Elastoplastic response and recoil of lattice structures under hyperbolic hardening

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Abstract

Elastoplastic response and recoil analyses for hexagonal honeycomb lattice structures are presented when hardening is described by a hyperbolic law. By exploiting the translational symmetry of the problem, the analysis is reduced to that of a thin beam under combined bending and axial loading coupled with the kinematics of lattice deformation and its relationship with cell wall deformation. A closed-form solution for the load-curvature relationship of a beam with rectangular cross-section is obtained. A systematic study of beam response, as the stress-strain curve of the constituent material approaches an ideal elastic-perfectly plastic law, is presented. The analysis is then applied to an infinite honeycomb sheet under remote tensile load to obtain the apparent non-linear structural response. Apparent recoil of such a lattice material upon unloading is also calculated in closed form, when unloading is assumed to take place along a linear stress-strain curve. The analytical results are in excellent agreement with the numerical calculations.

*Keywords*: Elastoplastic Analysis; Hyperbolic Hardening; Lattice Structures; Recoil Analysis

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1. Introduction

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## Several natural and synthetic materials show internal microstructure consisting of voids, or randomly or periodically interconnected struts. They are commonly referred to as cellular materials, if they do not show regularity, or lattices, if the geometric features are translationally repeated in space. Their main advantage lies in their high strength to weight ratio, a property frequently desired in applications within several sectors of industry, such as transport, aerospace, naval architecture and biomedical engineering [1-3]. In addition, their excellent energy absorption capability makes them suitable for crashworthiness applications, such as helmets and protective gear for sport [4]. Due to the wide applicability of such periodic porous structures, the prediction of their mechanical performances holds the key to their industrial future.

## The mechanical characterisation of lattice structures has attracted the attention of researchers since the middle of the 20th century. Abd El-Sayed et al. [5] presented the first analysis for *in-plane* elastic properties of planar structures with hexagonal cells. They derived expressions for the elastic moduli using energy methods, more specifically Castigliano's theorem. Few years later, Gibson et al. [6] gave the first comprehensive and detailed analysis of the *in-plane* mechanical properties of hexagonal lattice structures. Furthermore, they investigated the *out-of-plane* properties of lattice structures and they extended the analysis to three-dimensional foams [7]. Zhu and Chen [8] derived an analytical relationship between the elastic constants of honeycomb sheets and their relative density for honeycombs with struts showing thickness variations. Gibson et al. [4] further extended the elastic analysis to the estimation of the collapse limit of the cell using plastic hinge theory.

## Before the failure is reached, the response of lattice structures becomes nonlinear when they undergo plastic deformation. Zhu [9] presented a study of elasto-plastic response of plates under pure bending moment while assuming material hardening described by a power law. They posed honeycomb structures as a context for the applicability of their analysis, which is limited to the study of energy absorbed in plastic hinges and curvature of a single cell wall, while ignoring the stretch deformation. It is well known that cell walls in hexagonal honeycombs do not deflect with a constant curvature along their length. This is a consequence of the translational and reflective symmetry of the lattice; one could show that a cell wall deflect transversely with a point of inflection at its centre (i.e. it possesses zero curvature there), the curvature being maximum at the ends. The linear moment variation along the cell wall is due to a combined moment and transverse forces present at its ends – this is consistent with the free body diagram of a typical cell wall (see figure 7 in Appendix). In the present work, we (i) use a hyperbolic material hardening model (as described later in section 2) for cell wall deformation, (ii) respect the cell wall mechanics that accounts for linear variation of bending moment along its length – this leads to a non-constant curvature along the length of the cell walls, (iii) relate lattice kinematics with the cell wall deformation in order to analytically calculate apparent cellular response under hyperbolic hardening (iv) account for axial loading in the cell wall that shifts the neutral axis, (v) calculate lattice spring back upon unloading, (vi) consider two possible cross-sectional shapes (rectangular and circular) for the cell walls, while carrying out all the analysis in the spirit of elasto-plastic thin beam-column mechanics. To the best of our knowledge all these features for a remotely loaded honeycomb have not been considered previously. The closest analysis is one presented by Bonfanti and Bhaskar [10,11]. However the constitutive relationship in the analysis of [10] does not consider material hardening – instead an elastic-perfectly plastic stress-strain relationship has been used there.

## A combination of experimental and numerical approaches has been used to study the plastic behaviour of honeycombs under compression by Papka et al. [12-13] and Mangipudi et al. [14]. The effects of cell irregularity on the elasto-plastic response of bending dominated lattices has been studied numerically by Sotomayor [15]. Thus, a review of literature reveals that analytical consideration to lattice response under plastic deformation is very limited. The present work provides an analysis of periodic honeycombs for a nonlinear hardening material. A hyperbolic hardening model [16] that smoothly matches with the linear elastic part is used.

## Graphene is a two-dimensional sheet of carbon atoms arranged in a honeycomb configuration [17], which resemble the structure of the lattice sheet considered here. However, the physical origin of the mechanical properties of graphene is very different compared to such 2D cellular material. Unlike 2D honeycomb materials, where the mechanical properties are directly related to those of the constituent material and the geometry of the unit cell, those of graphene are mainly imputable to the interaction among carbon atoms [18]. While there are similarities in the deformation of graphene sheets with that of macroscopic honeycombs with regards to their lattice geometry and kinematics, the deformation mechanism within graphene and the cellular material considered here are very different.

## Although the analysis of plastically deforming structures has been studied for some time, it has recently attracted the attention of several researchers especially in the area of metal forming [19-20]. Several approaches to predict elasto-plastic response of beams are available [21-22]; however, most of them assume the constituent material as elastic-perfectly plastic. This simplification is frequently used since it facilitates the mathematics, yet it includes all the feature of plasticity problems. Solutions for beams of rectangular cross-section with the bending moment varying linearly [23] or quadratically [24-25] along its length are available. In many engineering applications, combined bending moment and axial load are applied to the structure. This affects the position of the neutral axis; therefore, the stress distribution over the cross-section is not symmetric. Such an effect under plastic deformation is taken into account analytically only by Yu and Johnson [23] for the rectangular cross-section. The effects of non-linear hardening on the elasto-plastic response of a beam with rectangular cross-section under pure constant moment has been presented by Rimovski [26]. Recently, Bonfanti et. al. [16] presented an analysis to study the influence of the axial force in manufacturing processes while considering the hardening effects. This has been possible by introducing an improved mathematical description of the material stress-strain curve. A piecewise [16] function was used to describe the stress-strain relationship there: the elastic part is modelled using a straight line, while the nonlinear hardening is a translated rectangular hyperbola that has tangent at the transition point matching with the linear elastic part. The application of such analysis was limited to the study of beam bending during manufacturing processes.

## The aim of the work presented here is to develop a new analysis to calculate the elastoplastic response of lattice structures when subjected to remote tensile loading when the constitutive behavior is described by a hyperbolic hardening law – this constitutive model was introduced recently in a metal forming context in [16]. The analysis takes into consideration the non-linear effects due to the material hardening and the stiffening effect due to the application of an axial load. The cell wall deformation kinematics and the equilibrium are incorporated into the mechanics of the lattice structure respecting the non-constant curvature of the cell walls resulting in an apparent non-linear response of the structured sheet. Effects of the cross-sectional shape of the cell walls on the permanent deformation after unloading are also quantified.

## The paper is organized as follows. In Section 2, plastic analysis of a single cell-wall with rectangular cross section is introduced. Following this, the calculation of springback is carried out in Section 3. The mechanical response of a hexagonal lattice sheet under remote tensile stress, together with a systematic study of a single cell-wall response when the material curve approaches that of an elastic-perfectly plastic material, are presented in Section 4. In Section 4, the elasto-plastic response of a honeycomb whose struts have a rectangular cross section is compared with that of circular cross section while considering the hardening effects. Further, the analytical solution is compared to the numerical solution obtained using Finite Element Analysis. Finally, concluding remarks are made in Section 5.

## 2. Elastoplastic analysis of a cell wall with rectangular cross-section

Consider a honeycomb sheet loaded remotely along the *x-*direction, as shown in Fig. 1(a). Due to the symmetry afforded by the infinite lattice, the study of the overall structural response subjected to elasto-plastic deformation, is reduced to the study of a single inclined cell wall [10]. However, as opposed to [10], where strain hardening is not considered and an elastic-perfectly plastic material is assumed instead, here we consider a smooth nonlinear material hardening law. Because of the translational symmetry and also two-fold rotational symmetry, the deflected shape of the inclined cell walls possesses two-fold rotational symmetry about their centres. This means the presence of a point of inflection, hence a zero curvature, at the centre of a deformed cell wall The mechanics of a single cell wall can, therefore, be further reduced its half from an end to its centre. Half of a cell wall can be modelled as a cantilever beam with combined transverse and axial loads. The cell-wall is made of an elasto-plastic material with non-linear hardening (see Fig. 1b). The plastic behaviour of the bulk material is described by a translated rectangular hyperbola which must be continuous and differentiable at the transition from the elastic to the plastic regime. This constitutive relation is expressed as

, (1)

where *E* is the Young's modulus, y is the yield stress, *A*, *B* and *D* are the coefficients of the hyperbola and and are the asymptotes. Full details of such a material model proposed by us before, but employed in a completely different context, are provided in [16]. Assume half of a cell-wall with rectangular cross-section of width *b* and thickness *t* which to be subjected respectively to transverse force *P* and an axial force *N*, as shown in Fig. 1(c). The moment *M* along the cell wall varies linearly, from zero at the tip of half of the beam (which is also the centre of the cell wall) to the maximum *M\* = Pl* at the root. In line with the thin beam theory, assume that the cross-sections remain plane and perpendicular to the neutral axis during the elasto-plastic deformation. Geometric non-linearities are neglected by assuming that deflections are small compared to *t*, the thickness of the cell wall. The cell wall is also assumed to be stress-free before loading.

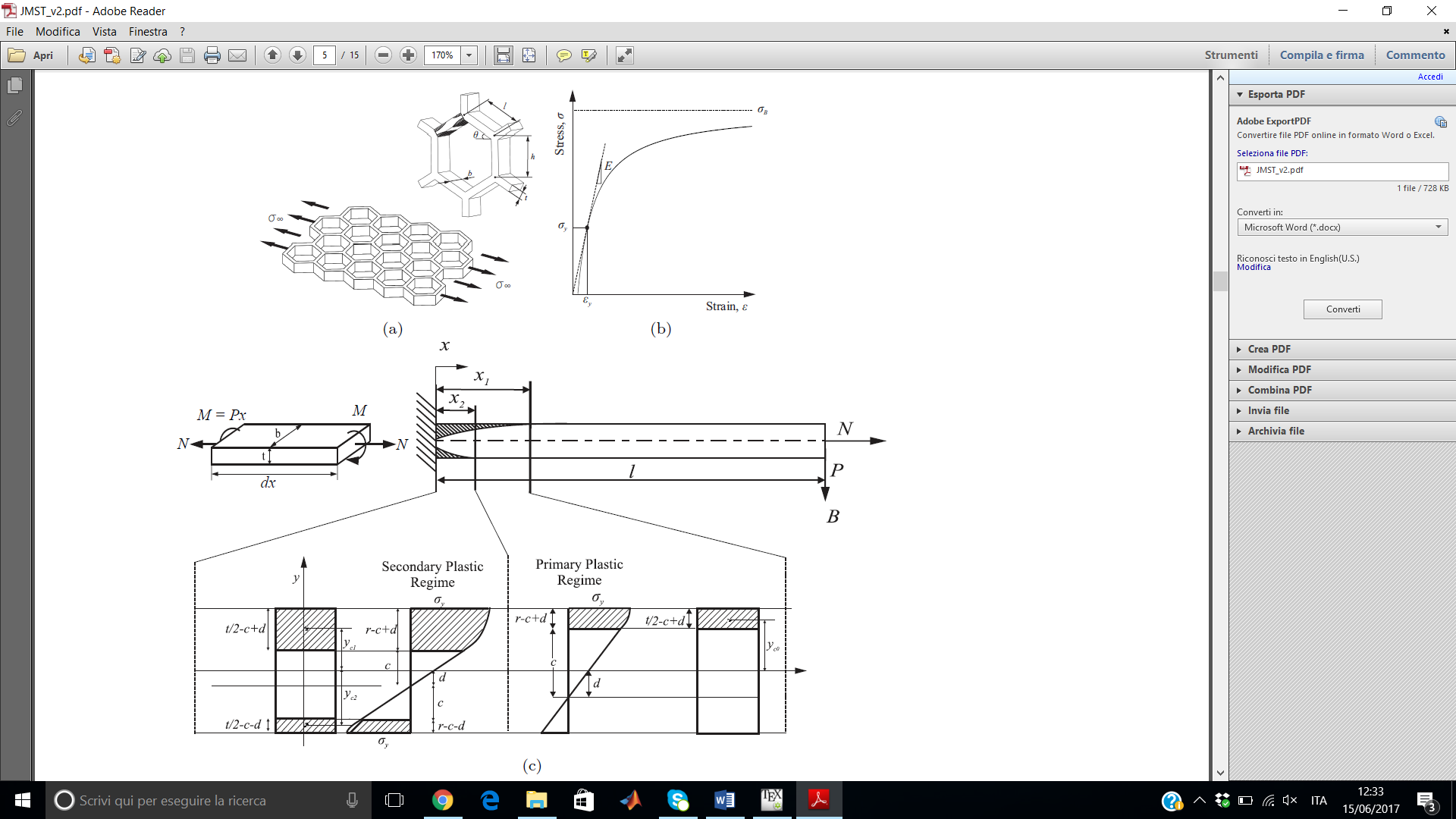


Fig. 1. (a) Infinite honeycomb sheet subjected to remote tensile stress. (b) Hyperbolic hardening model. (c) Cantilever beam with axial and shear components. On the left side, a section of the beam subjected to pure bending moment due to the transverse force and the tensile axial force is shown. The hatched areas are the plastic zones that spread along the beam while the load is increased. Due to the asymmetric stress distribution, the two sides of the beam do not undergo plastic deformation at the same time. The two types of stress distribution when the cross section is under plastic deformation are schematically shown below.

When a beam is loaded transversally, the stress within the cross-section varies smoothly between the maximum tensile and compressive stress. The neutral axis is located where the stress is zero—thus there are no strains. For symmetric sections of isotropic straight beams, the neutral axis is at the geometric centroid. However, its location changes if an axial load is applied. Due to this shift, the two parts of the sections do not experience a symmetric state of stress. Given the bending moment *M* and axial force *N* along half of the cell wall—the axial position of a cross-section along the cell wall is identified by *x*. The stress distribution in each cross-section can be of qualitatively three different types, each requiring analysis within regime:

* completely elastic, where all fibres in the cross-section elastically deformed which corresponds to ;
* primary plastic—PI, where only one side of the beam starts plastically deforming (Fig. 1(c), second from the left on the bottom). This regime spans ;
* secondary plastic—PII, where both sides of the beam plastically deform (Fig. 1(c), first from the left on the bottom). This regime spans .

The stress distribution in each cross-section is obtained by drawing a stress profile of the same shape as that of the stress-strain curve of the constituent material from the neutral axis [23]. This information is insufficient to fully determine the stress profile when the stress distribution is asymmetric about the geometrical centre line. Two geometric parameters are introduced to fully describe the shape of the stress distribution: (i) the offset between neutral axis and the first fiber that has undergone plastic deformation, *c* and (ii) the distance of the neutral axis from the geometric centroid of the cross-section, *d*.

We now introduce the following dimensionless parameters for moment, axial force and curvature

(2)

where *, ,*  are respectively the initial yield quantities for moment, axial force and curvature. For small deflections,

(3)

where *c* is the offset between the neutral axis and the first yielded fibre, as shown in Fig. 1(c).

2.1 Elastic regime (Er)

In this regime, the deformation is purely elastic. Because of the axial component, the stress distribution is asymmetric and the neutral axis does not go through the centroid of the section. Using the superposition principle, valid in the linear elastic regime, it can be shown that the load-curvature relationship is given by

(4)

for [23] Hence, is the boundary beyond which the treatment must account for plasticity.

2.2 Primary plastic regime (PI)

When a cross-section is subjected to combined axial force and bending moment, fibres parallel to the longitudinal direction of the cell wall start yielding asymmetrically only on one side because of the shifting of the neutral axis. The stress does not remain constant at the value of the yield stress , but it increases nonlinearly as dictated by the material constitutive law given by Eq. 1. The stress distribution through the thickness in the primary plastic regime is shown in Fig. 1(c), the second sub-figure from the left at the bottom. The axial force and moment equilibrium equations are used to determine the values of the two unknowns, *c* and *d*. Axial equilibrium requires

(5)

The first two terms are the contributions arising from the two triangular areas under linear deformation, while the integral represents the total axial force obtained by summing the distributed stress over the area under plastic deformation. Similarly, imposition of the moment equilibrium over the cross-section leads to

. (6)

By solving the non-linear system given by Eqs. (5)-(6), the values of *c* and *d* shown in Fig. 1(c) are obtained. Once the value of *c* is known, then the dimensionless curvature can be calculated from equation Eq. 3. It can be shown that the cell wall will start yielding on both sides when *.* This condition represents the boundary of the primary plastic regime and secondary plastic regime.

2.3 Secondary plastic regime (PII)

If we further increase the loading beyond the limit of the primary plastic regime, both sides of the cell wall undergo plastic deformation. The corresponding stress profile is qualitatively shown in the first sub-figures in Fig. 1(c), at the bottom. The force and momentum equilibria are imposed to determine the geometric parameters *c* and *d*, as done previously for the *PI*. By summing the contribution given by the two plastically deformed parts of the cross-section, the axial equilibrium requires

(7)

The contributions given by the two areas under elastic deformation are equal and opposite in sign. Therefore, the effects compensate and they do not generate any net force in the above equation. Note that for cross-sections of other shapes (e.g. circular), the linear stress distribution gives rise to a net force due to the varying shape in the *y-*direction. Similarly, the moment equilibrium given by the upper and lower part of the cross-section under plastic deformation leads to

. (8)

As before, the system of Eqs. (7)-(8) is solved for the unknowns *c* and *d*. By applying Eq. (3), the dimensionless curvature is obtained.

## 3. Springback of a cell wall subjected to pure bending and axial force

Consider a cell wall modelled as a thin beam that has undergone plastic deformation studied before. When the structure is unloaded, the elastic strain due to the axial force and the pure bending moment is recovered. As stated by Yu and Johnson [23] for an analysis carried out in the context of metal forming, respecting the conditions under which this model has been developed, the unloading can be assumed to be completely elastic, thus no yield effect occurs during the spring back. Therefore, the final dimensionless curvature is given by

(9)

where the value of is calculated by solving the systems of equations obtained previously.

The above analytical expression gives us the final curvature and bending moment relationship for different axial loadings. We will first obtain the elasto-plastic response of a single cell wall before combining this cell wall response with the kinetic of infinite lattices in Section 4. The numerical results for a single cell wall response, under remote loading of an infinite lattice, are plotted in Fig. 2(a). The shaded regions show the combination of normalised axial force and bending moment that lead to primary plastic state of stress. By examining the plot of such relationship (Fig. 2), it can be noticed that the amount of recoil decreases when the axial force *N* increases.

Fig. 2(b) relates to the final curvature of a beam with circular cross-section made of a material showing hyperbolic hardening. If the same axial force *n* and bending moment *m* are applied to the cell wall, the rectangular cross-section shows a lower springback and, therefore, a higher final curvature. As measured by the shape factor , that is the ratio of the collapse moment and the plastic moment, the circular cross-section (1.70) can store more energy than the rectangular cross-section (1.50), for equal area.

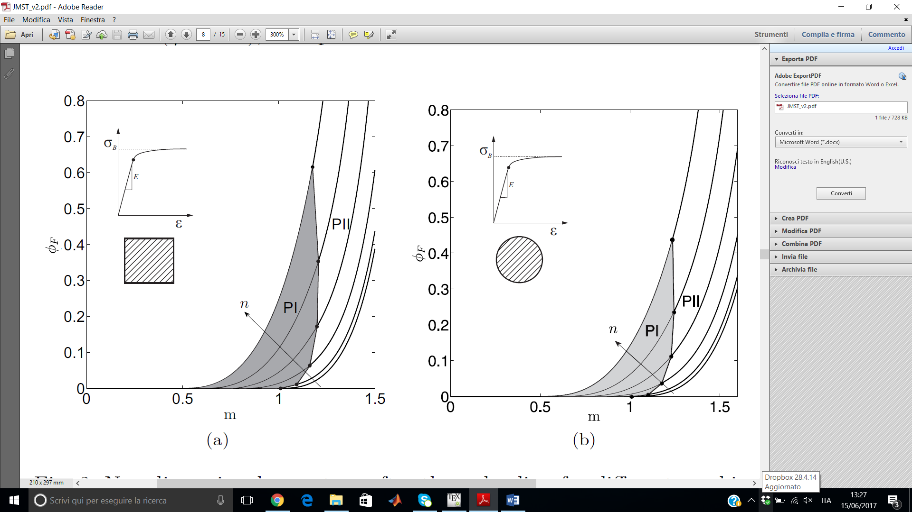


Fig. 2. Non-dimensional curvature after the unloading for different combinations of *m* and *n*. The non-dimensional axial force, *n*, is 0.0, 0.1, 0.2, 0.3, 0.4 and 0.5, respectively from the right to the left lines. The primary plastic region and the secondary plastic region are demarcated by the dark solid lines. (a) Hyperbolic hardening with rectangular cross-section (*E* = 209 GPa, = 240 MPa, = 200 MPa shown by a dot) and (b) Hyperbolic hardening material with circular cross-section.

## 4. In-plane elasto-plastic response of lattices

Having presented the elasto-plastic analysis of a single cell wall under combined axial and bending loading, when the cell wall material is described by hyperbolic hardening, we are now in a position to calculate the response of a honeycomb lattice. Unlike previous studies on honeycombs, where either linear elastic response is sought, or the collapse stress is calculated [4], here we obtain the detailed non-linear response when material hardening is considered.

Consider first a single cell wall with both axial and transverse load (see Fig. 1c). To simplify the analysis, the maximum bending moment at the root of the beam is expressed in the non-dimensional form *,* where *P* is the transverse load. Introduce the non-dimensional co-ordinate along the beam and the non-dimensional transverse deflection as and , where *w(x)* is the beam deflection and *l* the length of the beam. The non-dimensional curvature is now given by

. (10)

The complete cell wall deflection is obtained by integrating the curvature expressions derived in the previous sections, and applying the geometric boundary conditions at the fixed end, thus and . The final deflected shape of the cantilever beam after the unloading is found by subtracting the elastic deflection from the expression of , leading to The integration was performed by using the left-sided difference scheme for second order derivatives.

Since there are no known results for cell walls with rectangular cross-sections under hyperbolic hardening, we validate our analytical results with those presented by Yu and Johnson [23] who presented results for the elastic-perfectly plastic case. The validation is provided by gradually turning the corner of the hyperbola sharp, so as to mimic the non-smooth elasto-plastic material model of [23]. This ideal material cannot be exactly simulated with the hyperbolic hardening model since a singularity appears in the stress-strain curve; however, a really good approximation is obtained (see inset of Fig. 3). The parameters of the hyperbola obtained from the genetic algorithm which attempts to fit a hyperbola to an elatic-perfectly plastic material curve by minimizing the mean square error between the bilinear curve and the hyperbola. This process of mimicking the elaso-plastic stress-strain curve results in a hyperbola with a sharp elbow. The constants that appear in equation (1), after this curve fitting are obtained as , MPa and MPa. Note that the parameter *-B* represents the horizontal asymptote and for an elastic-perfectly-plastic ideal material it corresponds to the yield stress . The deformed shapes during the loading and unloading for two different combinations of and *n* have been plotted, as shown in Fig. 3. The thin lines are the results obtained by Yu and Johnson [23], while the thick solid and dash lines are those obtained using the analysis for a hyperbolic constitutive relationship with a sharp corner. The deformed shapes obtained by applying the current model closely resemble those of Yu and Johnson. The difference is due to the slightly imprecise approximation of the constitutive material law using the hyperbolic model. When no axial force is applied, the neutral axis is located at the middle of the section; therefore, primary and secondary plastic regimes start simultaneously. This results in an equal value of *x1* and *x2* (Fig. 1c). When and *n* = 0, which exactly matches Yu and Johnson result [23].

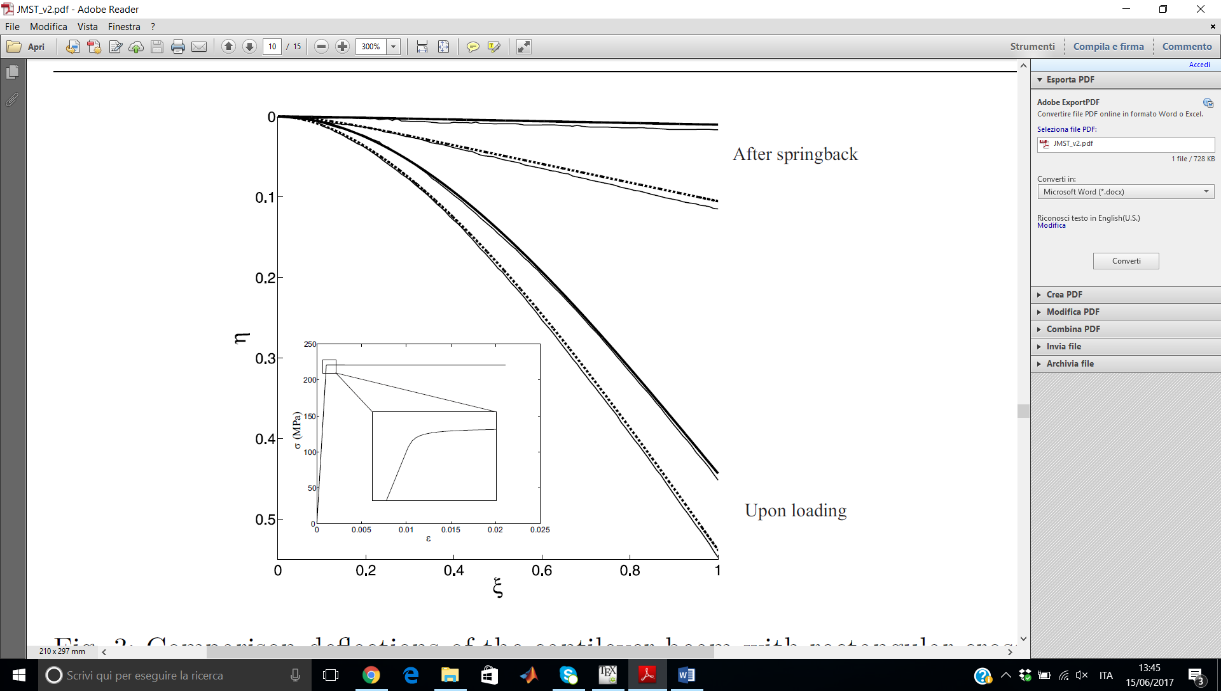


Fig. 3. Comparison deflections of the cantilever beam with rectangular cross-section between the current model (thick lines) and Yu and Johnson model (thin lines) [23]. The thick solid lines correspond to *m\**=1.3 and *n*=0.3, while the thick dash lines to *m\**=1.3 and *n*=0.0. The beam is made of an elastic-perfectly-plastic material, which is modelled using the hyperbola. Note the sharp corner that gives the visual appearance of the elastic-perfectly-plastic material, however this is deceptive (see inset).

In addition to provide a validation, this approach also affords exploration of a systematic asymptotic behaviour as the corner is increasingly sharpened towards the elastic-perfectly plastic piecewise linear curve. The influence of the material parameters on the tip deflection of the cell wall is now studied. By keeping the Young's modulus constant, the yield stress is gradually increased until the elastic-perfectly plastic material is reached. The influence of this change on the tip deflection of half of the cell wall is shown in Fig. 4(a). While increasing the yield stress towards the horizontal asymptote, the tip deflection reduces because of the higher energy stored by the material (proportional to the area under the material curve). The influence of the material change on the springback ratio (ratio between final curvature after springback and curvature upon loading) of the beam tip is plotted in Fig. 4(b). It can be seen that for materials with a lower yield stress the spring back reduces, in fact the amount of deformation that is recoverable decreases.

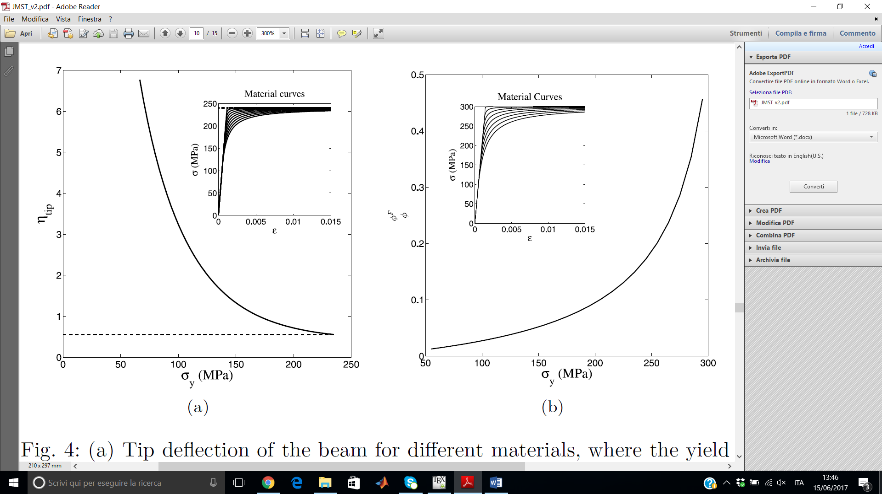


Fig. 4. (a) Tip deflection of the beam for different materials, where the yield stress is gradually increased (b) Tip spring back ratio of the beam for different materials, where the yield stress is gradually increased.

The effective elasto-plastic properties of the lattice structure can be inferred from the response of the tilted struts. A comprehensive study of this subject for two-dimensional lattice structures has been presented by the authors in [10]. However, the effects of material hardening were not taken into consideration. The mechanical response of a lattice sheet is related to the strut response through the geometric parameters. For a hexagonal cell of edge length *l*, height *h* and internal angle (see, Fig. 1a), strain along and across the direction of the load application are given by and , where is the deflection of the tip of inclined members with respect to their roots, measured transverse to them (see Fig. 7 in the Appendix for clarity, first sub-figure on the right). Having developed the response of a beam with rectangular cross-section, we are now in a position to quantitatively present the apparent non-linear response of such structured material. The relation between remote stress and the load applied at the end of the strut is given by and by projecting it along the *x* and *y* direction we obtain the transverse *P* and axial component *N* (see Fig. 1c), which are respectively given by

, (11)

where is the stress applied at infinity as shown in Fig. 1(a). The analytical model presented above is now used for the calculation of the plastic response of repetitive structures without ignoring the effects of the axial force and the material hardening. The overall lattice response is plotted in Fig. 5. When remote stress is applied, the initial phase is purely elastic. Gibson et. al. [4] studied the apparent modulus of elasticity when the cross-section of the struts is rectangular. The values thus obtained for the apparent modulus match up to 4 significant figures the well known benchmark results of Gibson and Ashby [4] that are valid for the initial linear part of the response only. The qualitative behaviour of the apparent response of the infinite lattice is the same as that obtained by assuming an elastic-perfectly-plastic material. This proves that such assumption simplifies the analysis without compromising the final result.

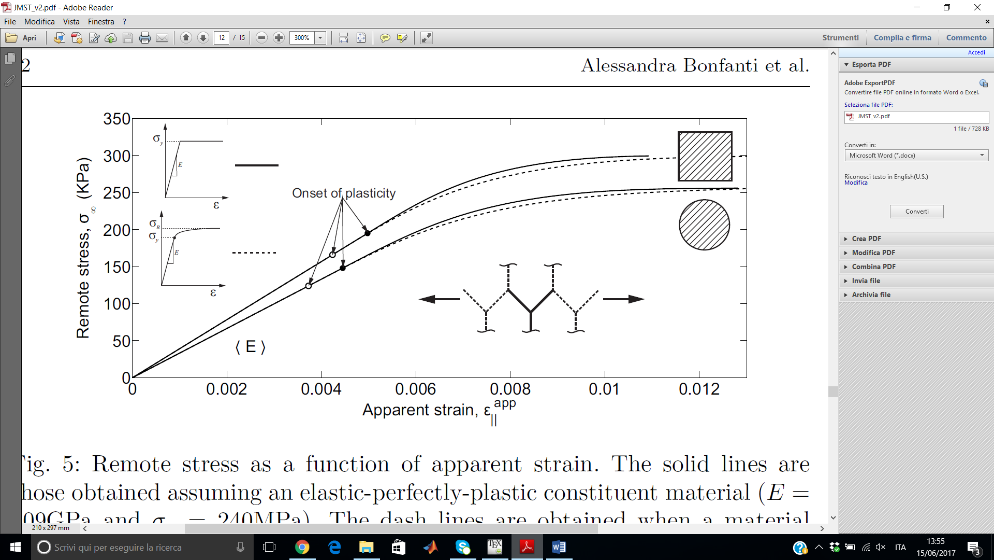


Fig. 5. Remote stress as a function of apparent strain. The solid lines are those obtained assuming an elastic-perfectly-plastic constituent material (*E*=209 GPa and = 240 MPa). The dash lines are obtained when a material with hyperbolic hardening is considered (*E*=209 GPa, = 200 MPa and = 240 MPa). A regular hexagonal cell is assumed, therefore *l*=*h* and . The two cross-sections have the same total area.

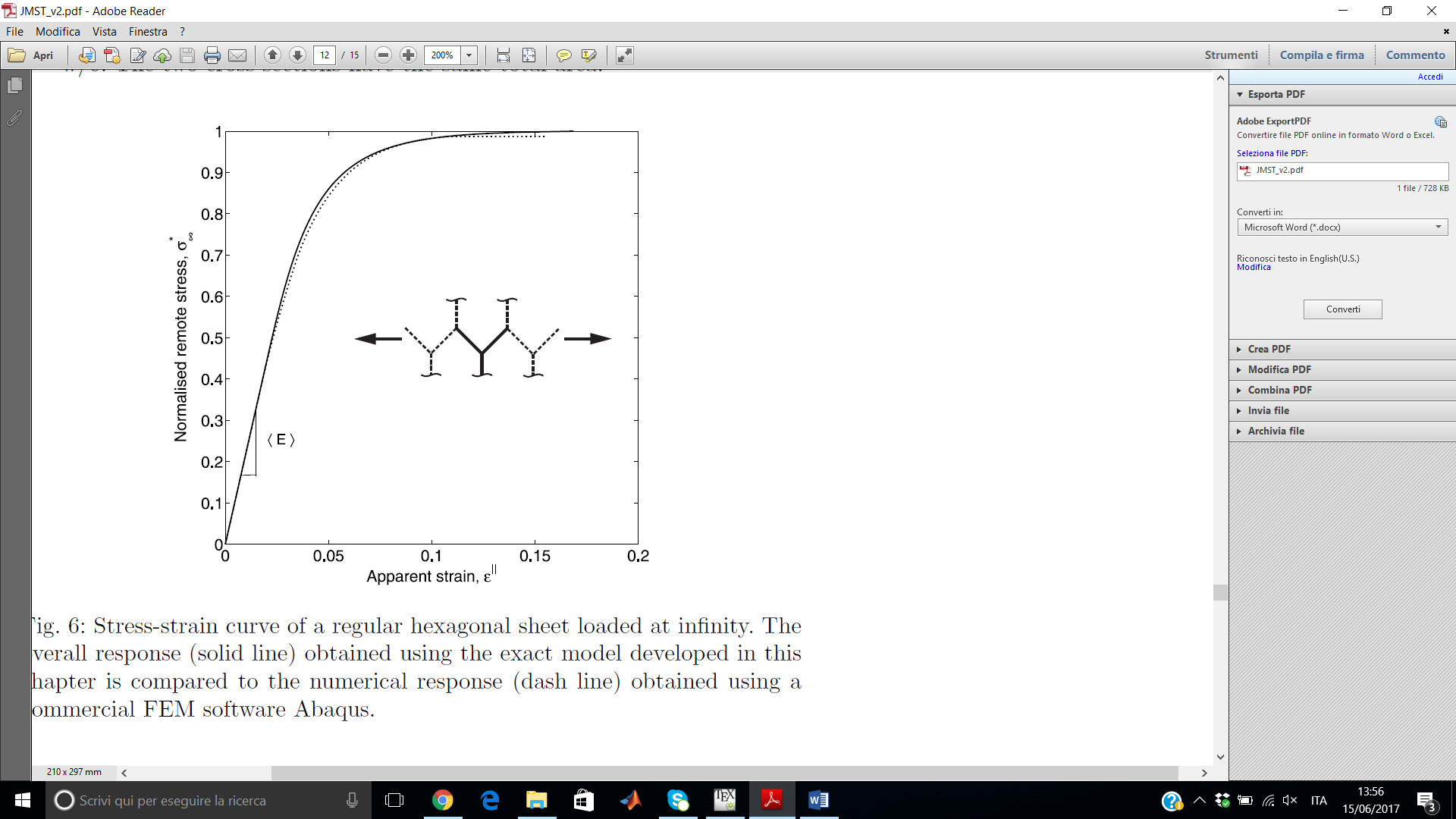


Fig. 6. Stress-strain curve of a regular hexagonal sheet loaded at infinity. The overall response (solid line) obtained using the exact model developed in this chapter is compared to the numerical response (dash line) obtained using a commercial FEM software Abaqus.

The overall lattice response plotted in Fig. 6 refers to an infinite regular honeycombs sheet made of an elasto-plastic material with hyperbolic hardening loaded at infinity. Thus mechanical behaviour is calculated by applying the analytical model developed above for a cantilever beam with inclined tip load. The normalised remote stress , where is the remote stress and is the value of the remote stress which corresponds to the cell wall cross-section at the joints becoming fully plastic, is plotted against the strain in the direction parallel to loading. The analytical response is compared with that obtained from the FEM analysis resulting in an excellent agreement, as shown in Fig. 6. The unit cell (see inset of Fig. 6 has been modelled as a 3D structure using an 8-node brick element (C3D8I) within the commercial finite element code ABAQUS.

We compare the response of a honeycomb whose members have rectangular cross-section with one whose members have circular cross section. By using beam theory to model the cell walls, we are able to calculate the apparent modulus of elasticity when the struts have a circular cross-section. The expression is given by (more details in Appendix A). The values of the apparent elastic modulus obtained using the above closed form expression for low values of remote stress match up to 4 significant figures with the values obtained from the linear part of the two curves in Fig. 5. Upon further loading, deformations become plastic which is associated with the non-linear part of the response in Fig. 5 (the rectangular cross-section and the circular cross-section have the same total area). Note that the tangent modulus at the onset of plasticity shows continuity for both the constituent materials assumed. This is because the plastic zones smoothly spread along the struts and through the thickness.

5. Conclusions

Analytical calculation of the elasto-plastic response of an infinite honeycomb sheet and its elastic spring back have been developed here. The present analysis provides a closed form solution of the problem that includes the hardening effects due to plasticity as described by a hyperbolic dependence of the stress-strain curve past yielding, together with the effects of stretch due to the axial force. Results for the apparent stress in the cellular solid as a non-linear function of the apparent strain are presented for the first time. The analysis presented here relates the distribution of the stress in each cross-section to the transverse displacement by imposing the force and moment equilibrium. In this calculation, the translation of the neutral axis due to the application of the axial force is considered, thus the stiffening effects due to an axial force are included in the results. We can see that the qualitative behaviour of the apparent response of a lattice structure obtained while assuming an elastic-perfectly-plastic constitutive material or a hyperbolic hardening model are similar. An interesting upshot of our analysis here is that despite choosing a fairly sophisticated representation of the non-linear stress strain curve, as afforded by the hyperbolic hardening model, the lattice response closely mimics that obtained using the simple elastic-perfectly-plastic material model. While it was possible to analytically obtain response for the material model in equation (1) here, it may not be possible to do the same with other non-linear material models, where the same simplicity may be offered by the bilinear material model. To conclude, this work presented the first analytical solution for a lattice structure subjected to elasto-plastic deformation due to bending and axial force which also includes the hardening effects. Also, results for spring back of honeycombs with non-linear hyperbolic hardening models are presented here for the first time.

Appendix A: Linear mechanics of two-dimensional honeycomb with circular cross-section

Linear elastic analysis of honeycombs for cell walls of rectangular cross-section is well known [6]. Here we adapt the analysis for honeycombs with cell walls having circular cross-sections. Assume an infinite honeycomb under tensile loading. The linear elastic response of honeycomb is primarily caused by bending of the inclined members, as shown in Fig. 7. Each wall is modelled as a beam of radius *r* and Young's modulus *E*. The Eulero-Bernulli beam formulation is used throughout the work; therefore, the shear deformation and axial extension or compression are neglected.

The horizontal force acting at the end of the cell wall is given by

, (12)

where is the remote stress applied to the honeycomb, *h* the high of the straight members, *l* the length of the inclined struts, the angle between the straight and tilted members and *r* the radius of the cross-section of the struts. Note that because of the symmetry is equal to zero.

From standard beam theory, the wall deflection is given by

, (13)

where is the second moment of inertia for a circular cross-section.

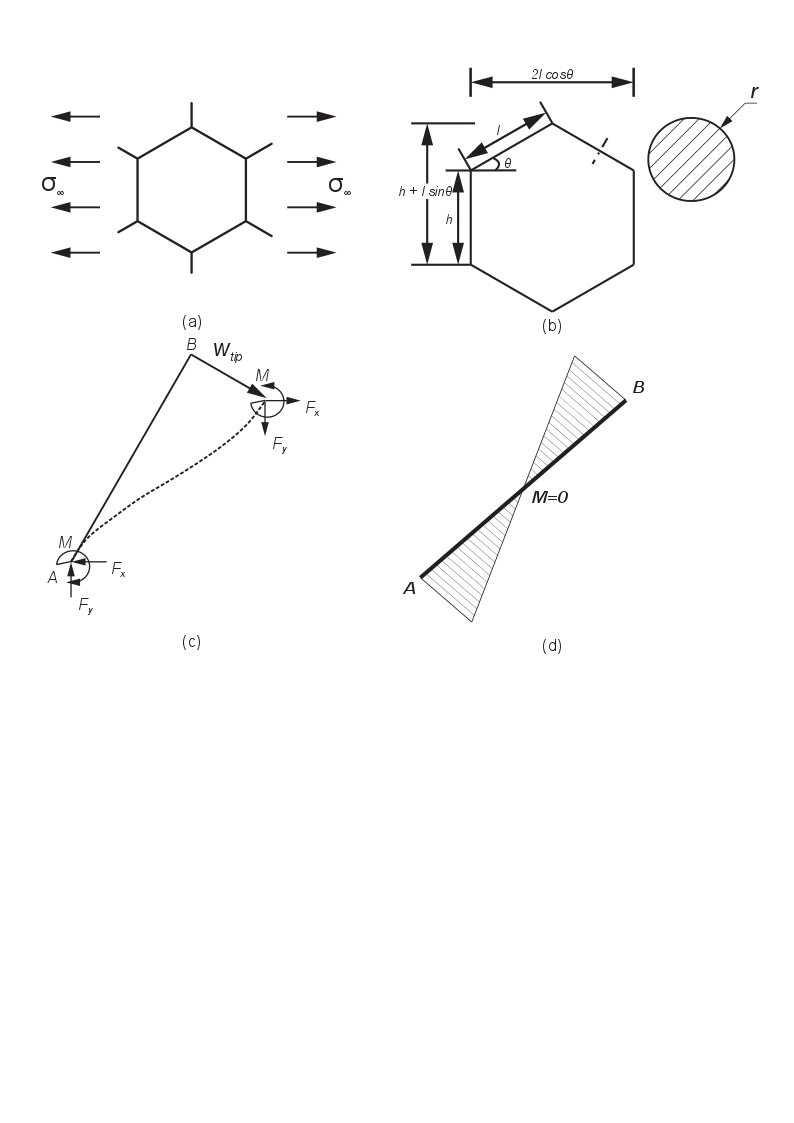


Fig. 7. Cell wall deformation under remote tensile loading along the horizontal *x-*direction.

The projection along the *x*-direction gives a strain equal to

*.* (14)

The apparent Young's modulus parallel to *x-*direction is giving

. (15)

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