A HOMOTOPICAL GENERALISATION OF THE BESTVINA-BRADY CONSTRUCTION

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Abstract. By using the notion of polyhedral products $(\underline{X}, \underline{A})^K$, we recognise the Bestvina-Brady construction [4] as the fundamental group of the homotopy fibre of $(S^1,*)^L \to (S^1,*)^K = S^1$, where L is a flag complex and K is a one vertex complex. We generalise their construction by studying the homotopy fibre F of $(S^1,*)^L \to (S^1,*)^K$ for an arbitrary simplicial complex L and K an $(m-1)$-dimensional simplex. We describe the homology of F, its fixed points, and maximal invariant quotients for coordinate subgroups of $\mathbb{Z}^m$. This generalises the work of Leary and Saadetoğlu [13] who studied the case when $m = 1$.

1. Introduction

Some of the interesting and well-studied properties of geometric groups are their homological finiteness properties [5]. For a finite flag complex L, Bestvina and Brady [4] constructed a group H_L which is type $\mathrm{FP}(\mathbb{Z})$ but not finitely representable. The homological finiteness properties of the group H_L are determined by the homotopy type of the underlying simplicial complex L. Their construction led to many exotic examples where different finiteness properties were achieved by varying the homotopy type of L. To a finite flag complex L, a right-angled Artin group, G_L is associated. The kernel H_L of the map $\phi\colon G_L \to \mathbb{Z}$ which maps the generators to 1 is now known as a Bestvina-Brady group. One of the main results of their work shows that H_L is finitely presented if and only if L is simply connected.

Leary and Saadetğlu [13] studied (co)homological properties of the groups H_L using techniques from algebraic topology. They recognised the construction of Bestvina and Brady as a pullback. Specifically, for a simplicial complex L, Leary and Saadetğlu construct an H-space $T_L \subset (S^1)^m$ whose fundamental group is the group G_L. The multiplication map on S^1 reduces to $\mu\colon T_L \to S^1$ which in turn induces the map ϕ on the fundamental groups. The pullback of the universal cover of S^1 along μ, denoted $\widetilde{T}_L$, is thus a space whose fundamental group is H_L. In this way, the techniques from CAT(0) geometry used by Bestvina and Brady are bypassed in favour of methods from algebraic topology which are more effective in studying homotopy and homological invariants. Notably, the Leary-Saadetğlu approach allows one

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to define an analogue of the Bestvina-Brady group H_L for any simplicial complex L.

In this paper, we recognise that this pullback $\widetilde{T}_L$ is the homotopy fibre of a map between the polyhedral product $(S^1,*)^L$ and S^1. In the early 2000s the unified description of moment-angle complexes and Davis-Januszkiewicz spaces [6] led to the definition of polyhedral products which brought homotopy theoretical techniques into the study of toric topology. The moment-angle complex $\mathcal{Z}_K$ associated to a simplicial complex K is a topological analogue of a projective toric variety, and the Davis-Januszkiewicz space $DJ(K)$ is its homotopy orbit space. As such, the study of polyhedral products links topology, geometry and combinatorics. Questions such as how to combinatorially decompose the suspension of a polyhedral product [1], how to desuspend such a description [7–11] and how to combinatorially describe when the moment-angle complex is a co-H space [2, 3] are some examples of work in the area. These results are obtained by studying homotopy fibrations of polyhedral products. Previously, these homotopy fibrations were constructed by considering homotopy fibrations of topological pairs and passing to polyhedral products by fixing a simplicial complex K. For example, the inclusion $\iota\colon (\mathbb{C}P^\infty,*) \to (\mathbb{C}P^\infty,\mathbb{C}P^\infty)$ and a simplicial complex K on m-vertices induce the homotopy fibration sequence

$$(D^2,S^1)^K \to (\mathbb{C}P^\infty,*)^K \to (\mathbb{C}P^\infty,\mathbb{C}P^\infty)^K \cong BT^m$$

where the total space is the Davis-Januszkiewicz space $DJ(K)$ and the homotopy fibre is the moment-angle complex $\mathcal{Z}_K$. This highlights the fact that $DJ(K)$ is the homotopy orbit space of Z_K by T^m-action.

In this paper we take a novel approach by fixing a topological monoid (X,A) and studying the properties of the homotopy fibre of $(X,A)^L \to (X,A)^K$ induced by a simplicial map $f\colon L \to K$ of finite simplicial complexes. When K is a vertex, f a constant map and $(X,A) = (S^1,*)$, the fibre specialises to Leary and Saadetğlu's space $\widetilde{T}_L$ and therefore captures the Bestvina-Brady groups. With $(X,A) = (S^1,*)$ we extend the results of [13] to the case where K is an $(m-1)$-simplex for $m \geq 1$, and $f\colon L \to \Delta^{m-1}$ is a simplicial map. In this case the homotopy fibre is denoted by $\widetilde{T}_{(f,L)}$. Our goal is to gain a combinatorial description of $\widetilde{T}_{(f,L)}$ in terms of the underlying information of L and f.

Our main theorems are Theorem 3.6 and Theorem 3.9, which, for a simplicial complex L and a simplicial map $f\colon L \to \Delta^{m-1}$ show that there is a short exact sequence which computes the homology of the homotopy fibre $\widetilde{T}_{(f,L)}$ where the other terms in the sequence are expressed in terms of L and the map f.

The description of homology we obtain lends itself nicely to understanding the action of the fundamental group of $(S^1)^m$. For the group $\mathbb{Z}^m$ and its coordinate subgroups, we compute the fixed points of the homology of $\widetilde{T}_{(f,L)}$

in Proposition 4.3 and describe the maximal $\mathbb{Z}_j$-invariant quotient groups under strong conditions on L.

To clarify the techniques of the paper, in the final section we provide some examples. We explicitly calculate the homology of $\widetilde{T}_{(f,L)}$ for a chosen simplicial complex L, and several different maps $f\colon L \to \Delta^1$. For these particular examples, we give the homotopy type of $\widetilde{T}_{(f,L)}$.

2. Polyhedral Products

An abstract simplicial complex K on a set of vertices $V(K)$ is a collection of subsets $I \subset V(K)$ such that if $I \in K$ then any subset of I belongs to K. We assume that the empty set $\emptyset$ belongs to K and K is a finite complex. We may also denote the vertex set of K by $[m] = \{1, 2, \ldots, m\}$ if $|V(K)| = m$.

For a simplicial complex K on m vertices and an m-tuple of topological pairs $((X_1, A_1), \ldots, (X_m, A_m))$, define the *polyhedral product* $(\underline{X}, \underline{A})^K$ as

$$(\underline{X}, \underline{A})^K = \bigcup_{\sigma \in K} (\underline{X}, \underline{A})^\sigma$$

$$\text{where} \qquad (\underline{X}, \underline{A})^\sigma = \prod_{i=1}^{m} Y_i \qquad \text{and} \qquad Y_i = \begin{cases} X_i & \text{if } i \in \sigma \\ A_i & \text{if } i \notin \sigma. \end{cases}$$

For example, when K is an $(m-1)$-simplex, $(\underline{X}, \underline{A})^K$ is the m-fold product $X_1 \times \cdots \times X_m$. When K is a disjoint union of m points, $(\underline{X}, *)^K$ is the wedge $X_1 \vee \cdots \vee X_m$.

Let $g_i\colon (X_i, A_i) \to (Y_i, B_i)$ be a map of topological pairs. Then the product map $g_1 \times \ldots \times g_m\colon \prod_i X^i \to \prod_i Y^i$ restricts to a map

$$g_*\colon (\underline{X}, \underline{A})^K \to (\underline{Y}, \underline{B})^K$$

and $(g \circ h)_* = g_* \circ h_*$.

A simplicial inclusion $\iota\colon L \to K$ induces a map of polyhedral products

$$\iota_*\colon (\underline{X}, \underline{A})^L \to (\underline{X}, \underline{A})^K$$

which is also an inclusion of topological spaces. Therefore, we can consider the polyhedral product construction $(\underline{X}, \underline{A})^K$ as a functor of two variables: topological pairs and continuous maps between them, and simplicial complexes and simplicial inclusions.

When all topological pairs are the same, that is, when $(X_1, A_1) = \ldots = (X_m, A_m) = (X, A)$, we denote the polyhedral product $(\underline{X}, \underline{A})^K$ by $(X, A)^K$.

The next property of polyhedral products is the one we are most concerned with in this paper. Let X be a commutative topological monoid, and A a sub-monoid. Assume that $f\colon L \to K$ is a simplicial map between simplicial complexes on vertex sets $[m']$ and $[m]$, respectively. Then the map $\hat{f}\colon X^{m'} \to X^m$ defined by $\hat{f}(x)_j = \prod_{i \mid f(i)=j} x_i$ restricts to a map

$$f\colon (X, A)^L \to (X, A)^K. \tag{2.1}$$

From now on, we denote both the map between simplicial complexes and the induced map between polyhedral products by f.

3. The space $\widetilde{T}_{(f,L)}$ and its homology

In this paper we study a polyhedral product $(S^1,*)^L$ as a functor of simplicial complexes L and arbitrary simplicial maps keeping the topological pair $(S^1,*)$ fixed. We denote $(S^1,*)^L$ by T_L to make the notation closer to the one used by Bestvina and Brady [4] and Leary and Saadetoğlu [13].

For any simplicial map $f\colon L \to K$, we study the homotopy fibre of the map

$$f\colon T_L \to T_K$$

where f is the map defined in (2.1) for the topological pair $(S^1,*)$ using the multiplicative structure on S^1. A homotopy approach allows us to define the homotopy fibre for any simplicial complexes K and L and any simplicial map, generalising the Bestvina-Brady construction which is a special case when L is a flag complex, K a vertex and f the constant map to the vertex K.

In particular, when K is an $(m-1)$-simplex Δ^{m-1}, we study the homology and homological properties of the homotopy fibre $\widetilde{T}_{(f,L)}$ of the map $T_L \to (S^1)^m$ induced by a simplicial map $f\colon L \to \Delta^{m-1}$. This space can be obtained, as it was in [13] for the case $K = \Delta^0$, by pulling back the universal cover of $T_K = (S^1)^m$ via f,

$$\begin{array}{ccc} \widetilde{T}_{(f,L)} & \longrightarrow & \mathbb{R}^m \\ \downarrow & & \downarrow \\ T_L & \xrightarrow{\ f\ } & (S^1)^m. \end{array}$$

The fundamental group of $(S^1)^m$ acts on the universal cover $\mathbb{R}^m$, and hence acts on $\widetilde{T}_{(f,L)}$. We denote by G the free abelian group $\pi_1((S^1)^m) \cong \mathbb{Z}^m$, and by G_j its jth coordinate subgroup with generator z_j. Finally, for $I \subset V(K)$, denote

$$\widehat{G}_I := \prod_{j\in V(K)\setminus I} G_j.$$

In this section, we compute the homology of $\widetilde{T}_{(f,L)}$ with coefficients in an abelian group. At this point, we assume the coefficient group to be an abelian group A and omit it from the notation. We begin by computing the cellular chain complex of $\widetilde{T}_{(f,L)}$ in terms of the augmented chain complex $(C^+_*(L), d_L)$ of L. To do so we recall a cellular structure of $T_L \subset (S^1)^{m'}$ as written in [13].

Let the unit circle S^1 come with a CW-structure consisting of one 0-cell and one 1-cell (a 1-dimensional cube). For $V = V(L)$, let $T(V)$ denote the product $T(V) = \prod_{v\in V} S^1 \cong (S^1)^{m'}$ with a cubical product CW-structure.

In this CW-structure on $(S^1)^{m'}$ the n-cells are in bijection with n-element subsets of V. If $\sigma \subset V$, the closure in $(S^1)^{m'}$ of the cell corresponding to σ is equal to $(S^1,*)^{\sigma}$, and consists of all the cells corresponding to subsets of σ. There is a bijection between simplicial complexes whose vertex set is contained in V and non-empty subcomplexes of $T(V)$ (see for example [12]). A non-empty simplicial complex L corresponds to the complex $T_L = (S^1,*)^L$.

For the remainder of the paper, we fix L to be any finite simplicial complex, $K = \Delta^{m-1}, m \geq 1$, and $f\colon L \to \Delta^{m-1}$.

Proposition 3.1. *The cellular chain complex $C_*(\widetilde{T}_{(f,L)})$ is isomorphic to $\mathbb{Z}[G] \otimes C^+_{*-1}(L)$ with differential*

$$d(1 \otimes v_{i_0} \dots v_{i_n}) = \sum_{k=0}^{n} (-1)^k (1 - z_{f(k)}) \otimes v_{i_0} \dots \widehat{v}_{i_k} \dots v_{i_n}.$$

Proof. Since $\widetilde{T}_{(f,L)}$ is defined as a pullback of the universal cover over $(S^1)^m$ along the map $f\colon T_L \to (S^1)^m$, we need to lift the cellular structure of T_L. The cubical n-cells in T_L with opposite faces identified are in bijective correspondence with the $(n-1)$-simplices in L. An $(n-1)$-simplex $\sigma^{n-1} = \{v_0, \dots, v_{n-1}\}$ in L lifts to a free G-orbit in $\widetilde{T}_{(f,L)}$. After lifting, the ith opposite pair of faces are not identified but differ by the translation action of $z_{f(i)}$. Note that for $j \in V(K)$, all $f^{-1}(j)$ opposite pairs of faces differ by the translation action of z_j. The n-cells of $\widetilde{T}_{(f,L)}$ correspond to $G \times \{\sigma^{n-1} \mid \sigma^{n-1} \in L\}$. To give a G-equivariant bijection between n-cells of $\widetilde{T}_{(f,L)}$ and $G \times \{\sigma^{n-1} \mid \sigma^{n-1} \in L\}$, we choose a representative of each orbit as follows. Fix a 0-cell, τ, in $\widetilde{T}_{(f,L)}$. In each orbit of higher dimensional cells, pick the orbit representative that has τ as a 0-cell, but does not have $z_j^{-1}\tau$ as a 0-cell for any $j \in V(K)$. Thus $C_n(\widetilde{T}_{(f,L)})$ is isomorphic to $\mathbb{Z}[G] \otimes C^+_{n-1}(L)$.

To describe the boundary map d of $C_*(\widetilde{T}_{(f,L)})$, it suffices to specify what happens on $\mathbb{Z}[G] \otimes C^+_0(L)$. For this, we lift v_i and then take its boundary. In this way, we see that $d(1 \otimes v_i) = (1 - z_{f(i)}) \otimes \emptyset$. The boundary map of $C_*(\widetilde{T}_{(f,L)})$ now follows. □

For a vertex $v_i \in L$, note that $d(1 \otimes v_i) = (1 - z_{f(i)}) \otimes \emptyset$. Associated to v_i, the generator $z_{f(i)}$ is called the *weight* of v_i.

Corollary 3.2. *For any simplicial complex L and any abelian group A,*

$$H_0(\widetilde{T}_{(f,L)}; A) \cong \mathbb{Z}[\widehat{G}_{f(V(L))}] \otimes A.$$

Proof. Denote by $A_\emptyset$ a copy of A generated by $\emptyset$. As $Z_0(\widetilde{T}_{(f,L)}; A) = \mathbb{Z}[G] \otimes A_\emptyset$ and $B_0(\widetilde{T}_{(f,L)}; A) = \bigoplus_{j \in f(V(L))} (1 - z_j)\mathbb{Z}[G] \otimes A_\emptyset$, the result follows. □

Recall that for any subset $I \subset V(K)$ the restriction of K to I is called a *full subcomplex* and is denoted K_I. For our complex L, we denote by $L_{f^{-1}(j)}$

the full subcomplex of L on all the vertices mapped by f to $j \in V(K)$. Using the full subcomplexes $L_{f^{-1}(j)}$, the cycles and boundaries of $\widetilde{T}_{(f,L)}$ admit a decomposition that allows a nice description of homology.

Definition 3.3. *For $x \in C_n^+(L_{f^{-1}(j)})$, define the image of the coning vertices of x to be*

$$\mathrm{coneIm}(x) = \{i \in V(K) \setminus \{j\} \mid \exists v \in f^{-1}(i) \text{ such that } vx \in C_{n+1}^+(L)\}.$$

Let a *geometric cone* be a chain x such that $\mathrm{coneIm}(x)$ is non-empty.

For a fixed $j \in V(K)$, the module $X_n^+(L) \cap C_n^+(L_{f^{-1}(j)})$ where X can be either boundaries or cycles in L will be decomposed by collecting chains whose geometrical representatives are coned off by vertices of L which are mapped into I. Notice that the module $Z_n^+(L) \cap C_n^+(L_{f^{-1}(j)})$ is just n-cycles on $L_{f^{-1}(j)}$. The boundaries $B_n^+(L) \cap C_n^+(L_{f^{-1}(j)})$ might contain $B_n^+(L_{f^{-1}(j)})$ as a proper subset, that is,

$$\begin{aligned} &B_n^+(L) \cap C_n^+(L_{f^{-1}(j)}) = \\ &\langle x \in Z_n^+(L_{f^{-1}(j)}) \mid x \in B_n^+(L_{f^{-1}(j)}) \text{ or } \mathrm{coneIm}(x) \neq \emptyset \rangle. \end{aligned} \tag{3.1}$$

Here we emphasise that some boundaries which lie in the full subcomplex $L_{f^{-1}(j)}$ are boundaries of chains in L which are not strictly in $L_{f^{-1}(j)}$. See Example 2 in Section 5 for an illustration.

For some boundary $b \in B_n^+(L) \cap C_n^+(L_{f^{-1}(j)})$ which is not a boundary of a chain in $C_{n+1}^+(L_{f^{-1}(j)})$, and some chain $c \in C_p^+(L)$, we have $d_L(cb) = d_L(c)b + cd_L(b) = d_L(c)b$. In order for b to be a boundary of cb, c has to be a vertex.

Definition 3.4. *For a fixed $j \in V(K)$ and any subset $I \subset V(K) \setminus \{j\}$, define the module $\mathrm{geoCone}^I Z_n^+(L_{f^{-1}(j)})$ as a module complement of*

$$\langle x \in Z_n^+(L_{f^{-1}(j)}) \mid I \subsetneq \mathrm{coneIm}(x) \rangle$$

in

$$\langle x \in Z_n^+(L_{f^{-1}(j)}) \mid I \subset \mathrm{coneIm}(x) \rangle.$$

For boundaries, define

$$\mathrm{geoCone}^I B_n^+(L_{f^{-1}(j)}) = B_n^+(L) \cap C_n^+(L_{f^{-1}(j)}) \cap \mathrm{geoCone}^I Z_n^+(L_{f^{-1}(j)}).$$

If any integer multiple of a chain represents a geometric cone, the original chain is a geometric cone as well. This observation implies that for $I \subset V(K) \setminus \{j\}$ the module

$$\langle x \in Z_n^+(L_{f^{-1}(j)}) \mid I \subsetneq \mathrm{coneIm}(x) \rangle$$

is a pure submodule of the finitely generated free abelian group

$$\langle x \in Z_n^+(L_{f^{-1}(j)}) \mid I \subset \mathrm{coneIm}(x) \rangle,$$

so the module complement $\mathrm{geoCone}^I Z_n^+(L_{f^{-1}(j)})$ exists. Notice that the chains in $\mathrm{geoCone}^I Z_n^+(L_{f^{-1}(j)})$ represent geometric cones over I, but there might be chains in $Z_n^+(L_{f^{-1}(j)})$ that are geometric cones over I and not elements of $\mathrm{geoCone}^I Z_n^+(L_{f^{-1}(j)})$. For instance, the sum of two geometric cones for disjoint cone images J, J' properly containing I can be a geometric cone over I.

It follows from the definition that

$$Z_n^+(L_{f^{-1}(j)}) = \bigoplus_{I \subset V(K)\setminus\{j\}} \mathrm{geoCone}^I Z_n^+(L_{f^{-1}(j)})$$

as well as

$$\mathrm{geoCone}^I B_n^+(L_{f^{-1}(j)}) \subseteq \mathrm{geoCone}^I Z_n^+(L_{f^{-1}(j)}).$$

For $I \neq \emptyset$ and $x \in Z_n^+(L_{f^{-1}(j)})$ with $\mathrm{coneIm}(x) = I$, there exists a coning vertex $v \in f^{-1}(I)$ such that $vx \in C_{n+1}^+(L)$. Then $d_L(vx) = d_L(v)x - vd_L(x) = x$. This implies that

$$\mathrm{geoCone}^I Z_n^+(L_{f^{-1}(j)}) \subset B_n^+(L) \cap C_n^+(L_{f^{-1}(j)})$$

and thus $\mathrm{geoCone}^I Z_n^+(L_{f^{-1}(j)}) = \mathrm{geoCone}^I B_n^+(L_{f^{-1}(j)})$ for non-empty subsets I of $V(K) \setminus \{j\}$. This proves the following statement.

Lemma 3.5. *For a fixed $j \in V(K)$,*

$$B_n^+(L) \cap C_n^+(L_{f^{-1}(j)}) = \bigoplus_{I \subset V(K)\setminus\{j\}} \mathrm{geoCone}^I B_n^+(L_{f^{-1}(j)}).$$

In the case where the boundaries of the subcomplexes determine the boundaries of $\widetilde{T}_{(f,L)}$, the decomposition using geometric cones allows us to compute homology.

Theorem 3.6. *For any simplicial complex L, any abelian group A, and $n \geq 1$ assume that*

$$B_{n-1}^+(L) \cap C_{n-1}^+(L_{f^{-1}(j)}) = B_{n-1}^+(L_{f^{-1}(j)}), \quad 1 \leq j \leq m. \tag{3.2}$$

Then there are short exact sequences of $\mathbb{Z}[G]$-modules:

$$0 \to N \to H_n(\widetilde{T}_{(f,L)}; A) \to \mathbb{Z}[G] \otimes \widetilde{H}_{n-1}(L; A) \to 0 \tag{3.3}$$

$$0 \to Q \to H_n(\widetilde{T}_{(f,L)}; A) \to M \to 0 \tag{3.4}$$

where

$$N = \bigoplus_{j=1}^{m} \Big(\bigoplus_{I \subset V(K) \setminus \{j\}} \mathbb{Z}[\widehat{G}_{I \cup j}] \otimes \mathrm{geoCone}^I B^+_{n-1}(L_{f^{-1}(j)}) \Big)$$

$$M = \bigoplus_{j=1}^{m} \Big(\bigoplus_{I \subset V(K) \setminus \{j\}} \mathbb{Z}[\widehat{G}_{I \cup j}] \otimes \mathrm{geoCone}^I Z^+_{n-1}(L_{f^{-1}(j)}) \Big)$$

$$Q = \bigoplus_{j=1}^{m} \Big(\bigoplus_{I \subset V(K) \setminus \{j\}} \bigoplus_{i \in I \cup j} (1 - z_i)\mathbb{Z}[G] \otimes \frac{\mathrm{geoCone}^I Z^+_{n-1}(L_{f^{-1}(j)})}{\mathrm{geoCone}^I B^+_{n-1}(L_{f^{-1}(j)})} \Big) \oplus \left(\mathbb{Z}[G] \otimes \frac{Z^{\mathrm{mixed}}_{n-1}}{B^{\mathrm{mixed}}_{n-1}} \right)$$

and $Z^{\mathrm{mixed}}_{n-1}, B^{\mathrm{mixed}}_{n-1}$ *are modules representing contributions of cycles and boundaries from all simplices of* L *that are not fully contained in a full subcomplex* $L_{f^{-1}(j)}$ *for some* $1 \leq j \leq m$.

If $A = R$ *is a ring and* $\widetilde{H}_{n-1}(L; R)$ *is projective, the first sequence splits. Further, the second sequence always admits an* R*-module splitting.*

Proof. We begin by describing the cycles $Z_n = Z_n(\widetilde{T}_{(f,L)})$ and boundaries $B_n = B_n(\widetilde{T}_{(f,L)})$ of $\widetilde{T}_{(f,L)}$. Recall that the chain complex on $\widetilde{T}_{(f,L)}$ is $C_*(\widetilde{T}_{(f,L)}) \cong \mathbb{Z}[G] \otimes C^+_{*-1}(L)$. For a vertex $v_i \in L$, recall that $d(1 \otimes v_i) = (1 - z_{f(i)}) \otimes \emptyset$ and the generator $z_{f(i)}$ is the weight of v_i.

Given a cycle in $\widetilde{T}_{(f,L)}$, $\sum_i p_i((z_1, \ldots, z_m)) \otimes \sigma_i$, we have that $\sum_i \sigma_i$ is a cycle in L. However, given a cycle $c \in Z_n(L)$, it is not necessarily the case that $1 \otimes c$ is a cycle in Z_n. To rectify this, we need to include weights. The weights are determined linearly by the differential. For example, let σ be a cycle in L given by $v_1v_2 + v_2v_3 + v_3v_1$. Then the associated cycle in $\widetilde{T}_{(f,L)}$ is given by $(1 - z_{f(3)}) \otimes v_1v_2 + (1 - z_{f(1)}) \otimes v_2v_3 + (1 - z_{f(2)}) \otimes v_3v_1$.

In the case that all of the weights are the same, that is, $(1 - z_j) \otimes \sigma$, we can factor out the weights so that $1 \otimes \sigma$ is a cycle in $\widetilde{T}_{(f,L)}$. The above observations imply that

$$Z_n \cong \mathbb{Z}[G] \otimes Z^+_{n-1}(L).$$

In particular, the cycles have the following decomposition

$$Z_n \cong \bigoplus_{j=1}^{m} \big(\mathbb{Z}[G] \otimes Z^+_{n-1}(L_{f^{-1}(j)}) \big) \oplus (\mathbb{Z}[G] \otimes Z^{\mathrm{mixed}}_{n-1})$$

where Z^{mixed}_{n-1} is the $\mathbb{Z}$-module complement of $\bigoplus_{j=1}^{m} \big(Z^+_{n-1}(L_{f^{-1}(j)}) \big)$ inside $Z^+_{n-1}(L)$. It consists of cycles on vertices mapped by f to more than one vertex in K.

We can further decompose $Z_{n-1}(L_{f^{-1}(j)})$ using $\mathrm{geoCone}^I Z^+_{n-1}(L_{f^{-1}(j)})$ by taking into account the existence of geometric cones over cycles in $L_{f^{-1}(j)}$.

$$Z_n \cong \bigoplus_{j=1}^{m} \left(\bigoplus_{I \subset V(K)\setminus\{j\}} \mathbb{Z}[G] \otimes \mathrm{geoCone}^I Z^+_{n-1}(L_{f^{-1}(j)}) \right) \oplus (\mathbb{Z}[G] \otimes Z^{\mathrm{mixed}}_{n-1}).$$

By assumption (3.2), the geoCone decomposition of cycles induces the following decomposition on boundaries

$$B_n \cong \bigoplus_{j=1}^{m} \Big(\bigoplus_{I \subset V(K)\setminus\{j\}} \bigoplus_{i \in I \cup j} (1-z_i)\mathbb{Z}[G] \otimes \mathrm{geoCone}^I B^+_{n-1}(L_{f^{-1}(j)}) \Big) \\ \oplus (\mathbb{Z}[G] \otimes B^{\mathrm{mixed}}_{n-1})$$

where B^{mixed}_{n-1} is the $\mathbb{Z}$-module complement of $\bigoplus_{j=1}^m \big(B^+_{n-1}(L) \cap C^+_{n-1}(L_{f^{-1}(j)})\big)$ inside $B^+_{n-1}(L)$. It consists of boundaries with vertices mapped by f to more than one vertex in K.

We define $\mathbb{Z}[G]$-modules B'_n and Z'_n such that $B_n \subset B'_n \subset Z_n$ and $B_n \subset Z'_n \subset Z_n$ which induce short exact sequences

$$0 \to \frac{B'_n}{B_n} \to H_n(\widetilde{T}_{(f,L)}) \to \frac{Z_n}{B'_n} \to 0 \tag{3.5}$$

$$0 \to \frac{Z'_n}{B_n} \to H_n(\widetilde{T}_{(f,L)}) \to \frac{Z_n}{Z'_n} \to 0 \tag{3.6}$$

and prove the theorem.

Let B'_n be

$$B'_n = \mathbb{Z}[G] \otimes \left(\bigoplus_{j=1}^{m} (B^+_{n-1}(L_{f^{-1}(j)})) \oplus B^{\mathrm{mixed}}_{n-1} \right)$$

and define Z'_n as

$$Z'_n = \bigoplus_{j=1}^{m} \Big(\bigoplus_{I \subset V(K)\setminus\{j\}} (\bigoplus_{i \in I \cup j} (1-z_i)\mathbb{Z}[G]) \otimes \mathrm{geoCone}^I Z^+_{n-1}(L_{f^{-1}(j)}) \Big) \\ \oplus (\mathbb{Z}[G] \otimes Z^{\mathrm{mixed}}_{n-1}).$$

By calculating the quotient groups in (3.5) and (3.6), we obtain the first two sequences in the statement.

The $\mathbb{Z}[G]$-modules and maps that appear in the short exact sequences also admit an R-module structure which commutes with the G-action. If $\widetilde{H}_{n-1}(L;R)$ is R-projective, then $\mathbb{Z}[G] \otimes \widetilde{H}_{n-1}(L;R)$ is $R[G]$-projective, and so the first short exact sequence of $R[G]$-modules splits. □

One important instance where the boundaries of the subcomplexes determine the boundaries of $\widetilde{T}_{(f,L)}$ is when L is a disjoint union of $L_{f^{-1}(j)}$ for $1 \le j \le m$.

Proposition 3.7. *If L is a disjoint union of $L_{f^{-1}(j)}$ for $1 \leq j \leq m$, then sequence* (3.4) *reduces to the short exact sequence*
(3.7)
$$0 \to \bigoplus_{j=1}^{m} \mathbb{Z}[G] \otimes \widetilde{H}_{n-1}(L_{f^{-1}(j)}) \to H_n(\widetilde{T}_{(f,L)}; A) \to \bigoplus_{j=1}^{m} \mathbb{Z}[\widehat{G}_j] \otimes Z^{+}_{n-1}(L_{f^{-1}(j)}) \to 0.$$

Proof. When L is a disjoint union of subcomplexes $L_{f^{-1}(j)}$, there are no simplices on vertices which are sent to different vertices in K. Observe further that the only non-trivial geometric cone is over $I = \emptyset$ and is this case $\text{geoCone}^{\emptyset} Z^{+}_{n-1}(L_{f^{-1}(j)}) = Z^{+}_{n-1}(L_{f^{-1}(j)})$ as well as $\text{geoCone}^{\emptyset} B^{+}_{n-1}(L_{f^{-1}(j)}) = B^{+}_{n-1}(L_{f^{-1}(j)})$. □

In the general situation, where $B^{+}_{n}(L_{f^{-1}(j)}) \subset B^{+}_{n}(L) \cap C^{+}_{n}(L_{f^{-1}(j)})$, there may be boundaries of L supported by chains $C^{+}_{n}(L_{f^{-1}(j)})$ that are not boundaries in $L_{f^{-1}(j)}$. In this case, the geometric cones are not sufficient for describing the boundaries of $\widetilde{T}_{(f,L)}$. Geometric cones only detect the weights z_l where $l \neq j$. To determine whether boundaries with weights z_j exist, we need a finer decomposition of the boundaries of L in $L_{f^{-1}(j)}$. This finer decomposition will be determined by an algebraic condition on geometric cones.

Definition 3.8. *Let $j \in V(K)$ be fixed. Assume that $B^{+}_{n}(L_{f^{-1}(j)})$ is a direct summand of $Z^{+}_{n}(L_{f^{-1}(j)})$.*

For any subset $\{j\} \subset I \subset V(K)$, define the module $\text{algCone}^{I} B^{+}_{n}(L_{f^{-1}(j)})$ *as a module complement of*
$$\langle x \in B^{+}_{n}(L_{f^{-1}(j)}) \mid I \setminus \{j\} \subsetneq \text{coneIm}(x) \rangle$$
in
$$\langle x \in B^{+}_{n}(L_{f^{-1}(j)}) \mid I \setminus \{j\} \subset \text{coneIm}(x) \rangle.$$

Now for $\emptyset \neq I \subset V(K) \setminus \{j\}$, the module $\text{algCone}^{I} B^{+}_{n}(L_{f^{-1}(j)})$ *is defined as a module complement of*
$$\langle x \in B^{+}_{n}(L) \cap C^{+}_{n}(L_{f^{-1}(j)}) \mid I \subsetneq \text{coneIm}(x) \text{ or } x \in B^{+}_{n}(L_{f^{-1}(j)}) \text{ with } \text{coneIm}(x) = I \rangle$$
in
$$\langle x \in B^{+}_{n}(L) \cap C^{+}_{n}(L_{f^{-1}(j)}) \mid I \subset \text{coneIm}(x) \rangle.$$
Furthermore, note that $\text{algCone}^{\emptyset} B^{+}_{n}(L_{f^{-1}(j)})$ *is the trivial module.*

For cycles, define the algebraic cones as follows. For $\emptyset \neq I \subset V(K)$, set
$$\text{algCone}^{I} Z^{+}_{n}(L_{f^{-1}(j)}) = \text{algCone}^{I} B^{+}_{n}(L_{f^{-1}(j)})$$
and let $\text{algCone}^{\emptyset} Z^{+}_{n}(L_{f^{-1}(j)})$ *be a module complement of*
$$\bigoplus_{\emptyset \neq I \subset V(K)} \text{algCone}^{I} Z^{+}_{n}(L_{f^{-1}(j)})$$

in $Z_n^+(L_{f^{-1}(j)})$.

The decomposition given in (3.1) can be refined in terms of algebraic cones as

$$
\begin{aligned}
&B_n^+(L) \cap C_n^+(L_{f^{-1}(j)}) \\
&= \langle x \in Z_n^+(L_{f^{-1}(j)}) \mid x \in B_n^+(L_{f^{-1}(j)}) \text{ or } \operatorname{coneIm}(x) \neq \emptyset \rangle \\
&= \textstyle\bigoplus_{I \subset V(K)} \operatorname{algCone}^I B_n^+(L_{f^{-1}(j)}).
\end{aligned} \tag{3.8}
$$

Furthermore, the assumptions of the definition imply that $B_n^+(L) \cap C_n^+(L_{f^{-1}(j)})$ is a direct summand of $Z_n^+(L_{f^{-1}(j)})$ so the module complement $\operatorname{algCone}^{\emptyset} Z_n^+(L_{f^{-1}(j)})$ is well-defined.

Just as for the geometric cones, for algebraic cones, we have that

$$\operatorname{algCone}^I B_n^+(L_{f^{-1}(j)}) \subset \operatorname{algCone}^I Z_n^+(L_{f^{-1}(j)})$$

for all $I \subset V(K)$.

Algebraic cones allow us to further decompose the homology of $\widetilde{T}_{(f,L)}$.

Theorem 3.9. *Let R be a PID and let $n \geq 1$. For any simplicial complex L such that $\widetilde{H}_{n-1}(L_{f^{-1}(j)}; R)$ is projective for each j, there are short exact sequences of $R[G]$-modules:*

$$0 \to N \to H_n(\widetilde{T}_{(f,L)}; R) \to \mathbb{Z}[G] \otimes \widetilde{H}_{n-1}(L; R) \to 0 \tag{3.9}$$

$$0 \to Q \to H_n(\widetilde{T}_{(f,L)}; R) \to M \to 0 \tag{3.10}$$

where

$$
\begin{aligned}
N &= \bigoplus_{j=1}^{m} \left(\bigoplus_{I \subset V(K)} \mathbb{Z}[\widehat{G}_I] \otimes \operatorname{algCone}^I B_{n-1}^+(L_{f^{-1}(j)}) \right) \\
M &= \bigoplus_{j=1}^{m} \left(\bigoplus_{I \subset V(K)} \mathbb{Z}[\widehat{G}_I] \otimes \operatorname{algCone}^I Z_{n-1}^+(L_{f^{-1}(j)}) \right) \\
Q &= \bigoplus_{j=1}^{m} \left(\bigoplus_{I \subset V(K)} \bigoplus_{i \in I} (1 - z_i)\mathbb{Z}[G] \otimes \frac{\operatorname{algCone}^I Z_{n-1}^+(L_{f^{-1}(j)})}{\operatorname{algCone}^I B_{n-1}^+(L_{f^{-1}(j)})} \right) \\
&\quad \oplus \mathbb{Z}[G] \otimes \frac{Z_{n-1}^{\text{mixed}}}{B_{n-1}^{\text{mixed}}}
\end{aligned}
$$

and $Z_{n-1}^{\text{mixed}}, B_{n-1}^{\text{mixed}}$ arise from terms with mixed weights. If $\widetilde{H}_{n-1}(L; R)$ is projective, the first sequence splits. Further, the second sequence always admits an R-module splitting.

Proof. As in the proof of Theorem 3.6, we seek descriptions of the cycles and boundaries of $\widetilde{T}_{(f,L)}$ and to define $\mathbb{Z}[G]$-modules Z'_n and B'_n such that

$B_n \subset B'_n \subset Z_n$ and $B_n \subset Z'_n \subset Z_n$. Recall that the cycles $Z_n = Z_n(\widetilde{T}_{(f,L)})$ decompose as

$$Z_n \cong \bigoplus_{j=1}^{m} \left(\mathbb{Z}[G] \otimes Z^{+}_{n-1}(L_{f^{-1}(j)})\right) \oplus (\mathbb{Z}[G] \otimes Z^{\text{mixed}}_{n-1})$$

where $\mathbb{Z}[G] \otimes Z^{\text{mixed}}_{n-1}$ is the $\mathbb{Z}[G]$-module complement consisting of cycles on vertices mapped by f to more than one vertex in K.

Notice that the condition of $B^{+}_n(L_{f^{-1}(j)})$ being a direct summand of $Z^{+}_n(L_{f^{-1}(j)})$ is satisfied because of the projectivity of the homology of the $L_{f^{-1}(j)}$ complexes. Therefore we decompose $Z_{n-1}(L_{f^{-1}(j)})$ using $\text{algCone}^I Z^{+}_{n-1}(L_{f^{-1}(j)})$.

$$Z_n \cong \bigoplus_{j=1}^{m} \left(\bigoplus_{I \subset V(K)} \mathbb{Z}[G] \otimes \text{algCone}^I Z^{+}_{n-1}(L_{f^{-1}(j)}) \right) \oplus (\mathbb{Z}[G] \otimes Z^{\text{mixed}}_{n-1}).$$

Similarly, for boundaries B_n we have

$$\begin{aligned} B_n \cong & \bigoplus_{j=1}^{m} \Big(\bigoplus_{I \subset V(K)} \bigoplus_{i \in I} (1 - z_i)\mathbb{Z}[G] \otimes \text{algCone}^I B^{+}_{n-1}(L_{f^{-1}(j)}) \Big) \\ & \oplus \mathbb{Z}[G] \otimes B^{\text{mixed}}_{n-1} \end{aligned}$$

where $\mathbb{Z}[G] \otimes B^{\text{mixed}}_{n-1}$ is the $\mathbb{Z}[G]$-module complement consisting of boundaries on vertices mapped by f to more than one vertex in K.

We define $\mathbb{Z}[G]$-modules B'_n and Z'_n such that $B_n \subset B'_n \subset Z_n$ and $B_n \subset Z'_n \subset Z_n$ which induce short exact sequences

$$0 \to \frac{B'_n}{B_n} \to H_n(\widetilde{T}_{(f,L)}) \to \frac{Z_n}{B'_n} \to 0 \tag{3.11}$$

$$0 \to \frac{Z'_n}{B_n} \to H_n(\widetilde{T}_{(f,L)}) \to \frac{Z_n}{Z'_n} \to 0. \tag{3.12}$$

Let B'_n be

$$B'_n = \mathbb{Z}[G] \otimes \left(\bigoplus_{j=1}^{m} (B^{+}_{n-1}(L) \cap C^{+}_{n-1} L_{f^{-1}(j)}) \oplus B^{\text{mixed}}_{n-1} \right)$$

and define Z'_n as

$$\begin{aligned} Z'_n = & \bigoplus_{j=1}^{m} \Big(\bigoplus_{I \subset V(K)} (\bigoplus_{i \in I} (1 - z_i)\mathbb{Z}[G]) \otimes \text{algCone}^I Z^{+}_{n-1}(L_{f^{-1}(j)}) \Big) \\ & \oplus (\mathbb{Z}[G] \otimes Z^{\text{mixed}}_{n-1}). \end{aligned}$$

Clearly $B_n \subset B'_n \subset Z_n$ and $B_n \subset Z'_n \subset Z_n$ and we can rewrite the short exact sequences of (3.11) and (3.12) as in the statement of the theorem.

The decomposition in equation (3.8) and the fact that $B'_n \cong \mathbb{Z}[G] \otimes B_{n-1}(L)$ are useful for calculating the quotients.

The $\mathbb{Z}[G]$-modules and maps that appear in the short exact sequences also admit an R-module structure which commutes with the G-action. If $\widetilde{H}_{n-1}(L;R)$ is R-projective, then $\mathbb{Z}[G]\otimes\widetilde{H}_{n-1}(L;R)$ is $R[G]$-projective, and so the first short exact sequence of $R[G]$-modules splits. □

We remark in the situation where the assumptions of both Theorem 3.6 and Theorem 3.9 are satisfied, the sequences (3.3) and (3.9) are identical, although the sequences (3.4) and (3.10) are only isomorphic. In all cases, N in sequence (3.3) can be rewritten as

$$\bigoplus_{j=1}^{m}\left(\bigoplus_{I\subset V(K)}\mathbb{Z}[\widehat{G}_I]\otimes\mathrm{algCone}^I B^+_{n-1}(L_{f^{-1}(j)})\right)$$

which is exactly the term N in sequence (3.9).

4. Group Actions on Homology of $\widetilde{T}_{(f,L)}$

In this section we identify the points in $H_n(\widetilde{T}_{(f,L)})$ which are left fixed by the action of the group G and its coordinate subgroups, and compute their maximal invariant quotients. We also give a combinatorial description of the rank of $H_n(\widetilde{T}_{(f,L)})$.

Theorem 3.9 facilitates the following computation of fixed points.

Proposition 4.1. *Let $n\geq 1$. When K has only one vertex,*

$$H_n(\widetilde{T}_{(f,L)})^G = B^+_{n-1}(L).$$

When K has more than one vertex, and the assumptions of Theorem 3.6 or Theorem 3.9 are satisfied, then

$$H_n(\widetilde{T}_{(f,L)})^G = \bigoplus_{j=1}^{m}\left(\mathrm{algCone}^{V(K)}B^+_{n-1}(L_{f^{-1}(j)})\right).$$

Proof. Since the fixed point functor is left exact, the short exact sequence (3.9) induces the exact sequence

$$0\to N^G\to H_n(\widetilde{T}_{(f,L)})^G\to(\mathbb{Z}[G]\otimes\widetilde{H}_{n-1}(L))^G\to\dots$$

from which the statement follows. □

Corollary 4.2. *Let L be a disjoint union of $L_{f^{-1}(j)}$ for $1\leq j\leq m$. The non-trivial G-fixed points exist only in the case when K has one vertex, that is, $m=1$,*

$$H_n(\widetilde{T}_{(f,L)})^G = B^+_{n-1}(L).$$

Proof. When L is a disjoint union of $L_{f^{-1}(j)}$ complexes, there are no geometric cones. The only non-trivial algCone are over the singleton set $\{j\}$. In Proposition 4.1, the only summands are algCone over the whole set $V(K)$. Therefore $V(K)=\{j\}$ for some $1\leq j\leq m$. □

Proposition 4.3. *When the assumptions of Theorem 3.6 or Theorem 3.9 are satisfied, then the G_ℓ-fixed points of $H_n(\widetilde{T}_{(f,L)})$ are given by*

$$H_n(\widetilde{T}_{(f,L)})^{G_\ell} = \bigoplus_{j=1}^{m} \left(\bigoplus_{\substack{I \subset V(K) \\ \ell \in I}} \mathbb{Z}[\widehat{G}_I] \otimes \mathrm{algCone}^I B^+_{n-1}(L_{f^{-1}(j)}) \right).$$

Proof. From the short exact sequence (3.9), the G_ℓ-fixed points are isomorphic to N^{G_ℓ}. □

Corollary 4.4. *Let L be a disjoint union of $L_{f^{-1}(j)}$ for $1 \leq j \leq m$. Then the G_ℓ-fixed points in $H_n(\widetilde{T}_{(f,L)})$ are given by*

$$H_n(\widetilde{T}_{(f,L)})^{G_\ell} = \mathbb{Z}[\widehat{G}_l] \otimes B^+_{n-1}(L_{f^{-1}(l)}).$$

Proof. When L is a disjoint union of complexes $L_{f^{-1}(j)}$, the only non-empty cone over $L_{f^{-1}(j)}$ is from the singleton set $\{j\}$, and Proposition 4.3 reduces to the statement here. □

Turning our attention to invariant quotients, the homological decompositions of $\widetilde{T}_{(f,L)}$ we obtained are not sufficient for calculating them for general complexes L. However, we can generalize the results of Leary and Saadetoğlu [13] to L being a disjoint union of $L_{f^{-1}(j)}$.

Proposition 4.5. *Let L be a disjoint union of $L_{f^{-1}(j)}$ for $1 \leq j \leq m$. Then the largest G-invariant quotient of $H_n(\widetilde{T}_{(f,L)})$ is*

$$H_n(\widetilde{T}_{(f,L)})_G \cong Z_{n-1}(L).$$

Proof. To compute the maximal G-invariant quotient of $H_n(\widetilde{T}_{(f,L)})$ start with the exact sequence

$$0 \to B_n \to Z_n \to H_n(\widetilde{T}_{(f,L)}) \to 0$$

and apply the right exact invariant quotient functor $H_0(G; -)$. One obtains an exact sequence

$$H_0(G; B_n) \to Z_{n-1}(L) \to H_n(\widetilde{T}_{(f,L)})_G \to 0.$$

Now by the short exact sequence (3.7), we get that $Z_{n-1}(L)$ is a quotient of $H_n(\widetilde{T}_{(f,L)})$. □

Proposition 4.6. *Let L be a disjoint union of points. Then the maximal G_l-invariant quotients are*

$$H_0(\widetilde{T}_{(f,L)})_{G_l} = \begin{cases} \mathbb{Z}[\widehat{G}_{f(V(L))}] & \text{if } \ell \in f(V(L)) \\ 0 & \text{otherwise} \end{cases}$$

$$H_1(\widetilde{T}_{(f,L)})_{G_l} = \mathbb{Z}[\widehat{G}_l] \otimes Z_0(L)$$

$$H_i(\widetilde{T}_{(f,L)})_{G_l} = 0 \text{ for } i \geq 2.$$

Proof. Since L is a 0-dimensional simplicial complex, $H_i(\widetilde{T}_{(f,L)}) = 0$ for $i \geq 2$.

To compute G_l-invariants of $H_1(\widetilde{T}_{(f,L)})$ consider the exact sequence

$$0 \to B_i \to Z_i \to H_i(\widetilde{T}_{(f,L)}) \to 0$$

and apply $H_0(G_l; -)$. Since there are no non-trivial boundaries in the 1-chains $C_1(L)$, we obtain an isomorphism

$$0 \to \mathbb{Z}[\widehat{G}_\ell] \otimes Z_0(L) \to H_1(\widetilde{T}_{(f,L)})_{G_\ell} \to 0.$$

To see that $\mathbb{Z}[\widehat{G}_\ell] \otimes Z_0(L)$ is the maximal quotient notice that $H_1(\widetilde{T}_{(f,L)}) \cong \mathbb{Z}[G] \otimes Z_0(L)$ which projects down to $\mathbb{Z}[\widehat{G}_\ell] \otimes Z_0(L)$.

By Corollary 3.2, $H_0(\widetilde{T}_{(f,L)}) \cong \mathbb{Z}[\widehat{G}_{f(V(L))}]$. Now the G_l-invariants can be calculated directly.

□

Similarly to T_L, the Betti numbers of $\widetilde{T}_{(f,L)}$ involve combinatorial features of L. To determine them precisely, these need to be weighted by the group G.

To an n-dimensional simplicial complex K we associate a face vector $f^K = (f_0, f_1, \ldots, f_n)$, where f_i is the number of i-dimensional faces of K. Denote by $f_i^{L_{f^{-1}(j)}}$ the face vector of $L_{f^{-1}(j)}$.

Proposition 4.7. *Let L be a disjoint union of connected simplicial complexes $L_{f^{-1}(j)}$, $1 \leq j \leq m$ such that $H_*(L; R) = 0$ for $* > 0$ over a ring R. Then for $n > 0$, the rank of $H_n(\widetilde{T}_{(f,L)}; R)$ is*

$$\operatorname{rank} H_n(\widetilde{T}_{(f,L)}) = \sum_{j=1}^{m} |\mathbb{Z}^{m-1}| \left(\sum_{l=0}^{n} (-1)^{n+l} f_{l-1}^{L_{f^{-1}(j)}} \right).$$

Proof. Since $H_*(L; R) = 0$ for $* > 0$, sequence (3.3) implies that for $n > 1$

$$H_n(\widetilde{T}_{(f,L)}) = \bigoplus_{j=1}^{m} \left(\mathbb{Z}[\widehat{G}_j] \otimes B_{n-1}^+(L_{f^{-1}(j)}) \right).$$

From sequence (3.7),

$$H_1(\widetilde{T}_{(f,L)}) \cong \bigoplus_{j=1}^{m} \left(\mathbb{Z}[\widehat{G}_j] \otimes Z_0^+(L_{f^{-1}(j)}) \right).$$

Since $L_{f^{-1}(j)}$ is a connected full subcomplex of L, $\widetilde{H}_*(L_{f^{-1}(j)}) \cong 0$. This implies that

$$H_n(\widetilde{T}_{(f,L)}) \cong \bigoplus_{j=1}^{m} \left(\mathbb{Z}[\widehat{G}_j] \otimes Z_{n-1}^+(L_{f^{-1}(j)}) \right) \cong \bigoplus_{j=1}^{m} \left(\mathbb{Z}[\widehat{G}_j] \otimes B_{n-1}^+(L_{f^{-1}(j)}) \right),$$

$$C_n^+(L_{f^{-1}(j)}) \cong Z_{n-1}^+(L_{f^{-1}(j)}) \oplus Z_n^+(L_{f^{-1}(j)}). \tag{4.1}$$

Since $Z_n^+(L_{f^{-1}(j)})$ is a free R-module, by iterating equation (4.1), the rank of $Z_n^+(L_{f^{-1}(j)})$ is given by

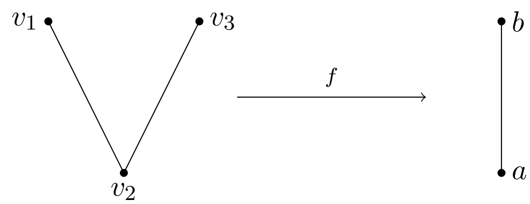
$$\operatorname{rank} Z_n^+(L_{f^{-1}(j)}) = \sum_{\ell=0}^{n} (-1)^{n+\ell} f_{\ell-1}^{L_{f^{-1}(j)}}.$$

□

5. Examples

In this section we present several examples to illustrate key technical points. By working with a single choice of simplicial complexes K and L but various maps f, we highlight the role of the map.

Let $L = \{\emptyset, \{v_1\}, \{v_2\}, \{v_3\}, \{v_1, v_2\}, \{v_2, v_3\}\}$ and K be a 1-simplex on the vertex set $\{a, b\}$.



For all choices f, following Proposition 3.1, the groups $C_*(\widetilde{T}_{(f,L)})$ are isomorphic to $\mathbb{Z}[\mathbb{Z}_a \times \mathbb{Z}_b] \otimes C_{*-1}^+(L; R)$.

Example 1. Let $f_1 \colon L \to K$ be given by $f_1(v_1) = f_1(v_2) = f_1(v_3) = a$. The differential is given by

$$\begin{aligned} &d(1 \otimes v_1) = d(1 \otimes v_2) = d(1 \otimes v_3) = (1 - z_a) \otimes \emptyset \\ &d(1 \otimes v_1 v_2) = (1 - z_a) \otimes (v_2 - v_1) \\ &d(1 \otimes v_2 v_3) = (1 - z_a) \otimes (v_3 - v_2). \end{aligned}$$

The non-trivial groups of cycles and boundaries are

$$\begin{aligned} Z_0 &= \langle 1 \otimes \emptyset \rangle \cong \mathbb{Z}[\mathbb{Z}_a \times \mathbb{Z}_b] \otimes R_\emptyset \\ B_0 &= \langle (1 - z_a) \otimes \emptyset \rangle \cong (1 - z_a)\mathbb{Z}[\mathbb{Z}_a \times \mathbb{Z}_b] \otimes R_\emptyset \\ Z_1 &= \langle 1 \otimes (v_2 - v_1), 1 \otimes (v_3 - v_2) \cong \mathbb{Z}[\mathbb{Z}_a \times \mathbb{Z}_b] \otimes (R_{v_2 - v_1} \oplus R_{v_3 - v_2}) \\ B_1 &= \langle (1 - z_a) \otimes (v_2 - v_1), (1 - z_a) \otimes (v_3 - v_2) \rangle \\ &\cong (1 - z_a)\mathbb{Z}[\mathbb{Z}_a \times \mathbb{Z}_b] \otimes (R_{v_2 - v_1} \oplus R_{v_3 - v_2}) \end{aligned}$$

and hence

$$H_0(\widetilde{T}_{(f_1,L)}) = \mathbb{Z}[\mathbb{Z}_b] \otimes R_\emptyset \text{ and } H_1(\widetilde{T}_{(f_1,L)}) = \mathbb{Z}[\mathbb{Z}_b] \otimes (R_{v_2 - v_1} \oplus R_{v_3 - v_2}).$$

A simple computation following Corollaries 4.2 and 4.4 shows that

$$H_*(\widetilde{T}_{(f_1,L)})^{\mathbb{Z}_a \times \mathbb{Z}_b} = 0, H_*(\widetilde{T}_{(f_1,L)})^{\mathbb{Z}_b} = 0, \text{ and } H_*(\widetilde{T}_{(f_1,L)})^{\mathbb{Z}_a} = H_*(\widetilde{T}_{(f_1,L)})$$

which can also be seen directly from the homology calculation.

Moreover, following Proposition 4.5 the maximal $(\mathbb{Z}_a\times\mathbb{Z}_b)-$invariant quotient is

$$H_0(\widetilde{T}_{(f_1,L)})_{\mathbb{Z}_a\times\mathbb{Z}_b}\cong R_{\emptyset},\quad H_1(\widetilde{T}_{(f_1,L)})_{\mathbb{Z}_a\times\mathbb{Z}_b}\cong R_{v_2-v_1}\oplus R_{v_3-v_2}.$$

In general, the homotopy type of $\widetilde{T}_{(f,L)}$ is difficult to determine. In this case, however, the complex L is the pushout of the inclusions $L_2\hookrightarrow L_1$ and $L_2\hookrightarrow L_3$, where $L_1=\{\emptyset,\{v_1\},\{v_2\},\{v_1,v_2\}\}$, $L_2=\{\emptyset,\{v_2\}\}$ and $L_3=\{\emptyset,\{v_3\},\{v_2\},\{v_3,v_2\}\}$ which allows us to explicitly calculate the homotopy type of $\widetilde{T}_{(f_1,L)}$. The corresponding polyhedral products are $T_{L_1}=S^1_1\times S^1_2$, $T_{L_2}=S^1_2$ and $T_{L_3}=S^1_2\times S^1_3$. Now T_L is a pushout of the inclusions of T_{L_2} into T_{L_1} and T_{L_3}. After finding the homotopy fibre of the restrictions of f_1 to T_{L_1},T_{L_2},T_{L_3}, we have the following commutative diagram

$$\begin{array}{ccccc}
\Omega S^1_b\times S^1_1 & \longleftarrow & \Omega S^1_b & \longrightarrow & \Omega S^1_b\times S^1_3\\
\downarrow & & \downarrow & & \downarrow\\
T_{L_1} & \longleftarrow & T_{L_2} & \longrightarrow & T_{L_3}\\
& \searrow & \downarrow & \swarrow & \\
& & S^1_a\times S^1_b & &
\end{array}$$

where the vertical maps are homotopy fibrations. Because we are working over a constant diagram, the homotopy colimit and homotopy limit commute. Therefore, $\widetilde{T}_{(f_1,L)}$ is the homotopy colimit of the diagram $\Omega S^1_b\times(S^1\leftarrow *\rightarrow S^1)$ and thus

$$\widetilde{T}_{(f_1,L)}\simeq(S^1\vee S^1)\times\Omega S^1.$$

Since all the vertices of L are mapped to the vertex a in K, the map f_1 factors as the constant map to $\{a\}$ composed with the inclusion map of the vertex a into K. It can be shown that the homotopy fibre of $T_L\rightarrow S^1_a$ is $S^1\vee S^1$. The effect of having a ghost vertex in $f(L)$, that is, a vertex whose pre-image by f is empty, is that a factor of ΩS^1 appears in $\widetilde{T}_{(f,L)}$. This can be seen in homology by the existence of $\mathbb{Z}[\mathbb{Z}_b]$.

Example 2. Let $f_2\colon L\rightarrow K$ be given by $f_2(v_1)=f_2(v_3)=b, f_2(v_2)=a$. The differential is given by

$$\begin{aligned}
&d(1\otimes v_1)=d(1\otimes v_3)=(1-z_b)\otimes\emptyset\\
&d(1\otimes v_2)=(1-z_a)\otimes\emptyset\\
&d(1\otimes v_1v_2)=(1-z_b)\otimes v_2-(1-z_a)\otimes v_1\\
&d(1\otimes v_2v_3)=(1-z_a)\otimes v_3-(1-z_b)\otimes v_2.
\end{aligned}$$

The non-trivial groups of cycles and boundaries are

$$\begin{aligned} Z_0 &= \langle 1 \otimes \emptyset \rangle \cong \mathbb{Z}[\mathbb{Z}_a \times \mathbb{Z}_b] \otimes R_\emptyset \\ B_0 &= \langle (1-z_a) \otimes \emptyset, (1-z_b) \otimes \emptyset \rangle \\ Z_1 &= \langle 1 \otimes (v_3 - v_1), (1-z_b) \otimes v_2 - (1-z_a) \otimes v_1 \rangle \\ B_1 &= \langle (1-z_a) \otimes (v_3 - v_1), (1-z_b) \otimes v_2 - (1-z_a) \otimes v_1 \rangle \end{aligned}$$

and thus the non-trivial homology groups are

$$H_0(\widetilde{T}_{(f_2,L)}) = R_\emptyset \text{ and } H_1(\widetilde{T}_{(f_2,L)}) = \mathbb{Z}[\mathbb{Z}_b] \otimes R_{v_3-v_1}.$$

In this example, we have $L_{f_2^{-1}(a)} = \{v_2\}$ and $L_{f_2^{-1}(b)} = \{\{v_1\}, \{v_3\}\}$. There are no non-trivial boundaries in either of these full subcomplexes. However, the cones are detecting boundaries in L. Here,

$$\text{geoCone}^{\{a\}} B_0^+(L_{f_2^{-1}(b)}) = \text{algCone}^{\{a\}} B_0^+(L_{f_2^{-1}(b)}) = R_{v_3-v_1}$$

and all other cones over the boundaries are trivial. The generator $(1-z_a) \otimes (v_3 - v_1)$ of B_1 is an example where a chain $v_3 - v_1$ is a boundary of L contained in $L_{f_2^{-1}(b)}$ but itself is not a boundary of $L_{f_2^{-1}(b)}$. Notice that the generator $(1-z_b) \otimes v_2 - (1-z_a) \otimes v_1$ is an instance of a cycle with mixed weights from $\mathbb{Z}[G] \otimes Z_0^{\text{mixed}}$.

The fixed points are given by

$$\begin{aligned} &H_0(\widetilde{T}_{(f_2,L)})^{\mathbb{Z}_a \times \mathbb{Z}_b} = H_0(\widetilde{T}_{(f_2,L)})^{\mathbb{Z}_a} = H_0(\widetilde{T}_{(f_2,L)})^{\mathbb{Z}_b} = H_0(\widetilde{T}_{(f_2,L)}) \\ &H_1(\widetilde{T}_{(f_2,L)})^{\mathbb{Z}_a} = H_1(\widetilde{T}_{(f_2,L)}) \end{aligned}$$

and all other groups are trivial.

The homotopy type of $\widetilde{T}_{(f_2,L)}$ can be determined in the same manner as in Example 1. Here, the restrictions of f_2 to T_{L_1} and T_{L_3} are identity maps and $\widetilde{T}_{(f_2,L)}$ is the homotopy pushout $* \leftarrow \Omega S_b^1 \rightarrow *$. Therefore $\widetilde{T}_{(f_2,L)} \simeq \Sigma\Omega S^1$.

Example 3. Let $f_3 \colon L \to K$ be given by $f_3(v_1) = b, f_3(v_2) = f_3(v_3) = a$. The differential is given by

$$\begin{aligned} &d(1 \otimes v_1) = (1-z_b) \otimes \emptyset \\ &d(1 \otimes v_2) = d(1 \otimes v_3) = (1-z_a) \otimes \emptyset \\ &d(1 \otimes v_1v_2) = (1-z_b) \otimes v_2 - (1-z_a) \otimes v_1 \\ &d(1 \otimes v_2v_3) = (1-z_a) \otimes (v_3 - v_2). \end{aligned}$$

Then

$$\begin{aligned} Z_0 &= \langle 1 \otimes \emptyset \rangle \cong \mathbb{Z}[\mathbb{Z}_a \times \mathbb{Z}_b] \otimes R_\emptyset \\ B_0 &= \langle (1-z_a) \otimes \emptyset, (1-z_b) \otimes \emptyset \rangle \\ Z_1 &= \langle 1 \otimes (v_3 - v_2), (1-z_b) \otimes v_2 - (1-z_a) \otimes v_1 \rangle \\ B_1 &= \langle (1-z_a) \otimes (v_3 - v_2), (1-z_b) \otimes v_2 - (1-z_a) \otimes v_1 \rangle \end{aligned}$$

and thus the homology groups are

$$H_0(\widetilde{T}_{(f_3,L)}) = R_\emptyset \text{ and } H_1(\widetilde{T}_{(f_3,L)}) = \mathbb{Z}[\mathbb{Z}_b] \otimes R_{v_3-v_2}.$$

In this example, we have $L_{f_3^{-1}(a)} = \{\{v_2, v_3\}\}$ and $L_{f_3^{-1}(b)} = \{\{v_1\}\}$. There is a non-trivial boundary $v_3 - v_2$ in $L_{f_3^{-1}(a)}$. Here,

$$\text{geoCone}^{\emptyset} B_0^+(L_{f_3^{-1}(a)}) = \text{algCone}^{\{a\}} B_0^+(L_{f_3^{-1}(a)}) = R_{v_3 - v_2}$$

and all other cones over the boundaries are trivial.

The results of the fixed point computations are the same as in the Example 2.

Since $\widetilde{T}_{(f_3,L)}$ is the homotopy pushout of $\Omega S^1_b \to *$ and the inclusion $\Omega S^1_b \Omega \to S^1_b \times S^1_3$, we have $\widetilde{T}_{(f_3,L)} \simeq \Omega S^1 \ltimes S^1$.

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