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Minima of multi-Higgs potentials with triplets of $\Delta(3n^2)$ and $\Delta(6n^2)$

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ABSTRACT

We analyse the minima of scalar potentials for multi-Higgs models where the scalars are arranged as either one triplet or two triplets of the discrete symmetries A_4, S_4, $\Delta(27)$, $\Delta(54)$, as well as $\Delta(3n^2)$ and $\Delta(6n^2)$ with $n > 3$. The results should be useful for both multi-Higgs models involving electroweak doublets and multi-flavon models involving electroweak singlets, where in both cases the fields transform as triplets under some non-Abelian discrete symmetry.

1. Introduction

Following the discovery of the Higgs boson of the Standard Model (SM), it remains an intriguing possibility that there are more scalar bosons waiting to be discovered. Indeed, many extensions beyond the Standard Model include additional scalars, whether electroweak $SU(2)_L$ doublets in multi-Higgs doublet models or $SU(2)_L$ singlets typically found for example in flavour models in order to break some family symmetry. Given this, it is important to catalogue the minima of potentials including several scalars. In general this is a technically difficult task, which simplifies somewhat for simple cases where the potential is controlled by a large discrete symmetry.

In this paper we consider potentials of scalars which transform as triplets under various non-Abelian discrete family symmetries. The potentials we consider therefore involve up to six scalar $SU(2)_L$ doublets or singlets. We follow a progressive method that relies on considering which degrees of freedom become physical as the symmetry of the potential decreases when adding terms. We find a list of minima (not necessarily exhaustive) for potentials with one and two scalar triplets of A_4, S_4, $\Delta(27)$, $\Delta(54)$, and $\Delta(3n^2)$ and $\Delta(6n^2)$ with $n > 3$.[1] These symmetries [1,2] are typically used in multi-Higgs doublet models [3–9] and as family symmetries [10,11].

It is worth explaining the motivation for studying potentials with one or two triplets invariant under these groups. For example, in models in which the flavour group is broken to the largest possible subgroup in the neutrino sector, $Z_2 \times Z_2$, called direct models, the only discrete groups which are not excluded by measurements, are members of the $\Delta(6n^2)$ series $n \geq 10$, or of a closely related series [10,12]. If the requirement of breaking to this maximal residual symmetry of $Z_2 \times Z_2$ is weakened, groups of the $\Delta(3n^2)$ series are used in realistic models, with the flavour symmetry broken differently in the various fermionic sectors, which requires several scalar multiplets [13,14]. Further, if the flavour triplet is a Higgs doublet, in supersymmetric models you would be required to have two triplets. The same applies for extensions of the two Higgs doublet model where each of the doublets is made into a flavour triplet.

The explicit CP properties of all these potentials were analysed recently with invariant methods [15]. We start with potentials of one triplet, many of which had been studied in [16,17], and their minima found in [18] with a geometric method developed in [19,20]. Minimisation methods are also reviewed in [21]. We then consider the two triplet cases based on the results of the one triplet cases. Minima related by symmetries of the potential form sets of related Vacuum Expectation Values (VEVs) referred to as orbits. When the symmetry of a potential is decreased by adding terms, this has the effect of splitting larger orbits into several smaller orbits. In addition, we mostly disregard the magnitude of the VEVs and focus mainly on their alignments.

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[1] Note that while $Z_3 = \Delta(3)$ and $S_3 = \Delta(6)$, both of these groups have no three-dimensional irreducible representations, which is why we chose not to consider them in this analysis.

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We emphasize that while the one-triplet cases of A_4, S_4, $\Delta(27)$, $\Delta(54)$ have been discussed previously in the literature, we not only list them in the following for completeness, but also as illustration of the splitting of orbits. Of the two-triplet cases only A_4 has been studied previously in the literature.

The layout of the remainder of the paper is as follows. We start by going through the potentials with one triplet and list their minima throughout Section 2, with $\Delta(6n^2)$ and $\Delta(3n^2)$ with $n > 3$ in Section 2.1, S_4 in 2.2, A_4 in 2.3, $\Delta(54)$ and $\Delta(27)$ in 2.4. We use these results to then find minima for two triplet potentials in Section 3, with $\Delta(6n^2)$ and $\Delta(3n^2)$ with $n > 3$ respectively in Sections 3.1 and 3.2, S_4 and A_4 in 3.3 and 3.4, $\Delta(54)$ and $\Delta(27)$ in 3.5 and 3.6. We conclude in Section 4.

2. Potentials and VEVs with one triplet

We use cycl. to denote the cyclic permutations, and h.c. to indicate the hermitian conjugate. In addition to the discrete symmetries, the potentials of $SU(2)_L$ singlets are invariant under additional $U(1)$ symmetries to eliminate tri-linear terms, making the potentials similar to those for $SU(2)_L$ doublets. As we assume the VEVs of $SU(2)_L$ doublets preserve $U(1)_{em}$ the analysis of VEVs for singlets and doublets thus becomes interchangeable. For presentational simplicity we list the VEV directions in flavour space for $SU(2)_L$ singlets. We emphasize that our results are therefore applicable to multi-Higgs potentials, as well as to the analogous multi-flavon potentials with some additional symmetry (we note the additional symmetry needs only to forbid tri-linear terms to do this – often a Z_N symmetry will suffice).

2.1. One triplet of $\Delta(3n^2)$ or equivalently of $\Delta(6n^2)$, with $n > 3$

The simplest potential we consider is that of one triplet of $\Delta(3n^2)$ with $n > 3$, which is the same as for one triplet of $\Delta(6n^2)$ with ($n > 3$). This potential wasn't studied in [17,18], as the renormalisable potential is invariant under a continuous symmetry (this potential has additional continuous symmetries, cf. Eq. (2.8)). For $SU(2)_L$ singlets φ_i, where $i = 1, 2, 3$ is a flavour index, the potential is

$$V_{\Delta(3n^2)}(\varphi) = V_{\Delta(6n^2)}(\varphi) \equiv V_0(\varphi) \tag{2.1}$$

$$V_0(\varphi) = -m_\varphi^2 \sum_i \varphi_i\varphi^{*i} + r\left(\sum_i \varphi_i\varphi^{*i}\right)^2 + s\sum_i (\varphi_i\varphi^{*i})^2 \,. \tag{2.2}$$

For electroweak $SU(2)_L \times U(1)_Y$ doublets $H = (h_{1\alpha}, h_{2\beta}, h_{3\gamma})$, the respective version is

$$V_{\Delta(3n^2)}(H) = V_{\Delta(6n^2)}(H) \equiv V_0(H) \tag{2.3}$$

$$V_0(H) = -m_h^2 \sum_{i,\alpha} h_{i\alpha}h^{*i\alpha} + s\sum_{i,\alpha,\beta}(h_{i\alpha}h^{*i\alpha})(h_{i\beta}h^{*i\beta}) + \sum_{i,j,\alpha,\beta}\left[r_1(h_{i\alpha}h^{*i\alpha})(h_{j\beta}h^{*j\beta}) + r_2(h_{i\alpha}h^{*i\beta})(h_{j\beta}h^{*j\alpha})\right] . \tag{2.4}$$

where the Greek letters denote the $SU(2)_L$ indices. Note that for $n \geq 4$, here, and later for two triplets, it does not matter if $n/3$ is integer or not. While the representation content of the $\Delta(3n^2)$ and $\Delta(6n^2)$ groups changes, with the renormalisable potential not having cubic couplings, no new couplings arise when $n/3$ is an integer compared to when it is not.

Minima can be obtained analytically, and for $m_\varphi \neq 0$ the VEVs belong to four classes. Each class contains many directions, which are physically equivalent due to the internal symmetries of the potentials. These classes of VEVs connected by the symmetry group are also called orbits. As representatives one can take the following directions (omitting for simplicity any possible phases for now)

$$\begin{pmatrix}0\\0\\0\end{pmatrix}, \quad \begin{pmatrix}1\\0\\0\end{pmatrix} v_1, \quad \begin{pmatrix}1\\1\\0\end{pmatrix} v_2, \quad \begin{pmatrix}1\\1\\1\end{pmatrix} v_3 \tag{2.5}$$

where

$$v_1^2 = \frac{m_\varphi^2}{2r+2s},\ v_2^2 = \frac{m_\varphi^2}{4r+2s},\ v_3^2 = \frac{m_\varphi^2}{6r+2s}. \tag{2.6}$$

When $SU(2)_L$ doublets are considered, r is replaced with $r_1 + r_2$. There are regions of parameter space where each of these can be the global minimum.

The potential in Eq. (2.2) can be split into two invariants with parameters m_φ and r, invariant under all of $U(3)$, and the term with parameter s, invariant under $((U(1) \times U(1)) \rtimes S_3) \times U(1) =: \Delta(6\infty^2) \times U(1)$ (the third $U(1)$ was imposed to keep the potential even and is not needed for $SU(2)_L$ doublets).

$$V_0 = V_{U(3)} + V_{\Delta(6\infty^2)\times U(1)}. \tag{2.7}$$

The minima of $V_{U(3)}$ all fall into one large orbit, represented e.g. by $(1, 0, 0)$, connected to other alignments by arbitrary unitary transformations. The effect of the rest of the potential makes it less symmetric and splits the big orbit into several orbits in which the direction of the VEV becomes physical (but phases remain unphysical).

The flavour symmetries of V_0 (which relate each of the representative VEVs to the rest of their respective orbits) are generated by

$$\begin{pmatrix}0&1&0\\0&0&1\\1&0&0\end{pmatrix}, \begin{pmatrix}0&0&1\\0&1&0\\1&0&0\end{pmatrix}, \begin{pmatrix}e^{i\alpha}&0&0\\0&e^{i\alpha}&0\\0&0&e^{i\alpha}\end{pmatrix}, \begin{pmatrix}e^{i\beta}&0&0\\0&1&0\\0&0&e^{-i\beta}\end{pmatrix}, \begin{pmatrix}1&0&0\\0&e^{i\gamma}&0\\0&0&e^{-i\gamma}\end{pmatrix}, \tag{2.8}$$

where α, β, γ are arbitrary phases, which we now consider explicitly in order to verify which are physical. Additionally, the potential is automatically invariant under canonical CP transformations, which we denotes as CP_0 and is associated with a unit matrix in flavour space (a 3×3 unit matrix in the case of one triplet) [15]. Note that the alignments in Eq. (2.5) all conserve canonical CP. The orbits of alignments of this potential are, together with the representatives which had been chosen above

$$\left\{\begin{pmatrix}e^{i\eta}\\0\\0\end{pmatrix}, \begin{pmatrix}0\\e^{i\eta}\\0\end{pmatrix}, \begin{pmatrix}0\\0\\e^{i\eta}\end{pmatrix}\right\} \to \begin{pmatrix}1\\0\\0\end{pmatrix},$$
$$\left\{\begin{pmatrix}e^{i\eta}\\e^{i\zeta}\\0\end{pmatrix}, \text{permut.}\right\} \to \begin{pmatrix}1\\1\\0\end{pmatrix}, \left\{\begin{pmatrix}e^{i\eta}\\e^{i\zeta}\\e^{i\theta}\end{pmatrix}\right\} \to \begin{pmatrix}1\\1\\1\end{pmatrix}. \tag{2.9}$$

This can be compared directly with Eq. (2.5).

2.2. One triplet of S_4

The potential of one triplet of S_4 is

$$V_{S_4}(\varphi) = V_0(\varphi) + V_{S_4\times U(1)}(\varphi) \tag{2.10}$$

with

$$V_{S_4\times U(1)}(\varphi) = c\left[\left(\varphi_1\varphi_1\varphi^{*3}\varphi^{*3} + \text{cycl.}\right) + \text{h.c.}\right]. \quad (2.11)$$

$$V_{S_4}(H) = V_0(H) + c\sum_{\alpha,\beta}\left[\left(h_{1\alpha}h_{1\beta}h^{*3\alpha}h^{*3\beta} + \text{cycl.}\right) + \text{h.c.}\right], \quad (2.12)$$

for $SU(2)_L$ singlets and doublets respectively, and c is real as S_4 forces the invariant multiplying it to be real (in contrast to A_4).

The flavour symmetries of the S_4 potential are generated by

$$\begin{pmatrix}0&1&0\\0&0&1\\1&0&0\end{pmatrix}, \begin{pmatrix}0&0&1\\0&1&0\\1&0&0\end{pmatrix}, \begin{pmatrix}e^{i\alpha}&0&0\\0&e^{i\alpha}&0\\0&0&e^{i\alpha}\end{pmatrix}, \begin{pmatrix}-1&0&0\\0&1&0\\0&0&-1\end{pmatrix}, \begin{pmatrix}1&0&0\\0&-1&0\\0&0&-1\end{pmatrix}. \quad (2.13)$$

The potential is automatically invariant under canonical CP. The elements of the orbits of the potential of one triplet of $\Delta(6n^2)$, Eq. (2.9), become the following orbits:

$$\begin{pmatrix}e^{i\eta}\\0\\0\end{pmatrix} \to \begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}e^{i\eta}\\e^{i\zeta}\\0\end{pmatrix} \to \begin{pmatrix}1\\e^{i\zeta'}\\0\end{pmatrix} \text{ with } \zeta' \in [0,\pi] \quad (2.14)$$

and

$$\begin{pmatrix}e^{i\eta}\\e^{i\zeta}\\e^{i\theta}\end{pmatrix} \to \begin{pmatrix}1\\e^{i\zeta''}\\e^{i\theta'}\end{pmatrix} \text{ with } \zeta'' \in [0,\pi] \text{ and } \theta' \in [0,2\pi]. \quad (2.15)$$

Essentially, phases that were unphysical for $\Delta(6n^2)$ can become physical. Minimizing just the phase-dependent part, $V_{S_4\times U(1)}$, with respect to phases reveals the global minima

$$\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}1\\1\\1\end{pmatrix}, \begin{pmatrix}\pm1\\\omega\\\omega^2\end{pmatrix}, \begin{pmatrix}1\\i\\0\end{pmatrix}, \quad (2.16)$$

which were re-phased to match their appearance in [18]. The appearance of complex phases in the VEVs above does not mean that CP is violated and in fact it is actually preserved by this potential. While the CP transformation that contains only complex conjugation of the VEV (sometimes called canonical CP) is indeed not preserved, combinations of canonical CP and other internal symmetries are preserved. This is sufficient for CP to be conserved in physical processes, cf. [22]. The internal transformations playing these roles in the case discussed here are

$$\begin{pmatrix}1&&\\&&1\\&1&\end{pmatrix} \text{ and } \begin{pmatrix}1&&\\&-1&\\&&-1\end{pmatrix} \quad (2.17)$$

for $(\pm1,\omega,\omega^2)$ and $(1,i,0)$, respectively.

An elegant method of eliminating these complications is via CP-odd basis invariants, and such an analysis has been done in [23].

2.3. One triplet of A_4

The potential of one triplet of A_4 is an extension of the potential of one triplet of $\Delta(3n^2)$ by a term that is invariant only under $A_4 \times U(1)$:

$$V_{A_4}(\varphi) = V_0(\varphi) + V_{A_4\times U(1)}(\varphi), \quad (2.18)$$

$$V_{A_4\times U(1)}(\varphi) = c\left(\varphi_1\varphi_1\varphi^{*3}\varphi^{*3} + \varphi_2\varphi_2\varphi^{*1}\varphi^{*1} + \varphi_3\varphi_3\varphi^{*2}\varphi^{*2}\right) + h.c., \quad (2.19)$$

$$V_{A_4}(H) = V_0(H) + \sum_{\alpha,\beta}\left[c\left(h_{1\alpha}h_{1\beta}h^{*3\alpha}h^{*3\beta} + \text{cycl.}\right) + \text{h.c.}\right], \quad (2.20)$$

respectively for $SU(2)_L$ singlets and doublets, and c can now be complex (in contrast to S_4). The full flavour-type symmetries of the full potential V_{A_4} are generated by

$$\begin{pmatrix}0&1&0\\0&0&1\\1&0&0\end{pmatrix}, \begin{pmatrix}e^{i\alpha}&0&0\\0&e^{i\alpha}&0\\0&0&e^{i\alpha}\end{pmatrix}, \begin{pmatrix}-1&0&0\\0&1&0\\0&0&-1\end{pmatrix}, \begin{pmatrix}1&0&0\\0&-1&0\\0&0&-1\end{pmatrix}. \quad (2.21)$$

In addition, the potential has a CP symmetry CP_{23} generated e.g. by a CP transformation associated with the flavour space matrix [15]

$$X_{23} = \begin{pmatrix}1&0&0\\0&0&1\\0&1&0\end{pmatrix}. \quad (2.22)$$

Under these symmetries, the elements of the orbits of the potential of one triplet of $\Delta(6n^2)$, Eq. (2.9), fall into the following orbits:

$$\begin{pmatrix}e^{i\eta}\\0\\0\end{pmatrix} \to \begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}e^{i\eta}\\e^{i\zeta}\\0\end{pmatrix} \to \begin{pmatrix}1\\e^{i\zeta'}\\0\end{pmatrix} \text{ with } \zeta' \in [0,\pi], \quad (2.23)$$

$$\begin{pmatrix}e^{i\eta}\\e^{i\zeta}\\e^{i\theta}\end{pmatrix} \to \begin{pmatrix}1\\e^{i\zeta''}\\e^{i\theta'}\end{pmatrix} \text{ with } \zeta'' \in [0,\pi] \text{ and } \theta' \in [0,2\pi]. \quad (2.24)$$

Up to two phases can become physical, which can be determined by minimizing the parts of the potential that depend on them, i.e. $V_{A_4\times U(1)}$. For the alignment $(1, e^{i\zeta'}, 0)$,

$$V_{A_4\times U(1)}[(1, e^{i\zeta'}, 0)] = ce^{2i\zeta'} + c^*e^{-2i\zeta'}, \quad (2.25)$$

such that $\zeta' = -\text{Arg}(c)/2 \bmod \pi$. For the alignment $(1, e^{i\zeta''}, e^{i\theta'})$ we get

$$(\zeta'', \theta') = (0,0), (5\pi/3, \pi/3), (2\pi/3, 4\pi/3), \quad (2.26)$$

meaning we have minima $(1,1,1)$, $(1,\omega,\omega^2)$ and $(1,-\omega,-\omega^2)$, defining $\omega = e^{2\pi i/3}$. An overall sign can be absorbed via rephasing, such that one obtains the full list of possible global alignments from [18] for one triplet of A_4:

$$\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}1\\1\\1\end{pmatrix}, \begin{pmatrix}\pm1\\\omega\\\omega^2\end{pmatrix}, \begin{pmatrix}1\\e^{i\zeta'}\\0\end{pmatrix}. \quad (2.27)$$

2.4. One triplet of $\Delta(27)$, or, equivalently of $\Delta(54)$

The potential of one triplet of $\Delta(27)$ and $\Delta(54)$ is an extension of the $\Delta(3n^2)$ by a term that is invariant under $\Delta(54)$

$$V_{\Delta(27)}(\varphi) = V_{\Delta(54)}(\varphi) = V_0(\varphi) + \left[d\left(\varphi_1\varphi_1\varphi^{*2}\varphi^{*3} + \text{cycl.}\right) + \text{h.c.}\right], \quad (2.28)$$

and the respective $SU(2)_L$ doublet version is

$$V_{\Delta(27)}(H) = V_{\Delta(54)}(H) = V_0(H) + \sum_{\alpha,\beta}\left[d\left(h_{1\alpha}h_{1\beta}h^{*2\alpha}h^{*3\beta} + \text{cycl.}\right) + \text{h.c.}\right]. \quad (2.29)$$

The potential in general violates CP explicitly. One may of course impose a CP symmetry. In [18] the two types of CP-symmetry that are normally considered consistent with the flavour-type symmetry of the potential are analysed. The 12 CP symmetries listed in [24] for $\Delta(27)$ are reduced to 6 in the context of $\Delta(54)$, e.g. canonical CP_0 and the CP symmetry associated with X_{23} become related. Of the 6 remaining, 3 restrict the phase of parameter d in the potential, the other 3 enforce a relation between parameters, such as $2s = (d + d^*)$ if imposing CP symmetry associated with the flavour matrix

$$X_4 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{pmatrix} \tag{2.30}$$

The full flavour-type symmetries of this potential, are generated by

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} e^{i\alpha} & 0 & 0 \\ 0 & e^{i\alpha} & 0 \\ 0 & 0 & e^{i\alpha} \end{pmatrix}, \begin{pmatrix} \omega & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \omega^2 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix}. \tag{2.31}$$

The orbits of one triplet of $\Delta(6n^2)$, Eq. (2.9) become

$$(1,0,0),\ (1, e^{i\zeta'}, 0),\ (1, e^{i\zeta''}, e^{i\theta'}). \tag{2.32}$$

The phases can now be physical. The phase-dependent part

$$V_{\Delta(54)\times U(1)} = \left[d \left(\varphi_1 \varphi_1 \varphi^{*2} \varphi^{*3} + \text{cycl.} \right) + \text{h.c.} \right] \tag{2.33}$$

yields simply zero for the alignment $(1, e^{i\zeta'}, 0)$, thus ζ' remains unphysical and $(1,1,0)$ is at least a local minimum of the potential, as it was already a possible global minimum of V_0. For $(1, e^{i\zeta''}, e^{i\theta'})$, one obtains the alignments $(1,1,1)$, $(1,1,\omega)$, $(1,\omega,\omega)$ (or equivalently $(1,1,\omega^2)$). As for any value of Arg(d), one of $(1,1,1)$, $(1,1,\omega)$, $(1,1,\omega^2)$ makes $V_{\Delta(54)\times U(1)}$ negative, we verify that $(1,1,0)$ is never a global minimum. We have thus obtained the full list of global minima by [18]:

$$(1,0,0), (1,1,1), (1,1,\omega), (1,1,\omega^2). \tag{2.34}$$

With canonical CP (CP_0), the last two VEVs are related whereas with the type of CP with matrix X_4 the last two VEVs in Eq. (2.34) become part of the same orbit and also the first two VEVs in Eq. (2.34) become part of the same orbit (separate from the last two VEVs).

3. Potentials and some VEVs with two triplets

Potentials of two triplets have two sets of terms for each triplet by themselves and also cross terms

$$V(\varphi, \varphi') = V(\varphi) + V'(\varphi') + V_c(\varphi, \varphi'). \tag{3.1}$$

In the cases we consider, the two triplets transform identically under the symmetry, making $V(\varphi)$ and $V'(\varphi')$ functionally identical.

The complete set of orbits for minima are known for the single triplet cases above, and we can proceed to two triplet potentials by analysing which degrees of freedom of $V(\varphi) + V'(\varphi')$ can become physical when the symmetry of the potential is reduced (e.g. by the cross-terms in $V_c(\varphi, \varphi')$). We omit the magnitudes of the VEVs, which are in general different for the two triplets.

It is convenient to define

$$\begin{aligned} V_1(\varphi, \varphi') = &+ \tilde{r}_1 \left(\sum_i \varphi_i \varphi^{*i} \right) \left(\sum_j \varphi'_j \varphi'^{*j} \right) \\ &+ \tilde{r}_2 \left(\sum_i \varphi_i \varphi'^{*i} \right) \left(\sum_j \varphi'_j \varphi^{*j} \right) \\ &+ \tilde{s}_1 \sum_i \left(\varphi_i \varphi^{*i} \varphi'_i \varphi'^{*i} \right) \\ &+ \tilde{s}_2 \left(\varphi_1 \varphi^{*1} \varphi'_2 \varphi'^{*2} + \varphi_2 \varphi^{*2} \varphi'_3 \varphi'^{*3} + \varphi_3 \varphi^{*3} \varphi'_1 \varphi'^{*1} \right) \\ &+ i\,\tilde{s}_3 \Big[(\varphi_1 \varphi'^{*1} \varphi'_2 \varphi^{*2} + \text{cycl.}) \\ &\qquad - (\varphi^{*1} \varphi'_1 \varphi'^{*2} \varphi_2 + \text{cycl.}) \Big], \end{aligned} \tag{3.2}$$

$$\begin{aligned} V_1(H, H') = &\sum_{i,j,\alpha,\beta} \left[\tilde{r}_{11} h_{i\alpha} h^{*i\alpha} h'_{j\beta} h'^{*j\beta} + \tilde{r}_{12} h_{i\alpha} h'^{*j\alpha} h'_{j\beta} h^{*i\beta} \right] \\ &+ \sum_{i,j,\alpha,\beta} \left[\tilde{r}_{21} h_{i\alpha} h'^{*i\alpha} h'_{j\beta} h^{*j\beta} + \tilde{r}_{22} h_{i\alpha} h^{*j\alpha} h'_{j\beta} h'^{*i\beta} \right] \\ &+ \sum_{i,\alpha,\beta} \left[\tilde{s}_{11} h_{i\alpha} h^{*i\alpha} h'_{i\beta} h'^{*i\beta} + \tilde{s}_{12} h_{i\alpha} h'^{*i\alpha} h'_{i\beta} h^{*i\beta} \right] \\ &+ \sum_{\alpha,\beta} [\tilde{s}_{21} (h_{1\alpha} h^{*1\alpha} h'_{2\beta} h'^{*2\beta} + \text{cycl.}) \\ &\qquad + \tilde{s}_{22} (h_{1\alpha} h'^{*2\alpha} h'_{2\beta} h^{*1\beta} + \text{cycl.})] \\ &+ i\tilde{s}_{31} \sum_{\alpha,\beta} [(h_{1\alpha} h'^{*1\alpha} h'_{2\beta} h^{*2\beta} + \text{cycl.}) \\ &\qquad - (h^{*1\alpha} h'_{1\alpha} h'^{*2\beta} h_{2\beta} + \text{cycl.})] \\ &+ i\tilde{s}_{32} \sum_{\alpha,\beta} [(h_{1\alpha} h^{*2\alpha} h'_{2\beta} h'^{*1\beta} + \text{cycl.}) \\ &\qquad - (h^{*1\alpha} h_{2\alpha} h'^{*2\beta} h'_{1\beta} + \text{cycl.})], \end{aligned} \tag{3.3}$$

$$\begin{aligned} V_2(\varphi, \varphi') = &\ \tilde{r}_1 \left(\sum_i \varphi_i \varphi^{*i} \right) \left(\sum_j \varphi'_j \varphi'^{*j} \right) \\ &+ \tilde{r}_2 \left(\sum_i \varphi_i \varphi'^{*i} \right) \left(\sum_j \varphi'_j \varphi^{*j} \right) \\ &+ \tilde{s}_1 \sum_i \left(\varphi_i \varphi^{*i} \varphi'_i \varphi'^{*i} \right), \end{aligned} \tag{3.4}$$

$$\begin{aligned} V_2(H, H') = &\sum_{i,j,\alpha,\beta} \left[\tilde{r}_{11} h_{i\alpha} h^{*i\alpha} h'_{j\beta} h'^{*j\beta} + \tilde{r}_{12} h_{i\alpha} h'^{*j\alpha} h'_{j\beta} h^{*i\beta} \right] \\ &+ \sum_{i,j,\alpha,\beta} \left[\tilde{r}_{21} h_{i\alpha} h'^{*i\alpha} h'_{j\beta} h^{*j\beta} + \tilde{r}_{22} h_{i\alpha} h^{*j\alpha} h'_{j\beta} h'^{*i\beta} \right] \\ &+ \sum_{i,\alpha,\beta} \left[\tilde{s}_{11} h_{i\alpha} h^{*i\alpha} h'_{i\beta} h'^{*i\beta} + \tilde{s}_{12} h_{i\alpha} h'^{*i\alpha} h'_{i\beta} h^{*i\beta} \right]. \end{aligned} \tag{3.5}$$

3.1. Two triplets of $\Delta(6n^2)$ with $n>3$

The potentials of two triplets of $\Delta(6n^2)$ are, using the definitions of V_0 and V_2 previously,

$$V_{\Delta(6n^2)}(\varphi,\varphi') = V_0(\varphi) + V_0'(\varphi') + V_2(\varphi,\varphi')\,, \tag{3.6}$$

$$V_{\Delta(6n^2)}(H,H') = V_0(H) + V_0'(H') + V_2(H,H')\,. \tag{3.7}$$

The orbits for one triplet of $\Delta(6n^2)$ are in Eq. (2.9), and we can obtain minima for the two triplet case by combining any two members of any orbit (not just the representatives), and then check which phases are unphysical. The symmetries of the two triplet potential are generated by simultaneous transformations of both triplets under $\Delta(6\infty^2)$ and by separate $U(1)$ phases acting on each triplet,

$$\begin{pmatrix} e^{i\alpha} & & \\ & e^{i\alpha} & \\ & & e^{i\alpha}\end{pmatrix} \oplus \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1\end{pmatrix}, \text{ and} \\ \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1\end{pmatrix} \oplus \begin{pmatrix} e^{i\alpha'} & & \\ & e^{i\alpha'} & \\ & & e^{i\alpha'}\end{pmatrix}, \tag{3.8}$$

as well as an overall canonical CP transformation. Accounting for these, we get the following combinations of orbits:

$$(e^{i\eta},0,0),(e^{i\eta'},0,0) \to (1,0,0),(1,0,0) \tag{3.9}$$

$$(e^{i\eta},0,0),(0,e^{i\eta'},0) \to (1,0,0),(0,1,0) \tag{3.10}$$

$$(e^{i\eta},0,0),(e^{i\eta'},e^{i\zeta'},0) \to (1,0,0),(1,1,0) \tag{3.11}$$

$$(e^{i\eta},0,0),(0,e^{i\eta'},e^{i\zeta'}) \to (1,0,0),(0,1,1) \tag{3.12}$$

$$(e^{i\eta},0,0),(e^{i\eta'},e^{i\zeta'},e^{i\theta'}) \to (1,0,0),(1,1,1) \tag{3.13}$$

$$(e^{i\eta},e^{i\zeta},0),(0,e^{i\zeta'},e^{i\theta'}) \to (1,1,0),(0,1,1) \tag{3.14}$$

$$(e^{i\eta},e^{i\zeta},0),(e^{i\eta'},e^{i\zeta'},0) \to (1,1,0),(1,e^{i\zeta'},0) \tag{3.15}$$

$$(e^{i\eta},e^{i\zeta},0),(e^{i\eta'},e^{i\zeta'},e^{i\theta'}) \to (1,1,0),(1,e^{i\zeta'},1) \tag{3.16}$$

$$(e^{i\eta},e^{i\zeta},e^{i\theta}),(e^{i\eta'},e^{i\zeta'},e^{i\theta'}) \to (1,1,1),(1,e^{i\zeta'},e^{i\theta'}). \tag{3.17}$$

The phases can now be fixed by the minimisation of the phase-dependent part of the potential. We get for $(1,1,0),(1,e^{i\zeta'},0)$ and for $(1,1,0),(1,e^{i\zeta'},1)$ that $\zeta'=0$ for $r_2'>0$ and $\zeta'=\pi$ for $r_2<0$, i.e.:

$$(1,1,0),(1,\pm1,0) \text{ and } (1,1,0),(1,\pm1,1) \tag{3.18}$$

different sign choices are different orbits. For $(1,1,1),(1,e^{i\zeta'},e^{i\theta'})$, we get for $r_2<1$ the orbit

$$(1,1,1),(1,1,1) \tag{3.19}$$

and for $r_2>0$ the orbits

$$(1,1,1),(1,\omega,\omega^2) \text{ and } (1,1,1),(1,\omega^2,\omega). \tag{3.20}$$

3.2. Two triplets of $\Delta(3n^2)$ with $n>3$

The potentials for two triplets of $\Delta(3n^2)$ with $n>3$ are

$$V_{\Delta(3n^2)}(\varphi,\varphi') = V_0(\varphi) + V_0'(\varphi') + V_1(\varphi,\varphi'), \tag{3.21}$$

$$V_{\Delta(3n^2)}(H,H') = V_0(H) + V_0'(H') + V_1(H,H')\,. \tag{3.22}$$

The single triplet orbits of $\Delta(3n^2)$ and $\Delta(6n^2)$ were the same and listed in Eq. (2.9). The difference to the previous potential lies in the fact that the full symmetries of $V_{\Delta(3n^2)}$ only allow for cyclic permutations, i.e. only

$$\begin{pmatrix} & 1 & \\ & & 1 \\ 1 & & \end{pmatrix} \oplus \begin{pmatrix} & 1 & \\ & & 1 \\ 1 & & \end{pmatrix}, \tag{3.23}$$

in addition to all phase symmetries arising from $\Delta(3n^2)$ and Eq. (3.8). This two triplet potential has no automatic CP symmetry. Compared to $\Delta(6n^2)$, several orbits split, but interchanging the first and second triplet allow us to reduce the number of distinct such orbits to $(1,0,0),(1,1,0)$ and $(1,0,0),(1,0,1)$. Besides Eqs. (3.9)–(3.14), Eqs. (3.19), (3.20), and the two orbits above that arise from splitting known orbits, two other new orbits can arise due to the lacking CP symmetry, namely

$$(1,1,0),(1,e^{i\zeta'},0) \text{ and } (1,1,0),(1,e^{i\zeta'},1) \tag{3.24}$$

with $\zeta'=\arctan(\tilde{r}_2/\tilde{s}_3)$ a function of $\tilde{s}_3$ and $\tilde{r}_2$ in contrast to the situation with a $\Delta(6n^2)$ symmetry, where $\zeta'=0,\pi$, depending on the value of $\tilde{r}_2$. The pair $(1,1,0),(1,e^{i\zeta'},0)$ and the pair $(1,1,0),(1,e^{i\zeta'},1)$ have the same ζ'. When special CP symmetries are imposed, then ζ' can be forced to take special values again.

3.3. Two triplets of S_4

The potentials for two triplets of S_4 are

$$\begin{aligned} V_{S_4}(\varphi,\varphi') = {} & V_0(\varphi) + V_0'(\varphi') + V_2(\varphi,\varphi') + \\ & + c\left[\left(\varphi_1\varphi_1\varphi^{*3}\varphi^{*3} + \text{cycl.}\right) + \text{h.c.}\right] \\ & + c'\left[\left(\varphi_1'\varphi_1'\varphi'^{*3}\varphi'^{*3} + \text{cycl.}\right) + \text{h.c.}\right] \\ & + \tilde{c}\left[\left(\varphi_1\varphi_1'\varphi^{*3}\varphi'^{*3} + \text{cycl.}\right) + \text{h.c.}\right], \end{aligned} \tag{3.25}$$

$$\begin{aligned} V_{S_4}(H,H') = {} & V_0(H) + V_0'(H') + V_2(H,H') \\ & + \sum_{\alpha,\beta} c\left[\left(h_{1\alpha}h_{1\beta}h^{*3\alpha}h^{*3\beta} + \text{cycl.}\right) + \text{h.c.}\right] \\ & + \sum_{\alpha,\beta} c'\left[\left(h'_{1\alpha}h'_{1\beta}h'^{*3\alpha}h'^{*3\beta} + \text{cycl.}\right) + \text{h.c.}\right] \\ & + \sum_{\alpha,\beta} \tilde{c}_1\left[\left(h_{1\alpha}h^{*3\alpha}h'_{1\beta}h'^{*3\beta} + \text{cycl.}\right) + \text{h.c.}\right] \\ & + \sum_{\alpha,\beta} \tilde{c}_2\left[\left(h_{1\alpha}h'^{*3\alpha}h'_{1\beta}h^{*3\beta} + \text{cycl.}\right) + \text{h.c.}\right]. \end{aligned} \tag{3.26}$$

For one triplet, the symmetry generators are in Eq. (2.13), and the potential has an automatic CP symmetry. The single triplet orbits are

$$\left\{\begin{pmatrix} \pm e^{i\alpha} \\ 0 \\ 0 \end{pmatrix}\right\}, \left\{\begin{pmatrix} (-1)^k e^{i\alpha} \\ (-1)^l e^{i\alpha} \\ (-1)^{k+l} e^{i\alpha} \end{pmatrix}\right\}, \left\{\begin{pmatrix} (-1)^k e^{i\alpha} \\ \omega(-1)^l e^{i\alpha} \\ \omega^2(-1)^{k+l} e^{i\alpha} \end{pmatrix}\right\}, \\ \left\{\begin{pmatrix} -(-1)^k e^{i\alpha} \\ \omega(-1)^l e^{i\alpha} \\ \omega^2(-1)^{k+l} e^{i\alpha} \end{pmatrix}\right\}, \left\{\begin{pmatrix} 0 \\ \pm e^{i\alpha} \\ \pm i e^{i\alpha} \end{pmatrix}\right\}, \tag{3.27}$$

where the sign choices represent independent orbits. We combine the single triplet orbits to obtain the two triplet orbit representatives

$$(1,0,0),(1,0,0) \tag{3.28}$$
$$(1,0,0),(0,1,0) \tag{3.29}$$
$$(1,0,0),(1,0,i) \tag{3.30}$$
$$(1,0,0),(0,1,i) \tag{3.31}$$
$$(1,0,0),(1,1,1) \tag{3.32}$$
$$(1,0,0),(1,\omega^2,\omega) \tag{3.33}$$
$$(1,0,i),(1,0,\pm i) \tag{3.34}$$
$$(1,0,i),(1,i,0) \tag{3.35}$$
$$(1,0,i),(1,1,1) \tag{3.36}$$
$$(1,0,i),(1,\omega^2,\pm\omega) \tag{3.37}$$
$$(1,1,1),(1,1,\pm 1) \tag{3.38}$$
$$(1,1,1),(1,\pm\omega^2,\omega) \tag{3.39}$$
$$(1,\omega^2,\omega),(1,\omega^2,\pm\omega) \tag{3.40}$$
$$(1,\omega^2,\omega),(1,-\omega,-\omega^2) \tag{3.41}$$
$$(1,\omega^2,\omega),(1,\omega,\omega^2) \tag{3.42}$$

3.4. Two triplets of A_4

The potentials for two triplets of A_4 are

$$\begin{aligned} V_{A_4}(\varphi,\varphi') = {} & V_0(\varphi)+V_0'(\varphi')+V_1(\varphi,\varphi')+ \\ & +\left[c\left(\varphi_1\varphi_1\varphi^{*3}\varphi^{*3}+\text{cycl.}\right)+\text{h.c.}\right] \\ & +\left[c'\left(\varphi_1'\varphi_1'\varphi'^{*3}\varphi'^{*3}+\text{cycl.}\right)+\text{h.c.}\right] \\ & +\left[\tilde{c}\left(\varphi_1\varphi_1'\varphi^{*3}\varphi'^{*3}+\text{cycl.}\right)+\text{h.c.}\right], \end{aligned} \tag{3.43}$$

$$\begin{aligned} V_{A_4}(H,H') = {} & V_0(H)+V_0'(H')+V_1(H,H') \\ & +\sum_{\alpha,\beta}\left[c\left(h_{1\alpha}h_{1\beta}h^{*3\alpha}h^{*3\beta}+\text{cycl.}\right)\right. \\ & \left.\quad + c'\left(h'_{1\alpha}h'_{1\beta}h'^{*3\alpha}h'^{*3\beta}+\text{cycl.}\right)+\text{h.c.}\right] \\ & +\sum_{\alpha,\beta}\left[\tilde{c}_1(h_{1\alpha}h^{*3\alpha}h'_{1\beta}h'^{*3\beta}+\text{cycl.})\right. \\ & \left.\quad +\tilde{c}_2(h_{1\alpha}h'^{*3\alpha}h'_{1\beta}h^{*3\beta}+\text{cycl.})+\text{h.c.}\right]. \end{aligned} \tag{3.44}$$

The symmetry generators for one triplet are in Eq. (2.21), and the orbit representatives are

$$\begin{aligned} &\left\{\begin{pmatrix}\pm e^{i\alpha}\\0\\0\end{pmatrix}\right\},\left\{\begin{pmatrix}(-1)^k e^{i\alpha}\\(-1)^l e^{i\alpha}\\(-1)^{k+l}e^{i\alpha}\end{pmatrix}\right\},\left\{\begin{pmatrix}(-1)^k e^{i\alpha}\\\omega(-1)^l e^{i\alpha}\\\omega^2(-1)^{k+l}e^{i\alpha}\end{pmatrix}\right\}, \\ &\left\{\begin{pmatrix}-(-1)^k e^{i\alpha}\\\omega(-1)^l e^{i\alpha}\\\omega^2(-1)^{k+l}e^{i\alpha}\end{pmatrix}\right\},\left\{\begin{pmatrix}0\\\pm e^{i\alpha}\\\pm e^{i\alpha+i\beta}\end{pmatrix}\right\}. \end{aligned} \tag{3.45}$$

For two triplets we get

$$(1,0,0),(1,0,0) \tag{3.46}$$
$$(1,0,0),(0,1,0) \tag{3.47}$$
$$(1,0,0),(1,e^{i\alpha'},0) \tag{3.48}$$
$$(1,0,0),(0,1,e^{i\alpha'}) \tag{3.49}$$
$$(1,0,0),(e^{i\alpha'},0,1) \tag{3.50}$$
$$(1,0,0),(1,1,1) \tag{3.51}$$
$$(1,0,0),(1,\omega,\omega^2) \tag{3.52}$$
$$(1,e^{i\alpha},0),(1,\pm e^{i\alpha'},0) \tag{3.53}$$
$$(1,e^{i\alpha},0),(0,1,e^{i\alpha'}) \tag{3.54}$$
$$(1,e^{i\alpha},0),(e^{i\alpha'},0,1) \tag{3.55}$$
$$(1,e^{i\alpha},0),(1,\pm 1,1) \tag{3.56}$$
$$(1,e^{i\alpha},0),(1,\pm\omega,\omega^2) \tag{3.57}$$
$$(1,1,1),(1,1,\pm 1) \tag{3.58}$$
$$(1,1,1),(1,\omega,\pm\omega^2) \tag{3.59}$$
$$(1,\omega,\omega^2),(1,\omega,\pm\omega^2) \tag{3.60}$$

where α and α' are fixed by the respective one-triplet parts of the two-triplet potential, as in Eq. (2.25). Note that $(1,0,0),(e^{i\alpha'},1,0)$ as well as $(1,0,0),(0,e^{i\alpha'},1)$ and $(1,0,0),(1,0,e^{i\alpha'})$, are part of the above orbits due to the separate rephasing symmetries of each triplet.

3.5. Two triplets of $\Delta(54)$

The potentials for two triplets of $\Delta(54)$ are

$$\begin{aligned} V_{\Delta(54)}(\varphi,\varphi') = {} & V_0(\varphi)+V_0'(\varphi')+V_2(\varphi,\varphi') \\ & +\left[d\left(\varphi_1\varphi_1\varphi^{*2}\varphi^{*3}+\text{cycl.}\right)+\text{h.c.}\right] \\ & +\left[d'\left(\varphi_1'\varphi_1'\varphi'^{*2}\varphi'^{*3}+\text{cycl.}\right)+\text{h.c.}\right] \\ & +\tilde{d}_1\left[\left(\varphi_1\varphi_1'\varphi^{*2}\varphi'^{*3}+\text{cycl.}\right)\right. \\ & \left.\quad +\left(\varphi_1\varphi_1'\varphi^{*3}\varphi'^{*2}+\text{cycl.}\right)\right]+\text{h.c.}, \end{aligned} \tag{3.61}$$

$$\begin{aligned} V_{\Delta(54)}(H,H') = {} & V_0(H)+V_0'(H')+V_2(H,H') \\ & +\sum_{\alpha,\beta}\left[d\left(h_{1\alpha}h_{1\beta}h^{*2\alpha}h^{*3\beta}+\text{cycl.}\right)\right. \\ & \left.\quad +d'\left(h'_{1\alpha}h'_{1\beta}h'^{*2\alpha}h'^{*3\beta}+\text{cycl.}\right)+\text{h.c.}\right] \\ & +\sum_{\alpha,\beta}\left[\tilde{d}_{11}(h_{1\alpha}h^{*2\alpha}h'_{1\beta}h'^{*3\beta}+\text{cycl.})\right. \\ & \left.\quad +\tilde{d}_{12}(h_{1\alpha}h'^{*3\alpha}h'_{1\beta}h^{*2\beta}+\text{cycl.})+\text{h.c.}\right] \\ & +\sum_{\alpha,\beta}\left[\tilde{d}_{11}(h_{1\alpha}h^{*3\alpha}h'_{1\beta}h'^{*2\beta}+\text{cycl.})\right. \\ & \left.\quad +\tilde{d}_{12}(h_{1\alpha}h'^{*2\alpha}h'_{1\beta}h^{*3\beta}+\text{cycl.})+\text{h.c.}\right]. \end{aligned} \tag{3.62}$$

This potential has no automatic CP symmetries. We can write

$$\begin{aligned} V_{\Delta(54)}(\varphi,\varphi') = {} & V_{\Delta(6n^2)}(\varphi)+V'_{\Delta(6n^2)}(\varphi')+V_{\Delta(54)}(\varphi) \\ & +V'_{\Delta(54)}(\varphi') \qquad (3.63) \\ & +V_{c,\Delta(6n^2)}(\varphi,\varphi')+V_{c,\Delta(54)}(\varphi,\varphi'). \qquad (3.64) \end{aligned}$$

The orbits of VEVs for one triplet are

$$\{\begin{pmatrix}\omega^k e^{i\alpha}\\0\\0\end{pmatrix}, \text{perm.}\}, \{\begin{pmatrix}\omega^k e^{i\alpha}\\\omega^l e^{i\alpha}\\\omega^{2k+2l}e^{i\alpha}\end{pmatrix}, \text{perm.}\},$$
$$\{\begin{pmatrix}\omega^k e^{i\alpha}\\\omega^l e^{i\alpha}\\\omega^{2k+2l+1}e^{i\alpha}\end{pmatrix}, \text{perm.}\}, \{\begin{pmatrix}\omega^k e^{i\alpha}\\\omega^l e^{i\alpha}\\\omega^{2k+2l+2}e^{i\alpha}\end{pmatrix}, \text{perm.}\}. \tag{3.65}$$

In addition to the direct sum of the generators in Eq. (2.31), the potential is invariant under separate phase symmetries for each triplet, as in Eq. (3.8). By combining single triplet orbits and then eliminating unphysical degrees of freedom we get

$$(1,0,0),(1,0,0) \tag{3.66}$$

$$(1,0,0),(0,1,0) \tag{3.67}$$

$$(1,0,0),(1,1,1) \tag{3.68}$$

$$(1,0,0),(1,1,\omega) \tag{3.69}$$

$$(1,0,0),(1,1,\omega^2) \tag{3.70}$$

$$(1,1,\omega^i),(\omega^{k'-k},\omega^{l'-l},\omega^{2k'+2l'-2k-2l+i'}) \tag{3.71}$$

where the last case has several orbits labelled by i and i'. We note that phase differences between the two triplets are physical.

3.6. Two triplets of $\Delta(27)$

The potentials for two triplets of $\Delta(27)$ are

$$\begin{aligned} V_{\Delta(27)}(\varphi,\varphi') = {} & V_0(\varphi) + V_0'(\varphi') + V_1(\varphi,\varphi') \\ & + \left[d\left(\varphi_1\varphi_1\varphi^{*2}\varphi^{*3} + \text{cycl.}\right) + \text{h.c.}\right] \\ & + \left[d'\left(\varphi_1'\varphi_1'\varphi'^{*2}\varphi'^{*3} + \text{cycl.}\right) + \text{h.c.}\right] \\ & + \left[\tilde{d}_1\left(\varphi_1\varphi_1'\varphi^{*2}\varphi'^{*3} + \text{cycl.}\right) + \text{h.c.}\right] \\ & + \left[\tilde{d}_2\left(\varphi_1\varphi_1'\varphi^{*3}\varphi'^{*2} + \text{cycl.}\right) + \text{h.c.}\right], \end{aligned} \tag{3.72}$$

$$\begin{aligned} V_{\Delta(27)}(H,H') = {} & V_0(H) + V_0'(H') + V_1(H,H') + \\ & + \sum_{\alpha,\beta}\left[d\left(h_{1\alpha}h_{1\beta}h^{*2\alpha}h^{*3\beta} + \text{cycl.}\right)\right. \\ & \left. + d'\left(h'_{1\alpha}h'_{1\beta}h'^{*2\alpha}h'^{*3\beta} + \text{cycl.}\right) + \text{h.c.}\right] \\ & + \sum_{\alpha,\beta}\left[\tilde{d}_{11}(h_{1\alpha}h^{*2\alpha}h'_{1\beta}h'^{*3\beta} + \text{cycl.})\right. \\ & \left. + \tilde{d}_{12}(h_{1\alpha}h'^{*3\alpha}h'_{1\beta}h^{*2\beta} + \text{cycl.}) + \text{h.c.}\right] \\ & + \sum_{\alpha,\beta}\left[\tilde{d}_{21}(h_{1\alpha}h^{*3\alpha}h'_{1\beta}h'^{*2\beta} + \text{cycl.})\right. \\ & \left. + \tilde{d}_{22}(h_{1\alpha}h'^{*2\alpha}h'_{1\beta}h^{*3\beta} + \text{cycl.}) + \text{h.c.}\right]. \end{aligned} \tag{3.73}$$

The one triplet VEVs for $\Delta(27)$ are the same as for $\Delta(54)$, so the VEV pairs generated by our method are similar to the case of two triplets of $\Delta(54)$. There is however the permutation generator that is missing in $\Delta(27)$, and therefore several orbits split with respect to $\Delta(54)$. In addition to Eqs. (3.66)–(3.71), we get independent orbits:

$$(1,0,0),(1,1,\omega) \tag{3.74}$$

$$(1,0,0),(1,\omega,1) \tag{3.75}$$

$$(1,0,0),(1,1,\omega^2) \tag{3.76}$$

$$(1,0,0),(1,\omega^2,1). \tag{3.77}$$

4. Conclusions

In this paper we have analysed the minima of scalar potentials for multi-Higgs models, where the scalars are arranged as either one triplet or two triplets of the discrete symmetries $\Delta(3n^2)$ and $\Delta(6n^2)$ with $n = 2$ (A_4, S_4), $n = 3$ ($\Delta(27)$, $\Delta(54)$) and $n > 3$. We have found the minima with a technique where we consider by steps the symmetry of parts of the potential and progressively add terms that reduce the symmetry, minimizing them in turn. We have identified cases where phases are physical. This does not guarantee that such cases spontaneously violate CP (such cases are known e.g. in A_4). Whether the minima spontaneously violate CP or not is a non-trivial issue, which is discussed in a separate work [23]. The results in this paper should be useful for both multi-Higgs models involving electroweak doublets and multi-flavon models involving electroweak singlets, where in both cases the fields transform as triplets under some non-Abelian discrete symmetry.

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