S1. Homogenization

In this supplementary section we detail the mathematical steps taken to homogenize equations (2.10). The starting point is to observe that there are two very different length scales present L_x, the macro-scale, and L_y the micro-scale. Any order 1 change on the length scale L_x will result in a change of size ϵ on the scale L_y. We can formalise this by assuming that all dependent variables are a function of both the small scale $\boldsymbol{y}$ and the large scale $\boldsymbol{x}$. In addition we follow Lee and Mei [1] and write the arbitrary field f as $f(\boldsymbol{x}) = f(\boldsymbol{x} + \boldsymbol{u}(\boldsymbol{x}), \boldsymbol{y})$, where $\boldsymbol{u}$ is the displacement vector. Expanding the spatial derivative using the chain rule we obtain

$$\boldsymbol{\nabla} = \epsilon^{-1}\boldsymbol{\nabla}_y + \left[(\boldsymbol{\nabla}_x\boldsymbol{u})\,\boldsymbol{\nabla}_y + \boldsymbol{\nabla}_x\right]. \tag{S1.1}$$

Here $\boldsymbol{\nabla}_x$ is the gradient operator on the macro-scale and $\boldsymbol{\nabla}_y$ is the gradient opperator on the micro-scale. The term $(\boldsymbol{\nabla}_x\boldsymbol{u})\,\boldsymbol{\nabla}_y$ in equation (S1.1) comes from the distortion of the medium due to elastic deformations. Using equation (S1.1) we can write the strain as $e(\boldsymbol{v}) = \epsilon^{-1}e_y(\boldsymbol{v}) + e_x(\boldsymbol{v}) + e_u(\boldsymbol{v})$ where

$$e_x(\boldsymbol{v}) = (\boldsymbol{\nabla}_x\boldsymbol{v}) + (\boldsymbol{\nabla}_x\boldsymbol{v})^T, \tag{S1.2}$$

$$e_y(\boldsymbol{v}) = (\boldsymbol{\nabla}_y\boldsymbol{v}) + (\boldsymbol{\nabla}_y\boldsymbol{v})^T, \tag{S1.3}$$

$$e_u(\boldsymbol{v}) = \left[(\boldsymbol{\nabla}_x\boldsymbol{u})\,\boldsymbol{\nabla}_y\boldsymbol{v}\right] + \left[(\boldsymbol{\nabla}_x\boldsymbol{u})\,\boldsymbol{\nabla}_y\boldsymbol{v}\right]^T. \tag{S1.4}$$

The final, nonlinear strain term comes from the nonlinear derivative in equation (S1.1). We make the expansions

$$\boldsymbol{u}^m = \sum_{k=0}^{\infty}\epsilon^k\boldsymbol{u}_k^m, \quad \boldsymbol{u}^s = \sum_{k=0}^{\infty}\epsilon^k\boldsymbol{u}_k^s, \quad \boldsymbol{v}^a = \sum_{k=0}^{\infty}\epsilon^k\boldsymbol{v}_k^a, \quad p^w = \sum_{k=0}^{\infty}\epsilon^k p_k^w, \quad p^a = \sum_{k=0}^{\infty}\epsilon^k p_k^a. \tag{S1.5}$$

In order to account for the moving geometry we perturb the surface normals [2,3]

$$\hat{\boldsymbol{n}}^\alpha = \hat{\boldsymbol{n}}_0^\alpha + \epsilon\boldsymbol{n}_1^\alpha + O(\epsilon^2) \tag{S1.6}$$

where $\alpha = \{ms, am, as\}$. We will consider the precise form of $\boldsymbol{n}_1$ at a later stage.

(a) Expansion $O(\epsilon^{-2})$

We substitute equations (S1.5) and (S1.6) into equations (2.10) and collecting terms of $O(\epsilon^{-2})$ we obtain

$$\boldsymbol{\nabla}_y\cdot\sigma_0^m = 0 \qquad \boldsymbol{y}\in\mathfrak{B}_m \tag{S1.7a}$$

$$\sigma_0^m = e_y(\boldsymbol{u}_0^m) + \frac{\nu}{1-2\nu}\boldsymbol{\nabla}_y\cdot\boldsymbol{u}_0^m I, \qquad \boldsymbol{y}\in\mathfrak{B}_m, \tag{S1.7b}$$

$$\nabla_y^2 p_0^w = 0, \qquad \boldsymbol{y}\in\mathfrak{B}_m, \tag{S1.7c}$$

in the mixed phase,

$$\boldsymbol{\nabla}_y p_0^a = 0, \qquad \boldsymbol{y}\in\mathfrak{B}_a, \tag{S1.7d}$$

in the air phase, and

$$\sum_{i=1}^{2}\left(\hat{\boldsymbol{\tau}}_i\cdot e_y(\boldsymbol{u}_0^s)\cdot\hat{\boldsymbol{e}}_j\right)^2 = 0, \qquad \boldsymbol{y}\in\partial\mathfrak{B}_s, \tag{S1.7e}$$

$$\int_{\partial\mathfrak{B}_{as,j}} \hat{\boldsymbol{n}}_0^{as}\cdot\sigma_0^{s,j}\,d\boldsymbol{y}+\int_{\partial\mathfrak{B}_{ms,j}} \hat{\boldsymbol{n}}_0^{ms}\cdot\sigma_0^{s,j}\,d\boldsymbol{y}=0, \tag{S1.7f}$$

in the solid. The leading order boundary conditions are

$$\hat{\boldsymbol{n}}_0^{ms}\cdot\boldsymbol{\nabla}_y p_0^w=0, \qquad \boldsymbol{y}\in\partial\mathfrak{B}_{ms}, \tag{S1.7g}$$

$$\hat{\boldsymbol{n}}_0^{am}\cdot\boldsymbol{\nabla}_y p_0^w=0, \qquad \boldsymbol{y}\in\partial\mathfrak{B}_{am}, \tag{S1.7h}$$

$$\boldsymbol{u}_0^m-\boldsymbol{u}_0^s=\lambda^s\hat{\boldsymbol{n}}_0^{ms}\cdot e_x(\boldsymbol{u}_0^m)\left(I-\hat{\boldsymbol{n}}_0^{ms}\hat{\boldsymbol{n}}_0^{ms}\right), \qquad \boldsymbol{x}\in\partial\mathfrak{B}_{ms}, \tag{S1.7i}$$

$$\hat{\boldsymbol{n}}_0^{ms}\cdot\sigma_0^m=\hat{\boldsymbol{n}}_0^{ms}\cdot\sigma_0^s, \qquad \boldsymbol{y}\in\partial\mathfrak{B}_{ms}, \tag{S1.7j}$$

$$\hat{\boldsymbol{n}}_0^{am}\cdot\sigma_0^m=0, \qquad \boldsymbol{y}\in\partial\mathfrak{B}_{am}, \tag{S1.7k}$$

$$\hat{\boldsymbol{n}}_0^{as}\cdot\sigma_0^s=0, \qquad \boldsymbol{y}\in\partial\mathfrak{B}_{as}. \tag{S1.7l}$$

Equations (S1.7) have solution $\sigma_0^m=0$, $\sigma_0^s=0$, $\boldsymbol{u}_0^m=\boldsymbol{u}_0^m(\boldsymbol{x},t)$, $p_0^w=p_0^w(\boldsymbol{x},t)$, $p_0^a=p_0^a(\boldsymbol{x},t)$ and, hence

$$\boldsymbol{v}_0^\alpha=\frac{\partial\boldsymbol{u}_0^\alpha}{\partial t}, \tag{S1.8}$$

for $\alpha=\{m,s\}$. Physically this tells us that the dominant part of the displacement vectors, velocities and pressures are constant on the mico-scale. From this point on we shall write the leading order displacement and velocities in terms of the mixed phase displacement, *i.e.*, $\boldsymbol{u}_0^m$ and $\frac{\partial\boldsymbol{u}_0^m}{\partial t}$.

(b) Expansion $O(\epsilon^{-1})$

Collecting terms at $O(\epsilon^{-1})$ and using results from $O(\epsilon^{-2})$ we obtain three independent problems for $\boldsymbol{u}_1^m$, p_1^w and $\boldsymbol{v}_0^a$. The problem for $\boldsymbol{v}_0^a$ is

$$\boldsymbol{\nabla}_y\cdot\sigma_2^a-\boldsymbol{\nabla}_y p_1^a=\boldsymbol{\nabla}_x p_0^a+g^w\delta_3\hat{\boldsymbol{e}}_3, \qquad \boldsymbol{y}\in\mathfrak{B}_a, \tag{S1.9a}$$

$$\sigma_2^a=\delta_1 e_y(\boldsymbol{v}_0^a), \qquad \boldsymbol{y}\in\mathfrak{B}_a, \tag{S1.9b}$$

$$\boldsymbol{\nabla}_y\cdot\boldsymbol{v}_0^a=0, \qquad \boldsymbol{y}\in\mathfrak{B}_a, \tag{S1.9c}$$

$$\boldsymbol{v}_0^a=\frac{\partial\boldsymbol{u}_0^m}{\partial t}, \qquad \boldsymbol{y}\in\partial\mathfrak{B}_{as}\cup\partial\mathfrak{B}_{am}, \tag{S1.9d}$$

with solution

$$\boldsymbol{v}_0^a=\frac{\partial\boldsymbol{u}_0^m}{\partial t}+\frac{1}{\delta_1}\sum_{k=1}^{3}\boldsymbol{\zeta}_k\hat{\boldsymbol{e}}_k\cdot\left(\boldsymbol{\nabla}_x p_0^a+g^w\delta_3\hat{\boldsymbol{e}}_3\right),\quad p_1^a=\sum_{k=1}^{3}\omega_k^a\hat{\boldsymbol{e}}_k\cdot\left(\boldsymbol{\nabla}_x p_0^a+g^w\delta_3\hat{\boldsymbol{e}}_3\right), \tag{S1.10}$$

where $\boldsymbol{\zeta}_k$ and ω_k^a satisfy the cell problem

$$\nabla_y^2\boldsymbol{\zeta}_k-\boldsymbol{\nabla}_y\omega_k^a=\hat{\boldsymbol{e}}_k, \qquad \boldsymbol{y}\in\mathfrak{B}_a, \tag{S1.11a}$$

$$\boldsymbol{\nabla}_y\cdot\boldsymbol{\zeta}_k=0, \qquad \boldsymbol{y}\in\mathfrak{B}_a, \tag{S1.11b}$$

$$\boldsymbol{\zeta}_k=0, \qquad \boldsymbol{y}\in\partial\mathfrak{B}_{as}\cup\partial\mathfrak{B}_{am}. \tag{S1.11c}$$

This is the standard first order problem for single phase Darcy flow. The problem for p_1^w is

$$\nabla_y^2 p_1^w=0, \qquad \boldsymbol{y}\in\mathfrak{B}_m, \tag{S1.12a}$$

$$\hat{\boldsymbol{n}}_0^{ms}\cdot\boldsymbol{\nabla}_y p_1^w+\hat{\boldsymbol{n}}_0^{ms}\cdot\left(\boldsymbol{\nabla}_x p_0^w+g^w\hat{\boldsymbol{e}}_3\right)=0, \qquad \boldsymbol{y}\in\partial\mathfrak{B}_{ms}, \tag{S1.12b}$$

$$\hat{\boldsymbol{n}}_0^{am}\cdot\boldsymbol{\nabla}_y p_1^w+\hat{\boldsymbol{n}}_0^{am}\cdot\left(\boldsymbol{\nabla}_x p_0^w+g^w\hat{\boldsymbol{e}}_3\right)=0,\qquad \boldsymbol{y}\in\partial\mathfrak{B}_{am},\tag{S1.12c}$$

with solution

$$p_1^w=\sum_{k=1}^{3}\omega_k^w\hat{\boldsymbol{e}}_k\cdot\left(\boldsymbol{\nabla}_x p_0^w+g^w\hat{\boldsymbol{e}}_3\right),\tag{S1.13}$$

where ω_k^w satisfies the cell problem

$$\nabla_y^2\omega_k^w=0,\qquad \boldsymbol{y}\in\mathfrak{B}_m,\tag{S1.14a}$$

$$\hat{\boldsymbol{n}}_0^{ms}\cdot\boldsymbol{\nabla}_y\omega_k^w+\hat{\boldsymbol{n}}_0^{ms}\cdot\hat{\boldsymbol{e}}_k=0,\qquad \boldsymbol{y}\in\partial\mathfrak{B}_{ms},\tag{S1.14b}$$

$$\hat{\boldsymbol{n}}_0^{am}\cdot\boldsymbol{\nabla}_y\omega_k^w+\hat{\boldsymbol{n}}_0^{am}\cdot\hat{\boldsymbol{e}}_k=0,\qquad \boldsymbol{y}\in\partial\mathfrak{B}_{am}.\tag{S1.14c}$$

Again this problem is recognizable from diffusion problems. The problem for $\boldsymbol{u}_1^m$ is

$$\boldsymbol{\nabla}_y\cdot\sigma_1^m=0,\qquad \boldsymbol{y}\in\mathfrak{B}_m,\tag{S1.15a}$$

$$\sigma_1^m=e_y(\boldsymbol{u}_1^m)+e_x(\boldsymbol{u}_0^m)+\frac{\nu}{1-2\nu}\left(\boldsymbol{\nabla}_y\cdot\boldsymbol{u}_1^m+\boldsymbol{\nabla}_x\cdot\boldsymbol{u}_0^m\right)I,\qquad \boldsymbol{y}\in\mathfrak{B}_m,\tag{S1.15b}$$

$$\sum_{i=1}^{2}\left(\hat{\boldsymbol{\tau}}_i\cdot e_y(\boldsymbol{u}_1^s)\cdot\hat{\boldsymbol{e}}_j+\hat{\boldsymbol{\tau}}_i\cdot e_x(\boldsymbol{u}_0^m)\cdot\hat{\boldsymbol{e}}_j\right)^2=0,\qquad \boldsymbol{y}\in\partial\mathfrak{B}_s,\tag{S1.15c}$$

$$\int_{\partial\mathfrak{B}_{ms,j}}\left[\hat{\boldsymbol{n}}_0^{ms}\cdot\sigma_1^m-\hat{\boldsymbol{n}}_0^{ms}p_0^w\right]\,d\boldsymbol{y}-\int_{\partial\mathfrak{B}_{as,j}}\hat{\boldsymbol{n}}_0^{as}p_0^a\,d\boldsymbol{y}=0,\tag{S1.15d}$$

with boundary conditions

$$\hat{\boldsymbol{n}}_0^{am}\cdot\sigma_1^m-\hat{\boldsymbol{n}}_0^{am}p_0^w=-\hat{\boldsymbol{n}}_0^{am}(p_0^a-p^c),\qquad \boldsymbol{y}\in\partial\mathfrak{B}_{am},\tag{S1.15e}$$

$$\boldsymbol{u}_1^m-\boldsymbol{u}_1^s=\lambda^s\hat{\boldsymbol{n}}_0^{ms}\cdot\left[e_y(\boldsymbol{u}_1^m)+e_x(\boldsymbol{u}_0^m)\right]\left(I-\hat{\boldsymbol{n}}_0^{ms}\hat{\boldsymbol{n}}_0^{ms}\right),\qquad \boldsymbol{y}\in\partial\mathfrak{B}_{ms}.\tag{S1.15f}$$

This is a coupled system of equations for the displacement of the mixed and solid phases. Integrating equation (S1.15a) over $\mathfrak{B}_m$ and applying the divergence theorem we obtain

$$\int_{\partial\mathfrak{B}_{am}}\hat{\boldsymbol{n}}_0^{am}\left(p_0^a-p_0^w-p^c\right)\,d\boldsymbol{y}+\sum_j\int_{\partial\mathfrak{B}_{ms,j}}\hat{\boldsymbol{n}}_0^{ms}p_0^w\,d\boldsymbol{y}+\sum_j\int_{\partial\mathfrak{B}_{as,j}}\hat{\boldsymbol{n}}_0^{as}p_0^a\,d\boldsymbol{y}=\quad 0,\tag{S1.16}$$

which, by using $\partial\mathfrak{B}_a=\partial\mathfrak{B}_{am}\cup\sum_j\partial\mathfrak{B}_{as,j}$, $\partial\mathfrak{B}_m=\partial\mathfrak{B}_{am}\cup\sum_j\partial\mathfrak{B}_{ms,j}$ and noting that the integral of a surface normal round a closed loop is zero, we can write as

$$\int_{\partial\mathfrak{B}_{am}}\hat{\boldsymbol{n}}_0^{am}p^c\,d\boldsymbol{y}=0.\tag{S1.17}$$

This condition can be interpreted as an statement of mechanical equilibrium. It states that the total force on the disconnected particles must be zero. We note that this is the total force on all solid particles not on an individual solid particle. There are several ways in which this condition can be met, these are illustrated in figure 2. The first way this condition can be met is simply if there is no air present in the soil, see figure 2A. Secondly, if the solid particle surface is completely wetted then the air–mixed phase interface will form a closed surface. Hence, the integral in equation (S1.17) will be automatically zero, see figure 2B. Finally, if the air–mixed phase interface is naturally arranged such that all contributions from the capillary pressure cancel out, see figure 2C.

This theory describes perturbations about a steady state, hence, for any real geometry we would expect that this condition is naturally met. However, from a practical point of view this theory could be applied to a geometry obtained via imaging. It is likely that, due to imaging

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artifacts, this condition will not be satisfied. Hence, in order to ensure that the cell problems obtained from equations (S1.15) have a solution we rewrite equation (S1.15*e*) as

$$\hat{\boldsymbol{n}}_0^{am} \cdot \sigma_1^m - \hat{\boldsymbol{n}}_0^{am} p_0^w = -\hat{\boldsymbol{n}}_0^{am} p_0^a + \hat{\boldsymbol{n}}_0^{am} p^c - \frac{1}{||\partial\mathfrak{B}_{am}||}\int_{\partial\mathfrak{B}_{am}} \hat{\boldsymbol{n}}_0^{am} p^c \, d\boldsymbol{y} \quad \boldsymbol{y} \in \partial\mathfrak{B}_{am}, \tag{S1.18}$$

where for an arbitrary domain ξ we define

$$||\xi|| = \int_\xi 1 \, d\boldsymbol{y}. \tag{S1.19}$$

This introduces the normalized error

$$\boldsymbol{E} = \frac{1}{||\partial\mathfrak{B}_{am}||}\int_{\partial\mathfrak{B}_{am}} \hat{\boldsymbol{n}}_0^{am} \, d\boldsymbol{y}. \tag{S1.20}$$

We note that if the error defined by equation (S1.20) is zero then we also have

$$\int_{\partial\mathfrak{B}_{as,j}} \hat{\boldsymbol{n}}_0^{as} \, d\boldsymbol{y} = 0 \quad \int_{\partial\mathfrak{B}_{ms,j}} \hat{\boldsymbol{n}}_0^{ms} \, d\boldsymbol{y} = 0. \tag{S1.21}$$

Hence, we find equations (S1.15) have the solution

$$\boldsymbol{u}_1^m = \sum_{p=1,q=1}^{3} \boldsymbol{\kappa}_{pq}^u \hat{\boldsymbol{e}}_p \cdot e_x(\boldsymbol{u}_0^m) \cdot \hat{\boldsymbol{e}}_q + \boldsymbol{\kappa}^p \left(p_0^w + p^c - p_0^a\right), \tag{S1.22}$$

$$\boldsymbol{u}_1^s = \sum_{p=1,q=1}^{3} \boldsymbol{\gamma}_{pq}^u \hat{\boldsymbol{e}}_p \cdot e_x(\boldsymbol{u}_0^m) \cdot \hat{\boldsymbol{e}}_q + \boldsymbol{\gamma}^p \left(p_0^w + p^c - p_0^a\right), \tag{S1.23}$$

where $\boldsymbol{\kappa}_{pq}^u$ and $\boldsymbol{\gamma}_{pq}^u$ satisfy

$$\boldsymbol{\nabla}_y \cdot \alpha_{pq}^u = 0, \qquad \boldsymbol{y} \in \mathfrak{B}_m, \tag{S1.24a}$$

$$\alpha_{pq}^u = e_y(\boldsymbol{\kappa}_{pq}^u) + \frac{\nu}{1-2\nu}\left(\boldsymbol{\nabla}_y \cdot \boldsymbol{\kappa}_{pq}^u\right) I, \qquad \boldsymbol{y} \in \mathfrak{B}_m, \tag{S1.24b}$$

$$\sum_{i=1}^{2}\left[\hat{\boldsymbol{\tau}}_i \cdot e_y(\boldsymbol{\gamma}_{pq}^u) \cdot \hat{\boldsymbol{e}}_j + \hat{\boldsymbol{\tau}}_i \cdot \frac{1}{2}\left(\hat{\boldsymbol{e}}_p\hat{\boldsymbol{e}}_q + \hat{\boldsymbol{e}}_q\hat{\boldsymbol{e}}_p\right) \cdot \hat{\boldsymbol{e}}_j\right]^2 = 0, \qquad \boldsymbol{y} \in \partial\mathfrak{B}_s, \tag{S1.24c}$$

$$\int_{\partial\mathfrak{B}_{ms,j}} \hat{\boldsymbol{n}}_0^{ms} \cdot \alpha_{pq}^u|_j \, d\boldsymbol{y} = 0, \tag{S1.24d}$$

$$\boldsymbol{\kappa}_{pq}^u - \boldsymbol{\gamma}_{pq}^u = \lambda^s \hat{\boldsymbol{n}}_0^{ms}\left[e_y(\boldsymbol{\kappa}_{pq}^u) + \hat{\boldsymbol{e}}_p\hat{\boldsymbol{e}}_q\right]\left[I - \hat{\boldsymbol{n}}_0^{ms}\hat{\boldsymbol{n}}_0^{ms}\right], \qquad \boldsymbol{y} \in \partial\mathfrak{B}_{ms}, \tag{S1.24e}$$

$$\hat{\boldsymbol{n}}_0^{am} \cdot \left[\alpha_{pq}^u + \hat{\boldsymbol{e}}_p\hat{\boldsymbol{e}}_q + \frac{\nu}{1-2\nu}\frac{\hat{\boldsymbol{e}}_p \cdot \hat{\boldsymbol{e}}_q}{2}\right] = 0, \qquad \boldsymbol{y} \in \partial\mathfrak{B}_{am}, \tag{S1.24f}$$

and $\boldsymbol{\kappa}^p$ and $\boldsymbol{\gamma}^p$ satisfy

$$\boldsymbol{\nabla}_y \cdot \alpha^p = 0, \qquad \boldsymbol{y} \in \mathfrak{B}_m, \tag{S1.25a}$$

$$\alpha^p = e_y(\boldsymbol{\kappa}^p) + \frac{\nu}{1-2\nu}\left(\boldsymbol{\nabla}_y \cdot \boldsymbol{\kappa}^p\right) I, \qquad \boldsymbol{y} \in \mathfrak{B}_m, \tag{S1.25b}$$

$$\sum_{i=1}^{2}\left(\hat{\boldsymbol{\tau}}_i \cdot e_y(\boldsymbol{\gamma}^p) \cdot \hat{\boldsymbol{e}}_j\right)^2 = 0, \qquad \boldsymbol{y} \in \partial\mathfrak{B}_s, \tag{S1.25c}$$

$$\int_{\partial\mathfrak{B}_{ms,j}} \hat{\boldsymbol{n}}_0^{ms} \cdot \alpha^p \, d\boldsymbol{y} = 0, \tag{S1.25d}$$

$$\boldsymbol{\kappa}^p - \boldsymbol{\gamma}^p = \lambda^s \hat{\boldsymbol{n}}_0^{ms} e_y(\boldsymbol{\kappa}^p)\left[I - \hat{\boldsymbol{n}}_0^{ms}\hat{\boldsymbol{n}}_0^{ms}\right], \qquad \boldsymbol{y} \in \partial\mathfrak{B}_{ms}, \tag{S1.25e}$$

$$\hat{\boldsymbol{n}}_0^{am} \cdot \alpha^p = \hat{\boldsymbol{n}}_0^{am} - \boldsymbol{E}, \qquad \boldsymbol{y} \in \partial\mathfrak{B}_{am}, \qquad \text{(S1.25}f\text{)}$$

where we have added the normalised error to equation (S1.25*f*) to ensure that the cell problem (S1.25) is well posed. Finally, we obtain the velocity expansion at $O(\epsilon^{-1})$:

$$\boldsymbol{v}_1^\alpha = \frac{\partial \boldsymbol{u}_1^\alpha}{\partial t} + \frac{\partial \boldsymbol{u}_0^m}{\partial t} \cdot \left[\boldsymbol{\nabla}_y \boldsymbol{u}_1^\alpha + \boldsymbol{\nabla}_x \boldsymbol{u}_0^\alpha\right], \qquad \text{(S1.26)}$$

for $\alpha = \{m, s, a, w\}$; we will use equation (S1.26) at $O(\epsilon^0)$. Equations (S1.11), (S1.14), (S1.24) and (S1.25) capture the effect of the micro-scale geometry. This information will be used at higher order to obtain an averaged set of equations which describe the poro-elastic material as a continuum.

(c) Expansion $O(\epsilon^0)$

Expanding the relevant equations to $O(\epsilon^0)$ we obtain

$$\begin{aligned}
&\boldsymbol{\nabla}_y \cdot \sigma_2^m + \left[\boldsymbol{\nabla}_x + \left(\boldsymbol{\nabla}_x \boldsymbol{u}_0^m\right) \cdot \boldsymbol{\nabla}_y\right] \cdot \sigma_1^m - \boldsymbol{\nabla}_y p_1^w - \boldsymbol{\nabla}_x p_0^w \\
&\qquad = g^w \left[\phi + \delta_2(1-\phi)\right] \hat{\boldsymbol{e}}_3, \qquad \boldsymbol{y} \in \mathfrak{B}_m, \qquad \text{(S1.27}a\text{)}
\end{aligned}$$

$$\begin{aligned}
&\boldsymbol{\nabla}_y \cdot \boldsymbol{v}_1^m + \boldsymbol{\nabla}_x \cdot \frac{\partial \boldsymbol{u}_0^m}{\partial t} = \boldsymbol{\nabla}_y \cdot \left\{\left[\boldsymbol{\nabla}_x + \left(\boldsymbol{\nabla}_x \boldsymbol{u}_0^m\right) \cdot \boldsymbol{\nabla}_y\right] p_1^w + \boldsymbol{\nabla}_y p_2^w\right\} \\
&\qquad + \left[\boldsymbol{\nabla}_x + \left(\boldsymbol{\nabla}_x \boldsymbol{u}_0^m\right) \cdot \boldsymbol{\nabla}_y\right] \cdot \left(\boldsymbol{\nabla}_x p_0^w + \boldsymbol{\nabla}_y p_1^w + g^w \hat{\boldsymbol{e}}_3\right), \qquad \boldsymbol{y} \in \mathfrak{B}_m, \qquad \text{(S1.27}b\text{)}
\end{aligned}$$

$$\begin{aligned}
\sigma_2^m = {} & e_y(\boldsymbol{u}_2^m) + e_x(\boldsymbol{u}_1^m) + e_u(\boldsymbol{u}_1^m) \\
& + \frac{\nu}{1-2\nu}\left(\boldsymbol{\nabla}_y \cdot \boldsymbol{u}_2^m + \left[\boldsymbol{\nabla}_x + \left(\boldsymbol{\nabla}_x \boldsymbol{u}_0^m\right) \cdot \boldsymbol{\nabla}_y\right] \cdot \boldsymbol{u}_1^m\right) I, \qquad \boldsymbol{y} \in \mathfrak{B}_m, \qquad \text{(S1.27}c\text{)}
\end{aligned}$$

$$\boldsymbol{\nabla}_y \cdot \boldsymbol{v}_1^a + \left[\boldsymbol{\nabla}_x + \left(\boldsymbol{\nabla}_x \boldsymbol{u}_0^m\right) \cdot \boldsymbol{\nabla}_y\right] \cdot \boldsymbol{v}_0^a = 0, \qquad \boldsymbol{y} \in \mathfrak{B}_a, \qquad \text{(S1.27}d\text{)}$$

$$\begin{aligned}
&\int_{\partial\mathfrak{B}_{ms,j}} \left[\hat{\boldsymbol{n}}_0^{ms} \cdot \sigma_2^{s,j} + \boldsymbol{n}_1^{ms} \cdot \sigma_1^{s,j}\right] d\boldsymbol{y} + \int_{\partial\mathfrak{B}_{ms,j}} \boldsymbol{n}_0^{ms} \left(\boldsymbol{y} \cdot \boldsymbol{\nabla}_x\right) \sigma_1^{s,j}\, dy \\
&+ \int_{\partial\mathfrak{B}_{as,j}} \left[\hat{\boldsymbol{n}}_0^{as} \cdot \sigma_2^{s,j} + \boldsymbol{n}_1^{as} \cdot \sigma_1^{s,j}\right] d\boldsymbol{y} + \int_{\partial\mathfrak{B}_{as,j}} \boldsymbol{n}_0^{as} \left(\boldsymbol{y} \cdot \boldsymbol{\nabla}_x\right) \sigma_1^{s,j}\, dy \qquad \text{(S1.27}e\text{)} \\
&= -g^w \delta_4 \int_{\mathfrak{B}_{s,j}} dy \hat{\boldsymbol{e}}_3,
\end{aligned}$$

the additional terms in equation (S1.27*e*) come from the multiple scales expansion of the integral constraint [4]. We also expand the relevant boundary conditions

$$\begin{aligned}
&\hat{\boldsymbol{n}}_0^{am} \cdot \left\{\boldsymbol{\nabla}_y p_2^w + \left[\boldsymbol{\nabla}_x + \left(\boldsymbol{\nabla}_x \boldsymbol{u}_0^m\right) \cdot \boldsymbol{\nabla}_y\right] p_1^w\right\} \\
&\qquad + \boldsymbol{n}_1^{am} \cdot \left\{\boldsymbol{\nabla}_y p_1^w + \boldsymbol{\nabla}_x p_0^w\right\} = 0, \qquad \boldsymbol{y} \in \partial\mathfrak{B}_{am}, \qquad \text{(S1.27}f\text{)}
\end{aligned}$$

$$\begin{aligned}
&\hat{\boldsymbol{n}}_0^{ms} \cdot \left\{\boldsymbol{\nabla}_y p_2^w + \left[\boldsymbol{\nabla}_x + \left(\boldsymbol{\nabla}_x \boldsymbol{u}_0^m\right) \cdot \boldsymbol{\nabla}_y\right] p_1^w\right\} \\
&\qquad + \boldsymbol{n}_1^{ms} \cdot \left\{\boldsymbol{\nabla}_y p_1^w + \boldsymbol{\nabla}_x p_0^w\right\} = 0, \qquad \boldsymbol{y} \in \partial\mathfrak{B}_{ms}, \qquad \text{(S1.27}g\text{)}
\end{aligned}$$

$$\hat{\boldsymbol{n}}_0^{ms} \cdot \left[\sigma_2^s - \sigma_2^m\right] + \boldsymbol{n}_1^{ms} \cdot \left[\sigma_1^s - \sigma_1^m\right] + \hat{\boldsymbol{n}}_0^{ms} p_1^w + \boldsymbol{n}_1^{ms} p_0^w = 0, \qquad \boldsymbol{y} \in \partial\mathfrak{B}_{ms}, \qquad \text{(S1.27}h\text{)}$$

$$\begin{aligned}
&\hat{\boldsymbol{n}}_0^{am} \cdot \left[\sigma_2^a - \sigma_2^m\right] - \hat{\boldsymbol{n}}_0^{am} \left[p_1^a - p_1^w\right] \\
&- \boldsymbol{n}_1^{am} \cdot \sigma_1^m - \boldsymbol{n}_1^{am} \left[p_0^a - p_0^w - p^c\right] = 0, \qquad \boldsymbol{y} \in \partial\mathfrak{B}_{am}, \qquad \text{(S1.27}i\text{)}
\end{aligned}$$

$$\hat{\boldsymbol{n}}_0^{as} \cdot \left[\sigma_2^a - \sigma_2^s\right] - \hat{\boldsymbol{n}}_0^{as} p_1^a + \boldsymbol{n}_0^{as} \cdot \left[\sigma_1^a - \sigma_1^s\right] - \boldsymbol{n}_1^{as} p_0^a = 0, \qquad \boldsymbol{y} \in \partial\mathfrak{B}_{as}, \qquad \text{(S1.27}j\text{)}$$

$$\boldsymbol{v}_1^m - \boldsymbol{v}_1^a = -\lambda^a \hat{\boldsymbol{n}}_0^{am} \left[\mathring{e}_y(\boldsymbol{u}_1^m) + \mathring{e}_x(\boldsymbol{u}_0^m)\right] \left[I - \hat{\boldsymbol{n}}_0^{am} \hat{\boldsymbol{n}}_0^{am}\right], \qquad \boldsymbol{y} \in \partial\mathfrak{B}_{am}, \qquad \text{(S1.27}k\text{)}$$

$$\boldsymbol{v}_1^m - \boldsymbol{v}_1^s = \lambda^s \hat{\boldsymbol{n}}_0^{ms} \left[\mathring{e}_y(\boldsymbol{u}_1^m) + \mathring{e}_x(\boldsymbol{u}_0^m)\right] \left[I - \hat{\boldsymbol{n}}_0^{ms} \hat{\boldsymbol{n}}_0^{ms}\right], \qquad \boldsymbol{y} \in \partial\mathfrak{B}_{ms}, \qquad \text{(S1.27}l\text{)}$$

$$\boldsymbol{v}_1^s = \boldsymbol{v}_1^a, \qquad \boldsymbol{y} \in \partial\mathfrak{B}_{as}. \qquad \text{(S1.27}m\text{)}$$

To obtain the macroscopic equations averaged over the micro-scale we integrate equations (S1.27*a*), (S1.27*b*) and (S1.27*d*) in turn. We start with equation (S1.27*b*) as this offers the least algabric complexity whilst allowing us to introduce the tools needed to obtain the remaining equations. Integrating equation (S1.27*b*) over $\mathfrak{B}_m$ we find

$$\int_{\mathfrak{B}_m} \left[\boldsymbol{\nabla}_y \cdot \boldsymbol{v}_1^m + \boldsymbol{\nabla}_x \cdot \frac{\partial \boldsymbol{u}_0^m}{\partial t}\right] d\boldsymbol{y} = \int_{\mathfrak{B}_m} \boldsymbol{\nabla}_y \cdot \left\{\left[\boldsymbol{\nabla}_x + \left(\boldsymbol{\nabla}_x \boldsymbol{u}_0^m\right) \cdot \boldsymbol{\nabla}_y\right] p_1^w + \boldsymbol{\nabla}_y p_2^w\right\} d\boldsymbol{y}$$
$$+ \int_{\mathfrak{B}_m} \left[\boldsymbol{\nabla}_x + \left(\boldsymbol{\nabla}_x \boldsymbol{u}_0^m\right) \cdot \boldsymbol{\nabla}_y\right] \cdot \left(\boldsymbol{\nabla}_x p_0^w + \boldsymbol{\nabla}_y p_1^w + g^w \hat{\boldsymbol{e}}_3\right) d\boldsymbol{y}. \tag{S1.28}$$

We first deal with the right hand side; by applying the divergence theorem to the first integral we can rewrite equation (S1.28) as

$$\int_{\mathfrak{B}_m} \left[\boldsymbol{\nabla}_y \cdot \boldsymbol{v}_1^m + \boldsymbol{\nabla}_x \cdot \frac{\partial \boldsymbol{u}_0^m}{\partial t}\right] d\boldsymbol{y} = \sum_j \int_{\partial \mathfrak{B}_{ms,j}} \hat{\boldsymbol{n}}_0^{ms} \cdot \left\{\left[\boldsymbol{\nabla}_x + \left(\boldsymbol{\nabla}_x \boldsymbol{u}_0^m\right) \cdot \boldsymbol{\nabla}_y\right] \cdot p_1^w + \boldsymbol{\nabla}_y p_2^w\right\} d\boldsymbol{y}$$
$$- \int_{\partial \mathfrak{B}_{am}} \hat{\boldsymbol{n}}_0^{am} \cdot \left\{\left[\boldsymbol{\nabla}_x + \left(\boldsymbol{\nabla}_x \boldsymbol{u}_0^m\right) \cdot \boldsymbol{\nabla}_y\right] \cdot p_1^w + \boldsymbol{\nabla}_y p_2^w\right\} d\boldsymbol{y}$$
$$+ \int_{\mathfrak{B}_m} \boldsymbol{\nabla}_x \cdot \left(\boldsymbol{\nabla}_x p_0^w + \boldsymbol{\nabla}_y p_1^w + g^w \hat{\boldsymbol{e}}_3\right) d\boldsymbol{y} + \int_{\mathfrak{B}_m} \left[\left(\boldsymbol{\nabla}_x \boldsymbol{u}_0^m\right) \cdot \boldsymbol{\nabla}_y\right] p_1^w \, d\boldsymbol{y}. \tag{S1.29}$$

Using equation (S1.27*g*) in the first integral on the right hand side, equation (S1.27*f*) in the second integral on the right hand side and the transport theorem on the third integral on the right hand side we find

$$\int_{\mathfrak{B}_m} \left[\boldsymbol{\nabla}_y \cdot \boldsymbol{v}_1^m + \boldsymbol{\nabla}_x \cdot \frac{\partial \boldsymbol{u}_0^m}{\partial t}\right] d\boldsymbol{y} = -\sum_j \int_{\partial \mathfrak{B}_{ms,j}} \left[\boldsymbol{n}_1^{ms} + \left(\boldsymbol{\nabla}_x \boldsymbol{r}^{ms}\right) \cdot \hat{\boldsymbol{n}}_0^{ms}\right] \cdot \boldsymbol{J} \, d\boldsymbol{y}$$
$$+ \int_{\partial \mathfrak{B}_{am}} \left[\boldsymbol{n}_1^{am} + \left(\boldsymbol{\nabla}_x \boldsymbol{r}^{am}\right) \cdot \hat{\boldsymbol{n}}_0^{am}\right] \cdot \boldsymbol{J} \, d\boldsymbol{y} + \boldsymbol{\nabla}_x \cdot \int_{\mathfrak{B}_m} \boldsymbol{J} \, d\boldsymbol{y} + \int_{\mathfrak{B}_m} \left[\left(\boldsymbol{\nabla}_x \boldsymbol{u}_0^m\right) \cdot \boldsymbol{\nabla}_y\right] p_1^w \, d\boldsymbol{y}, \tag{S1.30}$$

where $\boldsymbol{r}^\alpha = \boldsymbol{y}|_{\partial \mathfrak{B}_\alpha}$ for $\alpha = \{ms, am\}$ and

$$\boldsymbol{J} = \boldsymbol{\nabla}_x p_0^w + \boldsymbol{\nabla}_y p_1^w + g^w \hat{\boldsymbol{e}}_3. \tag{S1.31}$$

To proceed we consider the representation of $\boldsymbol{n}_1^\alpha$ and $\left(\boldsymbol{\nabla}_x \boldsymbol{r}^\alpha\right) \cdot \hat{\boldsymbol{n}}_0^\alpha$ in more detail. We will show that the terms induced through application of the transport theorem exactly cancel the additional terms which come from the surface normal expansion. In other words, the additional terms which come from the boundary expansion will be equal and opposite to those which come from the domain expansion. We represent the mixed phase using the level set function χ^m which is zero on the mixed phase boundary, $\chi^m(\boldsymbol{r}^\alpha) = 0$. Hence, by differentiating χ^m the surface normals can be written as

$$\hat{\boldsymbol{n}}^\alpha = \left.\frac{\epsilon^{-1} \boldsymbol{\nabla}_y \chi^m + \boldsymbol{\nabla}_x \chi^m}{\left|\epsilon^{-1} \boldsymbol{\nabla}_y \chi^m + \boldsymbol{\nabla}_x \chi^m\right|}\right|_{\boldsymbol{y} \in \partial \mathfrak{B}_\alpha} \approx \hat{\boldsymbol{n}}_0^\alpha + \epsilon \boldsymbol{n}_1^\alpha, \tag{S1.32}$$

where

$$\hat{\boldsymbol{n}}_0^\alpha = \left.\frac{\boldsymbol{\nabla}_y \chi^m}{\left|\boldsymbol{\nabla}_y \chi^m\right|}\right|_{\boldsymbol{y} \in \partial \mathfrak{B}_\alpha}, \quad \hat{\boldsymbol{n}}_1^\alpha = \left.\left(\frac{\boldsymbol{\nabla}_x \chi^m}{\left|\boldsymbol{\nabla}_y \chi^m\right|} - \hat{\boldsymbol{n}}_0^\alpha \frac{\hat{\boldsymbol{n}}_0^\alpha \cdot \boldsymbol{\nabla}_x \chi^m}{\left|\boldsymbol{\nabla}_y \chi^m\right|}\right)\right|_{\boldsymbol{y} \in \partial \mathfrak{B}_\alpha}. \tag{S1.33}$$

Differentiating χ^m on the long space scale at the boundaries and dividing by $\left|\boldsymbol{\nabla}_y \chi^m\right|$ we find

$$\frac{\boldsymbol{\nabla}_x \chi^m}{\left|\boldsymbol{\nabla}_y \chi^m\right|} + \left(\boldsymbol{\nabla}_x \boldsymbol{r}^\alpha\right) \cdot \frac{\boldsymbol{\nabla}_y \chi^m}{\left|\boldsymbol{\nabla}_y \chi^m\right|} = 0, \tag{S1.34}$$

which we can write as

$$\left.\frac{\boldsymbol{\nabla}_x\chi^m}{|\boldsymbol{\nabla}_y\chi^m|}\right|_{\partial\mathfrak{B}_{ms}} = -(\boldsymbol{\nabla}_x\boldsymbol{r}^{ms})\cdot\hat{\boldsymbol{n}}_0^{ms}, \quad \left.\frac{\boldsymbol{\nabla}_x\chi^m}{|\boldsymbol{\nabla}_y\chi^m|}\right|_{\partial\mathfrak{B}_{am}} = (\boldsymbol{\nabla}_x\boldsymbol{r}^{am})\cdot\hat{\boldsymbol{n}}_0^{am}, \tag{S1.35}$$

using equations (S1.33) and (S1.35) in equation (S1.30) we obtain the identity

$$\int_{\partial\mathfrak{B}_\alpha}\left[\boldsymbol{n}_1^\alpha + \left(\boldsymbol{\nabla}_x\boldsymbol{r}^\alpha\right)\cdot\hat{\boldsymbol{n}}_0^\alpha\right]\cdot\boldsymbol{J}\,d\boldsymbol{y} = -\int_{\partial\mathfrak{B}_\alpha}\frac{\hat{\boldsymbol{n}}_0^\alpha\cdot\boldsymbol{\nabla}_x\chi^m}{|\boldsymbol{\nabla}_y\chi^m|}\hat{\boldsymbol{n}}_0^\alpha\cdot\boldsymbol{J}\,d\boldsymbol{y} \tag{S1.36}$$

or, using equation (S1.12*c*) for $\alpha = am$ or (S1.12*b*) for $\alpha = ms$ we find

$$\int_{\partial\mathfrak{B}_\alpha}\left[\boldsymbol{n}_1^\alpha + \left(\boldsymbol{\nabla}_x\boldsymbol{r}^\alpha\right)\cdot\hat{\boldsymbol{n}}_0^\alpha\right]\cdot\boldsymbol{J}\,d\boldsymbol{y} = 0. \tag{S1.37}$$

Hence, by applying the same procedure to the left hand side of equation (S1.28), we can rewrite equation (S1.30) in index notation as

$$\begin{aligned}
&||\mathfrak{B}_m||\frac{\partial^2 u_{0j}^m}{\partial t\partial x_j} + A_{pq}^u\frac{\partial}{\partial t}\left[\frac{\partial u_{0p}^m}{\partial x_q} + \frac{\partial u_{0q}^m}{\partial x_p}\right] + A^p\frac{\partial}{\partial t}\left(p^c + p_0^w - p_0^a\right)\\
&+\frac{\partial u_{0i}^m}{\partial t}\left\{\mathcal{A}_{ipq}^u\left[\frac{\partial u_{0p}^m}{\partial x_q} + \frac{\partial u_{0q}^m}{\partial x_p}\right] + \mathcal{A}_i^p\left(p^c + p_0^w - p_0^a\right)\right\}\\
&=\left[\frac{\partial}{\partial x_j}K_{jk}^w + \frac{\partial u_{0i}^m}{\partial x_j}\gamma_{ijk}^w\right]\left[\frac{\partial p_0^w}{\partial x_k} + g^w\delta_{k3}\right],
\end{aligned} \tag{S1.38}$$

where

$$\begin{aligned}
K_{ij}^w &= \int_{\mathfrak{B}_m}\delta_{ij} + \frac{\partial\omega_j}{\partial y_i}\,d\boldsymbol{y}, \quad & A_{pq}^u &= \int_{\mathfrak{B}_m}\boldsymbol{\nabla}_y\cdot\boldsymbol{\kappa}_{pq}^u\,d\boldsymbol{y}, \quad & A^p &= \int_{\mathfrak{B}_m}\boldsymbol{\nabla}_y\cdot\boldsymbol{\kappa}^p\,d\boldsymbol{y},\\
\gamma_{ijk}^w &= \int_{\mathfrak{B}_m}\frac{\partial^2\omega_k^w}{\partial y_i\partial y_j}\,d\boldsymbol{y}, \quad & \mathcal{A}_{ipq}^u &= \int_{\mathfrak{B}_m}\frac{\partial}{\partial y_i}\boldsymbol{\nabla}_y\cdot\boldsymbol{\kappa}_{pq}^u\,d\boldsymbol{y}, \quad & \mathcal{A}_i^p &= \int_{\mathfrak{B}_m}\frac{\partial}{\partial y_i}\boldsymbol{\nabla}_y\cdot\boldsymbol{\kappa}^p\,d\boldsymbol{y}.
\end{aligned} \tag{S1.39}$$

Integrating equation (S1.27*a*) over $\mathfrak{B}_m$ and adding equation (S1.9*a*) integrated over $\mathfrak{B}_a$ and equation (S1.27*e*) we obtain

$$\begin{aligned}
&\int_{\mathfrak{B}_m}\left[\boldsymbol{\nabla}_y\cdot\left(\sigma_2^m - p_1^w I\right) + \boldsymbol{\nabla}_x\cdot\left(\sigma_1^m - p_0^w I\right)\right]\,d\boldsymbol{y} + \int_{\mathfrak{B}_m}\left[\left(\boldsymbol{\nabla}_x\boldsymbol{u}_0^m\right)\cdot\boldsymbol{\nabla}_y\right]\sigma_1^m\,d\boldsymbol{y}\\
&+\int_{\mathfrak{B}_a}\left[\boldsymbol{\nabla}_y\cdot\left(\sigma_2^a - p_1^a I\right) - \boldsymbol{\nabla}_x p_0^a\right]\,d\boldsymbol{y}\\
&+\sum_j\left\{\int_{\partial\mathfrak{B}_{ms,j}}\left[\hat{\boldsymbol{n}}_0^{ms}\cdot\sigma_2^{s,j} + \boldsymbol{n}_1^{ms}\cdot\sigma_1^{s,j}\right]\,d\boldsymbol{y} + \int_{\partial\mathfrak{B}_{as,j}}\left[\hat{\boldsymbol{n}}_0^{as}\cdot\sigma_2^{s,j} + \boldsymbol{n}_1^{as}\cdot\sigma_1^{s,j}\right]\,d\boldsymbol{y}\right\}\\
&= g^w\hat{\boldsymbol{e}}_3\left\{\int_{\mathfrak{B}_m}\left[\phi + \delta_2(1-\phi)\right]\,d\boldsymbol{y} + \int_{\mathfrak{B}_a}\delta_3\,d\boldsymbol{y} + \int_{\mathfrak{B}_s}\delta_4\,d\boldsymbol{y}\right\}.
\end{aligned} \tag{S1.40}$$

Applying the divergence and transport theorems we find

$$
\begin{aligned}
&\sum_j \int_{\partial\mathfrak{B}_{ms,j}} \left\{\hat{\boldsymbol{n}}_0^{ms}\cdot\left(\sigma_2^m - p_1^w I\right) - \left[\left(\boldsymbol{\nabla}_x \boldsymbol{r}^{ms}\right)\cdot\hat{\boldsymbol{n}}_0^{ms}\right]\cdot\left(\sigma_1^m - p_0^w I\right)\right\}\, d\boldsymbol{y}\\
&-\int_{\partial\mathfrak{B}_{am}} \left\{\hat{\boldsymbol{n}}_0^{am}\cdot\left(\sigma_2^m - p_1^w I\right) - \left[\left(\boldsymbol{\nabla}_x \boldsymbol{r}^{am}\right)\cdot\hat{\boldsymbol{n}}_0^{am}\right]\cdot\left(\sigma_1^m - p_0^w I\right)\right\}\, d\boldsymbol{y}\\
&+\boldsymbol{\nabla}_x\cdot\int_{\mathfrak{B}_m}\left(\sigma_1^m - p_0^w I\right)\, d\boldsymbol{y} + \int_{\mathfrak{B}_m}\left[\left(\boldsymbol{\nabla}_x \boldsymbol{u}_0^m\right)\cdot\boldsymbol{\nabla}_y\right]\sigma_1^m\, d\boldsymbol{y} - \boldsymbol{\nabla}_x\int_{\mathfrak{B}_a} p_0^a\, d\boldsymbol{y}\\
&+\int_{\partial\mathfrak{B}_{am}} \left\{\hat{\boldsymbol{n}}_0^{am}\cdot\left(\sigma_2^a - p_1^a I\right) - \left[\left(\boldsymbol{\nabla}_x \boldsymbol{r}^{am}\right)\cdot\hat{\boldsymbol{n}}_0^{am}\right] p_0^a\right\}\, d\boldsymbol{y}\\
&+\sum_j\int_{\partial\mathfrak{B}_{as,j}} \left\{\hat{\boldsymbol{n}}_0^{as}\cdot\left(\sigma_2^a - p_1^a I\right) - \left[\left(\boldsymbol{\nabla}_x \boldsymbol{r}^{as}\right)\cdot\hat{\boldsymbol{n}}_0^{as}\right] p_0^a\right\}\, d\boldsymbol{y}\\
&+\sum_j\left\{\int_{\partial\mathfrak{B}_{ms,j}}\left[\hat{\boldsymbol{n}}_0^{ms}\cdot\sigma_2^{s,j} + \boldsymbol{n}_1^{ms}\cdot\sigma_1^{s,j}\right] d\boldsymbol{y} + \int_{\partial\mathfrak{B}_{as,j}}\left[\hat{\boldsymbol{n}}_0^{as}\cdot\sigma_2^{s,j} + \boldsymbol{n}_1^{as}\cdot\sigma_1^{s,j}\right] d\boldsymbol{y}\right\}\\
&= g^w\hat{\mathbf{e}}_3\left\{||\mathfrak{B}_m||\left[\phi + \delta_2(1-\phi)\right] + ||\mathfrak{B}_a||\delta_3 + ||\mathfrak{B}_s||\delta_4\right\}.
\end{aligned}
\tag{S1.41}
$$

By representing the air and mixed phase domains through a level set function and using equations (S1.27*h*), (S1.27*i*), (S1.27*j*) and (S1.15*e*), we obtain the simplified expression

$$
\begin{aligned}
&\boldsymbol{\nabla}_x\cdot\int_{\mathfrak{B}_m}\left(\sigma_1^m - p_0^w I\right)\, d\boldsymbol{y} + \int_{\mathfrak{B}_m}\left[\left(\boldsymbol{\nabla}_x \boldsymbol{u}_0^m\right)\cdot\boldsymbol{\nabla}_y\right]\sigma_1^m\, d\boldsymbol{y} - \boldsymbol{\nabla}_x\int_{\mathfrak{B}_a} p_0^a\, d\boldsymbol{y}\\
&+\boldsymbol{\nabla}_x\cdot\int_{\partial\mathfrak{B}_{ms}}\left(\boldsymbol{y}\hat{\boldsymbol{n}}_0^{ms}\right)\left(\sigma_1^m - p_0^w I\right)\, d\boldsymbol{y} - \boldsymbol{\nabla}_x\int_{\partial\mathfrak{B}_{as}}\left(\boldsymbol{y}\hat{\boldsymbol{n}}_0^{as}\right) p_0^a\, d\boldsymbol{y}\\
&= g^w\hat{\mathbf{e}}_3\left\{||\mathfrak{B}_m||\left[\phi + \delta_2(1-\phi)\right] + ||\mathfrak{B}_a||\delta_3 + ||\mathfrak{B}_s||\delta_4\right\}
\end{aligned}
\tag{S1.42}
$$

which, using equation (S1.22), can be written in component notation as

$$
\begin{aligned}
&\left[\frac{\partial}{\partial x_i}C^u_{ijkl} + \frac{\partial u^m_{0p}}{\partial x_i}\mathcal{H}^u_{ijklp}\right]\left[\frac{\partial u^m_{0k}}{\partial x_l} + \frac{\partial u^m_{0l}}{\partial x_k}\right] + \left[\frac{\partial}{\partial x_i}C^p_{ij} + \frac{\partial u^m_{0p}}{\partial x_i}\mathcal{H}^p_{ijp}\right]\left(p_0^w + p^c - p_0^a\right)\\
&-\frac{\partial}{\partial x_i}\left[\left(||\mathcal{B}_m||\delta_{ij} - \int_{\partial\mathfrak{B}_{ms}}\hat{n}^{ms}_{0j}y_i\, dy\right)p_0^w + \left(||\mathcal{B}_a||\delta_{ij} - \int_{\partial\mathfrak{B}_{as}}\hat{n}^{as}_{0j}y_i\, dy\right)p_0^a\right] = g^{eff}\hat{\mathbf{e}}_3,
\end{aligned}
\tag{S1.43}
$$

where

$$
\begin{aligned}
C^u_{ijkl} = &\int_{\mathfrak{B}_m}\left[\frac{\partial\kappa^u_{kli}}{\partial y_j} + \frac{\partial\kappa^u_{klj}}{\partial y_i} + \delta_{ik}\delta_{jl} + \frac{\nu}{1-2\nu}\left(\delta_{ij}\delta_{kl} + \boldsymbol{\nabla}_y\cdot\boldsymbol{\kappa}^u_{kl}\delta_{ij}\right)\right] d\boldsymbol{y}\\
&-\int_{\partial\mathfrak{B}_{ms}} y_i\hat{n}^{ms}_{0q}\left[\frac{\partial\kappa^u_{klq}}{\partial y_j} + \frac{\partial\kappa^u_{klj}}{\partial y_q} + \delta_{qk}\delta_{jl} + \frac{\nu}{1-2\nu}\left(\delta_{qj}\delta_{kl} + \boldsymbol{\nabla}_y\cdot\boldsymbol{\kappa}^u_{kl}\delta_{qj}\right)\right] d\boldsymbol{y},
\end{aligned}
\tag{S1.44a}
$$

$$
\mathcal{H}^u_{ijklp} = \int_{\mathfrak{B}_m}\frac{\partial}{\partial y_p}\left[\frac{\partial\kappa^u_{kli}}{\partial y_j} + \frac{\partial\kappa^u_{klj}}{\partial y_i} + \frac{\nu}{1-2\nu}\boldsymbol{\nabla}_y\cdot\boldsymbol{\kappa}^u_{kl}\delta_{ij}\right] d\boldsymbol{y},
\tag{S1.44b}
$$

$$
\begin{aligned}
C^p_{ij} = &\int_{\mathfrak{B}_m}\left[\frac{\partial\kappa^p_i}{\partial y_j} + \frac{\partial\kappa^p_j}{\partial y_i} + \frac{\nu}{1-2\nu}\boldsymbol{\nabla}_y\cdot\boldsymbol{\kappa}^p\delta_{ij}\right] d\boldsymbol{y}\\
&-\int_{\partial\mathfrak{B}_{ms}} y_i\hat{n}^{ms}_{0q}\left[\frac{\partial\kappa^p_q}{\partial y_j} + \frac{\partial\kappa^p_j}{\partial y_q} + \frac{\nu}{1-2\nu}\boldsymbol{\nabla}_y\cdot\boldsymbol{\kappa}^p\delta_{qj}\right] d\boldsymbol{y},
\end{aligned}
\tag{S1.44c}
$$

$$
\mathcal{H}^p_{ijp} = \int_{\mathfrak{B}_m}\frac{\partial}{\partial y_p}\left[\frac{\partial\kappa^p_i}{\partial y_j} + \frac{\partial\kappa^p_j}{\partial y_i} + \frac{\nu}{1-2\nu}\boldsymbol{\nabla}_y\cdot\boldsymbol{\kappa}^p\delta_{ij}\right] d\boldsymbol{y},
\tag{S1.44d}
$$

and

$$
g^{eff} = g^w\left\{||\mathfrak{B}_m||\left[\phi + \delta_2(1-\phi)\right] + ||\mathfrak{B}_a||\delta_3 + ||\mathfrak{B}_s||\delta_4\right\}.
\tag{S1.44e}
$$

Finally we integrate equation (S1.27*d*) over $\mathfrak{B}_a$

$$\int_{\mathfrak{B}_a} \left[\nabla_x \cdot \boldsymbol{v}_0^a + \nabla_y \cdot \boldsymbol{v}_1^a + \left[\left(\nabla_x \boldsymbol{u}_0^m \right) \cdot \nabla_y \right] \cdot \boldsymbol{v}_0^a \right] \, d\boldsymbol{y} = 0. \tag{S1.45}$$

As before we apply the divergence and transport theorems to obtain

$$\begin{aligned} &\sum_j \int_{\partial \mathfrak{B}_{as,j}} \left\{ \hat{\boldsymbol{n}}_0^{as} \cdot \boldsymbol{v}_1^s - \left[\left(\nabla_x \boldsymbol{r}^{as} \right) \cdot \hat{\boldsymbol{n}}_0^{as} \right] \cdot \frac{\partial \boldsymbol{u}_0^m}{\partial t} \right\} \, d\boldsymbol{y} \\ &+ \int_{\partial \mathfrak{B}_{am}} \left\{ \hat{\boldsymbol{n}}_0^{am} \cdot \boldsymbol{v}_1^m - \left[\left(\nabla_x \boldsymbol{r}^{am} \right) \cdot \hat{\boldsymbol{n}}_0^{am} \right] \cdot \frac{\partial \boldsymbol{u}_0^m}{\partial t} \right\} \, d\boldsymbol{y} \\ &+ \nabla_x \cdot \int_{\mathfrak{B}_a} \boldsymbol{v}_0^a \, d\boldsymbol{y} + \int_{\mathfrak{B}_a} \left[\left(\nabla_x \boldsymbol{u}_0^m \right) \cdot \nabla_y \right] \cdot \boldsymbol{v}_0^a \, d\boldsymbol{y} = 0, \end{aligned} \tag{S1.46}$$

by defining the air boundary $\partial \mathfrak{B}_a = \partial \mathfrak{B}_{as,j} \cup \partial \mathfrak{B}_{am} \, \forall j$ we can rewrite this as

$$\begin{aligned} &\sum_j \int_{\partial \mathfrak{B}_{as,j}} \hat{\boldsymbol{n}}_0^{as} \cdot \boldsymbol{v}_1^s \, d\boldsymbol{y} + \int_{\partial \mathfrak{B}_{am}} \hat{\boldsymbol{n}}_0^{am} \cdot \boldsymbol{v}_1^m \, d\boldsymbol{y} - \int_{\partial \mathfrak{B}_a} \left[\left(\nabla_x \boldsymbol{r}^a \right) \cdot \hat{\boldsymbol{n}}_0^a \right] \cdot \frac{\partial \boldsymbol{u}_0^m}{\partial t} \, d\boldsymbol{y} \\ &+ \nabla_x \cdot \int_{\mathfrak{B}_a} \boldsymbol{v}_0^a \, d\boldsymbol{y} + \int_{\mathfrak{B}_a} \left[\left(\nabla_x \boldsymbol{u}_0^m \right) \cdot \nabla_y \right] \cdot \boldsymbol{v}_0^a \, d\boldsymbol{y} = 0. \end{aligned} \tag{S1.47}$$

We note that the integral of $\hat{\boldsymbol{n}}_0^a$ around a closed surface is zero, hence, for the integral of $\hat{\boldsymbol{n}}^a$ to be zero round a closed surface the integral of $\boldsymbol{n}_1^a$ must also be zero. Hence, we can write

$$\sum_j \int_{\partial \mathfrak{B}_{as,j}} \hat{\boldsymbol{n}}_0^{as} \cdot \boldsymbol{v}_1^s \, d\boldsymbol{y} + \int_{\partial \mathfrak{B}_{am}} \hat{\boldsymbol{n}}_0^{am} \cdot \boldsymbol{v}_1^m \, d\boldsymbol{y} + \nabla_x \cdot \int_{\mathfrak{B}_a} \boldsymbol{v}_0^a \, d\boldsymbol{y} + \int_{\mathfrak{B}_a} \left[\left(\nabla_x \boldsymbol{u}_0^m \right) \cdot \nabla_y \right] \cdot \boldsymbol{v}_0^a \, d\boldsymbol{y} = 0. \tag{S1.48}$$

Using equations (S1.22) and (S1.23), we can write

$$\begin{aligned} &||\mathfrak{B}_a|| \frac{\partial^2 \boldsymbol{u}_{0j}^m}{\partial t \partial x_j} + B_{pq}^u \frac{\partial}{\partial t} \left[\frac{\partial u_{0p}^m}{\partial x_q} + \frac{\partial u_{0q}^m}{\partial x_p} \right] + B^p \frac{\partial}{\partial t} \left(p^c + p_0^w - p_0^a \right) \\ &+ \frac{\partial u_{0i}^m}{\partial t} \left\{ \mathcal{B}_{ipq}^u \left[\frac{\partial u_{0p}^m}{\partial x_q} + \frac{\partial u_{0q}^m}{\partial x_p} \right] + \mathcal{B}_i^p \left(p^c + p_0^w - p_0^a \right) \right\} \\ &= - \left[\frac{\partial}{\partial x_j} K_{jk}^a + \frac{\partial u_{0i}^m}{\partial x_j} \mathcal{K}_{ijk}^a \right] \left[\frac{\partial p_0^a}{\partial x_k} + g^w \delta_3 \delta_{k3} \right], \end{aligned} \tag{S1.49}$$

where

$$K_{ij}^a = \frac{1}{\delta_1} \int_{\mathfrak{B}_a} \boldsymbol{\zeta}_i \cdot \hat{\boldsymbol{e}}_j \, d\boldsymbol{y}, \tag{S1.50a}$$

$$\mathcal{K}_{ijk}^a = \frac{1}{\delta_1} \int_{\mathfrak{B}_a} \frac{\partial \zeta_{kj}}{\partial y_i} \, d\boldsymbol{y}, \tag{S1.50b}$$

$$B_{pq}^u = \int_{\partial \mathfrak{B}_{am}} \hat{\boldsymbol{n}}_0^{am} \cdot \boldsymbol{\kappa}_{pq}^u \, d\boldsymbol{y} + \sum_j \int_{\partial \mathfrak{B}_{as,j}} \hat{\boldsymbol{n}}_0^{am} \cdot \boldsymbol{\gamma}_{pq}^u \, d\boldsymbol{y}, \tag{S1.50c}$$

$$\mathcal{B}_{ipq}^u = \int_{\partial \mathfrak{B}_{am}} \frac{\partial}{\partial y_i} \hat{\boldsymbol{n}}_0^{am} \cdot \boldsymbol{\kappa}_{pq}^u \, d\boldsymbol{y} + \sum_j \int_{\partial \mathfrak{B}_{as,j}} \frac{\partial}{\partial y_i} \hat{\boldsymbol{n}}_0^{am} \cdot \boldsymbol{\gamma}_{pq}^u \, d\boldsymbol{y}, \tag{S1.50d}$$

$$B^p = \int_{\partial \mathfrak{B}_{am}} \hat{\boldsymbol{n}}_0^{am} \cdot \boldsymbol{\kappa}^p \, d\boldsymbol{y} + \sum_j \int_{\partial \mathfrak{B}_{as,j}} \hat{\boldsymbol{n}}_0^{am} \cdot \boldsymbol{\gamma}^p \, d\boldsymbol{y}, \tag{S1.50e}$$

$$\mathcal{B}_i^p = \int_{\partial \mathfrak{B}_{am}} \frac{\partial}{\partial y_i} \left(\hat{\boldsymbol{n}}_0^{am} \cdot \boldsymbol{\kappa}^p \right) \, d\boldsymbol{y} + \sum_j \int_{\Gamma_{as,j}} \frac{\partial}{\partial y_i} \left(\hat{\boldsymbol{n}}_0^{am} \cdot \boldsymbol{\gamma}^p \right) \, d\boldsymbol{y}. \tag{S1.50f}$$

(S1.50*g*)

In summary we have found three averaged equations (S1.38), (S1.43) and (S1.49) for p_0^a, p_0^w and $\boldsymbol{u}_0^m$. These equations are parametrised by 16 tensor and scalar quantities (S1.39), (S1.44) and (S1.50) based on the 4 cell problems (S1.11), (S1.14), (S1.24) and (S1.25) In addition the relative effect of gravity on each phase is captured through 3 parameters based on the micro-scale geometry.

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