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Golden Littlest Seesaw

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Abstract

We propose and analyse a new class of Littlest Seesaw models, with two right-handed neutrinos in their diagonal mass basis, based on preserving the first column of the Golden Ratio mixing matrix. We perform an exhaustive analysis of all possible remnant symmetries of the group A_5 which can be used to enforce various vacuum alignments for the flavon controlling solar mixing, for two simple cases of the atmospheric flavon vacuum alignment. The solar and atmospheric flavon vacuum alignments are enforced by *different* remnant symmetries. We examine the phenomenological viability of each of the possible Littlest Seesaw alignments in A_5, which preserve the first column of the Golden ratio mixing matrix, using figures and extensive tables of benchmark points and comparing our predictions to a recent global analysis of neutrino data. A benchmark model is constructed based on $A_5 \times Z_6 \times Z_5 \times Z'_5$.

1. Introduction

Massive neutrinos together with neutrino oscillations has been firmly established, and it is unique experimental evidence for physics beyond the standard model. All the three lepton mixing angles θ_{12}, θ_{13} and θ_{23} and the mass squared differences $\delta m^2 \equiv m_2^2 - m_1^2$ and

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$\Delta m^2 \equiv m_3^2 - (m_1^2 + m_2^2)/2$ has been precisely measured in a large number of neutrino oscillation experiments. At present the 3σ ranges of these mixing parameters are determined to be [1]

$$\begin{aligned}
&0.250 \leq \sin^2\theta_{12} \leq 0.354, \quad 0.0190 \leq \sin^2\theta_{13} \leq 0.0240, \quad 0.381 \leq \sin^2\theta_{23} \leq 0.615, \\
&6.93 \times 10^{-5}\ \mathrm{eV}^2 \leq \delta m^2 \leq 7.96 \times 10^{-5}\ \mathrm{eV}^2, \\
&2.411 \times 10^{-3}\ \mathrm{eV}^2 \leq \Delta m^2 \leq 2.646 \times 10^{-3}\ \mathrm{eV}^2 \,, \qquad (1.1)
\end{aligned}$$

for normal ordering (NO) neutrino mass spectrum, and similar results are obtained for inverted ordering (IO) spectrum. Non-Abelian discrete finite groups have been widely used to explain the lepton mixing angles as well as CP violating phases, see Refs. [2–7] for reviews.

The most appealing possibility for the origin of neutrino mass seems to be the seesaw mechanism which, in its original formulation, involves heavy right-handed Majorana neutrinos [8]. The most minimal version of the seesaw mechanism involves one [9] or two right-handed neutrinos [10]. In order to reduce the number of free parameters still further to the smallest number possible, and hence increase predictivity, various approaches to the two right-handed neutrino seesaw model have been suggested, such as postulating one [11] or two [12] texture zeroes, however such two texture zero models are now phenomenologically excluded [13] for the case of a normal neutrino mass hierarchy considered here.

The minimal successful seesaw scheme with normal hierarchy is called the Littlest Seesaw (LS) model [14–16], although in fact, it represents a class of models. The LS models may be defined as two right-handed neutrino models with particularly simple patterns of Dirac mass matrix elements in the basis where both the charged lepton mass matrix and the two-right-handed neutrino mass matrix are diagonal. The Dirac mass matrix typically involves only one texture zero, but the number of parameters is reduced dramatically since each column of this matrix is controlled by a single parameter. In practice this is achieved by introducing a Non-Abelian discrete family symmetry, which is spontaneously broken by flavon fields with particular vacuum alignments governed by remnant subgroups of the family symmetry. Unlike the direct symmetry approach, where a common residual flavour and remnant CP symmetry is assumed in the neutrino sector, the Littlest Seesaw approach assumes a *different* residual flavour symmetry is preserved by each flavon, in the diagonal mass basis of two right-handed neutrinos, leading to a highly predictive set of possible alignments.

For example, in the original LS model [14–16], the lepton mixing matrix is predicted to be of the TM1 form in which the first column of the tri-bimaximal mixing matrix is preserved, but with the reactor angle and CP phases fixed by the same two parameters which fix the neutrino masses. This leads to a highly constrained model which is remarkably consistent with current data, but which can be tested in forthcoming neutrino experiments [17]. The LS approach may also be incorporated into grand unified models [18]. The success of the LS approach, raises the question of whether it is confined to TM1 mixing, or is of more general applicability. The present paper aims to address this question by considering a different mixing scheme within the same approach, namely the golden ratio (GR) mixing pattern [19,20].

In this paper, we shall propose another viable class of LS models, namely the golden Littlest seesaw (GLS). Although the golden ratio mixing [19,20] is excluded by the measurement of largish reactor mixing angle, the first column of U_{GR} may still be compatible with the experimental data. Inspired by the success of the LS approach for TM1 mixing, we would like to also preserve the first column vector of the GR mixing pattern in our GLS model. We shall perform an exhaustive analysis of all possible remnant symmetries of the group A_5 which can be used to enforce various vacuum alignments for the flavon controlling solar mixing, for two simple cases of the atmospheric flavon vacuum alignment, analogous to the procedure suggested in the LS approach based on S_4. For each possibility we examine the phenomenological viability of the alignment, using figures and extensive benchmark points, comparing our predictions to a recent global analysis of neutrino data.

The layout of this paper is as follows. In section 2 we briefly review direct and indirect model building approach based on the group A_5. In section 3 we present the golden Littlest seesaw approach, where two right-handed neutrinos are introduced and the Dirac mass matrix are controlled by flavon vacuum alignments which respect various remnant symmetries of A_5. The phenomenological viability of each case for a discrete choice of phase parameters are examined. In section 4, we present a benchmark golden Littlest seesaw model based on the flavor symmetry $A_5 \times Z_6 \times Z_5 \times Z'_5$, the discrete group A_5 is spontaneously broken to desired residual subgroups G_l, $G_{\rm atm}$ and $G_{\rm sol}$ due to non-vanishing vacuum expectation values of some flavons. In addition, the model fixes the parameters $x = 2i\phi^2 \sin\frac{2\pi}{5}$ and $\eta = 0, \pi$. Section 1 concludes the paper. We report some details of the group theory of A_5 along with the Clebsch–Gordan coefficients in Appendix A. The second possible golden Littlest seesaw model and the resulting predictions for neutrino masses and lepton mixing angles are given in Appendix B.

2. Direct and indirect approaches in A_5 flavor symmetry

In order to understand more clearly the idea of the golden Littlest seesaw, we shall first recapitulate the direct approach to the golden ratio mixing from A_5 and the idea of Littlest seesaw. Before the measurement of the reactor mixing angle, the golden ratio (GR) mixing pattern [19, 20] was a good leading order approximation and it predicted a zero reactor angle $\theta_{13} = 0$, maximal atmospheric mixing angle $\theta_{23} = 45°$ and a solar mixing angle given by $\cot\theta_{12} = \phi$, where $\phi = (1+\sqrt{5})/2$ is the golden ratio. The explicit form of the golden mixing matrix is given by

$$U_{GR} = \begin{pmatrix} -\sqrt{\frac{\phi}{\sqrt{5}}} & \sqrt{\frac{1}{\sqrt{5}\phi}} & 0 \\ \sqrt{\frac{1}{2\sqrt{5}\phi}} & \sqrt{\frac{\phi}{2\sqrt{5}}} & -\frac{1}{\sqrt{2}} \\ \sqrt{\frac{1}{2\sqrt{5}\phi}} & \sqrt{\frac{\phi}{2\sqrt{5}}} & \frac{1}{\sqrt{2}} \end{pmatrix}. \tag{2.1}$$

We shall denote the three columns of U_{GR} as $\Phi_{1,2,3}$,

$$\Phi_1 = \sqrt{\frac{1}{2\sqrt{5}\phi}}\begin{pmatrix} -\sqrt{2}\phi \\ 1 \\ 1 \end{pmatrix}, \qquad \Phi_2 = \sqrt{\frac{1}{2\sqrt{5}\phi}}\begin{pmatrix} \sqrt{2} \\ \phi \\ \phi \end{pmatrix}, \qquad \Phi_3 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}. \tag{2.2}$$

The interplay between A_5 flavor symmetry and lepton mixing has been extensively studied in the literature [21–30]. In the direct approach of flavor symmetry model building, it has been shown that the golden ratio mixing pattern can be naturally reproduced [21] if the flavor group A_5 is broken to the Z_5^T subgroup in the charged lepton sector and to Klein subgroup $K_4^{(S,T^3ST^2ST^3)}$ in the neutrino sector. Here the superscript of a subgroup denotes its generator (or generators). The group theory of A_5 as well as its Clebsch–Gordan coefficients are listed in Appendix A. The A_5 group has two three-dimensional representations **3** and **3**′. We find that the representation matrices of the generators S and T in **3**′ exactly coincide with those of $T^3ST^2ST^3$ and T^2 respectively in **3**. This implies that the set of all matrices describing the representations **3** and **3**′ are the same. Therefore the same results would be obtained no matter if the left-handed leptons transform as **3** or **3**′ of A_5. Without loss of generality, we shall assign the three generations of left-handed leptons to the triplet **3** in the following.

The indirect model building approach [4] is an interesting alternative to the direct approach. In the indirect approach, the original flavor symmetry is completely broken in the neutrino sector, and the residual symmetry $Z_2 \times Z_2$ of the neutrino mass matrix arises accidentally. The basic idea of the indirect approach is to effectively promote the columns of the Dirac mass matrix to fields which transform as triplets under the flavour symmetry. We assume that the Dirac mass matrix can be written as $m_D = (a\Phi_{\text{atm}}, b\Phi_{\text{sol}}, c\Phi_{\text{dec}})$ where the columns are proportional to triplet Higgs or flavon fields with particular vacuum alignments and a, b, c are three constants of proportionality. It is convenient to work in the basis where the right-handed neutrino mass matrix are diagonal with the mass eigenvalues equal to M_{atm}, M_{sol} and M_{dec}. Then the light neutrino mass matrix given by the seesaw formula is

$$m_\nu = a^2 \frac{\Phi_{\text{atm}}\Phi_{\text{atm}}^T}{M_{\text{atm}}} + b^2 \frac{\Phi_{\text{sol}}\Phi_{\text{sol}}^T}{M_{\text{sol}}} + c^2 \frac{\Phi_{\text{dec}}\Phi_{\text{dec}}^T}{M_{\text{dec}}}, \tag{2.3}$$

where we have dropped an overall minus sign which is physically irrelevant. The lepton mixing matrix is exactly the golden mixing pattern U_{GR} for the alignment $\Phi_{\text{atm}} \propto \Phi_3$, $\Phi_{\text{sol}} \propto \Phi_2$ and $\Phi_{\text{dec}} \propto \Phi_1$ [31].

The Littlest seesaw combines two right-handed neutrinos model with the indirect approach [9]. In this framework, two right-handed neutrinos N_R^{atm} and N_R^{sol} are introduced, and the third right-handed neutrino is assumed to be almost decoupled (i.e. $M_{\text{dec}} \gg M_{\text{atm}}, M_{\text{sol}}$). N_R^{atm} dominantly contributes to the seesaw mechanism and is mainly responsible for the atmospheric neutrino mass m_3. N_R^{sol} is sub-dominant and is mainly responsible for the solar neutrino mass m_2 while the lightest neutrino mass m_1 is zero in this limit. In the flavor basis where both the charged lepton mass matrix and the right-handed neutrino Majorana mass matrix are diagonal, the generic Littlest seesaw Lagrangian can be written as

$$\mathcal{L} = -y_{\text{atm}}\bar{L}.\phi_{\text{atm}}N_R^{\text{atm}} - y_{\text{sol}}\bar{L}.\phi_{\text{sol}}N_R^{\text{sol}} - \frac{1}{2}M_{\text{atm}}\overline{(N_R^{\text{atm}})^c}N_R^{\text{atm}} - \frac{1}{2}M_{\text{sol}}\overline{(N_R^{\text{sol}})^c}N_R^{\text{sol}} + \text{h.c.}, \tag{2.4}$$

where L denotes the electroweak lepton doublets which are unified into a triplet representation of the flavor symmetry group, the flavons ϕ_{sol} and ϕ_{atm} can be either Higgs fields transforming as triplets under the flavour symmetry, or combinations of a single Higgs electroweak doublet together with triplet flavons. Then Φ_{atm} and Φ_{sol} in Eq. (2.3) arise from the vacuum expectation values (VEVs) of ϕ_{sol} and ϕ_{atm} respectively.

3. Golden Littlest seesaw

The Littlest seesaw approach [15] assumes that both vacuum alignments Φ_{sol} and Φ_{atm} are orthogonal to Φ_1, in order to preserve the first column of the mixing matrix. Thus we shall choose Φ_{atm} to be either Φ_2 or Φ_3, and take Φ_{sol} to be a general vector orthogonal to Φ_1, as illustrated in Fig. 1. Furthermore we shall fix the alignment of Φ_{sol} by appealing to remnant symmetry, which is a generalisation of the direct approach. To be more specific, we assume that the A_5 group is broken to the abelian subgroup $G_l = Z_5^T$ in the charged lepton sector, the vacuum alignments Φ_{atm} and Φ_{sol} preserve different residual symmetries G_{atm} and G_{sol} respectively while the A_5 flavor symmetry is completely broken in the entire neutrino sector. The GLS approach is

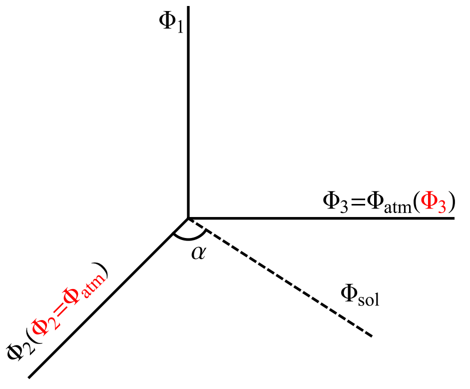

Fig. 1. The vacuum alignment in the Littlest seesaw model. Φ_1, Φ_2 and Φ_3 are the three columns of the golden ratio mixing matrix. The alignment vector $\Phi_{\rm atm}$ is either Φ_2 or Φ_3, and $\Phi_{\rm sol}$ is a general vector orthogonal to Φ_1.

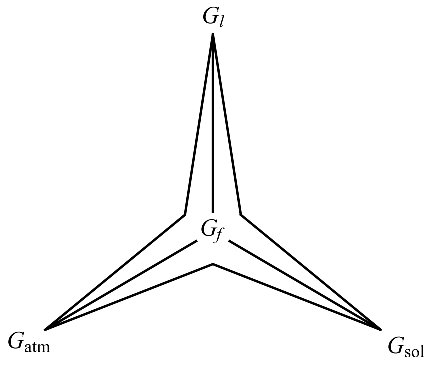

Fig. 2. A sketch of the indirect model building approach, where the charged lepton preserves a residual subgroup G_l, and the neutrino vacuum alignments $\Phi_{\rm atm}$ and $\Phi_{\rm sol}$ are enforced by the residual symmetries $G_{\rm atm}$ and $G_{\rm sol}$ respectively.

schematically illustrated in Fig. 2. In our GLS model, as stated above the alignment vector $\Phi_{\rm sol}$ is orthogonal to Φ_1, its most general form is

$$\Phi_{\rm sol} \propto \left(\sqrt{2},\ \phi + x,\ \phi - x\right)^T . \tag{3.1}$$

We find there are five possible values of x related to certain residual subgroups of A_5,

$$\begin{aligned}
&x = 0, \qquad G_{\rm sol} = Z_2^{T^3ST^2ST^3}, \\
&x = 2i\phi^2 \sin\frac{2\pi}{5}, \qquad G_{\rm sol} = Z_3^{ST^2ST^3}, \\
&x = -2i\phi^2 \sin\frac{2\pi}{5}, \qquad G_{\rm sol} = Z_3^{T^3ST^2S}, \\
&x = -2i \sin\frac{\pi}{5}, \qquad G_{\rm sol} = Z_5^{TST^2}, \\
&x = 2i \sin\frac{\pi}{5}, \qquad G_{\rm sol} = Z_5^{T^2ST},
\end{aligned} \tag{3.2}$$

where the generators of $G_{\rm sol}$ are represented by

$$T^3ST^2ST^3 = \frac{1}{\sqrt{5}}\begin{pmatrix} -1 & \sqrt{2} & \sqrt{2} \\ \sqrt{2} & -1/\phi & \phi \\ \sqrt{2} & \phi & -1/\phi \end{pmatrix},$$

$$ST^2ST^3 = \frac{1}{\sqrt{5}}\begin{pmatrix} -1 & \sqrt{2} & \sqrt{2} \\ \sqrt{2}e^{\frac{4i\pi}{5}} & e^{-\frac{i\pi}{5}}/\phi & e^{\frac{4i\pi}{5}}\phi \\ \sqrt{2}e^{-\frac{4i\pi}{5}} & e^{-\frac{4i\pi}{5}}\phi & e^{\frac{i\pi}{5}}/\phi \end{pmatrix},$$

$$T^3ST^2S = \frac{1}{\sqrt{5}}\begin{pmatrix} -1 & \sqrt{2}e^{\frac{4i\pi}{5}} & \sqrt{2}e^{-\frac{4i\pi}{5}} \\ \sqrt{2} & e^{-\frac{i\pi}{5}}/\phi & e^{-\frac{4i\pi}{5}}\phi \\ \sqrt{2} & e^{\frac{4i\pi}{5}}\phi & e^{\frac{i\pi}{5}}/\phi \end{pmatrix},$$

$$TST^2 = \frac{1}{\sqrt{5}}\begin{pmatrix} 1 & \sqrt{2}e^{-\frac{i\pi}{5}} & \sqrt{2}e^{\frac{i\pi}{5}} \\ \sqrt{2}e^{-\frac{3i\pi}{5}} & e^{\frac{i\pi}{5}}\phi & e^{-\frac{2i\pi}{5}}/\phi \\ \sqrt{2}e^{\frac{3i\pi}{5}} & e^{\frac{2i\pi}{5}}/\phi & e^{-\frac{i\pi}{5}}\phi \end{pmatrix}, \tag{3.3}$$

$$T^2ST = \frac{1}{\sqrt{5}}\begin{pmatrix} 1 & \sqrt{2}e^{-\frac{3i\pi}{5}} & \sqrt{2}e^{\frac{3i\pi}{5}} \\ \sqrt{2}e^{-\frac{i\pi}{5}} & e^{\frac{i\pi}{5}}\phi & e^{\frac{2i\pi}{5}}/\phi \\ \sqrt{2}e^{\frac{i\pi}{5}} & e^{-\frac{2i\pi}{5}}/\phi & e^{-\frac{i\pi}{5}}\phi \end{pmatrix}$$

for the triplet **3**. Accordingly the vacuum alignment of the solar flavon $\phi_{\rm sol}$ is:

$$\begin{aligned}
&x = 0, \qquad \Phi_{\rm sol} = \left(\sqrt{2},\ \phi,\ \phi\right)^T, \\
&x = 2i\phi^2\sin\frac{2\pi}{5}, \qquad \Phi_{\rm sol} = \left(\sqrt{2},\ 2\phi^2e^{2i\pi/5},\ 2\phi^2e^{-2i\pi/5}\right)^T, \\
&x = -2i\phi^2\sin\frac{2\pi}{5}, \qquad \Phi_{\rm sol} = \left(\sqrt{2},\ 2\phi^2e^{-2i\pi/5},\ 2\phi^2e^{2i\pi/5}\right)^T, \\
&x = -2i\sin\frac{\pi}{5}, \qquad \Phi_{\rm sol} = \left(\sqrt{2},\ 2e^{-i\pi/5},\ 2e^{i\pi/5}\right)^T, \\
&x = 2i\sin\frac{\pi}{5}, \qquad \Phi_{\rm sol} = \left(\sqrt{2},\ 2e^{i\pi/5},\ 2e^{-i\pi/5}\right)^T.
\end{aligned} \tag{3.4}$$

In our framework, another alignment vector $\Phi_{\rm atm}$ is assumed to be along the direction of Φ_3 or Φ_2. In the following, we shall firstly discuss the case of $\Phi_{\rm atm} \propto \Phi_3$, another alignment $\Phi_{\rm atm} \propto \Phi_2$ is studied in Appendix B. For this case, the vacuum $\Phi_{\rm atm}$ reads as

$$\Phi_{\rm atm} \propto (0,\ 1,\ -1)^T, \tag{3.5}$$

which is invariant under the action of the $Z_2^{T^3ST^2ST^3S}$ subgroup. Consequently the Dirac neutrino mass matrix M_D and the right-handed neutrino heavy Majorana mass matrix M_N are given by[1]

$$M_D = \begin{pmatrix} 0 & \sqrt{2}b \\ -a & (\phi - x)b \\ a & (\phi + x)b \end{pmatrix}, \qquad M_N = \begin{pmatrix} M_{\rm atm} & 0 \\ 0 & M_{\rm sol} \end{pmatrix} \tag{3.6}$$

[1] The contraction of two triplets into singlet is $(\alpha\beta) = \alpha_1\beta_1 + \alpha_2\beta_3 + \alpha_3\beta_2$, where $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ and $\beta = (\beta_1, \beta_2, \beta_3)$ denote two A_5 triplets.

Integrating out the right-handed neutrinos, the light effective Majorana neutrino mass matrix is approximately given by the seesaw formula

$$m_\nu = -M_D M_N^{-1} M_D^T$$
$$= m_a \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{pmatrix} + m_b e^{i\eta} \begin{pmatrix} 2 & \sqrt{2}(\phi - x) & \sqrt{2}(x+\phi) \\ \sqrt{2}(\phi - x) & (x-\phi)^2 & -x^2+\phi+1 \\ \sqrt{2}(x+\phi) & -x^2+\phi+1 & (x+\phi)^2 \end{pmatrix}, \quad (3.7)$$

where $m_a = |a|^2/M_{\text{atm}}$, $m_b = |b|^2/M_{\text{sol}}$, the relative phase $\eta = \arg(b^2/a^2)$, and an overall phase of m_ν has been omitted. Therefore four parameters m_a, m_b, x and η describe both the neutrino flavor mixing and neutrino masses. One can check that neutrino mass matrix m_ν of Eq. (3.7) satisfies

$$m_\nu \begin{pmatrix} -\sqrt{\frac{\phi}{\sqrt{5}}} \\ \sqrt{\frac{1}{2\sqrt{5}\phi}} \\ \sqrt{\frac{1}{2\sqrt{5}\phi}} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}. \quad (3.8)$$

This implies that the column vector $(-\sqrt{\frac{\phi}{\sqrt{5}}}, \sqrt{\frac{1}{2\sqrt{5}\phi}}, \sqrt{\frac{1}{2\sqrt{5}\phi}})^T$ is an eigenvector of m_ν with a zero eigenvalue. As a result, the first column of the PMNS mixing matrix exactly coincides with the GR mixing pattern, and the corresponding light neutrino mass vanishes $m_1 = 0$. In order to diagonalize the above neutrino mass matrix, we firstly perform a golden ratio transformation and obtain

$$m'_\nu = U_{GR}^T m_\nu U_{GR} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & y & z \\ 0 & z & w \end{pmatrix} \quad (3.9)$$

where

$$y = 2\sqrt{5}\,\phi m_b\, e^{i\eta},$$
$$z = 2x\sqrt{\phi+2}\, m_b\, e^{i\eta},$$
$$w = |w| e^{i\phi_w} = 2(m_a + x^2\, m_b\, e^{i\eta})\,. \quad (3.10)$$

The neutrino mass matrix m_ν in Eq. (3.9) by diagonalized through the standard procedure, as shown in Ref. [32]. We have

$$U'^T_\nu m'_\nu U'_\nu = \text{diag}(0, m_2, m_3)\,, \quad (3.11)$$

where the unitary matrix U'_ν can be written as

$$U'_\nu = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta\, e^{i(\psi+\rho)/2} & \sin\theta\, e^{i(\psi+\sigma)/2} \\ 0 & -\sin\theta\, e^{i(-\psi+\rho)/2} & \cos\theta\, e^{i(-\psi+\sigma)/2} \end{pmatrix}. \quad (3.12)$$

We find the light neutrino masses $m_{2,3}$ are

$$m_2^2 = \frac{1}{2}\left[|y|^2 + |w|^2 + 2|z|^2 - \frac{|w|^2 - |y|^2}{\cos 2\theta}\right],$$
$$m_3^2 = \frac{1}{2}\left[|y|^2 + |w|^2 + 2|z|^2 + \frac{|w|^2 - |y|^2}{\cos 2\theta}\right] \quad (3.13)$$

The rotation angle θ is determined to be

$$\sin 2\theta = \frac{-2iz\,e^{-i\eta}\sqrt{|y|^2+|w|^2-2|y||w|\cos(\phi_w-\eta)}}{\sqrt{(|w|^2-|y|^2)^2+4|z|^2\left[|y|^2+|w|^2-2|y||w|\cos(\phi_w-\eta)\right]}},$$
$$\cos 2\theta = \frac{|w|^2-|y|^2}{\sqrt{(|w|^2-|y|^2)^2+4|z|^2\left[|y|^2+|w|^2-2|y||w|\cos(\phi_w-\eta)\right]}}. \tag{3.14}$$

The phases ψ, ρ and σ are given by

$$\sin\psi = \frac{|y|-|w|\cos(\phi_w-\eta)}{\sqrt{|y|^2+|w|^2-2|y||w|\cos(\phi_w-\eta)}},$$
$$\cos\psi = \frac{|w|\sin(\phi_w-\eta)}{\sqrt{|y|^2+|w|^2-2|y||w|\cos(\phi_w-\eta)}},$$
$$\sin\rho = -\frac{(m_2^2-|z|^2)\cos\eta-|y||w|\cos\phi_w}{m_2\sqrt{|y|^2+|w|^2-2|y||w|\cos(\phi_w-\eta)}},$$
$$\cos\rho = \frac{-(m_2^2-|z|^2)\sin\eta+|y||w|\sin\phi_w}{m_2\sqrt{|y|^2+|w|^2-2|y||w|\cos(\phi_w-\eta)}},$$
$$\sin\sigma = -\frac{(m_3^2-|z|^2)\cos\eta-|y||w|\cos\phi_w}{m_3\sqrt{|y|^2+|w|^2-2|y||w|\cos(\phi_w-\eta)}},$$
$$\cos\sigma = \frac{-(m_3^2-|z|^2)\sin\eta+|y||w|\sin\phi_w}{m_3\sqrt{|y|^2+|w|^2-2|y||w|\cos(\phi_w-\eta)}}. \tag{3.15}$$

Thus the lepton mixing matrix is determined to be

$$U = U_{GR}U'_\nu = \sqrt{\frac{1}{2\sqrt{5}\phi}}\begin{pmatrix} -\sqrt{2}\phi & \sqrt{2}\cos\theta & \sqrt{2}e^{i\psi}\sin\theta \\ 1 & \phi\cos\theta+\sqrt{\phi+2}\sin\theta\,e^{-i\psi} & \phi\sin\theta\,e^{i\psi}-\sqrt{\phi+2}\cos\theta \\ 1 & \phi\cos\theta-\sqrt{\phi+2}\sin\theta\,e^{-i\psi} & \phi\sin\theta\,e^{i\psi}+\sqrt{\phi+2}\cos\theta \end{pmatrix}P_\nu\,, \tag{3.16}$$

with

$$P_\nu = \mathrm{diag}(1, e^{i(\psi+\rho)/2}, e^{i(-\psi+\sigma)/2})\,. \tag{3.17}$$

The most general leptonic mixing matrix in the two right-handed neutrino model can be parameterized as

$$U = \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta_{CP}} \\ -s_{12}c_{23}-c_{12}s_{13}s_{23}e^{i\delta_{CP}} & c_{12}c_{23}-s_{12}s_{13}s_{23}e^{i\delta_{CP}} & c_{13}s_{23} \\ s_{12}s_{23}-c_{12}s_{13}c_{23}e^{i\delta_{CP}} & -c_{12}s_{23}-s_{12}s_{13}c_{23}e^{i\delta_{CP}} & c_{13}c_{23} \end{pmatrix}\mathrm{diag}(1, e^{i\frac{\beta}{2}}, 1), \tag{3.18}$$

where $c_{ij} \equiv \cos\theta_{ij}$, $s_{ij} \equiv \sin\theta_{ij}$, δ_{CP} is the Dirac CP violation phase and β is the Majorana CP phase. Note that a second Majorana phase is needed if the lightest neutrino is not massless. Then we can extract the expressions for the lepton mixing angles as follows

$$\sin^2\theta_{13} = \frac{\sin^2\theta}{\sqrt{5}\phi}, \quad \sin^2\theta_{12} = \frac{\cos^2\theta}{\sqrt{5}\phi - \sin^2\theta},$$
$$\sin^2\theta_{23} = \frac{1}{2} - \frac{\sqrt{3+4\phi}\,\sin 2\theta\cos\psi}{2(\sqrt{5}\phi - \sin^2\theta)}. \tag{3.19}$$

Eliminating the free parameter θ, we see that a sum rule between the solar mixing angle θ_{12} and the reactor mixing angle θ_{13} is satisfied,

$$\cos^2\theta_{12}\cos^2\theta_{13} = \frac{\phi}{\sqrt{5}}. \tag{3.20}$$

Using the best fit value of $\sin^2\theta_{13} = 0.0215$, we find for the solar mixing angle

$$\sin^2\theta_{12} \simeq 0.261, \tag{3.21}$$

which is within the 3σ region [1]. As regards the CP violation, two weak basis invariants J_{CP} [33] and I_1 [34] associated with the CP phases δ_{CP} and β respectively can be defined,

$$J_{CP} = \Im(U_{11}U_{33}U^*_{13}U^*_{31}) = \frac{1}{8}\sin 2\theta_{12}\sin 2\theta_{13}\sin 2\theta_{23}\cos\theta_{13}\sin\delta_{CP},$$
$$I_1 = \Im(U^2_{12}U^{*2}_{13}) = \frac{1}{4}\sin^2\theta_{12}\sin^2 2\theta_{13}\sin(\beta + 2\delta_{CP}). \tag{3.22}$$

For the mixing pattern in Eq. (3.18), these CP invariants turn out to be

$$J_{CP} = \frac{\sin 2\theta\sin\psi}{4\sqrt{5(\phi+2)}}, \qquad I_1 = \frac{1}{20\phi^2}\sin^2 2\theta\sin(\rho - \sigma). \tag{3.23}$$

Since J_{CP} and all the three mixing angles depend on only two parameters θ and ψ, we can derive the following sum rule among the Dirac CP phase δ_{CP} and mixing angles

$$\cos\delta_{CP} = \frac{(\phi+2)(1+\sin^2\theta_{13}) - 5\cos^2\theta_{13}}{2\sqrt{(\phi+2)(5\cos^2\theta_{13} - \phi - 2)}}\csc\theta_{13}\cot 2\theta_{23}. \tag{3.24}$$

For maximal atmospheric mixing angle $\theta_{23} = \pi/4$, this sum rule predicts $\cos\delta_{CP} = 0$ which corresponds to maximal CP violation $\delta_{CP} = \pm\pi/2$. The mixing angles, CP phases and mass ratio m_2/m_3 depend on the x, η and $r \equiv m_b/m_a$ while m_2 and m_3 depend on all the four input parameters x, η, m_a and m_b. By comprehensively scanning over the parameter space of η and r, we find that the experimental data on the mixing angles and mass squared splittings can be accommodated only for the values of $x = \pm 2i\phi^2\sin\frac{2\pi}{5}$. In Table 1 we present the predictions for the mixing angles and CP violation phases for some benchmark values of the parameters η and r. It is remarkable that both atmospheric mixing angle and Dirac phase are maximal for $\eta = 0$, all the mixing angles and mass ratio m^2_2/m^2_3 lie in the experimentally preferred 3σ ranges except that the reactor angle θ_{13} is a bit smaller. This tiny discrepancy is expected to be easily resolved in an explicit model with small corrections or by the renormalization group corrections [35]. Notice that the same predictions for the mixing angles and maximal δ_{CP} can be obtained from the approach of combining A_5 flavor symmetry with generalized CP [22,29,30], but we have additional prediction for the neutrino masses here even if the CP symmetry is not introduced in the present context. We can check that the neutrino mass matrix m_ν in Eq. (3.7) has the following symmetry properties

$$m_\nu(\eta, x = \pm 2i\phi^2\sin 2\pi/5) = P^T_{23}m_\nu(\eta, x = \mp 2i\phi^2\sin 2\pi/5)P_{23},$$
$$m_\nu(\eta, x = \pm 2i\phi^2\sin 2\pi/5) = m^*_\nu(-\eta, x = \mp 2i\phi^2\sin 2\pi/5), \tag{3.25}$$

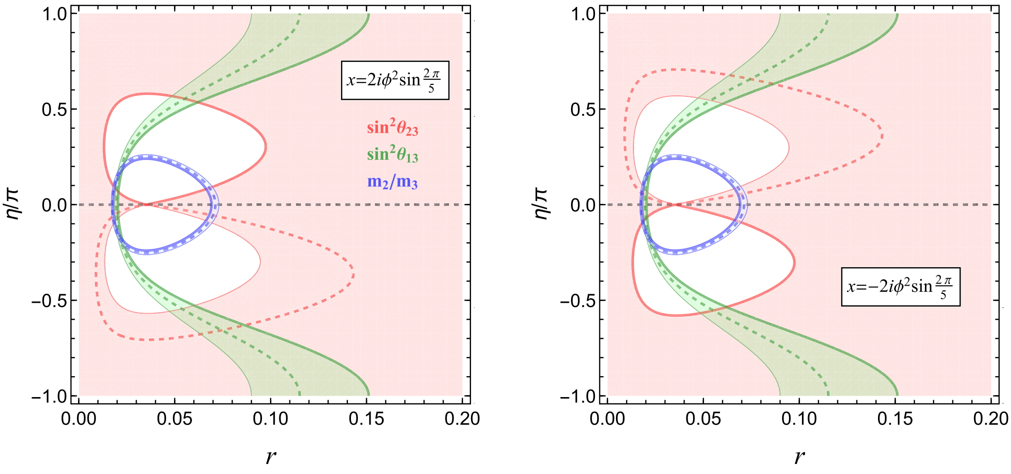

Fig. 3. Contour plots of $\sin^2\theta_{13}$, $\sin^2\theta_{23}$ and m_2/m_3 in the $\eta - r$ plane for the golden Littlest seesaw with $\Phi_{\text{atm}} \propto \Phi_3$. Here we take $x = 2i\phi^2\sin(2\pi/5)$ and $x = -2i\phi^2\sin(2\pi/5)$ for which the solar vacuum alignment Φ_{sol} preserves the residual symmetry $G_{\text{sol}} = Z_3^{ST^2ST^3}$ and $G_{\text{sol}} = Z_3^{T^3ST^2S}$ respectively. The 3σ upper (lower) bounds of the lepton mixing angles are labelled with thick (thin) solid curves, and the dashed contour lines represent the corresponding best fit values. The 3σ ranges as well as the best fit values of the mixing angles are adapted from [1]. The black contour line refers to maximal atmospheric mixing angle with $\sin^2\theta_{23} = 0.5$.

with

$$P_{23} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} . \tag{3.26}$$

As a consequence, the same reactor and solar mixing angles are obtained for $x = 2i\phi^2\sin\frac{2\pi}{5}$ and $x = -2i\phi^2\sin\frac{2\pi}{5}$, while the atmospheric angle changes from θ_{23} to $\pi/2 - \theta_{23}$ and the Dirac phase changes from δ_{CP} to $\pi + \delta_{CP}$. Moreover, all the lepton mixing angles are kept intact and the signs of all CP violation phases are reversed under the transformation $x \to -x$ and $\eta \to -\eta$. For the fixed value of $x = \pm 2i\phi^2\sin\frac{2\pi}{5}$, all the mixing angles, CP phases and mass ratio m_2^2/m_3^2 are fully determined by r and η, and the correct neutrino mass m_2 can be achieved for certain values of m_b. We show how these mixing parameters vary in the plane η versus r in Fig. 3. It can be seen that the measured values of the mixing angles and the neutrino masses can be accommodated for certain choices of η and r.

4. A benchmark model for the golden Littlest seesaw

We have tabulated many simple admissible values of η in Table 1. It is remarkable that the trivial value $\eta = 0$ leads to exactly maximal atmospheric mixing angle and maximal Dirac CP phase which is preferred by the present experimental data from T2K [36] and NOvA [37], the experimental data on solar and reactor angles and neutrino masses can be accommodated as well. In this section, we shall present a supersymmetric A_5 model which can realize the above golden Littlest seesaw scheme. In order to understand the origin of the phase $\eta = 0$ we shall also impose a CP symmetry compatible with A_5 at high energy scale. The CP transformations can not be defined arbitrarily in the presence of a flavor symmetry, and certain consistency conditions have

Table 1
Predictions for all the lepton mixing angles, CP violation phases and m_2^2/m_3^2 in the golden Littlest seesaw with $\Phi_{\text{atm}} \propto \Phi_3$. Here we choose many benchmark values for the parameters η and r.

η	r	x	$\sin^2\theta_{13}$	$\sin^2\theta_{12}$	$\sin^2\theta_{23}$	δ_{CP}/π	β/π	m_2^2/m_3^2
0	0.0177	$\pm 2i\phi^2 \sin\frac{2\pi}{5}$	0.0164	0.264	0.5	∓ 0.5	0	0.0309
$\pm\frac{\pi}{11}$	0.0185	$\pm 2i\phi^2 \sin\frac{2\pi}{5}$	0.0174	0.264	0.614	∓ 0.331	∓ 0.210	0.0302
$\pm\frac{\pi}{11}$	0.0185	$\mp 2i\phi^2 \sin\frac{2\pi}{5}$	0.0175	0.264	0.385	± 0.670	∓ 0.211	0.0304
$\pm\frac{\pi}{12}$	0.0183	$\pm 2i\phi^2 \sin\frac{2\pi}{5}$	0.0172	0.264	0.605	∓ 0.345	∓ 0.192	0.0303
$\pm\frac{\pi}{12}$	0.0184	$\mp 2i\phi^2 \sin\frac{2\pi}{5}$	0.0173	0.264	0.394	± 0.655	∓ 0.192	0.0305
$\pm\frac{\pi}{13}$	0.0182	$\pm 2i\phi^2 \sin\frac{2\pi}{5}$	0.0171	0.264	0.597	∓ 0.357	∓ 0.176	0.0304
$\pm\frac{\pi}{13}$	0.0183	$\mp 2i\phi^2 \sin\frac{2\pi}{5}$	0.0172	0.264	0.402	± 0.643	∓ 0.177	0.0306
$\pm\frac{\pi}{14}$	0.0182	$\pm 2i\phi^2 \sin\frac{2\pi}{5}$	0.0170	0.264	0.591	∓ 0.368	∓ 0.163	0.0304
$\pm\frac{\pi}{14}$	0.0182	$\mp 2i\phi^2 \sin\frac{2\pi}{5}$	0.0171	0.264	0.409	± 0.632	∓ 0.164	0.0307
$\pm\frac{\pi}{15}$	0.0181	$\pm 2i\phi^2 \sin\frac{2\pi}{5}$	0.0169	0.264	0.585	∓ 0.377	∓ 0.152	0.0305
$\pm\frac{\pi}{15}$	0.0181	$\mp 2i\phi^2 \sin\frac{2\pi}{5}$	0.0170	0.264	0.415	± 0.623	∓ 0.152	0.0307
$\pm\frac{\pi}{16}$	0.0180	$\pm 2i\phi^2 \sin\frac{2\pi}{5}$	0.0168	0.264	0.580	∓ 0.385	∓ 0.142	0.0305
$\pm\frac{\pi}{16}$	0.0181	$\mp 2i\phi^2 \sin\frac{2\pi}{5}$	0.0169	0.264	0.420	± 0.616	∓ 0.142	0.0308
$\pm\frac{\pi}{17}$	0.0180	$\pm 2i\phi^2 \sin\frac{2\pi}{5}$	0.0168	0.264	0.575	∓ 0.391	∓ 0.134	0.0306
$\pm\frac{\pi}{17}$	0.0180	$\mp 2i\phi^2 \sin\frac{2\pi}{5}$	0.0169	0.264	0.425	± 0.609	∓ 0.134	0.0308
$\pm\frac{\pi}{18}$	0.0180	$\pm 2i\phi^2 \sin\frac{2\pi}{5}$	0.0167	0.264	0.571	∓ 0.398	∓ 0.126	0.0306
$\pm\frac{\pi}{18}$	0.0180	$\mp 2i\phi^2 \sin\frac{2\pi}{5}$	0.0168	0.264	0.429	± 0.603	∓ 0.126	0.0308
$\pm\frac{\pi}{19}$	0.0179	$\pm 2i\phi^2 \sin\frac{2\pi}{5}$	0.0167	0.264	0.567	∓ 0.403	∓ 0.119	0.0306
$\pm\frac{\pi}{19}$	0.0180	$\mp 2i\phi^2 \sin\frac{2\pi}{5}$	0.0168	0.264	0.432	± 0.597	∓ 0.119	0.0308
$\pm\frac{\pi}{20}$	0.0179	$\pm 2i\phi^2 \sin\frac{2\pi}{5}$	0.0167	0.264	0.564	∓ 0.408	∓ 0.113	0.0306
$\pm\frac{\pi}{20}$	0.0179	$\mp 2i\phi^2 \sin\frac{2\pi}{5}$	0.0167	0.264	0.436	± 0.592	∓ 0.113	0.0308
$\pm\frac{\pi}{21}$	0.0179	$\pm 2i\phi^2 \sin\frac{2\pi}{5}$	0.0166	0.264	0.561	∓ 0.412	∓ 0.108	0.0306
$\pm\frac{\pi}{21}$	0.0179	$\mp 2i\phi^2 \sin\frac{2\pi}{5}$	0.0167	0.264	0.439	± 0.588	∓ 0.108	0.0308
$\pm\frac{\pi}{22}$	0.0179	$\pm 2i\phi^2 \sin\frac{2\pi}{5}$	0.0166	0.264	0.558	∓ 0.416	∓ 0.103	0.0307
$\pm\frac{\pi}{22}$	0.0179	$\mp 2i\phi^2 \sin\frac{2\pi}{5}$	0.0167	0.264	0.442	± 0.584	∓ 0.103	0.0309
$\pm\frac{\pi}{23}$	0.0179	$\pm 2i\phi^2 \sin\frac{2\pi}{5}$	0.0166	0.264	0.556	∓ 0.420	∓ 0.0982	0.0307
$\pm\frac{\pi}{23}$	0.0179	$\mp 2i\phi^2 \sin\frac{2\pi}{5}$	0.0167	0.264	0.444	± 0.580	∓ 0.0983	0.0309
$\pm\frac{2\pi}{23}$	0.0184	$\pm 2i\phi^2 \sin\frac{2\pi}{5}$	0.0173	0.264	0.610	∓ 0.338	∓ 0.201	0.0302
$\pm\frac{2\pi}{23}$	0.0184	$\mp 2i\phi^2 \sin\frac{2\pi}{5}$	0.0174	0.264	0.390	± 0.662	∓ 0.201	0.0305

to be satisfied [38–41]. It turns out that the viable CP transformations which can be consistently combined with A_5 flavor symmetry are of the same form as the flavor symmetry transformations in our working basis [22]. The model employs an auxiliary symmetry $Z_6 \times Z_5 \times Z_5'$ which are necessary to eliminate unwanted couplings, to ensure the needed vacuum alignment and to reproduce the observed charged lepton mass hierarchies. The three families of the electroweak lepton doublets L are unified into a triplet of A_5 while the right-handed charged leptons e^c, μ^c, τ^c, the right-handed neutrinos ν^c_{atm}, ν^c_{sol} and the two Higgs doublets H_u, H_d are all singlets of A_5. The relevant flavon fields and how they transform under the flavor symmetry $A_5 \times Z_6 \times Z_5 \times Z_5'$ are collected in Table 2. We shall consider three different sets of flavons, one responsible for the breaking to G_l, one for the breaking to G_{atm} and one for the breaking to G_{sol}. Now we proceed to

Table 2
The lepton, Higgs and flavon superfields and their transformation properties under the flavor symmetry $A_5 \times Z_6 \times Z_5 \times Z'_5$, where $\omega_5 \equiv e^{2\pi i/5}$ and $\omega_6 \equiv e^{2\pi i/6}$. In addition, we assume a standard $U(1)_R$ symmetry under which all lepton fields carry a unit charge while the Higgs and flavons have zero charge.

	L	e^c	μ^c	τ^c	$\nu^c_{\rm atm}$	$\nu^c_{\rm sol}$	$H_{u,d}$	ϕ_l	φ_l	ψ_l	$\Delta_{\rm atm}$	$\varphi_{\rm atm}$	$\phi_{\rm atm}$	$\xi_{\rm atm}$	$\phi_{\rm sol}$	$\chi_{\rm sol}$	$\xi_{\rm sol}$
A_5	**3**	**1**	**1**	**1**	**1**	**1**	**1**	**3**	**3′**	**5**	**4**	**3′**	**3**	**1**	**3**	**5**	**1**
Z_6	1	ω_6	ω_6^2	ω_6^5	1	1	1	ω_6	ω_6^2	ω_6^2	1	1	1	1	1	1	1
Z_5	1	1	ω_5^4	ω_5	ω_5	1	1	ω_5^4	ω_5^3	ω_5^3	ω_5^3	ω_5^4	ω_5^4	ω_5^3	1	1	1
Z'_5	1	1	1	1	1	ω_5	1	1	1	1	1	1	1	1	ω_5^4	ω_5^3	ω_5^3

Table 3
The driving fields and their transformation properties under the flavor symmetry $A_5 \times Z_6 \times Z_5 \times Z'_5$, where $\omega_5 \equiv e^{2\pi i/5}$ and $\omega_6 \equiv e^{2\pi i/6}$.

	σ^0	η^0	ψ^0	Δ^0	Δ'^0	$\phi^0_{\rm atm}$	χ^0	χ'^0
A_5	**1**	**4**	**5**	**4**	**4**	**3**	**5**	**5**
Z_6	ω_6^4	ω_6^3	ω_6^4	1	1	1	1	1
Z_5	ω_5^2	ω_5^3	ω_5^2	ω_5^4	ω_5^2	ω_5^3	1	1
Z'_5	1	1	1	1	1	1	ω_5^2	ω_5^4

show that the desired vacua $\Phi_{\rm atm} \propto (0, -1, 1)^T$ and $\Phi_{\rm sol} \propto \left(\sqrt{2}, 2\phi^2 e^{-2i\pi/5}, 2\phi^2 e^{2i\pi/5}\right)^T$ can be really accomplished.

4.1. Vacuum alignment

We shall exploit the supersymmetric F-term alignment mechanism to generate the appropriate vacuum alignments of the flavor symmetry breaking flavons. The necessary driving fields and their transformation properties under $A_5 \times Z_6 \times Z_5 \times Z'_5$ are given in Table 3. The driving fields are indicated with the superscript "0" and they carry two unit $U(1)_R$ charge. The assignments of the flavon and driving fields are properly chosen such that the vacua of the charged lepton flavons, atmospheric neutrino flavons and solar neutrino flavons are aligned separately at the renormalizable level.

We can read out the most general renormalizable driving superpotential invariant under $A_5 \times Z_6 \times Z_5 \times Z'_5$ as follows,

$$w_d = w_d^l + w_d^{\rm atm} + w_d^{\rm sol}\,, \tag{4.1}$$

with

$$\begin{aligned}
w_d^l &= M_\psi \left(\psi^0 \psi_l\right) + f_1 \left(\psi^0 \phi_l \phi_l\right) + f_2 \left(\sigma^0 \phi_l \phi_l\right) + f_3 \left(\eta^0 \phi_l \varphi_l\right) + f_4 \left(\eta^0 \phi_l \psi_l\right)\,,\\
w_d^{\rm atm} &= g_1 \left(\Delta^0 \Delta_{\rm atm} \Delta_{\rm atm}\right) + g_2 \left(\Delta^0 \Delta_{\rm atm}\right) \xi_{\rm atm} + M_\Delta \left(\Delta'^0 \Delta_{\rm atm}\right) + g_3 \left(\Delta'^0 \phi_{\rm atm} \varphi_{\rm atm}\right)\\
&\quad + g_4 \left(\phi^0_{\rm atm} \phi_{\rm atm}\right) \xi_{\rm atm} + g_5 \left(\phi^0_{\rm atm} \Delta_{\rm atm} \varphi_{\rm atm}\right)\,,\\
w_d^{\rm sol} &= M_\chi \left(\chi^0 \chi_{\rm sol}\right) + h_1 \left(\chi^0 \phi_{\rm sol} \phi_{\rm sol}\right) + h_2 \left(\chi'^0 \chi_{\rm sol}\right) \xi_{\rm sol} + h_3 \left(\chi'^0 \left(\chi_{\rm sol} \chi_{\rm sol}\right)_{\mathbf{5_1}}\right)\\
&\quad + h_4 \left(\chi'^0 \left(\chi_{\rm sol} \chi_{\rm sol}\right)_{\mathbf{5_2}}\right)\,,
\end{aligned} \tag{4.2}$$

where we indicate with $(\ldots)$ the contraction into a trivial singlet **1** and $(\ldots)_{\mathbf{R}}$ a contraction into the A_5 irreducible representation **R**. Notice that the mass parameters M_ψ, M_Δ, M_χ and all coupling constants f_i, g_i, h_i are real because we impose CP as a symmetry of the model. The vacuum configuration of the charged lepton flavons ϕ_l, φ_l and ψ_l is determined by the superpotential w_d^l, and the corresponding F-term conditions are

$$\begin{aligned}
\frac{\partial w_d^l}{\partial \sigma^0} &= f_2\left(\phi_{l,1}^2 + 2\phi_{l,2}\phi_{l,3}\right) = 0,\\
\frac{\partial w_d^l}{\partial \psi_1^0} &= M_\psi \psi_{l,1} + 2 f_1\left(\phi_{l,1}^2 - \phi_{l,2}\phi_{l,3}\right) = 0\\
\frac{\partial w_d^l}{\partial \psi_2^0} &= M_\psi \psi_{l,5} - 2\sqrt{3} f_1 \phi_{l,1}\phi_{l,3} = 0\\
\frac{\partial w_d^l}{\partial \psi_3^0} &= M_\psi \psi_{l,4} + \sqrt{6} f_1 \phi_{l,3}^2 = 0\\
\frac{\partial w_d^l}{\partial \psi_4^0} &= M_\psi \psi_{l,3} + \sqrt{6} f_1 \phi_{l,2}^2 = 0\\
\frac{\partial w_d^l}{\partial \psi_5^0} &= M_\psi \psi_{l,2} - 2\sqrt{3} f_1 \phi_{l,1}\phi_{l,2} = 0\\
\frac{\partial w_d^l}{\partial \eta_1^0} &= f_3\left(\sqrt{2}\phi_{l,3}\varphi_{l,1} + \phi_{l,2}\varphi_{l,3}\right) + f_4\left(-2\sqrt{2}\phi_{l,1}\psi_{l,5} - \phi_{l,2}\psi_{l,4} + \sqrt{6}\phi_{l,3}\psi_{l,1}\right) = 0,\\
\frac{\partial w_d^l}{\partial \eta_2^0} &= f_3\left(-\sqrt{2}\phi_{l,1}\varphi_{l,3} - \phi_{l,2}\varphi_{l,2}\right) + f_4\left(\sqrt{2}\phi_{l,1}\psi_{l,4} + 3\phi_{l,2}\psi_{l,3} - 2\phi_{l,3}\psi_{l,5}\right) = 0,\\
\frac{\partial w_d^l}{\partial \eta_3^0} &= f_3\left(-\sqrt{2}\phi_{l,1}\varphi_{l,2} - \phi_{l,3}\varphi_{l,3}\right) + f_4\left(-\sqrt{2}\phi_{l,1}\psi_{l,3} + 2\phi_{l,2}\psi_{l,2} - 3\phi_{l,3}\psi_{l,4}\right) = 0,\\
\frac{\partial w_d^l}{\partial \eta_4^0} &= f_3\left(\sqrt{2}\phi_{l,2}\varphi_{l,1} + \phi_{l,3}\varphi_{l,2}\right) + f_4\left(2\sqrt{2}\phi_{l,1}\psi_{l,2} - \sqrt{6}\phi_{l,2}\psi_{l,1} + \phi_{l,3}\psi_{l,3}\right) = 0.
\end{aligned} \tag{4.3}$$

The solution to these equations is

$$\langle\phi_l\rangle = \left(0, v_{\phi_l}, 0\right), \quad \langle\varphi_l\rangle = \left(0, v_{\varphi_l}, 0\right), \quad \langle\psi_l\rangle = \left(0, 0, v_{\psi_l}, 0, 0\right), \tag{4.4}$$

where

$$v_{\varphi_l} = -\frac{3\sqrt{6} f_1 f_4 v_{\phi_l}^2}{f_3 M_\psi}, \qquad v_\psi = -\frac{\sqrt{6} f_1 v_{\phi_l}^2}{M_\psi}, \tag{4.5}$$

with v_{ϕ_l} undetermined. We notice that the alignments of ϕ_l, φ_l and ψ_l don't change their directions under the transformation of the generator T, they pick up a phase factor ω_5, ω_5^2 and ω_5^2 respectively. However, these three directions preserve the symmetry Z_5^D which is the diagonal subgroup of $Z_5^T \subset A_5$ and the auxiliary Z_5 symmetry. As we shall see below, this residual subgroup Z_5^D is responsible for guaranteeing a diagonal charged lepton mass matrix. In a similar way, the F-terms of the driving field Δ^0 are given by

$$\begin{aligned}
\frac{\partial w_d^{\rm atm}}{\partial \Delta_1^0} &= g_1\left(\Delta_{{\rm atm},2}^2+2\Delta_{{\rm atm},1}\Delta_{{\rm atm},3}\right)+g_2\xi_{\rm atm}\Delta_{{\rm atm},4}=0\,,\\
\frac{\partial w_d^{\rm atm}}{\partial \Delta_2^0} &= g_1\left(\Delta_{{\rm atm},4}^2+2\Delta_{{\rm atm},1}\Delta_{{\rm atm},2}\right)+g_2\xi_{\rm atm}\Delta_{{\rm atm},3}=0\,,\\
\frac{\partial w_d^{\rm atm}}{\partial \Delta_3^0} &= g_1\left(\Delta_{{\rm atm},1}^2+2\Delta_{{\rm atm},3}\Delta_{{\rm atm},4}\right)+g_2\xi_{\rm atm}\Delta_{{\rm atm},2}=0\,,\\
\frac{\partial w_d^{\rm atm}}{\partial \Delta_4^0} &= g_1\left(\Delta_{{\rm atm},3}^2+2\Delta_{{\rm atm},2}\Delta_{{\rm atm},4}\right)+g_2\xi_{\rm atm}\Delta_{{\rm atm},1}=0\,,
\end{aligned}\tag{4.6}$$

which lead to the vacuum alignments of Δ and $\xi_{\rm atm}$ as follow

$$\langle\Delta_{\rm atm}\rangle=\left(v_{\Delta_{\rm atm}},v_{\Delta_{\rm atm}},v_{\Delta_{\rm atm}},v_{\Delta_{\rm atm}}\right),\qquad \langle\xi_{\rm atm}\rangle=v_{\xi_{\rm atm}},\qquad v_{\Delta_{\rm atm}}=-\frac{g_2}{3g_1}v_{\xi_{\rm atm}}\,.\tag{4.7}$$

The minimization equations for the atmospheric neutrino flavons are given by

$$\begin{aligned}
\frac{\partial w_d^{\rm atm}}{\partial \Delta_1'^0} &= M_\Delta\Delta_{{\rm atm},4}+g_3\left(\phi_{{\rm atm},2}\varphi_{{\rm atm},3}+\sqrt{2}\phi_{{\rm atm},3}\varphi_{{\rm atm},1}\right)=0\,,\\
\frac{\partial w_d^{\rm atm}}{\partial \Delta_2'^0} &= M_\Delta\Delta_{{\rm atm},3}+g_3\left(-\phi_{{\rm atm},2}\varphi_{{\rm atm},2}-\sqrt{2}\phi_{{\rm atm},1}\varphi_{{\rm atm},3}\right)=0\,,\\
\frac{\partial w_d^{\rm atm}}{\partial \Delta_3'^0} &= M_\Delta\Delta_{{\rm atm},2}+g_3\left(-\phi_{{\rm atm},3}\varphi_{{\rm atm},3}-\sqrt{2}\phi_{{\rm atm},1}\varphi_{{\rm atm},2}\right)=0\,,\\
\frac{\partial w_d^{\rm atm}}{\partial \Delta_4'^0} &= M_\Delta\Delta_{{\rm atm},1}+g_3\left(\phi_{{\rm atm},3}\varphi_{{\rm atm},2}+\sqrt{2}\phi_{{\rm atm},2}\varphi_{{\rm atm},1}\right)=0\,,\\
\frac{\partial w_d^{\rm atm}}{\partial \phi_{{\rm atm},1}^0} &= g_4\xi_{\rm atm}\phi_{{\rm atm},1}-\sqrt{2}g_5\left(\Delta_{{\rm atm},2}\varphi_{{\rm atm},3}+\Delta_{{\rm atm},3}\varphi_{{\rm atm},2}\right)=0\,,\\
\frac{\partial w_d^{\rm atm}}{\partial \phi_{{\rm atm},2}^0} &= g_4\xi_{\rm atm}\phi_{{\rm atm},3}+g_5\left(\Delta_{{\rm atm},1}\varphi_{{\rm atm},3}-\Delta_{{\rm atm},2}\varphi_{{\rm atm},2}+\sqrt{2}\Delta_{{\rm atm},4}\varphi_{{\rm atm},1}\right)=0\,,\\
\frac{\partial w_d^{\rm atm}}{\partial \phi_{{\rm atm},3}^0} &= g_4\xi_{\rm atm}\phi_{{\rm atm},2}+g_5\left(-\Delta_{{\rm atm},3}\varphi_{{\rm atm},3}+\Delta_{{\rm atm},4}\varphi_{{\rm atm},2}+\sqrt{2}\Delta_{{\rm atm},1}\varphi_{{\rm atm},1}\right)=0\,.
\end{aligned}\tag{4.8}$$

Considering the already aligned directions of Δ in Eq. (4.7), we find an extremum solution to the above equations

$$\langle\phi_{\rm atm}\rangle=\left(0,v_{\phi_{\rm atm}},-v_{\phi_{\rm atm}}\right),\qquad \langle\varphi_{\rm atm}\rangle=\left(0,v_{\varphi_{\rm atm}},-v_{\varphi_{\rm atm}}\right)\,.\tag{4.9}$$

The VEVs $v_{\phi_{\rm atm}}$, $v_{\varphi_{\rm atm}}$ and $v_{\xi_{\rm atm}}$ are related through

$$v_{\varphi_{\rm atm}}^2=-\frac{g_4}{2g_3g_5}M_\Delta v_{\xi_{\rm atm}},\qquad v_{\phi_{\rm atm}}=\frac{2g_2g_5}{3g_1g_4}v_{\varphi_{\rm atm}}\,.\tag{4.10}$$

We can check that the VEVs of $\phi_{\rm atm}$, $\varphi_{\rm atm}$, $\Delta_{\rm atm}$ and $\xi_{\rm atm}$ are eigenvectors of the element $T^3ST^2ST^3S$ corresponding to the eigenvalue 1, therefore the vacuum of the atmospheric neutrino flavon preserves the subgroup $Z_2^{T^3ST^2ST^3S}$. Moreover the ratio $v_{\phi_{\rm atm}}^2/v_{\xi_{\rm atm}}=-2g_2^2g_5M_\Delta/(9g_1^2g_3g_4)$ is real. For the solar neutrino flavons, the F-flatness gives rise to

$$
\begin{aligned}
\frac{\partial w_d^{\rm sol}}{\partial \chi_1^0} &= M_\chi \chi_{{\rm sol},1} + 2h_1\left(\phi_{{\rm sol},1}^2 - \phi_{{\rm sol},2}\phi_{{\rm sol},3}\right) = 0\,,\\
\frac{\partial w_d^{\rm sol}}{\partial \chi_2^0} &= M_\chi \chi_{{\rm sol},5} - 2\sqrt{3}h_1\phi_{{\rm sol},1}\phi_{{\rm sol},3} = 0\,,\\
\frac{\partial w_d^{\rm sol}}{\partial \chi_3^0} &= M_\chi \chi_{{\rm sol},4} + \sqrt{6}h_1\phi_{{\rm sol},3}^2 = 0\,,\\
\frac{\partial w_d^{\rm sol}}{\partial \chi_4^0} &= M_\chi \chi_{{\rm sol},3} + \sqrt{6}h_1\phi_{{\rm sol},2}^2 = 0\,,\\
\frac{\partial w_d^{\rm sol}}{\partial \chi_5^0} &= M_\chi \chi_{{\rm sol},2} - 2\sqrt{3}h_1\phi_{{\rm sol},1}\phi_{{\rm sol},2} = 0\,,\\
\frac{\partial w_d^{\rm sol}}{\partial \chi_1^{\prime 0}} &= h_2\xi_{\rm sol}\chi_{{\rm sol},1} + 2h_3\left(\chi_{{\rm sol},1}^2 + \chi_{{\rm sol},2}\chi_{{\rm sol},5} - 2\chi_{{\rm sol},3}\chi_{{\rm sol},4}\right)\\
&\quad + 2h_4\left(\chi_{{\rm sol},1}^2 - 2\chi_{{\rm sol},2}\chi_{{\rm sol},5} + \chi_{{\rm sol},3}\chi_{{\rm sol},4}\right) = 0\,,\\
\frac{\partial w_d^{\rm sol}}{\partial \chi_2^{\prime 0}} &= h_2\xi_{\rm sol}\chi_{{\rm sol},5} + 2h_3\left(\chi_{{\rm sol},1}\chi_{{\rm sol},5} + \sqrt{6}\chi_{{\rm sol},2}\chi_{{\rm sol},4}\right)\\
&\quad + h_4\left(-4\chi_{{\rm sol},1}\chi_{{\rm sol},5} + \sqrt{6}\chi_{{\rm sol},3}^2\right) = 0\,,\\
\frac{\partial w_d^{\rm sol}}{\partial \chi_3^{\prime 0}} &= h_2\xi_{\rm sol}\chi_{{\rm sol},4} + h_3\left(-4\chi_{{\rm sol},1}\chi_{{\rm sol},4} + \sqrt{6}\chi_{{\rm sol},5}^2\right)\\
&\quad + 2h_4\left(\chi_{{\rm sol},1}\chi_{{\rm sol},4} + \sqrt{6}\chi_{{\rm sol},2}\chi_{{\rm sol},3}\right) = 0\,,\\
\frac{\partial w_d^{\rm sol}}{\partial \chi_4^{\prime 0}} &= h_2\xi_{\rm sol}\chi_{{\rm sol},3} + h_3\left(-4\chi_{{\rm sol},1}\chi_{{\rm sol},3} + \sqrt{6}\chi_{{\rm sol},2}^2\right)\\
&\quad + 2h_4\left(\chi_{{\rm sol},1}\chi_{{\rm sol},3} + \sqrt{6}\chi_{{\rm sol},4}\chi_{{\rm sol},5}\right) = 0\,,\\
\frac{\partial w_d^{\rm sol}}{\partial \chi_5^{\prime 0}} &= h_2\xi_{\rm sol}\chi_{{\rm sol},2} + 2h_3\left(\chi_{{\rm sol},1}\chi_{{\rm sol},2} + \sqrt{6}\chi_{{\rm sol},3}\chi_{{\rm sol},5}\right)\\
&\quad + h_4\left(-4\chi_{{\rm sol},1}\chi_{{\rm sol},2} + \sqrt{6}\chi_{{\rm sol},4}^2\right) = 0\,,
\end{aligned}
\tag{4.11}
$$

from which we can extract the vacuum expectation values for $\phi_{\rm sol}$, $\xi_{\rm sol}$ and $\chi_{\rm sol}$ as follow

$$
\begin{aligned}
&\langle\phi_{\rm sol}\rangle = \left(\sqrt{2}, 2\phi^2 e^{2i\pi/5}, 2\phi^2 e^{-2i\pi/5}\right) v_{\phi_{\rm sol}}\,, \qquad \langle\xi_{\rm sol}\rangle = v_{\xi_{\rm sol}}\,,\\
&\langle\chi_{\rm sol}\rangle = \left(\sqrt{6}, 2e^{2i\pi/5}/\phi, 2e^{-i\pi/5}\phi, 2e^{i\pi/5}\phi, 2e^{-2i\pi/5}/\phi\right) v_{\chi_{\rm sol}}
\end{aligned}
\tag{4.12}
$$

up to symmetry transformations belonging to A_5. Furthermore, the VEVs $v_{\phi_{\rm sol}}$, v_χ and $v_{\xi_{\rm sol}}$ are related via

$$
v_{\xi_{\rm sol}} = 24(5+2\sqrt{5})\frac{h_1(h_3-h_4)v_{\phi_{\rm sol}}^2}{h_2 M_\chi}, \qquad v_{\chi_{\rm sol}} = 2\sqrt{6}(2+\sqrt{5})\frac{h_1 v_{\phi_{\rm sol}}^2}{M_\chi}\,. \tag{4.13}
$$

The vacuum configuration of $\phi_{\rm sol}$, $\chi_{\rm sol}$ and $\xi_{\rm sol}$ shown in Eq. (4.12) is the most general one invariant under the subgroup $Z_2^{ST^2ST^3}$. Notice that the ratio $v^2_{\phi_{\rm sol}}/v_{\xi_{\rm sol}} = (5-2\sqrt{5})h_2M_\chi/(120h_1(h_3-h_4))$ is real in our setup with imposed CP symmetry. It is worth to observe that since the three abelian factors in $Z_6\times Z_5\times Z_5'$ are complementary for the charged lepton flavons $\{\phi_l,\varphi_l,\psi_l\}$, the atmospheric neutrino flavons $\{\Delta_{\rm atm},\varphi_{\rm atm},\phi_{\rm atm},\xi_{\rm atm}\}$ and the solar neutrino flavons $\{\phi_{\rm sol},\chi_{\rm sol},\xi_{\rm sol}\}$, the interaction between these three sectors can arise only at a relative order $1/\Lambda^2$. As a consequence, the vacuum alignments in Eqs. (4.4), (4.7), (4.9), (4.12) are independent up to $1/\Lambda^2$.

4.2. The model

With the symmetries and superfields listed in Table 2, we can then write down the most relevant operators for charged lepton masses

$$\begin{aligned}
w_l =& \frac{y_\tau}{\Lambda}\tau^c\,(L\phi_l)\,H_d + \frac{y_{\mu,1}}{\Lambda^2}\mu^c\left(L\,(\varphi_l\psi_l)_{\mathbf{3}}\right)H_d + \frac{y_{\mu,2}}{\Lambda^2}\mu^c\left(L\,(\psi_l\psi_l)_{\mathbf{3}}\right)H_d \\
&+ \frac{y_{e,1}}{\Lambda^3}e^c\,(L\phi_l)\,(\varphi_l\varphi_l)\,H_d + \frac{y_{e,2}}{\Lambda^3}e^c\left((L\phi_l)_{\mathbf{5}}\,(\varphi_l\varphi_l)_{\mathbf{5}}\right)H_d \\
&+ \frac{y_{e,3}}{\Lambda^3}e^c\left((L\phi_l)_{\mathbf{3}}\,(\varphi_l\psi_l)_{\mathbf{3}}\right)H_d + \frac{y_{e,4}}{\Lambda^3}e^c\left((L\phi_l)_{\mathbf{5}}\,(\varphi_l\psi_l)_{\mathbf{5}}\right)H_d \\
&+ \frac{y_{e,5}}{\Lambda^3}e^c\,(L\phi_l)\,(\psi_l\psi_l)\,H_d + \frac{y_{e,6}}{\Lambda^3}e^c\left((L\phi_l)_{\mathbf{3}}\,(\psi_l\psi_l)_{\mathbf{3}}\right)H_d \\
&+ \frac{y_{e,7}}{\Lambda^3}e^c\left((L\phi_l)_{\mathbf{5}}\,(\psi_l\psi_l)_{\mathbf{5_1}}\right)H_d + \frac{y_{e,8}}{\Lambda^3}e^c\left((L\phi_l)_{\mathbf{5}}\,(\psi_l\psi_l)_{\mathbf{5_2}}\right)H_d\,,
\end{aligned} \tag{4.14}$$

where all the couplings y_τ, $y_{\mu,1}$, $y_{\mu,2}$ and $y_{e,i}$ $(i=1,\ldots,8)$ are real since we impose CP as a symmetry on the model. After the electroweak and flavor symmetry breaking, taking into account the alignment of ϕ_l, φ_l and ψ_l in Eq. (4.4), we find the charged lepton mass matrix is diagonal and the charged lepton masses are given by

$$m_l = \begin{pmatrix} -\sqrt{2}\,\frac{v_{\phi_l}}{\Lambda^3}\left(3y_{e,2}v^2_{\varphi_l} + (y_{e,3}-\sqrt{3}y_{e,4})v_{\varphi_l}v_{\psi_l} + 3y_{e,8}v^2_{\psi_l}\right) & 0 & 0 \\ 0 & -\sqrt{2}y_{\mu,1}\frac{v_{\varphi_l}v_{\psi_l}}{\Lambda^2} & 0 \\ 0 & 0 & y_\tau\frac{v_{\phi_l}}{\Lambda} \end{pmatrix} v_d\,, \tag{4.15}$$

where $v_d = \langle H_d\rangle$. The symmetry $Z_6\times Z_5$ imposes different powers of ϕ_l, φ_l and ψ_l for the electron, muon and tauon terms. Therefore the hierarchy of masses is naturally obtained. As already pointed out, the $Z_6\times Z_5\times Z_5'$ charge assignments in Table 2 determine that the nontrivial higher order corrections arise only at the relative order $1/\Lambda^3$ with respect to the terms already considered in w_l. Hence higher order corrections are completely negligible.

As for the neutrino sector, the light neutrino masses are generated via the type-I seesaw mechanism with two right-handed neutrinos. With the charge assignments in Table 2, the lowest dimensional operator responsible for neutrino masses are

$$w_\nu = \frac{y_{\rm atm}}{\Lambda}\,(L\phi_{\rm atm})\,H_u\nu^c_{\rm atm} + \frac{y_{\rm sol}}{\Lambda}\,(L\phi_{\rm sol})\,H_u\nu^c_{\rm sol} + x_{\rm atm}\nu^c_{\rm atm}\nu^c_{\rm atm}\xi_{\rm atm} + x_{\rm sol}\nu^c_{\rm sol}\nu^c_{\rm sol}\xi_{\rm sol}\,, \tag{4.16}$$

where the four coupling constants $y_{\rm atm}$, $y_{\rm sol}$, $x_{\rm atm}$ and $x_{\rm sol}$ are real because of CP conservation. Inserting the vacuum alignments in Eq. (4.9) and Eq. (4.12), we obtain the Dirac and right-handed Majorana matrices

$$M_D = \begin{pmatrix} 0 & \sqrt{2} y_{\rm sol} v_{\phi_{\rm sol}}/\Lambda \\ -y_{\rm atm} v_{\phi_{\rm atm}}/\Lambda & 2\phi^2 e^{-2i\pi/5} y_{\rm sol} v_{\phi_{\rm sol}}/\Lambda \\ y_{\rm atm} v_{\phi_{\rm atm}}/\Lambda & 2\phi^2 e^{2i\pi/5} y_{\rm sol} v_{\phi_{\rm sol}}/\Lambda \end{pmatrix} v_u, \qquad M_N = \begin{pmatrix} x_{\rm atm} v_{\xi_{\rm atm}} & 0 \\ 0 & x_{\rm sol} v_{\xi_{\rm sol}} \end{pmatrix}, \tag{4.17}$$

where $v_u = \langle H_u \rangle$. After applying the seesaw formula, the effective light neutrino mass matrix can be written as

$$m_\nu = -\frac{v_u^2}{\Lambda^2}\left[\frac{y_{\rm atm}^2 v_{\phi_{\rm atm}}^2}{x_{\rm atm} v_{\xi_{\rm atm}}} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{pmatrix} + \frac{y_{\rm sol}^2 v_{\phi_{\rm sol}}^2}{x_{\rm sol} v_{\xi_{\rm sol}}} \begin{pmatrix} 2 & 2\sqrt{2}\phi^2 e^{-2i\pi/5} & 2\sqrt{2}\phi^2 e^{2i\pi/5} \\ 2\sqrt{2}\phi^2 e^{-2i\pi/5} & 4\phi^4 e^{-4i\pi/5} & 4\phi^4 \\ 2\sqrt{2}\phi^2 e^{2i\pi/5} & 4\phi^4 & 4\phi^4 e^{4i\pi/5} \end{pmatrix}\right]. \tag{4.18}$$

We see that this neutrino mass matrix is of the same form as Eq. (3.7) but with fixed value $x = 2i\phi^2 \sin\frac{2\pi}{5}$. Furthermore we can read out the parameters m_a and $m_b e^{i\eta}$,

$$m_a = -\frac{y_{\rm atm}^2 v_{\phi_{\rm atm}}^2}{x_{\rm atm} v_{\xi_{\rm atm}}} \frac{v_u^2}{\Lambda^2}, \qquad m_b e^{i\eta} = -\frac{y_{\rm sol}^2 v_{\phi_{\rm sol}}^2}{x_{\rm sol} v_{\xi_{\rm sol}}} \frac{v_u^2}{\Lambda^2}. \tag{4.19}$$

As we have shown in section 4.1, both ratios $v_{\phi_{\rm atm}}^2/v_{\xi_{\rm atm}}$ and $v_{\phi_{\rm sol}}^2/v_{\xi_{\rm sol}}$ are fixed to be real in our setup. Consequently the phase η is trivial, to be more specific, η is equal to 0 for the combination $g_3 g_4 g_5 h_1 h_2 (h_3 - h_4) x_{\rm atm} x_{\rm sol} M_\Delta M_\chi < 0$ and π for $g_3 g_4 g_5 h_1 h_2 (h_3 - h_4) x_{\rm atm} x_{\rm sol} M_\Delta M_\chi > 0$. Thus the neutrino mass matrix satisfies the $\mu\tau$ reflection symmetry such that both atmospheric mixing angle and Dirac CP phase are maximal [42], and the experimental data on lepton mixing angles and neutrino masses can be accommodated, as displayed in Table 1. In summary, the golden Littlest seesaw neutrino mass matrix is exactly reproduced in the present model.

5. Conclusion

The Littlest Seesaw approach assumes that a *different* residual flavour symmetry is preserved by each flavon, in the diagonal mass basis of two right-handed neutrinos, leading to a highly predictive set of possible flavon alignments for the charged leptons and neutrinos. The Littlest seesaw model can thereby give a successful description of both neutrino mixing and the light neutrino masses in terms of four input parameters. The case of S_4, discussed in earlier work, leads to the lepton mixing matrix being predicted to be of the TM1 form. The neutrino mass spectrum is normal ordered and the lightest neutrino is massless. Moreover, CP violation in neutrino oscillation and leptogenesis arises from a unique single phase such that they are closely related. Therefore the Littlest seesaw model is quite predictive and attractive.

In this work, we have investigated whether the Littlest seesaw is confined to TM1 mixing, or is of more general applicability. We have performed a comprehensive analysis of possible lepton mixing which can be derived from the A_5 flavor symmetry group within the paradigm of the Littlest seesaw. The general principle of the Littlest seesaw is that different sectors of the Lagrangian preserve different residual subgroups of the flavor symmetry [15]. This idea is illustrated in Fig. 2. If the residual symmetry of the charged lepton sector is $G_l = Z_5^T$ which enforces the diagonality of the charged lepton mass matrix in the T generator diagonal basis, the subgroup $G_{\rm atm} = Z_2^{T^3ST^2ST^3S}$ or $G_{\rm atm} = Z_2^{T^3ST^2ST^3}$ is preserved by the atmospheric flavon, and solar flavon ϕ_{sol} breaks the flavor group A_5 into $G_{\mathrm{sol}} = Z_3^{ST^2ST^3}$ or $G_{\mathrm{sol}} = Z_3^{T^3ST^2S}$, the first column of the golden ratio mixing matrix is preserved. The experimental data on the lepton mixing angles and neutrino masses can be accommodated for certain values of the input parameters m_a, m_b and η except that the reactor angle θ_{13} is predicted to rather close to its 3σ boundary. This could be easily reconciled with the experimental results in an explicit model with small subleading corrections or by considering the third almost decoupled right-handed neutrino. Moreover, many numerical benchmark examples are found. The most remarkable point is $\eta = 0$ for $G_{\mathrm{atm}} = Z_2^{T^3ST^2ST^3S}$ and $\eta = \pi$ for $G_{\mathrm{atm}} = Z_2^{T^3ST^2ST^3}$, then both Dirac CP phase δ_{CP} and the atmospheric mixing angle θ_{23} would be exactly maximal. This mixing pattern is previously predicted in the semidirect approach of combining A_5 flavor symmetry with generalized CP [22,29,30], but here we have additional prediction for the neutrino masses. Furthermore, we mention that the most simple scenario to obtain maximal θ_{23} and δ_{CP} is imposing a $\mu\tau$ reflection symmetry which permutes a muon neutrino (antineutrino) and a tau antineutrino (neutrino) in the charged lepton diagonal basis [42]. However, the neutrino mixing angles θ_{12} and θ_{13} as well as neutrino masses are not subject to any constraint from the $\mu\tau$ reflection symmetry. In the context of golden Littlest seesaw, the A_5 flavor symmetry enforces the first column of the mixing matrix to be $\sqrt{\frac{1}{2\sqrt{5}\phi}}\left(-\sqrt{2}\phi, 1, 1\right)^T$, such that the sum rule of Eq. (3.20) between θ_{12} and θ_{13} is predicted.

Inspired by the model independent analysis, we have constructed a benchmark golden Littlest seesaw model based on the discrete group $A_5 \times Z_6 \times Z_5 \times Z'_5$. The necessary vacuum alignments needed to achieve the remnant symmetries is dynamically realized via the supersymmetric F-term alignment mechanism. The charged lepton mass hierarchy is correctly reproduced in our model, because the electron, muon and tau masses arise from operators with one, two and three flavons respectively. The model fixes the parameter $x = 2i\phi^2 \sin\frac{2\pi}{5}$, and the relative phase $\eta = 0, \pi$ is determined by spontaneous CP violation. As a consequence, both atmospheric mixing angle and Dirac CP phase are predicted to be maximal, the other mixing angles as well as neutrino masses are compatible with experimental data. Moreover, the predictions are protected from higher order corrections by the full symmetry of the model.

In conclusion, the Littlest seesaw is a general and predictive framework of explaining neutrino masses and lepton mixing. All the results of this paper only depend on the assumed residual symmetries and they are independent of the underlying mechanism which dynamically realizes the required vacuum alignments. Since all CP violation phases are completely fixed in the golden Littlest seesaw model, another interesting question is whether the observed baryon asymmetry of the universe can be generated through leptogenesis and the resulting constraints on the right-handed neutrino masses. The relevant work will appear elsewhere.

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Appendix A. Group theory of A_5

A_5 is the group of even permutations of five objects, and it has $5!/2 = 60$ elements. Geometrically it is the symmetry group of a regular icosahedron. A_5 group can be generated by two generators S and T which satisfy the multiplication rules [21]:

$$S^2 = T^5 = (ST)^3 = 1\,. \tag{A.1}$$

The A_5 group has five irreducible representations: one singlet representation **1**, two three-dimensional representations **3** and **3′**, one four-dimensional representation **4** and one five-dimensional representation **5**. In this paper we shall work in the generator T diagonal basis. The generators S and T in the five different irreducible representations are chosen as follows,

$$\begin{aligned}
&\mathbf{1}: \quad S = 1\,, && T = 1\,,\\
&\mathbf{3}: \quad S = \frac{1}{\sqrt{5}}\begin{pmatrix} 1 & -\sqrt{2} & -\sqrt{2} \\ -\sqrt{2} & -\phi & 1/\phi \\ -\sqrt{2} & 1/\phi & -\phi \end{pmatrix}, && T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega_5 & 0 \\ 0 & 0 & \omega_5^4 \end{pmatrix},\\
&\mathbf{3}': \quad S = \frac{1}{\sqrt{5}}\begin{pmatrix} -1 & \sqrt{2} & \sqrt{2} \\ \sqrt{2} & -1/\phi & \phi \\ \sqrt{2} & \phi & -1/\phi \end{pmatrix}, && T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega_5^2 & 0 \\ 0 & 0 & \omega_5^3 \end{pmatrix},\\
&\mathbf{4}: \quad S = \frac{1}{\sqrt{5}}\begin{pmatrix} 1 & 1/\phi & \phi & -1 \\ 1/\phi & -1 & 1 & \phi \\ \phi & 1 & -1 & 1/\phi \\ -1 & \phi & 1/\phi & 1 \end{pmatrix}, && T = \begin{pmatrix} \omega_5 & 0 & 0 & 0 \\ 0 & \omega_5^2 & 0 & 0 \\ 0 & 0 & \omega_5^3 & 0 \\ 0 & 0 & 0 & \omega_5^4 \end{pmatrix},\\
&\mathbf{5}: \quad S = \frac{1}{5}\begin{pmatrix} -1 & \sqrt{6} & \sqrt{6} & \sqrt{6} & \sqrt{6} \\ \sqrt{6} & 1/\phi^2 & -2\phi & 2/\phi & \phi^2 \\ \sqrt{6} & -2\phi & \phi^2 & 1/\phi^2 & 2/\phi \\ \sqrt{6} & 2/\phi & 1/\phi^2 & \phi^2 & -2\phi \\ \sqrt{6} & \phi^2 & 2/\phi & -2\phi & 1/\phi^2 \end{pmatrix}, && T = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & \omega_5 & 0 & 0 & 0 \\ 0 & 0 & \omega_5^2 & 0 & 0 \\ 0 & 0 & 0 & \omega_5^3 & 0 \\ 0 & 0 & 0 & 0 & \omega_5^4 \end{pmatrix},
\end{aligned} \tag{A.2}$$

where $\omega_5 = e^{2\pi i/5}$ is the fifth root of unit. Then we can easily obtain the character table of A_5 group which is reported in Table 4. When performing explicit calculations of A_5 invariant in model building, we need the Kronecker products and Clebsch–Gordan coefficients. The Kronecker products between various representations are of the following form:

$$\begin{aligned}
&\mathbf{1}\otimes\mathbf{R}=\mathbf{R}\otimes\mathbf{1}=\mathbf{R}, \quad \mathbf{3}\otimes\mathbf{3}=\mathbf{1}\oplus\mathbf{3}\oplus\mathbf{5}, \quad \mathbf{3}'\otimes\mathbf{3}'=\mathbf{1}\oplus\mathbf{3}'\oplus\mathbf{5}, \quad \mathbf{3}\times\mathbf{3}'=\mathbf{4}\oplus\mathbf{5},\\
&\mathbf{3}\otimes\mathbf{4}=\mathbf{3}'\oplus\mathbf{4}\oplus\mathbf{5}, \quad \mathbf{3}'\otimes\mathbf{4}=\mathbf{3}\oplus\mathbf{4}\oplus\mathbf{5}, \quad \mathbf{3}\otimes\mathbf{5}=\mathbf{3}\oplus\mathbf{3}'\oplus\mathbf{4}\oplus\mathbf{5},\\
&\mathbf{3}'\otimes\mathbf{5}=\mathbf{3}\oplus\mathbf{3}'\oplus\mathbf{4}\oplus\mathbf{5}, \quad \mathbf{4}\otimes\mathbf{4}=\mathbf{1}\oplus\mathbf{3}\oplus\mathbf{3}'\oplus\mathbf{4}\oplus\mathbf{5}, \quad \mathbf{4}\otimes\mathbf{5}=\mathbf{3}\oplus\mathbf{3}'\oplus\mathbf{4}\oplus\mathbf{5_1}\oplus\mathbf{5_2},\\
&\mathbf{5}\otimes\mathbf{5}=\mathbf{1}\oplus\mathbf{3}\oplus\mathbf{3}'\oplus\mathbf{4_1}\oplus\mathbf{4_2}\oplus\mathbf{5_1}\oplus\mathbf{5_2},
\end{aligned} \tag{A.3}$$

where **R** represents any irreducible representation of A_5, and $\mathbf{4_1}$, $\mathbf{4_2}$, $\mathbf{5_1}$ and $\mathbf{5_2}$ stand for the two **4** and two **5** representations that appear in the Kronecker products. In the following we list the Clebsch–Gordan coefficients in our basis. We shall use α_i to indicate the elements of the first representation of the product and β_i to indicate those of the second representation. The subscript "S" ("A") refers to symmetric (antisymmetric) combinations.

Table 4
The character table of the A_5 group, where ϕ represents the golden ratio $\phi = \frac{1+\sqrt{5}}{2}$.

R	Conjugacy classes				
	$1C_1$	$15C_2$	$20C_3$	$12C_5$	$12C_5'$
1	1	1	1	1	1
3	3	−1	0	ϕ	$1-\phi$
3′	3	−1	0	$1-\phi$	ϕ
4	4	0	1	−1	−1
5	5	1	−1	0	0

- $\mathbf{3}\otimes\mathbf{3}=\mathbf{1_S}\oplus\mathbf{3}_A\oplus\mathbf{5}_S$

$$\mathbf{1_S}\sim\alpha_1\beta_1+\alpha_2\beta_3+\alpha_3\beta_2\,,$$
$$\mathbf{3}_A\sim\begin{pmatrix}\alpha_2\beta_3-\alpha_3\beta_2\\ \alpha_1\beta_2-\alpha_2\beta_1\\ \alpha_3\beta_1-\alpha_1\beta_3\end{pmatrix},$$
$$\mathbf{5}_S\sim\begin{pmatrix}2\alpha_1\beta_1-\alpha_2\beta_3-\alpha_3\beta_2\\ -\sqrt{3}(\alpha_1\beta_2+\alpha_2\beta_1)\\ \sqrt{6}\alpha_2\beta_2\\ \sqrt{6}\alpha_3\beta_3\\ -\sqrt{3}(\alpha_1\beta_3+\alpha_3\beta_1)\end{pmatrix}. \tag{A.4}$$

- $\mathbf{3}'\otimes\mathbf{3}'=\mathbf{1}_S\oplus\mathbf{3}'_A\oplus\mathbf{5}_S$

$$\mathbf{1}_S\sim\alpha_1\beta_1+\alpha_2\beta_3+\alpha_3\beta_2\,,$$
$$\mathbf{3}'_A\sim\begin{pmatrix}\alpha_2\beta_3-\alpha_3\beta_2\\ \alpha_1\beta_2-\alpha_2\beta_1\\ \alpha_3\beta_1-\alpha_1\beta_3\end{pmatrix},$$
$$\mathbf{5}_S\sim\begin{pmatrix}2\alpha_1\beta_1-\alpha_2\beta_3-\alpha_3\beta_2\\ \sqrt{6}\alpha_3\beta_3\\ -\sqrt{3}(\alpha_1\beta_2+\alpha_2\beta_1)\\ -\sqrt{3}(\alpha_1\beta_3+\alpha_3\beta_1)\\ \sqrt{6}\alpha_2\beta_2\end{pmatrix}. \tag{A.5}$$

- $\mathbf{3}\otimes\mathbf{3}'=\mathbf{4}\oplus\mathbf{5}$

$$\mathbf{4}\sim\begin{pmatrix}\sqrt{2}\alpha_2\beta_1+\alpha_3\beta_2\\ -\sqrt{2}\alpha_1\beta_2-\alpha_3\beta_3\\ -\sqrt{2}\alpha_1\beta_3-\alpha_2\beta_2\\ \sqrt{2}\alpha_3\beta_1+\alpha_2\beta_3\end{pmatrix},$$

$$\mathbf{5}\sim\begin{pmatrix}\sqrt{3}\alpha_1\beta_1\\ \alpha_2\beta_1-\sqrt{2}\alpha_3\beta_2\\ \alpha_1\beta_2-\sqrt{2}\alpha_3\beta_3\\ \alpha_1\beta_3-\sqrt{2}\alpha_2\beta_2\\ \alpha_3\beta_1-\sqrt{2}\alpha_2\beta_3\end{pmatrix}. \tag{A.6}$$

- $\mathbf{3}\otimes\mathbf{4}=\mathbf{3}'\oplus\mathbf{4}\oplus\mathbf{5}$

$$\mathbf{3}'\sim\begin{pmatrix}-\sqrt{2}(\alpha_2\beta_4+\alpha_3\beta_1)\\ \sqrt{2}\alpha_1\beta_2-\alpha_2\beta_1+\alpha_3\beta_3\\ \sqrt{2}\alpha_1\beta_3+\alpha_2\beta_2-\alpha_3\beta_4\end{pmatrix},$$

$$\mathbf{4}\sim\begin{pmatrix}\alpha_1\beta_1-\sqrt{2}\alpha_3\beta_2\\ -\alpha_1\beta_2-\sqrt{2}\alpha_2\beta_1\\ \alpha_1\beta_3+\sqrt{2}\alpha_3\beta_4\\ -\alpha_1\beta_4+\sqrt{2}\alpha_2\beta_3\end{pmatrix},$$

$$\mathbf{5}\sim\begin{pmatrix}\sqrt{6}(\alpha_2\beta_4-\alpha_3\beta_1)\\ 2\sqrt{2}\alpha_1\beta_1+2\alpha_3\beta_2\\ -\sqrt{2}\alpha_1\beta_2+\alpha_2\beta_1+3\alpha_3\beta_3\\ \sqrt{2}\alpha_1\beta_3-3\alpha_2\beta_2-\alpha_3\beta_4\\ -2\sqrt{2}\alpha_1\beta_4-2\alpha_2\beta_3\end{pmatrix}. \tag{A.7}$$

- $\mathbf{3}'\otimes\mathbf{4}=\mathbf{3}\oplus\mathbf{4}\oplus\mathbf{5}$

$$\mathbf{3}\sim\begin{pmatrix}-\sqrt{2}(\alpha_2\beta_3+\alpha_3\beta_2)\\ \sqrt{2}\alpha_1\beta_1+\alpha_2\beta_4-\alpha_3\beta_3\\ \sqrt{2}\alpha_1\beta_4-\alpha_2\beta_2+\alpha_3\beta_1\end{pmatrix},$$

$$\mathbf{4}\sim\begin{pmatrix}\alpha_1\beta_1+\sqrt{2}\alpha_3\beta_3\\ \alpha_1\beta_2-\sqrt{2}\alpha_3\beta_4\\ -\alpha_1\beta_3+\sqrt{2}\alpha_2\beta_1\\ -\alpha_1\beta_4-\sqrt{2}\alpha_2\beta_2\end{pmatrix},$$

$$\mathbf{5}\sim\begin{pmatrix}\sqrt{6}(\alpha_2\beta_3-\alpha_3\beta_2)\\ \sqrt{2}\alpha_1\beta_1-3\alpha_2\beta_4-\alpha_3\beta_3\\ 2\sqrt{2}\alpha_1\beta_2+2\alpha_3\beta_4\\ -2\sqrt{2}\alpha_1\beta_3-2\alpha_2\beta_1\\ -\sqrt{2}\alpha_1\beta_4+\alpha_2\beta_2+3\alpha_3\beta_1\end{pmatrix}. \tag{A.8}$$

- $\mathbf{3}\otimes\mathbf{5}=\mathbf{3}\oplus\mathbf{3}'\oplus\mathbf{4}\oplus\mathbf{5}$

$$\mathbf{3}\sim\begin{pmatrix}-2\alpha_1\beta_1+\sqrt{3}\alpha_2\beta_5+\sqrt{3}\alpha_3\beta_2\\ \sqrt{3}\alpha_1\beta_2+\alpha_2\beta_1-\sqrt{6}\alpha_3\beta_3\\ \sqrt{3}\alpha_1\beta_5-\sqrt{6}\alpha_2\beta_4+\alpha_3\beta_1\end{pmatrix},$$

$$\mathbf{3}' \sim \begin{pmatrix} \sqrt{3}\alpha_1\beta_1 + \alpha_2\beta_5 + \alpha_3\beta_2 \\ \alpha_1\beta_3 - \sqrt{2}\alpha_2\beta_2 - \sqrt{2}\alpha_3\beta_4 \\ \alpha_1\beta_4 - \sqrt{2}\alpha_2\beta_3 - \sqrt{2}\alpha_3\beta_5 \end{pmatrix},$$

$$\mathbf{4} \sim \begin{pmatrix} 2\sqrt{2}\alpha_1\beta_2 - \sqrt{6}\alpha_2\beta_1 + \alpha_3\beta_3 \\ -\sqrt{2}\alpha_1\beta_3 + 2\alpha_2\beta_2 - 3\alpha_3\beta_4 \\ \sqrt{2}\alpha_1\beta_4 + 3\alpha_2\beta_3 - 2\alpha_3\beta_5 \\ -2\sqrt{2}\alpha_1\beta_5 - \alpha_2\beta_4 + \sqrt{6}\alpha_3\beta_1 \end{pmatrix},$$

$$\mathbf{5} \sim \begin{pmatrix} \sqrt{3}(\alpha_2\beta_5 - \alpha_3\beta_2) \\ -\alpha_1\beta_2 - \sqrt{3}\alpha_2\beta_1 - \sqrt{2}\alpha_3\beta_3 \\ -2\alpha_1\beta_3 - \sqrt{2}\alpha_2\beta_2 \\ 2\alpha_1\beta_4 + \sqrt{2}\alpha_3\beta_5 \\ \alpha_1\beta_5 + \sqrt{2}\alpha_2\beta_4 + \sqrt{3}\alpha_3\beta_1 \end{pmatrix}. \tag{A.9}$$

- $\mathbf{3}' \otimes \mathbf{5} = \mathbf{3} \oplus \mathbf{3}' \oplus \mathbf{4} \oplus \mathbf{5}$

$$\mathbf{3} \sim \begin{pmatrix} \sqrt{3}\alpha_1\beta_1 + \alpha_2\beta_4 + \alpha_3\beta_3 \\ \alpha_1\beta_2 - \sqrt{2}\alpha_2\beta_5 - \sqrt{2}\alpha_3\beta_4 \\ \alpha_1\beta_5 - \sqrt{2}\alpha_2\beta_3 - \sqrt{2}\alpha_3\beta_2 \end{pmatrix},$$

$$\mathbf{3}' \sim \begin{pmatrix} -2\alpha_1\beta_1 + \sqrt{3}\alpha_2\beta_4 + \sqrt{3}\alpha_3\beta_3 \\ \sqrt{3}\alpha_1\beta_3 + \alpha_2\beta_1 - \sqrt{6}\alpha_3\beta_5 \\ \sqrt{3}\alpha_1\beta_4 - \sqrt{6}\alpha_2\beta_2 + \alpha_3\beta_1 \end{pmatrix},$$

$$\mathbf{4} \sim \begin{pmatrix} \sqrt{2}\alpha_1\beta_2 + 3\alpha_2\beta_5 - 2\alpha_3\beta_4 \\ 2\sqrt{2}\alpha_1\beta_3 - \sqrt{6}\alpha_2\beta_1 + \alpha_3\beta_5 \\ -2\sqrt{2}\alpha_1\beta_4 - \alpha_2\beta_2 + \sqrt{6}\alpha_3\beta_1 \\ -\sqrt{2}\alpha_1\beta_5 + 2\alpha_2\beta_3 - 3\alpha_3\beta_2 \end{pmatrix},$$

$$\mathbf{5} \sim \begin{pmatrix} \sqrt{3}(\alpha_2\beta_4 - \alpha_3\beta_3) \\ 2\alpha_1\beta_2 + \sqrt{2}\alpha_3\beta_4 \\ -\alpha_1\beta_3 - \sqrt{3}\alpha_2\beta_1 - \sqrt{2}\alpha_3\beta_5 \\ \alpha_1\beta_4 + \sqrt{2}\alpha_2\beta_2 + \sqrt{3}\alpha_3\beta_1 \\ -2\alpha_1\beta_5 - \sqrt{2}\alpha_2\beta_3 \end{pmatrix}. \tag{A.10}$$

- $\mathbf{4} \otimes \mathbf{4} = \mathbf{1}_S \oplus \mathbf{3}_A \oplus \mathbf{3}'_A \oplus \mathbf{4}_S \oplus \mathbf{5}_S$

$$\mathbf{1}_S \sim \alpha_1\beta_4 + \alpha_2\beta_3 + \alpha_3\beta_2 + \alpha_4\beta_1 ,$$

$$\mathbf{3}_A \sim \begin{pmatrix} -\alpha_1\beta_4 + \alpha_2\beta_3 - \alpha_3\beta_2 + \alpha_4\beta_1 \\ \sqrt{2}(\alpha_2\beta_4 - \alpha_4\beta_2) \\ \sqrt{2}(\alpha_1\beta_3 - \alpha_3\beta_1) \end{pmatrix},$$

$$\mathbf{3}'_A \sim \begin{pmatrix} \alpha_1\beta_4+\alpha_2\beta_3-\alpha_3\beta_2-\alpha_4\beta_1 \\ \sqrt{2}(\alpha_3\beta_4-\alpha_4\beta_3) \\ \sqrt{2}(\alpha_1\beta_2-\alpha_2\beta_1) \end{pmatrix},$$

$$\mathbf{4}_S \sim \begin{pmatrix} \alpha_2\beta_4+\alpha_3\beta_3+\alpha_4\beta_2 \\ \alpha_1\beta_1+\alpha_3\beta_4+\alpha_4\beta_3 \\ \alpha_1\beta_2+\alpha_2\beta_1+\alpha_4\beta_4 \\ \alpha_1\beta_3+\alpha_2\beta_2+\alpha_3\beta_1 \end{pmatrix},$$

$$\mathbf{5}_S \sim \begin{pmatrix} \sqrt{3}(\alpha_1\beta_4-\alpha_2\beta_3-\alpha_3\beta_2+\alpha_4\beta_1) \\ -\sqrt{2}\alpha_2\beta_4+2\sqrt{2}\alpha_3\beta_3-\sqrt{2}\alpha_4\beta_2 \\ -2\sqrt{2}\alpha_1\beta_1+\sqrt{2}\alpha_3\beta_4+\sqrt{2}\alpha_4\beta_3 \\ \sqrt{2}\alpha_1\beta_2+\sqrt{2}\alpha_2\beta_1-2\sqrt{2}\alpha_4\beta_4 \\ -\sqrt{2}\alpha_1\beta_3+2\sqrt{2}\alpha_2\beta_2-\sqrt{2}\alpha_3\beta_1 \end{pmatrix}. \tag{A.11}$$

- $\mathbf{4}\otimes\mathbf{5}=\mathbf{3}\oplus\mathbf{3}'\oplus\mathbf{4}\oplus\mathbf{5}_1\oplus\mathbf{5}_2$

$$\mathbf{3} \sim \begin{pmatrix} 2\sqrt{2}\alpha_1\beta_5-\sqrt{2}\alpha_2\beta_4+\sqrt{2}\alpha_3\beta_3-2\sqrt{2}\alpha_4\beta_2 \\ -\sqrt{6}\alpha_1\beta_1+2\alpha_2\beta_5+3\alpha_3\beta_4-\alpha_4\beta_3 \\ \alpha_1\beta_4-3\alpha_2\beta_3-2\alpha_3\beta_2+\sqrt{6}\alpha_4\beta_1 \end{pmatrix},$$

$$\mathbf{3}' \sim \begin{pmatrix} \sqrt{2}\alpha_1\beta_5+2\sqrt{2}\alpha_2\beta_4-2\sqrt{2}\alpha_3\beta_3-\sqrt{2}\alpha_4\beta_2 \\ 3\alpha_1\beta_2-\sqrt{6}\alpha_2\beta_1-\alpha_3\beta_5+2\alpha_4\beta_4 \\ -2\alpha_1\beta_3+\alpha_2\beta_2+\sqrt{6}\alpha_3\beta_1-3\alpha_4\beta_5 \end{pmatrix},$$

$$\mathbf{4} \sim \begin{pmatrix} \sqrt{3}\alpha_1\beta_1-\sqrt{2}\alpha_2\beta_5+\sqrt{2}\alpha_3\beta_4-2\sqrt{2}\alpha_4\beta_3 \\ -\sqrt{2}\alpha_1\beta_2-\sqrt{3}\alpha_2\beta_1+2\sqrt{2}\alpha_3\beta_5+\sqrt{2}\alpha_4\beta_4 \\ \sqrt{2}\alpha_1\beta_3+2\sqrt{2}\alpha_2\beta_2-\sqrt{3}\alpha_3\beta_1-\sqrt{2}\alpha_4\beta_5 \\ -2\sqrt{2}\alpha_1\beta_4+\sqrt{2}\alpha_2\beta_3-\sqrt{2}\alpha_3\beta_2+\sqrt{3}\alpha_4\beta_1 \end{pmatrix},$$

$$\mathbf{5}_1 \sim \begin{pmatrix} \sqrt{2}\alpha_1\beta_5-\sqrt{2}\alpha_2\beta_4-\sqrt{2}\alpha_3\beta_3+\sqrt{2}\alpha_4\beta_2 \\ -\sqrt{2}\alpha_1\beta_1-\sqrt{3}\alpha_3\beta_4-\sqrt{3}\alpha_4\beta_3 \\ \sqrt{3}\alpha_1\beta_2+\sqrt{2}\alpha_2\beta_1+\sqrt{3}\alpha_3\beta_5 \\ \sqrt{3}\alpha_2\beta_2+\sqrt{2}\alpha_3\beta_1+\sqrt{3}\alpha_4\beta_5 \\ -\sqrt{3}\alpha_1\beta_4-\sqrt{3}\alpha_2\beta_3-\sqrt{2}\alpha_4\beta_1 \end{pmatrix},$$

$$\mathbf{5}_2 \sim \begin{pmatrix} 2\alpha_1\beta_5+4\alpha_2\beta_4+4\alpha_3\beta_3+2\alpha_4\beta_2 \\ 4\alpha_1\beta_1+2\sqrt{6}\alpha_2\beta_5 \\ -\sqrt{6}\alpha_1\beta_2+2\alpha_2\beta_1-\sqrt{6}\alpha_3\beta_5+2\sqrt{6}\alpha_4\beta_4 \\ 2\sqrt{6}\alpha_1\beta_3-\sqrt{6}\alpha_2\beta_2+2\alpha_3\beta_1-\sqrt{6}\alpha_4\beta_5 \\ 2\sqrt{6}\alpha_3\beta_2+4\alpha_4\beta_1 \end{pmatrix}. \tag{A.12}$$

- $\mathbf{5}\otimes\mathbf{5}=\mathbf{1}_S\oplus\mathbf{3}_A\oplus\mathbf{3}'_A\oplus\mathbf{4}_{S,1}\oplus\mathbf{4}_{A,2}\oplus\mathbf{5}_{S,1}\oplus\mathbf{5}_{S,2}$

$$\mathbf{1}_S \sim \alpha_1\beta_1+\alpha_2\beta_5+\alpha_3\beta_4+\alpha_4\beta_3+\alpha_5\beta_2\,,$$

$$\mathbf{3}_A \sim \begin{pmatrix} \alpha_2\beta_5+2\alpha_3\beta_4-2\alpha_4\beta_3-\alpha_5\beta_2 \\ -\sqrt{3}\alpha_1\beta_2+\sqrt{3}\alpha_2\beta_1+\sqrt{2}\alpha_3\beta_5-\sqrt{2}\alpha_5\beta_3 \\ \sqrt{3}\alpha_1\beta_5+\sqrt{2}\alpha_2\beta_4-\sqrt{2}\alpha_4\beta_2-\sqrt{3}\alpha_5\beta_1 \end{pmatrix},$$

$$\mathbf{3}'_A \sim \begin{pmatrix} 2\alpha_2\beta_5-\alpha_3\beta_4+\alpha_4\beta_3-2\alpha_5\beta_2 \\ \sqrt{3}\alpha_1\beta_3-\sqrt{3}\alpha_3\beta_1+\sqrt{2}\alpha_4\beta_5-\sqrt{2}\alpha_5\beta_4 \\ -\sqrt{3}\alpha_1\beta_4+\sqrt{2}\alpha_2\beta_3-\sqrt{2}\alpha_3\beta_2+\sqrt{3}\alpha_4\beta_1 \end{pmatrix},$$

$$\mathbf{4}_{S,1} \sim \begin{pmatrix} 3\sqrt{2}\alpha_1\beta_2+3\sqrt{2}\alpha_2\beta_1-\sqrt{3}\alpha_3\beta_5+4\sqrt{3}\alpha_4\beta_4-\sqrt{3}\alpha_5\beta_3 \\ 3\sqrt{2}\alpha_1\beta_3+4\sqrt{3}\alpha_2\beta_2+3\sqrt{2}\alpha_3\beta_1-\sqrt{3}\alpha_4\beta_5-\sqrt{3}\alpha_5\beta_4 \\ 3\sqrt{2}\alpha_1\beta_4-\sqrt{3}\alpha_2\beta_3-\sqrt{3}\alpha_3\beta_2+3\sqrt{2}\alpha_4\beta_1+4\sqrt{3}\alpha_5\beta_5 \\ 3\sqrt{2}\alpha_1\beta_5-\sqrt{3}\alpha_2\beta_4+4\sqrt{3}\alpha_3\beta_3-\sqrt{3}\alpha_4\beta_2+3\sqrt{2}\alpha_5\beta_1 \end{pmatrix},$$

$$\mathbf{4}_{A,2} \sim \begin{pmatrix} \sqrt{2}\alpha_1\beta_2-\sqrt{2}\alpha_2\beta_1+\sqrt{3}\alpha_3\beta_5-\sqrt{3}\alpha_5\beta_3 \\ -\sqrt{2}\alpha_1\beta_3+\sqrt{2}\alpha_3\beta_1+\sqrt{3}\alpha_4\beta_5-\sqrt{3}\alpha_5\beta_4 \\ -\sqrt{2}\alpha_1\beta_4-\sqrt{3}\alpha_2\beta_3+\sqrt{3}\alpha_3\beta_2+\sqrt{2}\alpha_4\beta_1 \\ \sqrt{2}\alpha_1\beta_5-\sqrt{3}\alpha_2\beta_4+\sqrt{3}\alpha_4\beta_2-\sqrt{2}\alpha_5\beta_1 \end{pmatrix},$$

$$\mathbf{5}_{S,1} \sim \begin{pmatrix} 2\alpha_1\beta_1+\alpha_2\beta_5-2\alpha_3\beta_4-2\alpha_4\beta_3+\alpha_5\beta_2 \\ \alpha_1\beta_2+\alpha_2\beta_1+\sqrt{6}\alpha_3\beta_5+\sqrt{6}\alpha_5\beta_3 \\ -2\alpha_1\beta_3+\sqrt{6}\alpha_2\beta_2-2\alpha_3\beta_1 \\ -2\alpha_1\beta_4-2\alpha_4\beta_1+\sqrt{6}\alpha_5\beta_5 \\ \alpha_1\beta_5+\sqrt{6}\alpha_2\beta_4+\sqrt{6}\alpha_4\beta_2+\alpha_5\beta_1 \end{pmatrix},$$

$$\mathbf{5}_{S,2} \sim \begin{pmatrix} 2\alpha_1\beta_1-2\alpha_2\beta_5+\alpha_3\beta_4+\alpha_4\beta_3-2\alpha_5\beta_2 \\ -2\alpha_1\beta_2-2\alpha_2\beta_1+\sqrt{6}\alpha_4\beta_4 \\ \alpha_1\beta_3+\alpha_3\beta_1+\sqrt{6}\alpha_4\beta_5+\sqrt{6}\alpha_5\beta_4 \\ \alpha_1\beta_4+\sqrt{6}\alpha_2\beta_3+\sqrt{6}\alpha_3\beta_2+\alpha_4\beta_1 \\ -2\alpha_1\beta_5+\sqrt{6}\alpha_3\beta_3-2\alpha_5\beta_1 \end{pmatrix}. \tag{A.13}$$

Appendix B. Golden Littlest seesaw with $\Phi_{\rm atm} \propto \Phi_2$

In this Appendix, we shall consider a second possible golden Littlest seesaw model which corresponds to $\Phi_{\rm atm} \propto \Phi_2$. Similar to section 3, the most general from of the solar vacuum $\Phi_{\rm sol}$ is given by Eq. (3.1), and the atmospheric alignment vector takes the form

$$\Phi_{\rm atm} \propto \left(\sqrt{2},\ \phi,\ \phi\right)^T, \tag{B.1}$$

which preserves the residual symmetry $G_{\rm atm} = Z_2^{T^3ST^2ST^3}$. Subsequently we can read out the Dirac neutrino mass matrix M_D and the right-handed neutrino mass matrix M_N as

$$M_D = \begin{pmatrix} \sqrt{2}a & \sqrt{2}b \\ \phi a & (\phi-x)b \\ \phi a & (\phi+x)b \end{pmatrix}, \qquad M_N = \begin{pmatrix} M_{\rm atm} & 0 \\ 0 & M_{\rm sol} \end{pmatrix}, \tag{B.2}$$

which leads to the following low energy effective Majorana neutrino mass matrix

Table 5
Benchmark numerical results in the golden Littlest seesaw for the case of $\Phi_{\rm atm} \propto \Phi_2$ and $x = \pm 2i\phi^2 \sin(2\pi/5)$.

η	r	x	$\sin^2\theta_{13}$	$\sin^2\theta_{12}$	$\sin^2\theta_{23}$	δ_{CP}/π	β/π	m_2^2/m_3^2
π	0.675	$\pm 2i\phi^2 \sin\frac{2\pi}{5}$	0.0257	0.257	0.5	±0.5	0	0.0294
$\pm\frac{4\pi}{5}$	0.670	$\pm 2i\phi^2 \sin\frac{2\pi}{5}$	0.0282	0.255	0.535	±0.465	∓0.203	0.0293
$\pm\frac{4\pi}{5}$	0.669	$\mp 2i\phi^2 \sin\frac{2\pi}{5}$	0.0282	0.255	0.465	∓0.536	∓0.203	0.0294
$\pm\frac{5\pi}{6}$	0.671	$\pm 2i\phi^2 \sin\frac{2\pi}{5}$	0.0275	0.256	0.529	±0.469	∓0.169	0.0294
$\pm\frac{5\pi}{6}$	0.67	$\mp 2i\phi^2 \sin\frac{2\pi}{5}$	0.0274	0.256	0.47	∓0.531	∓0.169	0.0295
$\pm\frac{6\pi}{7}$	0.672	$\pm 2i\phi^2 \sin\frac{2\pi}{5}$	0.027	0.256	0.526	±0.473	∓0.145	0.0294
$\pm\frac{6\pi}{7}$	0.671	$\mp 2i\phi^2 \sin\frac{2\pi}{5}$	0.027	0.256	0.474	∓0.527	∓0.145	0.0295
$\pm\frac{7\pi}{8}$	0.673	$\pm 2i\phi^2 \sin\frac{2\pi}{5}$	0.0267	0.257	0.523	±0.476	∓0.127	0.0294
$\pm\frac{7\pi}{8}$	0.672	$\mp 2i\phi^2 \sin\frac{2\pi}{5}$	0.0267	0.257	0.477	∓0.524	∓0.127	0.0295
$\pm\frac{8\pi}{9}$	0.674	$\pm 2i\phi^2 \sin\frac{2\pi}{5}$	0.0265	0.257	0.52	±0.479	∓0.113	0.0294
$\pm\frac{8\pi}{9}$	0.673	$\mp 2i\phi^2 \sin\frac{2\pi}{5}$	0.0265	0.257	0.48	∓0.521	∓0.113	0.0295
$\pm\frac{9\pi}{10}$	0.674	$\pm 2i\phi^2 \sin\frac{2\pi}{5}$	0.0264	0.257	0.518	±0.481	∓0.101	0.0294
$\pm\frac{9\pi}{10}$	0.673	$\mp 2i\phi^2 \sin\frac{2\pi}{5}$	0.0263	0.257	0.482	∓0.519	∓0.101	0.0295
$\pm\frac{10\pi}{11}$	0.674	$\pm 2i\phi^2 \sin\frac{2\pi}{5}$	0.0262	0.257	0.517	±0.482	∓0.0922	0.0294
$\pm\frac{10\pi}{11}$	0.674	$\mp 2i\phi^2 \sin\frac{2\pi}{5}$	0.0262	0.257	0.483	∓0.518	∓0.0922	0.0294
$\pm\frac{11\pi}{12}$	0.674	$\pm 2i\phi^2 \sin\frac{2\pi}{5}$	0.0262	0.257	0.515	±0.484	∓0.0845	0.0294
$\pm\frac{11\pi}{12}$	0.674	$\mp 2i\phi^2 \sin\frac{2\pi}{5}$	0.0262	0.257	0.485	∓0.516	∓0.0845	0.0294
$\pm\frac{12\pi}{13}$	0.675	$\pm 2i\phi^2 \sin\frac{2\pi}{5}$	0.0261	0.257	0.514	±0.485	∓0.078	0.0294
$\pm\frac{12\pi}{13}$	0.674	$\mp 2i\phi^2 \sin\frac{2\pi}{5}$	0.0261	0.257	0.486	∓0.515	∓0.078	0.0294
$\pm\frac{13\pi}{14}$	0.675	$\pm 2i\phi^2 \sin\frac{2\pi}{5}$	0.0260	0.257	0.513	±0.486	∓0.0724	0.0294
$\pm\frac{13\pi}{14}$	0.674	$\mp 2i\phi^2 \sin\frac{2\pi}{5}$	0.0260	0.257	0.487	∓0.514	∓0.0724	0.0294
$\pm\frac{13\pi}{15}$	0.673	$\pm 2i\phi^2 \sin\frac{2\pi}{5}$	0.0268	0.256	0.524	±0.475	∓0.135	0.0294
$\pm\frac{13\pi}{15}$	0.672	$\mp 2i\phi^2 \sin\frac{2\pi}{5}$	0.0268	0.256	0.476	∓0.525	∓0.135	0.0295
$\pm\frac{14\pi}{15}$	0.675	$\pm 2i\phi^2 \sin\frac{2\pi}{5}$	0.0260	0.257	0.512	±0.487	∓0.0676	0.0294
$\pm\frac{14\pi}{15}$	0.674	$\mp 2i\phi^2 \sin\frac{2\pi}{5}$	0.0260	0.257	0.488	∓0.513	∓0.0676	0.0294

$$m_\nu = m_a \begin{pmatrix} 2 & \sqrt{2}\phi & \sqrt{2}\phi \\ \sqrt{2}\phi & \phi+1 & \phi+1 \\ \sqrt{2}\phi & \phi+1 & \phi+1 \end{pmatrix} + m_b e^{i\eta} \begin{pmatrix} 2 & \sqrt{2}(\phi-x) & \sqrt{2}(x+\phi) \\ \sqrt{2}(\phi-x) & (x-\phi)^2 & -x^2+\phi+1 \\ \sqrt{2}(x+\phi) & -x^2+\phi+1 & (x+\phi)^2 \end{pmatrix}, \tag{B.3}$$

with $m_a = |a|^2/M_{\rm atm}$, $m_b = |b|^2/M_{\rm sol}$ and $\eta = \arg(b^2/a^2)$. This model is rather predictive since only four parameters m_a, m_b, x and η can describe the entire neutrino sector. The symmetry relations in Eq. (3.25) are also satisfied in this case. The neutrino mass matrix in Eq. (B.3) can be block diagonalized by the GR mixing matrix,

$$m'_\nu = U_{GR}^T m_\nu U_{GR} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & y & z \\ 0 & z & w \end{pmatrix}, \tag{B.4}$$

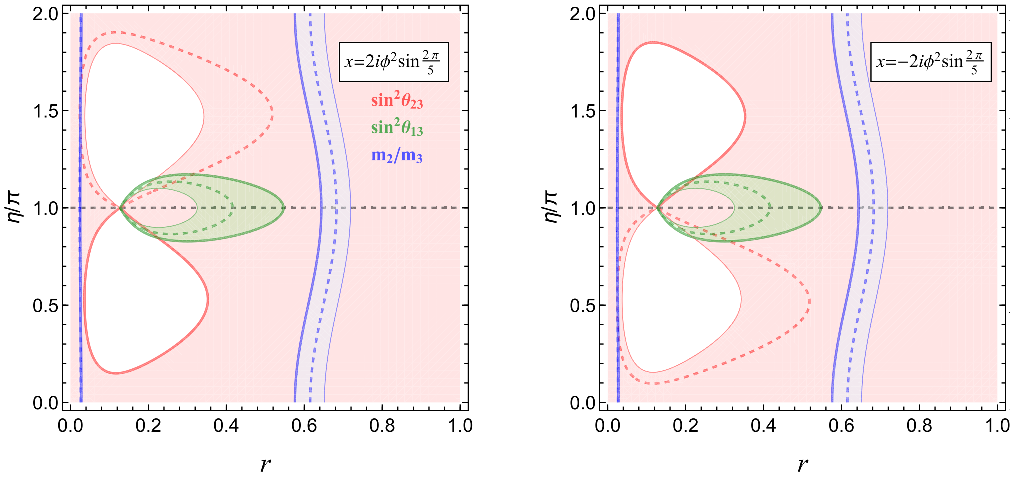

Fig. 4. Contour plots of $\sin^2\theta_{13}$, $\sin^2\theta_{23}$ and m_2/m_3 in the $\eta - r$ plane for the golden Littlest seesaw with $\Phi_{\rm atm} \propto \Phi_2$. Here we take $x = 2i\phi^2\sin(2\pi/5)$ and $x = -2i\phi^2\sin(2\pi/5)$ for which the solar vacuum alignment $\Phi_{\rm sol}$ preserves the residual symmetry $G_{\rm sol} = Z_3^{T^3ST^2S}$ and $G_{\rm sol} = Z_3^{ST^2ST^3}$ respectively. The 3σ upper (lower) bounds of the lepton mixing angles are labelled with thick (thin) solid curves, and the dashed contour lines represent the corresponding best fit values. The 3σ ranges as well as the best fit values of the mixing angles are adapted from [1]. The black contour line refers to maximal atmospheric mixing angle with $\sin^2\theta_{23} = 0.5$.

where

$$\begin{aligned} y &= |y|e^{i\phi_y} = 2\sqrt{5}\,\phi\left(m_a + m_b\,e^{i\eta}\right), \\ z &= 2x\sqrt{\phi+2}\,m_b\,e^{i\eta}, \\ w &= 2x^2\,m_b\,e^{i\eta}\,. \end{aligned} \tag{B.5}$$

Furthermore, m'_ν can be put into diagonal form by performing another unitary transformation

$$U'^T m'_\nu U' = \mathrm{diag}(0, m_2, m_3)\,, \tag{B.6}$$

with

$$U' = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta\,e^{i(\psi+\rho)/2} & \sin\theta\,e^{i(\psi+\sigma)/2} \\ 0 & -\sin\theta\,e^{i(-\psi+\rho)/2} & \cos\theta\,e^{i(-\psi+\sigma)/2} \end{pmatrix}, \tag{B.7}$$

where the parameters θ, ψ, ρ and σ are determined in terms of x, y, z defined in Eq. (B.5),

$$\begin{aligned} \sin 2\theta &= \frac{-2iz\,e^{-i\eta}\sqrt{|y|^2+|w|^2-2|y||w|\cos(\phi_y-\eta)}}{\sqrt{(|w|^2-|y|^2)^2+4|z|^2\left[|y|^2+|w|^2-2|y||w|\cos(\phi_y-\eta)\right]}}, \\ \cos 2\theta &= \frac{|w|^2-|y|^2}{\sqrt{(|w|^2-|y|^2)^2+4|z|^2\left[|y|^2+|w|^2-2|y||w|\cos(\phi_y-\eta)\right]}}, \\ \sin\psi &= \frac{|y|\cos(\phi_y-\eta)-|w|}{\sqrt{|y|^2+|w|^2-2|y||w|\cos(\phi_y-\eta)}}, \end{aligned}$$

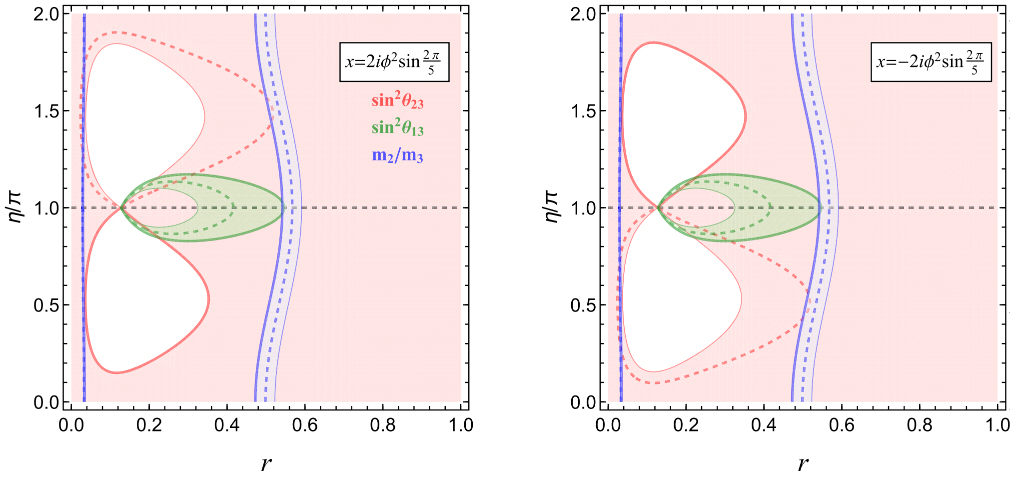

Fig. 5. Contour plots of $\sin^2\theta_{13}$, $\sin^2\theta_{23}$ and m_2/m_3 in the $\eta - r$ plane for the golden Littlest seesaw with $\Phi_{\text{atm}} \propto \Phi_2$. As an example, we assume that the decoupled alignment $\Phi_{\text{dec}} \propto \Phi_1$ which gives rise to $m_1 = 6 \times 10^{-3}$ eV.

$$\begin{aligned}
\cos\psi &= \frac{|y|\sin(\phi_y-\eta)}{\sqrt{|y|^2+|w|^2-2|y||w|\cos(\phi_y-\eta)}},\\
\sin\rho &= -\frac{(m_2^2-|z|^2)\cos\eta-|y||w|\cos\phi_y}{m_2\sqrt{|y|^2+|w|^2-2|y||w|\cos(\phi_y-\eta)}},\\
\cos\rho &= \frac{-(m_2^2-|z|^2)\sin\eta+|y||w|\sin\phi_y}{m_2\sqrt{|y|^2+|w|^2-2|y||w|\cos(\phi_y-\eta)}},\\
\sin\sigma &= -\frac{(m_3^2-|z|^2)\cos\eta-|y||w|\cos\phi_y}{m_3\sqrt{|y|^2+|w|^2-2|y||w|\cos(\phi_y-\eta)}},\\
\cos\sigma &= \frac{-(m_3^2-|z|^2)\sin\eta+|y||w|\sin\phi_y}{m_3\sqrt{|y|^2+|w|^2-2|y||w|\cos(\phi_y-\eta)}}.
\end{aligned} \tag{B.8}$$

The exact expressions for the neutrino masses are given by

$$\begin{aligned}
m_1^2 &= 0,\\
m_2^2 &= \frac{1}{2}\left[|y|^2+|w|^2+2|z|^2-\frac{|w|^2-|y|^2}{\cos 2\theta}\right],\\
m_3^2 &= \frac{1}{2}\left[|y|^2+|w|^2+2|z|^2+\frac{|w|^2-|y|^2}{\cos 2\theta}\right].
\end{aligned} \tag{B.9}$$

Furthermore, the charged lepton mass matrix is diagonal due to the Z_5^T residual symmetry. Therefore the lepton mixing matrix U is identical with the one of Eq. (3.16), whose first column is fixed to be that of the GR mixing matrix. Hence all the mixing angles and CP invariants are predicted to have the same form as those of Eq. (3.19) and Eq. (3.23) respectively, but their dependence on the input parameters m_a, m_b, η and x are different. The sum rules in Eq. (3.20) and Eq. (3.24) are satisfied as well. Detailed numerical analyses show that accordance with experimental data can be achieved for certain values of $r = m_b/m_a$ and η in the case of $x = \pm 2i\phi^2 \sin\frac{2\pi}{5}$, and the corresponding benchmark numerical results are listed in Table 5. The most interesting point is $\eta = \pi$ which predicts maximal atmospheric mixing and a maximal Dirac phase. The realistic values of $\sin^2\theta_{12}$ and m_2^2/m_3^2 can be obtained for $r = 1.486$ while the reactor angle is slightly a bit larger. This mixing pattern for $\eta = \pi$ can also be obtained from A_5 flavor symmetry and CP in the semidirect approach [22,29,30], the additional bonus in GLS is the prediction for neutrino masses. As discussed in above, all the mixing parameters as well as mass ratio m_2/m_3 depend only on η and r, this dependence is shown in Fig. 4.

If we further take into account the contribution of the third almost decoupled right-handed neutrino of mass M_{dec}, for example for the case of $\Phi_{\text{dec}} \propto \Phi_1$, the last term of Eq. (2.3) would contribute to the lightest neutrino mass $m_1 = c^2/M_{\text{dec}}$, while the neutrino mixing angles, CP violating phases and the other two neutrino masses are not changed. From Fig. 5, we can see that better agreement with experimental data can be achieved. The viable regions for $\sin^2\theta_{13}$, $\sin^2\theta_{23}$ and m_2/m_3 can overlap with each other.

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