutions discussed in this chapter, this analysis was already performed in previous publications. We therefore do not go into great detail and instead summarise and restate the existing results, putting more focus on the elements that we consider more relevant and important for our analysis. In particular, we need to compare solutions that have the same asymptotic charges *and* satisfy the same boundary conditions, including the boundary conditions of the dialton ϕ. Since bald solutions are compatible with any boundary condition for the scalar ϕ, they can potentially compete with any hairy solutions with the same asymptotic charges.

As was stressed when discussing the thermodynamic properties of each solutions, the appropriate thermodynamic potential to use when comparing competing solutions with the same asymptotic charges is the Helmholtz free energy density. For the bald solution this is given by equation (7.1.45)

$$\mathfrak{f} = m - \frac{2r_0^3}{\ell^2} + \frac{q_{\mathsf{e}}^2 + q_{\mathsf{m}}^2}{2r_0}. \tag{7.3.1}$$

Notice that the magnetic and electric charges enter the same way in the Helmholtz free energy and so the thermodynamic stability properties of the dyonic solutions are qualitatively equivalent to those of the corresponding purely electric solutions. Moreover, as it was pointed out in [88], planar black holes with axion charge are equivalent to black holes with horizons of constant negative curvature and no axion charge. As a result, the stability properties of the planar bald solutions of this theory are analogous to those of bald black holes with hyperbolic horizons, which have been studied for example in [106, 107, 108, 109] and we refer the interested reader to these publications for more details.

For the hairy solutions, the Helmholtz free energy density is given in equations 7.2.34,

$$\mathfrak{f} = p^2 v\sqrt{\alpha} + \frac{2v^3}{\ell^2}\frac{r_0^3\sqrt{\alpha}}{(r_0 + v\sqrt{\alpha})^3} - \frac{2r_0^3}{\ell^2} + \frac{q_{\mathsf{e}}^2 + q_{\mathsf{m}}^2}{2r_0}. \tag{7.3.2}$$

As for the bald solutions, the dependence of the Helmholtz free energy on the magnetic and electric charges is identical and so the thermodynamic stability properties of the hairy dyonic solutions of this theory are identical to those of the purely electric solutions studied in [88]. However, the constraint (7.2.3) implies that the temperature is not an independent thermodynamic variable for the hairy solutions, but rather a fixed function of the charge densities, namely

$$T(\rho, \mathcal{B}, \Pi) = \frac{\Pi}{2\pi\sqrt{2}}\sqrt{1 + 4\sqrt{\frac{\alpha}{1-\alpha}}\frac{\sqrt{\rho^2 + \mathcal{B}^2\ell^6}}{\Pi^2\ell^3}}. \tag{7.3.3}$$

This means that, for given charge densities, one can only compare the free energy of the hairy solutions with that of the bald ones at a *fixed* temperature, which considerably restricts the useful information one can extract from such an analysis. Nevertheless, this analysis was performed in [88] and reveals that at large temperatures the unbroken phase of bald black holes dominates, and as we lower the temperature (*together with the charge densities according to* (7.3.3)), the system undergoes a second order phase transition towards a phase of hairy black holes. As the temperature is lowered further, below the lower bound in (7.2.12), the hairy solution becomes dynamically unstable, while at an even lower temperature it ceases to exist.

The constraint (7.2.3) does not have a physical significance which makes us believe that there exist more general hairy solutions of this theory whose temperature is not determined by the charge densities. Such solutions would allow one to explore the full phase diagram of the theory. However, we have been unable to find this more general class analytically. It would be interesting to see if this more general class of hairy solutions can be found numerically.

Chapter 8

Theory II

The second theory we will study is described by the action (6.2.1) with the dialton potential and dialton–gauge field coupling given, respectively, by

$$\begin{aligned} V(\phi) &= \sigma_1 \mathrm{e}^{\frac{(d-2)(d-1)\delta^2-2}{2(d-1)\delta}\phi} + \sigma_2 \mathrm{e}^{\frac{2\phi}{\delta(1-d)}} + \sigma_3 \mathrm{e}^{(d-2)\delta\phi}, \\ Z(\phi) &= \mathrm{e}^{-(d-2)\delta\phi}, \qquad W(\phi) = 1, \end{aligned} \tag{8.0.1}$$

where,

$$\begin{aligned} \sigma_1 &= \sigma_0 \frac{8(d-2)(d-1)^2\delta^2}{d(2+(d-2)(d-1)\delta^2)^2}, & \sigma_2 &= \sigma_0 \frac{(d-2)^2(d-1)\delta^2(d(d-1)\delta^2-2)}{d(2+(d-2)(d-1)\delta^2)^2} \\ \sigma_3 &= -2\sigma_0 \frac{(d-2)^2(d-1)\delta^2-2d}{d(2+(d-2)(d-1)\delta^2)^2}, & \sigma_0 &= -\frac{d(d-1)}{\ell^2} = 2\Lambda. \end{aligned} \tag{8.0.2}$$

and δ is a free parameter. This action, and an analytic family of electrically charged black brane solutions were presented in appendix C of [89]. The expressions for the potential and dialton–gauge field coupling simplify if one restricts to $d = 3$ and trades the parameter δ for a new parameter ξ defined by

$$\delta = \sqrt{\frac{2-\xi}{\xi}}, \qquad 0 < \xi < 2. \tag{8.0.3}$$

In this case, the expressions for the potential and couplings become

$$\begin{aligned} V(\phi) &= -\frac{1}{\ell^2}\mathrm{e}^{-\sqrt{\frac{\xi}{2-\xi}}\phi}\left((2-\xi)(3-2\xi) + 4\xi(2-\xi)\mathrm{e}^{\frac{\phi}{\sqrt{\xi(2-\xi)}}} + \xi(2\xi-1)\mathrm{e}^{\frac{2\phi}{\sqrt{\xi(2-\xi)}}}\right), \\ Z(\phi) &= \mathrm{e}^{-\sqrt{\frac{2-\xi}{\xi}}\phi}, \qquad W(\phi) = 1, \end{aligned} \tag{8.0.4}$$

Comparing the above potential, (8.0.4), to the potential of theory I, (7.0.1), we observe that the two coincide when $\xi = 1/2$ (or equivalently $\xi = 3/2$) and $\alpha = 1$. In all other cases they do not match. However, using the language defined in section 6.3.2 and, in particular, in equation (6.3.17), the V_2 coefficient which gives the mass of the dialton, is the same for both potentials, irrespective of the values of ξ and α. Thus, the AdS masses of the scalar and conformal dimensions of the dual operators, are the same for the two theories. Moreover, the discussion in chapter 6, section 6.2.2, about the dialton boundary conditions, applies to this theory as well.

An interesting feature of the potential (8.0.4) is that it can be written globally in terms of a superpotential as

$$U'^2(\phi) - \frac{3}{4}U^2(\phi) = 2V(\phi), \tag{8.0.5}$$

where

$$U(\phi) = -\frac{2}{\ell}\left((2-\xi)e^{-\frac{\xi/2}{\sqrt{\xi(2-\xi)}}\phi} + \xi e^{\frac{(1-\xi/2)}{\sqrt{\xi(2-\xi)}}\phi}\right). \tag{8.0.6}$$

Moreover, for $\xi = 1/2$ (or equivalently with $\xi = 3/2$) and $\xi = 1$, this potential can be embedded in the $U(1)^4$ truncation of maximally supersymmetric gauge supergravity, including the gauge field with $Z(\phi)$ as given in (8.0.4).

8.1.1 The $\xi \to 0$ and $\xi \to 2$ limits

The limits $\xi \to 0$ and $\xi \to 2$ require separate treatment and correspond to special cases of Theory I. In order to consider these limiting cases one must first rescale the fields appropriately to ensure that the potential and couplings remain finite. In particular, to study the $\xi \to 0$ limit we set $\xi = \epsilon$ and redefine the dialton and the gauge field as

$$\phi \to \tilde{\phi} = \frac{\phi}{\sqrt{\epsilon}}, \qquad A \to \tilde{A} = \frac{A}{\sqrt{\epsilon}}. \tag{8.1.7}$$

In the limit $\epsilon \to 0$, the potential $V(\phi)$ becomes $-6/\ell^2 = 2\Lambda$ and the coupling of the dialton to the Maxwell field $Z(\phi) \to e^{-2\tilde{\phi}}$. Rewriting the action (6.2.1) in terms of the rescaled fields and noting that the kinetic terms for the dialton and gauge field acquire a factor of ϵ, in the limit $\epsilon \to 0$, we obtain

$$S_{\text{bulk}} = \int_{\mathcal{M}} \mathrm{d}^4x\sqrt{-G}\left(R - 2\Lambda - \frac{1}{2}\sum_{I=1}^{2}(\partial\psi_I)^2\right). \tag{8.1.8}$$

This is a consistent truncation of Theory I, obtained by setting the dialton, ϕ, and the gauge field, A, to zero.

Similarly, the limit $\xi \to 2$ is obtained by setting $\xi = 2-\epsilon$ and redefining the dialton as above. In this case one need not rescale the gauge field, which therefore survives the limit. Letting $\epsilon \to 0$ we have $V(\phi) \to 2\Lambda$ and $Z(\phi) \to 1$ and the action becomes

$$S_{\text{bulk}} = \int_{\mathcal{M}} \mathrm{d}^4x\sqrt{-G}\left(R - 2\Lambda - \frac{1}{2}\sum_{I=1}^{2}(\partial\psi_I)^2 - \frac{1}{4}F^2\right). \tag{8.1.9}$$

Once again, this is a consistent truncation of Theory I corresponding to setting $\phi = 0$, but keeping the gauge field. Since both limits $\xi \to 0$ and $\xi \to 2$ result in truncations of Theory I, we will not consider these cases further, except in a brief discussion about the bald solutions of theory II. Most of our subsequent analysis will instead focus on the cases $0 < \xi < 2$.

8.2 Bald Solutions

We begin our analysis of theory II with the discussion of bald solutions, i.e. solutions for which the dialton vanishes. The $Z(\phi)F^2$ term in the action means that the gauge field sources the dialton, as can be seen from the equation of motion for the dialton 6.3.1,

$$\Box\phi - V'(\phi) = \frac{1}{4}Z'(\phi)F^2. \tag{8.2.1}$$

It follows that there are two possible ways to turn off the dialton. Since the gauge field sources the dialton, one possibility is for neutral bald solutions, i.e. $A = 0$. In this case the solution is

$$\begin{aligned} ds^2 &= -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\vec{x}^2\,, & \psi_I &= px^I\,, \\ f(r) &= \frac{r^2}{\ell^2} - \frac{r_0^3}{r\ell^2} - \frac{p^2}{2}\left(1 - \frac{r_0}{r}\right)\,, & \phi &= 0, \qquad A = 0. \end{aligned} \tag{8.2.2}$$

This solution is the same as the bald solution of Theory I, (7.1.1), with appropriate identifications of the parameters. Moreover, we will see later that it corresponds to the $v_{\mathrm{e}}\,, \xi \to 0$ limit of the hairy solution of theory II.

In addition to the neutral bald solutions, it is possible to have charged ones as well, provided that the dialton and gauge field decouple. This means that $Z(\phi)$ must be constant. The only was this can be achieved is if $\xi = 2$. As we saw in the previous section, in this the theory is described by the action (8.1.9) and it is the consistent truncation of Theory I corresponding to setting $\phi = 0$. As such, it shares its bald solutions which were presented in chapter 7, section 7.1.

Thus, we find that the bald solutions of theory II are special cases of the bald solutions of theory I and, for this reason, we do not repeat the analysis and refer the reader to the relevant sections of the previous chapter.

8.3 Hairy Solution

8.3.1 Electric Solutions

In [89] the author obtained an analytic expression for a family of electrically charged hairy black hole solutions for theory II, generalised to any dimension d. In terms of the original parameter δ these are given by[1]

$$\begin{aligned}
ds^2 &= -f(r)h(r)^{\frac{-4}{2+(d-2)(d-1)\delta^2}}dt^2 + h(r)^{\frac{4}{(d-2)(2+(d-2)(d-1)\delta^2)}}\left(\frac{dr^2}{f(r)} + r^2 d\vec{x}^2_{d-1}\right),\\
f(r) &= \frac{r^2}{\ell^2}\left(h(r)^{\frac{4(d-1)}{(d-2)(2+(d-2)(d-1)\delta^2)}} - \frac{r_0^d}{r^d}h(r_0)^{\frac{4(d-1)}{(d-2)(2+(d-2)(d-1)\delta^2)}}\right) - \frac{p^2}{2(d-2)}\left(1-\frac{r_0^{d-2}}{r^{d-2}}\right),\\
\phi(r) &= \frac{-2(d-1)\delta}{2+(d-2)(d-1)\delta^2}\log h(r), \qquad h(r) = 1+\frac{v_{\mathsf{e}}}{r^{d-2}},\\
A(r) &= 2\sqrt{(d-1)v_{\mathsf{e}}}\,\frac{\sqrt{(d-2)\frac{r_0^{d+2}}{\ell^2}h(r_0)^{\frac{2(2-(d-2)^2(d-1)\delta^2)}{(d-2)(2+(d-1)(d-2)\delta^2)}} - \frac{p^2 r_0^d}{2h(r_0)}}}{(d-2)r_0^{d-1}h(r)\sqrt{2+(d-2)(d-1)\delta^2}}\left(1-\frac{r_0^{d-2}}{r^{d-2}}\right)\mathrm{d}t,\\
\psi_I &= px^I,
\end{aligned} \tag{8.3.1}$$

where the parameter v_{e} is identified with Q in [89]. This solution has a twofold advantage over the hairy solution of theory I: it is a solution for *any* dimension d, and there is no constraint among its parameters, i.e. it solves the equations of motion for any values of the parameters r_0, p and v_{e}. However, this parameterisation of the solution treats the electric charge density as a dependent parameter, expressed in terms of the radius of the horizon, r_0, the axion charge density p and the parameter v_{e} which, as we will see later, is proportional to the *vev* of the scalar operator dual to the dialton. This not only obscures the limiting process of taking the charge density to zero, but also it does not reflect the change of the sign of the gauge potential when the charge density changes sign.

Focusing on the four-dimensional case, i.e. $d = 3$, from now on, we will therefore introduce an alternative parameterisation of the solution (8.3.1), by introducing explicitly the charge density q_{e} as an independent parameter, in addition to replacing δ with ξ as in (8.0.3). With these modifications the solution takes the form

$$\begin{aligned}
ds^2 &= -f(r)h^{-\xi}(r)\mathrm{d}t^2 + h^{\xi}(r)\left(f^{-1}(r)\mathrm{d}r^2 + r^2\mathrm{d}\vec{x}^2\right),\\
f(r) &= \frac{r^2}{\ell^2}h^{2\xi}(r) - \frac{p^2}{2}h(r) - \frac{q_{\mathsf{e}}^2}{2\xi v_{\mathsf{e}} r} = \frac{r^2}{\ell^2}h^{2\xi}(r) - \frac{r_0^3}{r\ell^2}h^{2\xi}(r_0) - \frac{p^2}{2}\left(1-\frac{r_0}{r}\right),\\
\phi &= -\sqrt{\xi(2-\xi)}\log h(r), \qquad h(r) = 1+\frac{v_{\mathsf{e}}}{r},\\
A &= q_{\mathsf{e}}\left(\frac{1}{r_0 h(r_0)} - \frac{1}{r h(r)}\right)\mathrm{d}t, \qquad \psi_I = px^I,
\end{aligned} \tag{8.3.2}$$

where v_{e} can be expressed in terms of q_{e}, p and r_0 through the relation

$$v_{\mathsf{e}} = \frac{\ell^2 q_{\mathsf{e}}^2}{\xi r_0 h(r_0)\left(2r_0^2 h(r_0)^{2\xi-1} - \ell^2 p^2\right)}. \tag{8.3.3}$$

This identity should be viewed as an expression for $v_{\mathsf{e}}(r_0, q_{\mathsf{e}}, p)$ or conversely $r_0(v_{\mathsf{e}}, q_{\mathsf{e}}, p)$, but *not* $q_{\mathsf{e}}(r_0, v_{\mathsf{e}}, p)$, as in the original parameterisation (8.3.1). Note, in particular, that it

[1] This fixes a typo in equation C.5 of [89].

is clear from the form (8.3.2) of the solution that we can set $q_{\mathsf{e}} = 0$, while keeping $v_{\mathsf{e}} \neq 0$. In the parameterization in terms of r_0 this corresponds to a non-trivial scaling limit. Moreover, notice that the first expression for the blackening factor $f(r)$ in (8.3.2) implies that as long as $q_{\mathsf{e}} \neq 0$ we must necessarily have $v_{\mathsf{e}} \neq 0$, and hence the scalar field must have a non-trivial profile. In summary, $q_{\mathsf{e}} \neq 0$ implies that $v_{\mathsf{e}} \neq 0$, but the converse is not true.

8.3.2 Magnetic Solutions

We have been able to find in addition an analytical black brane solution that is purely magnetically charged and takes the form

$$\begin{aligned}
\mathrm{d}s^2 &= -f(r)h^{-(2-\xi)}(r)dt^2 + h^{2-\xi}(r)\left(\frac{\mathrm{d}r^2}{f(r)} + r^2\mathrm{d}\vec{x}^2\right), \\
\phi(r) &= -\sqrt{\xi(2-\xi)}\,\log h(r), \qquad h(r) = 1 + \frac{v_{\mathsf{m}}}{r}, \\
f(r) &= h(r)\left(\frac{r^2}{\ell^2} + \frac{(3-2\xi)v_{\mathsf{m}}r}{\ell^2}\left(1 + \frac{(1-\xi)v_{\mathsf{m}}}{r}\right) - \frac{p^2}{2} + \frac{q_{\mathsf{m}}^2}{2\xi v_{\mathsf{m}} r}\right), \\
A &= q_{\mathsf{m}}\, x\mathrm{d}y, \qquad \psi_I = p x^I.
\end{aligned} \tag{8.3.4}$$

In addition to the axion charge density p, this solution is parameterised by the magnetic charge density q_{m} as well as the independent parameter v_{m}, which as we will see later, is again related to the *vev* of the scalar operator dual to ϕ.

8.3.3 Black Brane Properties: Horizon and Extremality

Electric Solutions

Starting with the electric solution (8.3.2), we first need to know the range of parameters for which these solutions describe regular black branes, and to find the location of their event horizon. The solution has singularities at both $r = 0$ and $r = -v_{\mathsf{e}}$ (where $h(r)$ vanishes). To be regular, the solution must thus have an event horizon at a location r_0 such that $r_0 > 0$ and $r_0 > -v_{\mathsf{e}}$. Horizons correspond to the zeros of the function $f(r)$ and can be determined in terms of the electric and axion charge densities, respectively p and q_{e}, as well as the parameter v_{e}, by solving the (generically transcendental) equation

$$\frac{r_0^2}{\ell^2}h^{2\xi}(r_0) - \frac{p^2}{2}h(r_0) - \frac{q_{\mathsf{e}}^2}{2\xi v_{\mathsf{e}} r_0} = 0, \qquad h(r_0) = 1 + \frac{v_{\mathsf{e}}}{r_0}. \tag{8.3.5}$$

For $\xi = 1, 1/2, 3/2$, this equation is cubic in r_0 and the roots can be found explicitly, although they are still highly involved expressions.

The temperature of these black branes can be computed as usual by requiring that there is no conical singularity in the Euclidean section of the metric, giving

$$\begin{aligned} T &= \frac{1}{4\pi}\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{f(r;q_\mathsf{e},v_\mathsf{e},p)}{h^\xi(r)}\right)\bigg|_{r=r_0} \\ &= \frac{1}{4\pi\ell^2 h^\xi(r_0)}\left(2r_0h^{2\xi}(r_0) - v_\mathsf{e}\xi h^{2\xi-1}(r_0) + \frac{p^2\ell^2 v_\mathsf{e}(1-\xi)}{2r_0^2} + \frac{q_\mathsf{e}^2\ell^2(r+(1-\xi)v_\mathsf{e})}{2\xi v_\mathsf{e} r^3 h(r_0)}\right). \end{aligned} \tag{8.3.6}$$

Moreover, the entropy density is given by the area density of the event horizon,

$$s = \frac{r_0^2\, h^\xi(r_0)}{4G_\mathsf{N}} = 4\pi r_0^2\, h^\xi(r_0). \tag{8.3.7}$$

In contrast to the hairy solutions of Theory I, these solutions admit extremal limits, corresponding to the cases where the temperature (8.3.6) vanishes. Combining this condition with the defining equation $f(r_0) = 0$ for the horizon determines the location of the *extremal* horizon to be

$$r_0^\mathsf{ex} = \frac{\xi(2\xi-5)p^2v_\mathsf{e}^2 - 3q_\mathsf{e}^2 + \mathrm{sgn}(v_\mathsf{e})\sqrt{9q_\mathsf{e}^4 + \xi^2(2\xi-1)^2p^4v_\mathsf{e}^4 + 2\xi(2\xi+3)p^2q_\mathsf{e}^2v_\mathsf{e}^2}}{4\xi p^2 v_\mathsf{e}}. \tag{8.3.8}$$

Requiring in addition that $f(r_0^\mathsf{ex}) = 0$ gives the extremality condition, which can be expressed in the form $v_\mathsf{e}^\mathsf{ex}(q_\mathsf{e}, p)$. It is not possible to obtain this extremality condition analytically for generic ξ, but it can be done for specific values.[2] The simplest case is

$$\xi = 1, \qquad v_\mathsf{e} = -\sqrt{\frac{\ell\left(9\ell p^2q_\mathsf{e}^2 - \ell^3p^6 + (\ell^2p^4 - 6q_\mathsf{e}^2)^{3/2}\right)}{\ell^2p^4 - 8q_\mathsf{e}^2}}, \tag{8.3.9}$$

$$v_\mathsf{e} = \frac{\ell^2 q_\mathsf{e}^2}{\xi r_0 h(r_0)\left(2r_0^2 h(r_0)^{2\xi-1} - \ell^2p^2\right)}. \tag{8.3.10}$$

in which case the extremal horizon simplifies to (see (8.5.5))

$$r_0^\mathsf{ex} = -v_\mathsf{e} + \frac{(6q_\mathsf{e}^2 - \ell^2p^4)v_\mathsf{e}}{9q_\mathsf{e}^2 + p^2v_\mathsf{e}^2}. \tag{8.3.11}$$

However, as we shall see, the energy density for this solution, given in equation (8.4.2), is negative which implies that it is dynamically unstable (see section 8.5).

Magnetic Solutions

The magnetically charged solutions (8.3.4) can be studied similarly. The location r_0 of the horizon is determined by the equation

$$\frac{r_0^2}{\ell^2} + \frac{(3-2\xi)v_\mathsf{m}r_0}{\ell^2}\left(1 + \frac{(1-\xi)v_\mathsf{m}}{r_0}\right) - \frac{p^2}{2} + \frac{q_\mathsf{m}^2}{2\xi v_\mathsf{m} r_0} = 0, \tag{8.3.12}$$

[2]The extremality condition can also be determined by requiring that the discriminant of the polynomial $f(r)$ vanishes, in which case the horizon becomes a multiple root.

which in this case is cubic for arbitrary ξ. The generic expression for r_0, however, is still too lengthy to usefully reproduce here. The same applies to the temperature and entropy density, which are given by

$$T = \frac{1}{4\pi}\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{f(r;q_{\mathsf{m}},v_{\mathsf{m}},p)}{h^{2-\xi}(r)}\right)\Bigg|_{r=r_0} = \frac{h^{\xi-1}(r_0)}{8\pi\ell^2 r_0}\left(-\ell^2p^2+6r_0^2h^2(r_0)-8v_{\mathsf{m}}r_0\xi+2v_{\mathsf{m}}^2\xi(2\xi-5)\right),$$
$$s = \frac{r_0^2\,h^{2-\xi}(r_0)}{4G_{\mathsf{N}}} = 4\pi r_0^2\,h^{2-\xi}(r_0). \tag{8.3.13}$$

As for the electrically charged solutions, extremal solutions can be found, even analytically for specific values of the parameters. The simplest case is again

$$\xi = 1, \qquad v_{\mathsf{m}} = \sqrt{\frac{\ell\left(9\ell p^2 q_{\mathsf{m}}^2 - \ell^3 p^6 + (\ell^2p^4 - 6q_{\mathsf{m}}^2)^{3/2}\right)}{\ell^2p^4 - 8q_{\mathsf{m}}^2}}, \tag{8.3.14}$$

with the extremal horizon given by (see (8.5.13))

$$r_0^{\mathsf{ex}} = \frac{9\ell^2 q_{\mathsf{m}}^2 + \ell^2p^2v_{\mathsf{m}}^2}{2v_{\mathsf{m}}\left(3\ell^2p^2 + 2v_{\mathsf{m}}^2\right)}. \tag{8.3.15}$$

Again, the energy density (8.4.17) is negative for this solution, and so it is also dynamically unstable.

Scaling symmetry Finally, all the families of solutions presented above enjoy a scaling symmetry: they are left invariant by the scaling $(t,\vec{x},r) \to (\lambda t, \lambda\vec{x}, \lambda^{-1}r)$ of the coordinates, when accompanied by the rescaling $(p, v_{\mathsf{e/m}}, q_{\mathsf{e/m}}) \to (\lambda^{-1}p, \lambda^{-1}v_{\mathsf{e/m}}, \lambda^{-2}q_{\mathsf{e/m}})$ of the parameters. Under such a rescaling, the temperature and entropy transform according to $T \to \lambda^{-1}T$ and $s \to \lambda^{-2}s$. This invariance will simplify the study of the phases of these black branes, since it allows us to scale away one of the parameters such as the axion charge density p, as long as it is non vanishing.

8.3.4 Fefferman–Graham Gauge and One–Point Functions

In this section we give the asymptotic expansions of the fields and use them to read off the one–point functions of the dual operators. These are obtained by repeating the procedure outlined in section 7.2.2 (see also 6.3 for the complete discussion) to the hairy solutions (8.3.2) and (8.3.4).

Electric Solutions

Fefferman–Graham expansion For the electrically charged black branes of equation (8.3.2) the transformation of the holographic coordinate is,

$$\begin{aligned} r(z) = & \frac{\ell^2}{z} - \frac{\xi v_{\rm e}}{2} + \frac{1}{8}\left(p^2 + \frac{v_{\rm e}^2}{2\ell^2}\xi(2-\xi)\right) z \\ & + \frac{1}{12\ell^4}\left(2r_0^3 h(r_0)^{2\xi} - \ell^2 p^2 (r_0 + \xi v_{\rm e}) - \frac{1}{3}\xi(1-\xi)(2-\xi)v_{\rm e}^3\right) z^2 \\ & - \frac{\xi v_{\rm e}}{16\ell^6}\left(r_0^3 h(r_0)^{2\xi} - \frac{1}{2}\ell^2 p^2 r_0 h(r_0) - \frac{1}{4}(1-\xi)^2(2-\xi)v_{\rm e}^3\right) z^3 + \mathcal{O}(z^4). \end{aligned} \tag{8.3.16}$$

Using this, the metric components are

$$\begin{aligned} g_{tt} = & -1 + z^2\left(\frac{p^2}{4\ell^2} + \frac{\xi(2-\xi)v_{\rm e}^2}{8\ell^4}\right) \\ & + z^3\left(\frac{2r_0^3}{3\ell^6}h^{2\xi}(r_0) - \frac{p^2}{3\ell^4}(r_0 + \xi v_{\rm e}) - \frac{\xi(1-\xi)(2-\xi)v_{\rm e}^3}{9\ell^6}\right) + \mathcal{O}\left(z^4\right), \end{aligned} \tag{8.3.17}$$

$$g_{zz} = 1, \tag{8.3.18}$$

$$\begin{aligned} g_{xx} = g_{yy} = & \ \ell^2 + z^2\left(\frac{p^2}{4} - \frac{\xi(2-\xi)v_{\rm e}^2}{8\ell^2}\right) \\ & + z^3\left(\frac{r_0^3}{3\ell^4}h^{2\xi}(r_0) - \frac{p^2}{6\ell^2}(r_0 + \xi v_{\rm e}) + \frac{\xi(1-\xi)(2-\xi)v_{\rm e}^3}{9\ell^4}\right) + \mathcal{O}\left(z^4\right) \end{aligned} \tag{8.3.19}$$

and the remaining fields,

$$\phi = -\sqrt{\xi(2-\xi)}\frac{v_{\rm e}}{\ell^2}z + (1-\xi)\sqrt{\xi(2-\xi)}\frac{v_{\rm e}^2}{2\ell^4}z^2 + \mathcal{O}(z^3), \tag{8.3.20}$$

$$A_t = \frac{q_{\rm e}}{r_0 h(r_0)} - \frac{q_{\rm e}}{\ell^2}z + \mathcal{O}(z^2), \tag{8.3.21}$$

with the axions the same as in the original coordinate system.

The dialton potential is,

$$V(\phi) = -\frac{6}{\ell^2} - z^2\frac{v_{\rm e}^2\xi\left(2-\xi\right)}{\ell^6} + z^3\frac{v_{\rm e}^3\xi\left(2-\xi\right)\left(1-\xi\right)}{\ell^8} + \mathcal{O}\left(z^4\right). \tag{8.3.22}$$

and the dialton–gauge field coupling is

$$Z(\phi) = 1 + z\frac{v_{\rm e}\left(2-\xi\right)}{\ell^2} + z^2\frac{v_{\rm e}^2\left(2-\xi\right)}{2\ell^4} - z^3\frac{v_{\rm e}(2-\xi)\left(6\ell^2p^2 - v_{\rm e}^2\xi\left(10-\xi\right)\right)}{48\ell^6} + \mathcal{O}\left(z^4\right). \tag{8.3.23}$$

Dilaton boundary conditions As we discussed in section 6.3.3, the theories we study satisfy mixed boundary conditions for the dialton. These are given by the function $J_f = -\ell^2\varphi_{(1)} - f'\left(\varphi_{(0)}\right) = -\ell^2\varphi_{(1)} - \vartheta\varphi_{(0)}^2$ which is kept fixed. Accordingly, $\delta J_f = -\ell^2\delta\varphi_{(1)} - 2\vartheta\varphi_{(0)}\delta\varphi_{(0)} = 0$.

Here we confirm that the dialton field of the electrically charged black branes satisfies a boundary condition of this form and we derive the coupling ϑ.

From equation (8.3.20) we have $\varphi_{(0)} = -\sqrt{\xi(2-\xi)}v_{\mathsf{e}}/\ell^2$ and $\varphi_{(1)} = (1-\xi)\sqrt{\xi(2-\xi)}v_{\mathsf{e}}^2/2\ell^4$ and, thus,

$$\varphi_{(1)} = \frac{1-\xi}{2\sqrt{\xi\,(2-\xi)}}\varphi_{(0)}^2. \tag{8.3.24}$$

Varying this expression we find

$$\delta\varphi_{(1)} = \frac{(1-\xi)}{\sqrt{\xi(2-\xi)}}\varphi_{(0)}\delta\varphi_{(0)}, \qquad \delta J_f = \left(-\ell^2\frac{(1-\xi)}{\sqrt{\xi(2-\xi)}} - 2\vartheta_{\mathsf{II}}^{\mathsf{e}}\right)\varphi_{(0)}\delta\varphi_{(0)}. \tag{8.3.25}$$

It follows that the solution (8.3.2) satisfies mixed boundary conditions for the dialton field with

$$\vartheta_{\mathsf{II}}^{\mathsf{e}} = -\frac{(1-\xi)\,\ell^2}{2\sqrt{\xi\,(2-\xi)}}. \tag{8.3.26}$$

Moreover, we have already seen that the *vev* of the dual operator is $\langle\mathcal{O}_\phi\rangle = \varphi_{(0)}$ and its conformal dimension is 1.

One-point functions The one-point functions for the field theory dual to the electrically charged black branes can be read off from the asymptotic expansions of the bulk fields, using the holographic relations in equation (6.3.126). In particular, we find

$$\begin{aligned}
&\langle\mathcal{O}_\phi\rangle = -\sqrt{\xi(2-\xi)}\frac{v_{\mathsf{e}}}{\ell^2}, \qquad\qquad \langle\mathcal{O}_{\psi_I}\rangle = 0,\\
&\langle J^i\rangle = \left(\frac{q_{\mathsf{e}}}{\ell^2},0,0\right),\\
&\langle T_{ij}\rangle = \begin{pmatrix} \frac{1}{\ell^4}2r_0^3h^{2\xi}(r_0) - \frac{1}{\ell^2}p^2(r_0+\xi v_{\mathsf{e}}) & 0 & 0\\ 0 & \frac{1}{\ell^2}r_0^3h^{2\xi}(r_0) - \frac{1}{2}p^2(r_0+\xi v_{\mathsf{e}}) & 0\\ 0 & 0 & \frac{1}{\ell^2}r_0^3h^{2\xi}(r_0) - \frac{1}{2}p^2(r_0+\xi v_{\mathsf{e}})\end{pmatrix}.
\end{aligned} \tag{8.3.27}$$

Notice that, in the limit v_{e}, $q_{\mathsf{e}} \to 0$, the above stress energy tensor coincides with the stress energy tensor of the bald solution of theory I, given in equation (7.1.30), with the identification $m = r_0^3/\ell^2 - p^2r_0/2$ of their parameters. This is in perfect agreement with the discussion in section 8.2 of bald black branes of theory II.

Magnetic Solutions

Fefferman-Graham expansion For the magnetic solutions (8.3.4) the transformation of the holographic coordinate is,

$$\begin{aligned}
r =& \frac{\ell^2}{z} - \frac{2-\xi}{2}v_{\mathsf{m}} + \frac{1}{8}\left(p^2 + \frac{v_{\mathsf{m}}^2}{2\ell^2}\xi(2-\xi)\right)z\\
&- \frac{1}{12\ell^2\xi v_{\mathsf{m}}}\left(q_{\mathsf{m}}^2 + p^2v_{\mathsf{m}}^2\xi(1-\xi) - \frac{v_{\mathsf{m}}^4}{3L^2}\xi(1-\xi)(6-14\xi+7\xi^2)\right)z^2\\
&- \frac{1}{32\ell^4}\left(q_{\mathsf{m}}^2 - \frac{v_{\mathsf{m}}^4}{2\ell^2}\xi(1-\xi)^2(2-\xi)\right)z^3 + \mathcal{O}(z^4).
\end{aligned} \tag{8.3.28}$$

Using this, the metric components are

$$g_{tt} = -1 + z^2\left(\frac{p^2}{4\ell^2} + \frac{\xi(2-\xi)v_{\mathsf{m}}^2}{8\ell^4}\right) + z^3\left(\frac{(1-\xi)(6-14\xi+7\xi^2)v_{\mathsf{m}}^3}{9\ell^6} - \frac{q_{\mathsf{m}}^2}{3\ell^4\xi v_{\mathsf{m}}} - \frac{(1-\xi)p^2 v_{\mathsf{m}}}{3\ell^4}\right) + \mathcal{O}(z^4), \quad (8.3.29)$$

$$g_{zz} = 1, \quad (8.3.30)$$

$$g_{xx} = g_{yy} = \ell^2 + z^2\left(\frac{p^2}{4} - \frac{\xi(2-\xi)v_{\mathsf{m}}^2}{8\ell^2}\right) + z^3\frac{(1-\xi)(3-10\xi+5\xi^2)v_{\mathsf{m}}^3}{9\ell^4} - \frac{q_{\mathsf{m}}^2}{6\ell^2\xi v_{\mathsf{m}}} - \frac{(1-\xi)p^2 v_{\mathsf{m}}}{6\ell^2} + \mathcal{O}(z^4), \quad (8.3.31)$$

and the remaining fields,

$$\phi = -\sqrt{\xi(2-\xi)}\frac{v_{\mathsf{m}}}{\ell^2}z + (1-\xi)\sqrt{\xi(2-\xi)}\frac{v_{\mathsf{m}}^2}{2\ell^4}z^2 + \mathcal{O}(z^3), \quad (8.3.32)$$

$$A_i = (0, 0, q_{\mathsf{m}}x), \quad (8.3.33)$$

with the axions the same as in the original coordinate system.

The dialton potential is,

$$V(\phi) = -\frac{6}{\ell^2} - z^2\frac{v_{\mathsf{m}}^2\xi\,(2-\xi)}{\ell^6} - z^3\frac{v_{\mathsf{m}}^3\xi\,(2-\xi)\,(1-\xi)}{\ell^8} + \mathcal{O}\left(z^4\right). \quad (8.3.34)$$

and the dialton–gauge field coupling is

$$Z(\phi) = 1 + z\frac{v_{\mathsf{m}}\,(2-\xi)}{\ell^2} + z^2\frac{v_{\mathsf{m}}^2\,(2-\xi)\,(3-2\xi)}{2\ell^4} + \mathcal{O}(z^3). \quad (8.3.35)$$

Dilaton boundary conditions Once more, the asymptotic expansion for the scalar ϕ determines the boundary conditions these black holes are compatible with. Comparing the relation $\varphi_{(1)} = \frac{1-\xi}{2\sqrt{\xi(2-\xi)}}\varphi_{(0)}^2$ between the two scalar modes with the condition that the single trace source for the dual scalar operator vanishes, i.e. $J_f = -\ell^2\varphi_{(1)} - \mathcal{F}'(\varphi_{(0)}) = 0$, determines that the multitrace deformation function $\mathcal{F}(\varphi_{(0)})$ is of the form (6.3.72) with

$$\vartheta_{\mathsf{II}}^{\mathsf{m}} = \frac{(1-\xi)\ell^2}{2\sqrt{\xi(2-\xi)}}. \quad (8.3.36)$$

Notice that this is the same as the boundary condition (8.3.26) for the electrically charged solutions, except for the sign.

One–point functions Once again, the one–point functions of the operators dual to the bulk fields of the magnetic solution, can be read off from the asymptotic expansions of the bulk fields, using the holographic relations given in equation 6.3.126. In particular, we find

$$\langle \mathcal{O}_{\Delta_-} \rangle = \varphi_{(0)} = -\sqrt{\xi(2-\xi)}\, v_{\mathsf{m}}/\ell^2, \qquad \langle J^i \rangle = (0,\,0,\,0)\,, \qquad \langle \mathcal{O}_{\psi_I} \rangle = 0,$$
$$\langle T_{ij} \rangle = \begin{pmatrix} \frac{2}{3\ell^4}(1-\xi)(1-2\xi)(3-2\xi)v_{\mathsf{m}}^3 - \frac{q_{\mathsf{m}}^2}{\ell^2\xi v_{\mathsf{m}}} - \frac{1}{\ell^2}(1-\xi)p^2 v_{\mathsf{m}} & 0 & 0 \\ 0 & \frac{2}{6\ell^2}(1-\xi)(1-2\xi)(3-2\xi)v_{\mathsf{m}}^3 - \frac{q_{\mathsf{m}}^2}{2\xi v_{\mathsf{m}}} - \frac{1}{2}(1-\xi)p^2 v_{\mathsf{m}} & 0 \\ 0 & 0 & \frac{2}{6\ell^2}(1-\xi)(1-2\xi)(3-2\xi)v_{\mathsf{m}}^3 - \frac{q_{\mathsf{m}}^2}{2\xi v_{\mathsf{m}}} - \frac{1}{2}(1-\xi)p^2 v_{\mathsf{m}} \end{pmatrix}. \tag{8.3.37}$$

8.4 Thermodynamics

In this section we use the definitions and relations discussed in section 6.4 to study the thermodynamic properties of the field theory dual to the hairy solutions (8.3.2) and (8.3.4), of theory II.

8.4.1 Electric Solutions

We begin with the discussion of the electrically charged black brane solutions given in equation (8.3.2). The temperature and entropy density of the field theory are equal to the corresponding quantities for the black hole, given by equations (8.3.6) and (8.3.7), respectively. Namely,

$$\begin{aligned} T &= \frac{1}{4\pi\ell^2 h^{\xi}(r_0)}\left(2r_0 h^{2\xi}(r_0) - v_{\mathsf{e}}\xi h^{2\xi-1}(r_0) + \frac{p^2\ell^2 v_{\mathsf{e}}(1-\xi)}{2r_0^2} + \frac{q_{\mathsf{e}}^2\ell^2(r+(1-\xi)v_{\mathsf{e}})}{2\xi v_{\mathsf{e}} r^3 h(r_0)}\right), \\ s &= \frac{r_0^2\, h^{\xi}(r_0)}{4G_{\mathsf{N}}} = 4\pi r_0^2\, h^{\xi}(r_0). \end{aligned} \tag{8.4.1}$$

The energy density, ε, chemical potential μ, charge density ρ and axionic charge density Π are

$$\begin{aligned} \varepsilon &= \ell^2 \left\langle T^{tt} \right\rangle = (1-\xi)p^2 v_{\mathsf{e}} + \frac{q_{\mathsf{e}}^2}{\xi v_{\mathsf{e}}}, \\ \mu &= \lim_{r\to\infty} A_t = \frac{\rho}{r_0 h(r_0)}, \qquad \rho = \ell^2 \left\langle J^t \right\rangle = q_e, \qquad \Pi = \frac{|p|}{\ell} \end{aligned} \tag{8.4.2}$$

where we have used the relations between bulk fields and boundary charges and potentials, discussed in section 6.4. We have not derived the axionic magnetisation ϖ conjugate to the axionic charge Π holographically. Instead we will derive it from the Gibbs free energy or grand canonical potential, through the standard thermodynamic relations. To obtain the Gibbs free energy we must compute the renormalised Euclidean on–shell action. The general expression for the Lorentzian on–shell action for the class of theories we are studying was derived in chapter 6 (equation (6.3.118)). In particular, there are four terms to consider, S_{Bulk} obtained by integrating the on–shell bulk action between the horizon and the UV cutoff $\bar{r}(\epsilon)$ (corresponding to $z=\epsilon$), the Gibbons–Hawking term

S_{GH}, the boundary action associated with the dialton boundary conditions S_f and the counterterm action S_{ct}. The expressions for all these terms can be found in section 6.3.4. The only thing left for us to do is evaluate them for the electrically charged solution studied. This is a trivial exercise for all but S_{Bulk} which we must still integrate along the holographic direction, from the horizon r_0, to a UV cutoff $\bar{r} = \bar{r}(\epsilon)$. At the end of this calculation we express $\bar{r}(\epsilon)$ in terms of the cutoff ϵ in the z coordinate using the $r(z)$ expansion (7.1.25) The resulting expression is

$$\begin{aligned} S_{\text{bulk}} &= \int_{r_0}^{\bar{r}(\text{e})} \mathrm{d}r \int \mathrm{d}^3x\, \sqrt{-G}\left(V - \frac{1}{4}F^2\right) \\ &= \int \mathrm{d}^3x \left[-\frac{2\ell^4}{\epsilon^3} - \frac{6\ell^2p^2 - \xi(2-\xi)v_{\text{e}}^2}{8\epsilon} + r_0\left(\frac{p^2}{2} + \frac{r_0^2}{\ell^2}h^{2\xi}(r_0)\right) + \mathcal{O}(\epsilon).\right] \end{aligned} \tag{8.4.3}$$

Combining this result with the S_f, S_{GH} and S_{ct} we finally obtain

$$S^E_{\text{ren}} = -S_{\text{ren}} = -\beta\mathcal{V}_2\left(\frac{r_0^3h^{2\xi}(r_0)}{\ell^2} + \frac{r_0p^2}{2}\right). \tag{8.4.4}$$

where $\mathcal{V}_2 = \text{Vol}(\mathbb{R}^2)$. This is of course infinite but we circumvent this issue by using densities. Finally, by Wick rotating we can perform the integral over Euclidean time to obtain the Gibbs free energy, $\mathcal{W}(T,\mu)$, and the corresponding free energy density $w(T,\mu)$,

$$\mathcal{W}(T,\mu) = -S_{\text{ren}}/\beta = w\mathcal{V}_2 = \left(\frac{r_0^3h^{2\xi}(r_0)}{\ell^2} + \frac{r_0p^2}{2}\right)\mathcal{V}_2. \tag{8.4.5}$$

Using the expressions for ε, T, s, μ and ρ one can verify the relation

$$w = \varepsilon - sT - \mu\rho, \tag{8.4.6}$$

as required. As a consistency check, one can also check that

$$s = -\left(\frac{\partial w}{\partial T}\right)_{\mu,\Pi}, \qquad \rho = -\left(\frac{\partial w}{\partial \mu}\right)_{T,\Pi}. \tag{8.4.7}$$

Moreover, by varying the free energy density with respect to axion charge density Π, keeping T and μ fixed, we obtain the thermodynamic potential conjugate to the axion charge density,

$$\varpi = -\left(\frac{\partial w}{\partial \Pi}\right)_{T,\mu} = 2p\ell\left(r_0 + \frac{v_{\text{e}}\xi}{2}\right). \tag{8.4.8}$$

Combining this with our previous results allows us to confirm the density first law (6.4.9),

$$\mathrm{d}\varepsilon = T\,\mathrm{d}s + \mu\,\mathrm{d}\rho - \varpi\,\mathrm{d}\Pi. \tag{8.4.9}$$

The pressure of the system, defined in equation (6.4.12), is given by

$$\mathcal{P} = -\left(\frac{\partial \mathcal{E}}{\partial \mathcal{V}_2}\right)_{S,Q_\mathsf{e},\Pi} = \langle T_{xx}\rangle + p^2\left(r_0 + \frac{v_\mathsf{e}\xi}{2}\right) = \langle T_{xx}\rangle + \frac{1}{2}\Pi\varpi, \tag{8.4.10}$$

where we have introduced the total energy $\mathcal{E} = \varepsilon\mathcal{V}_2$, electric charge $Q_\mathsf{e} = \rho\mathcal{V}_2$, entropy $S = s\mathcal{V}_2$. It is easy to verify that this is equal to $-w$, as required. Furthermore, combining the above result, one can check that they satisfy both the Gibbs–Duhem relation,

$$\varepsilon + \mathcal{P} = Ts + \mu\rho \tag{8.4.11}$$

and the first law (6.4.11)

$$\mathrm{d}\mathcal{E} = T\,\mathrm{d}S - \mathcal{P}\,\mathrm{d}\mathcal{V}_2 + \mu\,\mathrm{d}Q_\mathsf{e} - (\varpi\mathcal{V})\mathrm{d}\Pi - (\mathcal{M}\mathcal{V}_2)\,\mathrm{d}\mathcal{B}, \tag{8.4.12}$$

As an additional check, one can verify that the equation of state derived in section 6.4.1 holds, provided we set $\mathcal{B} = 0$ in equation (6.4.22). In particular, for the current solutions, one finds

$$\varepsilon = 2\mathcal{P} - \varpi\Pi. \tag{8.4.13}$$

Finally, in order to be able to compare solutions with the same charge densities, we need to derive an expression for the Helmholtz free energy density $\mathfrak{f}$, which is a function of T, ρ, Π. This is related to the Gibbs free energy by a Legendre transform with respect to the chemical potential μ,

$$\mathfrak{f} = w + \mu\rho = \frac{q_\mathsf{e}^2}{r_0 h(r_0)} - \frac{r_0^3 h^2(r_0)}{\ell^2} - \frac{r_0 p^2}{2}. \tag{8.4.14}$$

This concludes our current discussion of the electrically charged black branes of theory II. We will return to them in section 8.5 where we discuss the phase structure of the theory.

8.4.2 Magnetic Solutions

We now turn to the magnetically charged black brane solutions, given by equation (8.3.4), and repeat the analysis performed above. The temperature and entropy density of the field theory are equal to the corresponding quantities for the black hole, given in equation (8.3.13),

$$T = \frac{h^{\xi-1}(r_0)}{8\pi\ell^2 r_0}\left(-\ell^2p^2 + 6r_0^2h^2(r_0) - 8v_\mathsf{m}r_0\xi + 2v_\mathsf{m}^2\xi(2\xi-5)\right), \tag{8.4.15}$$

$$s = \frac{r_0^2\,h^{2-\xi}(r_0)}{4G_\mathsf{N}} = 4\pi r_0^2\,h^{2-\xi}(r_0). \tag{8.4.16}$$

Moreover, the energy density, ε, axionic charge density Π and magnetic charge density $\mathcal{B}$ are

$$\varepsilon = \frac{2}{3}(1-\xi)(2\xi-3)(2\xi-1)\frac{v_{\mathsf{m}}^3}{\ell^2} - (1-\xi)p^2 v_{\mathsf{m}} - \frac{q_{\mathsf{m}}^2}{\xi v_{\mathsf{m}}}, \tag{8.4.17}$$

$$\mathcal{B} = \frac{1}{\ell^3}F_{(0)xy} = \frac{q_{\mathsf{m}}}{\ell^3}, \qquad \Pi = \frac{|p|}{\ell}, \tag{8.4.18}$$

where we have used the relations between bulk fields and boundary charges and potentials, discussed in section 6.4. Once more, the axionic magnetisation ϖ conjugate to the axionic charge Π, as well as the magnetisation $\mathcal{M}$ conjugate to the magnetic charge, will be derived from the Gibbs free energy through the standard thermodynamic relations. We therefore proceed to compute the Gibbs free energy by computing the renormalised on–shell action, as we did for the electrically charged black branes. Using equation (6.3.118) as our reference, we begin by integrating the bulk on–shell action,

$$\begin{aligned} S_{\mathsf{bulk}} = \int_{r_0}^{\bar{r}(\epsilon)} \mathrm{d}r \int \mathrm{d}^3x\, \sqrt{-G}\left(V - \frac{1}{4}F^2\right) = \int \mathrm{d}^3x \Bigg[&-\frac{2\ell^4}{\epsilon^3} - \frac{6\ell^2p^2 - \xi(2-\xi)v_{\mathsf{m}}^2}{8\epsilon} \\ &+ \frac{1}{6\ell^2\xi v_{\mathsf{m}} r_0}\left(3\ell^2(p^2v_{\mathsf{m}}^2 r_0\xi(1-\xi) + q_{\mathsf{m}}^2(r_0 - v_{\mathsf{m}}\xi)) + 2v_{\mathsf{m}}r_0\xi(6r_0^3 + 9v_{\mathsf{m}}r_0^2(2-\xi)\right. \\ &\left.+3v_{\mathsf{m}}^2 r_0(6 - 7\xi + 2\xi^2) + v_{\mathsf{m}}^3(3 - 2\xi - 3\xi^2 + 2\xi^3))\right) + \mathcal{O}(\epsilon)\Bigg], \end{aligned} \tag{8.4.19}$$

where we have used the UV cutoff $\bar{r}(\epsilon)$ to regularise the integral, as explained above. Add to S_{Bulk} S_f, S_{GH} and S_{ct} and performing the remaining integrations, we finally obtain

$$\begin{aligned} S^E_{\mathsf{ren}} = -S_{\mathsf{ren}} = -\,\beta\mathcal{V}_2 \Bigg[&\frac{1}{\ell^2}\left(2r_0^3 + 3v_{\mathsf{m}}r_0^2(2-\xi) + 3v_{\mathsf{m}}^2 r_0(2-\xi)(1-\frac{2}{3}\xi) + v_{\mathsf{m}}^3(1-\xi^2)(1-\frac{2}{3}\xi)\right) \\ &+ \frac{q_{\mathsf{m}}^2}{2\xi v_{\mathsf{m}}}\left(1 - \frac{\xi v_{\mathsf{m}}}{r_0}\right) + \frac{1}{2}(1-\xi)p^2 v_{\mathsf{m}}\Bigg]. \end{aligned} \tag{8.4.20}$$

Hence, the Gibbs free energy, $\mathcal{W}(T,\mu)$, and the corresponding free energy density $w(T,\mu)$, are

$$\begin{aligned} \mathcal{W}(T,\mu) &= -S_{\mathsf{ren}}/\beta = w\mathcal{V}_2 \\ &= \mathcal{V}_2\left[\frac{1}{\ell^2}\left(2r_0^3 + 3v_{\mathsf{m}}r_0^2(2-\xi) + 3v_{\mathsf{m}}^2 r_0(2-\xi)(1-\frac{2}{3}\xi) + v_{\mathsf{m}}^3(1-\xi^2)(1-\frac{2}{3}\xi)\right)\right. \\ &\qquad \left. + \frac{q_{\mathsf{m}}^2}{2\xi v_{\mathsf{m}}}\left(1 - \frac{\xi v_{\mathsf{m}}}{r_0}\right) + \frac{1}{2}(1-\xi)p^2 v_{\mathsf{m}}\right]. \end{aligned} \tag{8.4.21}$$

Now that we have derived w, the magnetisation $\mathcal{M}$ and axionic magnetisation ϖ can also be computed,

$$\mathcal{M} = -\left(\frac{\partial w}{\partial \mathcal{B}}\right)_{T,\Pi} = -\frac{q_{\mathsf{m}}\ell^3}{r_0}, \qquad \varpi = -\left(\frac{\partial w}{\partial \Pi}\right)_{T,\mu} = 2p\ell\left(r_0 + \frac{(2-\xi)v_{\mathsf{m}}}{2}\right). \tag{8.4.22}$$

Combining these results with the expressions for ε, T, s, $\mathcal{B}$ and w, it is straight forward to confirm that the energy density is equal to the Legendre transform of the free energy density with respect to T (the chemical potential is zero in this case),

$$\varepsilon(s, \mathcal{B}, \Pi) = w + Ts, \tag{8.4.23}$$

as well as the density first law (6.4.9),

$$\mathrm{d}\varepsilon = T\mathrm{d}s - \varpi \mathrm{d}\Pi - \mathcal{M}\mathrm{d}\mathcal{B}. \tag{8.4.24}$$

The pressure in this case is

$$\mathcal{P} = -\left(\frac{\partial \mathcal{E}}{\partial \mathcal{V}_2}\right)_{\mathcal{S},\mathcal{B},\Pi} = \langle T_{xx}\rangle + \frac{1}{2}\Pi\varpi + \mathcal{M}\mathcal{B}, \tag{8.4.25}$$

where $\mathcal{E}$ and $\mathcal{S}$ are defined as in the discussion of the electically charged branes. Once more it is related to the transverse components of the stress tensor. Using this result one can check that the Gibbs-Duhem relation (6.4.14), and the first law (6.4.11) both hold, and that the equation of state of the system is of the form discussed in 6.4.1, as anticipated. Finally, we note that the Helmholtz free energy coincides with the Gibbs free energy in this case since there is no chemical potential.

8.5 Thermodynamic Stability and Phase Transitions

In this section we address the question of the phase structure of the theory. We want to compare solutions with the same charge densities and which satisfy the same boundary conditions. The first requirement implies that if we want to compare bald and hairy solutions, we must restrict to neutral solutions only since, as we saw in section 8.2, in order to set the dialton equal to zero consistently, we must also have vanishing gauge field. However, for a given temperature and finite charge density, there are up to three hairy solutions with different radii and scalar *vevs*, which compete thermodynamically, giving rise to an intricate phase diagram. We will focus on these solutions and, moreover, we will restrict our analysis to $\xi = 1$ for which we are able to obtain analytic solutions. The discussion is straightforwardly applicable to $\xi \neq 1$ but in those case one has to solve transcedental equations numerically. An important thing to note regarding the $\xi = 1$ solutions is that the multitrace deformation coupling $\vartheta_{\parallel}^{\mathtt{e/m}}$ vanishes and thus, the resulting theory satisfies Neumann boundary conditions.

We begin our analysis by deriving analytic expressions that will help us understand the phase structure of the theory. We then proceed to study the scalar *vev*, energy density, radius and Helmholtz free energy of the solutions as functions of the temperature, at fixed charge densities. The Helmholtz free energy tells us which solution is thermodynamically preferred whereas the energy density determines whether a solution is dynamically stable; it was shown in [2] that, for these solutions, dynamical stability is equivalent to positivity of the energy density.

We begin by looking at the electric solutions. The analysis of the magnetic solutions follows after.

8.5.1 Phases of Electric Solutions

For the hairy solutions of our theory, the temperature is an independent variable and it can be used to explore the full phase diagram. To do this, we begin by deriving an analytic expression that can be used to determine the solutions of the theory in terms of physical parameters. In particular, using the defining equation for the horizon, $f(r_0) = 0$,

$$\frac{r_0^2}{\ell^2} h^2(r_0) - \frac{p^2}{2} h(r_0) - \frac{q_{\rm e}^2}{2 v_{\rm e} r_0} = 0, \tag{8.5.1}$$

we obtain two different expressions for the temperature. The two different expressions for T correspond to using the equation $f(r_0) = 0$ in two different ways. The first expression is obtained by solving (8.5.1) for $\frac{q_e^2}{2v_e r_0}$ and substituting the result in the expression for the temperature (8.3.6), with $\xi = 1$,

$$T = \frac{1}{4\pi\ell^2}\left(2r_0 + v_{\rm e} + \frac{q_{\rm e}^2 \ell^2}{2 v_{\rm e} r_0^2 h^2(r_0)}\right), \tag{8.5.2}$$

This yields

$$\tau = 3r_0 + v_{\rm e} - \frac{p^2 \ell^2}{2(r_0 + v_{\rm e})}, \tag{8.5.3}$$

where we have defined $\tau \equiv 4\pi\ell^2 T$. Alternatively, we can solve (8.5.1) for $\frac{r_0^2}{\ell^2} h^2(r_0)$ and substitute in (8.5.2), finding

$$\tau = 2r_0 + v_{\rm e} + \frac{q_{\rm e}^2 r_0}{q_{\rm e}^2 + p^2 v_{\rm e}(r_0 + v_{\rm e})}. \tag{8.5.4}$$

Eliminating the quadratic terms in r_0 we obtain an explicit formula for the radius of the horizon as a function of the charge densities, the temperature and the scalar *vev* parameter $v_{\rm e}$, namely

$$r_0 = -v_{\rm e} + \frac{(6q_{\rm e}^2 - \ell^2 p^4) v_{\rm e} + 3 q_{\rm e}^2 \tau}{9 q_{\rm e}^2 + p^2 v_{\rm e}(v_{\rm e} - \tau)}. \tag{8.5.5}$$

Finally, inserting this result in either (8.5.3) or (8.5.4) we obtain the *characteristic curve*

$$(\ell^2 p^4 - 8q_{\rm e}^2) v_{\rm e}^4 + (2\ell^4 p^6 - \ell^2 p^4 \tau^2 - 18\ell^2 p^2 q_{\rm e}^2 + 6 q_{\rm e}^2 \tau^2) v_{\rm e}^2 + 2 q_{\rm e}^2 \tau^3 v_{\rm e} - 27 \ell^2 q_{\rm e}^4 = 0, \tag{8.5.6}$$

which is an equation relating physical observables only.

The expression (8.5.5) for the horizon radius and the characteristic curve are valid everywhere except in the regime where $q_{\rm e} \to 0$ and $v_{\rm e} \to 0$ simultaneously, in which case the manipulations that lead to (8.5.5) and (8.5.6) starting from (8.5.3) and (8.5.4) break down. This limit must be taken so that $v_{\rm e}/q_{\rm e}^2$ is kept fixed and corresponds to the bald solutions that only exist at zero charge density. Being a quartic equation in v_{e}, the characteristic equation admits four roots at fixed temperature and fixed electric and axionic charge densities. At most three of these roots are real and have $r_0 \geq 0$: we checked numerically that there is always either a root corresponding to a singular metric with $r_0 < 0$, or at least two complex conjugate roots.

Restricting for now our analysis to small charges and using equations (8.5.3) and (8.5.4), we find that, in this limit, there are three physical black brane solutions whose scalar *vev* parameter and horizon radius are given by

$$\text{Red:} \qquad v_{\mathsf{e}} = -\sqrt{\tau^2 - \tau_c^2} + \mathcal{O}(q_{\mathsf{e}}^2), \qquad r_0 = \frac{1}{2}\left(\tau + \sqrt{\tau^2 - \tau_c^2}\right) + \mathcal{O}(q_{\mathsf{e}}^2), \qquad \tau > \tau_c, \tag{8.5.7a}$$

$$\text{Blue:} \qquad v_{\mathsf{e}} = \frac{-q_{\mathsf{e}}^2 \ell^2}{r_0(2r_0^2 - p^2\ell^2)} + \mathcal{O}(q_{\mathsf{e}}^4), \quad r_0 = \frac{1}{6}\left(\tau + \sqrt{\tau^2 + 3\tau_c^2}\right) + \mathcal{O}(q_{\mathsf{e}}^2), \qquad \tau > \tau_c, \tag{8.5.7b}$$

$$\text{Orange:} \quad v_{\mathsf{e}} = \begin{cases} \frac{-q_{\mathsf{e}}^2\ell^2}{r_0(2r_0^2 - p^2\ell^2)} + \mathcal{O}(q_{\mathsf{e}}^4), \\ \sqrt{\tau^2 - \tau_c^2} + \mathcal{O}(q_{\mathsf{e}}^2), \end{cases} \qquad r_0 = \begin{cases} \frac{1}{6}\left(\tau + \sqrt{\tau^2 + 3\tau_c^2}\right) + \mathcal{O}(q_{\mathsf{e}}^2), & \tau < \tau_c, \\ \frac{1}{2}\left(\tau - \sqrt{\tau^2 - \tau_c^2}\right) + \mathcal{O}(q_{\mathsf{e}}^2), & \tau > \tau_c, \end{cases} \tag{8.5.7c}$$

where we have colour coded each solution. Here

$$\tau_c = 4\pi\ell^2 T_c = \sqrt{2} p\ell, \tag{8.5.8}$$

denotes the critical temperature at *zero* charge density. As we will see below, the true critical temperature increases slightly with increasing charge density. Nonetheless, we find it convenient to use τ_c as a reference temperature at arbitrary charge density. An important observation regarding the above perturbative solutions, (8.5.7), is that, near the critical temperature τ_c, both the blue and orange solutions have a pole in v_{e}. To see this note that as $\tau \to \tau_c^+$, $r_0^{\texttt{blue}} \to \tau_c/2 = p\ell/\sqrt{2}$ and hence $v_{\mathsf{e}}^{\texttt{blue}} \to \infty$. The same is observed for the orange solution for $\tau \to \tau_c^-$. Moreover, the orange solution is discontinuous at τ_c. These, however, are not true features of the solutions but consequences of the break down of perturbation theory. If one uses instead the characteristic curve (8.5.6) to find the analytic solutions then he finds that all curves are in fact smooth and the corners only exist in the strict q_{e} limit. This is demonstrated in figure 8.5.2, which shows that all curves are in fact smooth.

In order to understand the behaviour of these solutions, we fix ρ and plot, as a function of temperature, the energy density and the Helmholtz free energy, given in equations (8.4.2) and (8.4.14), respectively, by setting $\xi = 1$. We also plot the radius and scalar *vev* of each solution. The first two plots allow us to determine the dynamical and thermodynamical stability of the solutions whereas the latter two provide an insight into the physical characteristics of the dominating solution. Figures 8.5.1, 8.5.3 and 8.5.4 show plots of

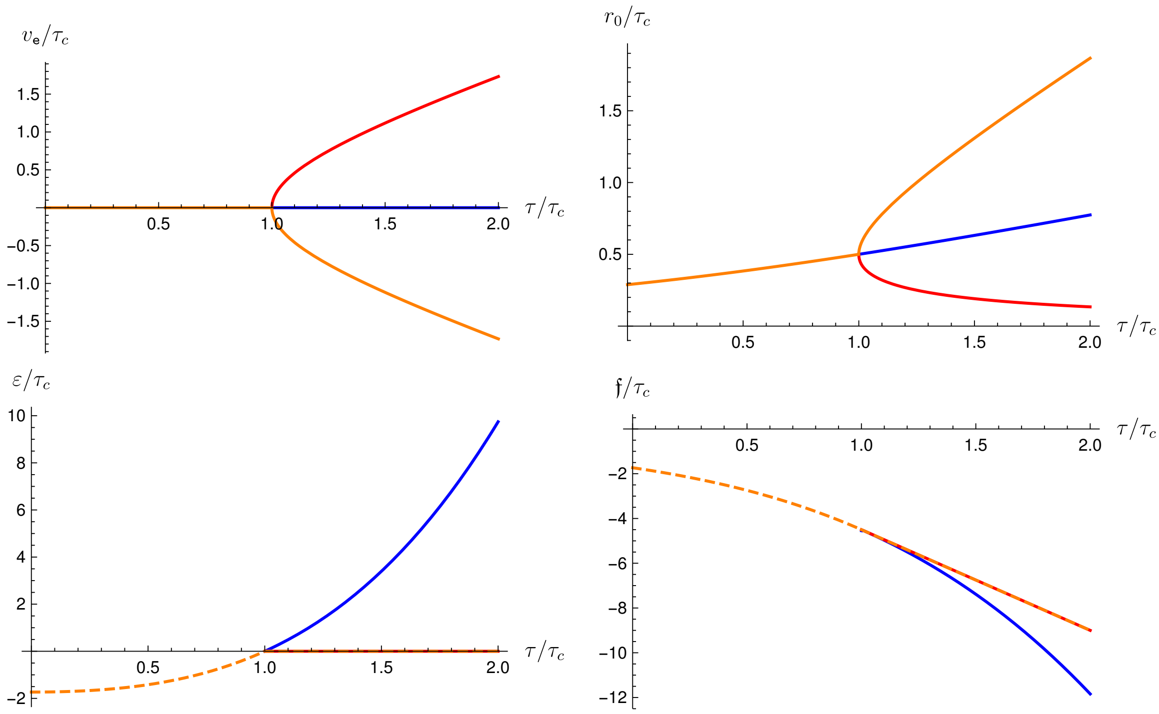


Figure 8.5.1: Plot of the perturbative solutions (8.5.7) for $\rho = 0.003\rho_c$, together with the corresponding energy ε and Helmholtz free energy $\mathfrak{f}$ densities.

these quantities for the three solutions for increasing–but small–charge densities and figure 8.5.5 shows a plot for large ρ. Note that the plots in 8.5.1 and 8.5.1 correspond to the same value of ρ. Studying these plots, we observe that several features of these solutions persist at higher charge densities, but, there are also qualitative changes as the charge density is increased. In particular, studying these plots, we observe that below the critical charge density

$$|\rho| < \rho_c = \frac{\ell^3}{2\sqrt{2}}\,\Pi^2 = \frac{p^2\ell}{2\sqrt{2}}, \tag{8.5.9}$$

there always exists one solution for all temperatures (orange), while two additional solutions appear above a (charge density dependent) critical temperature that equals τ_c at zero charge density. For $|\rho| > \rho_c$ the orange solution disappears, leaving only the other two branches above the critical temperature. There are no solutions above the critical charge density and below the critical temperature. This is depicted in the left plot in figure 8.5.6. The right plot in figure 8.5.6 shows the number of solutions as a function of temperature and electric chemical potential, instead of charge density. Notice that there is no critical value for the chemical potential, which reflects the fact that $|q_e(\mu, p, T)|$ for the orange solution is bounded by ρ_c.

The dynamic and thermodynamic stability properties of the solutions can be read off respectively the energy density and Helmholtz free energy density plots in figures 8.5.1, 8.5.3, 8.5.4 and 8.5.5. From the energy density plots we deduce that the orange solution is always dynamically unstable, while the blue and red solutions are always dynamically stable. Above T_c, however, in the limit of vanishing charge density the red solution becomes marginally stable while the orange one becomes marginally unstable. Moreover, from the free energy plots follows that when they coexist, the blue solution is thermodynamically stable, while both the red and the orange solutions are thermodynamically unstable. Nevertheless, the orange solution has the largest radius, while the red solution has the smallest radius. We will see below that this last property is reversed in the magnetically charged solutions.

Putting everything together, we conclude that the electric solutions describe in general two distinct phases, as shown in figure 8.5.7. Phase I corresponds to the orange solutions and is the only possible phase below the critical temperature and critical charge density. Phase II corresponds to the blue solutions and it dominates above the critical temperature, for any value of the charge density. There is no regime of parameters where the red solutions are thermodynamically dominant. At non zero charge density the Helmholtz free energy jumps at the critical temperature and so this is a *zeroth* order phase transition. Zeroth order phase transitions have been predicted in the context of superfluidity and superconductivity and are related to the presence of metastable states [110, 111], as well as in higher dimensional black holes [112]. As the charge density approaches zero, however, the jump of the free energy across the critical temperature goes to zero, but at the same time its derivative is continuous across T_c and, hence, the transition becomes second order. However, since the solutions of phase I are dynamically unstable, this phase diagram is presumably not the complete picture. There are probably other solutions that are thermodynamically and dynamically stable below the critical temperature, that also continue to exist above the critical charge density. It would be interesting to identify these solutions. The general structure of the physical solutions that emerges from the plots in figures 8.5.1, 8.5.3, 8.5.4 and 8.5.5 is as follows. Below the critical charge density

$$|\rho| < \rho_c = \frac{\ell^3}{2\sqrt{2}}\,\Pi^2 = \frac{p^2\ell}{2\sqrt{2}}, \tag{8.5.10}$$

there always exists one solution for all temperatures (orange), while two additional solutions appear above a (charge density dependent) critical temperature that equals T_c at zero charge density. For $|\rho| > \rho_c$ the orange solution disappears, leaving only the other two branches above the critical temperature. There are no solutions above the critical charge density and below the critical temperature. This is depicted in the left plot in figure 8.5.6. The right plot in figure 8.5.6 shows the number of solutions as a function of temperature and electric chemical potential, instead of charge density. Notice that there is no critical value for the chemical potential, which reflects the fact that $|q_{\mathsf{e}}(\mu, p, T)|$ for the orange solution is bounded by ρ_c.

The dynamic and thermodynamic stability properties of the solutions can be read off respectively the energy density and Helmholtz free energy density plots in figures 8.5.1, 8.5.3, 8.5.4 and 8.5.5. From the energy density plots we deduce that the orange solution

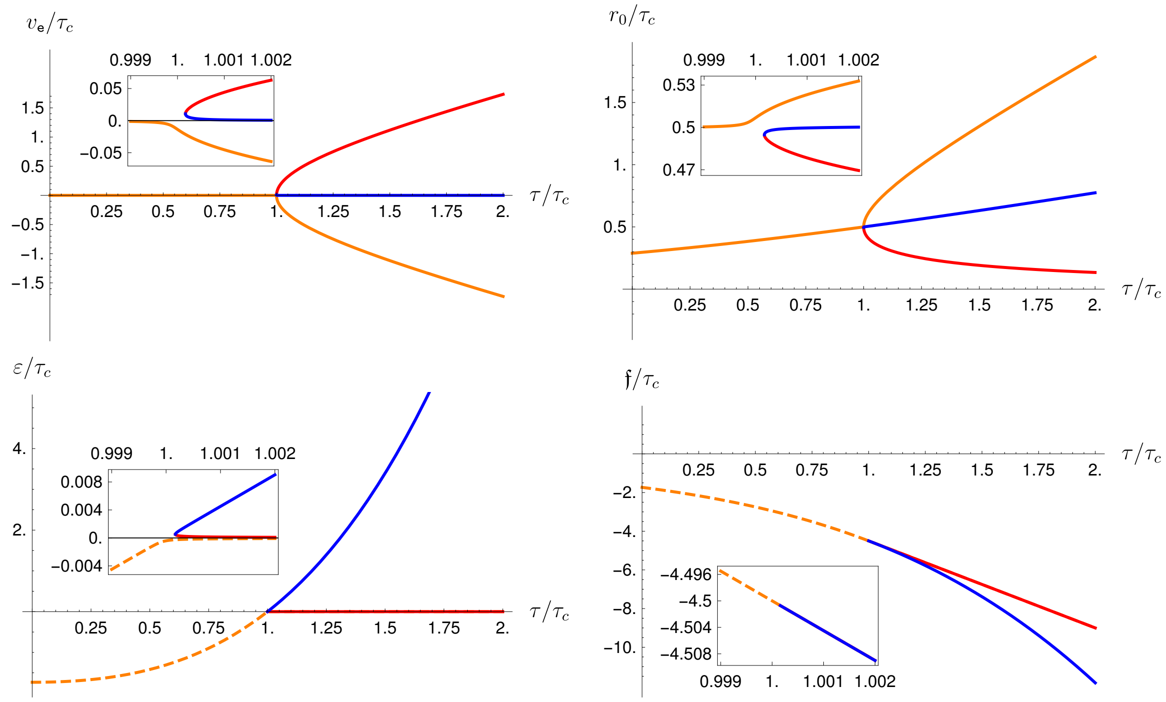


Figure 8.5.2: Plot of the solutions of (8.5.6) for $\rho = 0.003\rho_c$, together with the corresponding energy ε and Helmholtz free energy $\mathfrak{f}$ densities. Notice that the solutions look identical to the perturbative ones in figure 8.5.2 for the same charge density, but zooming in near the critical temperature shows that the exact solutions are in fact smooth.

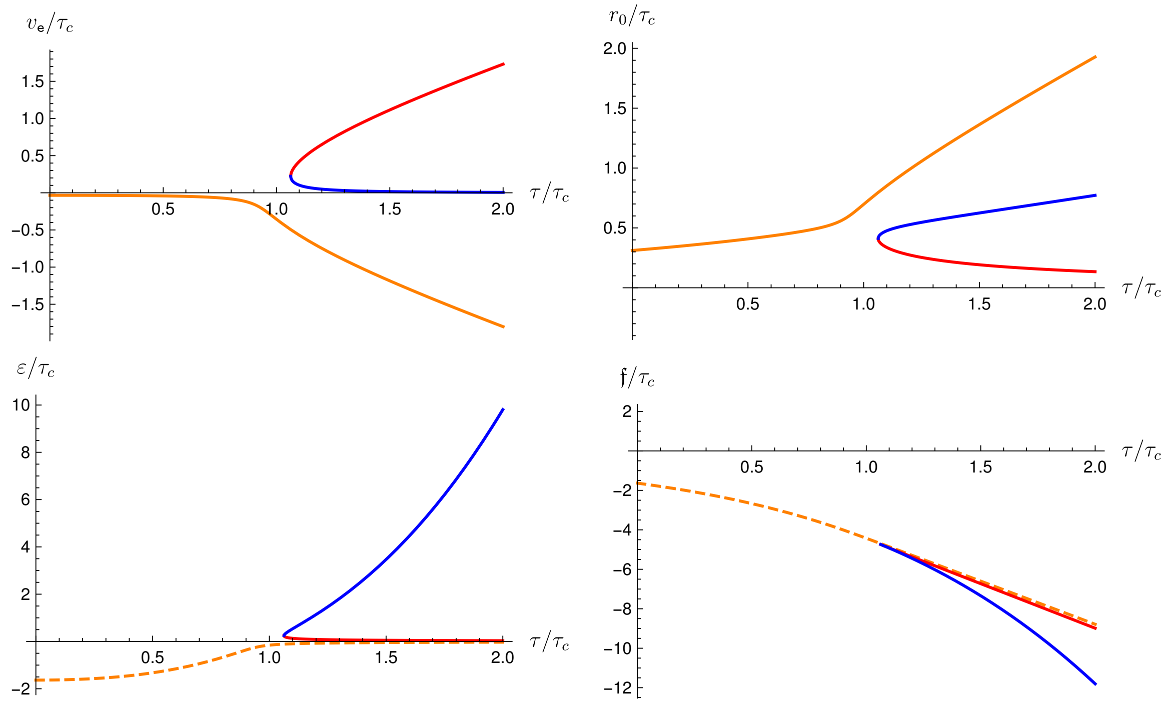


Figure 8.5.3: Plot of the solutions of (8.5.14) for $\rho = 0.314\rho_c$, together with the corresponding energy ε and Helmholtz free energy $\mathfrak{f}$ densities.

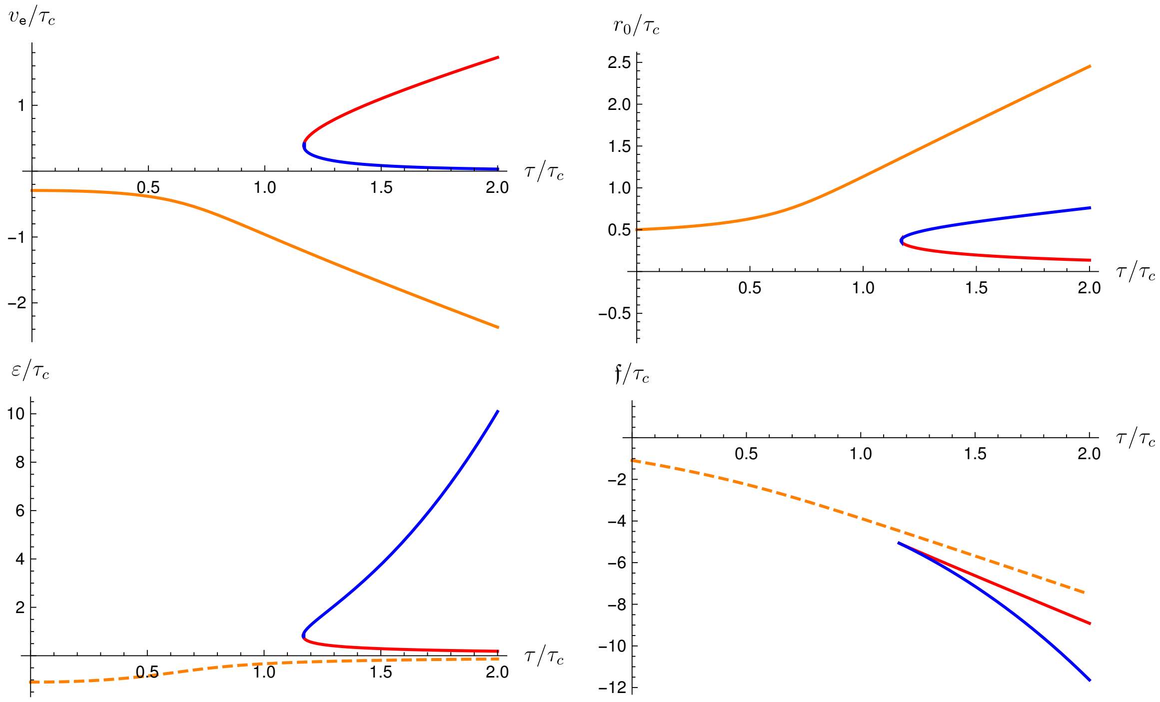


Figure 8.5.4: Plot of the solutions of (8.5.14) for $\rho = 0.786\rho_c$, together with the corresponding energy ε and Helmholtz free energy $\mathfrak{f}$ densities.

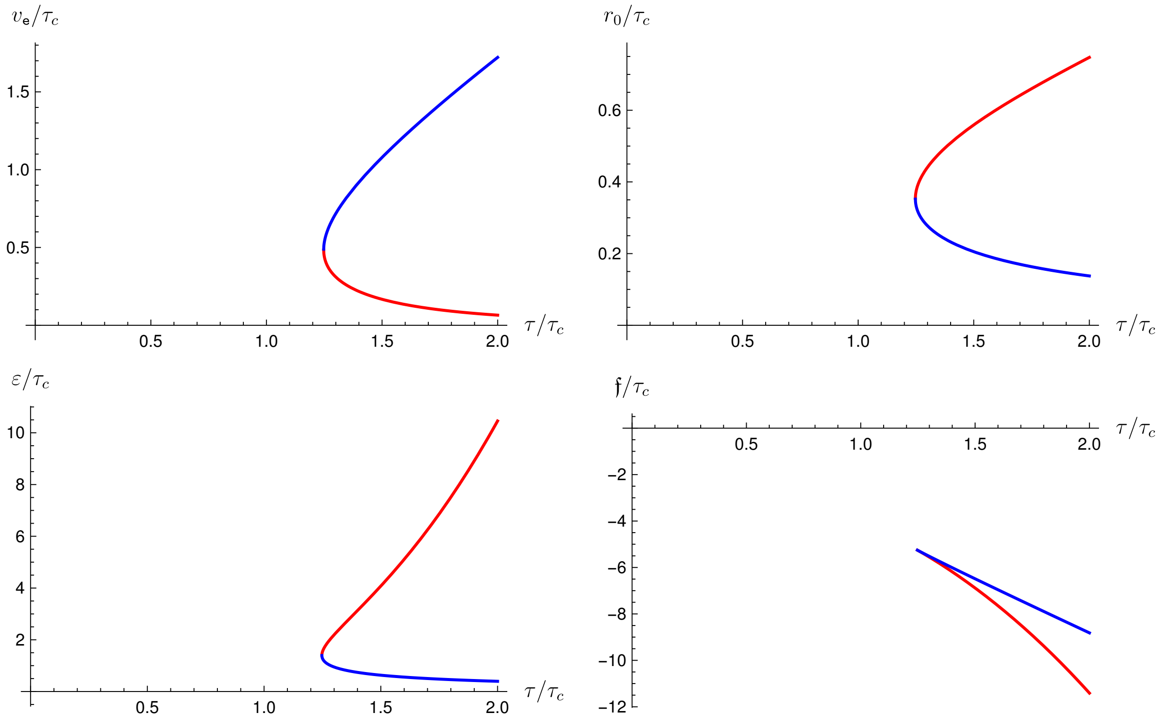


Figure 8.5.5: Plot of the solutions of (8.5.14) for $\rho = 1.1\rho_c$, together with the corresponding energy ε and Helmholtz free energy $\mathfrak{f}$ densities.

is always dynamically unstable, while the blue and red solutions are always dynamically stable. Above T_c, however, in the limit of vanishing charge density the red solution becomes marginally stable while the orange one becomes marginally unstable. Moreover, from the free energy plots follows that when they coexist, the blue solution is thermodynamically stable, while both the red and the orange solutions are thermodynamically unstable. Nevertheless, the orange solution has the largest radius, while the red solution has the smallest radius. We will see below that this last property is reversed in the magnetically charged solutions.

Putting everything together, we conclude that the electric solutions of theory II describe in general two distinct phases, as shown in figure 8.5.7. Phase I corresponds to the orange solutions and is the only possible phase below the critical temperature and critical charge density. Phase II corresponds to the blue solutions and it dominates above the critical temperature, for any value of the charge density. There is no regime of parameters where the red solutions are thermodynamically dominant. At non zero charge density the Helmholtz free energy jumps at the critical temperature and so this is a *zeroth* order phase transition. Zeroth order phase transitions have been predicted in the context of superfluidity and superconductivity and are related to the presence of metastable states [110, 111], as well as in higher dimensional black holes [112]. As the charge density approaches zero, however, the jump of the free energy across the critical temperature goes to zero, but at the same time its derivative is continuous across T_c and, hence, the transition becomes second order. However, since the solutions of phase I are dynamically unstable, this phase diagram is presumably not the complete picture. There are probably other solutions that are thermodynamically and dynamically stable below the critical temperature, that also continue to exist above the critical charge density. It would be interesting to identify these solutions.

8.5.2 Phases of Magnetic Solutions

The structure of the magnetic solutions is very similar to that of the electric ones, except for a few minor features that we are going to highlight. As for the electric solutions, we can process the temperature in two different ways leading to

$$\tau = 3r_0 + 2v_{\mathsf{m}} - \frac{\ell^2 p^2}{2r_0}, \tag{8.5.11}$$

and

$$\tau = 2v_{\mathsf{m}} - \frac{3\ell^2 q_{\mathsf{m}}^2 + \ell^2 p^2 v_{\mathsf{m}}(v_{\mathsf{m}} - 2r_0)}{2r_0 v_{\mathsf{m}}(v_{\mathsf{m}} + r_0)}, \tag{8.5.12}$$

for the rescaled temperature $\tau = 4\pi\ell^2 T$. At generic values of the parameters these again correspond to two quadratic equations for r_0. Eliminating the quadratic term in r_0 by a suitable linear combination of these expressions, we obtain the general expression for the horizon radius

$$r_0 = \frac{9\ell^2 q_{\mathsf{m}}^2 + \ell^2 p^2 v_{\mathsf{m}}(v_{\mathsf{m}} + \tau)}{2v_{\mathsf{m}}\left(3\ell^2 p^2 + (2v_{\mathsf{m}} - \tau)(v_{\mathsf{m}} + \tau)\right)}. \tag{8.5.13}$$

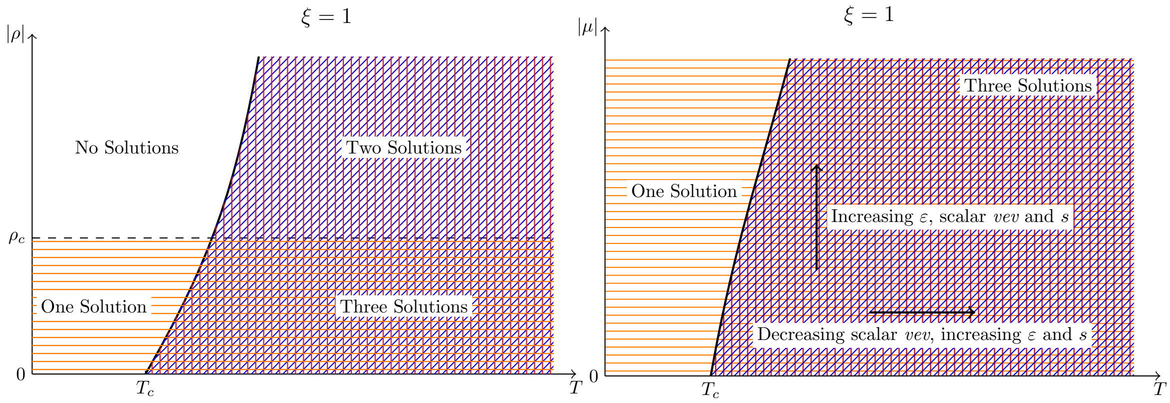


Figure 8.5.6: Number of electric solutions with $\xi = 1$ as a function of charge density ρ and temperature T (**left plot**), or chemical potential μ and temperature T (**right plot**). The plots apply to any fixed $|p| > 0$. There are three distinct solutions, indicated respectively by horizontal orange lines, red vertical lines, and blue diagonal lines. We refer to these solutions as 'orange', 'red' and 'blue', respectively. The orange solution exists for all temperatures provided the (absolute value of the) charge density is below the critical value $\rho_c = \ell p^2/2\sqrt{2}$. However, this solution is dynamically unstable since it always has negative energy density. Moreover, where it coexists with the red and blue solutions it has the largest radius, but it is thermodynamically unstable. The red and blue solutions appear simultaneously above a critical temperature $T_c \geq |p|/2\sqrt{2}\pi\ell$ and are always dynamically stable, since they have positive energy density. The blue solutions have larger radius than the red ones and are thermodynamically preferred.

Inserting this back in either (8.5.11) or (8.5.12) gives the characteristic curve

$$(\ell^2 p^4 - 8q_{\mathsf{m}}^2)v_{\mathsf{m}}^4 + (2\ell^4 p^6 - \ell^2 p^4 \tau^2 - 18\ell^2 p^2 q_{\mathsf{m}}^2 + 6q_{\mathsf{m}}^2\tau^2)v_{\mathsf{m}}^2 - 2q_{\mathsf{m}}^2\tau^3 v_{\mathsf{m}} - 27\ell^2 q_{\mathsf{m}}^4 = 0. \quad (8.5.14)$$

Notice that although the expression for the horizon (8.5.13) is different from the corresponding expression for the electric solutions in (8.5.5), the characteristic curves are identical under the map $q_{\mathsf{e}} \to q_{\mathsf{m}}$, $v_{\mathsf{e}} \to -v_{\mathsf{m}}$. It follows that the solutions for the scalar *vev* are identical to those for the electric solutions, except for an overall sign change.

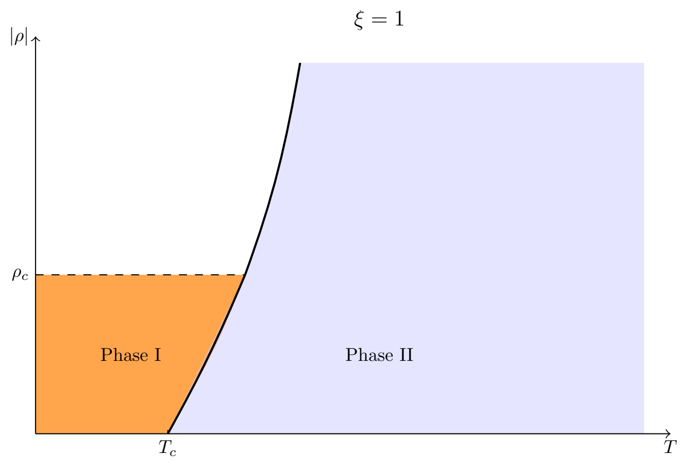


Figure 8.5.7: Phase diagram for the electric solutions with $\xi = 1$ as a function of charge density ρ and temperature T. Below a critical charge density ρ_c and a critical temperature T_c there is only one, dynamically unstable, black hole solution. This solution exists for arbitrary temperature, but it disappears above ρ_c. Above the critical temperature T_c there are either two or three solutions depending on whether $|\rho| > \rho_c$ or $|\rho| < \rho_c$ respectively. The two solutions that exist only above T_c are both dynamically stable and the one with larger radius is thermodynamically preferred. At the critical temperature T_c, therefore, there is a phase transition from a dynamically stable black hole with negative scalar *vev* above T_c, to a dynamically unstable solution with positive scalar *vev*. At non-zero charge density the Helmholtz free energy jumps at T_c and so this is a *zeroth* order transition. At zero charge density the free energy is continuous across T_c, as is its first derivative, and so the phase transition becomes second order at $\rho = 0$.

In particular, for small q_m the perturbative solutions take the form

$$\text{Red:} \quad v_\mathsf{m} = -\sqrt{\tau^2 - \tau_c^2} + \mathcal{O}(q_\mathsf{m}^2), \qquad r_0 = \frac{1}{2}\left(\tau + \sqrt{\tau^2 - \tau_c^2}\right) + \mathcal{O}(q_\mathsf{m}^2), \qquad \tau > \tau_c, \tag{8.5.15a}$$

$$\text{Blue:} \quad v_\mathsf{m} = \frac{-q_\mathsf{m}^2 \ell^2}{r_0(2r_0^2 - p^2\ell^2)} + \mathcal{O}(q_\mathsf{m}^4), \quad r_0 = \frac{1}{6}\left(\tau + \sqrt{\tau^2 + 3\tau_c^2}\right) + \mathcal{O}(q_\mathsf{m}^2), \qquad \tau > \tau_c, \tag{8.5.15b}$$

$$\text{Orange:} \quad v_\mathsf{m} = \begin{cases} \frac{-q_\mathsf{m}^2 \ell^2}{r_0(2r_0^2 - p^2\ell^2)} + \mathcal{O}(q_\mathsf{m}^4), \\ \sqrt{\tau^2 - \tau_c^2} + \mathcal{O}(q_\mathsf{m}^2), \end{cases} \qquad r_0 = \begin{cases} \frac{1}{6}\left(\tau + \sqrt{\tau^2 + 3\tau_c^2}\right) + \mathcal{O}(q_\mathsf{m}^2), & \tau < \tau_c, \\ \frac{1}{2}\left(\tau - \sqrt{\tau^2 - \tau_c^2}\right) + \mathcal{O}(q_\mathsf{m}^2), & \tau > \tau_c, \end{cases} \tag{8.5.15c}$$

where again $\tau_c = 4\pi\ell^2 T_c = \sqrt{2}p\ell$. As for the electric solutions, perturbation theory breaks down near τ_c, as can be seen from the plots in figure 8.5.8. Comparing with the corresponding plots for the electric solutions in figure 8.5.1 we see that at small charge densities the electric and magnetic solutions look identical, except that the orange and red branches of the solution are switched above the critical temperature. At higher

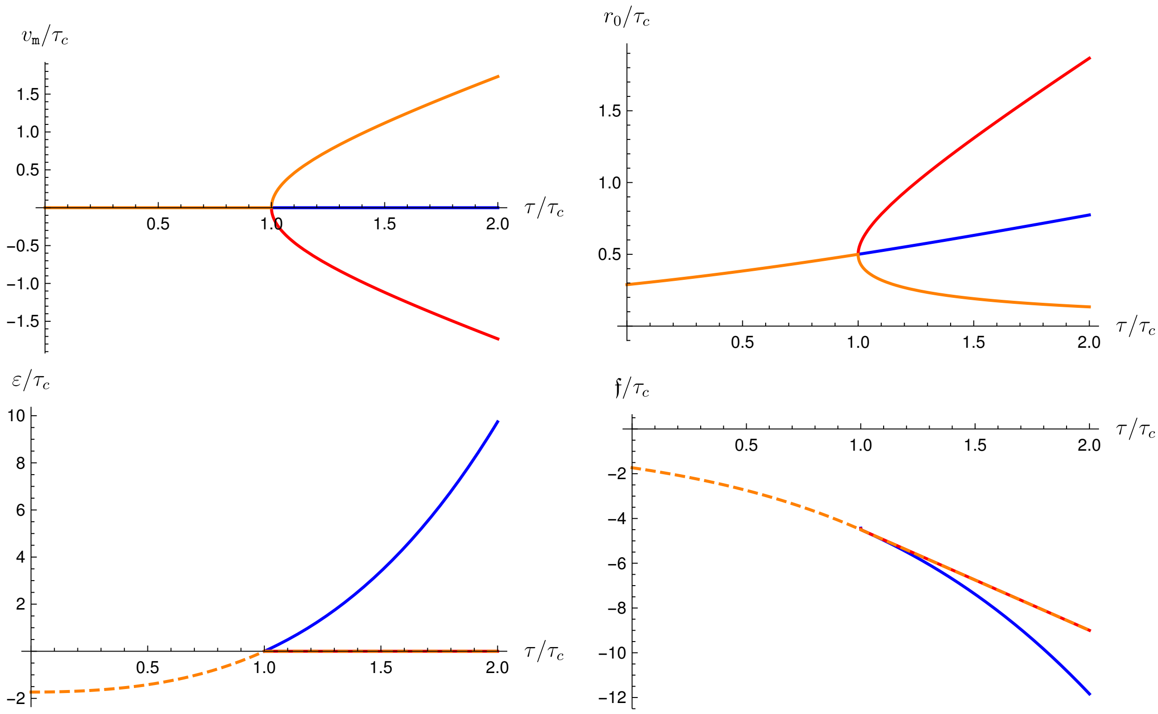


Figure 8.5.8: Plot of the perturbative solutions (8.5.15) for $\mathcal{B} = 0.003\mathcal{B}_c$, together with the corresponding energy ε and Helmholtz free energy $\mathfrak{f}$ densities.

charge densities the corners are again smoothened out as is shown in figures 8.5.9, 8.5.10 and 8.5.11.

As for the electric solutions, the orange branch exists for all temperatures but disappears above the critical magnetic field

$$|\mathcal{B}| < \mathcal{B}_c = \frac{\Pi^2}{2\sqrt{2}} = \frac{p^2}{2\sqrt{2}\ell^2}. \tag{8.5.16}$$

Above the critical temperature again there are two additional branches for any value of the magnetic field, while above the critical magnetic field and below the critical temperature there are no solutions. The number of solutions as a function of the temperature and the magnetic charge density matches the number of electric solutions as a function of of the temperature and the electric charge density depicted in figure 8.5.6. Moreover, the dynamic and thermodynamic stability properties of the three branches of solutions are identical to those of the electric solutions and hence the phase diagram in figure 8.5.7 applies equally well to the magnetic solutions upon replacing the charge density ρ with the magnetic field $\mathcal{B}$. As can be seen from the plots of the solutions, the only difference between the electric and magnetic ones is that the value of the scalar *vev* is opposite, while above the critical temperature the red branch of the electric solutions has the smallest radius and the orange has the largest, while the reverse holds for the magnetic solutions.

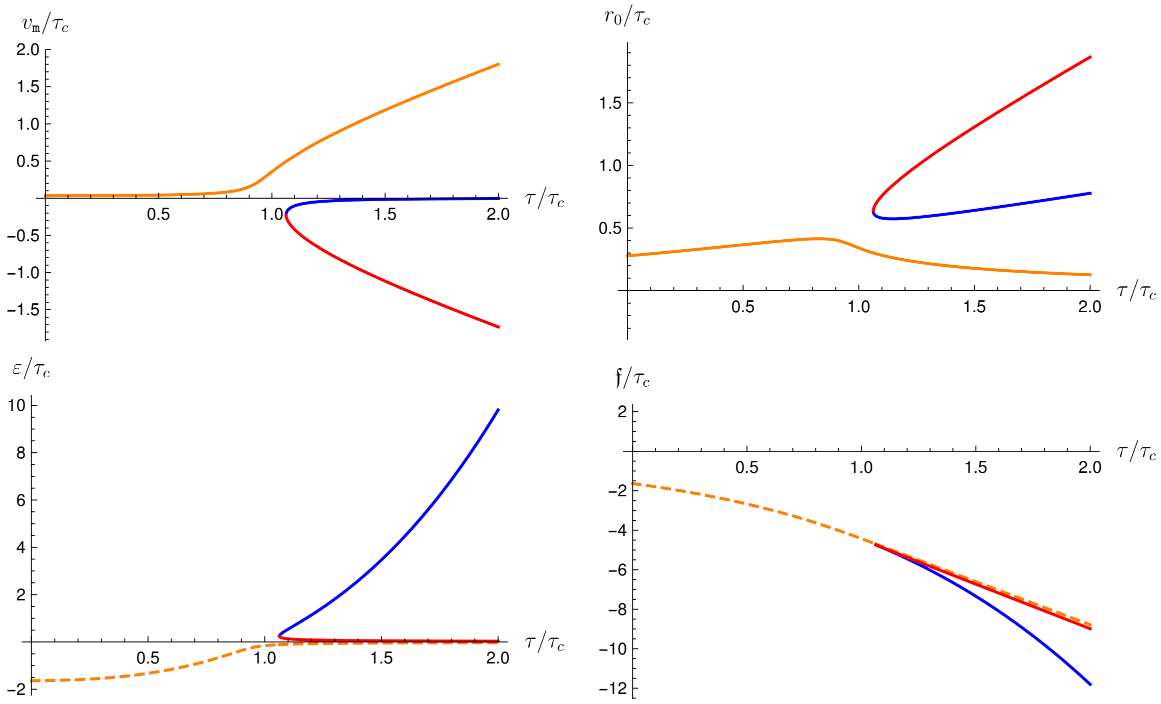


Figure 8.5.9: Plot of the solutions of (8.5.14) for $\mathcal{B} = 0.314\mathcal{B}_c$, together with the corresponding energy ε and Helmholtz free energy $\mathfrak{f}$ densities.

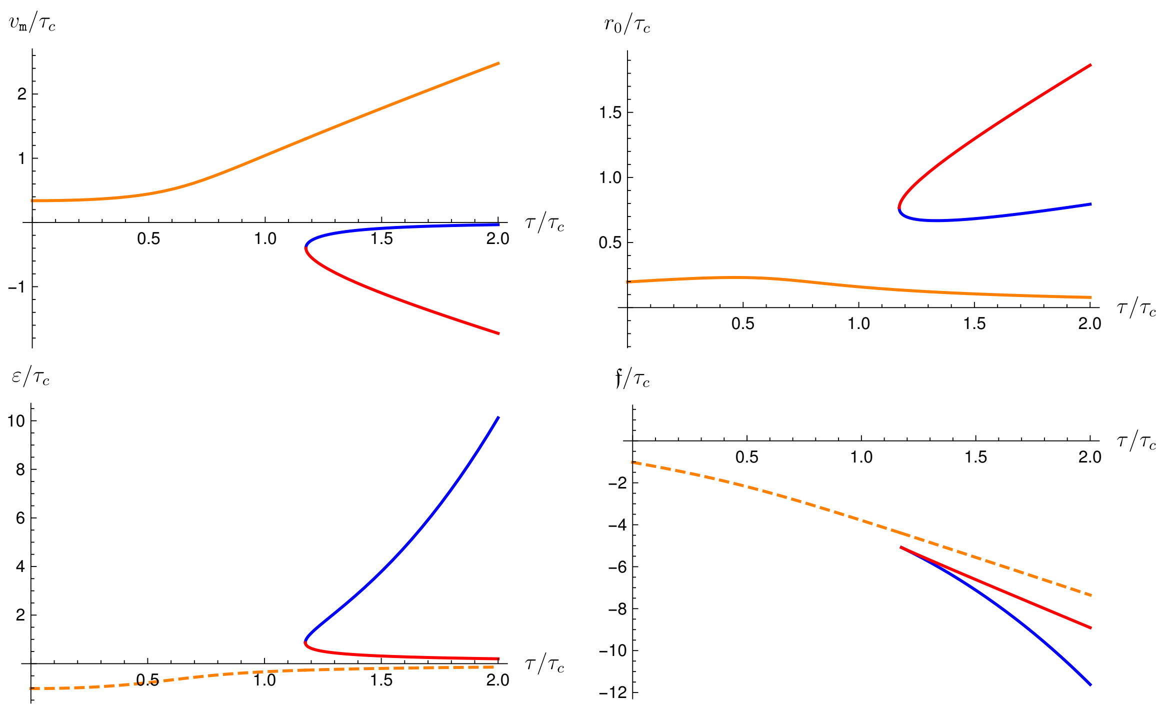


Figure 8.5.10: Plot of the solutions of (8.5.14) for $\mathcal{B} = 0.786\mathcal{B}_c$, together with the corresponding energy ε and Helmholtz free energy $\mathfrak{f}$ densities.

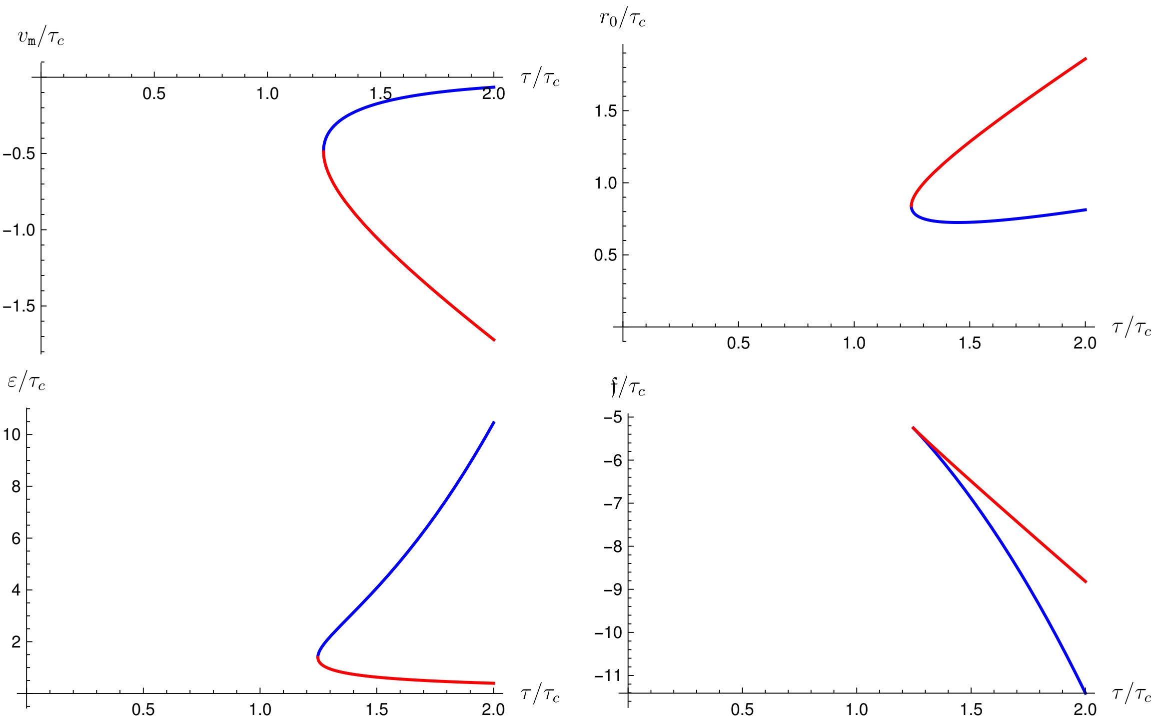


Figure 8.5.11: Plot of the solutions of (8.5.14) for $\mathcal{B} = 1.1\mathcal{B}_c$, together with the corresponding energy ε and Helmholtz free energy $\mathfrak{f}$ densities.

Chapter 9

Discussion

The second part of this thesis was dedicated to the study of a family of theories which admit electrically and/or magnetically charged black brane solutions with additional axionic charges and a running scalar that we called the dialton.

We focused on particular solutions for which the axions admit a linear profile in the boundary directions and, consequently, they explicitly break scale invariance. Moreover, they act as vacuum energy for the horizon, giving rise to topological black holes with a flat horizon but which also possess the additional length scale usually associated with the radius of hyperbolic black holes. In the context of black hole charges, the axions constitute primary hair [83]. In particular, they are zero forms with the operators dual to them satisfying global Ward identities, and they carry topological charges. These charges enter the thermodynamic relations and they *modify the first law*.

The other scalar field supported by the black branes we studied, namely the dialton, constitutes secondary hair of the branes. As such, contrary to the axions, it does not enter the thermodynamic relations as an independent charge. However, for the solutions we studied, the dialton satisfies mixed boundary conditions and, consequently, it modifies the bulk on–shell action that one would obtain had the more conventional Dirichlet boundary conditions been imposed [82]. This modification leads, in turn, to a corresponding modification of the holographic stress tensor and of the associated conserved charges and free energy [30]. In this part of the thesis we performed the full asymptotic analysis and derived the holographic dictionary in the presence of such mixed boundary conditions for the running scalar. In doing so we demonstrated that, when the modifications associated with the mixed boundary conditions are correctly accounted for, the theories satisfy the standard thermodynamic relations and, in particular, the first law, without having to add new charges associated with the scalar.

The holographic dictionary and and thermodynamic relations were first derived for a general class of solutions for the family of theories under investigation, subject to mixed boundary conditions for the dialton and axions with linear profiles along the boundary directions but otherwise unrestricted. In chapter 7 we revisited the exact black holes found in [87], which do not have a running profile for the dialton, as well as those obtained in [88] which have a running dialton. We use the general results obtained to derive the thermodynamic properties of the dyonic version of these black holes and verify that they satisfy the expected relations. A natural next step would have been to study the thermodynamic stability and dynamic stability of these solutions which do compete. However, this had already been done in [87, 88] and was therefore not repeated.

In chapter 8 we applied our general results to another family of known analytic solutions, found in [89], as well as a their magnetic analogue which first appeared in [2]. Using the derived thermodynamic relations we studied the thermodynamic stability and phase structure of a subfamily of these solution for which the coupling for the multitrace deformation vanishes and the dialton satisfies Neumann boundary conditions. The reason for focusing on these solutions is the fact that in this case the analysis of the phase structure can be done analytically. Furthermore, for this theory, the dialton and the gauge field are coupled, meaning that bald solutions are necessarily electrically and magnetically neutral. Consequently they do not compete with the hairy solutions at non zero charge density. Nevertheless, for a given non zero charge density and temperature, there are up to three hairy solutions with different horizon radii and scalar *vevs* that compete thermodynamically, giving rise to an intricate phase structure.

Firstly, we observed that at non–zero charge density there exist a critical temperature, that depends on the charge density, above which there are three hairy black holes, two with positive energy density–a large and a small black hole–and one with negative. This result is identical for both the electrically charged black holes as well as their magnetic analogue, with the only difference being that for the electric solutions the black hole with negative energy density is the largest of the three, while for magnetically charged solutions it is the smallest. As was demonstrated in [2], where the authors derived the quantum effective potential for the dialton, dynamical stability of the solutions against quantum fluctuations of the dialton is equivalent to having positive energy density. Hence, the negative energy density solution is not dynamically stable. Returning to the phase diagram, lowering the temperature such that T_c is approached from above, the two positive energy black holes converge and cease to exist below T_c. However, at non-zero charge density the negative energy solution at the critical temperature has a lower energy density than the other two solutions, which are therefore metastable. Moreover, the larger of the two black holes with positive energy density has the smallest free energy and is therefore thermodynamically favored above T_c. Accordingly, the free energy is discontinuous at T_c, leading to a zeroth order phase transition. As the charge density is tuned to zero the negative and positive energy solutions all converge as T_c is approached from above, with their energy density approaching zero from below and above respectively, which leads to a regular second order phase transition at T_c.

Putting everything together, we found that both the electric and magnetic solutions of this theory describe in general two distinct phases. Below a charge density dependent

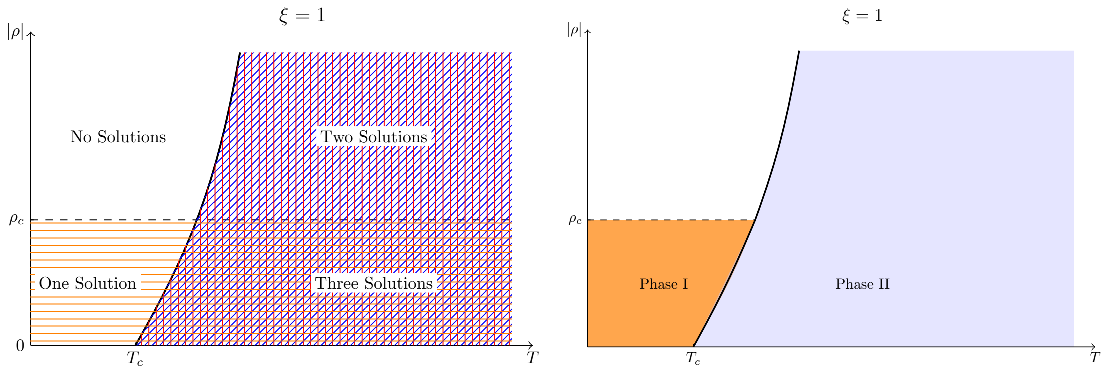


Figure 9.0.1: Number of solutions with $\xi = 1$ as a function of charge density ρ and temperature T (left) and corresponding phase diagram (right). These figures are the same for both the electric and magnetic solutions of the theory for $\xi = 1$.

critical temperature and a critical charge density there is only one black hole. This corresponds to phase I which has negative energy density, indicating that this phase is dynamically unstable with regards to dialton fluctuations. However, this phase has positive specific heat in this region of the phase space. Above the critical temperature and below the critical charge density there are three black holes. As the charge density increases, the negative energy density black hole of phase I disappears and we are left with only two solutions. In this region of the phase space we found that there are no phase transitions and one solution always dominates. This solution has positive energy density and specific heat. At non zero charge density the Helmholtz free energy jumps at the critical temperature and so this is a *zeroth* order phase transition. Zeroth order phase transitions have been predicted in the context of superfluidity and superconductivity and are related to the presence of metastable states [110, 111], as well as in higher dimensional black holes [112]. As the charge density approaches zero, however, the jump of the free energy across the critical temperature goes to zero, but at the same time its derivative is continuous across T_c and, hence, the transition becomes second order. Figure 9.0.1 show the number of solutions and phase space of the theory with $\xi = 1$ as a function of the charge density ρ and temperature T. These figures apply for both the electic and magnetic solutions.

A number of open questions and future directions remain. Firstly, since the solutions of phase I have negative energy density, we conclude that they are dynamically unstable and hence this phase diagram is presumably not the complete picture. However, we also noted that these solutions have positive heat capacity, something that poses an interesting puzzle. Most likely, there are other solutions that are thermodynamically and dynamically stable below the critical temperature, that also continue to exist above the critical charge density. It would be interesting to identify these solutions.

Secondly, another potential next step is to seek exact dyonic solutions of this theory. However, it seems likely that such solutions can only exist for the special case $\xi = 1$, since only in that case the scalar boundary conditions for the electric and magnetic solutions coincide. Moreover, we found that the relative size of the radii of two of the three solutions–when these coexist–are opposite for the electric and magnetic solutions, suggesting that, for a dyonic black brane solution, there will only exist one solution in the corresponding region of the phase space. Below the critical temperature, however, the single, negative energy solution should continue to exist even for a dyonic black brane. For $p = 0$ dyonic solutions of Theory II have been found in [113]. It would be interesting to generalise these solutions to non-zero axion charge.

Finally, we have focused only on properties of the background solutions, without discussing fluctuations around them. It would be very interesting, for example, to compute the thermoelectric and Hall conductivities for these black branes.

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