UNCOUNTABLY MANY QUASI-ISOMETRY CLASSES OF GROUPS OF TYPE FP

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Abstract. In [8] one of the authors constructed uncountable families of groups of type FP and of n-dimensional Poincaré duality groups for each $n \geq 4$. We show that the groups constructed in [8] comprise uncountably many quasi-isometry classes. We deduce that for each $n \geq 4$ there are uncountably many quasi-isometry classes of acyclic n-manifolds admitting free cocompact properly discontinuous discrete group actions.

1. Introduction

Throughout this article, the phrase 'continuously many' will be used to describe sets having the cardinality of the real numbers. In [8] one of the authors exhibited continuously many isomorphism types of groups of type FP, extending the work of Bestvina and Brady [1], who constructed the first examples of groups of type FP that are not finitely presented. We extend these results still further, by showing that the groups constructed in [8] fall into continuously many quasi-isometry classes.

Bestvina-Brady associate a group BB_L to each finite flag complex in such a way that the homological properties of the group BB_L are controlled by those of the flag complex L. In particular, in the case when L is acyclic but not contractible, BB_L is type FP but not finitely presented. In [8], a group $G_L(S)$ is associated to each connected finite flag complex L and each set $S \subseteq \mathbb{Z}$ in such a way that the homological properties of $G_L(S)$ are controlled by those of L and its universal cover, $\widetilde{L}$. In the case when L and $\widetilde{L}$ are both acyclic, each $G_L(S)$ is type FP. The construction of $G_L(S)$ generalizes that of BB_L, and in particular $G_L(\mathbb{Z})$ is BB_L. Our first main theorem is as follows.

Theorem 1.1. *For each finite connected flag complex L that is not simply-connected, there are continuously many quasi-isometry classes of groups $G_L(S)$.*

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The invariant that we rely on to distinguish the groups $G_L(S)$ is the invariant that was introduced by Bowditch [2] in his construction of continuously many quasi-isometry classes of 2-generator groups. (Grigorchuk's construction of such a family, using growth rate to distinguish the groups [7], is no help to us because the groups $G_L(S)$ all have exponential growth.) Our account of Bowditch's invariant differs from his and may be of independent interest.

To a graph Γ Bowditch associates a set $H(\Gamma)$ of natural numbers, consisting of the lengths of loops in Γ that are *taut* in the sense that they are not consequences of shorter loops. He describes the relationship between $H(\Gamma)$ and $H(\Gamma')$ in the case when Γ and Γ' are quasi-isometric. When Γ is the Cayley graph associated to a group presentation satisfying the $C'(1/6)$ small cancellation condition, the set $H(\Gamma)$ is equal to the set of lengths of the relators of the presentation.

Our proof involves estimating the set $H(\Gamma(S))$, where $\Gamma(S)$ is the Cayley graph associated to the natural generating set for $G_L(S)$. The natural presentation for $G_L(S)$ contains relators whose lengths are parametrized by the absolute values of the members of S, but it also contains many relators of length 3, and does not satisfy the $C'(1/6)$ condition. To apply Bowditch's technique we need a lower bound for the word lengths of elements in the kernel of the map $G_L(S) \to G_L(T)$ for $S \subseteq T$, in terms of $T - S$. The Cayley graph $\Gamma(S)$ embeds naturally in a CAT(0) cubical complex. Our lower bound on word length uses this embedding and an easy lemma concerning maps between CAT(0) spaces, which will be proved in Section 3. In the statement, the singular set for a map consists of all points at which it is not a local isometry.

Lemma 1.2. *Let $f : X \to Y$ be a continuous map of CAT(0) metric spaces, and suppose that $x \neq x'$ but $f(x) = f(x')$. Then the distance $d_X(x, x')$ is at least the sum of the distances from x and x' to the singular set for f.*

The so-called 'Davis trick' [5, 6] allows one to embed groups of type FP as retracts of Poincaré duality groups, and enabled the construction of continuously many isomorphism types of Poincaré dualtiy groups [8, Thm. 18.1]. In the final section we strengthen this result in two ways.

Corollary 1.3. *For each $n \geq 4$ there are continuously many quasi-isometry classes of non-finitely presented n-dimensional Poincaré duality groups.*

Corollary 1.4. *For each $n \geq 4$ there is a closed aspherical n-manifold admitting continuously many quasi-isometry classes of regular acyclic covers.*

To establish these results, we study the behaviour of the Bowditch length spectrum under some semi-direct product constructions that arise when implementing the Davis trick. In contrast to the geometric methods used throughout the rest of the article, the proof of Theorem 6.2, which is the main result of the final section, is purely algebraic. It would be interesting to have a geometric proof of this theorem and conversely to have algebraic proofs of our results concerning $G_L(S)$.

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2. Background

There are various types of Cayley graphs, but we shall need just one type which we shall call the *simplicial Cayley graph* $\Gamma(G, S)$ associated to the group G and generating set S. This is the simplicial graph with vertex set G and edge set the 2-element sets of the form $\{g, gs\}$ for some $s \in S$. This definition could be made for any $S \subseteq G$; the fact that S generates G is equivalent to the graph $\Gamma(G, S)$ being connected. Any simplicial graph with a free, transitive G-action on its vertex set is isomorphic as a graph with G-action to $\Gamma(G, S)$ for some S. The action of G on the edges of $\Gamma(G, S)$ is free if and only if S contains no element of order two.

Next we recall some material from [2] concerning quasi-isometries and Bowditch's taut loop length spectrum. Bowditch's article [2] does not mention homotopies, 2-complexes or the fundamental group, all of which play crucial roles in our account of his work. We believe that some readers will benefit from our different but equivalent account. For this reason we restate some of his results in our terms, and encourage the interested reader to try to prove them before consulting [2].

Each graph Γ that we consider will be connected, simplicial, and will be viewed as a metric space via the path metric d_Γ, in which each edge has length one. The induced metric on the vertex set of a Cayley graph $\Gamma(G, S)$ is thus the S-word length metric on the group G. For $k > 0$ an integer, recall that a function $f : X \to Y$ between metric spaces is *k-Lipschitz* if $d_Y(f(x), f(x')) \leq k.d_X(x, x')$ for all $x, x' \in X$. Following Bowditch [2], we say that graphs Γ and Λ are *k-quasi-isometric* if there exist a pair of k-Lipschitz maps of vertex sets $\phi : V(\Gamma) \to V(\Lambda)$ and $\psi : V(\Lambda) \to V(\Gamma)$ so that $d_\Gamma(x, \psi\circ\phi(x)) \leq k$ for each vertex x of Γ and similarly $d_\Lambda(y, \phi\circ\psi(y)) \leq k$ for each vertex y of Λ. Graphs are *quasi-isometric* if they are k-quasi-isometric for some integer $k > 0$.

We remark that the above definition is not the standard one; see for example [3, I.8.14] for the standard definition of a quasi-isometry between metric spaces. We leave it as an exercise to check that graphs Γ, Λ are quasi-isometric as above if and only if the metric spaces (Γ, d_Γ) and (Λ, d_Λ) are quasi-isometric in the usual sense.

An edge loop of length l in a (simplicial) graph Γ is a sequence $v_0, \ldots, v_l$ of vertices such that $v_0 = v_l$ and $\{v_{i-1}, v_i\}$ is an edge for $1 \leq i \leq l$. For a graph Γ and an integer constant l, let Γ_l denote the 2-complex whose 1-skeleton is the geometric realization of Γ, with one 2-cell attached to each edge loop in Γ of length strictly less than l. An edge loop of length l in Γ is said to be *taut* if it is not null-homotopic in Γ_l. Bowditch's *taut loop length spectrum* $H(\Gamma)$ for the graph Γ is the set of lengths of taut loops.

We are interested in the 2-complex Γ_l only to define taut loops: if Γ' is any subcomplex with $\Gamma \subseteq \Gamma' \subseteq \Gamma_l$ so that the induced map on fundamental groups $\pi_1(\Gamma') \to \pi_1(\Gamma_l)$ is an isomorphism, then an edge loop is taut if and only if it is not null-homotopic in Γ'.

Bowditch defines subsets $H, H' \subseteq \mathbb{N}$ to be *k-related* if for all $l \geq k^2+2k+2$, whenever $l \in H$ then there is some $l' \in H'$ with $l/k \leq l' \leq lk$ and vice-versa. He then proves

Lemma 2.1. *If (connected) graphs Γ and Λ are k-quasi-isometric, then $H(\Gamma)$ and $H(\Lambda)$ are k-related.*

In our terms, the lemmas that Bowditch uses to prove the above result are as follows.

Lemma 2.2. *$H(\Gamma)$ is equal to the set of $l \in \mathbb{N}$ for which the induced map of fundamental groups $\pi_1(\Gamma_l) \to \pi_1(\Gamma_{l+1})$ is not an isomorphism.*

For any fixed l, let $i_{\Gamma,l}$ denote the inclusion of Γ in the 2-complex Γ_l.

Lemma 2.3. *If $\phi : \Gamma \to \Lambda$ and $\psi : \Lambda \to \Gamma$ are k-Lipschitz maps whose restrictions to vertex sets define a k-quasi-isometry between Γ and Λ, then for any $l \geq k^2 + 2k + 2$ there are homotopies*

$$i_{\Gamma,l} \circ \psi \circ \phi \simeq i_{\Gamma,l} \quad \text{and} \quad i_{\Lambda,l} \circ \phi \circ \psi \simeq i_{\Lambda,l}.$$

Next we review some material from [1, 8]. A *flag complex* or *clique complex* is a simplicial complex in which every finite set of mutually adjacent vertices spans a simplex. For the remainder of this section, L will denote a finite flag complex. Let $\mathbb{T}$ denote the circle $\mathbb{R}/\mathbb{Z}$, viewed as a CW-complex with one vertex at $0 + \mathbb{Z} \in \mathbb{R}/\mathbb{Z}$ and one edge. For a finite set V let $\mathbb{T}^V$ denote the product $\prod_{v \in V} \mathbb{T}_v$, where $\mathbb{T}_v$ denotes a copy of $\mathbb{T}$. Non-empty subcomplexes of $\mathbb{T}^V$ are in bijective correspondence with simplicial complexes with vertex set contained in V. If L is a finite flag complex with vertex set V, let $\mathbb{T}_L$ denote the corresponding subcomplex of $\mathbb{T}^V$. This complex is aspherical, and its fundamental group is the *right-angled Artin group* A_L associated to L, with generators corresponding to the vertices of L, subject only to the relations that v and w commute whenever $\{v, w\}$ is an edge of L.

The universal covering space X_L of $\mathbb{T}_L$ has a natural cubical structure, and is a CAT(0) cubical complex. The additive group structure in $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ defines a map $l : \mathbb{T}^V \to \mathbb{T}$, and hence a map $l_L : \mathbb{T}_L \to \mathbb{T}$. Define $\widetilde{\mathbb{T}}_L$ to be the regular covering of $\mathbb{T}_L$ induced by pulling back the universal covering of $\mathbb{T}$ along l_L. The Bestvina-Brady group BB_L is defined to be the fundamental group $\pi_1(\widetilde{\mathbb{T}}_L)$, or equivalently the kernel of the map $A_L \to \mathbb{Z}$ of fundamental groups induced by $l : \mathbb{T}_L \to \mathbb{T}$. Bestvina and Brady showed that many properties of BB_L are determined by properties of L. In the case when L is acyclic but not simply connected, BB_L is type FP but not finitely presented [1].

Let $f_L : X_L \to \mathbb{R}$ be the map of universal coverings induced by l_L. This map has the following properties: if we identify each n-cube of X_L with $[0,1]^n$ then its restriction to each n-cube is equal to an affine map; the image of each vertex of X_L is an integer; the image of each n-cube of X_L is an interval of length n. We view f_L as defining a height function on X_L. There is a regular covering map $X_L \to \widetilde{\mathbb{T}}_L$, with covering group BB_L.

In [8] this is generalized, under the extra assumption that L be connected. For each set $S \subseteq \mathbb{Z}$, a CAT(0) cubical complex $X_L^{(S)}$ is defined, together with a regular branched covering map $X_L^{(S)} \to \widetilde{\mathbb{T}}_L$, and the group $G_L(S)$ is by definition the covering group for this covering. The only branch points of this covering are the vertices of $X_L^{(S)}$ whose height is not in S, and the stabilizer in $G_L(S)$ of each branch point is a subgroup isomorphic to the fundamental group $\pi_1(L)$. (In particular, the construction is non-trivial only when L is not simply-connected.) If $S \subseteq T \subseteq \mathbb{Z}$, there is a regular branched covering map $X_L^{(S)} \to X_L^{(T)}$, branched only at vertices of height in $T - S$, and the branched covering $X_L^{(S)} \to \mathbb{T}_L$ factors through this. If $S \subseteq T$ then there is a surjective group homomorphism $G_L(S) \to G_L(T)$, and the branched covering map $X_L^{(S)} \to X_L^{(T)}$ is equivariant for this homomorphism. The group $G_L(\mathbb{Z})$ is equal to BB_L.

The height function on X_L induces a $G_L(S)$-invariant height function on $X_L^{(S)}$ for each $S \subseteq \mathbb{Z}$. Since $\widetilde{\mathbb{T}}_L$ has only one vertex of each integer height, the group $G_L(S)$ acts transitively on the vertices of $X_L^{(S)}$ of each height. The intersection of the 2-skeleton of $X_L^{(S)}$ and the 0-level set (i.e., the points of height 0) is a simplicial graph Γ whose 0-skeleton is the vertices of height 0. Orbits of edges in Γ correspond to A_L-orbits of squares in X_L, or equivalently to edges of L. If $0 \in S$ then $G_L(S)$ acts freely on Γ, and so Γ can be identified with a simplicial Cayley graph for $G_L(S)$. This gives a natural choice of generators for $G_L(S)$ when $0 \in S$, in bijective correspondence with the directed edges of L. Under the composite map $G_L(S) \to G_L(\mathbb{Z}) = BB_L \to A_L$ the element corresponding to the directed edge from vertex x to vertex y maps to the element xy^{-1}. To give a presentation for $G_L(S)$ with this generating set, we first fix a finite collection Ω of directed loops in L that normally generates $\pi_1(L)$. In other words, if one attaches discs to L along the loops in Ω, one obtains a simply-connected complex. Three families of relators occur in this presentation, which we call $P(L, \Omega)$:

- (Edge relations) for each directed edge a with opposite edge $\overline{a}$, the relation $a\overline{a} = 1$;
- (Triangle relations) for each directed triangle (a, b, c) in L the relations $abc = 1$ and $a^{-1}b^{-1}c^{-1} = 1$;
- (Long cycle relations) for each $n \in S - \{0\}$ and each $(a_1, \ldots, a_l) \in \Omega$ the relation $a_1^n a_2^n \cdots a_l^n = 1$.

Another crucial property of these presentations is that only the long relations corresponding to $n \in S$ hold in $G_L(S)$: if $(a_1, \ldots, a_l) \in \Omega$ is not null-homotopic in L, then $a_1^n a_2^n \cdots a_l^n \neq 1$ for $n \notin S \cup \{0\}$.

We close by giving some references for more general background material. For CAT(0) spaces we suggest [3], and for homological finiteness conditions such as the FP property we suggest [4]. Each of these topics is also covered briefly in the appendices to [6], which is our recommended source for Coxeter groups.

3. Bounding word lengths by CAT(0) distances

Our first task is to establish Lemma 1.2, as stated in the Introduction. Recall that a singular point of a map between CAT(0) spaces is a point at which the map is not a local isometry.

Proof. (Lemma 1.2). As in the statement, let $x, x' \in X$ be distinct points such that $f(x) = f(x')$, and suppose that the geodesic arc γ from x to x' does not pass through the singular set. In this case, $f \circ \gamma$ is a locally geodesic arc in Y, whose end points are both equal to $f(x)$. In a CAT(0) space any locally geodesic arc is a geodesic arc, and the unique geodesic arc from $f(x)$ to $f(x') = f(x)$ is the constant arc of length 0. This contradiction shows that γ must pass through the singular set. The claim follows. □

Lemma 3.1. *Let L be a finite connected flag complex of dimension d. For any $S \subseteq \mathbb{Z}$ the distance from the 0-level set in the CAT(0) space $X = X_L^{(S)}$ to a vertex of height n is $|n|/\sqrt{d+1}$.*

Proof. By symmetry it suffices to consider the case $n > 0$. Let γ be a path starting at a vertex v of height n, and moving at unit speed in X to the 0-level set, and let $f : X \to \mathbb{R}$ denote the height function on X. Minimizing the length of γ is equivalent to maximizing the speed of descent, i.e., minimizing the derivative of $f \circ \gamma$.

The initial direction of travel of the path γ can be represented by a point of the link, $\mathrm{Lk}_X(v)$, of v in X. This is a simplicial complex in which each m-cube C of X that is incident on v contributes one $(m-1)$-simplex, consisting of the unit tangent vectors at v that point into C.

If we identify an m-cube C of X with $[0,1]^m$, then f restricted to C is equal to $(t_1, \ldots, t_m) \mapsto t_1 + t_2 + \cdots + t_m + r$ for some integer r. The gradient of f on the cube C is the vector $(1, 1, \ldots, 1)$, of length $\sqrt{m}$. Thus any path γ of fastest descent leaves v travelling in the direction of the long diagonal of a cube C of maximal dimension whose highest vertex is v.

If $\sqrt{m} < n$ then the path will reach the unique lowest vertex v' of C before it reaches the 0-level set; at this vertex a new choice of cube C' of maximal dimension with v' as its highest vertex should be made.

If w is any vertex of X then the cubes of X that have w as their highest vertex correspond to a subcomplex of $\mathrm{Lk}_X(w)$ called the descending or $\downarrow$-link, $\mathrm{Lk}_X^{\downarrow}(w)$. Each descending link $\mathrm{Lk}_X^{\downarrow}(w)$ is isomorphic to either L or its universal covering space $\widetilde{L}$, depending only on whether the height of w lies in S. In particular, each descending link has dimension equal to d, the dimension of L.

It follows that we can always find at least one unit speed path γ starting at v with constant rate of descent $\sqrt{d+1}$ and there is no path descending faster. Hence the distance from v to the 0-level set is $n/\sqrt{d+1}$ as claimed. □

For $S \subseteq \mathbb{Z}$, define $m(S) := \min\{|n| : n \in S\}$.

Lemma 3.2. *Suppose that L is d-dimensional and that $0 \in S \subseteq T \subseteq \mathbb{Z}$, and take the standard generating set for $G_L(S)$ and $G_L(T)$. The word length of any non-identity element in the kernel of the map $G_L(S) \to G_L(T)$ is at least $m(T-S)\sqrt{2/(d+1)}$.*

Proof. The Cayley graph $\Gamma(S)$ for $G_L(S)$ is embedded in the 0-level set in $X := X_L^{(S)}$, and similarly $\Gamma(T)$ is embedded in the 0-level set in $Y := X_L^{(T)}$. Moreover the branched covering map $X \to Y$ induces the natural quotient map $\Gamma(S) \to \Gamma(T)$. Let v be a height 0 vertex of X. Each standard generator for $G_L(S)$ is represented by the diagonal of a square of X so for any $g \in G_L(S)$ the triangle inequality tells us that the word length $l(g)$ satisfies $d_X(v, gv) \leq \sqrt{2}l(g)$. Now g is in the kernel of the map to $G_L(T)$ if and only if gv and v map to the same vertex of Y. Singular points for the map $X \to Y$ are vertices w whose heights lie in $T-S$, and by Lemma 3.1 these have $d_X(v,w) \geq m(T-S)/\sqrt{d+1}$. By Lemma 1.2 it follows that $d_X(v, gv) \geq 2m(T-S)/\sqrt{d+1}$ and hence $l(g) \geq d_X(v, gv)/\sqrt{2} \geq m(T-S)\sqrt{2/(d+1)}$. □

4. Digression on convexity

The arguments used in the previous section can be used to show that the 0-level sets are very rarely convex or even quasi-convex. The material in this section is not needed for our main theorem.

Corollary 4.1. *The 0-level set in $X_L^{(S)}$ is convex if and only if L is a single simplex. In this case $X_L^{(S)} = X_L$ does not depend on S.*

Proof. If L is a d-simplex then $X_L^{(S)} = X_L$ is a copy of $\mathbb{R}^{d+1}$ and the 0-level set is an affine subspace. For the converse, if L is any flag complex other than a single simplex, then L will contain at least two maximal simplices. If v is any vertex of $X_L^{(S)}$ of height one, the directions defined by the barycentres of these two maximal simplices give distinct geodesic paths from v to the 0-level set that are both locally distance minimizing, with end points x, x' of height 0. Within the geodesic arc from x to x', at most one point can locally minimize distance to v; the assumption that this geodesic arc lies in the 0-level set leads to a contradiction. □

Corollary 4.2. *If either L contains two simplices of maximal dimension d, or $\widetilde{L}$ does and $\mathbb{Z}-S$ is infinite, then the 0-level set in $X_L^{(S)}$ is not quasi-convex.*

Proof. We give only a sketch. Let $\theta = \theta(d)$ be the angle in $\mathbb{R}^{d+1}$ between the vector $(1, 1, \ldots, 1)$ and one of the coordinate hyperplanes. Let v be a vertex of height N in $X := X_L^{(S)}$. If L has a unique simplex of dimension d, then N should be chosen in $\mathbb{Z} - S$, otherwise N may be arbitrary. In this case there are two distance-minimizing geodesic paths from v to the 0-level set, with end points x and x' as above, corresponding to leaving v in the directions given by the barycentres of two distinct d-dimensional simplices of the descending link. We view x, x' and other points that depend on them as functions of N. The angle at v between these two geodesics is at least the constant 2θ. The geodesic triangle with vertices x, x' and v is isosceles with angle at least 2θ between the two equal sides. The length of the equal sides is $N/\sqrt{d+1}$. If y is the midpoint of the geodesic arc from x to x', it follows that $d_X(v, y) \le N\cos(\theta)/\sqrt{d+1}$. By increasing N this distance can be made arbitrarily smaller than $N/\sqrt{d+1}$, the distance from v to the 0-level set. Hence for any k, there is an N so that y is not in the k-neighbourhood of the 0-level set. Thus the 0-level set is not quasi-convex. $\square$

5. Taut loop length spectra for $G_L(S)$

Throughout this section we fix a finite connected non-simply connected flag complex L. For S a subset of $\mathbb{Z}$ containing 0, let $\Gamma(S)$ denote the Cayley graph of $G_L(S)$ with respect to the standard generators. We analyze the taut loop length spectrum $H(\Gamma(S))$. As in [2], it will be convenient to assume that elements of S grow quickly, which we do as follows.

Define $\alpha = \alpha(L)$ by $\alpha = \sqrt{2/(d+1)}$, where d is the dimension of L. For a finite set Ω of loops in L that normally generates $\pi_1(L)$, let $\beta(L, \Omega)$ be the maximum of the lengths of the loops in Ω, and define $\beta = \beta(L)$ to be the minimum value of $\beta(L, \Omega)$ over all such Ω. Choose an integer constant $C = C(L)$ so that $C > \beta/\alpha$ and $C\alpha > 3$. For F any subset of $\mathbb{N}$, define $S(F) = \{0\} \cup \{C^{2^n} : n \in F\}$. With these definitions we prove an analogue of [2, Proposition 1].

Proposition 5.1. *If F, F' are subsets of $\mathbb{N}$ so that $\Gamma(S(F))$ and $\Gamma(S(F'))$ are quasi-isometric then the symmetric difference of F and F' is finite.*

Theorem 1.1 follows immediately from this Proposition. To prove the Proposition we first describe $H(\Gamma(S(F)))$.

Theorem 5.2. *For any $F \subseteq \mathbb{N}$, the set $H(\Gamma(S(F)))$ is contained in the disjoint union $\{3\} \cup \bigcup_{n\in\mathbb{N}} [\alpha C^{2^n}, \beta C^{2^n}]$. The set $H(\Gamma(S(F))) \cap [\alpha C^{2^n}, \beta C^{2^n}]$ is non-empty if and only if $n \in F$. Also 3 is in $H(\Gamma(S(F)))$ if and only if L is not 1-dimensional.*

Proof. The choice of C ensures that $3 < \alpha C^{2^0}$ and that for all n, $\beta C^{2^n} < \alpha C^{2^{n+1}}$, which implies that the union is disjoint. Since $\Gamma = \Gamma(S(F))$ is a simplicial Cayley graph, the edge relations $a\overline{a} = 1$ do not contribute to $H(\Gamma)$, but the triangle relations imply that $3 \in H(\Gamma)$ whenever L has dimension at least two. If $F = \emptyset$ then the presentation $P(L, \Omega)$ contains only relations of length at most 3, so $H(\Gamma)$ is either empty if L is 1-dimensional or is equal to $\{3\}$ otherwise.

It remains to establish three statements

- If $n \in F$ then $H(\Gamma(S(F))) \cap [\alpha C^{2^n}, \beta C^{2^n}] \neq \emptyset$;
- If $n \notin F$ then $H(\Gamma(S(F))) \cap [\alpha C^{2^n}, \beta C^{2^n}] = \emptyset$;

- If $k > 3$ and $k \notin \bigcup_{n\in\mathbb{N}}[\alpha C^{2^n}, \beta C^{2^n}]$ then $k \notin H(\Gamma(S(F)))$.

The second and third of these statements can be grouped together into a single fourth statement:

- If $k > 3$ and $k \notin \bigcup_{n\in F}[\alpha C^{2^n}, \beta C^{2^n}]$ then $k \notin H(\Gamma(S(F)))$.

For the first statement, let $F' = F - \{n\}$, and consider the covering map $\Gamma(S(F')) \to \Gamma(S(F))$. The group $G_L(S(F'))$ acts freely on $\Gamma(S(F'))$, so we may attach free orbits of 2-cells to $\Gamma(S(F'))$ to make a simply-connected Cayley 2-complex Δ. Now let K be the kernel of the map $G_L(S(F')) \to G_L(S(F))$, or equivalently the covering group for the regular covering $\Gamma(S(F')) \to \Gamma(S(F))$. The quotient Δ/K is a 2-complex with 1-skeleton the graph $\Gamma(S(F))$ and fundamental group K. We know that any non-identity element of K has word length at least αC^{2^n} and that there is a non-identity element of K of word length βC^{2^n}. The shortest non-identity element of K defines a loop in $\Gamma(S(F)) \subseteq \Delta/K$ that must be taut, since it is not null-homotopic in Δ/K whereas every strictly shorter loop in $\Gamma(S(F))$ is null-homotopic in Δ/K.

It remains to prove the fourth statement. Fix an integer $k > 3$ that is not an element of $\bigcup_{n\in F}[\alpha C^{2^n}, \beta C^{2^n}]$. Choose $n \in \mathbb{N}$ maximal so that $\beta C^{2^n} < k$, if such n exists, and define $n := -1$ in the case when $3 < k < \alpha C$. Now let $F' := F \cap [0, n]$, where by definition $[0, -1] = \emptyset$. Once again, consider the covering map $\Gamma(S(F')) \to \Gamma(S(F))$. Since every relator in the presentation for $G_L(S(F'))$ has length at most βC^{2^n}, we may build a Cayley 2-complex Δ with 1-skeleton $\Gamma(S(F'))$ in which each 2-cell is attached to a loop of length at most βC^{2^n}. Now suppose that γ is a loop of length k in $\Gamma(S(F))$. If γ lifts to a loop in $\Gamma(S(F'))$ then it cannot be taut, since every loop in $\Gamma(S(F'))$ is null-homotopic in Δ. If on the other hand γ lifts to a non-closed path in $\Gamma(S(F'))$ then it corresponds to a non-identity element of the kernel of the map $G_L(S(F')) \to G_L(S(F))$ of word length at most k. But the shortest element in the kernel of this map has length at least αC^{2^m}, where m is the least element of $S(F) - S(F')$. By choice of n, we have that $k \leq \beta C^{2^m}$, and by hypothesis $k \notin [\alpha C^{2^m}, \beta C^{2^m}]$. This contradiction shows that the loop γ cannot be taut. $\square$

Proof. (Proposition 5.1). For $l \in [\alpha C^{2^n}, \beta C^{2^n}]$ and $l' \in [\alpha C^{2^{n+m}}, \beta C^{2^{n+m}}]$ for some $m > 0$, we have that $l'/l \geq C^{2^n-1}$. By Lemma 2.1, since $\Gamma(S(F))$ and $\Gamma(S(F'))$ are quasi-isometric, we see that $H(\Gamma(S(F)))$ and $H(\Gamma(S(F')))$ are k-related. Now if n is in the symmetric difference of F and F' then $C^{2^n-1} < k$. $\square$

6. Poincaré duality groups and semi-direct products

A *right-angled Coxeter group* is a group W admitting a presentation in which the only relators are that each generator has order two and that certain pairs of generators commute. A simplicial graph K gives rise to a right-angled Coxeter group W_K, with generators the set $V(K)$ of vertices of K and as commuting pairs the ends of each element of the edge set $E(K)$. As we consider cases when K is infinite, we start with a well-known lemma that will help us reduce to the finite case.

Lemma 6.1. *Let K' be any full subgraph of the simplicial graph K. The Coxeter group $W' = W_{K'}$ is a retract of the Coxeter group $W = W_K$.*

Proof. Let $V := V(K)$ and $V' := V(K')$. The inclusion of V' into V extends to define a group homomorphism $i : W' \to W$. The function $\pi : V \to V' \cup \{1\}$ defined by $\pi(v') = v'$ for $v' \in V'$ and $\pi(v) = 1$ for $v \notin V'$ extends to a group homomorphism $\pi : W \to W'$ and the composite $\pi \circ i : W' \to W'$ is the identity. □

Now suppose that a group G acts by automorphisms on the graph K. This induces an action of G on W_K by automorphisms, permuting the given generators for W_K, and so there is a semidirect product group $J = W_K \rtimes G$. Identify G with its image inside the semidirect product J. A choice of generating set for G together with a choice of G-orbit representatives in $V(K)$ gives rise to a generating set for J.

Now suppose that $S \mapsto G(S)$ is a functor from the category of subsets of $\mathbb{Z}$ with inclusions as morphisms to the category of finitely generated groups and surjective homomorphisms; for example $S \mapsto G_L(S)$ is such a functor for any connected finite flag complex L. Suppose further that $G(\emptyset)$ acts freely cocompactly on a (simplicial) graph $K(\emptyset)$ in such a way that any two vertices in the same $G(\emptyset)$-orbit are at edge path distance at least four. For $S \subseteq \mathbb{Z}$, define $K(S)$ to be the quotient of $K(\emptyset)$ by the kernel of the map $G(\emptyset) \to G(S)$, so that $G(S)$ acts freely cocompactly on the graph $K(S)$.

For $S \subseteq \mathbb{Z}$, define $J(S)$ to be the semidirect product $W_{K(S)} \rtimes G(S)$. Then $S \mapsto J(S)$ is another functor from subsets of $\mathbb{Z}$ and inclusions to finitely generated groups and surjective group homomorphisms. Fix a finite generating set for $J(\emptyset)$ consisting of a finite generating set for $G(\emptyset)$ and a set V' of $G(\emptyset)$-orbit representatives in $V(K(\emptyset))$. As generating set for $J(S)$, take the image of our given generating set for $J(\emptyset)$, and as generating set for $G(S)$ take the image of our given generating set for $G(\emptyset)$. For each S, the generating set for $G(S)$ is a subset of the generating set for $J(S)$, and its complement consists of generators that are in the kernel of the map $J(S) \to G(S)$.

It will be useful to have a presentation for $J(S)$ in terms of our generating set. Since $G(\emptyset)$ acts freely on the graph $K(\emptyset)$, the Coxeter relators between all of the generators for $W_{K(\emptyset)}$ are consequences of a finite set of relators indexed by the orbit representatives of vertices and edges in $K(\emptyset)$. To describe these relations, we choose a set E' of $G(\emptyset)$-orbit representatives of edges in $K(\emptyset)$, in such a way that each $e \in E'$ is incident on at least one $v \in V'$. For each $u \in V(K(\emptyset))$, let $g_u \in G(\emptyset)$ be the unique element such that $g_u.u \in V'$. Now define an integer N_1 by

$$N_1 := \max\{l(g_u)\colon u \text{ is incident on some edge in } E'\},$$

where $l(g)$ denotes the word length of $g \in G(\emptyset)$. The relations in our presentation for $J(S)$ are of the following kinds:

- v^2 for each $v \in V'$;
- relators $(vg_u u g_u^{-1})^2$, where $e \in E'$, u and v are the vertices incident on e and $v \in V'$;

- the relators in a presentation for $G(S)$.

Note that $J(S)$ is finitely presented whenever $G(S)$ is, and that the relators of the second kind are of length at most $4N_1 + 4$.

In the theorem below we write $l_{J(S)}$ and $l_{G(S)}$ for the word length with respect to these generating sets.

Theorem 6.2. *Take notation and hypotheses as in the paragraphs above, and define $N := 2N_1$. For all $S \subseteq T \subseteq \mathbb{Z}$, if $w \in J(S)$ is in the kernel of $J(S) \to J(T)$ and $l_{J(S)}(w) > N$ then there is $g \in G(S) - \{1\}$ with $l_{G(S)}(g) \leq l_{J(S)}(w)$ so that g is in the kernel of $G(S) \to G(T)$.*

Proof. In the case when w is not in the kernel of the map $J(S) \to G(S)$, we may take $g = \overline{w}$, the image of w, since this element is in the kernel of $G(S) \to G(T)$ and $l_{G(S)}(g) \leq l_{J(S)}(w)$.

Before starting the remaining (more difficult) case, we recall Tits' solution to the word problem for a right-angled Coxeter group [6, Theorem 3.4.2]. (This is usually only stated for the finitely generated case, but the general case follows via Lemma 6.1.) If $w = v_1 v_2 \cdots v_l$ is a word in the standard generators for a right-angled Coxeter group that represents the identity, then w can be reduced to the trivial word using some sequence of moves of two types:

- if v, v' are Coxeter generators that commute, replace the subword vv' by $v'v$;
- replace a subword vv by the trivial subword.

The kernel of the map $J(S) \to G(S)$ is the right-angled Coxeter group $W(S) := W_{K(S)}$. Let w be in the kernel of this map as well as in the kernel of the map $J(S) \to J(T)$. Pick a shortest word in the generators for $J(S)$ representing w, and write this word in the form $w = h_0 v_1 h_1 v_2 h_2 \cdots h_{n-1} v_n h_n$, where each $v_i \in V'$, each $h_i \in G(S)$, so that $l_{J(S)}(w) = n + \sum_{i=0}^{n} l_{G(S)}(h_i)$. Now define $g_i = h_0 h_1 \cdots h_{i-1}$ for $1 \leq i \leq n$. We have that

$$w = h_0 v_1 h_1 v_2 h_2 \cdots h_{n-1} v_n h_n = (g_1 v_1 g_1^{-1})(g_2 v_2 g_2^{-1}) \cdots (g_n v_n g_n^{-1}).$$

This second expression for w will not be reduced in general, but each subword $g_i v_i g_i^{-1}$ is equal to one of the standard Coxeter generators for the subgroup $W(S)$. By hypothesis w is non-trivial in $W(S)$ but is in the kernel of the map $W(S) \to W(T)$. Hence there must be a Tits move that can be applied to the image of this expression in $W(T)$ that cannot be applied to the same expression in $W(S)$.

If there is a Tits move of the second type that can be applied in $W(T)$ but not in $W(S)$, this implies that there exist i and j with $1 \leq i < j \leq n$ so that $g_i = g_j \in G(T)$, but $g_i \neq g_j \in G(S)$. If on the other hand there is a Tits move of the first type that can be applied in $W(T)$ but not in $W(S)$, there exist $1 \leq i < j \leq n$ so that $g_i = g_j g_u \in G(T)$ but $g_i \neq g_j g_u \in G(S)$, where g_u is one of the elements that takes a vertex of some edge in E' to a vertex in V', and so by definition of N_1, $l_{G(S)}(g_u) \leq l_{G(\emptyset)}(g_u) \leq N_1$.

Define an element of $J(S)$ by $w' := v_i h_i v_{i+1} \cdots h_{j-1} v_j$, where i and j are as above. Since the expression $w = h_0 v_1 h_1 v_2 h_2 \cdots h_n v_n$ is of minimal length in terms of our generators for $J(S)$, the length of the defining expression for

w' is also minimal. But $l_{J(S)}(w')$ is greater than or equal to the length of its image in $G(S)$, $h_i h_{i+1} \cdots h_{j-1} = g_i^{-1} g_j$. Thus $l_{J(S)}(w') \geq l_{G(S)}(g_i^{-1} g_j)$. Depending on which sort of Tits move was applied, $g_i^{-1} g_j$ is either a non-trivial element of the kernel of the map $G(S) \to G(T)$ or differs from such an element by some g_u. Let g be this element of the kernel, and note that $l_{G(S)}(g) \leq l_{G(S)}(g_i^{-1} g_j) + N_1$. Since w maps to the identity element of $G(S)$ and the image in $G(S)$ of w' is a path whose endpoints are at distance at least $l_{G(S)}(g_i^{-1} g_j)$, we see that $l_{J(S)}(w) \geq 2 l_{G(S)}(g_i^{-1} g_j)$. Putting these inequalities together we obtain $l_{G(S)}(g) \leq l_{J(S)}(w)/2 + N_1$. Since we may assume that $l_{J(S)}(w) > N = 2N_1$, this implies that $l_{G(S)}(g) \leq l_{J(S)}(w)$ as required. □

Now we specialize to the case of interest; the case when $G(S)$ is the group $G_L(S)$ for some finite flag complex L that is connected but not simply connected. In this case, Theorem 6.2 allows us to prove an analogue of Theorem 5.2 for the semidirect product $J(S) := W_{K(S)} \rtimes G_L(S)$. We define constants $\alpha = \alpha(L)$, $\beta := \beta(L)$ and $C := C(L)$ as in the previous section, and for $F \subseteq \mathbb{N}$ we define $S(F) \subseteq \mathbb{N}$ as before. Finally, we define $M := 4N_1 + 4$, which depends on both L and on the action of $G(\emptyset)$ on $K(\emptyset)$, and we denote by $\Lambda(S(F))$ the Cayley graph $\Gamma(J(S(F)))$.

Theorem 6.3. *For any $F \subseteq \mathbb{N}$, the set $H(\Lambda(S(F)))$ is contained in the union $[0, M] \cup \bigcup_{n \in \mathbb{N}} [\alpha C^{2^n}, \beta C^{2^n}]$.*

If $\alpha C^{2^n} > M$, then $H(\Lambda(S(F))) \cap [\alpha C^{2^n}, \beta C^{2^n}]$ is non-empty if and only if $n \in F$.

Proof. Since the presentation for $G_L(S(\emptyset))$ has only relators of length at most 3, we see that our presentation for $J(S(\emptyset))$ consists of relators of length at most M. This verifies the claim in the case when $F = \emptyset$. As in the proof of Theorem 5.2, it suffices to verify two claims:

- If $n \in F$ and $\alpha C^{2^n} > M$ then $H(\Lambda(S(F))) \cap [\alpha C^{2^n}, \beta C^{2^n}] \neq \emptyset$;
- If $k > M$ and $k \notin \bigcup_{n \in F} [\alpha C^{2^n}, \beta C^{2^n}]$, then $k \notin H(\Lambda(S(F)))$.

These statements can be verified exactly as in Theorem 5.2. For the first, we consider $F' := F - \{n\}$ and look at the covering map $\Lambda(S(F')) \to \Lambda(S(F))$. Attach free $J(S)$-orbits of 2-cells to $\Lambda(S(F'))$ to make a simply-connected Cayley complex Δ, and let K' be the kernel of the map $J(S(F')) \to J(S(F))$. The quotient Δ/K' has 1-skeleton $\Lambda(S(F))$ and fundamental group K'. As before, we know that there is an element of $K' \cap G(S(F'))$ of length βC^{2^n} and that any non-identity element of this subgroup has length at least αC^{2^n}. Since $M = 4N_1 + 4 > N = 2N_1$, Theorem 6.2 tells us that the word length of any non-identity element of K' is also at least αC^{2^n}. Now the shortest non-identity element of K' defines a taut loop in the required range.

For the second statement, given such a k, take n maximal so that $\beta C^{2^n} < k$, let $F' := F \cap [0, n]$, and consider the covering map $\Lambda(S(F')) \to \Lambda(S(F))$. As before, we can build a Cayley 2-complex Δ with 1-skeleton $\Lambda(S(F'))$ in which each 2-cell is attached to a loop of length at most $\max\{M, \beta C^{2^n}\}$. If γ is a loop in $\Lambda(S(F))$ of length k, then either γ is not taut, or the lift of γ to $\Lambda(S(F'))$ is a non-closed path. In the second case one obtains a non-identity element in the kernel of the map $J(S(F')) \to J(S(F))$ of length at most k. Since $k > M > N$, Theorem 6.2 tells us that there is a non-identity element of length at most k in the kernel of the map $G(S(F')) \to G(S(F))$, which cannot happen. This contradiction shows that γ cannot be taut. □

The next two corollaries follow easily by the same proofs as in the previous section.

Corollary 6.4. *If F, F' are subsets of $\mathbb{N}$ so that $\Lambda(S(F))$ and $\Lambda(S(F'))$ are quasi-isometric, the symmetric difference of F and F' is finite.*

Corollary 6.5. *For any L that is not simply-connected, and any graph $K(\emptyset)$ with a free $G_L(\emptyset)$-action, there are continuously many quasi-isometry classes of groups $J(S)$.*

The reason why Corollary 6.5 is of value concerns the use of the Davis trick [6] to construct non-finitely presented Poincaré duality groups, as described in [8, Sec. 18]. The starting point is a 2-complex L for which each $G_L(S)$ is type FP; for this group there is a finite 2-complex that is an Eilenberg-Mac Lane space $K(G_L(\emptyset), 1)$. For any $n \geq 4$, one can find a compact n-manifold V with boundary that is also a $K(G_L(\emptyset), 1)$. Now let K be the 1-skeleton of the barycentric subdivision of a triangulation of the boundary of V, and let $K(\emptyset)$ be the 1-skeleton of the induced triangulation of the boundary of the universal cover of V, with $G_L(\emptyset)$ acting via deck transformations. For this choice of $K(\emptyset)$, the group $J(\emptyset)$ contains a finite-index torsion-free subgroup J' that is the fundamental group of a closed aspherical n-manifold M, and such that regular covering $M(S)$ of M with fundamental group the kernel of $J' \to J(S)$ is acyclic for each $S \subseteq \mathbb{Z}$. One deduces that each $J(S)$ contains a finite-index torsion-free subgroup that is a Poincaré duality group of dimension n. Since the inclusion of a finite-index subgroup is always a quasi-isometry, Corollary 6.5 implies Corollary 1.3 as stated in the introduction.

It remains to prove Corollary 1.4 from the introduction. This follows from the above discussion by the Schwarz-Milnor Lemma [3, I.8.19], which tells us that the acyclic covering manifold $M(S)$ of M is quasi-isometric to the group $J(S)$.

References

[1] M. Bestvina and N. Brady, *Morse theory and finiteness properties of groups*, Invent. Math. **129** (1997), 445–470.

[2] B. H. Bowditch, *Continuously many quasiisometry classes of 2-generator groups*, Comm. Math. Helv. **73** (1998), 232–236.

[3] M. R. Bridson and A. Haefliger, *Metric spaces of non-positive curvature*, Grundlehren der Mathematischen Wissenschaften **319**, Springer-Verlag (1999).

[4] K. S. Brown, *Cohomology of Groups*, Grad. Texts in Math. **87**, Springer-Verlag (1982).

[5] M. W. Davis, *The cohomology of a Coxeter group with group ring coefficients*, Duke Math. J. **91** (1998) 297–314.

[6] M. W. Davis, *The geometry and topology of Coxeter groups*, London Mathematical Series Monographs Series **32**, Princeton University Press (2008).

[7] R. I. Grigorchuk, *Degrees of growth of finitely generated groups and the theory of invariant means* (English translation) Math. U.S.S.R. Izv. **25** (2) (1985) 259–300.

[8] I. J. Leary, *Uncountably many groups of type FP*, Proc. London Math. Soc. (3) **117** (2018), 246–276.

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