Governing equations for the amplitudes of interacting modes

The slowly varying complex amplitudes of the breathing $\tilde{a}_{00} = a_{00}\exp(\mathrm{i}\omega_0 t)$ and distortion modes $\tilde{a}_{lm} = a_{lm}\exp(\mathrm{i}\omega_l t)$ satisfy the equations that have the form [1]:

$$\frac{d\tilde{a}_{00}}{dt} = \left[\mathrm{i}(\omega-\omega_0)-\gamma_0\right]\tilde{a}_{00} - \mathrm{i}C_{ll0}\sum_{m=-l}^{l}(-1)^m\tilde{a}^*_{lm}\tilde{a}_{l-m} + \frac{\sqrt{\pi}R_0^2 P_m}{\left(2\rho_0 R_0^3\omega_0\right)^{1/2}},$$

$$\frac{d\tilde{a}_{lm}}{dt} = \left[\mathrm{i}(\omega/2-\omega_l)-\gamma_l\right]\tilde{a}_{lm} - 2\mathrm{i}C_{ll0}(-1)^m\tilde{a}_{00}\tilde{a}^*_{l-m}$$

$$+2C_{n'll}\sum_{m'=-n'}^{n'}(-1)^{m'}\overline{Y_{n'm'}Y_{l-m}Y_{lm-m'}}\,\tilde{a}_{n'm'}\tilde{a}^*_{lm-m'},$$

$$\frac{d\tilde{a}_{n'm'}}{dt} = \left[\mathrm{i}(\omega-\omega_{n'})-\gamma_{n'}\right]\tilde{a}_{n'm'} + 2\mathrm{i}C_{n'll}\sum_{m_1=-l}^{l}(-1)^{m_1}\overline{Y_{n'-m'}Y_{lm_1}Y_{lm'-m_1}}\,\tilde{a}_{lm_1}\tilde{a}_{lm'-m_1}, \qquad (1)$$

where $C_{ll0} = (2^7\pi)^{-1/2}(4l-1)\omega_l\left(\rho_0\omega_0 R_0^3\right)^{-1/2}R_0^{-1}$ is the coupling coefficient in the energy of interaction of the breathing and distortion modes and $\overline{A} = (4\pi)^{-1}\int A\sin\vartheta d\vartheta d\alpha$ is the averaging over the solid angle. The coupling coefficient in the energy of interaction of the distortion modes $C_{n'll}$ has the form

$$C_{n'll} = \frac{\omega_l}{\left(2^5\rho_0 R_0^5\omega_{n'}\right)^{1/2}}\left\{\frac{(n'+1)^{1/2}}{(l+1)}\left[\frac{n'(n'+1)}{2} - 3(l+1) + \frac{2}{3}\frac{(l+1)}{(l+2)(l-1)}\right]\right.$$

$$\left.+\frac{\omega_{n'}}{\omega_l(n'+1)^{1/2}}\left[\left[n'(n'+1) - 2(n'+1)(l-1) + 4(l+1) + \frac{(n'+1)(l+1)\omega_2^2}{9\omega_{n'}\omega_l}\right]\right]\right\}. \qquad (2)$$

The canonical equations of motion (1) describe evolution of the complex amplitudes of the monopole mode $\tilde{a}_{00}$, the parametrically unstable distortion mode $\tilde{a}_{lm}$ and the partner of the unstable mode in resonant triad $\tilde{a}_{n'm'}$. The resonant triad is formed by two unstable waves with the same frequencies ω_l interacting to form a wave of higher frequency $\omega_{n'} \approx 2\omega_l$. These equations are obtained by differentiation of the Hamiltonian with respect to a^*_{00}, a^*_{lm}, $a^*_{n'm'}$ and retaining only the resonant terms having the same time evolution as a_{00}, a_{lm}, $a_{n'm'}$.

The damping of the breathing mode γ_0 and the distortion modes of order l and n' (i.e. γ_l and $\gamma_{n'}$ are included in this model. The damping factor for the breathing mode $\gamma_0 \approx \omega^2 R_0/2c + \left(2\nu/R_0^2\right) + 3(\gamma-1)\left(\omega_0/2R_0\right)\left(D/2\omega\right)^{1/2}$ is the sum of radiation damping, viscous damping and damping owing to thermal diffusion, as estimated by a linear analysis, where D is the diffusivity coefficient. The damping factor for the distortion modes is given by Lamb's formula $\gamma_l = (l+2)(2l+1)\nu/R_0^2$. It is assumed in the both cases that thermal and viscous lengths are smaller than the bubble radius R_0.

[1] A. O. Maksimov and T. G. Leighton, "Pattern formation on the surface of a bubble driven by an acoustic field," Proc. R. Soc. A **468**, 57–75 (2012).