An Algebraic Characterization of O-Minimal and Weakly O-Minimal MV-Chains

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Abstract

We present an algebraic characterization of both o-minimal and weakly o-minimal MV-chains by showing that a linearly ordered MV-algebra is (1) o-minimal if and only if it is finite or divisible, and (2) weakly o-minimal if and only if its first-order theory admits quantifier elimination in the language $\langle \oplus, ^*, 0\rangle$ if and only if $Rad(\mathbf{A})$ is a divisible monoid and $\mathbf{A}/Rad(\mathbf{A})$ is either finite or divisible.

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1. Introduction

A totally ordered structure $\mathbf{A}$ in a signature $\mathcal{L}$ is called o-minimal, whenever every set defined on its domain A by a first-order $\mathcal{L}$-formula $\phi(x)$ is a finite union of points and open intervals with endpoints in A [19, 20]. The study of o-minimality has led to an extensive and deep study of model-theoretic, topological and algebraic properties of several classes of ordered structures, such as ordered divisible Abelian groups, real closed fields, and

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their expansions [20]. The class of o-minimal ordered Abelian groups has been completely characterized and it coincides with the class of ordered divisible Abelian groups [19], which also are the only ordered Abelian groups having elimination of quantifiers in the language of ordered groups $\langle +, -, 0, < \rangle$ [14].

MV-algebras are a variety of structures that provide the equivalent algebraic semantics for the infinitely valued Łukasiewicz calculus [3]. One of the most remarkable properties of MV-algebras is their tight relation with lattice-ordered groups. In fact, the category of MV-algebras, with morphisms corresponding to object homomorphisms, is equivalent to the category of Abelian lattice-ordered groups with strong unit [18], with morphisms corresponding to homomorphisms preserving the strong unit. In particular, each linearly ordered MV-algebra is isomorphic to a structure definable on the unit interval of a unique (up to isomorphism) ordered Abelian group with strong unit [2, 3].

Given the connection between linearly ordered MV-algebras and ordered Abelian groups, it is worth asking if a similar characterization for o-minimal MV-chains can be given, whether it requires any form of divisibility, and whether they enjoy quantifier elimination in the language of linearly ordered MV-algebras $\mathcal{L}_{\mathrm{MV}} = \langle \oplus, ^*, 0 \rangle$.

In this work, we achieve this goal and provide a complete algebraic characterization of o-minimal MV-chains:

Theorem 1. *Let* $\mathbf{A}$ *be any* MV*-chain in the language* $\mathcal{L}_{\mathrm{MV}} = \langle \oplus, ^*, 0 \rangle$*. Then the following are equivalent:*

(1) $\mathbf{A}$ *is o-minimal.*
(2) $\mathbf{A}$ *is finite or divisible.*

Unlike ordered groups, however, the class of o-minimal MV-chains cannot be characterized in terms of elimination of quantifiers in $\mathcal{L}_{\mathrm{MV}}$. In fact, while each o-minimal MV-chain has a theory that admits quantifier elimination in $\mathcal{L}_{\mathrm{MV}}$, the converse is not true in general (see the proof of Theorem 2). To obtain such a characterization, we rely, instead, on the notion of weak o-minimality.

A totally ordered structure $\mathbf{A}$ in a signature $\mathcal{L}$ is called weakly o-minimal, whenever every set defined on its domain A by a first-order $\mathcal{L}$-formula $\phi(x)$ is a finite union of convex sets in A [5]. While o-minimal structures are obviously also weakly o-minimal, the converse is not generally true (see [15]). Still, in the case of ordered groups, both notions coincide. In particular any ordered Abelian group $\mathbf{G}$ is o-minimal if and only if it is weakly o-minimal, if and only if it is divisible, if and only if it has elimination of quantifiers in the language $\langle +, -, 0, < \rangle$ [15].

As for MV-chains, relying on the concept of weak o-minimality makes it possible to provide both a model-theoretic characterization in terms of quantifier elimination and an algebraic characterization.

Theorem 2. *Let $\mathbf{A}$ be any MV-chain, and let $\mathsf{Th}(\mathbf{A})$ be the first-order theory of $\mathbf{A}$ in the language $\mathcal{L}_{\mathrm{MV}} = \langle \oplus, ^*, 0 \rangle$. Then the following are equivalent:*

(1) $\mathbf{A}$ *is weakly o-minimal.*
(2) $\mathsf{Th}(\mathbf{A})$ *has elimination of quantifiers in* $\mathcal{L}_{\mathrm{MV}}$.
(3) $Rad(\mathbf{A})$ *is divisible, and* $\mathbf{A}/Rad(\mathbf{A})$ *is finite or divisible.*

This paper is organized as follows. In the next section, we provide some background information about MV-algebras, along with the main model-theoretic concepts that will be used in this work. In Section 3, we prove that certain classes of MV-chains have quantifier elimination in the language $\langle \oplus, ^*, 0 \rangle$. In Section 4, we shed light on the connection between quantifier elimination and weak o-minimality, and give a full algebraic characterization of both properties, leading to a proof of Theorem 2. Finally, on the basis of those results, we offer a characterization of o-minimality by giving a proof of Theorem 1.

2. Background Notions

In this section, we introduce the basic background notions we will make use of in the rest of the paper. An extensive and in-depth treatment of MV-algebras can be found in [3, 9][1], while, for a thorough and detailed presentation of Model Theory, the reader is advised to consult [10].

2.1. MV-Algebras

Definition 3. *An* MV-*algebra* $\mathbf{A}$ *is a structure* $(A, \oplus, ^*, 0)$ *of type* $\langle 2, 1, 0 \rangle$, *such that the following axioms are satisfied for every* $x, y \in A$:

[1] Notice that some of the results mentioned in this section only refer to the linearly ordered case. However, proper generalizations can be found for MV-algebras that are not necessarily totally ordered. The interested reader can consult [3, 9] and the references therein.

(MV1) $(A, \oplus, 0)$ *is an Abelian monoid,*
(MV2) $(x^*)^* = x$,
(MV3) $0^* \oplus x = 0^*$,
(MV4) $(x^* \oplus y)^* \oplus y = (y^* \oplus x)^* \oplus x$.

The class of MV-algebras forms a variety $\mathbb{MV}$ that is is generated by the algebra $[0,1]_{\text{MV}} = \langle [0,1], \oplus, ^*, 0\rangle$ over the real unit interval (see [2, 3]), where $x \oplus y$ is interpreted as $\min(x+y, 1)$ and x^* is interpreted as $1 - x$.

On each MV-algebra $\mathbf{A}$ we define

$$1 =: 0^*, \qquad nx := \overbrace{x \oplus \cdots \oplus x}^{n}, \qquad x \odot y =: (x^* \oplus y^*)^*,$$
$$x \ominus y =: x \odot y^*, \qquad d(x,y) =: (x \ominus y) \oplus (y \ominus x).$$

In the MV-algebra over $[0,1]$,

$$x \odot y = \max(0, x+y-1), \qquad x \ominus y = \max(0, x-y), \qquad d(x,y) = |x-y|.$$

For any two elements $x, y \in A$, we write $x \leq y$ iff $x \odot y^* = 0$. It follows that $\leq$ is a partial order. Every MV-algebra where the order relation $\leq$ is linear is called an MV-*chain*. Over $[0,1]_{\text{MV}}$, $\leq$ coincides with the usual order over real numbers.

There exists a strong connection between MV-chains and ordered Abelian groups. Indeed, let $\mathbf{G} = \langle G, +, -, 0_G, <\rangle$ be any ordered Abelian group, and for some positive element $u \in G$, let

$$G(u) = \{x \mid x \in G \text{ and } 0_G \leq x \leq u\}.$$

Define over $G(u)$ the operations

$$x \oplus y := \min(u, x+y); \qquad x^* := u - x.$$

Then, the structure $\Gamma(\mathbf{G}, u) = \langle G(u), \oplus, ^*, 0_G\rangle$ can be easily seen to be an MV-chain [2].

Conversely, for any linearly ordered MV-algebra $\mathbf{A} = \langle A, \oplus, ^*, 0\rangle$ define a structure $\Xi(\mathbf{A}) = \langle \Xi(A), +, -, 0_{\Xi(\mathbf{A})}, \leq_{\Xi(\mathbf{A})}\rangle$ where

$$\Xi(A) = \{(n, x) \mid n \in \mathbb{Z}, x \in A/\{1\}\},$$

and $0_{\Xi(\mathbf{A})} = (0, 0)$,

$$(n,x)+(m,y) = \begin{cases} (n+m, x\oplus y) & \text{if } x\oplus y<1 \\ (n+m+1, (x^*\oplus y^*)^*) & \text{if } x\oplus y=1 \end{cases},$$

$$-(n,x) = \begin{cases} (-n,0) & \text{if } x=0 \\ (-(n+1), x^*) & \text{if } 0<x<1 \end{cases},$$

$$(n,x) \leq_{\Xi(\mathbf{A})} (m,y) \quad \text{if} \quad n<m, \text{ or } n=m \text{ and } x\leq y.$$

The structure $\Xi(\mathbf{A})$ can be easily shown to be an ordered Abelian group, where $(1,0)$ is a strong unit [2] (i.e. for each $x \in \Xi(A)$, there exists an n such that $x \leq_{\Xi(\mathbf{A})} n(1,0)$).

Proposition 4 ([2])**.** *If* $\mathbf{A}$ *is an* MV*-chain, the mapping* $x \mapsto (0,x)$*, for all* $x \in A/\{1\}$*, and* $1 \mapsto (1,0)$*, is an isomorphism between* $\mathbf{A}$ *and* $\Gamma(\Xi(\mathbf{A}),(1,0))$*, and the element* $(1,0)\in\Xi(A)$ *is a strong unit for* $\Xi(\mathbf{A})$*. Moreover, given an ordered Abelian group* $\mathbf{G}$ *with a strong unit* u*,* $\Xi(\Gamma(\mathbf{G},u))$ *is isomorphic to* $\mathbf{G}$*.*

The above connection between MV-chains and ordered Abelian groups with strong unit goes well beyond the linearly ordered case. In fact, Mundici [18] proved that there exists an equivalence between the category of MV-algebras with homomorphisms and the category of lattice-ordered Abelian groups with strong unit, with homomorphisms preserving the strong unit.

Definition 5. *An* MV*-chain* $\mathbf{A}$ *is called* divisible *when* $\mathbf{A}\cong\Gamma(\mathbf{G},u)$ *and* $\mathbf{G}$ *is an ordered divisible Abelian group.*

Given an MV-algebra $\mathbf{A}$, a nonempty set $I\subseteq A$ is called an *ideal* if the following properties are satisfied for all $x,y\in A$: (1) $x\leq y$ and $y\in I$ imply $x\in I$; (2) x, $y\in I$ implies $x\oplus y\in I$. An ideal I is called *proper* if $I\neq A$. An ideal I is *maximal* iff it is proper and there is no proper ideal J of $\mathbf{A}$ such that $I\subset J$. An MV-algebra is *simple*, if $\{0\}$ is the only proper ideal. Moreover, every simple MV-chain is isomorphic to a subalgebra of $[0,1]_{\mathrm{MV}}$ (see [3]).

Definition 6. *The* radical *of an* MV*-algebra* $\mathbf{A}$*, denoted by* $Rad(\mathbf{A})$*, is the intersection of the maximal ideals of* $\mathbf{A}$*.*

For every MV-chain $\mathbf{A}$, $\mathbf{A}/Rad(\mathbf{A})$ is a simple MV-chain isomorphic to a subchain of $[0,1]_{\mathrm{MV}}$. An MV-chain $\mathbf{A}$ is called *radical retractive* iff there is a homomorphism $\mathfrak{f} : \mathbf{A}/Rad(\mathbf{A}) \to \mathbf{A}$ such that $\mathfrak{p} \circ \mathfrak{f}$ is the identity map, and $\mathfrak{p}$ is the canonical homomorphism from $\mathbf{A}$ onto $\mathbf{A}/Rad(\mathbf{A})$.

Notice that $\langle Rad(\mathbf{A}), \oplus, 0\rangle$ is a monoid.

Definition 7. *$Rad(\mathbf{A})$ is called* divisible *when $\langle Rad(\mathbf{A}), \oplus, 0\rangle$ is divisible as a monoid.*

Given an element x in an MV-algebra $\mathbf{A}$, $ord(x)$, the order of x, is defined to be the smallest integer such that $nx = 1$, if such an n exists, and $ord(x) = \infty$ otherwise. Every MV-chain $\mathbf{A}$ has only one maximal ideal that coincides with $Rad(\mathbf{A})$, which is exactly the set $\{x \in A \mid x \neq 0, nx \leq x^*\}$, for all $n \in \mathbb{N}$.

Let
$$S_n = \left\{0, \frac{1}{n}, \ldots, \frac{n-1}{n}, 1\right\}.$$
The structure $\mathbf{S}_n = \langle S_n, \oplus, ^*, 0\rangle$, where $\oplus$ and * are the restrictions to S_n of the operations defined over $[0,1]_{\mathrm{MV}}$, is a called a *finite* MV-chain.

Definition 8. *Let $\mathbf{A}$ be an* MV*-chain. The* order *of $\mathbf{A}$ is defined by*

$$ord(\mathbf{A}) = n \text{ iff } \mathbf{A} \cong \mathbf{S}_n.$$

Whenever $\mathbf{A} \ncong \mathbf{S}_n$, $ord(\mathbf{A}) = \infty$.

The rank *of $\mathbf{A}$ is defined by*

$$rank(\mathbf{A}) = ord(\mathbf{A}/Rad(\mathbf{A})).$$

MV-chains of finite rank were characterized by Komori in [12] (see also [3]), distinguishing between simple and non-simple structures. Simple MV-chains of finite rank are exactly finite MV-chains.

Proposition 9 ([12])**.** *Let $\mathbf{A}$ be a simple* MV*-chain of rank n. Then $\mathbf{A} \cong \mathbf{S}_n$.*

Let $\mathbb{Z}\vec{\times}\mathbf{G}$ be the lexicographic product of the group of integers $\mathbb{Z}$ and an ordered Abelian group $\mathbf{G}$. It is easily seen that $\mathbf{A} = \Gamma(\mathbb{Z}\vec{\times}\mathbf{G}, (n, g))$, where $g \in G$, is an MV-chain [3]. Moreover,

$$Rad(\mathbf{A}) = \{(0, x) \mid 0 \leq x \in G\} \neq \{(0, 0)\},$$

and $\mathbf{A}$ is a non-simple MV-chain of rank n. All non-simple MV-chains of finite rank are exactly of this form:

Proposition 10 ([12]). $\mathbf{A}$ *is a non-simple* MV*-chain of rank* n *iff* $\mathbf{A} \cong \Gamma(\mathbb{Z}\vec{\times}\mathbf{G}, (n, g))$*, for some ordered Abelian group* $\mathbf{G}$*.*

Non-simple radical retractive MV-chains of finite rank play a special role. In fact we have:

Proposition 11 ([6, 12]). *Let* $\mathbf{A}$ *be a non-simple* MV*-chain of rank* n*.* $\mathbf{A}$ *is radical retractive iff* $\mathbf{A} \cong \Gamma(\mathbb{Z}\vec{\times}\mathbf{G}, (n, 0))$*, for some ordered Abelian group* $\mathbf{G}$*. Moreover, every non-simple* MV*-chain of rank* n *is embeddable into a non-simple radical retractive* MV*-chain of the same rank.*

For a non-simple chain $\Gamma(\mathbb{Z}\vec{\times}\mathbf{G}, (n, g))$, the embedding into its related radical retractive structure $\Gamma(\mathbb{Z}\vec{\times}\mathbf{G}, (n, 0))$ is given by the map

$$\mathfrak{h}(x, y) = (x, ny - xg)$$

(see [6]). Notice that when $\mathbf{G}$ is an ordered divisible Abelian group, the mapping $\mathfrak{h}$ actually is an isomorphism.

Proposition 12. *Let* $\mathbf{A} = \Gamma(\mathbb{Z}\vec{\times}\mathbf{G}, (n, g))$ *be any non-simple* MV*-chain of rank* n*, where* $\mathbf{G}$ *is an ordered divisible Abelian group. Then,* $\mathbf{A}$ *is isomorphic to the radical retractive* MV*-chain* $\mathbf{B} = \Gamma(\mathbb{Z}\vec{\times}\mathbf{G}, (n, 0))$*.*

Proof. We know that $\mathfrak{h}(x, y) = (x, ny - xg)$ is an embedding from $\mathbf{A}$ into $\mathbf{B}$. It is easy to see that the mapping $\mathfrak{f}(x, y) = (x, (y + xg)/n)$ is an embedding from $\mathbf{B}$ into $\mathbf{A}$, and, moreover, the composition $\mathfrak{f} \circ \mathfrak{h}$ coincides with the identity mapping. □

As shown by Komori in [12], every proper subvariety of MV-algebras is generated by a finite set of MV-chains of finite rank:

Theorem 13 ([12]). *If* $\mathbb{V}$ *is a proper subvariety of* $\mathbb{MV}$*, then there exists two finite sets* X *and* Y *of integers* ≥ 1*, such that* $X \cup Y$ *is non-empty and*

$$\mathbb{V} = \mathbb{V}(\{\mathbf{S}_m \mid m \in X\} \cup \{\Gamma(\mathbb{Z}\vec{\times}\mathbb{Z}, (n, 0)) \mid n \in Y\}).$$

2.2. Model-Theoretic Notions

Definition 14. *Let* Th *be a first-order theory in some language* $\mathcal{L}$*. Then:*

(1) *We say that* Th *admits* elimination of quantifiers (QE) *in* $\mathcal{L}$ *if for every formula* $\phi(\overline{x})$ *there is a quantifier-free formula* $\psi(\overline{x})$ *that is provably equivalent to* $\phi(\overline{x})$ *in* Th*.*

(2) Th *is said to be* model-complete *if every embedding between models of* Th *is elementary, i.e.: for any* $\mathbf{A}, \mathbf{B} \models \mathsf{Th}$*, every embedding* $\mathfrak{f} : \mathbf{A} \to \mathbf{B}$*, every* $\mathcal{L}$*-formula* $\phi(x_1, \dots, x_m)$*, and* $a_1, \dots, a_m \in A$*,*

$$\mathbf{A} \models \phi(a_1, \dots, a_m) \text{ iff } \mathbf{B} \models \phi(\mathfrak{f}(a_1), \dots, \mathfrak{f}(a_m)).$$

(3) *Two* $\mathcal{L}$*-structures* $\mathbf{A}, \mathbf{B}$ *are said to be* elementarily equivalent *if, for every* $\mathcal{L}$*-sentence* ϕ*,* $\mathbf{A} \models \phi$ *iff* $\mathbf{B} \models \phi$*.*

(4) *Given a structure* $\mathbf{A}$ *in a signature* $\mathcal{L}$*, a set* $X \subseteq A$ *is said to be* (parametrically) definable *in* $\mathbf{A}$*, if there exists a formula* $\phi(x)$ *in* $\mathcal{L}$*, with parameters from* $\mathbf{A}$*, such that* $X = \{a \mid \mathbf{A} \models \phi(a)\}$*.*

Proposition 15 (Corollary 3.1.6, [17])**.** *Let* Th *be a theory in a given language* $\mathcal{L}$*.* Th *has quantifier elimination if and only if for all quantifier-free formulas* $\phi(\overline{v}, w)$*, if* $\mathbf{M}, \mathbf{N} \models \mathsf{Th}$*,* $\mathbf{A}$ *is a common substructure of* $\mathbf{M}$ *and* $\mathbf{N}$*,* $\overline{a} \in A$*, and there is* $b \in M$ *such that* $\mathbf{M} \models \phi(\overline{a}, b)$*, then there is* $c \in N$ *such that* $\mathbf{N} \models \phi(\overline{a}, c)$*.*

Definition 16 (**O-Minimality** [20, 19])**.** *A linearly ordered structure* $\mathbf{A} = \langle A, <, \dots \rangle$ *is said to be* o-minimal *if every parametrically definable subset of* A *is a finite union of points and open intervals in* A*, with endpoints in* $A \cup \{-\infty, +\infty\}$*. A first-order theory* Th *is said to be o-minimal if every model of* Th *is o-minimal.*

Definition 17 (**Weak O-Minimality** [5, 15])**.** *A linearly ordered structure* $\mathbf{A} = \langle A, <, \dots \rangle$ *is said to be* weakly o-minimal *if every parametrically definable subset of* A *is a finite union of convex sets in* A*. A first-order theory* Th *is said to be weakly o-minimal if every model of* Th *is weakly o-minimal.*

3. Quantifier Elimination

In this section, we are going to see that the structures belonging to certain classes of MV-chains have QE in the language $\mathcal{L}_{\mathrm{MV}} = \langle \oplus, ^*, 0 \rangle$. Notice that the order relation $<$ is actually definable in $\mathcal{L}_{\mathrm{MV}}$. In fact, in every MV-chain $\mathbf{A}$, for all $x, y \in A$:

$$x < y \text{ iff } \neg(y^* \oplus x = 0^*).$$

The goal is to prove that, given an MV-chain $\mathbf{A}$, if $Rad(\mathbf{A})$ is divisible and $\mathbf{A}/Rad(\mathbf{A})$ is finite or divisible, then $\mathsf{Th}(\mathbf{A})$ has QE in $\mathcal{L}_{\mathrm{MV}}$. We begin by showing that any MV-chain $\mathbf{A}$ satisfying this property belongs to one of the following classes:

(a) finite MV-chains;

(b) non-simple MV-chains of finite rank $\Gamma(\mathbb{Z}\vec{\times}\mathbf{G},(n,0))$, where $\mathbf{G}$ is a ordered divisible Abelian group;

(c) divisible MV-chains.

Lemma 18. *Let* $\mathbf{A}$ *be an* MV*-chain.*

(1) *If* $Rad(\mathbf{A})$ *is divisible and* $\mathbf{A}/Rad(\mathbf{A})$ *is finite, then either* $\mathbf{A}\cong\mathbf{S}_n$ *or* $\mathbf{A}\cong\Gamma(\mathbb{Z}\times\mathbf{G},(n,0))$*, where* $\mathbf{G}$ *is divisible.*

(2) *If* $Rad(\mathbf{A})$ *is divisible and* $\mathbf{A}/Rad(\mathbf{A})$ *is divisible, then* $\mathbf{A}$ *is divisible.*[2]

Proof. (1) If $\mathbf{A}/Rad(\mathbf{A})$ is finite, then, trivially, $\mathbf{A}$ has finite rank. In particular, if $Rad(\mathbf{A})$ is divisible, then either

(a) $Rad(\mathbf{A})$ is $\{0\}$ and $\mathbf{A}$ is simple and so, by Proposition 9, it is isomorphic to a finite MV-chain $\mathbf{S}_n$, or

(b) by Proposition 10, $\mathbf{A}\cong\Gamma(\mathbb{Z}\times\mathbf{G},(n,g))$, with $\mathbf{G}$ divisible, which in turn, by Proposition 12, is isomorphic to $\Gamma(\mathbb{Z}\times\mathbf{G},(n,0))$.

(2) Suppose that $Rad(\mathbf{A})$ is divisible and $\mathbf{A}/Rad(\mathbf{A})$ is divisible. Since $\mathbf{A}\cong\Gamma(\mathbf{G},u)$, for some ordered Abelian group $\mathbf{G}$, without any loss of generality, we treat $\mathbf{A}$ directly as $\Gamma(\mathbf{G},u)$, i.e. we assume $\mathbf{A}$ to be the MV-chain defined over the interval $[0,u]$ in $\mathbf{G}$.[3]

Our aim is to prove that $\mathbf{G}$ is divisible. In fact, if that is the case, then $\mathbf{A}$ is divisible as well.

By [3, Theorem 7.2.2], the mapping

$$\mathfrak{q}: I\mapsto\mathfrak{q}(I)=\{x\in G\mid\min(\max(x,-x),u))\in I\}$$

defines an isomorphism between the set of ideals of $\mathbf{A}$ and the set of ideals of $\mathbf{G}$. By [3, Lemma 7.3.2], the set

$$\mathfrak{q}(Rad(\mathbf{A}))=\{x\in G\mid\max(x,-x)\in Rad(\mathbf{A})\}$$

is the ideal of $\mathbf{G}$ associated to $Rad(\mathbf{A})$ under $\mathfrak{q}$, and is an ordered subgroup of $\mathbf{G}$. In particular, $Rad(\mathbf{A})$ coincides with the set of non-negative elements of $\mathfrak{q}(Rad(\mathbf{A}))$, i.e.

$$Rad(\mathbf{A})=\{x\in\mathfrak{q}(Rad(\mathbf{A}))\mid x\geq 0\}.$$

[2]Notice that this fact is also a consequence of a more general result from [7].

[3]This proof makes heavy use of several results from [3]. The reader interested in the details should consult the references mentioned above, and, in particular, [3, Chapter 7].

Consequently, if $Rad(\mathbf{A})$ is a divisible monoid, $\mathfrak{q}(Rad(\mathbf{A}))$ is a divisible subgroup of $\mathbf{G}$.
By [3, Theorem 7.2.4],

$$\mathbf{A}/Rad(\mathbf{A}) \cong \Gamma(\mathbf{G}/\mathfrak{q}(Rad(\mathbf{A})), u/\mathfrak{q}(Rad(\mathbf{A}))).$$

Since, $\mathbf{A}/Rad(\mathbf{A})$ is divisible, $\mathbf{G}/\mathfrak{q}(Rad(\mathbf{A}))$ is a divisible group.
The functor Γ (along with its adjoint Ξ) defines an equivalence between the category of ordered Abelian groups with strong unit and the category of MV-chains, both with homomorphisms (see Section 2 and [3]). So, given the canonical homomorphism $\mathfrak{p} : \mathbf{A} \to \mathbf{A}/Rad(\mathbf{A})$, there exists a unique homomorphism $\mathfrak{v} : \mathbf{G} \to \mathbf{G}/\mathfrak{q}(Rad(\mathbf{A}))$ such that $\Gamma(\mathfrak{v}) = \mathfrak{p}$:

$$\begin{array}{ccc} \mathbf{A} & \xrightarrow{\ \mathfrak{p}\ } & \mathbf{A}/Rad(\mathbf{A}) \\ \Gamma \uparrow & & \uparrow \Gamma \\ \mathbf{G} & \xrightarrow[\ \mathfrak{v}\ ]{} & \mathbf{G}/\mathfrak{q}(Rad(\mathbf{A})) \end{array}$$

As shown above, $\mathbf{G}/\mathfrak{q}(Rad(\mathbf{A}))$ is divisible. Therefore, for every $x \in G$ and for every $n \geq 1$, there exists some $y \in G$, such that $\mathfrak{v}(x) = n(\mathfrak{v}(y))$. This means that $x - ny \in \mathfrak{q}(Rad(\mathbf{A}))$. Since $\mathfrak{q}(Rad(\mathbf{A}))$ is divisible, there exists some z such that $x - ny = nz$, and so $x = n(y + z)$. Consequently, $\mathbf{G}$ is a divisible group, which implies that $\mathbf{A}$ is a divisible MV-chain.

□

3.1. Finite MV-Chains

Recall that a structure $\mathbf{A}$ is called *ultrahomogeneous* whenever every isomorphism between any of its subalgebras can be extended to an automorphism of $\mathbf{A}$. A finite structure is ultrahomogeneous if and only if its first-order theory has QE (see [10, Corollary 8.4.2]).

If $\mathbf{A}$ is a finite MV-chain $\mathbf{S}_n$, in $\mathcal{L}_{\mathrm{MV}}$, then it is easy to check that $\mathbf{A}$ is ultrahomogeneous. Indeed, the only isomorphism between subalgebras of $\mathbf{S}_n$ is the identity mapping. Consequently, for any finite MV-chain $\mathbf{S}_n$, $\mathsf{Th}(\mathbf{S}_n)$ has quantifier elimination in $\mathcal{L}_{\mathrm{MV}}$.

This result was first given by Baaz and Veith in [1]. Here we offer a different and direct proof of the same fact.

Lemma 19. *Let* $\mathbf{A}$ *be any finite* MV*-chain* $\mathbf{S}_n$*.* $\mathsf{Th}(\mathbf{A})$ *has quantifier elimination in* $\mathcal{L}_{\mathrm{MV}}$*.*

Proof. We are going to see that, in every finite MV-chain $\mathbf{S}_n$, each single element of

$$S_n = \left\{0, \frac{1}{n}, \ldots, \frac{n-1}{n}, 1\right\}$$

is definable by a quantifier-free formula.

We show that for each $0 \leq i \leq n$, where i and n are coprime, there exists an MV-term $p_{i,n}(x)$ such that

$$x = \tfrac{i}{n} \qquad \text{if and only if} \qquad p_{i,n}(x) = 0.$$

If i and n are not coprime, then

$$x = \tfrac{i}{n} \qquad \text{if and only if} \qquad p_{j,m}(x) = 0,$$

where j and m are coprime and $\frac{i}{n} = \frac{j}{m}$.

Let $q_{i,n}$ and $r_{i,n}$ denote the quotient and the remainder, respectively, of the Euclidean division of n by i.

For $i = 0$, trivially,

$$p_{0,n}(x) = x.$$

When $i = 1$,

$$p_{1,n}(x) = d(((n-1)x)^*, x).$$

It is easy to check that

$$x = \tfrac{1}{n} \qquad \text{if and only if} \qquad d(((n-1)x)^*, x) = 0.$$

For $i \geq 2$, the proof proceeds by induction. For i and n coprime, let

$$p_{i,n}(x) = p_{((r_{i,n}),n)}((q_{i,n})x)^*).$$

Notice that $r_{i,n} < i$.

So, if $i = 2$, then

$$p_{2,n}(x) = p_{1,n}((q_{i,n}x)^*) = d(((n-1)((q_{i,n}x)^*))^*, (q_{i,n}x)^*).$$

For $i > 2$, the result follows by induction.

Finally, since every set X defined over S_n by a formula $\varphi(\overline{x})$ in $\mathcal{L}_{\mathrm{MV}}$ includes only finitely many elements, the above shows that $\varphi(\overline{x})$ is equivalent to a finite union of equations $p_{i,n}(x) = 0$ each defining an element of X. This concludes the proof. $\square$

3.2. Non-Simple MV-Chains of Finite Rank

Quantifier elimination also holds for all those MV-chains $\mathbf{A}$ of finite rank that are isomorphic to some MV-chain $\Gamma(\mathbb{Z}\vec{\times}\mathbf{G}, (n, 0))$ of an ordered Abelian group $\mathbb{Z}\vec{\times}\mathbf{G}$, where $\mathbf{G}$ is divisible. Any such $\mathbf{A} \cong \Gamma(\mathbb{Z}\vec{\times}\mathbf{G}, (n, 0))$ belongs to the variety $\mathbb{V}$ generated by $\Gamma(\mathbb{Z}\vec{\times}\mathbb{Z}, (n, 0))$ (see, [6, Theorem 7.2]). The theory $\mathsf{Th}(\mathbf{A})$ of $\Gamma(\mathbb{Z}\vec{\times}\mathbf{G}, (n, 0))$ is axiomatizable in $\mathcal{L}_{\mathrm{MV}}$ by taking the universal closure of the equations defining $\mathbb{V}$, the sentence defining the linearity of the order relation $<$, the sentence

$$\exists x \ \left(\underbrace{x \oplus \cdots \oplus x}_{n+1} = 1\right) \sqcap \left(\underbrace{x \oplus \cdots \oplus x}_{n} < 1\right),$$

(where $\sqcap$ denotes the classical conjunction), and the sentences

$$\forall x \exists y \ ((n+1)x < 1) \to (x = my),$$

for all $m \geq 1$, which state that the set of elements $\{(0, x) \mid x \in G\}$ is m-divisible.

To show quantifier elimination for $\mathsf{Th}(\mathbf{A})$, we are going to see that it is possible to interpret $\mathsf{Th}(\mathbf{A})$ into the theory of the ordered Abelian groups $\mathbb{Z}\vec{\times}\mathbf{G}$, where $\mathbf{G}$ is an ordered divisible Abelian group. Komori [11] showed that the theory $\mathsf{Th}(\mathbb{Z}\vec{\times}\mathbf{G})$ of any such group has QE in the language of Presburger Arithmetic (see [17])

$$\langle +, -, <, 0, 1, \{m|\}_{m \in \mathbb{N}} \rangle,$$

where each $m|$ is a unary predicate denoting the elements divisible by m, and 1 is interpreted as the element $(1, 0)$.[4]

We introduce an appropriate notion of interpretation, and show that $\mathsf{Th}(\mathbf{A})$ can be translated into $\mathsf{Th}(\mathbb{Z}\vec{\times}\mathbf{G})$. This will make it possible to prove that $\mathsf{Th}(\mathbf{A})$ inherits QE from $\mathsf{Th}(\mathbb{Z}\vec{\times}\mathbf{G})$.

[4]The same result was obtained by Weispfenning in [21, 22] as a consequence of a general characterization of quantifier eliminable ordered Abelian groups.

Let $\mathcal{L}$ be a signature of the form $\langle <, f_1, \ldots, f_n, c_1, \ldots, c_m \rangle$, where each f_i is a function symbol and each c_j is a constant symbol. $\mathcal{L}$ will be assumed to include no relation symbol but $<$. By an *unnested atomic formula* in $\mathcal{L}$ we mean one of the following formulas:

(i)	$x = y$,	$(x < y)$;	
(ii)	$x = c$,	$(x < c)$,	for some constant symbol $c \in \mathcal{L}$;
(iii)	$f(\overline{x}) = y$,	$(f(\overline{x}) < y)$,	for some function symbol $f \in \mathcal{L}$.

A formula is called unnested if all its atomic subformulas are unnested. Then it is an easy exercise to see ([10]):

Lemma 20. *For a first-order language $\mathcal{L} = \langle <, f_1, \ldots, f_n, c_1, \ldots, c_m \rangle$, every formula is equivalent to an unnested formula.*

The following definition sets what it means for a theory Th_1 in the language $\mathcal{L}_1$ to be interpretable in a theory Th_2 in the language $\mathcal{L}_2$.

Definition 21. *Let Th_1 and Th_2 be two theories in the languages $\mathcal{L}_1$ and $\mathcal{L}_2$, respectively.* Th_1 *is* interpretable *in* Th_2 *if*

- (i) *there exists an $\mathcal{L}_2$-formula $\chi(z)$,*
- (ii) *there exists a map $\sharp$ from the set of unnested atomic $\mathcal{L}_1$-formulas into the set of $\mathcal{L}_2$ formulas,*
- (iii) *there exists a map $\star$ from the set of models of Th_1 into the set of models of Th_2,*

such that, for every $\mathbf{M} \models \mathsf{Th}_1$, there exists a bijection

$$\mathfrak{h}_{\mathbf{M}} \colon M \to \{a \mid \mathbf{M}^\star \models \chi(a)\}$$

from the domain of $\mathbf{M}$ into the set defined by $\chi(z)$ over the domain of $\mathbf{M}^\star$, and, for all $\overline{b} \in M$ and each unnested atomic $\mathcal{L}_1$-formula ϕ,

$$\mathbf{M} \models \phi(\overline{b}) \qquad \text{iff} \qquad \mathbf{M}^\star \models \phi^\sharp(\mathfrak{h}_{\mathbf{M}}(\overline{b})).$$

The above definition together with Lemma 20 yields that the interpretation of Th_1 into Th_2 can be extended to arbitrary formulas.

Lemma 22. *Let* Th_1 *and* Th_2 *be two theories in the languages* $\mathcal{L}_1$ *and* $\mathcal{L}_2$*, respectively. Suppose that* Th_1 *is interpretable in* Th_2*. Then, for each* $\mathcal{L}_1$*-formula* $\phi(\overline{x})$ *there exists an* $\mathcal{L}_2$*-formula* $\phi^\sharp(\overline{x})$ *so that, for every* $\mathbf{M} \models \mathsf{Th}_1$ *and all* $\overline{b} \in M$*,*

$$\mathbf{M} \models \phi(\overline{b}) \qquad \text{iff} \qquad \mathbf{M}^\star \models \phi^\sharp(\mathfrak{h}_\mathbf{M}(\overline{b})).$$

Then, we can easily show:

Lemma 23. *Let* $\mathbf{A}$ *be an* MV*-chain of finite rank such that* $\mathbf{A} \cong \Gamma(\mathbb{Z}\vec{\times}\mathbf{G}, (n, 0))$ *where* $\mathbf{G}$ *is an ordered divisible Abelian group.* $\mathsf{Th}(\mathbf{A})$ *is interpretable into* $\mathsf{Th}(\mathbb{Z}\vec{\times}\mathbf{G})$*.*

Proof. We know that every $\mathbf{B} \models \mathsf{Th}(\mathbf{A})$ is (up to isomorphism) an MV-chain of the form $\Gamma(\mathbb{Z}\vec{\times}\mathbf{H}, (n, 0))$ where $\mathbf{H}$ is an ordered divisible Abelian group. Moreover, by [12], $\mathbb{Z}\vec{\times}\mathbf{H}$ is a model of $\mathsf{Th}(\mathbb{Z}\vec{\times}\mathbf{G})$. The domain of $\mathbf{B}$ is definable in $\langle +, -, <, 0, 1, \{m|\}_{m\in\mathbb{N}}\rangle$ over $\mathbb{Z}\vec{\times}\mathbf{H}$, with the formula

$$\chi(x) := ((x = 0) \vee (x = n1) \vee ((0 < x) \wedge (x < n1)),$$

and $\mathfrak{h}_\mathbf{B}$ corresponds to the isomorphism between $\mathbf{B}$ and $\Gamma(\mathbb{Z}\vec{\times}\mathbf{H}, (n, 0))$.

It is trivial to see that unnested formulas in $\mathcal{L}_{\mathrm{MV}}$ can be translated into formulas in $\langle +, -, <, 0, 1, \{m|\}_{m\in\mathbb{N}}\rangle$. Consequently, $\mathsf{Th}(\mathbf{A})$ is interpretable into $\mathsf{Th}(\mathbb{Z}\vec{\times}\mathbf{G})$. □

Now, we can prove:

Lemma 24. *Let* $\mathbf{A}$ *be an* MV*-chain of finite rank such that* $\mathbf{A} \cong \Gamma(\mathbb{Z}\vec{\times}\mathbf{G}, (n, 0))$ *where* $\mathbf{G}$ *is an ordered divisible Abelian group.* $\mathsf{Th}(\mathbf{A})$ *has quantifier elimination in* $\mathcal{L}_{\mathrm{MV}}$*.*

Proof. Let $\mathbf{C}, \mathbf{D} \models \mathsf{Th}(\mathbf{A})$ and $\mathbf{B}$ be a common substructure of $\mathbf{C}$ and $\mathbf{D}$. Suppose that for all quantifier-free formulas $\phi(\overline{v}, w)$, $\overline{b} \in B$, there is $c \in C$ such that $\mathbf{C} \models \phi\left(\overline{b}, c\right)$. By Proposition 15, we just need to show that there is $d \in D$ such that $\mathbf{D} \models \phi\left(\overline{b}, d\right)$.

Now, $\mathbf{C} \cong \Gamma(\mathbb{Z}\vec{\times}\mathbf{H}, (n, 0))$ and $\mathbf{D} \cong \Gamma(\mathbb{Z}\vec{\times}\mathbf{I}, (n, 0))$, where $\mathbf{H}$ and $\mathbf{I}$ are ordered divisible Abelian groups. Since $\mathbf{B} \subseteq \mathbf{C}, \mathbf{D}$, then $\mathbf{B}$ is isomorphic to the MV-chain $\Gamma(\mathbf{J}, (n, 0))$ of an ordered subgroup $\mathbf{J}$ of $\mathbb{Z}\vec{\times}\mathbf{H}$ and $\mathbb{Z}\vec{\times}\mathbf{I}$, with the same strong unit (see [3]).

By Lemma 23,

$$\mathbf{C} \models \phi(\overline{b}, c) \text{ iff } \mathbb{Z}\vec{\times}\mathbf{H} \models \phi^\sharp(\overline{b}, c).$$

$\mathsf{Th}(\mathbb{Z}\vec{\times}\mathbf{G})$ admits elimination of quantifiers in $\langle +, -, <, 0, 1, \{m|\}_{m\in\mathbb{N}}\rangle$, and so by Proposition 15 there is $d \in \mathbb{Z}\vec{\times}\mathbf{I}$ such that $\mathbb{Z}\vec{\times}\mathbf{I} \models \phi^\sharp(\bar{b}, d)$. By Lemma 23,

$$\mathbf{D} \models \phi(\bar{b}, d) \text{ iff } \mathbb{Z}\vec{\times}\mathbf{I} \models \phi^\sharp(\bar{b}, d),$$

and consequently, by Proposition 15, $\mathsf{Th}(\mathbf{A})$ has QE. □

3.3. Divisible MV-Chains

Finally, we deal with the theory of divisible MV-chains, i.e. those structures $\Gamma(\mathbf{G}, u)$, where $\mathbf{G}$ is an ordered divisible Abelian group and u a strong unit (see Definition 5). Divisible MV-chains are the models of the theory obtained by adding to the first-order theory of MV-chains the sentence $\forall x \forall y (x \leq y \vee y \leq x)$ plus the sentences

$$\forall x \exists y \; x = py, \qquad \exists x \; (p-1)x = x^*.$$

for each prime number p[5]. Note that these two sentences may be replaced by a single one: $\forall x \exists y ((p-1)y = x \ominus y)$, again for every prime number p.

The fact that for any divisible MV-chain $\mathbf{A}$, $\mathsf{Th}(\mathbf{A})$ has QE in $\mathcal{L}_{\mathrm{MV}}$ is well-known and different proofs can be found in [1, 4, 16].

Lemma 25. *Let* $\mathbf{A}$ *be any divisible* MV*-chain. Then* $\mathsf{Th}(\mathbf{A})$ *has quantifier elimination in* $\mathcal{L}_{\mathrm{MV}}$*.*

We now prove that certain algebraic conditions are sufficient to guarantee quantifier elimination in $\mathcal{L}_{\mathrm{MV}}$ for an MV-chain $\mathbf{A}$.

Theorem 26. *Let* $\mathbf{A}$ *be an* MV*-chain, and suppose that one of the following conditions holds:*

(1) *$Rad(\mathbf{A})$ is divisible and $\mathbf{A}/Rad(\mathbf{A})$ is finite.*
(2) *$Rad(\mathbf{A})$ is divisible and $\mathbf{A}/Rad(\mathbf{A})$ is divisible.*

Then $\mathsf{Th}(\mathbf{A})$ *has quantifier elimination in* $\mathcal{L}_{\mathrm{MV}}$*.*

---

[5]This axiomatization of the theory of divisible MV-chains was first given by Lacava and Saeli in [13].

Proof. By Lemma 18, we know that: (a) if $Rad(\mathbf{A})$ is divisible and $\mathbf{A}/Rad(\mathbf{A})$ is finite, then either $\mathbf{A}$ is isomorphic to a finite MV-chain $\mathbf{S}_n$, or $\mathbf{A} \cong \Gamma(\mathbb{Z} \times \mathbf{G}, (n, g))$, where $\mathbf{G}$ is divisible; (b) if $Rad(\mathbf{A})$ is divisible and $\mathbf{A}/Rad(\mathbf{A})$ is divisible, then $\mathbf{A}$ is divisible as well.

Lemma 19, Lemma 24 and Lemma 25 show that in all the above cases $\mathsf{Th}(\mathbf{A})$ has QE in $\mathcal{L}_{\mathrm{MV}}$. □

In the next section, we will see that the above conditions are not only sufficient, but also necessary for QE in $\mathcal{L}_{\mathrm{MV}}$.

4. Weak O-Minimality and O-Minimality: A Full Characterization

In this section, we make the link between QE and weak o-minimality clear, and give a full characterization of the latter for MV-chains. The characterization of o-minimal MV-chains will be built upon those results.

The next lemma shows that all MV-chains whose theory in $\mathcal{L}_{\mathrm{MV}}$ has QE must be weakly o-minimal.

Theorem 27. *Let* $\mathbf{A}$ *be an* MV*-chain. If* $\mathsf{Th}(\mathbf{A})$ *has quantifier elimination in* $\mathcal{L}_{\mathrm{MV}}$*, then* $\mathbf{A}$ *is weakly o-minimal.*

Proof. We begin by showing that every one-variable quantifier-free formula $\xi(x)$ in $\mathcal{L}_{\mathrm{MV}}$ defines a finite union of convex sets.

Indeed, every ordered Abelian group can be embedded into a divisible one (i.e. its divisible hull), consequently, every MV-chain a $\mathbf{A}$ is embeddable into a divisible MV-chain $\mathbf{B}$. Let $\mathfrak{f} : \mathbf{A} \to \mathbf{B}$ be such an embedding. Since an ordered Abelian group is divisible if and only if it is weakly o-minimal [15], $\mathbf{B}$ is trivially weakly o-minimal as well. Consequently, $\xi(x)$ defines over B a finite union of convex sets. Embeddings between structures preserve quantifier-free formulas (see [10, Theorem 2.4.1]), and so, for all $a \in A$:

$$\mathbf{A} \models \xi(a) \text{ iff } \mathbf{B} \models \xi(\mathfrak{f}(a)).$$

Therefore, $\xi(x)$ defines over A a finite union of convex sets.

Now, if $\mathsf{Th}(\mathbf{A})$ has QE in $\mathcal{L}_{\mathrm{MV}}$, then $\phi(x)$ is equivalent to a quantifier-free formula $\psi(x)$, which, by the above, defines a finite union of convex sets. Consequently, $\mathbf{A}$ is weakly o-minimal. □

From Lemma 26 and Theorem 27, we obtain that whenever $Rad(\mathbf{A})$ is divisible and either $\mathbf{A}/Rad(\mathbf{A})$ is finite or divisible, then $\mathbf{A}$ is weakly o-minimal. We now proceed to proving the converse. The proof requires, again, some preliminary lemmas.

Lemma 28. *Let* $\mathbf{A}$ *be an* MV*-chain. If* $\mathbf{A}$ *is weakly o-minimal, then* $Rad(\mathbf{A})$ *is divisible.*

Proof. Suppose that $Rad(\mathbf{A})$ is not n-divisible for some n, and let x be a positive element not divisible by n. Then, the sequence

$$(\dagger)\quad 0 < x < nx < (n+1)x < 2nx < (2n+1)x < \ldots < knx < (kn+1)x < \ldots$$

is an infinite alternating sequence of n-divisible and non-n-divisible elements of $Rad(\mathbf{A})$. Let $\phi_n(y)$ be the formula $\exists w\ y = nw$ defining over A the set of n-divisible elements. If $\mathbf{A}$ were weakly o-minimal, $\phi_n(y)$ would define on A a finite union of convex sets $\bigcup_i X_i$. So there would be a set X_j, containing infinitely many n-divisible elements of the sequence $(\dagger)$. If that was the case, then X_j would contain also elements that are not n-divisible. Therefore, $\mathbf{A}$ cannot be weakly o-minimal. □

Lemma 29. *Let* $\mathbf{A}$ *be an* MV*-chain. If* $\mathbf{A}$ *is weakly o-minimal, then* $\mathbf{A}/Rad(\mathbf{A})$ *is finite or divisible.*

Proof. Suppose that $\mathbf{A}/Rad(\mathbf{A})$ is infinite and not n-divisible for some n. Recall that, up to isomorphism, $\mathbf{A}/Rad(\mathbf{A})$ is a dense subalgebra of $[0,1]_{\mathrm{MV}}$ (see [3, Proposition 3.5.3]). We show that both n-divisible elements and non-n-divisible elements are dense in $\mathbf{A}/Rad(\mathbf{A})$.

Notice that there exist arbitrarily small, non-zero n-divisible elements of $\mathbf{A}/Rad(\mathbf{A})$: in fact, for every k there is an element $y \in \mathbf{A}/Rad(\mathbf{A})$ with $0 < y < \frac{1}{nk}$, so $0 < ny < \frac{1}{k}$ and ny is n-divisible. Then, since n-divisible elements are closed under multiples, the n-divisible elements of $\mathbf{A}/Rad(\mathbf{A})$ are dense in $\mathbf{A}/Rad(\mathbf{A})$.

Similarly, we have arbitrarily small non-n-divisible elements. In fact, we prove that for every $k > 1$, there is a non n-divisible element smaller than $\frac{1}{k}$. Indeed, let z be a non n-divisible element of $\mathbf{A}/Rad(\mathbf{A})$. If $z < \frac{1}{k}$, there is nothing to prove. Suppose then that $z > \frac{1}{k}$. By density we have an n-divisible d such that $z \ominus \frac{1}{k} < d < z$, so $0 < z \ominus d < \frac{1}{k}$ and $z \ominus d$ is not n-divisible.

Therefore, non-n-divisible elements are also dense in $\mathbf{A}/Rad(\mathbf{A})$: given $b \in \mathbf{A}/Rad(\mathbf{A})$ and given k, we can take a non-n-divisible $e < \frac{1}{2k}$ and an n-divisible d with $b \ominus \frac{1}{2k} < d < b \oplus \frac{1}{2k}$, hence $b \ominus \frac{1}{k} < d \oplus e < b \oplus \frac{1}{k}$, and $d \oplus e$ is not n-divisible.

The next claim will enable us to prove that $\mathbf{A}$ cannot be weakly o-minimal.

Claim 1. *Let* $\mathbf{A}$ *be an* MV*-chain and* $\mathfrak{p}$ *be the canonical homomorphism from* $\mathbf{A}$ *onto* $\mathbf{A}/Rad(\mathbf{A})$*. If* $\mathbf{A}$ *is weakly o-minimal, then, for all* $x \in A$*,* x *is* n*-divisible if and only if* $\mathfrak{p}(x) \in A/Rad(A)$ *is* n*-divisible as well.*

Proof of Claim 1. Let $\phi_n(y)$ be the formula $\exists w\ y = nw$ defining the set of n-divisible elements. Recall that in first-order structures positive formulas (i.e. formulas that do not contain any negated subformula) are preserved under surjective homomorphisms (see [10, Theorem 2.4.3]). Consequently, for all $a \in A$,

$$\text{if } \mathbf{A} \models \phi_n(a) \text{ then } \mathbf{A}/Rad(\mathbf{A}) \models \phi_n(\mathfrak{p}(a)).$$

So, if a is an n-divisible element of $\mathbf{A}$, $\mathfrak{p}(a)$ must be n-divisible in $\mathbf{A}/Rad(\mathbf{A})$.

To prove the converse, we follow the proof of Lemma 18(2). We know that there exists a unique homomorphism $\mathfrak{v} : \mathbf{G} \to \mathbf{G}/\mathfrak{q}(Rad(\mathbf{A}))$ such that $\Gamma(\mathfrak{v}) = \mathfrak{p}$. Suppose then that $\mathfrak{p}(x) \in A/Rad(A)$ is n-divisible. Clearly, this means $\mathfrak{v}(x)$ is n-divisible as an element of $\mathbf{G}/\mathfrak{q}(Rad(\mathbf{A}))$. Therefore, $\mathfrak{v}(x) = n(\mathfrak{v}(y))$, and $x - ny \in \mathfrak{q}(Rad(\mathbf{A}))$. Since $\mathbf{A}$ is weakly o-minimal, by Lemma 28, $Rad(\mathbf{A})$ is divisible, which implies, following again Lemma 18(2), that $\mathfrak{q}(Rad(\mathbf{A}))$ is a divisible subgroup of $\mathbf{G}$. Then, there exists some z such that $x - ny = nz$, and so $x = n(y + z)$. Consequently, x is n-divisible as an element of $\mathbf{G}$. It is easily seen that $x = n(y \oplus z)$, and, therefore, x is n-divisible in $\mathbf{A}$. $\square$

To conclude the proof of Lemma 29, recall that we are assuming that $\mathbf{A}/Rad(\mathbf{A})$ is infinite and not n-divisible for some n. We show that $\mathbf{A}$ cannot be weakly o-minimal. In fact, if $\mathbf{A}$ was weakly o-minimal, $\phi_n(y)$ would define a finite union of convex sets $\bigcup_{i=1}^{m_1} X_i$, and, similarly, its negation $\neg\phi_n(y)$ would define a finite union of convex sets $\bigcup_{j=1}^{m_2} Y_j$.

Since $\mathbf{A}/Rad(\mathbf{A})$ is infinite, $\mathfrak{p}$ is a surjective homomorphism, and

$$A = \bigcup_{i=1}^{m_1} X_i \cup \bigcup_{j=1}^{m_2} Y_j,$$

there must be a set J either from $\{X_1, \ldots, X_{m_1}\}$ or from $\{Y_1, \ldots. Y_{m_2}\}$ containing infinitely many elements, whose image $\mathfrak{p}(J)$ is a set also containing infinitely many elements.

Therefore, if $\mathbf{A}$ was weakly o-minimal, by Claim 1, $\mathfrak{p}(J)$ would contain only elements that are either all n-divisible or all non-n-divisible. This clearly contradicts the fact that both the set of n-divisible and the set non-n-divisible elements are dense in $\mathbf{A}/Rad(\mathbf{A})$. Hence, $\mathbf{A}$ cannot be weakly o-minimal. □

Therefore, we are now able to prove:

Theorem 2. *Let* $\mathbf{A}$ *be any* MV*-chain, and let* $\mathsf{Th}(\mathbf{A})$ *be the first-order theory of* $\mathbf{A}$ *in the language* $\mathcal{L}_{\mathrm{MV}} = \langle \oplus, ^*, 0 \rangle$*. Then the following are equivalent:*

(1) $\mathbf{A}$ *is weakly o-minimal.*
(2) $\mathsf{Th}(\mathbf{A})$ *has elimination of quantifiers in* $\mathcal{L}_{\mathrm{MV}}$*.*
(3) $Rad(\mathbf{A})$ *is divisible, and* $\mathbf{A}/Rad(\mathbf{A})$ *is finite or divisible.*

Proof. Lemma 26 shows that if $Rad(\mathbf{A})$ is divisible and $\mathbf{A}/Rad(\mathbf{A})$ is finite or divisible, then $\mathsf{Th}(\mathbf{A})$ has QE in $\mathcal{L}_{\mathrm{MV}}$. Theorem 27 proves that whenever $\mathsf{Th}(\mathbf{A})$ has QE in $\mathcal{L}_{\mathrm{MV}}$, $\mathbf{A}$ is weakly o-minimal. Finally, Lemma 28 and Lemma 29 show that whenever $\mathbf{A}$ is weakly o-minimal, $Rad(\mathbf{A})$ is divisible and $\mathbf{A}/Rad(\mathbf{A})$ is finite or divisible. □

We can now proceed to present a characterization of o-minimal MV-chains. Notice that since every o-minimal structure is also weakly o-minimal, by Theorem 2, every o-minimal MV-chain $\mathbf{A}$ has a theory $\mathsf{Th}(\mathbf{A})$ with elimination of quantifiers in $\mathcal{L}_{\mathrm{MV}}$, and is such that $Rad(\mathbf{A})$ is divisible, and $\mathbf{A}/Rad(\mathbf{A})$ is finite or divisible. However, the converse is not true. In fact, as shown in the proof of Theorem 1, every non-simple MV-chain of finite rank is not o-minimal, in spite of having QE in $\mathcal{L}_{\mathrm{MV}}$.

Theorem 1. *Let* $\mathbf{A}$ *be any* MV*-chain in the language* $\mathcal{L}_{\mathrm{MV}} = \langle \oplus, ^*, 0 \rangle$*. Then the following are equivalent:*

(1) $\mathbf{A}$ *is o-minimal.*
(2) $\mathbf{A}$ *is finite or divisible.*

Proof. Suppose that $\mathbf{A}$ is o-minimal. Then, trivially, $\mathbf{A}$ is weakly o-minimal, and, by Lemma 28 and Lemma 29, this means that $Rad(\mathbf{A})$ is divisible and $\mathbf{A}/Rad(\mathbf{A})$ is finite or divisible. Consequently, by Lemma 18, $\mathbf{A}$ is either divisible, or finite, or is a non-simple MV-chain of finite rank. The latter, however, is not possible. In fact, suppose that $\mathbf{A} \cong \Gamma(\mathbb{Z}\vec{\times}\mathbf{G}, (n, g))$. The formula

$$(n + 1)x < 1$$

defines a set that exactly coincides with $Rad(\mathbf{A})$, which obviously is a convex set but does not have an endpoint in A. Therefore, every non-simple MV-chain of finite rank cannot be o-minimal. Consequently, if $\mathbf{A}$ is o-minimal, it is either finite or divisible.

Conversely, if $\mathbf{A}$ is finite then it trivially is o-minimal. Moreover, if $\mathbf{A}$ is divisible then o-minimality immediately follows from the fact that $\mathbf{A}$ is isomorphic to the MV-chain $\Gamma(\mathbf{G}, u)$ for some ordered divisible Abelian group $\mathbf{G}$ (with strong unit), which is o-minimal.

This concludes the proof of the theorem. □

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