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"NOT TO LOSE THE CHAIN IN LEARNING MATHEMATICS": EXPERT TEACHING WITH VARIATION IN SHANGHAI

Liping Ding*, Keith Jones**, Lina Mei***, Svein Arne Sikko*

*Sør-Trøndelag University College, Norway; ** University of Southampton, UK;
***Shanghai Qingpu District Experimental Elementary School

This paper reports on an expert teacher's ideas and practice of teaching with variation that underpin her guidance of junior teachers in lesson design study in a research project in Shanghai, China. The data we analysed included the teacher's lesson plan, teaching references, classroom materials, together with the video of the lesson and its transcript. Using the framework of teaching with variation, we identified four types of variation: task variation, example variation, calculation method variation, and exercises variation. The findings help towards a deeper understanding of the complexity of teaching expertise valued in the Chinese mathematics classroom.

RATIONALE AND STUDY BACKGROUD

With an ongoing research focus on identifying effective ways of enhancing mathematics teaching within our teacher professional development (TPD) project in Shanghai (SH), China (see Ding et al., 2014), this paper focuses on the pedagogical ideas and practices of an expert teacher of mathematics in the local school context of Shanghai. Our rationale is that, as Li and Kaiser (2012) point out, understanding the conception and nature of teacher expertise in mathematics instruction remains quite limited. Our study aims to contribute better understanding the complexity of teaching expertise valued in the Chinese classroom context (Li, Huang, & Yang, 2011).

It has been known for some time that pedagogical approaches that limit learners to rote learning and that accentuate instrumental understanding have relatively poor long term effects, with learners not being able to apply knowledge to new situations nor in their everyday life as citizens (e.g., Skemp, 1976). To understand the complexity of teaching expertise valued in the Chinese classroom context, we focus on two aspects of the practice of an SH expert teacher of mathematics: one is the relationship between the teacher's leading teaching role and students' active learning role; the other is the relationship between conceptual and procedural knowledge in mathematics and in students' learning .

Teaching with variation (briefly called 'TwithV' in this paper) has long been widely practiced by mathematics teachers in China. Perhaps as a consequence, different notions of variation have been identified as characterizing the features of TwithV. For instance, Gu (1981) showed aspects of figural variation in teaching and learning mathematical concepts, developing independent thinking skills in problem solving and in establishing knowledge systems in geometry. Huang, Mok, and Leung (2006) identified classroom practice in SH in terms of implicit variation; that is, where the changes from the origins to their variations "have to be discerned by abstract and logical analysis by learners ...so that the conditions or strategies for applying relevant knowledge are implicit and not obvious" (p.265). Sun (2011) characterised the variation of problems in terms of "one problem multiple solution" and "one problem multiple changes" (p.65). Li, Peng, and Song (2011) identify that teaching algebra with variation involves "aspects of orientation of variation, types of variation, levels of variation, and variation exploration" (p.546).

To date, researchers have been largely engaged in tackling two key questions in studying TwithV in Chinese mathematics teaching. The first question is 'why an emphasis on variation should be made in mathematics teaching and learning'; the other question is 'how to design such variation for the effective teaching and learning of mathematics'. To the first question, Sun (2007) points out that TwithV enables students to appreciate the abstract nature of 'invariable in variation" of mathematical laws and to develop insight into the mathematical system by the idea of "applying the invariant concepts to the varied situations" (p.16, translated by the first author). The second question remains a challenge in mathematics education research and is our focus in this paper. We use the work of Gu (1981, 1994, 2014) as the theoretical framework within which we address our research question: how does an expert teacher help Grade 2 students to establish the internal relationship of new concepts and methods with previous ones by TwithV in a lesson on division with remainder?

THEORETICAL FRAMEWORK

In the '*Qingpu experiment study*' (a project led by Gu, in collaboration with a number of teachers and researchers, from 1977 to 1994 that focused on improving the effectiveness of teaching and learning of mathematics in Qingpu district, SH), Gu (1994) found that the most effective teachers were able to deliberately arrange what we might call multiple layers of teaching and learning. Here, the multiple layers refer to the *Xun Xu Jian Jin* principle of Confucius; that is, to make progress by following foundational principles such as the development of understanding from shallow to deep, the subject content from easy to difficult, the learning from simple to complicated, and the practice from single to complex tasks.

Based on this, Gu *et al*. (2004) identify and illustrate two forms of TwithV, namely *conceptual variation* (*CV*) and *procedural variation* (*PV*). Within *CV*, there are two means of variations: (1) concept variation (e.g., varying connotation of a concept); (2) non-concept variation (e.g., giving counterexamples). Thus, *CV* emphasizes understanding concepts from multiple perspectives. In contrast, *PV* highlights a hierarchical system in unfolding mathematics activities (e.g., different steps to arrive at a solution or different strategies to solve problems). In this paper we aim to develop a deeper insight into the use of *PV* by an SH expert teacher.

Here we note that in his most recent writing, Gu (2014) further explains that it is *PV* that plays a key role as *Pu Dian* (铺垫); that is, in setting up a proper distance between previous and new knowledge in students' learning. Akin to the notion of 'scaffolding', *Pu Dian* means to build up one or several layers so as to enable learners to complete tasks that they cannot complete independently. In this paper we aim, in particular, to develop a deeper understanding of how *PV* creates the 'proper distance' for all students in learning in the Chinese classroom context.

METHOD

Our lesson design study is being conducted through a school-based TPD in a local laboratory school located in the western suburb of SH. The overall approach is a form of the Action Education (AE) model developed by Gu and Wang (2003) (for more on our use of the AE model, see Ding et al., 2014). During the process of supporting one of the case teachers to redesign and re-implement her lesson plan according to TwithV (for the details of our study cycle see Ding et al., 2014), we noted that in her teaching, expert teacher Mei constantly addressed the idea "not to lose the chain in mathematics learning". As Mei gave an open lesson on the same mathematics topic and was video-recorded for both her school and her school district key junior teachers (those teachers considered as potentially effective young teachers by their schools) as part of the school-based TPD activity in 2010, it became interesting and possible for us to examine Mei's idea in her own teaching practice. The term 'expert teacher' in our study recognizes that Mei is not only an effective teacher in subject teaching, but that she also plays the multiple roles that are described by Yang (2014, p.271-2).

The data we present in this paper includes Mei's own lesson plan, teaching references and learning materials of the lesson (e.g., the textbook, worksheet), and the lesson video and transcript. Mei has over 30 years teaching experiences in elementary mathematics teaching in her school district. She has taken the leadership of the in-service elementary mathematics teachers TPD program at her school district level since 2009. Her school is a public school, and the school size in 2010 was about 2500 students (from grade 1 to grade 9), 56 classes and about 200 teachers (all subjects).

The class in this lesson was Grade 2 (students age 7-8 years old). The length of the lesson was 35 minutes. There were 44 students in the class. 25 of them were boys and the rest were girls. According to the school's regular learning assessment in mathematics, more than 80% of the students in this class were excellent at mathematics, 15% of them were good at mathematics and the last 5% also passed in all school tests. This means that there was not a student who was really weak in mathematics in this class. The lesson topic was division with remainder, which has remained one of the key and the most difficult topics in the SH reformed elementary mathematics curriculum.

The data was analysed through three main stages: (1) Mei's lesson plan, the textbook and teaching references and the video transcripts were carefully studied and key codes for analysing the lesson were developed as follows: the lesson structure (e.g., introduction activity, the main activity and exercise activity), the learning goals of the lesson (e.g., the key points and the difficult points of the lesson), teaching strategies (e.g., questioning, using concrete materials such as drawing or pictures, hands on experiments, and use of multiplication table), classroom interactions (e.g., teacher-whole class, teacher-individual, students in pairs). (2) We also developed codes to analyse the teaching tasks (e.g. solving problems, division operational procedure) and teaching strategies of the TwithV largely according to Gu et al. (2004). (3) We also referred to what Mei analysed of her lesson according to her instructional intention of TwithV that Gu has not yet sufficiently explained (e.g., the variation of tasks to tackle the individual differences in the class).

DATA ANALYSIS

We analysed Mei's TwithV in the observed lesson according to two key points of learning goals in Mei's lesson plan (see the left column in Table 1). Noticeably, Mei considered that the difficult point of students' learning was to correctly use the method of 'trying quotient' by the multiplication table (briefly called MT in this paper) in the operation of division with remainder. In the first place, we use Table 1 to outline the main lesson structure that focused on the two key points and the difficult point of learning. Then we focus on analysing of Mei's TwithV in relation to these points.

1. The observed lesson

Lesson structure & learning goals	Key teaching tasks	Examples of the task outcome
1. Learning goal in teaching: To know the new concept of "division with remainder" and to develop an understanding of the fact that 'a remainder is always smaller than a divisor'.	*Task 1*. A problem of sharing 12 peaches by 3 monkeys. *Task 2*. (1) Sharing 13 peaches by 3 monkeys. (2) Sharing 14 peaches by 3 monkeys. (3) Sharing 15 peaches by 3 monkeys. *Task 3*. (1) Sharing 17 strawberries by 4 friends. (2) Sharing 17 strawberries by 6 friends.	*Task 1*. 12÷3=4 *Task 2*. (1) 13÷3=4……1 (2) 14÷3=4……2 (3) 15÷3=5 or 15÷3=4……3? *Task 3*. (1) 17÷4=3……5 (2) 17÷6=2……5
2. Learning goal: To learn to correctly calculate the division with remainder when both divisor and remainder are one digital.	Task 4. Sharing 11 oranges by 4 friends, with the help of a picture. Task 5. Representing thinking method of how to operate '11÷3=?' (to use students' hands to respectively represent the quotient and remainder).	Task 4. 11÷4=2……3 Task 5. 11÷3=3……2 Task 6. 11÷5=2……1 Task 7. 11÷6=1……5

	Task 6. Exchanging ideas and representation with neighbour student of the operation of '11÷5=?'. Task 7. Without a picture, explaining the method of the operation of '11÷6=?'.	
3. Learning goal: To correctly use the method of 'trying quotient'.	*Task 8.* 31÷5=( ) …… ( ). To think: 31－ _ =_.	31÷5=(6) …… (1). To think: 31－30 =1.

Table 1: The lesson structure, learning goals and key teaching tasks

2. Teaching with variation

(1) The introduction of the concept of division with remainder

Teaching episode one: Task variation to generate conflict and interest in learning new concept.

In *Task 1* (see Table 1), we note that students already learned to use the division method to solve a problem in a situation involving the concepts such as 'equal' and 'sharing'. They were also able to correctly use the MT to get the quotient 4 in the division; that is, the previously learned procedural operation of division in relation with the MT, together with the concepts like 'equal', 'sharing', 'dividend', 'divisor' and 'quotient', are the "anchoring part of knowledge" (Gu et al., 2004, p.325) for students to be able actively to explore new knowledge/problem. The interaction between Mei and the class below shows that Mei deliberately helped students to establish such knowledge anchor for the new learning in *Task 1*.

Teacher (T): (asked the class) What do the numbers 12, 3, and 4 respectively mean?

Student1 (S1): (one student was invited to give his answer.) 12 means 12 peaches. 3 means 3 monkeys. 4 means each monkey got 4 peaches.

T: Very good. But how did you get the quotient 4? Why did you think so? (another student was invited to give the answer.)

S2: I used the statement (a brief way used by the class to mean the multiplication table), that is, three four is twelve (3×4=12).

T: Very good. Why did you think of this statement? What did you refer to?

S2: Because the divisor is 3, I therefore thought about the statement of 3. And, the dividend is 12, so I thought that three four is twelve.

The interaction above also shows how Mei helped students to develop relational understanding (Skemp, 1976) by making connections between division and multiplication explicit. Emphasizing that 12 ÷ 4 = 3 because 3 times 4 is 12 can be seen both as an essential part of a relational understanding of division and multiplication and as an essential part of the *PV*. By starting with a division task that students were already familiar with, Mei also deliberately enable students to develop their autonomy and to generate new interest in learning.

In *Task 2* (see Table 1), Mei carefully varied the dividend (added one more peach to the 12 peaches in *Task 1*), while the 3 monkeys (the divisor) and the question of sharing were kept unvaried. Such a task variation recognized students' early learning experience and enabled them to develop learning autonomy in new problem situation. Noticeably, Mei also used a set of same questions to facilitate such autonomy during the process of solving *Task 2* (1). For instance, "*what does 13 mean here? What do 3, 4 mean then? What does 1 mean? Why did you not divide this one peach?*".

Teaching episode two: Example variation to deepen understanding of new concept

To define the connotation of the concept and further understand the concept of 'remainder' and its relationship with divisor, Mei deliberately applied the non-concept variation of the *CV* (Gu *et al*., 2004, p. 318) in *Task 2*(3) (see Table 1). Firstly, Mei challenged students by a non-concept example '15÷3=4……3'. By comparing it with '15÷3=5', students were able to discern the fact that "a remainder should be smaller than a divisor".

Next, in *Task 3* (see Table 1), Mei purposefully requested students to explain why the remainder 5 is incorrect in '17÷4=3……5', while it is correct in '17÷6=2……5'. It appears that the teacher's leading role in varying examples and questions is necessary here as it is not natural for young students to make it explicit of their thinking process of the fact that 'a remainder is ALWAYS smaller than a divisor' automatically establish on their own.

(2) Develop mathematical thinking through the calculations

Teaching episode three: Calculation method variation to experience the process of mathematisation.

During the next four tasks (see Table 1), Mei deliberately varied the calculation methods (from the concrete (e.g., *Task 1*), the semi-concrete (*Task 4&5*), the semi-abstract (*Task 6*), to the abstract (*Task 7*); Gu et al. 2004, p. 330) to enable students to gradually experience the process of mathematisation. In such a process of accumulation, Mei constantly used the same set of questions such as "*what is the quotient? What is the remainder? What statement [of the MT] do you think? What do you think firstly [dividend or divisor]? How do you find this statement? How did you get the remainder? etc.*". Mei considered that the teacher's leading role is important to make individual students' implicit inner thinking process explicit and to enable different students at various levels of understanding to communicate of their thoughts and to learn from each other in the class.

(3) Improve calculation skill by using the method of 'trying quotient' by the MT

Teaching episode four: Exercises variation to improve mental calculation skill.

Mei also set up multiple layers of classroom exercises. In this paper, we focus on an analysis on one of the exercises, Task 8 (see Table 1). Here, Mei considered that the form "31－ _ =_" was essential to enable those students who had difficulty in making a direct shift from their early learned procedure of division without remainder to the newly-learned procedure of division with remainder. The "31－ _ =_" form can be considered as a stepping stone set up by Mei in establishing the 'proper distance' for those students in active learning. By thinking of "31－ _ =_", what became visible to these students was the intricate relationship amongst the dividend (31), the outcome of the right statement of the multiplication table (here, it meant 30 as 5×6=30), and the remainder (1 for 31-30=1).

DISCUSSION AND CONCLUSION

In this paper, we identified four types of variation underlying the expert teacher Mei's teaching idea to 'not to lose the chain in learning mathematics'. These are task variation, example variation, calculation method variation and exercises variation. In the first place, the example variation is the *CV* (Gu et al., 2004) for deepening students' understanding of the concept of 'remainder in division' and the fact 'a remainder is *always* smaller than a divisor'. Next, the task, calculation method and exercises variation consists of the multiple layers of *PV* (Gu et al., 2004).

We identified three layers that Mei carefully set up to create the 'proper distances' in learning for understanding the internal relationship of a new concept and method with previous ones through these three types of variation in *PV*. The first layer was to use task variation (e.g., see the variation from *Task 1* to *Task 2*) to enable students to develop learning autonomy in using the same method of the MT in trying the quotient in division with remainder. The second layer was to use the calculation method variation through four similar tasks (from *Task 4* to *Task 7*) to enable students not only to make a shift from physical objects to arithmetic forms, but also to make their individual inner thinking process explicit to their classmates in the class. The third layer was to set up the exercise variation to tackle students' learning difficulty in making a shift from their early learned procedure of division without remainder to the newly learned procedure of division with remainder.

Findings of our study highlight that the teacher's leading role is essential not only in engaging students in effective learning, but also developing their learning autonomy and motivation. As illustrated by the data analysis above, Mei not only used the four types of variation to create the proper learning distances for students to develop understanding of the new concept, fact and calculation procedure, but also skilfully used questioning strategies to create various kinds of classroom learning space (e.g., learning by individuals, learning between students, and learning between the teacher and the whole class). This use of four types of variation relates to Li *et al*. (2011) analysis of three broad categories of teaching expertise in the mathematics classroom context in China, namely teacher knowledge for teaching, mathematics-specific instruction and student-oriented approaches (p.190). As such, this paper is offered as a contribution towards developing a deeper understanding of the complexity of teaching expertise valued in the Chinese mathematics classroom. An important aspect for the teacher is 'not to lose the chain in learning mathematics'.

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