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A_∞-resolutions and Massey products on Koszul homology

by

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ABSTRACT

FACULTY OF SOCIAL, HUMAN, AND MATHEMATICAL SCIENCES
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This work presents a new approach to studying Massey products on Koszul homology and the Golod property using A_∞-algebras. In the first part we study rooted monomial rings which includes monomial rings whose Lyubeznik resolution is minimal. We give a combinatorial characterization of the Golod property for this class of monomial rings.

In the second part of this thesis we combine our approach with the power of algebraic Morse theory. In this way, we extend our approach to simplicially resolvable rings, that is, rings with minimal simplicial resolution. We show that for simplicially resolvable rings the Golod property is equivalent to the gcd condition. Lastly, we use our tools to give sufficient conditions for the existence of non-trivial Massey products in low degrees.

Contents

Academic Thesis: Declaration of Authorship

I, Robin Frankhuizen, declare that this thesis and the work presented in it are my own and has been generated by me as the result of my own original research.

Title of thesis: A_∞-resolutions and Massey products on Koszul homology.

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To Hennie and Frank

Chapter 1

Introduction

The main objects of study of this work are Massey products on Koszul homology. Massey products were first studied by Massey [34] in algebraic topology but have become ubiquitous in large parts of topology and algebra. Massey products on Koszul homology have become a topic of especial interest because of their connection with various central problems in algebra and topology. To motivate this work, we will discuss two such connections, one *algebraic* and one *topological*.

We first turn to algebra. Let $S = k[x_1, \ldots, x_m]$ be the polynomial algebra over a field k in m variables and let $I = (m_1, \ldots, m_r)$ be an ideal generated by monomials. In that case, S/I is called a *monomial* ring. Given a monomial ring $R = S/I$, the *Poincaré series* of R is defined as

$$P(R) = \sum_{j=0}^{\infty} \dim \operatorname{Tor}_j^R(k,k)t^j.$$

A general open-ended problem in commutative algebra is to describe or compute the Poicaré series for various classes of monomial rings R. Important progress on this question was made by Golod in the 70s when he proved the following theorem.

Theorem. Let R be a monomial ring. There is a coefficientwise inequality of power series

$$P(R) \leq \frac{(1+t)^m}{1 - t(\sum_{j=0}^{\infty} \dim \operatorname{Tor}_j^S(R,k)t^j - 1)}.$$

Moreover, equality holds if and only if all Massey products on the Koszul homology $\operatorname{Tor}^S(R,k)$ vanish.

In honor of this theorem a monomial ring R is called Golod if any of the following equivalent conditions hold.

1. There is an equality of power series

$$P(R) = \frac{(1+t)^m}{1 - t(\sum_{j=0}^{\infty} \dim \mathrm{Tor}_j^S(R,k)t^j - 1)}.$$

2. All Massey products on $\mathrm{Tor}^S(R,k)$ vanish.

However, in general it is hard to verify triviality of Massey products directly. Thus, we would like to find necessary and/or sufficient conditions for the Golod property that are easier to directly verify.

Next, we describe how Massey products on Koszul homology turn up in toric topology. Toric topology studies spaces that admit a 'nice' action of the n-torus. An important family of such spaces are the *moment-angle complexes.* To fix some notation, let D^2 denote the 2-disc and let S^1 be its bounding circle. Given a simplicial complex Δ on vertex set $[m] = \{1, \ldots, m\}$ and a simplex $\sigma \in \Delta$, define

$$(D^2, S^1)_\sigma = \prod_{i=1}^{m} Y_i$$

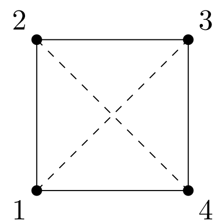
where

$$Y_i = \begin{cases} D^2 \text{ if } i \in \sigma \\ S^1 \text{ if } i \notin \sigma. \end{cases}$$

The *moment-angle complex* Z_Δ is defined as

$$Z_\Delta = \mathrm{colim}_{\sigma \in \Delta}(D^2, S^1)_\sigma$$

where the colimit is taken over the face category of Δ. For example, if Δ is the boundary of the square



then $Z_\Delta \cong S^3 \times S^3$. Moment-angle complexes are related to monomial rings via the following construction. Given a simplicial complex Δ on $[m]$, define the *Stanley-Reisner ring* (or *face ring*) $k[\Delta]$ of Δ by

$$k[\Delta] = k[x_1, \ldots, x_m]/(x_{i_1} \cdots x_{i_k} \mid \{i_1, \ldots, i_k\} \notin \Delta).$$

Continuing the example of the square above, we get

$$k[\Delta] = k[x_1, x_2, x_3 x_4]/(x_1 x_3, x_2 x_4).$$

In particular, Stanley-Reisner rings are monomial rings. The relation between Z_Δ and $k[\Delta]$ is described by the following theorem.

Theorem ([8], Theorem 4.5.4)**.** Let Δ be a simplicial complex on $[m]$. There is an isomorphism of graded algebras

$$H^*(Z_\Delta, k) \cong \mathrm{Tor}^{k[x_1,\dots,x_m]}(k[\Delta], k).$$

Therefore, studying Massey products on the singular cohomology of moment-angle complexes is the same as studying Massey products on the Koszul homology of the corresponding Stanley-Reisner rings. The main strategy of this work is to study Massey products via A_∞-algebras. The notion was originally introduced by Stasheff [46] in the 1960s in the context of algebraic topology. In the last few decades A_∞-algebras have found applications in various branches of mathematics such as geometry [16], algebra [47] and mathematical physics [29], [36].

An A_∞-algebra is a graded module A together with linear maps

$$\mu_n \colon A^{\otimes n} \to A$$

satisfying certain higher associativity relations called the Stasheff identities (2.3.2). A precise definition can be found in Definition 2.3.1. A rich source of A_∞-algebras is the Homotopy Transfer Theorem (Theorem 2.3.3) coupled with the Merkulov construction (Theorem 2.3.6). Suppose that A is a dg algebra and let B be a chain complex such that there exist maps

$$\begin{tikzcd} B \arrow[r, bend left, "i"] & A \arrow[l, bend left, "p"] \arrow[loop right, "\phi"] \end{tikzcd}$$

be chain maps such that $pi = 1_B$ and $ip - 1 = d\phi + \phi d$. Define maps

$$\lambda_n \colon A^{\otimes n} \to A$$

by letting λ_2 be the product of A and setting

$$\lambda_n = \sum_{s+t=n} (-1)^{s+1} \lambda_2(\phi\lambda_s \otimes \phi\lambda_t).$$

Next, define

$$\mu_n \colon B^{\otimes n} \to B$$

by

$$\mu_n = p \circ \lambda_n \circ i^{\otimes n}$$

then Theorem 2.3.6 tells us that the pair $(B, \{\mu_n\})$ is an A_∞-algebra. To be able to apply A_∞-algebras to the Koszul homology $\mathrm{Tor}^S(R, k)$ of a monomial ring $R = S/I$, we first need to find a suitable dg algebra and a suitable chain complex to apply the above procedure to. For this purpose, the Taylor resolution T is discussed in Construction 3.2.7. Let E denote the exterior algebra on generators $u_1, \dots, u_r$ corresponding to the generating monomials $m_1, \dots, m_r$ of I. Then the resolution T has underlying module $S \otimes_k E$. In general, T is not the smallest possible resolution. However, it can be shown that every monomial ring admits a unique resolution F that is as small as possible in the sense that

$$\mathrm{Tor}^S(R, k) \cong F \otimes k.$$

In particular, we can apply the above procedure. We prove the following theorem.

Theorem. Let R be a monomial ring and let F be the minimal free resolution of R. Fix some $n \in \mathbb{N}$. Then the following are equivalent.

1. For every $r \leq n$, all r-Massey products are trivial.
2. For every $r \leq n$, μ_r is minimal, that is $\mathrm{im}(\mu_r) \subseteq (x_1, \dots, x_m)F$.

In particular, R is Golod if and only if all μ_r are minimal.

The strategy for the rest of this work then is to study the maps μ_r instead of studying Massey products directly. To this end, we study minimal free resolutions in more depth. The following special type of resolution is due to Novik [41]. Given an monomial ideal $I = (m_1, \dots, m_r)$, we define $L(I)$ to be the set of all $\mathrm{lcm}(m_{i_1}, \dots, m_{i_k})$ where $1 \leq i_1 \leq \dots \leq i_k \leq r$ and $k = 1, \dots, r$.

Definition. A *rooting map* on L is a map

$$\pi \colon L \setminus \{\hat{0}\} \to \{m_1, \dots, m_r\}$$

such that

1. for every $m \in L$, $\pi(m)$ divides m
2. $\pi(m) = \pi(n)$ whenever $\pi(m)$ divides n and n divides m.

Now, let π be a rooting map and let $A \subseteq \{m_1, \dots, m_r\}$ be non-empty. Define $\pi(A) = \pi(\mathrm{lcm}(A))$. A set A is *unbroken* if $\pi(A) \in A$ and A is *rooted* if every non-empty $B \subseteq A$ is unbroken. Let $RC(L, \pi)$ denote the set of all rooted sets with respect to L and π. Let $F = F_{RC(L,\pi)}$ be the subcomplex of the Taylor resolution spanned by elements of

$RC(L, \pi)$. Then F is a free resolution of R. An important special case of this construction is the Lyubeznik resolution.

Definition. Let $I = (m_1, \ldots, m_r)$ be a monomial ideal and pick some total order $\prec$ on the m_i. After relabelling we may assume that $m_1 \prec m_2 \prec \cdots \prec m_r$. Define

$$\pi(S) = \min_{\prec}\{m_i \mid m_i \text{ divides } \operatorname{lcm}(S)\}.$$

Then π is easily seen to be a rooting map. The resolution associated $RC(L, \pi)$ is called the *Lyubeznik resolution*.

If the minimal free resolution of S/I is a resolution associated to $RC(L, \pi)$ for some rooting map π, then S/I is called a *rooted ring*. Similarly, if the Lyubeznik resolution of S/I is minimal then S/I is called a *Lyubeznik ring*. The main theorem of Chapter 4 is then the following.

Theorem. Let R be a rooted ring with rooting map π. Then the following are equivalent.

1. The ring R is Golod.
2. The product on $\operatorname{Tor}^S(R, k)$ vanishes.
3. The ring R is π-gcd, that is $\pi(m_i, m_j) \neq m_i, m_j$ whenever $\gcd(m_i, m_j) = 1$

As a corollary, we obtain the following special case.

Corollary. Let R be a Lyubeznik ring. Then the following are equivalent.

1. The ring R is Golod.
2. The product on $\operatorname{Tor}^S(R, k)$ vanishes.
3. The ring R is gcd, that is for all generators m_i and m_j with $\gcd(m_i, m_j) = 1$ there exists a $m_k \prec m_i, m_j$ such that m_k divides $\operatorname{lcm}(m_i, m_j)$.

To study the Golod property more generally, we turn to algebraic Morse theory. Algebraic Morse theory was first developed independently by Sköldberg [45] and Jöllenbeck and Welker [23] based on earlier work by Forman [12] as a way to minimize chain complexes. A *based complex* K is a chain complex K together with a direct sum decomposition

$$K_n = \bigoplus_{\alpha \in I_n} K_\alpha$$

where the I_n are pairwise disjoint. We will write $\alpha^{(n)}$ to indicate that $\alpha \in I_n$. Let $f \colon K \to K$ be a graded map. We write $f_{\beta,\alpha}$ for the component of f going from K_α to K_β, that is $f_{\beta,\alpha}$ is the composition

$$K_\alpha \longrightarrow K_m \xrightarrow{f} K_n \longrightarrow K_\beta$$

where $K_\alpha \to K_m$ is the inclusion and $K_n \to K_\beta$ the projection. Given a based complex K, define a directed weighted graph G_K with vertex set $\cup_n I_n$ and a directed weighted edge $\alpha \to \beta$ if $d_{\beta,\alpha} \neq 0$. In that case, the weight is $d_{\beta,\alpha}$. A subset of the edges is called a *Morse matching* if each vertex is in at most one of the edges, the weights $d_{\beta,\alpha}$ are invertible and a certain acyclicity condition is satisfied.

Given a dg algebra A, we show that every Morse matching $\mathcal{M}$ on the underlying complex of A induces an A_∞-algebra structure on the corresponding Morse complex $A^{\mathcal{M}}$. Moreover, we show that any two Morse matchings on A give rise to equivalent A_∞-algebras.

In Chapter 6 we apply the combination of algebraic Morse theory and A_∞-algebras to Massey products on Koszul homology. We show that the minimal free resolution F of any monomial ring R can always be obtained as the Morse complex of some Morse matching $\mathcal{M}$ on the Taylor resolution T. We also study how A_∞-structures behave under standard techniques from commutative algebra such as polarization.

Using these new tools we prove various new criteria for vanishing of Massey products and the Golod property. The main result is the following.

Theorem. Let $R = S/I$ be simplicially resolvable. Then the following are equivalent.

1. R is Golod
2. The product on $\operatorname{Tor}^S(R,k)$ is trivial.
3. I satisfies the gcd condition. That is, for any two generators m_1 and m_2 of I with $\gcd(m_1,m_2) = 1$ there exists a generator $m \neq m_1, m_2$ such that m divides $\operatorname{lcm}(m_1,m_2)$.
4. For $u, v \in \mathcal{M}_0$ we have $\operatorname{lcm}(u)\operatorname{lcm}(v) \neq \operatorname{lcm}(uv)$ whenever $uv \in \mathcal{M}_0$.

The conclusion of the theorem does not hold without the assumption that F is simplicial even if we assume that F is cellular. An explicit example is discussed in [25].

Next, we turn our attention to standard matchings. Standard matchings were first introduced in [22] as a special kind of Morse matching that is compatible with the product on the Taylor resolution T. In [22] it is claimed that standard matchings always exist but a counterexample was found in [25]. The following theorem shows that the existence of standard matchings is in fact a very strong assumption.

Theorem. Let R be a simplicially resolvable ring and let T denote the Taylor resolution. Suppose that T admits a standard matching. Then all higher Massey products on $\operatorname{Tor}^S(R,k)$ vanish.

Though the main part of this work consists of studying the vanishing of Massey products, our tools are also suitable for finding and computing non-trivial Massey products. As an example of this, we give some sufficient conditions for non-triviality of Massey products in low degrees.

Lastly, we present some preliminary work on representability of Massey products. More precisely, we show that if $\langle a_1, \ldots, a_n \rangle$ is a non-trivial Massey product on $\mathrm{Tor}^S(R,k)$ and $a \in \langle a_1, \ldots, a_n \rangle$, then there exists on A_∞-structure $\{\mu_n\}$ on $\mathrm{Tor}^S(R,k)$ such that

$$\mu_n(a_1 \otimes \cdots \otimes a_n) = \pm a.$$

Chapter 2

A_∞-algebras

2.1 Differential graded algebras

In this section we develop the necessary background on differential graded (co)algebras. No claims of originality are made for this section. Indeed, most of what follows is covered in standard textbooks on the topic like [20], [15] and [50].

Let R be a commutative ring with unit. Unless otherwise stated, all modules are taken to be R-modules and all unadorned tensor products are taken over R.

Definition 2.1.1. A *graded module* M is a direct sum

$$M = \bigoplus_{p \in \mathbb{Z}} M_p$$

of modules M_p. An element $x \in M_p$ is said to be *homogeneous of degree* p. We write $\deg(x)$ for the degree of a homogeneous element x.

Let M be a module. For $n \in \mathbb{Z}$, define a graded module $M[n]$ by setting

$$M[n]_p = \begin{cases} M & \text{if } p = n \\ 0 & \text{if } p \neq n \end{cases}.$$

We will identify the module M with the graded module $M[0]$. In this way, every module can be considered as a graded module.

Example 2.1.2. Let $k[x]$ denote the polynomial algebra over a field k. Then $k[x]$ is a graded module with components

$$k[x]_p = kx^p.$$

This particular grading is called the *standard* grading.

Definition 2.1.3. Let M and N be graded modules. A *graded map of degree* k from M to N

$$f\colon M \to N$$

is a family

$$f_p\colon M_p \to N_{p+k}$$

of linear maps. We write $\deg(f)$ for the degree of a graded map.

Example 2.1.4. Define a map

$$d\colon k[x] \to k[x]$$

by setting

$$d(x^p) = px^{p-1}.$$

Then d is a graded map of degree -1.

Graded maps of degree zero will usually be called *maps* or *morphisms*. The *identity map* $1_M\colon M \to M$ is the degree zero map consisting of the identity morphisms

$$1_p\colon M_p \to M_p$$

for all $p \in \mathbb{Z}$. Composition of graded maps is defined levelwise. That is to say, let $f\colon M \to N$ and $g\colon N \to P$ be graded maps of degree k and l,respectively Define

$$gf\colon M \to P$$

by setting

$$(gf)_p = g_{p+k}f_p\colon M_p \to P_{p+k+l}.$$

Note that $\deg(gf) = \deg(f) + \deg(g)$. In this way, graded modules and morphisms of graded modules form a category which will be denote by GrMod. In fact, it is easily seen that GrMod is an abelian category. Indeed, for a morphism $f\colon M \to N$ define the *kernel* $\ker(f)$ by

$$\ker(f)_p = \ker(f_p)$$

and the *cokernel* $\operatorname{coker}(f)$ by

$$\operatorname{coker}(f)_p = \operatorname{coker}(f_p).$$

Similarly, for graded modules M and N define the *product* $M \times N$ by

$$(M \times N)_p = M_p \times N_p$$

and the *coproduct* $M \oplus N$ by

$$(M \oplus N)_p = M_p \oplus N_p.$$

It is easily verified that GrMod together with the above data forms an abelian category.

The category GrMod also comes with a tensor product.

Definition 2.1.5. Let M and N be graded modules. The *tensor product* $M \bigotimes N$ is the graded module consisting of

$$(M \otimes N)_p = \bigoplus_{k+l=p} M_k \otimes N_l$$

for all $p \in \mathbb{Z}$.

Now, let $f\colon M \to N$ and $g\colon P \to Q$ be graded maps. Define a map

$$f \otimes g\colon M \otimes P \to N \otimes Q$$

by setting

$$(f \otimes g)(x \otimes y) = (-1)^{\deg(g)\deg(x)} f(x) \otimes g(y) \tag{2.1.1}$$

where $x \in M$ and $y \in P$.

Remark 2.1.6. The sign convention in equation (2.1.1) is an example of what is known as the *Koszul sign rule*. In fact, all signs in the graded world follow from this rule which states that if x and y swap then we multiply by $(-1)^{\deg(x)\deg(y)}$. Without this sign the tensor product of two chain complexes need not be a chain complex in general.

Definition 2.1.7. A *graded algebra* (A, μ) consists of a graded module A together with a graded morphism

$$\mu\colon A \otimes A \to A$$

called the multiplication that is associative. More precisely, the diagram

$$\begin{tikzcd}
A \otimes A \otimes A \arrow[r, "\mu \otimes 1"] \arrow[d, "1 \otimes \mu"'] & A \otimes A \arrow[d, "\mu"] \\
A \otimes A \arrow[r, "\mu"'] & A
\end{tikzcd}$$

is commutative.

For convenience, we usually write ab instead of $\mu(a \otimes b)$. Usually, we will assume that graded algebras are *unital*. That is to say, there exists an element $1 \in A$ such that

$$\mu(1 \otimes a) = a = \mu(a \otimes 1)$$

for all $a \in A$. Equivalently, there exists a map $\eta\colon R \to A$ such that the following two diagrams commute

$$
\begin{tikzcd}
R \otimes A \arrow[r, "\eta \otimes 1"] \arrow[dr, "\cong"'] & A \otimes A \arrow[d, "\mu"] \\
& A
\end{tikzcd}
\qquad
\begin{tikzcd}
A \otimes R \arrow[r, "1 \otimes \eta"] \arrow[dr, "\cong"'] & A \otimes A \arrow[d, "\mu"] \\
& A
\end{tikzcd}
$$

The equivalence of the two definitions follows from the fact that η is uniquely determined by its value on $1 \in R$.

Example 2.1.8. Let $k[x]$ be the polynomial algebra over some field k with the standard grading. Then $k[x]$ with the ordinary product of polynomials is a graded algebra.

We will also need to consider morphisms of graded algebras.

Definition 2.1.9. Let (A, μ_A) and (B, μ_B) be graded algebras. A *morphism of graded algebras*

$$f\colon A \to B$$

is a morphism $f\colon A \to B$ of graded modules such that the diagram

$$
\begin{tikzcd}
A \otimes A \arrow[r, "f \otimes f"] \arrow[d, "\mu_A"'] & B \otimes B \arrow[d, "\mu_B"] \\
A \arrow[r, "f"'] & B
\end{tikzcd}
$$

commutes. If A and B are unital then we additionally require that

$$
\begin{tikzcd}
& R \arrow[dl, "\eta_A"'] \arrow[dr, "\eta_B"] & \\
A \arrow[rr, "f"'] & & B
\end{tikzcd}
$$

commutes.

Definition 2.1.10. A *differential graded R-module* (or *dg R-module*) is a graded module M together with a degree ± 1 map

$$d\colon M \to M$$

such that $d^2 = 0$. The map d is called the *differential.* If $\deg(d) = -1$, M is called a *chain complex.* If $\deg(d) = 1$, M is called a cochain complex.

When the base ring R is clear from the context, we will simply speak about dg modules.

Definition 2.1.11. Let (M,d) and (N,d') be dg modules. A *map of degree* k

$$f\colon M \to N$$

consists of linear maps

$$f_p\colon M_p \to N_{p+k}$$

making the diagram

$$\begin{tikzcd}
\cdots \arrow[r, "d"] & M_{p+1} \arrow[r, "d"] \arrow[d, "f_{p+1}"'] & M_p \arrow[r, "d"] \arrow[d, "f_p"'] & M_{p-1} \arrow[r, "d"] \arrow[d, "f_{p-1}"] & \cdots \\
\cdots \arrow[r, "d'"'] & N_{p+k+1} \arrow[r, "d'"'] & N_{p+k} \arrow[r, "d'"'] & N_{p+k-1} \arrow[r, "d'"'] & \cdots
\end{tikzcd}$$

commute up to a sign $(-1)^k$. That is,

$$d'f = (-1)^k fd.$$

A *morphism of dg modules* or *chain map* is a map of degree zero.

Note that in the above definition we assume the differential has degree -1. The cochain version is obtained by inverting the horizontal arrows. We will denote the category of dg R-modules by DgMod_R.

Given a chain complex M, define modules Z_p and B_p as follows. Put

$$Z_p = \ker(d_p) \qquad \text{and} \qquad B_p = \operatorname{im}(d_{p+1}).$$

The elements of Z_p are called p-cycles and the elements of B_p are called p-boundaries. Since $d^2 = 0$, it follows that $B_p \subseteq Z_p$ for all $p \in \mathbb{Z}$. The quotient module

$$H_p = Z_p/B_p$$

is called the p-th *homology module.* Similarly, given a cochain complex M, define modules Z^p and B^p as follows. Put

$$Z^p = \ker(d_p) \qquad \text{and} \qquad B^p = \operatorname{im}(d_{p-1}).$$

The elements of Z^p are called p-cocycles and the elements of B^p are called p-coboundaries. Since $d^2 = 0$, it again follows that $B^p \subseteq Z^p$ for all $p \in \mathbb{Z}$. The quotient module

$$H^p = Z^p/B^p$$

is called the p-th *cohomology module*.

Now, let $f\colon M \to N$ be a dg map. Then by definition f preserves (co)cycles and (co)boundaries and thus induces a maps

$$H_p(f)\colon H_p(M) \to H_p(N)$$

in the chain complex case or maps

$$H^p(f)\colon H^p(M) \to H^p(N)$$

in the cochain complex case. It is easily seen that H_p and H^p are functors. For ease of notation, we will usually write f_* for $H_p(f)$ and $H^p(f)$.

Definition 2.1.12. Let

$$\begin{tikzcd} M \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] & N \end{tikzcd}$$

be dg morphisms. A *dg homotopy* from f to g is a map

$$\phi\colon M \to N$$

of degree $\deg(\phi) = -\deg(d)$ such that

$$f - g = d\phi + \phi d.$$

In that case, we will say that f and g are *homotopic* and we write $f \simeq g$.

Proposition 2.1.13. Let M and N be dg modules. Then the relation $\simeq$ is an equivalence relation on $\mathrm{DgMod}_R(M, N)$.

Proof. Let f be a dg morphism. Then f is homotopic to itself via the homotopy $\phi = 1$ and so $\simeq$ is reflexive. To see that $\simeq$ is symmetric, let f and g be dg morphisms and suppose that ϕ is a homotopy from f to g. It is clear that $-\phi$ is a homotopy from g to f. Lastly, let f, g and h be dg maps. Suppose that ϕ is a homotopy from f to g and ψ a homotopy from g to h. Then

$$\begin{aligned} d(\phi+\psi) + (\phi+\psi)d &= d\phi + \phi d + d\psi + \psi d \\ &= f - g + g - h \\ &= f - h \end{aligned}$$

and so f and h are homotopic. □

Proposition 2.1.14. Let

$$\begin{tikzcd} K \arrow[r, "k"] & M \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] & N \arrow[r, "h"] & P \end{tikzcd}$$

be dg morphisms and suppose that f and g are homotopic. Then fk is homotopic to gk and hf is homotopic to hg.

Proof. Let ϕ be a homotopy from f to g. Then, since k is a chain map, it follows that

$$\begin{aligned} d\phi k + \phi k d &= d\phi k + \phi d k \\ &= (d\phi + \phi d)k \\ &= (f-g)k \\ &= fk - gk. \end{aligned}$$

Therefore, fk is homotopic to gk. Similarly, since h is a chain map, we have

$$\begin{aligned} dh\phi + h\phi d &= dh\phi + h(f - g - d\phi) \\ &= dh\phi + hf - hg - hd\phi \\ &= hf - hg \end{aligned}$$

and so hf is homotopic to hg. □

Proposition 2.1.15. Let $f, g\colon M \to N$ be dg maps. Suppose that $f \simeq g$. Then $f_* = g_*$.

Proof. Let ϕ be a homotopy from f to g and let $[x] \in HM$. Compute

$$\begin{aligned} (f_* - g_*)[x] &= (f-g)_*[x] \\ &= [(f-g)x] \\ &= [(d\phi - \phi d)x] \\ &= [d\phi x] \\ &= 0. \end{aligned}$$

Consequently, $f_* = g_*$. □

Definition 2.1.16. Let

$$f\colon M \to N$$

be a dg map. Then f is called a *dg homotopy equivalence* if there exists a dg map

$$g\colon N \to N$$

and dg homotopies

$$\phi\colon gf \simeq 1_M \qquad \text{and} \qquad \psi\colon fg \simeq 1_N.$$

Proposition 2.1.17. Let $f\colon M \to N$ be a dg homotopy equivalence. Then

$$f_*\colon HM \to HN$$

is an isomorphism.

Proof. Since f is a homotopy equivalence, there exists a morphism $g\colon N \to M$ such that $fg \simeq 1_N$ and $gf \simeq 1_M$. By Proposition 2.1.15, we have

$$g_*f_* = 1_{HM} \qquad \text{and} \qquad f_*g_* = 1_{HN}.$$

Therefore, f_* is an isomorphism. □

Definition 2.1.18. Let (M, d) and (N, d') be dg modules. The *tensor product*

$$M \otimes N$$

of M and N is the dg module $(M \otimes N, d'')$ where $M \otimes N$ is the tensor product of the underlying graded modules and d'' is given by

$$d''(a \otimes b) = da \otimes b + (-1)^p a \otimes d'b$$

where $a \in M_p$ and $b \in N_q$.

Let $f\colon A \to B$ and $g\colon C \to D$ be dg morphisms. As for graded morphisms, define a map

$$f \otimes g\colon A \otimes C \to B \otimes D$$

by setting

$$(f \otimes g)(a \otimes c) = (-1)^{\deg(g)\deg(a)} f(a) \otimes g(c)$$

where $a \in A$ and $c \in C$.

We still need to check that $f \otimes g$ is a dg morphism. So let $a \in A_p$ and $c \in C_q$, then

$$\begin{aligned}
d(f \otimes g)(a \otimes c) &= d(fa \otimes gc) \\
&= dfa \otimes gc + (-1)^p fa \otimes dgc \\
&= fda \otimes gc + (-1)^p fa \otimes gdc \\
&= (f \otimes g)(da \otimes c) + (f \otimes g)((-1)^p a \otimes dc) \\
&= (f \otimes g)d(a \otimes c)
\end{aligned}$$

as required.

Definition 2.1.19. A *differential graded algebra* or *dg algebra* is a triple (A, d, μ) where

1. (A, d) is a dg module,

2. (A, μ) is a graded algebra and

3. $\mu\colon A \otimes A \to A$ is a morphism of dg modules.

Any (graded) algebra can be made into a dg algebra by setting $d = 0$.

Definition 2.1.20. A *morphism of dg algebras*

$$f\colon (A, d, \mu) \to (A', d', \mu')$$

is a morphism of dg modules

$$f\colon (A, d) \to (A', d')$$

such that

$$f\colon (A, \mu) \to (A', \mu')$$

is a morphism of graded algebras.

The category of dg algebras and dg algebra morphisms over a fixed ring R will be denote by DgAlg_R.

Definition 2.1.21. Let A be a dg algebra. A *derivation of degree* k is a degree k map

$$f\colon A \to A$$

such that

$$f(ab) = f(a)b + (-1)^{kp} a f(b)$$

for all $a \in A_p$ and $b \in A_q$.

The category DgAlg_R comes with a tensor product.

Definition 2.1.22. Let A and B be dg algebras. The *tensor product* $A \otimes B$ is defined to be the tensor product $A \otimes_R B$ of the underlying dg modules together with multiplication defined by

$$(a \otimes b)(a' \otimes b'') = (-1)^{\deg(a')\deg(b)} aa' \otimes bb'$$

In particular, the differential is given by

$$d(a \otimes b) = da \otimes b + (-1)^{\deg(a)} a \otimes db.$$

In later chapters we will also need the dual notions of what has been discussed so far.

Definition 2.1.23. A *graded coalgebra* (C, Δ) consists of a graded module C together with a morphism of graded modules

$$\Delta\colon C \to C \otimes C$$

called the *comultiplication* or *coproduct* such that the diagram

$$\begin{tikzcd}
C \arrow[r, "\Delta"] \arrow[d, "\Delta"'] & C \otimes C \arrow[d, "1 \otimes \Delta"] \\
C \otimes C \arrow[r, "\Delta \otimes 1"'] & C \otimes C \otimes C
\end{tikzcd}$$

commutes. We also say that Δ is coassociative.

Definition 2.1.24. A *morphism of graded coalgebras*

$$f\colon (C, \Delta) \to (C', \Delta')$$

is a morphism of graded modules

$$f\colon C \to C'$$

such that the diagram

$$\begin{tikzcd}
C \arrow[r, "f"] \arrow[d, "\Delta"'] & C' \arrow[d, "\Delta'"] \\
C \otimes C \arrow[r, "f \otimes f"'] & C' \otimes C'
\end{tikzcd}$$

commutes.

Example 2.1.25. Let $S = k[x]$ be the polynomial algebra. Then S can be considered as a graded module

$$S = \bigoplus_{n=0}^{\infty} kx^n.$$

Define a comultiplication

$$\Delta\colon S \to S \otimes S$$

by setting

$$\Delta x^n = \sum_{k+l=n} x^k \otimes x^l.$$

Then (S, Δ) is easily seen to be a graded coalgebra.

We will also need the dual notion of dg algebras.

Definition 2.1.26. A *differential graded coalgebra* or *dg coalgebra* is a triple (C, d, Δ) where

1. (C, d) is a dg module,

2. (C, Δ) is a graded coalgebra and

3. $\Delta \colon C \to C \otimes C$ is a morphism of dg modules.

Definition 2.1.27. A *morphism of dg coalgebras*

$$f \colon (C, d, \Delta) \to (C', d', \Delta')$$

is a morphism of dg modules

$$f \colon (C, d) \to (C', d')$$

such that

$$f \colon (C, \Delta) \to (C', \Delta')$$

is a morphism of graded coalgebras.

The category of dg algebras and dg algebra morphisms over a fixed ring R will be denote by $\mathrm{DgCoAlg}_R$.

Definition 2.1.28. Let C be a dg coalgebra. A *coderivation of degree* k is a degree k map

$$f \colon C \to C$$

such that the diagram

$$\begin{tikzcd} C \arrow[r, "\Delta"] \arrow[d, "f"'] & C \otimes C \arrow[d, "f \otimes 1 + 1 \otimes f"] \\ C \arrow[r, "\Delta"'] & C \otimes C \end{tikzcd}$$

commutes.

2.2 Massey products and formality

One of the central objects studied in this thesis are Massey products [34] which will be discussed in this section. If A is a dg algebra, then the homology HA is a graded algebra which we will think of HA as a dg algebra with trivial differential. Let (A, d) be a differential graded algebra. If $a \in A$, we write $\bar{a}$ for $(-1)^{\deg(a)+1}a$. Let $\alpha_1, \alpha_2 \in HA$. The length 2 *Massey product* $\langle \alpha_1, \alpha_2 \rangle$ is defined to be the product $\alpha_1 \alpha_2$ in the homology algebra HA. Since the definition of higher Massey products is slightly involved, we treat the triple Massey product separately so the pattern becomes clearer.

Definition 2.2.1. Let $\alpha_1, \alpha_2, \alpha_3 \in HA$ be homology classes with the property that

$$\alpha_1\alpha_2 = 0 = \alpha_2\alpha_3.$$

A *defining system* $\{a_{ij}\}$ consists of

1. Representing homogeneous cycles a_{01}, a_{12}, a_{23} of $\alpha_1, \alpha_2, \alpha_3$ respectively.
2. Elements a_{02} and a_{13} such that $d(a_{02}) = \bar{a}_{01}a_{12}$ and $d(a_{13}) = \bar{a}_{12}a_{23}$.

Note that the existence of a_{02} and a_{13} is guaranteed by the condition $\alpha_1\alpha_2 = 0 = \alpha_2\alpha_3$. The length 3 *Massey product* $\langle\alpha_1, \alpha_2, \alpha_3\rangle$ is defined as the set

$$\langle\alpha_1, \alpha_2, \alpha_3\rangle = \{[\bar{a}_{02}a_{23} + \bar{a}_{01}a_{13}] \mid \{a_{ij}\} \text{ is a defining system } \} \subseteq H^{s\pm 1}$$

where $s = \deg\alpha_1 + \deg\alpha_2 + \deg\alpha_3$.

A Massey product is called *trivial* if it contains the zero homology class $[0]$. Note that in the definition of the triple Massey product a_{02} and a_{13} are not uniquely defined. If a'_{02} also satisfies $d(a'_{02}) = \bar{a}_{01}a_{12}$ then the difference $a_{02} - a'_{02}$ lies in the *indeterminacy*

$$\alpha_1 H^{\deg\alpha_2+\deg\alpha_3-1}A + H^{\deg\alpha_1+\deg\alpha_2-1}\alpha_3$$

of the Massey product and similarly for a_{13}. Therefore, a Massey product is trivial if and only if its projection to $HA/(\alpha_1, \alpha_3)$ is zero.

Example 2.2.2. Let A be the exterior algebra on generators x, y, z of degree 1 and with differential d defined by

$$dx = 0 \qquad dy = 0 \qquad dz = xy$$

We claim that

$$\langle[x], [x], [y]\rangle$$

is a non-trivial Massey product on HA. Since $x^2 = 0$ and $xy = dz$, we have

$$[xz] \in \langle[x], [x], [y]\rangle$$

and clearly $[xz]$ is a nonzero cohomology class. Consequently, it is sufficient to show that the indeterminacy of $\langle[x], [x], [y]\rangle$ is zero. The indeterminacy of the Massey product is

$$[x]H^1A + [y]H^1A.$$

Note that H^1A is generated by $[x]$ and $[y]$ as a k-vectorspace. Since $[x]^2 = [y]^2 = [x][y] = 0$, it follows that

$$[x]H^1A + [y]H^1A = 0$$

and so the Massey product is non-trivial.

The higher Massey products are defined inductively as follows.

Definition 2.2.3. Let $\alpha_1, \ldots, \alpha_n \in HA$ be homology classes with the property that each length $j-i+1$ Massey product $\langle \alpha_i, \ldots, \alpha_j \rangle$ is defined and contains only zero for $1 \leq i < j \leq n$ and $j-i \leq n-2$. A *defining system* $\{a_{ij} \mid 0 \leq i < j \leq n, j-i \leq n-1\}$ consists of

1. For $i = 1, \ldots, n$, representing homogeneous cycles $a_{i-1,i}$ of the homology class α_i.
2. For $o \leq i < j \leq n, 2 \leq j-i \leq n-1$, elements a_{ij} such that

$$da_{ij} = \sum_{i<k<j} \bar{a}_{ik} a_{kj}.$$

Note that the existence is guaranteed by the condition that $\langle \alpha_i, \ldots, \alpha_j \rangle$ is defined and contains only zero for $1 \leq i < j \leq n$ and $j-i \leq n-2$. The length n *Massey product* $\langle \alpha_1, \ldots, \alpha_n \rangle$ is defined as the set

$$\langle \alpha_1, \ldots, \alpha_n \rangle = \{ [\sum_{0<i<n} \bar{a}_{0i} a_{in}] \mid \{a_{ij}\} \text{ is a defining system } \} \subseteq H^{s+2-n}$$

where $s = \sum_{i=1}^n \deg \alpha_i$.

A Massey product $\langle \alpha_1, \ldots, \alpha_n \rangle$ is said to be *trivial* if it contains zero.

Definition 2.2.4. A morphism of dg algebras $f\colon A \to B$ is called a *quasi-isomorphism* if the induced map on homology $f_*\colon HA \to HB$ is an isomorphism. Two dg algebras A and B are called quasi-isomorphic if there exist dg algebras $A_0, \ldots, A_n$ with $A_0 = A$ and $A_n = B$ as well as a zig-zag

$$A_0 \xleftarrow{f_1} A_1 \xrightarrow{f_2} \cdots \xleftarrow{f_{n-1}} A_{n-1} \xrightarrow{f_n} A_n$$

of quasi-isomorphisms f_i.

One of the main properties of dg algebras we will be studying is formality which we will introduce next.

Definition 2.2.5. A dg algebra A is said to be *formal* if A is quasi-isomorphic to its homology algebra HA considered as a dg algebra with trivial differential.

Example 2.2.6. Suppose that A is a dg algebra such that $H_i A = 0$ for all $i \neq 0$. Then A is formal. Indeed, let $A' \subseteq A$ be the subcomplex

$$A' = \begin{cases} A_i & \text{if } i > 0 \\ \ker(d\colon A_0 \to A_{-1}) & \text{if } i = 0 \\ 0 & \text{if } i < 0 \end{cases}$$

It is readily verified that A' is a dg algebra. Plainly, there is a zig-zag

$$A \xleftarrow{\ i\ } A' \xrightarrow{\ p\ } H_0 A$$

of quasi-isomorphisms where i is the inclusion and p is the projection.

Example 2.2.7. Let A be a dg algebra such that $HA \cong \mathbb{Z}[x]$ where x is a degree n generator. Then A is formal. To see this, pick a representing cycle a for x and define a quasi-isomorphism

$$f\colon (\mathbb{Z}[x], 0) \to (A, d)$$

by $x \mapsto a$.

Proposition 2.2.8. Let $f\colon A \to B$ be a morphism of dg algebras and let $u_1, \dots, u_n \in HA$ be such that $\langle u_1, \dots, u_n \rangle$ is defined. Then $\langle f_* u_1, \dots, f_* u_n \rangle$ is defined and there is an inclusion

$$f_* \langle u_1, \dots, u_n \rangle \subseteq \langle f_* u_1, \dots, f_* u_n \rangle. \tag{2.2.1}$$

Furthermore, if f is a quasi-isomorphism then (2.2.1) is an equality.

Proof. For simplicity of notation, we will only consider the case $n = 3$. Let $a, b, c \in A$ be representing cycles for u, v, w respectively. Choose $x, y \in A$ such that

$$dx = ab \text{ and } dy = bc.$$

Then

$$[xc + (-1)^{\deg u} ay] \in \langle u, v, w \rangle.$$

Since f is a morphism of dg algebras, it follows that

$$dfx = fdx = f(ab) = f(a)f(b)$$

and similarly $dfy = f(b)f(c)$. Consequently,

$$[f(x)f(c) + (-1)^{\deg u} f(a)f(y)] \in \langle f_* u, f_* v, f_* w \rangle$$

proving the first statement. For the second statement, suppose that f is a quasi-isomorphism. Then

$$\begin{aligned} f_*(uHA + HAw) &= f_*(u)f_*(HA) + f_*(HA)f_*(w) \\ &= f_*(u)HB + HBf_*(w) \end{aligned}$$

and so f preserves indeterminacies. So, (2.2.1) is an equality. □

Corollary 2.2.9. Let A be a dg algebra. If A is formal then all Massey products HA are trivial.

Proof. Since A is formal, A is quasi-isomorphic to HA as dg algebras. The result now follow from Proposition 2.2.8 since all Massey products vanish in the dg algebra HA with trivial differential. □

2.3 A_∞-algebras

The notion of an A_∞-algebra was first introduced by Stasheff [46] in the 1960s in the context of algebraic topology. Since their introduction A_∞-algebras have found applications in various branches of mathematics such as geometry [16], algebra [47] and mathematical physics [29], [36]. Though the following section aims to be self-contained, more extensive discussions can be found in Chapter 9 of [32] and in [28]. The exposition below follows that of [49], [28] and [33].

Suppose we are working in the category of chain complexes and we are given the following data. Let A and B be chain complexes and let

$$\begin{tikzcd} B \arrow[r, bend left, "i"] & A \arrow[l, bend left, "p"] \arrow[loop right, "\phi"] \end{tikzcd}$$

be a diagram where p and i are chain maps and ϕ is a degree 1 map such that $pi = 1_B$ and $ip - 1 = d\phi + \phi d$. In that case, it follows immediately that i and p are quasi-isomorphisms. Consequently, if A and B are *just* chain complexes then from the point of view of homology they are the same.

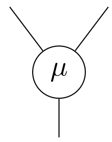
In practice, however, it often happens that A comes with additional algebraic structure. For example, suppose that A is a (graded) algebra. That is to say, A comes with a multiplication

$$\mu \colon A \otimes A \to A$$

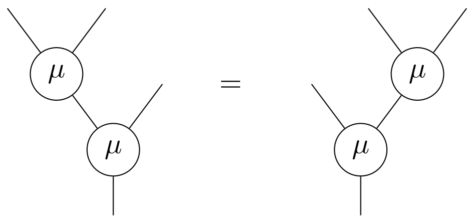
such that

$$\mu(\mu \otimes 1_A) = \mu(1_A \otimes \mu).$$

Graphically, we can represent μ by



and then associativity can be represented as



A natural question to ask is how this algebraic data combines with the above homotopy data. That is to say, we are interested in the question what (if any) algebraic structure we obtain on B.

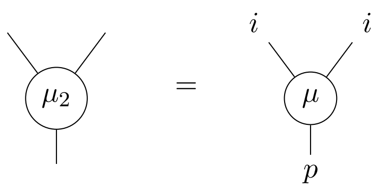
As a start, we can obtain a map

$$\mu_2 \colon B \otimes B \to B$$

by setting

$$\mu_2 = p\mu(i \otimes i).$$

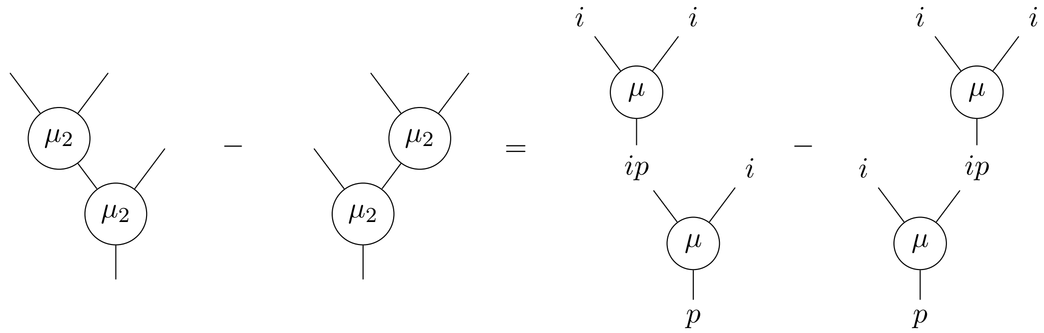
Graphically, we have



Since μ is associative, one might hope that μ_2 is associative as well. To see if this is the case, we compute the *associator*

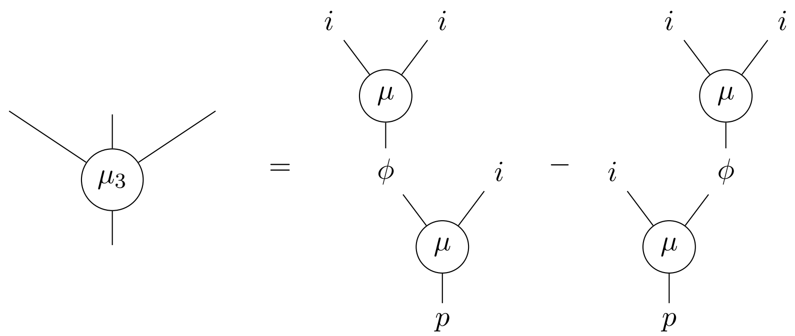
$$\mu_2(\mu_2 \otimes 1_B) - \mu_2(1_B \otimes \mu_2).$$

We have



Therefore, if i and p were inverse isomorphisms then μ_2 would be associative. However, in our setup ip is not equal to the identity but only homotopic to the identity. The idea then seems to be that μ_2 is 'associative up to homotopy'. The goal of this section will be to formalize this idea.

As a first step, define an element μ_3 by



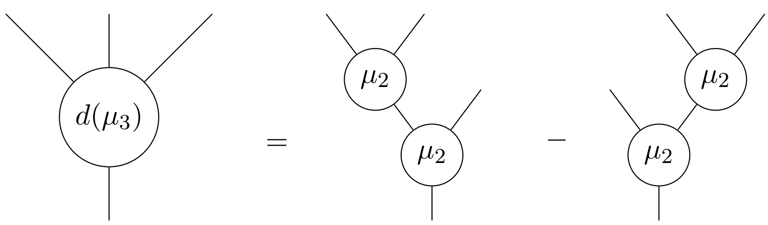
We would like to relate μ_3 to the associator $\mu_2(\mu_2 \otimes 1) - \mu_2(1 \otimes \mu_2)$. Recall that if X and Y are chain complexes, then $\mathrm{Hom}(X, Y)$ is the chain complex consisting of modules

$$\mathrm{Hom}(X,Y)_p = \{f \colon X \to Y \mid \deg(f) = p\}$$

together with differential

$$d(f) = d_Y f - (-1)^{\deg(f)} f d_X. \tag{2.3.1}$$

Observe that μ_3 is a degree 1 element of $\mathrm{Hom}(B^{\otimes 3}, B)$. A straightforward computation then shows that



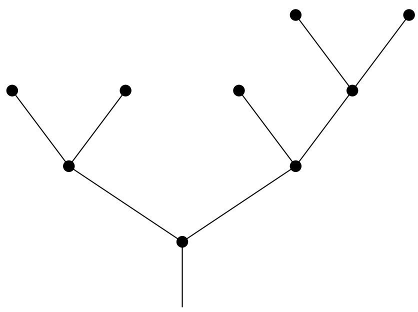
where $d(\mu_3)$ is computed using (2.3.1). So, the conclusion is that the associator vanishes on $\mathrm{Hom}(B^{\otimes 3}, B)$ up to the homotopy μ_3.

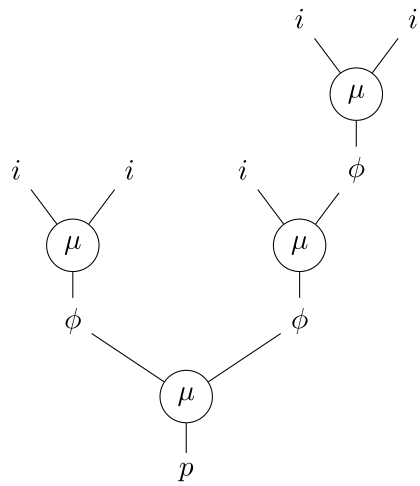
Of course, the next question one can ask is how μ_2 and μ_3 are related. By arguing similarly as above, we see that μ_2 and μ_3 are related only up to a homotopy $\mu_4 \in \mathrm{Hom}(B^{\otimes 4}, B)$ of degree 2. Instead of discussing μ_4 explicitly, we will skip ahead to the general case.

Given a planar binary tree T with n leaves we can define an operation $\mu_T \colon B^{\otimes n} \to B$ by the following procedure.

1. Apply i at the leaves of the tree,
2. apply $\pi \circ \mu$ for every vertex that is not the root or any of the leaves,
3. apply p at the root of the tree

and compose according to the tree T. The following picture shows an example of a tree T and the corresponding operation μ_T.

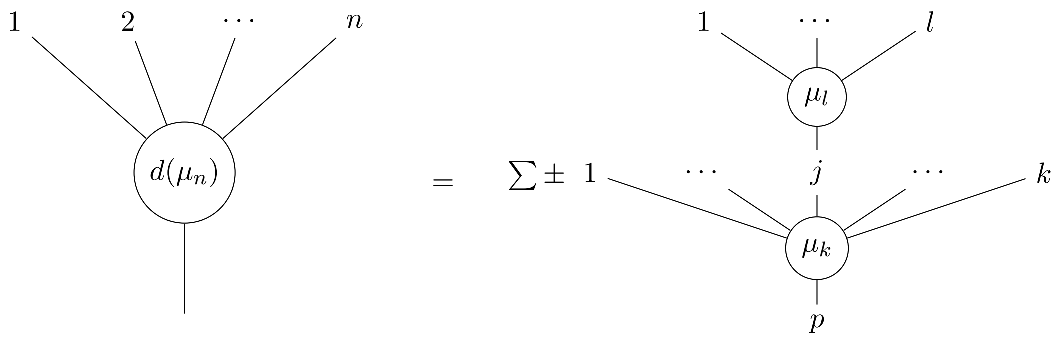




Define a map $\mu_n \colon B^{\otimes n} \to B$ by

$$\mu_n = \sum \pm \mu_T$$

where the sum runs over all planar binary trees with n leaves. Then the μ_n are linear maps of degree $n-2$ satisfying the following relations.



We make this relation into a definition as follows.

Definition 2.3.1. An *A_∞-algebra* consists of a graded module A together with linear maps $\mu_n \colon A^{\otimes n} \to A$ of degree $n-2$ satisfying the *Stasheff identities*

$$\sum (-1)^{r+st} \mu_u (1^{\otimes r} \otimes \mu_s \otimes 1^{\otimes t}) = 0 \tag{2.3.2}$$

where the sum runs over all decompositions $n = r+s+t$ with $r, t \geq 0$ and $s \geq 1$ and where $u = r+t+1$.

Before proceeding, we will consider what the Stasheff identities say in low degrees. If $n = 1$, then equation (2.3.2) says $\mu_1 \mu_1 = 0$. Therefore, (A, μ_1) is a chain complex. Next up, if $n = 2$ the equation becomes

$$\mu_1 \mu_2 = \mu_2 (\mu_1 \otimes 1 + 1 \otimes \mu_1)$$

and so μ_1 is a derivation with respect to the multiplication given by μ_2. Lastly, for $n = 3$ we obtain

$$\mu_2 (1 \otimes \mu_2 - \mu_2 \otimes 1) = \mu_1 \mu_3 + \mu_3 (\mu_1 \otimes 1 \otimes 1 + 1 \otimes \mu_1 \otimes 1 + 1 \otimes 1 \otimes \mu_1).$$

This means that the multiplication μ_2 is associative up to the homotopy μ_3. In particular, if $\mu_3 = 0$ then μ_2 is strictly associative and makes A into a differential graded algebra. Conversely, any dg algebra can be seen as an A_∞-algebra by setting $\mu_k = 0$ for all $k \geq 3$.

All A_∞-algebras will be assumed to be *strictly unital*, that is to say there exists an element $1 \in A$ that is a unit for μ_2 and for all $n \neq 2$ we have

$$\mu_n (a_1 \otimes \cdots \otimes a_n) = 0$$

whenever some $a_i = 1$.

Definition 2.3.2. Let (A, μ_n) and (B, ν_n) be two A_∞-algebras. A *morphism* of A_∞-algebras $f \colon A \to B$ is a family of linear maps

$$f_n \colon A^{\otimes n} \to B$$

of degree $1-n$ satisfying the *Stasheff morphism identities*

$$\sum(-1)^{r+st} f_u(1^r \otimes \mu_s \otimes 1^t) = \sum(-1)^w \nu_q(f_{i_1} \otimes \cdots \otimes f_{i_q}) \tag{2.3.3}$$

where the sum on the left runs over all decompositions $n = r+s+t$ and $u = r+t+1$ and the second sum runs over all $1 \leq q \leq n$ and all decompositions $n = i_1 + \cdots + i_q$. The sign on the right hand side is given by

$$w = (q-1)(i_1-1) + (q-2)(i_2-1) + \cdots + 2(i_{q-2}-1) + (i_{q-1} = 1).$$

If A and B are unital, we also require that $f_1(1) = 1$ and $f_n(a_1 \otimes \cdots \otimes a_n) = 0$ whenever $n > 1$ and some $a_i = 1$. Usually, we will only consider *strict morphisms* of A_∞-algebras, that is to say morphisms for which $f_n = 0$ whenever $n \neq 2$.

Given an A_∞-algebra A, the homology HA is defined as

$$HA = \ker \mu_1 / \operatorname{im} \mu_1.$$

From the Stasheff identities (2.3.2) it follows that

$$\mu_2 \colon A \otimes A \to A$$

induces a strictly associative multiplication on HA. A morphism f is called a *quasi-isomorphism* if f_1 is a quasi-isomorphism of chain complexes, that is to say that f_1 induces an isomorphism in homology. The following theorem is a basic but extremely useful result relating an A_∞-algebra to its homology HA.

Theorem 2.3.3 (Homotopy Transfer Theorem [24], see also [37])**.** Let $(A, \{\mu_n\})$ be an A_∞-algebra and HA its homology algebra. Suppose that HA is free. Then there exists an A_∞-algebra structure $\{\nu_n\}$ on the homology HA such that

1. $\nu_1 = 0$
2. $\nu_2 = H(\mu_2)$
3. there exists a quasi-isomorphism of A_∞-algebras $HA \to A$ lifting the identity on HA.

Moreover, this A_∞-algebra structure on HA is unique up to A_∞-isomorphism. Furthermore, if A has a unit then the structure and the quasi-isomorphism can be chosen to be unital.

In the introduction to this chapter we discussed how A_∞-algebras naturally arise by transferring dg algebra structures along homotopy equivalences. Our next goal is to formalize this idea. To achieve this goal, we will discuss an inductive construction of the ν_n that is due to Merkulov [37]. Given a dg algebra A, we are interested in the question of which subcomplexes of A admit an A_∞-structure that is derived from the multiplication on A. To answer this question, we need the notion of a *transfer diagram.*

Definition 2.3.4. Let A and B be a chain complexes. A *transfer diagram* is a diagram of the form

$$\begin{tikzcd} B \arrow[r, bend left, "i"] & A \arrow[l, bend left, "p"] \arrow[loop right, "\phi"] \end{tikzcd}$$

where $pi = 1_B$ and $1 - ip = d\phi + \phi d$.

Note that some authors use the term *strong deformation retract* for what we call a transfer diagram.

Proposition 2.3.5. Let A be a chain complex over a field. Then there exists a transfer diagram

$$\begin{tikzcd} HA \arrow[r, bend left, "i"] & A \arrow[l, bend left, "p"] \arrow[loop right, "\phi"] \end{tikzcd}$$

Proof. Write Z_n for the n-cycles of A and B_n for the n-boundaries of A. Since we are working over a field, all short exact sequences split. From the short exact sequence

$$Z_n \longrightarrow A_n \xrightarrow{d} B_{n-1}$$

it follows that

$$A_n \cong Z_n \oplus B_{n-1}.$$

Similarly, from exactness of

$$B_n \longrightarrow Z_n \longrightarrow H_n$$

we obtain

$$Z_n \cong H_n \oplus B_n.$$

Therefore, we have a splitting

$$A_n \cong B_n \oplus H_n \oplus B_{n-1}.$$

Define the maps i, p and ϕ as follows. Let

$$i\colon H_n \to B_n \oplus H_n \oplus B_{n-1}$$

be the inclusion and let

$$p\colon B_n \oplus H_n \oplus B_{n-1} \to H_n$$

be the projection. Lastly, let ϕ be the composition

$$\phi\colon A_{n-1} \cong B_{n-1} \oplus H_{n-1} \oplus B_{n-2} \to B_{n-1} \to B_n \oplus H_n \oplus B_{n-1} \cong A_n.$$

It is readily verified that

$$\begin{tikzcd} HA \arrow[r, bend left, "i"] & A \arrow[l, bend left, "p"] \arrow[loop right, "\phi"] \end{tikzcd}$$

is indeed a transfer diagram. □

Let A be a differential graded algebra and suppose that $B \subseteq A$ is a subcomplex such that there exists a transfer diagram

$$\begin{tikzcd} B \arrow[r, bend left, "i"] & A \arrow[l, bend left, "p"] \arrow[loop right, "\phi"] \end{tikzcd}$$

We can construct an A_∞-structure on B as follows. First we define auxiliary maps

$$\lambda_n\colon A^{\otimes n} \to A$$

of degree $2-n$ as follows. There is no λ_1 but we put formally $\phi\lambda_1 = -1_A$. The map λ_2 is just the multiplication of the dg algebra A. For $n > 2$, we define inductively

$$\lambda_n = \sum (-1)^{s+1} \lambda_2(\phi\lambda_s \otimes \phi\lambda_t) \tag{2.3.4}$$

where the sum runs over all decompositions $n = s + t$ with $s, t \geq 1$.

Now, define maps

$$\mu_n\colon B^{\otimes n} \to B$$

by setting

$$\mu_n = p\lambda_n i^{\otimes n}.$$

Note that this construction indeed recovers our informal construction from the beginning of this chapter. We have the following theorem.

Theorem 2.3.6 ([37], Theorem 3.4)**.** The complex B together with the maps μ_n form an A_∞-algebra.

For us the most interesting case is when $B = HA$. By Proposition 2.3.5, there exists a transfer diagram

$$\begin{tikzcd} HA \arrow[r, bend left, "i"] & A \arrow[l, bend left, "p"] \arrow[loop right, "\phi"] \end{tikzcd}$$

and so Theorem 2.3.6 gives an A_∞-structure on HA.

Lastly, we have the following proposition.

Proposition 2.3.7 ([33], Proposition 2.3)**.** Let A be a dg algebra over a field and let μ_n denote the A_∞-structure on HA obtained from Proposition 2.3.5 and Theorem 2.3.6. Define maps

$$f_n \colon (HA)^{\otimes n} \to A$$

by putting $f_n = -\phi\lambda_n$. Then f is a quasi-isomorphism of A_∞-algebras.

2.4 Homotopy theory of A_∞-algebras

The exposition in this section follows that of [28]. Before discussing the homotopy theory of A_∞-algebras, we will first consider an alternative definition of an A_∞-algebras in terms of dg coalgebras.

Let M be a graded module and let

$$\overline{T}M = \bigoplus_{n\geq 1} T^{\otimes n}$$

be the reduced tensor algebra on M. Define a comultiplication

$$\Delta \colon \overline{T}M \to \overline{T}M \otimes \overline{T}M$$

by putting

$$\Delta(x_1, \dots, x_n) = \sum_{1\leq i\leq n} (x_1, \dots, x_i) \otimes (x_i, \dots, x_n).$$

In this way, $(\overline{T}M, \Delta)$ becomes a graded coalgebra.

Given a graded coalgebra C, let $\mathrm{Coder}(C)$ denote the set of coderivations of C.

Lemma 2.4.1. Let M be a graded module and let $\pi \colon \overline{T}M \to M$ be the projection. Then the map

$$\pi_* \colon \mathrm{Coder}(\overline{T}M) \to \mathrm{Hom}(\overline{T}M, M)$$

defined by $\pi_*(c) = \pi \circ c$ is a bijection.

Proof. Define an inverse ρ of π_* as follows. Let $b \in \mathrm{Hom}(\overline{T}M, M)$ and write b_i for the restriction of b to $V^{\otimes i}$. Define

$$\rho\colon \mathrm{Hom}(\overline{T}M, M) \to \mathrm{Coder}(\overline{T}M)$$

by

$$\rho(b) = \sum_{i=0}^{n} \sum_{j=0}^{n-i} 1^{\otimes j} \otimes b_i \otimes 1^{\otimes (n-i-j)}.$$

Then ρ and π_* are inverses and we are done. □

If M is a graded module, then the *suspension* SM of M is defined to be the graded module with components

$$(SM)_p = M_{p+1}.$$

Now, define a bijection between families of maps

$$b_n\colon (SA)^{\otimes n} \to SA$$

and families of maps

$$\mu_n\colon A^{\otimes n} \to A$$

by requiring that the diagram

$$\begin{tikzcd}
(SA)^{\otimes n} \arrow[r, "b_n"] & SA \\
A^{\otimes n} \arrow[u, "s^{\otimes n}"] \arrow[r, "\mu_n"'] & A \arrow[u, "s"']
\end{tikzcd}$$

commutes for all $n \geq 1$. Here, $s\colon A \to SA$ is the degree -1 map defined by $a \mapsto a$. Note that the degree of μ_n is $-n+1-(-1) = 2-n$. The following lemma is then straightforward to verify.

Lemma 2.4.2. The following are equivalent

1. The maps $\mu_n\colon A^{\otimes n} \to A$ give an A_∞-structure on A.
2. The coderivation $b\colon \overline{T}SA \to \overline{T}SA$ satisfies $b^2 = 0$.
3. For all $n \geq 1$, we have
$$\sum b_u(1^r \otimes b_s \otimes 1^t) = 0$$
where the sum runs over all decompositions $n = r+s+t$ and $u = r+t+1$.

Similarly, we have the following lemma.

Lemma 2.4.3. Let A and B be A_∞-algebras. Then there is a bijection

$$\left\{ \begin{array}{l} A_\infty\text{-morphisms} \\ f\colon A \to B \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{l} \text{dg coalgebra morphisms} \\ f\colon \overline{T}SA \to \overline{T}SB \end{array} \right\}.$$

The main idea of this section is that instead of introducing a notion of homotopy directly for A_∞-algebras we define the concept for dg coalgebras. Two morphisms of A_∞-algebras are then said to be *homotopic* if the corresponding maps between dg coalgebras are homotopic in the following sense.

Definition 2.4.4. Let

$$\begin{tikzcd} A \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] & B \end{tikzcd}$$

be morphisms of A_∞-algebras and denote by

$$\begin{tikzcd} \overline{T}SA \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] & \overline{T}SB \end{tikzcd}$$

the corresponding morphisms of dg coalgebras. Then f is said to be *homotopic* to g if there exists a morphism of dg modules

$$H\,:\overline{T}SA \to \overline{T}SB$$

of degree -1 such that

$$\Delta H = F \otimes H + H \otimes G \qquad \text{and} \qquad F - G = bH + Hb.$$

In that case, we will write $f \simeq g$.

Lemma 2.4.5 ([43])**.** Let A and B be A_∞-algebras. The relation $\simeq$ is an equivalence relation on the set of all A_∞-morphisms from A to B.

Definition 2.4.6. Let $f\colon A \to B$ be a morphism of A_∞-algebras. Then f is said to be a *homotopy equivalence* if there exists an A_∞-morphism

$$g\colon B \to A$$

and homotopies

$$H\colon gf \simeq 1_A \qquad \text{and} \qquad K\colon fg \simeq 1_B.$$

Two A_∞-algebras A and B are called *homotopy equivalent* if there exists a homotopy equivalence

$$f\colon A \to B.$$

Proposition 2.4.7 ([43])**.** Let $f\colon A \to B$ be a morphism of A_∞-algebras. Then f is a quasi-isomorphism if and only if f is a homotopy equivalence.

Chapter 3

Monomial rings

3.1 Definitions and examples

Let $S = k[x_1, \ldots, x_m]$ be the polynomial algebra in m variables over a field k. A *monomial* in S is an element of the form

$$x_1^{a_1} \cdots x_m^{a_m}.$$

An ideal $I \subseteq S$ is called a *monomial ideal* if it is generated by monomials. If $I = (m_1, \ldots, m_r)$ is a monomial ideal in S and $f \in S$ is some polynomial, then $f \in I$ if and only if every term of f is divisible by at least one of the m_i. Note that by the Hilbert Basis Theorem Theorem 1.2 of [11]) a monomial ideal is necessarily finitely generated.

Notation 3.1.1. Whenever we write

$$I = (m_1, \ldots, m_r)$$

we will always assume that none of the m_i divide each other. Indeed, if m_i divides m_j then

$$(m_1, \ldots, m_r) = (m_1, \ldots, \widehat{m_j}, \ldots, m_r)$$

and we can just relabel the monomials.

We have the following lemma.

Lemma 3.1.2. Let I be a monomial ideal. Then there exist *unique* monomials $m_1, \ldots, m_r$ such that

$$I = (m_1, \ldots, m_r).$$

Proof. Suppose that

$$I = (m_1, \ldots, m_r) = (n_1, \ldots, n_s).$$

First note that $r = s$. Indeed, without loss of generality, assume that $r < s$. Then there exists $i \neq j$ such that n_i and n_j are divisible some m_k. First note that m_k is not divisible by n_i since otherwise we would have that n_i divides n_j. Similarly, m_k is not divisible by n_j. Since $m_k \in (n_1, \dots, n_s)$, it follows that there exists some n_l such that n_l divides m_k. But then n_l divides n_i as well which is a contradiction. Therefore, $r = s$ as required.

Let $i \in \{1, \dots, r\}$. By symmetry, it is sufficient to show that $m_i \in \{n_1, \dots, n_r\}$. By the observation above, there exists some $j \in \{1, \dots, r\}$ such that n_j divides m_i. Assume that $n_j \neq m_i$. Then, again, there exists some k such that m_k divides n_j. Consequently, m_k divides m_i which is a contradiction. □

Let I be a monomial ideal generated by (unique) monomials $m_1, \dots, m_r$. Then the set

$$\{m_1, \dots, m_r\}$$

is called the *minimal generating set* of I.

A monomial $x_1^{a_1} \cdots x_m^{a_m}$ is called *square-free* if $a_i \in \{0, 1\}$ for all $i = 1, \dots, m$. Square-free monomial ideals are of special interest because they correspond to simplicial complexes.

Definition 3.1.3. Let $[m] = \{1, \dots, m\}$. A *simplicial complex on* $[m]$ is a collection Δ of subsets $\sigma \subseteq [m]$ (called *simplices*) such that

1. $\emptyset \in \Delta$ and
2. if $\sigma \in \Delta$ and $\tau \subseteq \sigma$ then $\tau \in \Delta$.

We will denote by Δ^m the m-simplex. That is Δ^m consists of all subsets of $[m+1]$. Let Δ be a simplicial complex on $[m]$. A subset $\sigma \subseteq [m]$ is called a *missing face* if $\sigma \notin \Delta$. σ is called a *minimal missing face* if σ is a missing face and no $\tau \subsetneq \sigma$ is a missing face. We will write $\mathrm{mf}(\Delta)$ for the collection of minimal missing faces. Note that a simplicial complex is uniquely determined by its minimal missing faces.

Definition 3.1.4. Let Δ be a simplicial complex. Given a subset $\sigma \subseteq [m]$, write

$$x_\sigma = \prod_{i \in \sigma} x_i.$$

The *Stanley-Reisner ideal* of Δ is the ideal

$$I = (x_\sigma \mid \sigma \notin \Delta).$$

The *Stanley-Reisner ring* or *face ring* $k[\Delta]$ of Δ is the quotient

$$k[\Delta] = S/I$$

where I is the Stanley-Reisner ideal of Δ.

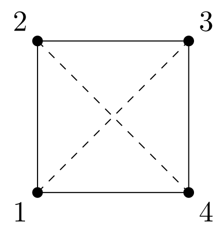
Let Δ be a simplicial complex. Observe that the Stanley-Reisner ideal is in fact generated by all X_σ with σ a minimal missing face.

Example 3.1.5. 1. If $\Delta = \Delta^{m-1}$, then clearly $k[\Delta]$ is just S.

2. If $\Delta = \{\{1\}, \dots, \{m\}\}$, then

$$k[\Delta] = k[x_1, \dots, x_m]/(x_i x_j \mid i \neq j).$$

3. Suppose Δ is the boundary of the square labeled as



Then

$$k[\Delta] = k[x_1, x_2, x_3, x_4]/(x_1 x_3, x_2 x_4).$$

Theorem 3.1.6 (Stanley-Reisner Correspondence)**.** There is a bijection

$$\{\text{square-free monomial ideals } I\} \longleftrightarrow \{\text{simplicial complexes } \Delta\}.$$

Proof. Let

$$I = (m_1, \dots, m_r)$$

be a square-free monomial ideal. For $i = 1, \dots, r$, put

$$\sigma_i = \{k \in [m] \mid x_k \text{ divides } m_i\}.$$

Let Δ be the simplicial complex with minimal missing faces σ_i where $i = 1, \dots, r$.

Conversely, if Δ is a simplicial complex then its Stanley-Reisner ideal is a square-free monomial ideal. $\square$

3.2 Homological algebra of monomial rings

Definition 3.2.1. Let $S = k[x_1, \dots, x_m]$ and let S/I be a monomial ring. A *free resolution* over S of S/I is an exact sequence

$$\cdots \xrightarrow{d} F_n \xrightarrow{d} F_{n-1} \xrightarrow{d} \cdots \xrightarrow{d} F_1 \xrightarrow{d} F_0 \xrightarrow{d} S/I \longrightarrow 0$$

where each F_i is a free S-module. Defining a grading on S/I by defining the degree of each x_i to be 1. A free resolution F of S/I is called *graded* if each F_i is a graded free module and each d_i is a graded map of degree 0.

In the sequel, we will always assume every free resolution is graded and so we will refer to them simply as free resolutions. As the following construction shows, free resolutions always exist.

$$\begin{tikzcd}
 & 0 \arrow[dr] & & 0 \arrow[dr] & & 0 & & \\
 & & \ker d_1 \arrow[ur] \arrow[dr] & & \ker d_0 \arrow[ur] \arrow[dr] & & & \\
\cdots \arrow[r] & F_2 \arrow[ur] \arrow[rr, dashed, "d_2"'] & & F_1 \arrow[ur] \arrow[rr, dashed, "d_1"'] & & F_0 \arrow[r, "d_0"] & S/I \arrow[r] & 0
\end{tikzcd}$$

Pick some epimorphism $d_0\colon F_0 \to S/I$ where F_0 is free. Then pick some epimorphism $F_1 \to \ker d_0$ where F_1 is also free. Define d_1 to be the composition

$$F_1 \to \ker d_0 \to F_0.$$

By iterating this process, we obtain free modules F_n and morphism $d_n\colon F_n \to F_{n-1}$ and it follows immediately that the sequence

$$\cdots \longrightarrow F_n \xrightarrow{d_n} F_{n-1} \xrightarrow{d_{n-1}} \cdots \xrightarrow{d_2} F_1 \xrightarrow{d_1} F_0 \xrightarrow{d_0} S/I \longrightarrow 0$$

is exact and hence forms a free resolution of S/I.

Definition 3.2.2. A free resolution

$$\cdots \longrightarrow F_n \xrightarrow{d_n} F_{n-1} \xrightarrow{d_{n-1}} \cdots \xrightarrow{d_2} F_1 \xrightarrow{d_1} F_0 \xrightarrow{d_0} S/I \longrightarrow 0$$

is called *minimal* if $d_n \otimes_S 1_k = 0$ for all $n \in \mathbb{N}$.

The following lemma is straightforward but will be used often.

Lemma 3.2.3. Let $F \to S/I$ be a free resolution. Then F is minimal if and only if

$$d_n(F_n) \subseteq (x_1, \ldots, x_m)F_{n-1}$$

for all $n \in \mathbb{N}$.

Proof. Let $n \in \mathbb{N}$. Since $F_n \otimes_S k \cong F_n/(x_1, \ldots, x_m)F_n$, we have a commutative diagram

$$\begin{tikzcd}
F_n \otimes_S k \arrow[r, "\cong"] \arrow[d, "d \otimes_S 1_k"'] & F_n/(x_1, \ldots, x_m)F_n \arrow[d, "d"] \\
F_{n-1} \otimes_S k \arrow[r, "\cong"'] & F_{n-1}/(x_1, \ldots, x_m)F_{n-1}.
\end{tikzcd}$$

Therefore, $d \otimes_S 1 = 0$ if and only if the right hand map is zero which is equivalent to

$$d_n(F_n) \subseteq (x_1, \ldots, x_m)F_{n-1}$$

as required. $\square$

The terminology may seem strange at first since one would expect a resolution to be minimal if we pick the smallest amount of generators for F_n in the construction above. The point of the following lemma is that this is indeed the case.

Lemma 3.2.4 ([11], Lemma 19.4)**.** Let

$$\cdots \longrightarrow F_n \xrightarrow{d_n} F_{n-1} \xrightarrow{d_{n-1}} \cdots \xrightarrow{d_2} F_1 \xrightarrow{d_1} F_0 \xrightarrow{d_0} S/I \longrightarrow 0$$

be a free resolution. Then F is minimal if and only if for all n a basis of F_{n-1} maps onto a minimal set of generators of $\operatorname{coker} d_n$.

Proof. We have exact sequences

$$\operatorname{im} d_n \xrightarrow{d_n} F_{n-1} \xrightarrow{p} \operatorname{coker} d_n$$

which induce exact sequences

$$\operatorname{im} d_n/J \operatorname{im} d_n \longrightarrow F_{n-1}/JF_{n-1} \xrightarrow{p'} \operatorname{coker} d_n/J \operatorname{coker} d_n$$

where $J = (x_1, \ldots, x_m)$. By Nakayama's Lemma (Corollary 4.8(b) of [11]), a basis of the vector space $\operatorname{coker} d_n/J \operatorname{coker} d_n$ corresponds to a minimal set of generators of $\operatorname{coker} d_n$. Therefore, a basis of F_{n-1} maps to a set of generators of $\operatorname{coker} d_n$ if and only if p' is an isomorphism. By exactness, p' is an isomorphism if and only if $\operatorname{im} d_n/J \operatorname{im} d_n = 0$ if and only if $\operatorname{im} d_n \subseteq JF_{n-1}$ as required. $\square$

Since we have

$$\operatorname{coker} d_n = F_{n-1}/\operatorname{im} d_n = F_{n-1}/ker d_{n-1} \cong \operatorname{im} d_{n-1} = \ker d_{n-2},$$

it follows that making a minimal choice for the cokernels corresponds to making a minimal choice in our above construction.

A *short trivial complex* is a chain complex of the form

$$0 \longrightarrow S \xrightarrow{1} S \longrightarrow 0.$$

A chain complex is called a *trivial complex* if it is the direct sum of short trivial complexes. We have the following lemma.

Lemma 3.2.5. Let R be a monomial ring. Then

1. there exists a minimal free resolution F of R
2. if F is a minimal free resolution of R and G is any other free resolution of R then there exists a trivial complex H such that $G = F \oplus H$ as chain complexes.

Proof. The first statement follows by Lemma 3.2.4 since we can always pick a minimal set of generators when constructing free resolutions. For a proof of the second statement see Theorem 7.5(b) of [42]. □

The following lemma will be occasionally used.

Lemma 3.2.6 ([42], Theorem 7.5(c))**.** Let R be a monomial ring. Up to isomorphism, there exists a unique minimal free resolution of R.

Proof. This follows immediately from Lemma 3.2.5. □

Let V be a vector space with basis $\{u_1, \ldots, u_r\}$. The *tensor algebra* on generators $u_1, \ldots, u_r$ is

$$T(V) = \bigoplus_n V^{\otimes n}$$

with multiplication given by concatenation. Let J be the ideal generated by all $v \otimes v$ with $v \in V$. The quotient $T(V)/J$ is called the *exterior algebra* on generators $u_1, \ldots, u_r$ and is denoted by $E(V)$.

Let $S = k[x_1, \ldots, x_m]$ and let $I = (m_1, \ldots, m_r)$ be a monomial ideal. As usual, write $R = S/I$. The following construction is due to Taylor [48].

Construction 3.2.7. Let E denote the exterior algebra on generators $u_1, \ldots, u_r$. The *Taylor resolution* T has underlying module $S \otimes_k E$. If $J = \{j_1 < \ldots < j_k\} \subseteq \{1, \ldots, r\}$, then we write $u_J = u_{j_1} \cdots u_{j_k}$. Furthermore, we put $m_J = \mathrm{lcm}(m_{j_1}, \ldots, m_{j_k})$. We will also write $J^i = \{j_1 < \ldots < \widehat{j_i} < \ldots < j_k\}$. Define $d \colon T \to T$ by

$$d(u_J) = \sum_{i=1}^{|J|} (-1)^{i+1} \frac{m_J}{m_{J^i}} u_{J^i}.$$

The Taylor resolution admits a multiplication defined by

$$u_I \cdot u_J = \begin{cases} \mathrm{sgn}(I,J) \frac{m_I m_J}{m_{I \cup J}} u_{I \cup J} & \text{if } I \cap J = \emptyset \\ 0 & \text{otherwise} \end{cases}$$

where $\mathrm{sgn}(I, J)$ is the sign of the permutation making $I \cup J$ into an increasing sequence.

Lemma 3.2.8. The Taylor resolution T is a dg algebra resolution of S/I.

Proof. First, we show that T is a complex, that is, $d^2 = 0$. Let $J = \{j_1 < \cdots < j_k\}$. Given $1 \leq p < q \leq k$, We will write

$$J^{p,q} = \{j_1 < \cdots < \widehat{j_p} < \cdots < \widehat{j_q} < \cdots < j_k\}.$$

On the basis element u_J we have

$$\begin{aligned}
d^2(u_J) &= \sum_{p=0}^{k} (-1)^{p+1} \frac{m_J}{m_{J^p}} d(u_{J^p}) \\
&= \sum_{p=0}^{k} \sum_{q<p} (-1)^{p+q+1} \frac{m_J}{m_{J^{p,q}}} u_{J^{p,q}} + \sum_{p=0}^{k} \sum_{q<p} (-1)^{p+q} \frac{m_J}{m_{J^{q,p}}} u_{J^{q,p}}.
\end{aligned}$$

These two sums cancel since after swapping p and q in the second sum it becomes the negative of the first. Therefore, $d^2 = 0$. The fact that the Taylor resolution is a free resolution follows from Theorem 3.2.14 below. Associativity and skew commutativity are immediate from the definition of the product on T. Lastly, we verify the Leibniz identity. Let $I = \{j_1 < \cdots < j_k\}$ and $J = \{j_{k+1} < \cdots j_{k+l}\}$. Then

$$\begin{aligned}
d(u_I \cdot u_J) &= \operatorname{sgn}(I, J) \frac{m_I m_J}{m_{I \cup J}} d(u_{I \cup J}) \\
&= \sum_{p=1}^{k+l} \operatorname{sgn}(I, J)(-1)^{p+1} \frac{m_I m_J}{m_{(I \cup J)^p}} u_{(I \cup J)^p} \\
&= \sum_{p=1}^{k} \operatorname{sgn}(I^p, J)(-1)^{p+1} \frac{m_I m_J}{m_{I^p \cup J}} u_{I^p \cup J} + \sum_{p=k+1}^{k+l} \operatorname{sgn}(I, J^p)(-1)^{p+1} \frac{m_I m_J}{m_{I \cup J^p}} u_{I \cup J^p} \\
&= \sum_{p=1}^{k} (-1)^{p+1} \frac{m_I}{m_{I^p}} u_{I^p} \cdot u_J + \sum_{q=1}^{l} (-1)^{q+1} \frac{m_J}{m_{J^q}} u_I \cdot u_{J^q} \\
&= d(u_I) \cdot u_J + (-1)^k u_I \cdot d(u_J)
\end{aligned}$$

$\square$

Example 3.2.9. Let $S = k[x, y]$ and let

$$R = k[x, y]/(x^2, xy, y^3).$$

The Taylor resolution of R is

$$0 \longrightarrow S \xrightarrow{d_3} S^3 \xrightarrow{d_2} S^3 \xrightarrow{d_1} S \longrightarrow R \longrightarrow 0$$

where

$$d_3 = \begin{pmatrix} y^2 \\ x \\ -1 \end{pmatrix} \qquad d_2 = \begin{pmatrix} -y & 0 & -y^3 \\ x & -y^2 & 0 \\ 0 & x & x^2 \end{pmatrix} \qquad d_1 = \begin{pmatrix} x^2 \\ xy \\ y^3 \end{pmatrix}^T$$

We note that the Taylor resolution is not minimal in general. Indeed, in the above example $\operatorname{im} d_3 \not\subseteq (x, y)S^3$. However, the following characterization follows readily from the definition of the differential d on T, as first observed in Lemma 4.14 of [3].

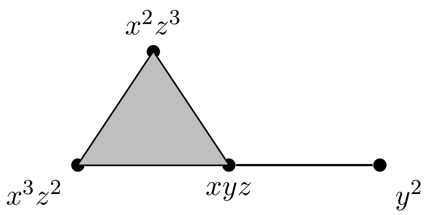
Lemma 3.2.10. Let $I = (m_1, \ldots, m_r)$ be a monomial ideal. Then the Taylor resolution of S/I is minimal if and only if m_i is not a divisor of m_J for all $i \in \{1, \ldots, r\}$ and $J \subseteq \{1, \ldots, r\} \setminus i$.

The following construction is due to Bayer, Peeva and Sturmfels [5]. Our exposition will follow that of Mermin [38]. Let $\{m_1, \ldots, m_r\}$ be a set of monomials. Fix some total order $\prec$ on $\{m_1, \ldots, m_r\}$. After relabelling we may assume that $m_1 \prec m_2 \prec \cdots \prec m_r$. Let I be the ideal generated by $\{m_1, \ldots, m_r\}$. Then the basis sets of the Taylor resolution T of S/I form the full simplex Δ^r on $\{m_1, \ldots, m_r\}$. Given a subcomplex $\Delta \subseteq \Delta^r$, let F_n be the submodule of T_n spanned by the n-simplices of Δ. It follows that $d(F_n) \subseteq F_{n-1}$ and so (F, d) is a subcomplex of (T, d). We will denote this chain complex by F_Δ.

Example 3.2.11. Let $S = k[x, y, z]$ and let

$$I = (x^2z^3, x^3z^2, xyz, y^2).$$

Let Δ be the following simplicial complex.



Then F_Δ is given by:

$$0 \longrightarrow S \xrightarrow{d_3} S^4 \xrightarrow{d_2} S^4 \xrightarrow{d_1} S \longrightarrow S/I \longrightarrow 0$$

where

$$d_3 = \begin{pmatrix} y \\ -x \\ z \\ 0 \end{pmatrix} \qquad d_2 = \begin{pmatrix} -x & -y & 0 & 0 \\ z & 0 & -y & 0 \\ 0 & xz^2 & x^2z & -y \\ 0 & 0 & 0 & xz \end{pmatrix} \qquad d_1 = \begin{pmatrix} x^2z^3 \\ x^3z^2 \\ xyz \\ y^2 \end{pmatrix}^T$$

Remark 3.2.12. The complex of Example 3.2.11 is an example of a general construction called the *Scarf complex*. Given a monomial ideal $I = (m_1, \ldots, m_r)$, define

$$\Delta = \{I \subseteq \{1, \ldots, r\} \mid m_I \neq m_J \text{ for all } J \subseteq \{1, \ldots, r\} \text{ other than } I\}.$$

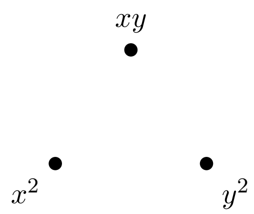
Then Δ is a simplicial complex and the complex F_Δ is called the Scarf complex of I. More details can be found in [5] or [42].

As the following example shows, F_Δ need not be a resolution of S/I.

Example 3.2.13. Let $S = k[x, y]$ and let

$$I = (x^2, xy, y^2).$$

Let Δ be the following simplicial complex.



The corresponding complex F_Δ is given by:

$$0 \longrightarrow S^3 \xrightarrow{d_1} S \longrightarrow S/I \longrightarrow 0$$

where

$$d_1 = \begin{pmatrix} x^2 & xy & y^2 \end{pmatrix}.$$

Since $\ker d_1 \neq 0$, F_Δ is not exact and hence is not a resolution.

However, we have the following theorem which characterizes which simplicial complexes give rise to resolutions. A simplicial complex Δ is called *acyclic* if $\tilde{H}_p(\Delta; k) = 0$ for all $p \in \mathbb{N}$.

Theorem 3.2.14 ([5], Lemma 2.2)**.** Let Δ be a simplicial complex on the vertex set $\{m_1, \ldots, m_r\}$ and define, for a monomial μ, a subcomplex

$$\Delta_\mu = \{J \in \Delta \mid m_J \text{ divides } \mu\}.$$

Then F_Δ is a resolution of R if and only if Δ_μ is either acyclic or empty for all monomials μ.

Definition 3.2.15. Let R be a monomial ring and let $F \to R$ be a free resolution. Then F is called a *simplicial resolution* if $F = F_\Delta$ for some simplicial complex Δ.

Remark 3.2.16. Note that if $\Delta' \subseteq \Delta$, then $F_{\Delta'}$ is a subcomplex of F_Δ. In particular, since each simplicial complex Δ is included in the full simplex on its vertex set, each simplicial resolution of S/I is a subcomplex of the Taylor resolution of S/I.

The following special type of simplicial resolution is due to Novik [41]. Given a monomial ideal $I = (m_1, \dots, m_r)$, we define the *lcm-lattice* $L(I)$ to be the set of all m_J where $J \subseteq \{1, \dots, r\}$. The set $L = L(I)$ admits a partial order given by divisibility. Then L forms a lattice under $a \vee b = \operatorname{lcm}(a, b)$ and $a \wedge b = \gcd(a, b)$. The lattice L has minimal element $\hat{0} = 1$ and maximal element $\hat{1} = \operatorname{lcm}(m_1, \dots, m_r)$.

Definition 3.2.17. A *rooting map* on L is a map

$$\pi\colon L \setminus \{\hat{0}\} \to \{m_1, \dots, m_r\}$$

such that

1. for every $m \in L$, $\pi(m)$ divides m
2. $\pi(m) = \pi(n)$ whenever $\pi(m)$ divides n and n divides m.

Now, let π be a rooting map and let $A \subseteq \{m_1, \dots, m_r\}$ be non-empty. Define $\pi(A) = \pi(\operatorname{lcm}(A))$. A set A is *unbroken* if $\pi(A) \in A$ and A is *rooted* if every non-empty $B \subseteq A$ is unbroken. The following lemma will be used later.

Lemma 3.2.18. Let $J_1 \subseteq \cdots \subseteq J_S \subseteq \{m_1, \dots, m_r\}$ be non-empty. Then

$$\pi(\operatorname{lcm}(\pi(J_1), \dots, \pi(J_S))) = \pi(J_s).$$

Proof. Indeed, we have

$$\pi(J_S) \mid \operatorname{lcm}(\pi(J_1), \dots, \pi(J_S)) \mid \operatorname{lcm}(J_1, \dots, J_s) = \operatorname{lcm}(J_s)$$

and so the result follows. □

Let $RC(L, \pi)$ denote the set of all rooted subsets with respect to L and π. Then $RC(L, \pi)$ is easily seen to be a simplicial complex on vertex set $\{m_1, \dots, m_r\}$ and we have the following result.

Theorem 3.2.19 ([41], Theorem 1)**.** Let $I = (m_1, \dots, m_r)$ be a monomial ideal and let L denote its lcm-lattice. Suppose that π is a rooting map on L. Then the chain complex $F_{RC(L,\pi)}$ associated to the simplicial complex $RC(L, \pi)$ is a free resolution of I.

An important special case of this construction is the Lyubeznik resolution which will be defined next.

Definition 3.2.20. Let $I = (m_1, \dots, m_r)$ be a monomial ideal and let $\prec$ be a total order on the set $\{m_1, \dots, m_r\}$. For $A \subseteq \{m_1, \dots, m_r\}$, define

$$\pi(A) = \min_{\prec}\{m_i \mid m_i \text{ divides } \operatorname{lcm}(A)\}.$$

Then π is easily seen to be a rooting map. The resolution associated $RC(L,\pi)$ is called the *Lyubeznik resolution* with respect to $\prec$.

If the minimal free resolution of S/I is a resolution associated to $RC(L,\pi)$ for some rooting map π, then I (respectively S/I) is called a *rooted ideal* (respectively a *rooted ring*). Similarly, if for some total order $\prec$ the Lyubeznik resolution of S/I with respect to $\prec$ is minimal then I (respectively S/I) is called a *Lyubeznik ideal* (respectively a *Lyubeznik ring*).

Example 3.2.21. Let $S = k[x,y,z]$ and let I be the ideal generated by $m_1 = xy$, $m_2 = yz$ and $m_3 = xz$. Order the generators as $m_1 \prec m_2 \prec m_3$. Let π be the rooting map of the Lyubeznik resolution as in Definition 3.2.20. Then the rooted sets are $\{m_1\}$, $\{m_2\}$, $\{m_3\}$, $\{m_1, m_2\}$ and $\{m_1, m_3\}$. So the Lyubeznik resolution is

$$S^2 \xrightarrow{\quad d_2 \quad} S^3 \xrightarrow{\quad d_1 \quad} S$$

where the differential is given by

$$d_1 = \begin{pmatrix} xy & yz & xz \end{pmatrix}$$

and

$$d_2 = \begin{pmatrix} -z & -z \\ x & 0 \\ 0 & y \end{pmatrix}.$$

In particular, the resolution is minimal and so I is a Lyubeznik ideal.

Definition 3.2.22. Let A be a commutative ring with unit and let M and N be A-modules. Let $F \to M$ be a free resolution of M by A-modules. Define A-modules $\mathrm{Tor}_p^A(M,N)$ by setting

$$\mathrm{Tor}_p^A(M,N) = H_p(F \otimes n).$$

The following theorem shows that the modules $\mathrm{Tor}_p^A(M,N)$ are well-defined.

Theorem 3.2.23 ([44], Corollary 6.21)**.** The modules $\mathrm{Tor}_p^A(M,N)$ do not depend on the choice of free resolution. That is to say, if F and G are free resolutions of M then there exists a natural isomorphism

$$H_p(F \otimes N) \cong H_p(G \otimes N).$$

Definition 3.2.24. Let R be a monomial ring. The *Tor-algebra* $\mathrm{Tor}^S(R,k)$ of R is defined

$$\mathrm{Tor}^S(R,k) = \bigoplus_{n=0}^{\infty} \mathrm{Tor}_n^S(R,k)$$

where the product is induced from the product of the Taylor resolution T of R. More precisely, if $[x], [y] \in \mathrm{Tor}^S(R,k)$ then their product is defined by

$$[x][y] = [xy]$$

where xy is the product in T.

Chapter 4

The Golod property for rooted rings

4.1 A_∞-resolutions and the Golod property

In this section we will introduce the Golod property and discuss how this property is related to A_∞-structures on the minimal free resolution of a monomial ring. We start off with some definitions. As usual, we write $S = k[x_1, \dots, x_m]$ and assume $I = (m_1, \dots, m_r)$ is a monomial ideal.

Definition 4.1.1. Given a monomial ring $R = S/I$ the *Poincaré series* of R is defined as

$$P(R) = \sum_{j=0}^{\infty} \dim \operatorname{Tor}_j^R(k,k)t^j.$$

The fact that $\dim \operatorname{Tor}_j^R(k,k)$ is finite for all j follows from Theorem 11.2 of [42]. A general open-ended problem in commutative algebra is to compute the Poincaré series for various R. A classic result of Serre states that there is always a coefficientwise upper bound

$$P(R) \leq \frac{(1+t)^m}{1 - t(\sum_{j=0}^{\infty} \dim \operatorname{Tor}_j^S(R,k)t^j - 1)}. \tag{4.1.1}$$

Proofs of this fact can be found in [19] and [2]. Note that it follows from Theorem 11.2 of [42] that $\dim \operatorname{Tor}_j^S(R,k)$ is finite for all $j \in \mathbb{N}$.

Definition 4.1.2. A monomial ring R is called *Golod* if (4.1.1) is an equality, that is, if there is an equality of power series

$$P(R) = \frac{(1+t)^m}{1 - t(\sum_{j=0}^{\infty} \dim \operatorname{Tor}_j^S(R,k)t^j - 1)}. \tag{4.1.2}$$

So, a ring R is Golod if and only if the Poincaré series $P(R)$ of R is maximal. The Golod property admits an equivalent description in terms of Massey products which will be crucial in what follows. The *Koszul homology* of a monomial ring R is $H(R) = \mathrm{Tor}^S(R, k)$.

Theorem 4.1.3 ([17], see also Section 4.2 of [19])**.** For any monomial ring R the following are equivalent

1. Equality (4.1.2) holds, that is R is Golod.

2. All Massey products on the Koszul homology are trivial.

Let $r \geq 2$. Following [26], we will say that a dg algebra A satisfies *condition* (B_r) if all k-ary Massey products are defined and contain only zero for all $k \leq r$. Recall the following lemma.

Lemma 4.1.4 ([35], Proposition 2.3)**.** Let A be a dg algebra satisfying (B_{r-1}). Then $\langle a_1, \ldots, a_r \rangle$ is defined and contains only one element for any choice $a_1, \ldots, a_r \in H(A)$.

Before continuing, we recall the construction of the Koszul complex.

Construction 4.1.5. Let E denote the exterior algebra on $u_1, \ldots, u_m$ where $\deg u_i = 1$. The *Koszul complex* K_S of S is the graded module $S \otimes_k E$ together with differential d defined by

$$d(u_{i_1} \cdots u_{i_k}) = \sum_{j=1}^{k} (-1)^{j+1} x_{i_j} u_{i_1} \cdots \widehat{u_{i_j}} \cdots u_{i_k}.$$

Define a multiplication on K_S by concatenation. Then K_S with this multiplication is a dg algebra.

Note that if $I = (x_1, \ldots, x_m)$ then the Taylor resolution T of S/I reduces to the Koszul complex.

Given a monomial ring R, the *Koszul dg algebra* K_R of R is defined to be

$$K_R = R \otimes_k E.$$

Note that K_S is a free resolution of k over S and so we have

$$\mathrm{Tor}^S(R, k) = H(R \otimes_S S \otimes_k E) = H(R \otimes_k E) = H(K_R).$$

We say that a monomial ring R satisfies (B_r) if the Koszul dg algebra K_R of R satisfies (B_r).

Lemma 4.1.6. Let R be a monomial ring. Then R is Golod if and only if R satisfies condition (B_r) for all $r \in \mathbb{N}$.

Proof. It is clear that if R satisfies condition (B_r) for every r then R is Golod. Conversely, suppose that R is Golod. We proceed by induction on r. The case $r = 2$ is trivial. So assume R satisfies (B_{r-1}). By Lemma 4.1.4, the Massey product $\langle a_1, \ldots, a_r \rangle$ is defined and contains only one element for any choice $a_1, \ldots, a_r \in \operatorname{Tor}^S(R, k)$. Since R is Golod, it follows by Theorem 4.1.3 that this element must be zero and so R satisfies (B_r). $\square$

In general it is very hard to study Massey products directly. However, A_∞-algebras provide a systematic way of studying Massey products in view of the following theorem.

Theorem 4.1.7 ([33], Theorem 3.1)**.** Let A be a differential graded algebra. Up to a sign, the higher A_∞-multiplications ν_n on HA from Theorem 2.3.3 give Massey products. That is to say, if $\alpha_1, \ldots, \alpha_n \in HA$ are homology classes such that the Massey product $\langle \alpha_1, \ldots, \alpha_n \rangle$ is defined then

$$\pm \nu_n(\alpha_1 \otimes \cdots \otimes \alpha_n) \in \langle \alpha_1, \ldots, \alpha_n \rangle.$$

Using Theorem 4.1.7, we can describe under what conditions the Massey products on $\operatorname{Tor}^S(R, k)$ vanish.

Definition 4.1.8. Let $F \to R$ be a free resolution and let $\alpha \colon F^{\otimes n} \to F$ be S-linear. Then α is said to be *minimal* if $\alpha \otimes_S 1_k = 0$.

Arguing similarly as in the proof of Lemma 3.2.3 we obtain the following.

Lemma 4.1.9. Let $F \to R$ be a free resolution and let $\alpha \colon F^{\otimes n} \to F$ be S-linear. Then α is minimal if and only if $\operatorname{im} \alpha \subseteq (x_1, \ldots, x_m)F$.

The following theorem will be one of our main tools in the rest of this work.

Theorem 4.1.10. Let $R = S/I$ be a monomial ring with minimal free resolution F. Let μ_n be an A_∞-structure on F and let $r \in \mathbb{N}$. Then R satisfies (B_r) if and only if μ_k is minimal for all $k \leq r$.

Proof. By Proposition 2.17 of [31], it follows that $\mu_n \otimes 1$ is an A_∞-structure on $F \otimes_S k$. Now, assume μ_k is minimal for all $k \leq r$. Since $\operatorname{Tor}^S(R, k)$ is the homology of the A_∞-algebra $F \otimes k$, the homotopy transfer theorem (Theorem 2.3.3) implies that $\operatorname{Tor}^S(R, k)$ inherits an A_∞-structure ν_n. Since F is minimal, $\operatorname{Tor}^S(R, k)$ is isomorphic to $F \otimes k$ and we can take $\nu_n = \mu_n \otimes 1$. Let $k \leq r$ and let $\alpha_1, \ldots, \alpha_k \in \operatorname{Tor}^S(R, k)$ be such that the Massey product $\langle \alpha_1, \ldots, \alpha_k \rangle$ is defined. It follows from Theorem 4.1.7 we have

$$\pm(\mu_k \otimes 1)(\alpha_1, \ldots, \alpha_k) \in \langle \alpha_1, \ldots, \alpha_k \rangle.$$

Since μ_k is minimal, we have $(\mu_k \otimes 1)(\alpha_1, \ldots, \alpha_k) = 0$. Therefore, $\langle \alpha_1, \ldots, \alpha_k \rangle$ is trivial and so R satisfies (B_r).
Conversely, assume that R satisfies (B_r). We need to show that μ_k is minimal for all $k \leq r$. For $k = 2$, we have $(\mu_2 \otimes 1)(a_1, a_2) = a_1 a_2$ but the product on $\operatorname{Tor}^S(R, k)$ is zero as R satisfies (B_r). Now, let $3 \leq k \leq r$. Since R satisfies (B_k), for all $a_1, \ldots, a_k$ the Massey product $\langle a_1, \ldots, a_k \rangle$ is defined and contains only zero. Since $(\mu_k \otimes 1)(a_1, \ldots, a_k) \in$

$\langle a_1, \ldots, a_k \rangle$ we have $(\mu_k \otimes 1)(a_1, \ldots, a_k) = 0$ for all $a_1, \ldots, a_k$. Consequently, μ_k is minimal as required. □

Corollary 4.1.11. Let $R = S/I$ be a monomial ring with minimal free resolution F. Let μ_n be an A_∞-structure on F. Then R is Golod if and only if μ_n is minimal for all $n \geq 1$.

Corollary 4.1.11 was first proved in [9] using different methods. The following immediate corollary to Corollary 4.1.11 is well-known, see for example Proposition 5.2.4(4) of [2] where it is proved using different methods.

Corollary 4.1.12 ([2], Proposition 5.2.4(4))**.** Let $R = S/I$ be a monomial ring with minimal free resolution F. If F admits the structure of a dg algebra, then R is Golod if and only if the product on $\mathrm{Tor}^S(R, k)$ vanishes.

As a special case of Corollary 4.1.12, we consider monomial rings R for which the Taylor resolution is minimal.

Corollary 4.1.13. Let $R = S/I$ be a monomial ring where $I = (m_1, \ldots, m_r)$. Suppose that the Taylor resolution T of R is minimal. Then the following are equivalent.

1. R is Golod.
2. $\gcd(m_i, m_j) \neq 1$ for all $i \neq j$.

Proof. By Corollary 4.1.12 it is sufficient to show that the second statement is equivalent to the vanishing of the product on $\mathrm{Tor}^S(R, k)$. By minimality of T, we have $\mathrm{Tor}^S(R, k) \cong T \otimes_S k$. Now, assume that the product on $\mathrm{Tor}^S(R, k)$ is not trivial. Then there exist $u_I, u_J \in T$ such that $I \cap J = \emptyset$ and

$$u_I u_J = \mathrm{sgn}(I, J) \frac{m_I m_J}{m_{I \cup J}} u_{I \cup J} \notin (x_1, \ldots, x_m)T.$$

Hence, we have

$$\frac{m_I m_J}{m_{I \cup J}} = 1.$$

It follows that $\gcd(m_I, m_J) = 1$. In particular, $\gcd(m_i, m_j) = 1$ for all $i \in I$ and $j \in J$. Conversely, assume that the product on $\mathrm{Tor}^S(R, k)$ is trivial and fix m_i and m_j. Since $u_i u_j = 0$, a similar argument to the above shows that $m_i m_j \neq \mathrm{lcm}(m_i, m_j)$. But then $\gcd(m_i, m_j) \neq 1$ for all i and j and so we are done. □

Corollary 4.1.13 was independently proved in [21] using different methods. This corollary shows that the class of monomial rings which are Golod and have minimal Taylor resolution is quite small. In fact, for flag complexes we have a complete description of this class. Recall that a simplicial complex is called a *flag complex* if every minimal missing face is an edge, i.e. a subset a cardinality two. In the next proposition we write $C(\Delta)$ for the cone on the simplicial complex Δ and we write $\sqcup$ for the disjoint union.

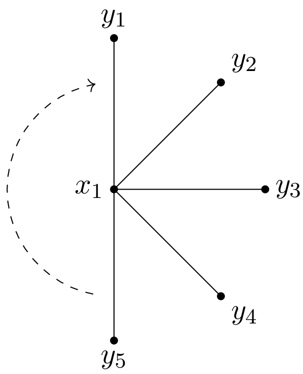
Proposition 4.1.14. Let Δ be a flag complex such that the Taylor resolution of $k[\Delta]$ is minimal. Then $k[\Delta]$ is Golod if and only if

$$\Delta = C^m\big(\Delta^n \sqcup *\big)$$

where $n = |\operatorname{mf}(\Delta)| - 1$ and $m = |V(\Delta)| - |\operatorname{mf}(\Delta)| - 1 = |V(\Delta)| - n - 2$.

Proof. First suppose that Δ is of the form $C^m\big(\Delta^n \sqcup *\big)$. Then the minimal missing faces are of the form $\{*, x_i\}$ where the x_i are the verices of the subcomplex Δ^n. Clearly, $\{*, x_i\} \cap \{*, x_i\} \neq \emptyset$ and so Δ vacuously satisfies the gcd condition. Since the Taylor resolution of Δ is minimal by assumption, it follows from Corollary 4.1.13 that $k[\Delta]$ is Golod.

For the converse, write $\operatorname{mf}(\Delta) = \{f_1, \ldots, f_r\}$. Since Δ is flag, we can write $f_i = \{x_i, y_i\}$. By Corollary 4.1.13 it follows that $\{x_1, y_1\} \cap \{x_2, y_2\} \neq \emptyset$. Without loss of generality we can assume that $x_1 = x_2$ since otherwise we can relabel the elements. Now, let $i \geq 3$ and consider $\{x_i, y_i\}$. Again, by Corollary 4.1.13 it follows that $\{x_1, y_1\} \cap \{x_i, y_i\} \neq \emptyset$. By relabelling x_i and y_i, we may assume that $x_i \in \{x_1, y_1\} \cap \{x_i, y_i\}$. If $x_i = y_1$ then $\{x_1, y_1\} \subset \{x_2, y_2\} \cup \{x_i, y_i\}$ which contradicts minimality. Therefore, $x_1 = \cdots = x_r$ and so the set of minimal missing faces looks like



Now, if $F \subseteq \{y_1, \ldots, y_r\}$ then F is a face of Δ. Indeed, otherwise it would be a minimal missing face as it does not contain any of the f_i. But this is impossible since $f_1, \ldots, f_r$ are all the minimal missing faces of Δ. Consequently, the full subcomplex $\Delta|_{[x_1, y_1, \ldots, y_r]}$ on $\{x_1, y_1, \ldots, y_r\}$ is isomorphic to $\{x_1\} \sqcup \Delta^{r-1}$.

Write $\{z_1, \ldots, z_m\}$ for the vertices of Δ not in $\Delta|_{[x_1, y_1, \ldots, y_r]}$ and note that $m = |V(\Delta)| - |\operatorname{mf}(\Delta)| - 1$. It is sufficient to establish that

$$C\big(\Delta|_{[x_1, y_1, \ldots, y_r, z_1, \ldots, z_{k-1}]}\big) \subseteq \Delta|_{[x_1, y_1, \ldots, y_r, z_1, \ldots, z_k]}$$

since the other inclusion clearly holds. Now, choose an element

$$\{z_k\} \cup F \in C\big(\Delta|_{[x_1, y_1, \ldots, y_r, z_1, \ldots, z_{k-1}]}\big).$$

Then F does not contain any of the f_i. Therefore, $F \in \Delta|_{[x_1,y_1,\ldots,y_r,z_1,\ldots,z_k]}$ since otherwise it would be a minimal missing face of Δ. Taking $k = m$ in the second statement proves the theorem. $\square$

Figure 4.1 illustrates what is going on geometrically. In fact, a complete classification of Golod flag complexes is given in Theorem 4.6 of [18]. The point of Proposition 4.1.14 is, then, to illustrate the restrictiveness of assuming minimality of the Taylor resolution.

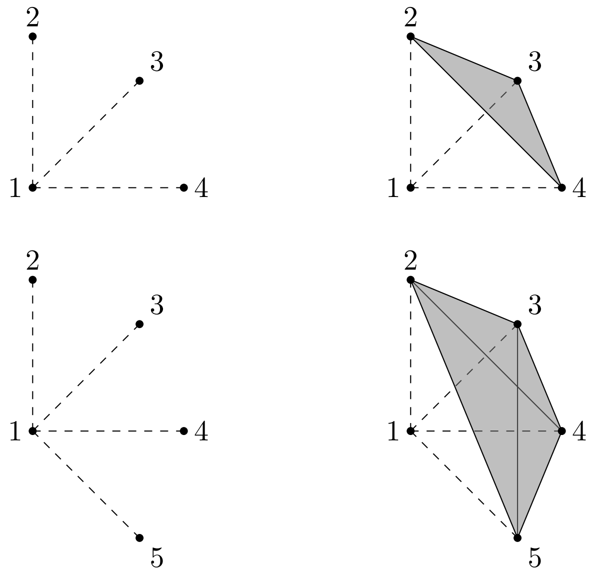


FIGURE 4.1: Proposition 4.1.14 for $n = 2, 3$.

4.2 Homotopy transfer on the Taylor resolution

Corollary 4.1.11 says that monomial rings with minimal dg algebra resolution are Golod if and only the product on $\operatorname{Tor}^S(S/I, k)$ vanishes. However, there exists monomial rings whose minimal resolution does not admit the structure of a dg algebra. The following example is due to Avramov [1].

Example 4.2.1 ([1], Example 1)**.** Let $S = k[x_1, x_2, x_3, x_4]$ and

$$I = (x_1^2, x_1x_2, x_2x_3, x_3x_4, x_4^2).$$

Then the minimal free resolution of S/I does not admit the structure of a dg algebra. The proof of this fact uses obstruction theory based upon the Avramov spectral sequence and is outside the scope of this thesis.

On the other hand, every free resolution of a monomial ring S/I admits an A_∞-structure [9]. In general, it is not clear how to obtain an explicit description of this A_∞-structure. Instead of considering general A_∞-structures on resolutions, we will consider only those that arise as a deformation of the dg algebra structure on the Taylor resolution. To make this idea precise we will use transfer diagrams of the Taylor resolution. In that case Theorem 2.3.6 tells us how to construct an A_∞-structure to which we may apply Corollary 4.1.11.

Recall from Definition 2.3.4 that for a chain complex A and a subcomplex $B \subseteq A$, a transfer diagram is a diagram of the form

$$\begin{tikzcd} B \arrow[r, bend left, "i"] & A \arrow[l, bend left, "p"] \arrow[loop right, "\phi"] \end{tikzcd}$$

where $pi = 1_B$ and $ip - 1 = d\phi + \phi d$. Given a transfer diagram with A a dg algebra, Theorem 2.3.6 applies and we record the following for future reference.

Corollary 4.2.2. Let A be a dg algebra and let

$$\begin{tikzcd} B \arrow[r, bend left, "i"] & A \arrow[l, bend left, "p"] \arrow[loop right, "\phi"] \end{tikzcd}$$

be a transfer diagram. Then the maps μ_n from Theorem 2.3.6 give the structure of an A_∞-algebra on B.

Let $R = S/I$ be a monomial ring. Denote by T the Taylor resolution of R and by F the minimal free resolution of R. We claim that there exists a transfer diagram

$$\begin{tikzcd} F \arrow[r, bend left, "i"] & T \arrow[l, bend left, "p"] \arrow[loop right, "\phi"] \end{tikzcd}$$

To see this. Note that by Lemma 3.2.5, we have that $T \cong F \oplus G$ where G is a trivial complex. Let p be the projection $F \oplus G \to F$ and i the inclusion $F \to F \oplus G$. Clearly, $pi = 1_F$. To show that ip is homotopic to 1_T, it is sufficient to show that 1_G is homotopic to zero as $1_T = 1_{F\oplus G} = 1_F \oplus 1_G$. For this, it is sufficient to show that the identity 1_H on the short trivial complex H

$$0 \longrightarrow Sb \xrightarrow{\ 1\ } Sa \longrightarrow 0$$

is homotopic to zero. Define $\psi(a) = b$. Then $d\psi a = db = a$ and $\psi d = 0$. Therefore, $1_H = d\psi + \psi d$ as required. Therefore, there exists a transfer diagram as above and so Corollary 4.2.2 gives an A_∞-structure on F. The following lemma relates μ_2 to the product on $\mathrm{Tor}^S(S/I, k)$.

Lemma 4.2.3. Let S/I be rooted with minimal free resolution $F \to S/I$ and let μ_n be the A_∞-algebra structure on F from Corollary 4.2.2. Then the map $\mu_2\colon F \otimes F \to F$ induces the product on $\mathrm{Tor}^S(S/I, k)$.

Proof. Consider the following diagram

$$\begin{tikzcd} F \otimes F \arrow[r, "\mu_2"] \arrow[d, "i \otimes i"'] & F \arrow[d, "i"] \\ T \otimes T \arrow[r, "\lambda_2"'] & T \end{tikzcd}$$

where T is the Taylor resolution. Since $\mu_2 = p \circ \lambda_2 \circ (i \otimes i)$, it follows that

$$i \circ \mu_2 = i \circ p \circ \lambda_2 \circ (i \otimes i) \simeq \lambda_2 \circ (i \otimes i).$$

Hence the above square is homotopy commutative. Consequently, by tensoring with the field k and taking homology we get a commutative square as follows.

$$\begin{tikzcd} \mathrm{Tor}^S(S/I, k)^{\otimes 2} \arrow[r, "(\mu_2)_*"] \arrow[d, "i_* \otimes i_*"'] & \mathrm{Tor}^S(S/I, k) \arrow[d, "i_*"] \\ \mathrm{Tor}^S(S/I, k)^{\otimes 2} \arrow[r, "(\lambda_2)_*"'] & \mathrm{Tor}^S(S/I, k). \end{tikzcd}$$

Since $i_* = 1$, we have $p_* = (i_*)^{-1} = 1$. Therefore, $(\mu_2)_* = (\lambda_2)_*$. Since λ_2 induces the product, the same holds for μ_2. $\square$

Remark 4.2.4. In fact, it can be shown that any map $m\colon F \otimes F \to F$ that covers the multiplication on S/I will induce the product on the Tor-algebra [7].

As above, let π be a rooting map and let F be the free resolution of S/I associated to $RC(L, \pi)$. Recall that F_n is the free S-module on u_J with $J \in L$ such that $|J| = n$. The remainder of this section is devoted to an explicit computation of the A_∞-algebra structure on F from Lemma 4.2.3 in terms of the rooting map π. Let T denote the Taylor resolution of S/I. We will write d for the differential of F. Define a second differential ∂ by

$$\partial u_J = \sum_{i=1}^{|J|} (-1)^{i+1} u_{J^i}$$

on a basis element u_J of F. If u_J is a basis element of F we define $[u_J] = \frac{1}{m_J} u_J$. Let $u_{J_1}, \ldots, u_{J_n}$ be rooted sets and $\alpha_1, \ldots, \alpha_n \in S$. Then for $u = \sum \alpha_k u_{J_k}$, we set $[u] = \sum \frac{\alpha_k}{m_{J_k}} u_{J_k}$. The following lemma will be used extensively.

Lemma 4.2.5. For any basis element u_J of F, we have

$$d[u_J] = [\partial u_J].$$

Proof. We have

$$\begin{aligned} d[u_J] &= \frac{1}{m_J} du_J \\ &= \frac{1}{m_J} \sum_{i=1}^{|J|} (-1)^{i+1} \frac{m_J}{m_{J^i}} u_{J^i} \\ &= \sum_{i=1}^{|J|} (-1)^{i+1} \frac{1}{m_{J^i}} u_{J^i} \\ &= \sum_{i=1}^{|J|} (-1)^{i+1} [u_{J^i}] \\ &= [\partial u_J]. \end{aligned}$$

□

Let π be a rooting map. For $u_J \in T$, define $\pi(u_J) = u_i$ if $\pi(m_J) = m_i$. An element $u_J \in T$ is said to be *rooted* if the corresponding m_J is rooted. Define a map $p' \colon T \to F$ as follows. Let $u \in T$ and write $u = u_{i_1} \cdots u_{i_k}$ for some $i_1 < \cdots < i_k$. For $q = 1, \ldots, k$, define $I_q = \{i_1, \ldots, i_q\}$. For a permutation $\sigma \in S_k$, put $u_{\sigma I_q} = u_{i_{\sigma(1)}} \cdots u_{i_{\sigma(q)}}$. We define

$$p'(u) = \sum_{\sigma \in S_k} \operatorname{sgn}(\sigma) \pi(u_{\sigma I_1}) \pi(u_{\sigma I_2}) \cdots \pi(u_{\sigma I_k}). \tag{4.2.1}$$

We need to verify that $\operatorname{im}(p') \subseteq F$. Let $\sigma \in S_k$, we need to verify that

$$\{\pi(m_{\sigma I_1}), \pi(m_{\sigma I_2}), \ldots, \pi(m_{\sigma I_k})\}$$

is rooted. Since

$$m_{\sigma I_1} \mid m_{\sigma I_2} \mid \cdots \mid m_{\sigma I_k}$$

it follows by Lemma 3.2.18 that for all $j_1, \ldots, j_l$ we have

$$\pi(\pi(m_{\sigma I_{j_1}}), \pi(m_{\sigma I_{j_2}}), \ldots, \pi(m_{\sigma I_{j_k}})) = \pi(m_{\sigma I_{j_k}}).$$

Therefore,

$$\{\pi(m_{\sigma I_1}), \pi(m_{\sigma I_2}), \ldots, \pi(m_{\sigma I_k})\}$$

is rooted and so $\operatorname{im}(p') \subseteq F$.

Lemma 4.2.6. The map p' is a chain map with respect to the differential ∂.

Proof. It is sufficient to prove the result for basis elements $u_I \in T$. Write $I = \{i_1, \dots, i_k\}$. We first show that

$$\partial p'(u_I) = \sum_{\sigma \in S_k} (-1)^{k+1} \operatorname{sgn}(\sigma) \pi(u_{\sigma I_1}) \cdots \pi(u_{\sigma I_{k-1}}).$$

We have

$$\begin{aligned}\partial p'(u_I) &= \sum_{\sigma \in S_k} \operatorname{sgn}(\sigma) \partial\big(\pi(u_{\sigma I_1}) \cdots \pi(u_{\sigma I_k})\big) \\ &= \sum_{\sigma \in S_k} \sum_{j=1}^{k} (-1)^{j+1} \operatorname{sgn}(\sigma) \pi(u_{\sigma I_1}) \cdots \widehat{\pi(u_{\sigma I_j})} \cdots \pi(u_{\sigma I_k}).\end{aligned}$$

Now, fix some $j < k$ and let τ_j be the transposition $(\sigma(j), \sigma(j+1))$. Then the summands indexed by σ and $\tau_j \sigma$ cancel. Indeed, if $q < j$ then τ_j acts as the identity on σI_q and so $u_{\sigma I_q} = u_{\tau_j \sigma I_q}$. On the other hand, if $q \geq j+1$ then the underlying sets of σI_q and $\tau_j \sigma I_q$ are the same. Since $\pi(u_J)$ depends only on the set J and not on the ordering we have

$$\pi(u_{\sigma I_q}) = \pi(u_{\tau_j \sigma I_q})$$

and so the summands indexed by σ and $\tau_j \sigma$ cancel. Note that since the map $\sigma \to \tau_j \sigma$ is an involution these permutations cancel in pairs. Therefore, we obtain

$$\partial p'(u_I) = \sum_{\sigma \in S_k} (-1)^{k+1} \operatorname{sgn}(\sigma) \pi(u_{\sigma I_1}) \cdots \pi(u_{\sigma I_{k-1}}).$$

For $\sigma \in S_k$, write

$$G_\sigma = \pi(u_{\sigma I_1}) \cdots \pi(u_{\sigma I_{k-1}})$$

and so

$$\partial p'(u_I) = \sum_{\sigma \in S_k} (-1)^{k+1} \operatorname{sgn}(\sigma) G_\sigma. \tag{4.2.2}$$

Next, we compute $p'\partial(u_I)$. For $j \in \{1, \dots, k\}$ and $\sigma \in S_{k-1}$, set $I_q(j) = I_q \setminus \{j\}$ and

$$F_{\sigma,j} = \pi(u_{\sigma I_1(j)}) \cdots \pi(u_{\sigma I_{j-1}(j)}) \pi(u_{\sigma I_{j+1}(j)}) \cdots \pi(u_{\sigma I_k(j)}).$$

Then

$$p'\partial u = \sum_{j=1}^{k} (-1)^{j+1} p'(u_{I_k(j)}) = \sum_{j=1}^{k} \sum_{\sigma \in S_{k-1}} (-1)^{j+1} \operatorname{sgn}(\sigma) F_{\sigma,j}. \tag{4.2.3}$$

Given $j \in \{1, \dots, k\}$, we can embed S_{k-1} into S_k by fixing j. Therefore, we have

$$p'\partial u = \sum_{j=1}^{k} \sum_{\sigma \in S_{k-1}} (-1)^{j+1} \operatorname{sgn}(\sigma) F_{\sigma,j} = \sum_{j=1}^{k} \sum_{\substack{\sigma \in S_k \\ \sigma(j)=j}} (-1)^{j+1} \operatorname{sgn}(\sigma) F_{\sigma,j}.$$

Now, fix $j \in \{1, \ldots, k\}$ and fix $\sigma \in S_k$ such that $\sigma(j) = j$. Define ρ to be the cycle $(j \cdots k)$ and let $\tau = \sigma\rho$. Then we have $G_\tau = F_{\sigma,j}$ and

$$(-1)^{k+1} \operatorname{sgn}(\tau) G_\tau = (-1)^{2k+j+1} G_{\sigma\rho} = (-1)^{j+1} \operatorname{sgn}(\sigma) F_{\sigma,j}.$$

Since both sums in (4.2.2) and (4.2.3) have $k!$ terms, it follows that they are equal. $\square$

Let $i\colon F \to T$ denote the inclusion.

Lemma 4.2.7. For all $u \in T$, we have

$$\pi(u) i p' \partial u = i p' u.$$

Proof. It is sufficient to prove the result for basis elements $u_I \in T$. Write $I = \{i_1, \ldots, i_k\}$. As in the proof of Lemma 4.2.6, we have

$$\partial p'(u_I) = \sum_{\sigma \in S_k} (-1)^{k+1} \operatorname{sgn}(\sigma) \pi(u_{\sigma I_1}) \cdots \pi(u_{\sigma I_{k-1}}).$$

Since p' is a chain map by Lemma 4.2.6, we have

$$\begin{aligned}
\pi(u_I) i p' \partial u_I &= \pi(u_I) \partial i p'(u_I) \\
&= \pi(u_I) \sum_{\sigma \in S_k} (-1)^{k+1} \operatorname{sgn}(\sigma) \pi(u_{\sigma I_1}) \cdots \pi(u_{\sigma I_{k-1}}) \\
&= \sum_{\sigma \in S_k} (-1)^{k+1+k-1} \operatorname{sgn}(\sigma) \pi(u_{\sigma I_1}) \cdots \pi(u_{\sigma I_{k-1}}) \pi(u_I) \\
&= \sum_{\sigma \in S_k} \operatorname{sgn}(\sigma) \pi(u_{\sigma I_1}) \cdots \pi(u_{\sigma I_{k-1}}) \pi(u_\sigma I_k) \\
&= i p'(u_I)
\end{aligned}$$

where we have used that $\pi(u_I) = \pi(u_{I_k}) = \pi(u_{\sigma I_k})$. $\square$

Note that the differential ∂ is compatible with the action of S_k in the sense that for all $\sigma \in S_k$ and $I = \{i_1 < \ldots < i_k\}$ we have

$$\partial(u_{\sigma I}) = \sum_{j=1}^{k} (-1)^{j+1} u_{i_{\sigma(1)}} \cdots u_{i_{\sigma(k)}}.$$

This fact is used in the following lemma.

Lemma 4.2.8. The composition ip' is chain homotopic to 1_T as chain maps $(T, \partial) \to (T, \partial)$.

Proof. Define $\phi'\colon T \to T$ by induction as follows. Set $\phi'_0 = \phi'_1 = 0$ and

$$\phi'_2(u_{i_1} u_{i_2}) = \pi(u_{i_1}, u_{i_2}) u_{i_1} u_{i_2}$$

where $\pi(u_{i_1}, u_{i_2}) = \pi(\{u_{i_1}, u_{i_2}\})$. For $k > 2$, write $u = u_{i_1} \cdots u_{i_k}$ and define

$$\phi'_k(u) = \pi(u)\big(u - \phi'_{k-1}(\partial u)\big).$$

We need to show that $1_T - ip' = \partial\phi' + \phi'\partial$. We proceed by induction on k. If $k = 1$, there is nothing to prove. If $k = 2$, we have

$$\begin{aligned} \partial\phi'_2(u_{i_1}u_{i_2}) &= \partial(\pi(u_{i_1}, u_{i_2})u_{i_1}u_{i_2}) \\ &= u_{i_1}u_{i_2} - \pi(u_{i_1}, u_{i_2})u_{i_2} + \pi(u_{i_1}, u_{i_2})u_{i_1} \\ &= (1_F - ip')(u_{i_1}u_{i_2}). \end{aligned}$$

Now, let $k > 2$. Using Lemma 4.2.7, we get

$$\begin{aligned} \partial\phi'_k(u) &= u - \phi'_{k-1}\partial u - \pi(u)(\partial u - \partial\phi'_{k-1}\partial u) \\ &= u - \phi'_{k-1}\partial u - \pi(u)\big(\partial u - \partial u + ip'\partial u + \phi'_{k-2}\partial^2 u\big) \\ &= u - \phi'_{k-1}\partial u - \pi(u)ip'\partial u \\ &= u - ip'u - \phi'_{k-1}\partial u \end{aligned}$$

which finishes the proof. □

Define a map $p\colon T \to F$ as follows. For $u_J \in T$, define

$$p(u_J) = m_J[p'(u_J)] \tag{4.2.4}$$

where p' is the map from (4.2.1). Then we have the following theorem.

Theorem 4.2.9. Let π be a rooting map for a monomial ideal I and let F be the resolution of S/I associated to π. Then there exists a strong deformation retract

$$\begin{tikzcd} F \arrow[r, bend left, "i"] & T \arrow[l, bend left, "p"] \arrow[loop right, "\phi"] \end{tikzcd}$$

where $i\colon F \to T$ is the inclusion and $p\colon T \to F$ is the map from (4.2.4).

Proof. Let $u_J \in T$ and define a map ϕ by

$$\phi(u_J) = m_J[\phi'(u_J)].$$

Then, using Lemmas 4.2.5 and 4.2.8, we have

$$\begin{aligned} d\phi(u_J) &= m_J d[\phi'(u_J)] \\ &= m_J[\partial\phi'(u_J)] \\ &= m_J[u_J - ip'u_J - \phi'\partial u_J] \\ &= u_J - ipu_J - \phi du_J \end{aligned}$$

and so 1_T and ip are homotopic. On the other hand, we clearly have $pi = 1_F$ which finishes the proof. □

4.3 The Golod property for rooted rings

Let $R = S/I$ be a rooted ring with rooting map π and minimal free resolution F. The purpose of this section is to provide necessary and sufficient conditions for R being Golod. We start with the following definition where we write $\pi(m_i, m_j)$ for $\pi(\{m_i, m_j\})$.

Definition 4.3.1. Let $R = S/I$ be a rooted ring with rooting map π. Write $I = (m_1, \dots, m_r)$. We say that R is *π-gcd* if $\pi(m_i, m_j) \neq m_i, m_j$ whenever $\gcd(m_i, m_j) = 1$.

The following lemma is straightforward but included for completeness.

Lemma 4.3.2. Let u_I and u_J be basis elements of T such that $\gcd(m_I, m_J) \neq 1$. Then

$$p\lambda_2(u_I \otimes u_J) \in (x_1, \dots, x_m)F.$$

Proof. Indeed, we have

$$\begin{aligned} p\lambda_2(u_I \otimes u_J) &= p\left(\frac{m_I m_J}{m_{I\cup J}} u_{I\cup J}\right) \\ &= \frac{m_I m_J}{m_{I\cup J}} p(u_{I\cup J}). \end{aligned}$$

By assumption

$$\frac{m_I m_J}{m_{I\cup J}} \neq 1$$

and so

$$p\lambda_2(u_I \otimes u_J) = \frac{m_I m_J}{m_{I\cup J}} p(u_{I\cup J}) \in (x_1, \dots, x_m)F.$$

□

We now come to the main theorem of this section.

Theorem 4.3.3. Let R be a rooted ring with rooting map π. Then the following are equivalent.

1. The ring R is Golod.
2. The product on $\mathrm{Tor}^S(R, k)$ vanishes.
3. The ring R is π-gcd.

Proof. The implication $1 \Rightarrow 2$ is immediate from the definition. We first prove $3 \Rightarrow 1$. Let F be the minimal free resolution of R. Then by Theorem 4.2.9 there is a strong deformation retract

$$\begin{tikzcd} F \arrow[r, bend left, "i"] & T \arrow[l, bend left, "p"] \arrow[loop right, "\phi"] \end{tikzcd}$$

where $i\colon F \to T$ is the inclusion and $p\colon T \to F$ is the map from (4.2.4). By Corollary 4.2.2, we obtain an A_∞-structure μ_n on F. From Theorem 4.1.11 it follows that it is sufficient to show that each μ_n is minimal. Recall that $\mu_n = p\lambda_n i^{\otimes n}$, where

$$\lambda_n = \sum_{\substack{s+t=n \\ s,t\geq 1}} (-1)^{s+1}\lambda_2(\phi\lambda_s \otimes \phi\lambda_t).$$

Therefore, it is sufficient to prove that $p\lambda_2$ maps into the maximal ideal. Let u_I and u_J be basis elements of T. We may assume that $\gcd(m_I, m_J) = 1$ since otherwise $p\lambda_2(u_I \otimes u_J) \in (x_1, \dots, x_m)F$ by Lemma 4.3.2. Write $I = \{i_1, \dots, i_k\}$ and $J = \{i_{k+1}, \dots, i_n\}$, where $n = k + l$. By definition of p we have

$$p(u_{i_1} \cdots u_{i_n}) = m[\sum_{\sigma\in S_n} \operatorname{sgn}(\sigma)\pi(u_{\sigma I_1}) \cdots \pi(u_{\sigma I_n})]$$

where $m = \operatorname{lcm}(m_{i_1}, \dots, m_{i_n}) = m_I m_J$ and

$$u_{\sigma I_p} = u_{i_{\sigma(1)}} \cdots u_{i_{\sigma(p)}}.$$

Write

$$\alpha_\sigma = \frac{m}{\operatorname{lcm}(\pi(m_{\sigma I_1}), \dots, \pi(m_{\sigma I_n}))}.$$

Then

$$p(u_{i_1} \cdots u_{i_n}) = \sum_{\sigma\in S_n} \operatorname{sgn}(\sigma)\alpha_\sigma\pi(u_{\sigma I_1}) \cdots \pi(u_{\sigma I_n}).$$

We need to show that $\alpha_\sigma \in (x_1, \dots, x_m)$ for all $\sigma \in S_n$. Suppose $\alpha_\sigma = 1$ for some $\sigma \in S_n$. Since $\lambda_2(u_I \otimes u_j)$ and $\lambda_2(u_J \otimes u_I)$ only potentially differ in sign, we may assume $i_{\sigma(1)} \in I$. Set

$$q = \min\{q' | i_{\sigma(q')} \in J\}.$$

By assumption, $\operatorname{lcm}(\pi(m_{\sigma I_1}), \dots, \pi(m_{\sigma I_n})) = m$ is divisible by $m_{i_{\sigma(q)}}$. Since

$$\gcd(m_{i_{\sigma(q)}}, m_I) = 1,$$

we have

$$\gcd(m_{i_{\sigma(q)}}, \pi(m_{\sigma I_k})) = 1$$

for all $k < q$. Therefore, $\operatorname{lcm}(\pi(m_{\sigma I_q}), \dots, \pi(m_{\sigma I_n}))$ is still divisible by $m_{i_{\sigma(q)}}$. We claim that

$$m_{i_{\sigma(q)}} \notin \{\pi(m_{\sigma I_q}), \dots, \pi(m_{\sigma I_n})\}.$$

Indeed, assume that $m_{i_{\sigma(q)}} = \pi(m_{\sigma I_s})$ for some $s \geq q$. We have that $\pi(m_{\sigma I_s}) = \pi(m_{i_{\sigma(1)}}, \dots, m_{i_{\sigma(s)}})$. Then

$$m_{i_{\sigma(q)}} \,|\, \mathrm{lcm}(m_{i_{\sigma(1)}}, m_{i_{\sigma(q)}}) \,|\, \mathrm{lcm}(m_{i_{\sigma(1)}}, \dots, m_{i_{\sigma(s)}})$$

and so $m_{i_{\sigma(q)}} = \pi(m_{i_{\sigma(1)}}, m_{i_{\sigma(q)}})$ since π is a rooting map. By definition of q, we have $\gcd(m_{i_{\sigma(1)}}, m_{i_{\sigma(q)}}) = 1$ so this contradicts I being π-gcd. Therefore

$$m_{i_{\sigma(q)}} \notin \{\pi(m_{\sigma I_q}), \dots, \pi(m_{\sigma I_n})\}.$$

Define

$$u = u_{i_{\sigma(q)}} \pi(u_{\sigma I_q}) \cdots \pi(u_{\sigma I_n}).$$

We claim that u is in F. To see that u is rooted, let

$$v \subseteq \{u_{i_{\sigma(q)}}, \pi(u_{\sigma I_q}), \dots, \pi(u_{\sigma I_n})\}.$$

If $u_{i_{\sigma(q)}} \notin v$ then there is nothing to prove as

$$\{\pi(u_{\sigma I_q}), \dots, \pi(u_{\sigma I_n})\}$$

is rooted. So, assume $u_{i_{\sigma(q)}} \in v$. We can write

$$v = u_{i_{\sigma(q)}} \pi(u_{\sigma I_{q_1}}) \cdots \pi(u_{\sigma I_{q_k}})$$

for some $q_i \geq q$. We have

$$\pi(u_{\sigma I_{q_k}}) | m_v | m_{\sigma I_{q_k}}$$

and so

$$\pi(v) = \pi(u_{\sigma I_{q_k}}) \in v.$$

Hence, u is rooted as claimed. Since $m_{i_{\sigma(q)}}$ divides $\mathrm{lcm}(\pi(m_{\sigma I_q}), \dots, \pi(m_{\sigma I_n}))$, it follows that

$$\begin{aligned} du &= \frac{\mathrm{lcm}(m_{i_{\sigma(q)}}, \pi(m_{\sigma I_q}), \dots, \pi(m_{\sigma I_n}))}{\mathrm{lcm}(\pi(m_{\sigma I_q}), \dots, \pi(m_{\sigma I_n}))} \pi(u_{\sigma I_q}) \cdots \pi(u_{\sigma I_n}) - d\big(\pi(u_{\sigma I_q}) \cdots \pi(u_{\sigma I_n})\big) \\ &= \pi(u_{\sigma I_q}) \cdots \pi(u_{\sigma I_n}) - d\big(\pi(u_{\sigma I_q}) \cdots \pi(u_{\sigma I_n})\big) \notin (x_1, \dots, x_m)F \end{aligned}$$

But this contradicts minimality of F.

Finally, we prove $2 \Rightarrow 3$. Since the product on $\mathrm{Tor}^S(R, k)$ is just $\mu_2 \otimes 1$, the product vanishes if and only if μ_2 is minimal. Let m_i and m_j be generators such that $\gcd(m_i, m_j) = 1$. Then

$$\mu_2(u_i, u_j) = \frac{\mathrm{lcm}(m_i, m_j)}{\mathrm{lcm}(\pi(m_i, m_j) m_i)} \pi(u_i, u_j) u_i - \frac{\mathrm{lcm}(m_i, m_j)}{\mathrm{lcm}(\pi(m_i, m_j) m_j)} \pi(u_i, u_j) u_j.$$

If $\pi(m_i, m_j) = m_j$, then
$$\frac{\operatorname{lcm}(m_i, m_j)}{\operatorname{lcm}(\pi(m_i, m_j)m_j)} = 1$$
which contradicts minimality of μ_2 and so $\pi(m_i, m_j) \neq m_j$. By the same argument, $\pi(m_i, m_j) \neq m_i$ and thus R is π-gcd. □

Using Theorem 4.3.3, we can reprove Proposition 3.6(1) of [26].

Corollary 4.3.4 ([26], Proposition 3.6(1))**.** Let R be a rooted ring and k' any field. Define $R' = R \otimes_k k'$. Then R is Golod if and only if R' is Golod.

Proof. Being rooted and being π-gcd are both independent of the base field. □

As a special case, we consider rings for which the Lyubeznik resolution is minimal. Let $I = (m_1, \ldots, m_r)$ be a monomial ideal and pick some total order $\prec$ on the m_i. After relabelling, we may assume that $m_1 \prec m_2 \prec \cdots \prec m_r$. Recall from Definition 3.2.20 that the Lyubeznik resolution is the rooted resolution with rooting map
$$\pi(A) = \min_{\prec}\{m_i \mid m_i \text{ divides } \operatorname{lcm}(A)\}.$$
To characterize Golodness for rings for which the Lyubeznik resolution is minimal the following definition is needed.

Definition 4.3.5. Let $I = (m_1, \ldots, m_r)$ be a Lyubeznik ideal with respect to the ordering
$$m_1 \prec m_2 \prec \cdots \prec m_r.$$
Then I is said to satisfy the *gcd condition* if for all generators m_i and m_j with $\gcd(m_i, m_j) = 1$ there exists a $m_k \prec m_i, m_j$ such that m_k divides $\operatorname{lcm}(m_i, m_j)$.

Remark 4.3.6. Note that the gcd condition as defined above is different from the gcd condition in [6].

Lemma 4.3.7. Let I be a Lyubeznik ideal. Then I is π-gcd if and only if I is gcd.

Proof. Let m_i and m_j be generators of I such that $\gcd(m_i, m_j) = 1$. First, suppose that I is π-gcd. Set $m_k = \pi(m_i, m_j)$. Since I is π-gcd, $m_k \neq m_i, m_j$. So, by definition of π, $m_k \prec m_i, m_j$ and m_k divides $\operatorname{lcm}(m_i, m_j)$. Therefore, I is gcd.
Conversely, if I is gcd then there exists some $m_k \prec m_i, m_j$ such that m_k divides $\operatorname{lcm}(m_i, m_j)$. But by definition of π, $\pi(m_i, m_j)$ is the smallest such m_k. Hence, $\pi(m_i, m_j) \preceq m_k \prec m_i, m_j$ and so I is π-gcd. □

Therefore, we immediately have the following consequence of Theorem 4.3.3.

Corollary 4.3.8. Let R be a Lyubeznik ring. Then the following are equivalent.

1. The ring R is Golod.

2. The product on $\operatorname{Tor}^S(R, k)$ vanishes.

3. The ring R is gcd.

Example 4.3.9. Let $S = k[x_1, \ldots, x_9]$ and let I be the ideal

$$(x_2x_5x_8, x_2x_3x_8x_9, x_5x_6x_7x_8, x_1x_2x_4x_5, x_1x_2x_3, x_4x_5x_6, x_7x_8x_9).$$

Label the generators by $u_1, \ldots, u_9$ and order them by $u_1 \prec u_2 \prec \cdots \prec u_9$. Let L be the Lyubeznik resolution with respect to the ordering $\prec$. Then L is easily seen to be minimal. The ideal I satisfies the gcd condition and so S/I is Golod.

Chapter 5

Algebraic Morse theory and A_∞-resolutions

5.1 Algebraic Morse theory

In this section we recall algebraic Morse theory that was independently developed by Sköldberg [45] and Jöllenbeck and Welker [23] based on earlier work by Forman [12] [13]. Our exposition follows that of [45].

Let R be a ring with unit. A *based complex* K is a chain complex (K, d) together with a direct sum decomposition

$$K_n = \bigoplus_{\alpha \in I_n} K_\alpha$$

where the I_n are pairwise disjoint. We will write $\alpha^{(n)}$ to indicate that $\alpha \in I_n$. Let $f\colon K \to K$ be a graded map. We write $f_{\beta,\alpha}$ for the component of f going from K_α to K_β, that is $f_{\beta,\alpha}$ is the composition

$$K_\alpha \longrightarrow K_m \xrightarrow{f} K_n \longrightarrow K_\beta$$

where $K_\alpha \to K_m$ is the inclusion and $K_n \to K_\beta$ the projection. Given a based complex K, define a directed graph G_K with vertex set $\cup_n I_n$ and a directed edge $\alpha \to \beta$ if $d_{\beta,\alpha} \neq 0$. We will only consider situations in which G_K is finite.

A *partial matching* on a directed graph $D = (V, E)$ is a subset A of the edges E such that no vertex is incident to more than one edge of A. We define a new directed graph $D^A = (V, E^A)$ by setting

$$E^A = (E \setminus A) \cup \{\beta \to \alpha \mid \alpha \to \beta \in A\}.$$

That is to say, D^A is the directed graph obtained from D by inverting all the edges in A.

We have the following definition.

Definition 5.1.1. A partial matching M on a directed graph G_K is a *Morse matching* if, for each edge $\alpha \to \beta$ in M, the component $d_{\beta,\alpha}$ is an isomorphism, and furthermore there is a well-founded partial order $\prec$ on each I_n such that $\gamma \prec \alpha$ whenever there is a path $\alpha^{(n)} \to \beta \to \gamma^{(n)}$ in G_K^M

A vertex in G_K^M that is not matched by M is called M-*critical* and we write M^0 for the set of M-critical vertices. Furthermore, define

$$M^- = \{\alpha \mid \beta \to \alpha \in M \text{ for some } \beta\}$$

and

$$M^+ = \{\alpha \mid \alpha \to \gamma \in M \text{ for some } \gamma\}.$$

We will also write

$$M_n^0 = M^0 \cap I_n \qquad M_n^- = M^- \cap I_n \qquad M_n^+ = M^+ \cap I_n$$

The following lemma gives a simpler characterization of Morse matchings.

Lemma 5.1.2 ([45], Lemma 1)**.** Let K be a based complex and let M be a partial matching on G_K such that $d_{\beta,\alpha}$ is an isomorphism whenever $\alpha \to \beta \in M$. Then M is a Morse matching if and only if G_K^M has no directed cycles.

Proof. First, assume that G_K^M contains no directed cycles. Given a vertex $u \in G_K^M$, set

$$l(u) = \max\{s \mid u^{(n)} = u_0^{(n)} u_1 u_2^{(n)} \cdots u_{s-1} u_s^{(n)} \text{ is a direct path in } G_K^M\}.$$

Since there are no directed cycles $l(u)$ is finite for all u. Define a well-founded partial order $\prec$ by

$$u^{(n)} \prec v^{(n)} \text{ if } l(u) < l(v).$$

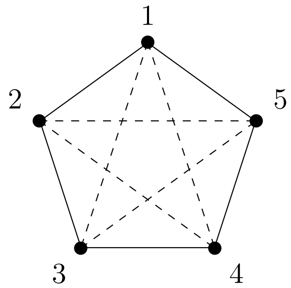
Therefore, M is a Morse matching. Conversely, suppose

$$u_1 u_2 \cdots u_n$$

is a directed cycle in G_K^M. Then $u_1 = u_n$. Since M is a partial matching, at most one of $u_i u_{i+1}$ and $u_{i+1} u_{i+2}$ is the reverse of an edge in M. Since $\deg(d) = -1$ and $u_1 = u_n$, it follows that the reverse of every second edge is in M. So if M is a Morse matching then $u_1 = u_n \prec u_1$ which is a contradiction. Thus, M is not a Morse matching. $\square$

Definition 5.1.3. Let K be a based complex. Denote the edges of G_K by E. A Morse matching M on G_K is called *maximal* if no proper super set $M \subseteq M' \subseteq E$ is a Morse matching.

Example 5.1.4. Let Δ be the 5-gon labeled as



The Stanley-Reisner ring of Δ is

$$k[\Delta] = k[x_1, x_2, x_3, x_4, x_5]/(x_1x_3, x_1x_4, x_2x_4, x_2x_5, x_3x_5).$$

Let $T \to k[\Delta]$ denote the Taylor resolution of $k[\Delta]$. Denote the generators respectively by $u_1, \dots, u_5$. Given $J = \{j_1, \dots, j_k\}$, write $u_J = u_{j_1} \cdots u_{j_k}$. Figure 5.1 depicts the graph G_T corresponding to I. Here, the red arrows (both solid and dashed) are invertible. The solid red arrows give an example of a Morse matching on T.

Given a Morse matching M on a based complex K, our next goal is to define a map $\phi\colon K \to K$ of degree 1 and show that it is a splitting homotopy in the sense of [4]. We recall the following definition from [4].

Definition 5.1.5. Let K be a chain complex and $\phi\colon K \to K$ a degree 1 map. Then ϕ is called a *splitting homotopy* if

$$\begin{aligned} \phi^2 &= 0, \\ \phi d \phi &= \phi. \end{aligned}$$

Define ϕ by induction as follows. If α is minimal with respect to $\prec$ and $x \in K_\alpha$, put

$$\phi(x) = \begin{cases} d^{-1}_{\alpha,\beta}(x) & \text{if } \beta \to \alpha \in M \text{ for some } \beta, \\ 0 & \text{otherwise.} \end{cases}$$

If α is not minimal with respect to $\prec$ and $x \in K_\alpha$, put

$$\phi(x) = \begin{cases} d^{-1}_{\alpha,\beta}(x) - \sum_{\substack{\beta \to \gamma \\ \gamma \neq \alpha}} \phi d_{\gamma,\beta} d^{-1}_{\alpha,\beta}(x) & \text{if } \beta \to \alpha \in M \text{ for some } \beta, \\ 0 & \text{otherwise.} \end{cases}$$

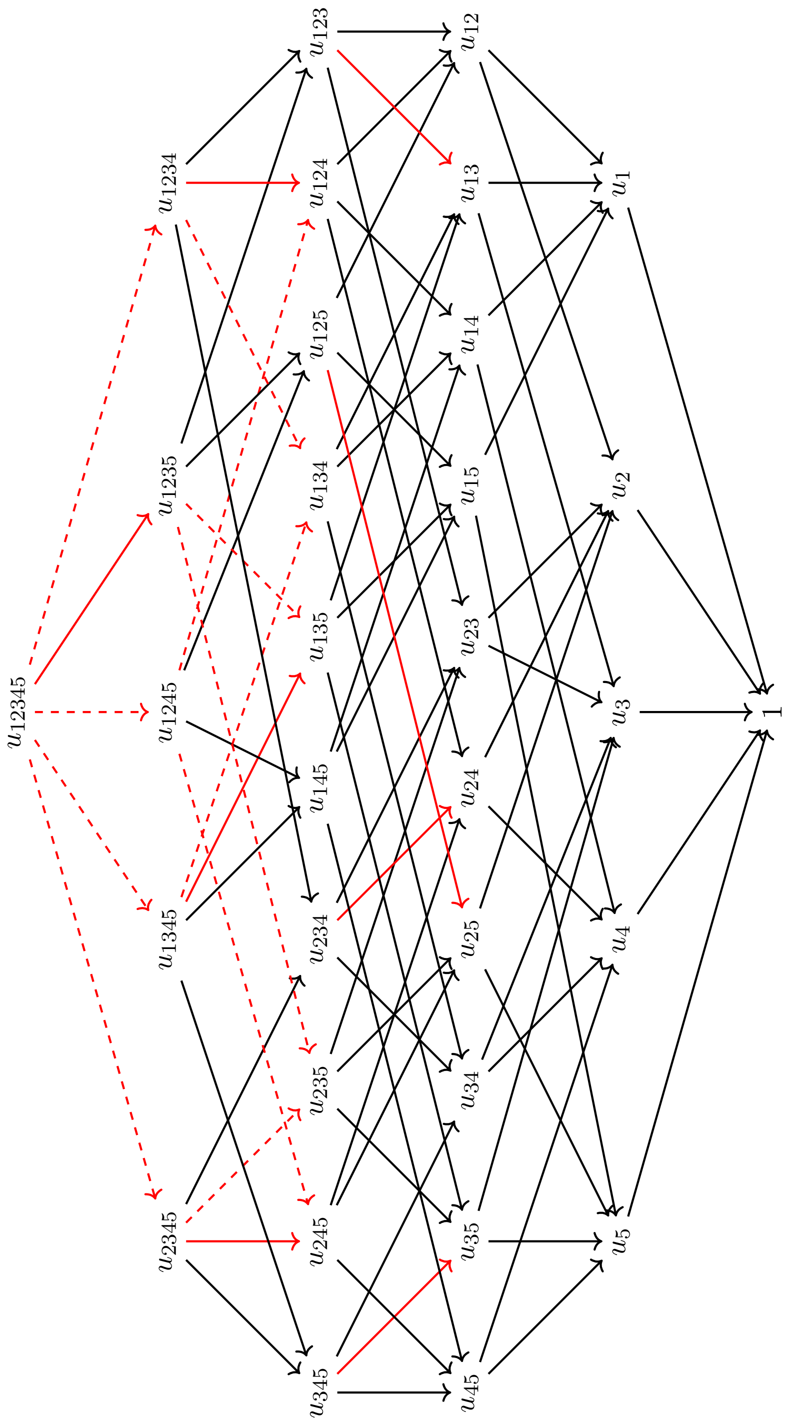


FIGURE 5.1: The graph G_T corresponding to the face ideal I of the 5-gon.

Note that for all γ in the last sum we have $\gamma \prec \alpha$ and so ϕ is well-defined. Observe that the second definition of ϕ is only relevant if there is a subgraph of the form

$$\begin{tikzcd} \beta \arrow[d] \arrow[dr] & \phi(\gamma) \arrow[d] \\ \alpha & \gamma \end{tikzcd}$$

where the red arrows are elements of the matching $\mathcal{M}$.

Lemma 5.1.6 ([45], Lemma 2)**.** Let M be a Morse matching on a based complex K. Then the map ϕ is a splitting homotopy.

Proof. We need to check that $\phi^2 = 0$ and that $\phi d \phi = \phi$. For the first statement, recall that every vertex of G_K is contained in at most one edge of M. Therefore, the statement follows immediately from the definition of ϕ.

For the second statement, proceed by induction on $\prec$. First, assume that α is minimal. If $\alpha \notin M^-$, then $\phi(x) = 0$ for all $x \in K_\alpha$ and hence also $\phi d \phi(x) = 0$. So, we may assume that $\alpha \in M^-$. By definition, this means that there exists some β such that

$$\beta \to \alpha \in M.$$

Compute

$$\phi d \phi(x) = \phi d d^{-1}_{\alpha,\beta}(x) = \phi d_{\alpha,\beta} d^{-1}_{\alpha,\beta} = \phi(x)$$

for all $x \in K_\alpha$. Next, suppose that α is not minimal. Again, if $\alpha \notin M^-$ then $\phi(x) = 0$ and we are done. Suppose $\alpha \in M^-$ and $\beta \to \alpha \in M$. We compute

$$\begin{aligned}
\phi d \phi(x) &= \phi d (d^{-1}_{\alpha,\beta} - \sum_{\substack{\beta\to\gamma \\ \gamma\neq\alpha}} \phi d_{\gamma,\beta} d^{-1}_{\alpha,\beta}(x)) \\
&= \phi d d^{-1}_{\alpha,\beta}(x) - \sum_{\substack{\beta\to\gamma \\ \gamma\neq\alpha}} \phi d \phi d_{\gamma,\beta} d^{-1}_{\alpha,\beta}(x) \\
&= \phi(d_{\alpha,\beta} d^{-1}_{\alpha,\beta}(x) + \sum_{\substack{\beta\to\gamma \\ \gamma\neq\alpha}} d_{\gamma,\beta} d^{-1}_{\alpha,\beta}(x)) - \sum_{\substack{\beta\to\gamma \\ \gamma\neq\alpha}} \phi d_{\gamma,\beta} d^{-1}_{\alpha,\beta}(x) \\
&= \phi d_{\alpha,\beta} d^{-1}_{\alpha,\beta}(x) + \sum_{\substack{\beta\to\gamma \\ \gamma\neq\alpha}} \phi d_{\gamma,\beta} d^{-1}_{\alpha,\beta}(x) - \sum_{\substack{\beta\to\gamma \\ \gamma\neq\alpha}} \phi d_{\gamma,\beta} d^{-1}_{\alpha,\beta}(x) \\
&= \phi(x)
\end{aligned}$$

Note that the penultimate equality holds since the sum runs over $\gamma \prec \alpha$ and so by induction $\phi d \phi(y) = \phi(y)$ for all $y \in K_\gamma$. $\square$

Define a map $p\colon K \to K$ by

$$p = 1_K - (\phi d + d\phi).$$

Lemma 5.1.7. The map p is an idempotent, that is, $p^2 = p$.

Proof. We compute

$$\begin{aligned} p^2 &= (1 - (\phi d + d\phi))^2 \\ &= 1 - 2(\phi d + d\phi) + \phi d\phi d + \phi dd\phi + d\phi\phi d + d\phi d\phi \\ &= 1 - 2(\phi d + d\phi) + \phi d + d\phi \\ &= 1 - (\phi d + d\phi) \\ &= p. \end{aligned}$$

□

From now on, let $L = \mathrm{im}(p)$. Note that we have

$$\begin{aligned} dp &= d(1 - \phi d - d\phi) \\ &= d - d\phi d \\ &= (1 - \phi d - d\phi)d \\ &= pd. \end{aligned}$$

So p is a chain map. Consequently, we have a splitting of chain complexes

$$K = \ker(p) \oplus L.$$

Lemma 5.1.8. There exists a transfer diagram

$$\begin{tikzcd} L \arrow[r, bend left, "i"] & K \arrow[l, bend left, "p"] \arrow[loop right, "\phi"] \end{tikzcd}$$

where i is the inclusion.

Proof. We first show that $\phi i = 0$. Indeed, we have

$$\phi p = \phi(1_K - (\phi d + d\phi)) = \phi - \phi^2 d - d\phi d = \phi - 0 - \phi = 0.$$

Since i is a chain map, it follows that

$$\phi di = \phi id = 0.$$

Therefore,

$$pi = (1 - \phi d - d\phi)i = 1.$$

By definition of p, we have $ip \simeq 1_L$ which finishes the proof. □

Lemma 5.1.9. Suppose K is a based complex and M is a Morse matching on G_K. Let ϕ denote the corresponding splitting homotopy. Then, for $x_\alpha \in K_\alpha$, we have

$$d\phi(x_\alpha) = \begin{cases} x_\alpha + \sum_{\beta \prec \alpha} y_\beta & \text{if } \alpha \in M^- \\ 0 & \text{otherwise} \end{cases}$$

where $y_\beta \in K_\beta$.

Proof. As before, we proceed by induction on $\prec$. First, assume that α is minimal. If $\alpha \notin M^-$ then $\phi(x_\alpha) = 0$ and so there is nothing to prove. So, assume $\alpha \in M^-$. This means that there exists some β and an edge $\beta \to \alpha$ in M. We have

$$d\phi x_\alpha = dd_{\alpha,\beta}^{-1}(x_\alpha) = d_{\alpha,\beta}d_{\alpha,\beta}^{-1}(x_\alpha) = x_\alpha.$$

For the induction step, assume α is not minimal. Clearly, $d\phi(x) = 0$ if $\alpha \notin M^-$ so assume $\alpha \in M^-$. Then there is some $\beta \to \alpha$ in M and we have

$$\begin{aligned} d\phi(x_\alpha) &= d(d_{\alpha,\beta}^{-1}(x_\alpha) - \sum_{\substack{\beta\to\gamma \\ \gamma\neq\alpha}} \phi d_{\gamma,\beta}d_{\alpha,\beta}^{-1}(x_\alpha)) \\ &= x_\alpha + \sum_{\substack{\beta\to\gamma \\ \gamma\neq\alpha}} d_{\gamma,\beta}d_{\alpha,\beta}^{-1}(x_\alpha) - d\phi \sum_{\substack{\beta\to\gamma \\ \gamma\neq\alpha}} d_{\gamma,\beta}d_{\alpha,\beta}^{-1}(x_\alpha). \end{aligned}$$

All terms in the right two sums satisfy $\gamma \prec \alpha$ proving the lemma. □

Lemma 5.1.10. Suppose K is a based complex and M is a Morse matching on G_K. Let ϕ denote the corresponding splitting homotopy. Then, for $x_\alpha \in K_\alpha$, we have

$$\phi d(x_\alpha) = \begin{cases} x_\alpha & \text{if } \alpha \in M^+ \\ \sum_{\beta \prec \alpha} y_\beta & \text{otherwise} \end{cases}$$

where $y_\beta \in K_\beta$.

Proof. As in the last two lemmas we proceed by induction on $\prec$. First, assume that α is minimal. If $\alpha \notin M^+$ then, since α is minimal, we have

$$\phi d(x_\alpha) = \phi \sum_{\alpha\to\beta} d_{\beta,\alpha}(x_\alpha) = 0.$$

On the other hand, if $\alpha \in M^+$ then there exists some $\alpha \to \beta \in M$ and we have

$$\phi d(x_\alpha) = \phi d_{\beta,\alpha}(x_\alpha) + \sum_{\substack{\alpha\to\gamma \\ \alpha\neq\beta}} \phi d_{\gamma,\alpha}(x_\alpha) = d_{\beta,\alpha}^{-1}d_{\beta,\alpha}(x_\alpha) = x_\alpha.$$

This proves the base case. For the induction step, assume α is not minimal. First, suppose $\alpha \notin M^+$. Then we have

$$\phi d(x_\alpha) = \phi \sum_{\alpha\to\beta} d_{\beta,\alpha} x_\alpha = \sum_{\substack{\alpha\to\beta\\ \gamma\to\beta\in M}} d_{\beta,\gamma}^{-1} d_{\beta,\alpha}(x_\alpha) - \phi \sum_{\substack{\alpha\to\beta\\ \gamma\to\beta\in M}} \sum_{\substack{\gamma\to\delta\\ \delta\neq\beta}} d_{\delta,\gamma} d_{\beta,\gamma}^{-1} d_{\beta,\alpha}(x_\alpha).$$

Note that for all δ we have $\delta \prec \alpha$ as required.

Now, suppose $\alpha \in M^+$ and let $\alpha \to \beta \in M$. Compute

$$\begin{aligned} \phi d(x_\alpha) &= \phi d_{\beta,\alpha}(x_\alpha) + \sum_{\substack{\alpha\to\gamma\\ \gamma\neq\beta}} \phi d_{\gamma,\alpha}(x_\alpha) \\ &= d_{\beta,\alpha}^{-1} d_{\beta,\alpha}(x_\alpha) - \sum_{\substack{\alpha\to\gamma\\ \gamma\neq\beta}} \phi d_{\gamma,\alpha}(x_\alpha) + \sum_{\substack{\alpha\to\gamma\\ \gamma\neq\beta}} \phi d_{\gamma,\alpha}(x_\alpha) \\ &= x_\alpha \end{aligned}$$

which finishes the proof. □

The following theorem is one of the central results of algebraic Morse theory.

Theorem 5.1.11 ([45], Theorem 1)**.** Let M be a Morse matching on a based complex K. Then the complexes K and $p(K)$ are homotopy equivalent. Furthermore, the map

$$p\colon \bigoplus_{\alpha\in M_n^0} K_\alpha \to L_n \tag{5.1.1}$$

is an isomorphism of modules for every $n \in \mathbb{N}$.

Proof. We start with the first statement. We have seen that p is a chain map and by definition p is homotopic to the identity map. Consequently, K and $p(K)$ are homotopy equivalent.

Next, we claim that

$$p(K) = p\Big(\bigoplus_{\gamma\in M^0} K_\gamma\Big). \tag{5.1.2}$$

It is suffcient to show that

$$p(K_\alpha) \subseteq p\Big(\bigoplus_{\gamma\in M^0} K_\gamma\Big)$$

for all α. We proceed by induction on $\prec$.

First, assume that α is minimal. If $\alpha \in M^0$, there is nothing to prove. If $\alpha \notin M^0$, then by Lemma 5.1.9 and Lemma 5.1.10 it follows that $p(x) = 0$ for all $x \in K_\alpha$.

Now, assume α is not minimal. Again, if $\alpha \in M^0$ then there is nothing to prove. So assume $\alpha \notin M^0$ and let $x \in K_\alpha$. Applying Lemmas 5.1.9 and 5.1.10, there exists a set

J such that $\gamma \prec \alpha$ for all $\gamma \in J$ and

$$p(x) = p^2(x) = p(\sum_{\gamma \in J} y_\gamma) = \sum_{\substack{\gamma \in J \\ \gamma \in M^0}} p(y_\gamma) + \sum_{\substack{\gamma \in J \\ \gamma \notin M^0}} p(y_\gamma)$$

where $y_\gamma \in K_\gamma$. Therefore, (5.1.2) follows by induction.

Next, we show that the map

$$p \colon \bigoplus_{\alpha \in M_n^0} K_\alpha \to K_n$$

is injective. Applying Lemmas 5.1.9 and 5.1.10, it follows that for all $x \in K_\alpha$ where $\alpha \in M^0$ we have

$$p(x) = x + \sum_{\gamma \in J} y_\gamma$$

where $y_\gamma \in K_\gamma$ and $\gamma \prec \alpha$ for all $\gamma \in J$. Since $\gamma \prec \alpha$, it follows that for degree reasons there can be no cancellation. Therefore, p is injective. □

Note that in general the isomorphism (5.1.1) is only an isomorphism of graded modules and not of chain complexes. In case the components corresponding to the critical vertices do form a subcomplex we have the following corollary.

Corollary 5.1.12 ([45], Corollary 2)**.** Suppose that M is a Morse matching on K such that

$$C = \bigoplus_{\alpha \in M_n^0} K_\alpha$$

is a subcomplex of K. Then K and C are homotopy equivalent.

It follows from Theorem 5.1.11 that C admits a differential $\tilde{d}$ such that $(C, \tilde{d})$ is isomorphic to (L, d). Indeed, define

$$q \colon K \to C$$

on $x \in K_\alpha$ by

$$q(x) = \begin{cases} x & \text{if } \alpha \in M_0 \\ 0 & \text{otherwise.} \end{cases}$$

The map q is called the *projection on the critical cells*. Next, $\tilde{d}$ by

$$\tilde{d} = q(d - d\phi d).$$

Then we have the following theorem.

Theorem 5.1.13 ([45], Theorem 2)**.** The complex $(C, \tilde{d})$ is homotopy equivalent to (K, d).

Proof. Applying Lemmas 5.1.9 and 5.1.10, it follows that $qp = 1$ on $\oplus_n C_n$. Now let $y \in p(K)$, say $y = p(x)$ where $x \in \oplus_{\alpha \in M^0} K_\alpha$. We have

$$pq(y) = pqp(x) = p(x) = y$$

and so $p(K)$ and C are isomorphic as graded modules. Define $\tilde{d} = qdp$. Then $p(K)$ and C become isomorphic as chain complexes. Now,

$$qdp = qd(1 - d\phi - \phi d) = q(d - d\phi d)$$

as required. Since $p(K)$ and K are homotopy equivalent by Theorem 5.1.11 the result follows. □

Definition 5.1.14. Let K be a complex and M a Morse matching on K. Let C be as in Corollary 5.1.12. The complex $(C, \tilde{d})$ is called the *Morse complex* of K associated to M. Given a Morse matching M on K, we will write G^M for the Morse complex of K associated to M.

5.2 A_∞-resolutions via algebraic Morse theory

The following theorem shows how Morse matchings on a differential graded algebra give A_∞-structures.

Theorem 5.2.1. Let A be a differential graded algebra and let M be a Morse matching on A. Let ϕ and p be as before and set $B = \mathrm{im}(p)$. Then B has the structure of an A_∞-algebra.

Proof. By Lemma 5.1.8 there exists a transfer diagram

$$\begin{tikzcd} L \arrow[r, bend left, "i"] & K \arrow[l, bend left, "p"] \arrow[loop right, "\phi"] \end{tikzcd}$$

where i is the inclusion. Therefore, the result follows from Theorem 2.3.6. □

Given two Morse matchings $\mathcal{M}_1$ and $\mathcal{M}_2$ on a dg algebra A, we want to know how the corresponding A_∞-structures $\{\mu_n^1\}$ and $\{\mu_n^2\}$ are related. We start with the following theorem.

Theorem 5.2.2 ([30])**.** Let $f\colon (V, d_V) \to (W, d_W)$ be a chain homotopy equivalence. Then any A_∞ structure on W transfers to an A_∞-structure on V such that f extends to an A_∞-morphism with $f_1 = f$ which is an A_∞-homotopy equivalence.

We have the following result.

Theorem 5.2.3. Let A be a dg algebra and $\mathcal{M}_1$ and $\mathcal{M}_2$ two Morse matchings on A. Put $A_i = \operatorname{im} p_i$ and denote by $\{\mu_n^i\}$ the corresponding A_∞-structure. Then there exists an A_∞-homotopy equivalence

$$f\colon (A_1, \{\mu_n^1\}) \to (A_2, \{\mu_n^2\}).$$

Proof. Consider the following commutative diagram

$$\begin{tikzcd}
A \arrow[r, "1"] \arrow[d, shift right, "p_1"'] & A \arrow[d, shift right, "p_2"'] \\
A_1 \arrow[u, shift right, "i_1"'] \arrow[r, shift left, "p_2 i_1"] & A_2. \arrow[u, shift right, "i_2"'] \arrow[l, shift left, "p_1 i_2"]
\end{tikzcd}$$

Here, p_i is defined as

$$p_i = 1 - d\phi_i - \phi_i d$$

where ϕ is the splitting homotopy associated to $\mathcal{M}_i$. By definition, we have

$$(p_2 i_1)(p_1 i_2) \simeq p_2 i_2 = 1_{A_2}$$

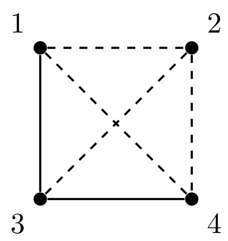
since $i_1 p_1$ is chain homotopic to the identity. Similarly, $(p_1 i_2)(p_2 i_1) \simeq 1_{A_1}$. Therefore, $p_2 i_1$ is a chain homotopy equivalence and so the result follows from the previous theorem. $\square$

Remark 5.2.4. Note that if $p_2 i_1$ is an isomorphism of chain complexes then it extends to an A_∞-isomorphism by a similar argument.

Before we proceed it will be instructive to look at a fully worked example. For this purpose, let $S = k[x_1, x_2, x_3, x_4]$ and let I be the ideal generated by

$$u_1 = x_1 x_2, \qquad u_2 = x_2 x_3, \qquad u_3 = x_2 x_4, \qquad u_4 = x_1 x_4.$$

That is to say, $R = S/I$ is the Stanley-Reisner ring of the following simplicial complex.



Let T denote the Taylor resolution of I. Figure 5.2 shows the graph G_T. In Figure 5.2 the red arrows are those for which $d_{\alpha,\beta}$ is an isomorphism and, hence, which are allowed to be in a Morse matching. The solid red arrows give a specific example of a Morse

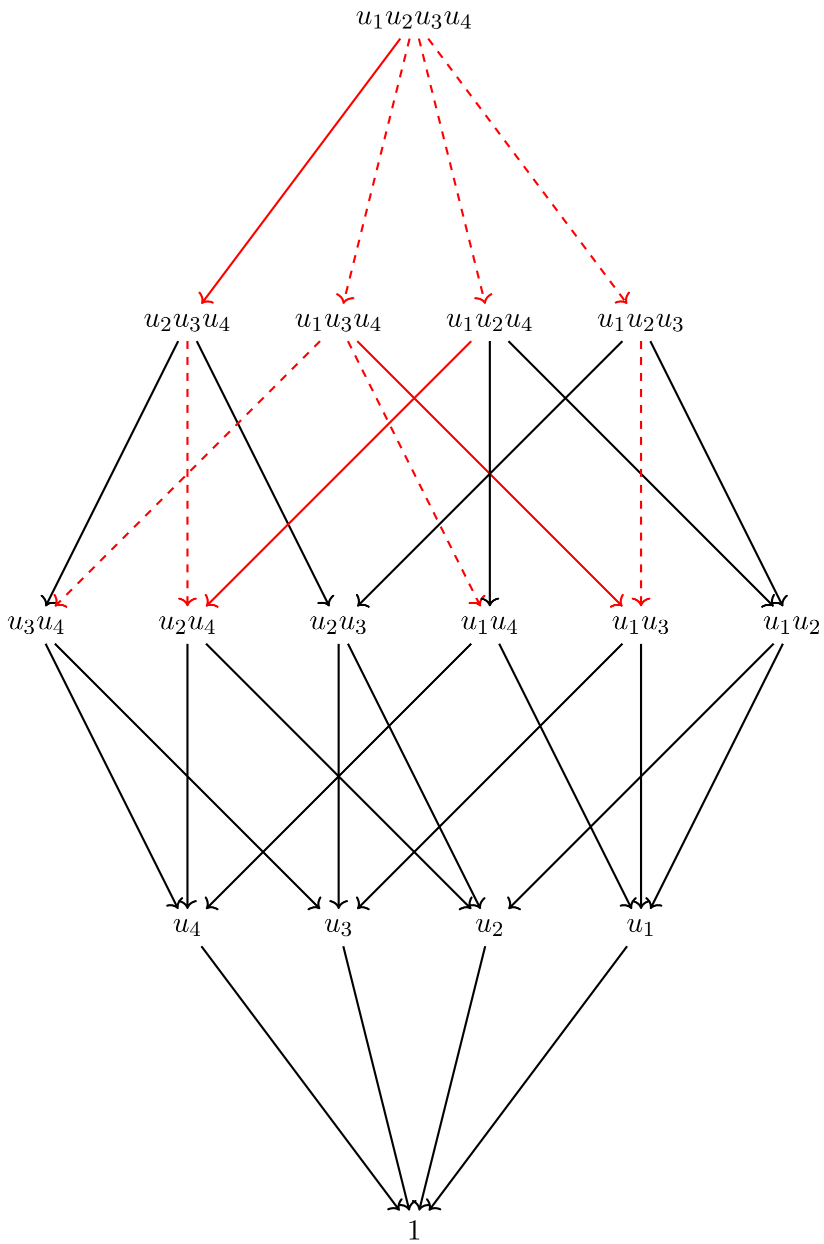


FIGURE 5.2: The graph G_T corresponding to the ideal I.

matching. Computing ϕ we obtain

$$\begin{cases} \phi(u_2u_4) = u_1u_2u_4, \\ \phi(u_1u_3) = u_1u_3u_4, \\ \phi(u_2u_3u_4) = u_1u_2u_3u_4. \end{cases}$$

In all other cases, ϕ is zero. For p, we compute

$$\begin{aligned} p(u_2u_4) &= (1 - d\phi - \phi d)(u_2u_4) \\ &= u_2u_4 - d(u_1u_2u_4) \\ &= u_2u_4 - u_2u_4 + x_3u_1u_4 - x_4u_1u_2 \\ &= x_3u_1u_4 - x_4u_1u_2. \end{aligned}$$

Similarly, we have

$$\begin{aligned} p(u_1u_3) &= (1 - d\phi - \phi d)(u_1u_3) \\ &= u_1u_3 - d(u_1u_3u_4) \\ &= u_1u_3 - u_3u_4 + u_1u_4 - u_1u_3 \\ &= u_1u_4 - u_3u_4 \end{aligned}$$

and

$$\begin{aligned} p(u_2u_3u_4) &= (1 - d\phi - \phi d)(u_2u_3u_4) \\ &= u_2u_3u_4 - d(u_1u_2u_3u_4) - x_3\phi(u_3u_4) + \phi(u_2u_4) - x_1\phi(u_2u_3) \\ &= u_2u_3u_4 - u_2u_3u_4 + u_1u_3u_4 - u_1u_2u_4 + u_1u_2u_3 + u_1u_2u_4 \\ &= u_1u_2u_3 + u_1u_3u_4. \end{aligned}$$

Further, we have $p(u_i) = u_i$ and $p(u_1u_2u_3u_4) = 0$. Also, we have

$$\begin{cases} p(u_3u_4) = u_3u_4, \\ p(u_2u_3) = u_2u_3, \\ p(u_1u_4) = u_1u_4, \\ p(u_1u_2) = u_1u_2 \end{cases}$$

and

$$\begin{cases} p(u_1u_3u_4) = 0, \\ p(u_1u_2u_4) = u_1u_2u_4 - \phi(u_2u_4 - x_3u_1u_4 + x_4u_1u_2) = 0. \end{cases}$$

Lastly, we have

$$\begin{aligned} p(u_1u_2u_3) &= (1 - d\phi - \phi d)(u_1u_2u_3) \\ &= u_1u_2u_3 - x_1\phi(u_2u_3) + x_3\phi(u_1u_3) - x_4\phi(u_1u_2) \\ &= u_1u_2u_3 + x_3u_1u_3u_4. \end{aligned}$$

Consequently, $\mathrm{im}(p)$ is equal to

$$0 \longrightarrow S \xrightarrow{d} S^4 \xrightarrow{d} S^4 \xrightarrow{d} S \longrightarrow 0$$

with basis

degree	generators
0	1
1	$u_1,\ u_2,\ u_3,\ u_4$
2	$u_1u_2,\ u_1u_4,\ u_2u_3,\ u_3u_4$
3	$u_1u_2u_3 + x_3u_1u_3u_4$

Figure 5.3 depicts the full table for the multiplication $\mu_2 = p\lambda_2$.

We have seen how Morse matchings give rise to A_∞-structures on $\operatorname{im}(p)$. Our next goal is to describe A_∞-structure on the actual Morse complex. This will allow us to study these structures in terms of the critical vertices of the Morse matching.

Let A be a differential graded algebra and let $\mathcal{M}$ be a Morse matching on A with corresponding splitting homotopy ϕ. Define

$$p = 1 - d\phi - \phi d$$

as before. We have seen that there is a transfer diagram

$$\begin{tikzcd} \operatorname{im}(p) \arrow[r, bend left, "i"] & A \arrow[l, bend left, "p"] \arrow[loop right, "\phi"] \end{tikzcd}$$

where i is the inclusion. Consequently, the multiplication $\lambda\colon A^2 \to A$ induces an A_∞-structure μ_n on $\operatorname{im}(p)$ via the Merkulov construction from Theorem 2.3.6. Let $A^{\mathcal{M}}$ denote the Morse complex. Then we have a diagram

$$\begin{tikzcd} A \arrow[rr, "1"] \arrow[d, shift right, "p"'] & & A \arrow[d, shift right, "q"'] \\ \operatorname{im}(p) \arrow[u, shift right, "i"'] \arrow[rr, shift left, "qi"] & & A^{\mathcal{M}} \arrow[u, shift right, "j"'] \arrow[ll, shift left, "pj"] \end{tikzcd}$$

where q is the projection on the critical cells. Recall from the proof of Theorem 5.1.13 that the differential $\tilde{d}$ on $A^{\mathcal{M}}$ can be rewritten as

$$\begin{aligned} \tilde{d} &= q(d - d\phi d)j \\ &= qidpj. \end{aligned}$$

By Theorem 2 of [45] it follows that $p_1 i_2$ is an isomorphism and hence we get the following corollary of Theorem 5.2.2.

	u_1	u_2	u_3	u_4	u_{12}	u_{14}	u_{23}	u_{34}
u_1	0	x_2u_{12}	$x_3u_{14} - x_2u_{34}$	x_1u_{14}	0	0	$x_2(u_{123} + u_{134})$	0
u_2		0	x_3u_{23}	$x_3u_{14} - x_4u_{12}$	0	0	0	$x_3(u_{123} + u_{134})$
u_3			0	x_4u_{34}	$x_3(u_{123} + u_{134})$	0	0	0
u_4				0	0	0	$x_4(u_{123} + u_{134})$	0
u_{12}					0	0	0	0
u_{14}						0	0	0
u_{23}							0	0
u_{34}								0

FIGURE 5.3: The multiplication μ_2 for the ideal I.

Corollary 5.2.5. Let A be a dg algebra and $\mathcal{M}$ a Morse matching. Then the Morse complex $A^{\mathcal{M}}$ has an A_∞-algebra structure ν_n such that there exists an isomorphism of A_∞-algebras

$$(A^{\mathcal{M}}, \nu_n) \to (\mathrm{im}(p), \mu_n).$$

Proof. Indeed, define ν_n by

$$\nu_n = qi\mu_n(pj)^{\otimes n}.$$

The required isomorphism is then the one induced by $p_1 i_2$ via Theorem 5.2.2. □

Chapter 6

Algebraic Morse theory and Massey products on Koszul homology

6.1 Algebraic Morse theory on the Taylor resolution

In this section, we apply algebraic Morse theory to the Taylor resolution T. First, we discuss one way of constructing Morse matchings on the Taylor resolution which is due to Jöllenbeck [22].

Given a basis element $u = u_{i_1} \cdots u_{i_p} \in T$, define an equivalence relation as follows. We say that u_{i_j} and u_{i_k} are equivalent if $\gcd(m_{i_j}, m_{i_k}) \neq 1$. In that case, we write $u_{i_j} \sim u_{i_k}$. The transitive closure of $\sim$ gives an equivalence relation on u and we write $\mathrm{cl}(u)$ for the number of equivalence classes. An arrow $u \to v$ in G_T is called *admissible* if $m_u = m_v$ and the Taylor differential d maps u to v with nonzero coefficient.

Construction 6.1.1. 1. Let $u \to v$ be an admissible arrow with $\mathrm{cl}(u) = \mathrm{cl}(v) = 1$ such that no proper subsets $u' \subset u$ and $v' \subset v$ define an admissible arrow $u' \to v'$ with $\mathrm{cl}(u) = 1$ and $\mathrm{cl}(v') = 1$. Define

$$\mathcal{M}_{11} = \{uw \to vw \mid \text{for each } w \text{ with } \gcd(m_w, m_u) = 1 = \gcd(m_w, m_v)\}.$$

To simplify notation, write $u \in \mathcal{M}_{11}$ if there exists v such that either $u \to v$ or $v \to u$ is in $\mathcal{M}_{11}$. Then $\mathcal{M}_{11}$ is an acyclic matching. Note that if $\gcd(m_u, m_v) = 1$ and $uv \in \mathcal{M}_{11}$ then $u \in \mathcal{M}_{11}$ or $v \in \mathcal{M}_{11}$. Consequently, the same procedure can repeated on the Morse complex $T^{\mathcal{M}_{11}}$. Therefore, we obtain a series of acyclic matchings $\mathcal{M}_1 = \cup_{i \geq 1} \mathcal{M}_{1i}$. After finitely many steps we obtain a complex such that for each admissible arrow $u \to v$ we have $\mathrm{cl}(u) \geq 1$ and $\mathrm{cl}(v) \geq 2$.

2. Let $u \to v$ be an admissible arrow in $T^{\mathcal{M}_1}$ with $\mathrm{cl}(u) = 1$ and $\mathrm{cl}(v)$ such that no proper subsets $u' \subset u$ and $v' \subset v$ define an admissible arrow $u' \to v'$ with $\mathrm{cl}(u) = 1$

and $\mathrm{cl}(v') = 2$. Define

$$\mathcal{M}_{21} = \{uw \to vw \mid \text{for each } w \text{ with } \gcd(m_w, m_u) = 1 = \gcd(m_w, m_v)\}.$$

By the same argument as before, this procedure can be repeated on the Morse complex $T^{\mathcal{M}_{21}}$. Consequently, we obtain a sequence of acyclic matchings $\mathcal{M}_2 = \cup_{i\geq 1} \mathcal{M}_{2i}$.

3. Continuing on we obtain a sequence of matching $\mathcal{M} = \cup_{i\geq 1} \mathcal{M}_i$.

Each admissible arrow is of the form $uw \to vw$ where $m_u = m_v$, $\gcd(m_u, m_w) = 1$, $\mathrm{cl}(u) = 1$ and $\mathrm{cl}(v) \geq 1$. Therefore, $(T^{\mathcal{M}}, \tilde{d})$ is the minimal free resolution of S/I.

The following lemma is immediate from the above construction.

Lemma 6.1.2. Let $\mathcal{M}$ be constructed as above. Then

1. for all arrows $u \to v$ in $\mathcal{M}$ we have $m_u = m_v$,
2. for all arrows in the Morse complex $T^{\mathcal{M}}$ we have $m_u \neq m_v$,
3. $\mathcal{M}_i$ is a sequence of acyclic matchings on the Morse complex $T^{\mathcal{M}_{<i}}$ where $\mathcal{M}_{<i} = \cup_{j<i} \mathcal{M}_j$,
4. for all arrows $u \to v$ we have $\mathrm{cl}(u) - \mathrm{cl}(v) = i - 1$ and $|v| + 1 = |u|$,

The following lemma is straightforward but will be used often.

Lemma 6.1.3. Let M be a Morse matching on the Taylor resolution T. Then M is maximal if and only if T^M is the minimal free resolution.

Proof. Clearly, if T^M is minimal then M cannot be extended and hence is maximal.

For the converse, if T^M is not minimal then there exists some component $d_{\alpha,\beta}$ which does not map into $(x_1, \ldots, x_m)T$. By definition of the Taylor differential this is only possible if $d_{\alpha,\beta} = \pm 1$ and so is invertible. Define M' by setting

$$M' = M \cup \{\alpha \to \beta\}.$$

Then it is easily seen that M' is a Morse matching. Therefore, M is not maximal. □

Corollary 6.1.4. Let R be a monomial ring and let $F \to R$ be the minimal free resolution of R. Then there exists a maximal Morse matching M on the Taylor resolution T such that $T^M = F$.

Proof. Let M be the matching obtained from Jöllenbeck's construction. Then T^M is maximal by Lemma 6.1.3. Since the minimal free resolution is unique by Lemma 3.2.6, we have $T^M = F$. □

Theorem 6.1.5. Let M_1 and M_2 be maximal Morse matchings on T and let $\{\mu_n^1\}$ and $\{\mu_n^2\}$ be the corresponding A_∞-algebra structures. Then there is an A_∞-isomorphism

$$(T^{M_1}, \{\mu_n^1\}) \cong (T^{M_2}, \{\mu_n^2\}).$$

Proof. From Lemmas 3.2.6 and 6.1.3 it follows that $T^{M_1} = T^{M_2}$. Therefore, we can apply Theorem 5.2.3 and Remark 5.2.4 and we get an A_∞-isomorphism

$$(T^{M_1}, \{\mu_n^1\}) \cong (T^{M_2}, \{\mu_n^2\})$$

which finishes the proof. □

Our next goal is to investigate how A_∞-structures behave under polarization. We briefly recall the polarization construction following the exposition in [42].

Definition 6.1.6. Let $f \in S$ be a monomial. Write $f = q_1 \cdots q_m$ where $q_i = x_i^{a_i}$. Define the *polarization* q_i^{pol} of q_i to be

$$q_i^{\mathrm{pol}} = \begin{cases} 1 & \text{if } a_i = 0 \\ x_i \prod_{1 \le j \le a_{i-1}} t_{i,j} & \text{if } a_i \neq 0 \end{cases}$$

and define the *polarization* f^{pol} to be

$$f^{\mathrm{pol}} = q_1^{\mathrm{pol}} \cdots q_m^{\mathrm{pol}}.$$

If $I = (m_1, \ldots, m_r)$ is an ideal, then the *polarization* I^{pol} of I is

$$I^{\mathrm{pol}} = (m_1^{\mathrm{pol}}, \ldots, m_r^{\mathrm{pol}}).$$

Note that if I is an ideal in $S = k[x_1, \ldots, x_m]$ then I^{pol} is an ideal in

$$S^{\mathrm{pol}} = k[t_{1,1}, \ldots, t_{1,p_1}, t_{2,1}, \ldots, t_{2,p_2}, \ldots, t_{m,1}, \ldots, t_{m,p_m}]$$

where

$$p_i = \max\{c \mid x_i^{c+1} \text{ divides one of } m_1, \ldots, m_r\}.$$

Now, set

$$\alpha = \{t_{j,i} - x_j \mid 1 \le j \le n, 1 \le i \le p_j\}.$$

We have

$$S^{\mathrm{pol}}/(I^{\mathrm{pol}} + (\alpha)) = S/I.$$

We call the process of factoring S^{pol} by (α) *depolarization*.

Example 6.1.7. Let $I = (x_1^3, x_2 x_3^2 x_4, x_3^3, x_1 x_4)$. Then

monomial	polarization
x_1^3	$x_1t_{1,1}t_{1,2}$
$x_2x_3^2x_4$	$x_2x_3t_{3,1}x_4$
x_3^3	$x_3t_{3,1}t_{3,2}$
x_1x_4	x_1x_4

and so we have

$$I^{\mathrm{pol}} = (x_1t_{1,1}t_{1,2}, x_2x_3t_{3,1}x_4, x_3t_{3,1}t_{3,2}, x_1, x_4)$$

and

$$S^{\mathrm{pol}} = k[x_1, t_{1,1}t_{1,2}, x_2, x_3, t_{3,1}, t_{3,2}, x_4].$$

Let F be the minimal free resolution of R and let $\{\mu_n\}_{n=1}^{\infty}$ be an A_∞-structure on F. By Corollary 6.1.4 it follows that there is some Morse matching M on T such that $T^M = F$. Consequently, F admits a basis consisting of basis sets u_I of the Tayor resolution. Let

$$u_I = \{u_{i_1}, \ldots, u_{i_k}\}$$

be such a basis set and define u_I^{pol} by

$$u_I^{\mathrm{pol}} = \{u_{i_1}^{\mathrm{pol}}, \ldots, u_{i_k}^{\mathrm{pol}}\}.$$

Here, we use that the basis elements of the Taylor resolution can be identified with subsets of the minimal generators $\{m_1, \ldots, m_r\}$ of I.

Let F^{pol} denote the free S^{pol}-module spanned by the u_I^{pol} where u_I is a basis element of F. Define a differential d^{pol} by

$$d^{\mathrm{pol}}u_I^{\mathrm{pol}} = (du_I)^{\mathrm{pol}}.$$

It is clear that F^{pol} is the minimal free resolution of $S^{\mathrm{pol}}/I^{\mathrm{pol}}$.

Our goal is to construct an A_∞-structure $\{\mu_n^{\mathrm{pol}}\}_{n=1}^{\infty}$ on the polarized resolution F^{pol}. Define a k-linear map $(-)^{\mathrm{pol}}\colon F \to F^{\mathrm{pol}}$ by setting

$$(fu_I)^{\mathrm{pol}} = f^{\mathrm{pol}}u_I^{\mathrm{pol}}.$$

Next, define the maps μ_n^{pol} by putting

$$\mu_n^{\mathrm{pol}}(u_{I_1}^{\mathrm{pol}}, \ldots, u_{I_n}^{\mathrm{pol}}) = (\mu_n(u_{I_1}, \ldots, u_{I_n}))^{\mathrm{pol}}.$$

Lemma 6.1.8. The pair $(F^{\mathrm{pol}}, \{\mu_n^{\mathrm{pol}}\}_{n=1}^{\infty})$ is an A_∞-algebra.

Proof. We compute

$$\begin{aligned}
&\mu_u^{\mathrm{pol}}(u_{I_1}^{\mathrm{pol}},\dots,u_{I_r}^{\mathrm{pol}},\mu_s^{\mathrm{pol}}(u_{I_{r+1}}^{\mathrm{pol}},\dots,u_{I_{r+s}}^{\mathrm{pol}}),u_{I_{r+s+1}}^{\mathrm{pol}},\dots,u_{I_n}^{\mathrm{pol}})\\
&=\mu_u^{\mathrm{pol}}(u_{I_1}^{\mathrm{pol}},\dots,u_{I_r}^{\mathrm{pol}},(\mu_s(u_{I_{r+1}},\dots,u_{I_{r+s}}))^{\mathrm{pol}},u_{I_{r+s+1}}^{\mathrm{pol}},\dots,u_{I_n}^{\mathrm{pol}})\\
&=(\mu_u(u_{I_1},\dots,u_{I_r},\mu_s(u_{I_{r+1}},\dots,u_{I_{r+s}}),u_{I_{r+s+1}},\dots,u_{I_n})^{\mathrm{pol}}.
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
\sum(-1)^{r+st}\mu_u^{\mathrm{pol}}(1^r\otimes\mu_s^{\mathrm{pol}}\otimes 1^t) &= (\sum(-1)^{r+st}\mu_u(1^r\otimes\mu_s\otimes 1^t))^{\mathrm{pol}}\\
&=0
\end{aligned}$$

where the second identity follows from the Stasheff identities for μ_n. □

Lemma 6.1.9. The map μ_n is minimal if and only if the map μ_n^{pol} is minimal.

Proof. The lemma is immediate from the observation that the coefficients of μ_n^{pol} are just that polarized coefficients of μ_n. □

The following corollary is an immediate application of Lemma 6.1.9.

Corollary 6.1.10. For any monomial ring R,

1. R satisfies condition (B_r) if and only of R^{pol} satisfies condition (B_r).

2. R is Golod if and only if R^{pol} is Golod.

Remark 6.1.11. The second statement of Corollary 6.1.10 was first proved in [14].

6.2 Simplicially resolvable monomial rings

A well-known result by Berglund and Jöllenbeck is the following.

Theorem 6.2.1 ([6], Theorem 5.1)**.** Let $R = S/I$ be a monomial ring. Then the following are equivalent.

1. R is Golod

2. The product on the Koszul homology $\mathrm{Tor}^S(R,k)$ vanishes.

However, in [25] Katthän presented the following counterexample to Theorem 6.2.1.

Theorem 6.2.2 ([25], Theorem 3.1)**.** Let k be a field and let $S = k[x_1,x_2,y_1,y_2,z]$. Let I be the ideal generated by

$$\begin{array}{ccc} m_1 = x_1x_2^2 & m_4 = x_1x_2y_1y_2 & m_7 = x_1y_1z \\ m_2 = y_1y_2^2 & m_5 = y_2^2z^2 & m_8 = x_2^2y_2^2z \\ m_3 = z^3 & m_6 = x_2^2z^2 & \end{array}$$

Then the product on $\operatorname{Tor}^S(R,k)$ is trivial but R is not Golod. More precisely, the Massey product $\langle m_1, m_2, m_3\rangle$ is non-trivial.

A natural question to ask, then, is under what additional assumptions Theorem 6.2.1 does hold. The main purpose of this section is to provide an answer to this question. Recall from Definition 3.2.15 that a resolution F is called simplicial if $F = F_\Delta$ for some simplicial complex Δ. We start with a definition.

Definition 6.2.3. A monomial ring R is called *simplicially resolvable* if the minimal free resolution of R is a simplicial resolution.

Lemma 6.2.4. Let R be a monomial ring. Then the following are equivalent.

1. R is simplicially resolvable
2. The Taylor resolution T of R admits a maximal Morse matching $\mathcal{M}$ such that the set $\mathcal{M}_0$ of $\mathcal{M}$-critical cells forms a simplicial complex.

Proof. If the second statement holds then $F_{\mathcal{M}_0} = T^{\mathcal{M}}$ is a simplicial resolution of R. Since $\mathcal{M}$ is maximal, it follows by Lemma 6.1.3 that $F_{\mathcal{M}_0}$ is minimal.

Conversely, assume that R is simplicially resolvable. Let $F = F_\Delta$ denote the minimal free resolution and let T denote the Taylor resolution. Then there exists a trivial complex $G = \oplus_{\alpha\in I} G_\alpha$ where

$$G_\alpha : \qquad 0 \longrightarrow Su_\alpha \xrightarrow{d_\alpha} Sv_\alpha \longrightarrow 0$$

with $d_\alpha(u_\alpha) = v_\alpha$. Define

$$\mathcal{M} = \{d_\alpha \colon u_\alpha \to v_\alpha \mid \alpha \in I\}$$

then $\mathcal{M}$ is Morse matching and $F_\Delta = T^{\mathcal{M}_0}$. Therefore, $\mathcal{M}_0 = \Delta$ and hence $\mathcal{M}_0$ is a simplicial complex. □

The following lemma is straightforward but crucial in what follows.

Lemma 6.2.5. Let R be simplicially resolvable and let $\mathcal{M}$ be a Morse matching on the Taylor resolution T of R as in Lemma 6.2.4. Let $\{\nu_n\}$ denote the corresponding A_∞-structure on the Morse complex $T^{\mathcal{M}}$. Then

$$\nu_n = q \circ \lambda_n \circ j^{\otimes n}$$

where $q\colon T \to T^{\mathcal{M}}$ is the projection on the critical cells, $j\colon T^{\mathcal{M}} \to T$ is the inclusion and $\lambda_n\colon T^{\otimes n} \to T$ is the auxiliary map from the Merkulov construction (2.3.4).

Proof. Recall from Corollary 5.2.5 that ν_n is given by

$$\nu_n = qi\mu_n pj = qip\lambda_n(ipj)^{\otimes n}.$$

Let d denote the differential of the Taylor resolution T and let $\tilde{d}$ denote the differential of the Morse complex $\tilde{d}$ from Definition 5.1.14. That is to say, $\tilde{d} = q(d - d\phi d)j$. Since R is simplicially resolvable, it follows by Lemma 6.2.4 the critical cells $\mathcal{M}_0$ form a simplicial complex. Therefore, if $u \in T$ is a critical then all $v \subseteq u$ are critical as well. Consequently, $d(T^{\mathcal{M}}) \subseteq T^{\mathcal{M}}$. Now, $p = 1 - d\phi - \phi d$ and $\phi(T^{\mathcal{M}}) = 0$ by definition. Hence, for $u \in T^{\mathcal{M}}$ we have

$$pj(x) = (1 - d\phi - \phi d)jx = x - d\phi x - \phi dx = x.$$

Consequently, the isomorphism $pj\colon T^{\mathcal{M}} \to \operatorname{im}(p)$ is just the identity. Hence so is its inverse qi. Therefore,

$$\nu_n = q \circ \lambda_n \circ j^{\otimes n}$$

as required. □

Our next goal is to investigate how the auxiliary maps $\lambda_n\colon T^n \to T$ of the Merkulov construction (Theorem 2.3.6) behave with respect to multidegrees. Recall that λ_2 is the product $\cdot$ of T and

$$\lambda_n = \sum_{\substack{s+t=n\\ s,t\geq 1}} (-1)^{s+1}\lambda_2(\phi\lambda_s \otimes \phi\lambda_t)$$

for all $n \geq 3$. We start with the following lemma.

Lemma 6.2.6. For all n and all $v_1, \dots, v_n \in T$, $\operatorname{lcm}(\lambda_n(v_1, \dots, v_n))$ divides $\operatorname{lcm}(v_1, \dots, v_n)$.

Proof. We prove the statement by induction on n. If $n = 2$, then

$$\lambda_2(v_1, v_2) = \begin{cases} \frac{\operatorname{lcm}(v_1)\operatorname{lcm}(v_2)}{\operatorname{lcm}(v_1 v_2)} v_1 v_2 & \text{if } v_1 \cap v_2 = \emptyset \\ 0 & \text{otherwise} \end{cases}$$

and so the result is clear. Next, assume the result holds for all degrees up to $n-1$. Fix some $k + l = n$. By assumption, $\operatorname{lcm}(\lambda_k(v_1, \dots, v_k))$ is a divisor of $\operatorname{lcm}(v_1, \dots, v_k)$. If $\phi\lambda_k(v_1, \dots, v_k) = 0$ then there is nothing to prove. Otherwise, there exists some $x \to \lambda_k(v_1, \dots, v_k) \in \mathcal{M}$. Then we can write

$$\phi\lambda_k(v_1, \dots, v_k) = x + \sum_{\alpha} y_\alpha$$

for some α. If $y_\alpha \neq 0$ for some α, then in the Morse graph G_T there is a subgraph

$$\begin{tikzcd} x \arrow[d] \arrow[dr] & y_\alpha \arrow[d] \\ \lambda_k(v_1,\dots,v_k) & z_\alpha \end{tikzcd}$$

where the red arrows are in the Morse matching $\mathcal{M}$. By definition of $\mathcal{M}$, we have $\mathrm{lcm}(x) = \mathrm{lcm}(\lambda_k(v_1,\dots,v_k))$ and $\mathrm{lcm}(y_\alpha) = \mathrm{lcm}(z_\alpha)$ for all α. Since $\mathrm{lcm}(z_\alpha)$ is a divisor of $\mathrm{lcm}(x)$, it follows that $\mathrm{lcm}(y_\alpha)$ is a divisor or $\mathrm{lcm}(\lambda_k(v_1,\dots,v_k)) = \mathrm{lcm}(v_1,\dots,v_k)$. The result now follows from the case $n = 2$. □

Our next goal is to give a lower bound for $\mathrm{cl}(\lambda_n(v_1,\dots,v_n))$. We have the following lemmma.

Lemma 6.2.7. Let $n \geq 2$ and let $v_1,\dots,v_n \in T$ with $\gcd(v_i,v_j) = 1$ for $i \neq j$. Then

$$\mathrm{cl}(\lambda_n(v_1,\dots,v_n)) \geq 2.$$

Proof. Fix some $k + l = n$ and let $v_1,\dots,v_n \in T$ with $\gcd(v_i,v_j) = 1$ for $i \neq j$. It is sufficient to show that

$$\gcd\big(\mathrm{lcm}(\phi\lambda_k(v_1,\dots,v_k)), \mathrm{lcm}(\phi\lambda_l(v_{k+1},\dots,v_n))\big) = 1.$$

By the previous lemma, it follows that $\mathrm{lcm}(\phi\lambda_k(v_1,\dots,v_k))$ is a divisor of $\mathrm{lcm}(v_1,\dots,v_k)$ and that $\mathrm{lcm}(\phi\lambda_l(v_{k+1},\dots,v_n))$ is a divisor of $\mathrm{lcm}(v_{k+1},\dots,v_n)$. Since

$$\gcd\big(\mathrm{lcm}(v_1,\dots,v_k), \mathrm{lcm}(v_{k+1},\dots,v_n)\big) = 1,$$

it follows that $\mathrm{cl}(\lambda_n(v_1,\dots,v_n)) \geq 2$ as desired. □

We now come to the first main theorem of this section.

Theorem 6.2.8. Let R be simplicially resolvable and let $T \to R$ denote the Taylor resolution. Suppose that the Taylor resolution admits a Morse matching $\mathcal{M}$ such that for all $u \in T$ with $\mathrm{cl}(u) \geq 2$ we have $u \in \mathcal{M}$. Then S/I is Golod.

Proof. Recall from the previous section that for a given Morse matching $\mathcal{M}$ we have a diagram

$$\begin{tikzcd} T \arrow[r, "1"] \arrow[d, shift right, "p"'] & T \arrow[d, shift right, "q"'] \\ F = \mathrm{im}(p) \arrow[u, shift right, "i"'] \arrow[r, shift left, "f"] & T^{\mathcal{M}} \arrow[u, shift right, "j"'] \arrow[l, shift left, "g"] \end{tikzcd}$$

where $f = qi$ and $g = pj$. As usual, we let $\lambda_k \colon T^{\otimes k} \to T$ denote the auxiliary maps from the Merkulov construction and $\mu_k \colon F^{\otimes k} \to F$ the A_∞-structure on F. That is,

$$\mu_k = p \circ \lambda_k \circ i^{\otimes k}.$$

Since R is simplicially resolvable, it follows from Lemma 6.2.5 that the maps

$$\begin{aligned}\nu_k &= qi \circ \mu_k \circ (pj)^{\otimes k} \\ &= q \circ \lambda_k \circ j^{\otimes k}\end{aligned}$$

give an A_∞-structure on the Morse complex $T^{\mathcal{M}}$. Suppose that the Massey product $\langle u_1, \dots, u_n \rangle$ is defined. It is sufficient to show that $\nu_n(u_1, \dots, u_n) \in (x_1, \dots, x_m)T^{\mathcal{M}}$. We may assume $u_1, \dots, u_n$ have pairwise trivial gcd since otherwise the Massey product will be trivial. By the previous lemma,

$$\mathrm{cl}(\lambda_n(v_1, \dots, v_n)) \geq 2.$$

Consequently,

$$\nu_n(u_1, \dots, u_n) = q\lambda_n(v_1, \dots, v_n) = 0$$

since q is the projection on the critical cells. □

We have the following lemma.

Lemma 6.2.9. Let R be simplicially resolvable. Then the following are equivalent.

1. The Taylor resolution T admits a Morse matching $\mathcal{M}$ such that for all $u \in T$ with $\mathrm{cl}(u) \geq 2$ we have $u \in \mathcal{M}$.
2. The product on Koszul homology is trivial.

Proof. For a given Morse matching $\mathcal{M}$ we have a diagram

$$\begin{tikzcd} T \arrow[r, "1"] \arrow[d, shift right, "p"'] & T \arrow[d, shift right, "q"'] \\ F = \mathrm{im}(p) \arrow[u, shift right, "i"'] \arrow[r, shift left, "f"] & T^{\mathcal{M}} \arrow[u, shift right, "j"'] \arrow[l, shift left, "g"] \end{tikzcd}$$

where $f = qi$ and $g = pj$. First, assume that T admits such a Morse matching and call it $\mathcal{M}$. Let $u, v \in F$. It is suffcient to show that $\mu_2(u, v) \in (x_1, \dots, x_m)F$. We may assume that $\gcd(m_u, m_v) = 1$ since otherwise $\lambda_2(u, v) \in (x_1, \dots, x_m)F$ and hence $\mu_2(u, v) \in (x_1, \dots, x_m)F$. Since $\gcd(m_u, m_v) = 1$, it follows that $\lambda_2(u, v) = uv$. Therefore,

$$\mu_2(u, v) = p(uv) = gq(uv).$$

But since $uv \in \mathcal{M}$ by assumption, we have $q(uv) = 0$ hence $\mu_2(u, v) = 0$. Therefore, the cup product is trivial.

For the converse, suppose that the first statement does not hold. Then for every Morse matching $\mathcal{M}$ there is some u with $\mathrm{cl}(u) \geq 2$ such that $u \notin \mathcal{M}$. So, fix some $\mathcal{M}$ and pick $u \notin \mathcal{M}$ with $\mathrm{cl}(u) \geq 2$. Since $\mathrm{cl}(u) \geq 2$, there exist v, w such that $u = \lambda_2(v, w)$. Since R

is simplicially resolvable, it follows by Lemma 6.2.4 that v and w are critical. Note that necessarily $\gcd(m_v, m_w) = 1$. We claim that

$$[v] \smile [w] = [u] \neq 0.$$

Let $\{\nu_n\}$ be the A_∞-structure on $T^{\mathcal{M}}$ corresponding to $\mathcal{M}$, that is

$$\nu_n = f \circ \mu_n \circ g^{\otimes n}.$$

Then $\nu_n \otimes 1 = \smile$ as $T^{\mathcal{M}}$ is minimal. Compute

$$\begin{aligned} \nu_2(v, w) &= f\mu_2(gv, gw) \\ &= fp\lambda_2(igv, igw) \\ &= q\lambda_2(jv, jw) \\ &= q(vw) \\ &= vw \end{aligned}$$

where the last step follows because vw is $\mathcal{M}$-critical by assumption. Hence $[v] \smile [w] = [u] \neq 0$ as desired. $\square$

We now come to the main theorem of this section.

Theorem 6.2.10. Let R be simplicially resolvable. Then R is Golod if and only if the product on Koszul homology is trivial.

Proof. If R is Golod then the product is trivial by definition. Conversely, if the product is trivial then it follows by the previous lemma that the Taylor resolution T admits a Morse matching $\mathcal{M}$ such that for all $u \in T$ with $\mathrm{cl}(u) \geq 2$ we have $u \in \mathcal{M}$. But this implies that all Massey products vanish by Theorem 6.2.8. $\square$

Recall that if $\mathcal{M}$ is a Morse matching then we denote by $\mathcal{M}_0$ the set of critical cells. We have the following result.

Theorem 6.2.11. Let $R = S/I$ be simplicially resolvable. Then the following are equivalent.

1. R is Golod
2. The product on $\mathrm{Tor}^S(R, k)$ is trivial.
3. I satisfies the gcd condition. That is, for any two generators m_1 and m_2 of I with $\gcd(m_1, m_2) = 1$ there exists a generator $m \neq m_1, m_2$ such that m divides $\mathrm{lcm}(m_1, m_2)$.
4. For $u, v \in \mathcal{M}_0$ we have $\mathrm{lcm}(u)\,\mathrm{lcm}(v) \neq \mathrm{lcm}(uv)$ whenever $uv \in \mathcal{M}_0$.

Proof. The equivalence 1 ⇔ 2 is Theorem 6.2.10. The equivalence 2 ⇔ 3 is well-known, see for example Lemma 2.4 of [25]. We prove 2 ⇔ 4. Since R is simplicially resolvable the product on $\mathrm{Tor}^S(R,k)$ is induced $q\lambda_2$. Assume the product is trivial and let $u,v \in \mathcal{M}_0$. Then either $\lambda_2(u,v) = 0$ or $q\lambda_2(u,v) \in (x_1,\dots,x_m)$. In the first case, $uv \notin \mathcal{M}_0$ by definition. In the second case, we have $uv \in \mathcal{M}_0$ and

$$q\lambda_2(u,v) = \frac{\mathrm{lcm}(u)\,\mathrm{lcm}(v)}{\mathrm{lcm}(uv)} q(uv).$$

So $q\lambda_2(u,v) \in (x_1,\dots,x_m)$ implies that $\mathrm{lcm}(u)\,\mathrm{lcm}(v) \neq \mathrm{lcm}(uv)$.

For the converse implication, let $u,v \in \mathcal{M}_0$. If $uv \notin \mathcal{M}_0$, then $q(uv) = 0$ and so $u \smile v = 0$. So, assume $uv \in \mathcal{M}_0$. Then $\mathrm{lcm}(u)\,\mathrm{lcm}(v) \neq \mathrm{lcm}(uv)$ and so $q\lambda_2(u,v) \in (x_1,\dots,x_m)$. Consequently, $u \smile v = 0$. □

The following examples show that the class of simplicially resolvable is quite expansive.

Example 6.2.12. Let I be a monomial ideal. Recall that I is called *strongly generic*[5] if no variable x_i occurs with the same nonzero exponent in two distinct minimal generators of I. By Theorem 3.2 of [5], it follows that S/I is simplicially resolvable.

We point out that in [5] it is claimed that the minimal free resolution of a strongly generic ideal always has a dg algebra structure. However, recently a counterexample to this claim was found in [27].

Example 6.2.13. For a monomial $m = x_1^{a_1}\cdots x_m^{a_m} \in k[x_1,\dots,x_m]$, write

$$\mathrm{supp}(m) = \{i \mid a_i \neq 0\}$$

for the *support* of m. A monomial ideal $I = (m_1,\dots,m_r)$ is called *generic*[39] if for any distinct m_i and m_j that have the same positive degree in some variable x_s there exists a third generator m_k such that m_k divides $\mathrm{lcm}(m_i,m_j)$ and

$$\mathrm{supp}\left(\frac{\mathrm{lcm}(m_i,m_j)}{m_k}\right) = \mathrm{supp}(\mathrm{lcm}(m_i,m_j)).$$

If I is generic then S/I is simplicially resolvable by Theorem 1.5 of [39].

Next, we want to investigate the vanishing on higher Massey products. First, recall the definition of a standard matching introduced in [22].

Definition 6.2.14 ([22], Definition 3.1)**.** Let $\mathcal{M} = \cup_{i\geq 0}\mathcal{M}_i$ be a sequence of matchings on the Taylor resolution T. Then $\mathcal{M}$ is called a *standard matching* if the following hold

1. for all arrows $u \to v$ in $\mathcal{M}$, we have $m_u = m_v$,
2. for all arrows in the Morse complex $T^{\mathcal{M}}$, we have $m_u \neq m_v$,

3. $\mathcal{M}_i$ is a sequence of acyclic matchings on the Morse complex $T^{\mathcal{M}_{<i}}$, where $\mathcal{M}_{<i} = \cup_{j<i} \mathcal{M}_j$,

4. for all arrows $u \to v$ in $\mathcal{M}_i$, we have $\mathrm{cl}(u) - \mathrm{cl}(v) = i - 1$ and $|v| + 1 = |u|$,

5. there exist $\mathcal{B}_i \subset \mathcal{M}_i$ such that

   (a) $\mathcal{M}_i = \mathcal{B}_i \cup \{u \cup w \to v \cup w \mid \gcd(m_u, m_w) = 1 \text{ and } u \to v \in \mathcal{B}_i\}$,

   (b) for all arrows $u \to v$ in $\mathcal{B}_i$, we have $\mathrm{cl}(u) = 1$ and $\mathrm{cl}(v) = i$.

Theorem 6.2.15. Let R be simplicially resolvable. Suppose that the Taylor resolution T admits a standard matching. Then all higher Massey products are trivial.

Proof. Let $\mathcal{M}$ be a standard matching on T. We obtain an A_∞-structure on the Morse complex $T^{\mathcal{M}}$ by

$$\nu_k = q \circ \mu_k \circ j^k.$$

Suppose that the Massey product $\langle u_1, \ldots, u_n \rangle$ is defined. It is sufficient to show that $\nu_n(u_1, \ldots, u_n) \in (x_1, \ldots, x_m)T^{\mathcal{M}}$. We may assume the u_i have pairwise trivial gcd since otherwise the Massey product will be trivial. We have

$$\lambda_n(u_1, \ldots, u_n) = \sum_{k+l=n} (-1)^{k+1} \lambda_2(\phi\lambda_k(u_1, \ldots, u_k), \phi\lambda_l(u_{k+1}, \ldots, u_n)).$$

Fix some k, l. We may assume that $\phi\lambda_k(u_1, \ldots, u_k) \neq 0$. Therefore, there exists some $x \to \lambda_k(u_1, \ldots, u_k) \in \mathcal{M}$. Then we can write

$$\phi\lambda_k(u_1, \ldots, u_k) = x + \sum_\alpha y_\alpha$$

for some α. If $y_\alpha \neq 0$ for some α, then in the Morse graph G_T there is a subgraph

$$\begin{tikzcd} x \arrow[d, red] \arrow[dr] & y_\alpha \arrow[d, red] \\ \lambda_k(u_1, \ldots, u_k) & z_\alpha \end{tikzcd}$$

where the red arrows are in the Morse matching $\mathcal{M}$. We have that

$$\mathrm{lcm}(\lambda_k(u_1, \ldots, u_k)) = \mathrm{lcm}(x)$$

and

$$\mathrm{lcm}(z_\alpha) = \mathrm{lcm}(y_\alpha).$$

Since $\mathrm{lcm}(z_\alpha)$ divides $\mathrm{lcm}(x)$, it follows that $\mathrm{lcm}(y_\alpha)$ divides $\mathrm{lcm}(\lambda_k(u_1, \ldots, u_k))$. Therefore, $\phi\lambda_l(u_{k+1}, \ldots, u_n)$ is disjoint from x and y_α. By definition of standard matching, it follows that

$$x\phi\lambda_l(u_{k+1}, \ldots, u_n) \to uv\phi\lambda_l(u_{k+1}, \ldots, u_n) \in \mathcal{M}$$

and

$$y_\alpha \phi\lambda_l(u_{k+1},\dots,u_n) \to z_\alpha \phi\lambda_l(u_{k+1},\dots,u_n) \in \mathcal{M}\,.$$

So,

$$q\lambda_2(\phi\lambda_k(u_1,\dots,u_k),\phi\lambda_l(u_{k+1},\dots,u_n)) = 0$$

since q is the projection on the $\mathcal{M}$-critical cells and elements $x\phi\lambda_l(u_{k+1},\dots,u_n)$ and $y_\alpha\phi\lambda_l(u_{k+1},\dots,u_n)$ are not $\mathcal{M}$-critical. Consequently,

$$\nu_n(u_1,\dots,u_n) = 0$$

and so the Massey product $\langle u_1,\dots,u_n\rangle$ is trivial as desired. □

As said before, in [6] it is claimed that the Golod property is equivalent to the vanishing of the product on $\mathrm{Tor}^S(R,k)$. However, in [25] a counterexample to this claim is given. The problem is to be found in [22] where it is claimed that standard matchings always exist. However, the following example due to Katthän [25] shows that this is not the case.

Let $S = k[x_1,x_2,x_3,x_4]$ and let I denote the ideal

$$I = (x_1^2, x_1x_2, x_2x_3, x_3x_4, x_4^2).$$

We will show that I does not admit a standard matching.

For ease of notation, denote the generators of I by

$$u_1 = x_1^2, \qquad u_2 = x_1x_2, \qquad u_3 = x_2x_3, \qquad u_4 = x_3x_4, \qquad u_5 = x_4^2.$$

Further, we will write $u_A = \prod_{i\in A} u_i$. Since any matching most contain all arrows $u \to v$ with $m_u = m_v$ for it to be standard, we first consider all possible such arrows. This produces the following list

$$u_{12345} \to u_{1345}$$
$$u_{12345} \to u_{1245}$$
$$u_{12345} \to u_{1235}$$

$$u_{2345} \to u_{245}$$
$$u_{2345} \to u_{235}$$

$$u_{1345} \to u_{135}$$
$$u_{1235} \to u_{135}$$

$$u_{1234} \to u_{134}$$
$$u_{1234} \to u_{124}$$

$$u_{345} \to u_{35}$$
$$u_{234} \to u_{24}$$
$$u_{123} \to u_{13}$$

These are the red arrows in Figure 6.1. Note that *any* standard matching must contain the arrows

$$u_{123} \to u_{13} \tag{6.2.1}$$

and

$$u_{345} \to u_{35} \tag{6.2.2}$$

since there are no other monomials in the relevant multidegrees. We have $\mathrm{cl}(u_{123}) = \mathrm{cl}(u_{345}) = 1$ and $\mathrm{cl}(u_{13}) = \mathrm{cl}(u_{35}) = 2$. Consequently, using the notation of Definition 6.2.14, the arrows (6.2.1) and (6.2.2) are in $\mathcal{B}_2$. Since $\gcd(m_{u_{123}}, m_{u_5}) = 1$, any standard matching containing the arrow $u_{123} \to u_{13}$ must also contain the arrow $u_{1235} \to u_{135}$ by property (5a) of Definition 6.2.14. Similarly, since $\gcd(m_{u_{345}}, m_{u_1}) = 1$ any standard matching containing $u_{345} \to u_{35}$ must also contain $u_{1345} \to u_{135}$. Therefore, any standard matching must contain $u_{1235} \to u_{135}$ and $u_{1345} \to u_{135}$. But this a contradiction as a matching can hit each set at most once. Consequently, I does not admit a standard matching.

Define a matching $\mathcal{M}$ by

$$u_{12345} \to u_{1235}$$
$$u_{2345} \to u_{245}$$
$$u_{1345} \to u_{135}$$
$$u_{1234} \to u_{124}$$
$$u_{345} \to u_{35}$$
$$u_{234} \to u_{24}$$
$$u_{123} \to u_{13}$$

These are the solid red arrows in Figure 6.1. This choice of $\mathcal{M}$ gives an acyclic matching satisfying the first four conditions of Definition 6.2.14.

Next, we want to use our tools to show that S/I has a nontrivial Massey product. The only Massey product that can possibly be nontrivial is $\langle u_1, u_2u_3\rangle$ since these are the only disjoint generators. By our earlier observation, the second definition of ϕ is irrelevant. We compute

$$\begin{aligned}
\mu_2(u_1, u_3) &= p\lambda_2(u_1, u_3) \\
&= p(u_1u_3) \\
&= (1 - d\phi - \phi d)(u_1u_3) \\
&= u_1u_3 + d(u_1u_2u_3) \\
&= u_1u_3 + x_1u_2u_3 - u_1u_3 + x_3u_1u_2 \\
&= x_1u_2u_3 + x_3u_1u_2
\end{aligned}$$

and

$$\begin{aligned}
\mu_2(u_3, u_5) &= p\lambda_2(u_3, u_5) \\
&= p(u_3u_5) \\
&= (1 - d\phi - \phi d)(u_3u_5) \\
&= u_3u_5 + d(u_3u_4u_5) \\
&= u_3u_5 + x_2u_4u_5 - u_3u_5 + x_4u_3u_4 \\
&= x_2u_4u_5 + x_4u_3u_4.
\end{aligned}$$

Therefore, in $\mathrm{Tor}^S(S/I, k)$ both u_1u_3 and u_3u_5 are zero and so the Massey product $\langle u_1, u_2u_3\rangle$ is defined.

To get rid of signs, we assume the characteristic of k is two. Then we have

$$\begin{aligned}
\lambda_3(u_1, u_3, u_5) &= \lambda_2(\phi\lambda_1 u_1, \phi\lambda_2(u_3, u_5)) + \lambda_2(\phi\lambda_2(u_1, u_3), \phi\lambda_1 u_5) \\
&= \lambda_2(u_1, \phi\lambda_2(u_3, u_5)) + \lambda_2(\phi\lambda_2(u_1, u_3), u_5) \\
&= \lambda_2(u_1, u_3u_4u_5)) + \lambda_2(u_1u_2u_3, u_5) \\
&= u_1u_3u_4u_5 + u_1u_2u_3u_5.
\end{aligned}$$

Now,

$$\begin{aligned}
p(u_1u_3u_4u_5) &= (1 - d\phi - \phi d)(u_1u_3u_4u_5) \\
&= u_1u_3u_4u_5 - x_1^2\phi(u_3u_4u_5) + x_2\phi(u_1u_4u_5) \\
&\quad - \phi(u_1u_3u_5) + x_4\phi(u_1u_3u_4) \\
&= u_1u_3u_4u_5 - u_1u_3u_4u_5 \\
&= 0
\end{aligned}$$

and

$$\begin{aligned}
p(u_1u_2u_3u_5) &= (1 - d\phi - \phi d)(u_1u_2u_3u_5) \\
&= u_1u_2u_3u_5 + d(u_1u_2u_3u_4u_5) - x_1\phi(u_2u_3u_5) \\
&\quad + \phi(u_1u_3u_5) - x_3\phi(u_1u_2u_5) + x_4^2\phi(u_1u_2u_3) \\
&= u_1u_2u_3u_5 + x_1u_2u_3u_4u_5 - u_1u_3u_4u_5 + u_1u_2u_4u_5 \\
&\quad - u_1u_2u_3u_5 + x_4u_1u_2u_3u_4 + u_1u_3u_4u_5 \\
&= x_1u_2u_3u_4u_5 + u_1u_2u_4u_5 + x_4u_1u_2u_3u_4.
\end{aligned}$$

Thus,

$$\mu_3(u_1, u_3u_5) = x_1u_2u_3u_4u_5 + u_1u_2u_4u_5 + x_4u_1u_2u_3u_4$$

which does not lie in the maximal ideal. Next, we show that the indeterminancy of $\langle u_1, u_3, u_5\rangle$ is zero. Again, it is sufficient to show that

$$(u_1, u_5) \cap \mathrm{Tor}_4^S(R, k) = 0.$$

So suppose $u_1 v \in \mathrm{Tor}_4^S(R, k)$. Since $\mathrm{mdeg}(u_1) = x_1^2$, it follows that $\mathrm{mdeg}(v) = x_2x_3x_4^2$. Since there are no critical cells of multidegree $x_2x_3x_4^2$, we get $v = 0$. Therefore, $\langle u_1, u_3, u_5\rangle = u_1u_2u_4u_5$ and so S/I has a nontrivial Massey product.

Remark 6.2.16. Since S/I has a non-trivial higher Massey product, it follows that there does not exist a dg algebra structure on the minimal free resolution of S/I. Indeed, S/I was the first example of such a monomial ring [1] but the original proof uses different methods to establish this.

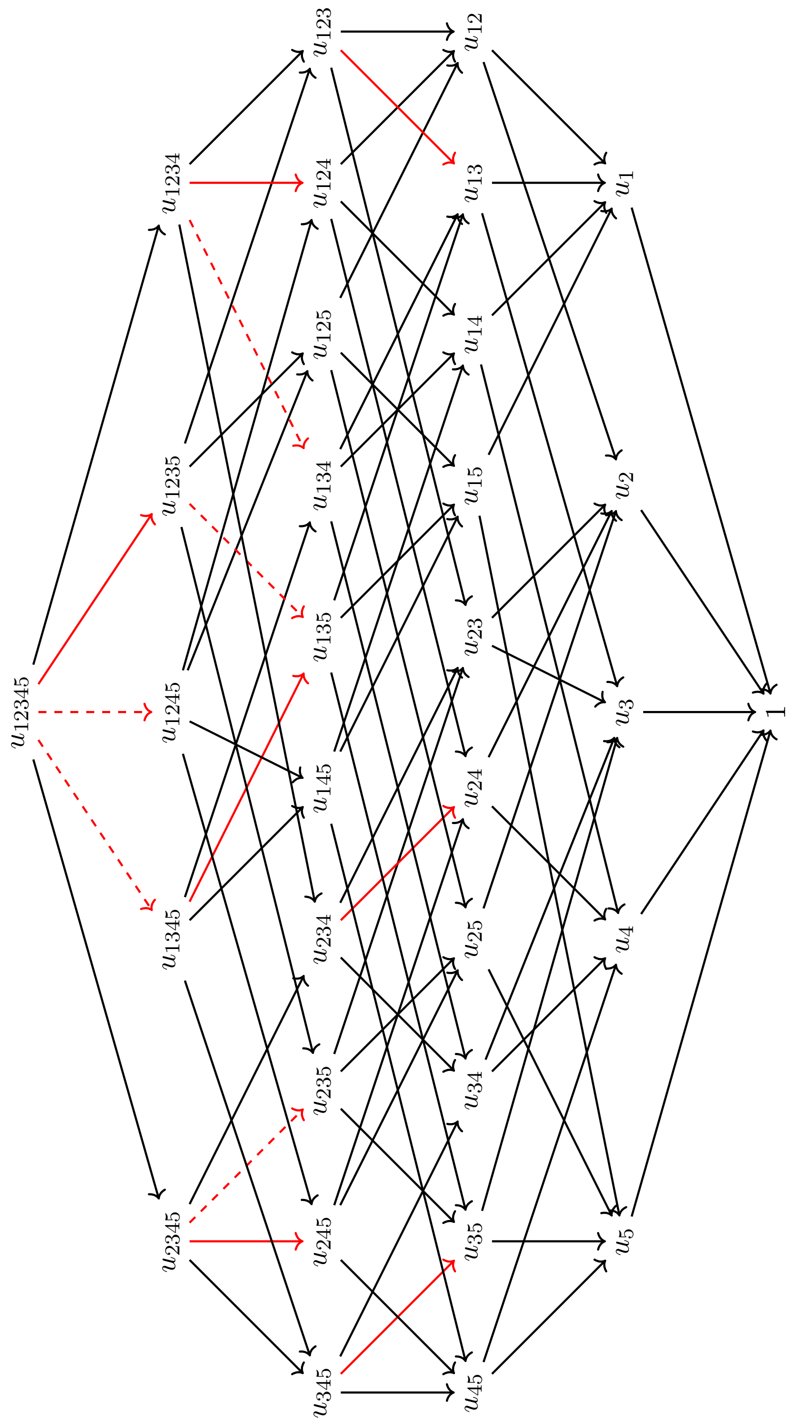


FIGURE 6.1: The graph G_T corresponding to the ideal I.

6.3 Massey products in low degrees

6.3.1 Denham-Suciu obstruction graphs

Up until now we have mainly been concerned with *vanishing* of Massey products. In this section we will use algebraic Morse theory to prove some results concerning the *non-vanishing* of Massey products. In what follows we write $\Delta^{(1)}$ for the 1-skeleton of a simplicial complex Δ on a vertex set $[m]$. Recall the following result by Denham and Suciu [10].

Theorem 6.3.1 ([10], Theorem 6.1.1)**.** Let Δ be a simplicial complex. The following statements are equivalent.

1. There exist classes $\alpha, \beta, \gamma \in \mathrm{Tor}^S(k[\Delta], k)$ for which the triple Massey product $\langle \alpha.\beta, \gamma \rangle$ is non-trivial.
2. The underlying graph of $\Delta^{(1)}$ contains an induced subgraph isomorphic to one of the five graphs of Figure 6.3.

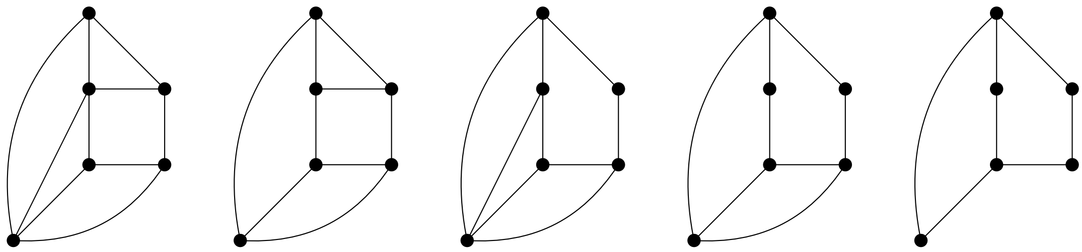
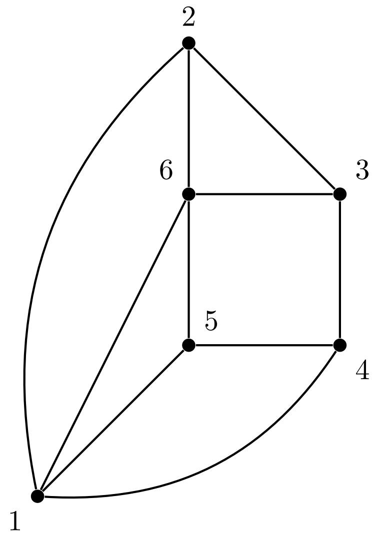


FIGURE 6.2: The five Denham-Suciu obstruction graphs

Our aim in this section is to use algebraic Morse theory to compute these Massey products and thereby giving a different proof of the implication $2 \Rightarrow 1$ of Theorem 6.3.1. By arguing as in [10], it is sufficient to prove the result for simplicial complexes Δ on six vertices. Consider a simplicial complex Δ having the following graph as its 1-skeleton.



Note that since $\Delta^{(1)}$ does not contain the full graph on four vertices it follows that $\dim \Delta = 2$. First assume that Δ contains all triangles. Then the Stanley-Reisner ideal of this complex is generated by

$$u_1 = x_1x_3, \qquad u_2 = x_2x_4, \qquad u_3 = x_2x_5, \qquad u_4 = x_3x_5, \qquad u_5 = x_4x_6.$$

The graph G_T corresponding to I is pictured in Figure 6.3.

By considering supports, we see that the only possible nontrivial Massey product is $\langle u_1, u_3, u_5\rangle$. To get rid of signs we will assume the characteristic of k is 2. We have

$$\begin{aligned}
\lambda_3(u_1, u_2, u_3) &= \lambda_2(\phi\lambda_1 u_1, \phi\lambda_2(u_3, u_5)) + \lambda_2(\phi\lambda_2(u_1, u_3), \phi\lambda_1 u_5) \\
&= \lambda_2(u_1, \phi(u_3u_5)) + \lambda_2(\phi(u_1u_3), u_5) \\
&= \lambda_2(u_1, u_2u_3u_5) + \lambda_2(u_1u_3u_4, u_5) \\
&= u_1u_2u_3u_5 + u_1u_3u_4u_5.
\end{aligned}$$

Now, we have

$$\begin{aligned}
p(u_1u_2u_3u_5) &= (1 - d\phi - \phi d)(u_1u_2u_3u_5) \\
&= u_1u_2u_3u_5 + x_1x_3\phi(u_2u_3u_4) \\
&+ \phi(u_1u_3u_5) + x_5\phi(u_1u_2u_5) + x_6\phi(u_1u_2u_3) \\
&= u_1u_2u_3u_5 + u_1u_2u_3u_5 \\
&= 0
\end{aligned}$$

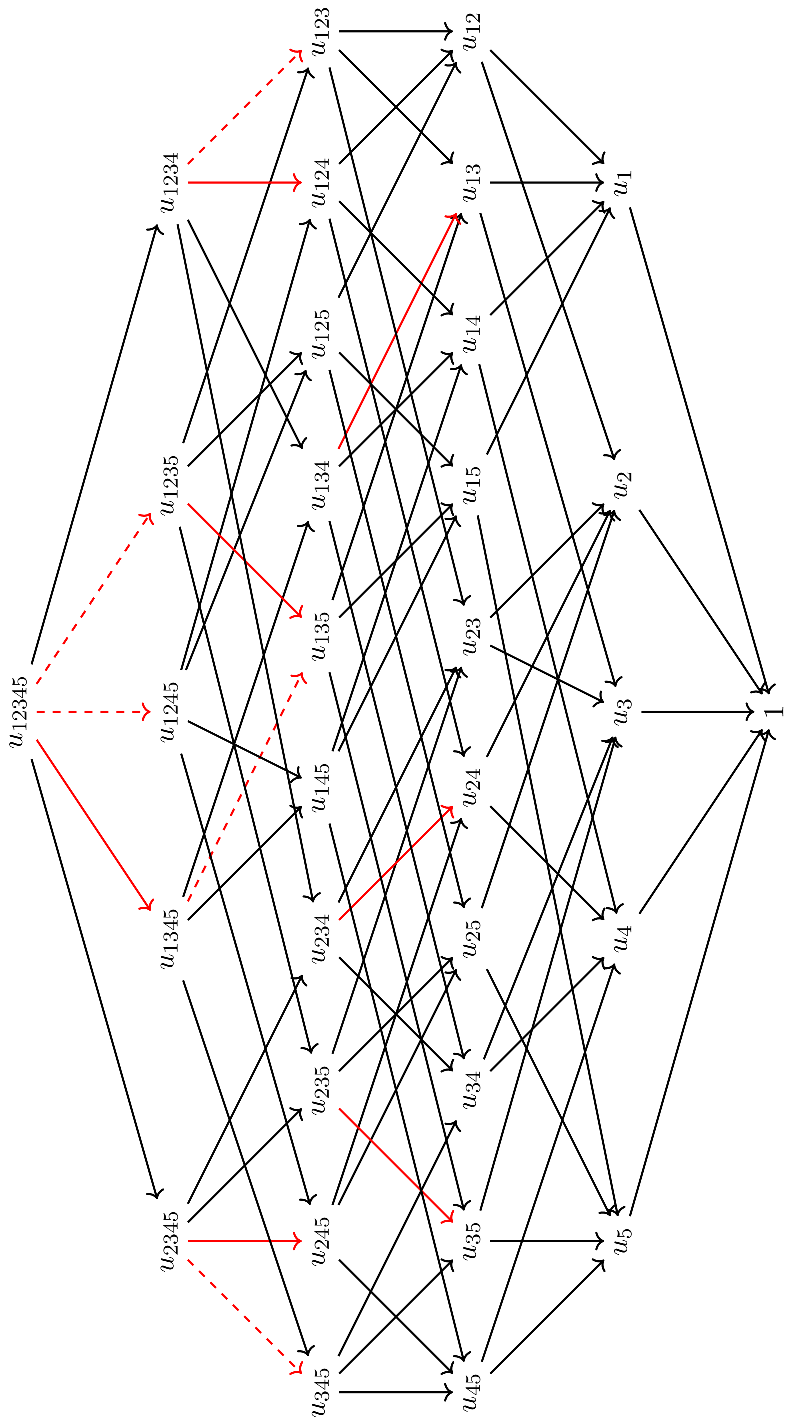


FIGURE 6.3: The graph G_T corresponding to the ideal I.

and

$$\begin{aligned}
p(u_1u_3u_4u_5) &= (1 - d\phi - \phi d)(u_1u_3u_4u_5) \\
&= u_1u_3u_4u_5 + d(u_1u_2u_3u_4u_5) + x_1\phi(u_3u_4u_5) \\
&+ x_2\phi(u_1u_4u_5) + \phi(u_1u_3u_5) + x_4x_6\phi(u_1u_3u_4) \\
&= u_1u_3u_4u_5 + x_1u_2u_3u_4u_5 + u_1u_3u_4u_5 + u_1u_2u_4u_5 \\
&+ u_1u_2u_3u_5 + x_6u_1u_2u_3u_4 + u_1u_2u_3u_5 \\
&= x_1u_2u_3u_4u_5 + u_1u_2u_4u_5 + x_6u_1u_2u_3u_4.
\end{aligned}$$

Consequently,

$$u_1u_2u_4u_5 \in \langle u_1, u_3, u_5\rangle.$$

To see that $\langle u_1, u_3, u_5\rangle$ is non-trivial, we compute the indeterminacy of the Massey product. That is, we need to compute

$$(u_1, u_5) \cap \mathrm{Tor}_4^S(R, k).$$

So, suppose

$$u_1v \in (u_1, u_5) \cap \mathrm{Tor}_4^S(R, k).$$

Since $\mathrm{mdeg}(u_1) = x_1x_3$, we must have $\mathrm{mdeg}(v) = x_2x_4x_5x_6$. However, there are no critical cells of that multidegree and so $v = 0$. The same argument applies to u_5 and so

$$(u_1, u_5) \cap \mathrm{Tor}_4^S(R, k) = 0.$$

Therefore, we have

$$\langle u_1, u_3, u_5\rangle = u_1u_2u_4u_5$$

and so in particular the Massey product is non-trivial.

Now, if Δ does not contain all triangles then the computation carries over and we get

$$u_1u_2u_4u_5 \in \langle u_1, u_3, u_5\rangle.$$

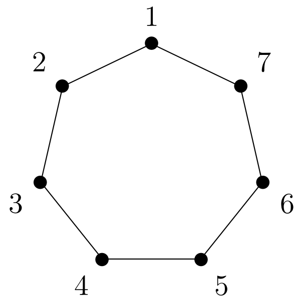
To see that the indeterminacy does not change, note that every triangle contains either the vertex 1 or 3. Therefore, adding triangles does not add any critical cells of multidegree $x_2x_4x_5x_6$ and so the indeterminacy remains the same. Thus, the Massey product remains non-trivial. The proof for the other four obstruction graphs is similar and will be omitted.

6.3.2 Massey products in low degrees

Let $\langle u_1, \cdots, u_n\rangle$ be a Massey product. If the homological degree of u_i is p_i, then $\langle u_1, \cdots, u_n\rangle$ is called a $(p_1, \ldots, p_n)$-Massey product. Our goal is to prove the following proposition.

Proposition 6.3.2. Let Δ be a simplicial complex. Suppose that the edge complement G of $\Delta^{(1)}$ contains a 7-cycle without cords. Then $\mathrm{Tor}^S(k[\Delta], k)$ has non-trivial (p, q, r)-Massey products for $(p, q, r) = (2, 1, 1), (1, 2, 1), (1, 1, 2)$.

Proof. Arguing as in [10], it is sufficient to prove the result for the case when the edge complement is given by



Therefore, we can assume that the Stanley-Reisner ideal I of Δ is generated by

$$\{u_i = x_i x_{i+1} \mid i = 1, \ldots, 6\} \cup \{u_7 = x_1 x_7\}.$$

Let $\mathcal{M}'$ be the matching consisting of $u_1u_2u_3u_4 \to u_1u_2u_4$, $u_4u_5u_6 \to u_4u_6$ as well as

$$\begin{array}{ll} u_2u_3u_4u_5u_6u_7 \to u_2u_4u_5u_6u_7 & u_1u_3u_4u_5u_6u_7 \to u_1u_3u_5u_6u_7 \\ u_1u_2u_4u_5u_6u_7 \to u_1u_2u_5u_6u_7 & u_1u_2u_3u_4u_6u_7 \to u_2u_3u_4u_6u_7 \\ u_1u_2u_3u_4u_5u_7 \to u_2u_3u_4u_5u_7 & u_1u_2u_3u_4u_5u_6 \to u_1u_2u_3u_4u_5 \end{array}$$

and

$$\begin{array}{lll} u_3u_4u_5u_6u_7 \to u_3u_5u_6u_7 & u_2u_3u_5u_6u_7 \to u_2u_3u_5u_7 & u_2u_3u_4u_5u_6 \to u_2u_4u_5u_6 \\ u_1u_4u_5u_6u_7 \to u_1u_4u_6u_7 & u_1u_3u_4u_6u_7 \to u_1u_3u_4u_6 & u_1u_3u_4u_5u_7 \to u_1u_3u_5u_7 \\ u_1u_3u_4u_5u_6 \to u_1u_3u_5u_6 & u_1u_2u_4u_6u_7 \to u_1u_2u_4u_6 & u_1u_2u_4u_5u_7 \to u_2u_4u_5u_7 \\ u_1u_2u_3u_6u_7 \to u_1u_3u_6u_7 & u_1u_2u_3u_4u_7 \to u_1u_3u_4u_7 & u_1u_2u_3u_4u_7 \to u_1u_2u_4u_7 \end{array}$$

$$u_1u_2u_3u_4u_5u_6u_7 \to u_1u_2u_3u_5u_6u_7$$

Next, extend $\mathcal{M}'$ to some maximal Morse matching $\mathcal{M}$. To get rid of signs we will work over $\mathbb{Z}/2$ but it is clear the computation carries over to any field. We have

$$\lambda_2(u_1u_2, u_4, u_6) = \lambda_2(\phi(u_1u_2u_4), u_6) + \lambda_2(u_1u_2, \phi(u_4u_6)) = u_1u_2u_3u_4u_6 + u_1u_2u_4u_5u_6.$$

Next,

$$\begin{aligned} p(u_1u_2u_3u_4u_6) &= (1 - d\phi + \phi d)(u_1u_2u_3u_4u_6) \\ &= u_1u_2u_3u_4u_6 + \phi(u_1u_3u_4u_6) + \phi(u_1u_2u_4u_6) + a \\ &= u_1u_2u_3u_4u_6 + u_1u_3u_4u_6u_7 + u_1u_2u_4u_6u_7 \end{aligned}$$

where $a = x_1\phi(u_2u_3u_4u_6) + x_5\phi(u_1u_2u_3u_6) + x_7\phi(u_1u_2u_3u_4)$. Since $a \in (x_1, \ldots, x_7)$, it will vanish after tensoring with k and hence can be ignored in the computation. Arguing similarly we get

$$p(u_1u_2u_4u_5u_6) = u_1u_2u_4u_5u_6 + u_1u_2u_4u_6u_7$$

and so

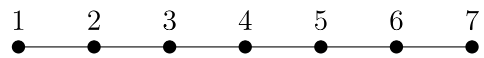
$$\mu_3(u_1u_2, u_4u, u_6) = u_1u_2u_3u_4u_6 + u_1u_3u_4u_6u_7 + u_1u_2u_4u_5u_6.$$

Observe that $u_1u_2u_3u_4u_6, u_1u_2u_4u_5u_6 \in (u_1u_2, u_6)$. Therefore, the Massey product $\langle u_1u_2, u_4, u_6\rangle$ is trivial if and only if $u_1u_3u_4u_6u_7 \in (u_1u_2, u_6)$ if and only if $u_1u_3u_4u_6u_7 \in (u_6)$. But by considering supports we see

$$u_1u3u_4u_7 \smile u_6 = 0$$

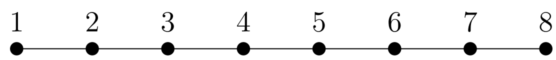
and therefore the Massey product is non-trivial. The proofs for $(p, q, r) = (1, 2, 1), (1, 1, 2)$ are similar and will be omitted. □

An interesting observation is that by contrast to Proposition 6.3.1 it is necessary to have a cycle in order for there to be Massey products of the form $(2, 1, 1)$, $(1, 2, 1)$ or $(1, 1, 2)$. Indeed, if the edge complement is



then the projective dimension of S/I is 4. Since the homological degree of Massey products of the form $(2, 1, 1)$, $(1, 2, 1)$ or $(1, 1, 2)$ is 5, they must all be trivial. The next example shows that adding edges does not produce non-trivial Massey products of the required form.

Example 6.3.3. Suppose the edge complement is given by



Then the ideal I is generated by

$$\{u_i = x_ix_{i+1} \mid i = 1, \ldots, 7\}.$$

We show that the $(2, 1, 1)$-Massey product $\langle u_1u_2, u_4, u_6\rangle$ is trivial. Let T denote the Taylor resolution of I. As usual, we will use $\mathbb{Z}/2$-coefficients. Our first goal is to compute

$$\lambda_3(u_1u_2, u_4, u_6) = \lambda_2(\phi\lambda_2(u_1u_2, u_4), u_6) + \lambda_2(u_1u_2, \phi\lambda_2(u_4, u_6)).$$

We have $\lambda_2(u_1u_2, u_4) = u_1u_2u_4$ and $\lambda_2(u_4, u_6) = u_4u_6$. Observe that the only invertible arrow $u \to u_1u_2u_4$ in G_T is $u_1u_2u_3u_4 \to u_1u_2u_4$. Therefore, we can restrict our attention to Morse matchings $\mathcal{M}$ such that $u_1u_2u_3u_4 \to u_1u_2u_4 \in \mathcal{M}$. For any such Morse matching $\mathcal{M}$ on T

$$\phi(u_1u_2u_4) = u_1u_2u_3u_4.$$

Arguing similarly, we get

$$\phi(u_4u_6) = u_4u_5u_6$$

and so

$$\lambda_3(u_1u_2, u_4, u_6) = u_1u_2u_3u_4u_6 + u_1u_2u_4u_5u_6.$$

Next, we need to compute

$$\mu_3(u_1u_2, u_4, u_6) = p\lambda_3(u_1u_2, u_4, u_6) = p(u_1u_2u_3u_4u_6) + p(u_1u_2u_4u_5u_6).$$

For this, let $\mathcal{M}'$ be the matching consisting of $u_1u_2u_3u_4 \to u_1u_2u_4$, $u_4u_5u_6 \to u_4u_6$ as well as

$$\begin{array}{cc}
u_1u_2u_3u_4u_5u_6 \to u_1u_3u_4u_5u_6 & u_1u_2u_3u_4u_5u_7 \to u_1u_2u_4u_5u_7 \\
u_1u_2u_3u_5u_6u_7 \to u_1u_2u_3u_5u_7 & u_1u_2u_4u_5u_6u_7 \to u_1u_2u_4u_6u_7 \\
u_1u_3u_4u_5u_6u_7 \to u_1u_3u_4u_5u_7 & u_2u_3u_4u_5u_6u_7 \to u_2u_3u_4u_5u_7 \\
u_1u_2u_3u_4u_5u_6u_7 \to u_1u_2u_3u_4u_6u_7 &
\end{array}$$

and

$$\begin{array}{ccc}
u_3u_4u_5u_6u_7 \to u_3u_5u_6u_7 & u_2u_4u_5u_6u_7 \to u_2u_4u_6u_7 & u_2u_3u_5u_6u_7 \to u_2u_3u_5u_7 \\
u_2u_3u_4u_5u_6 \to u_2u_4u_5u_6 & u_1u_4u_5u_6u_7 \to u_1u_4u_6u_7 & u_1u_3u_5u_6u_7 \to u_1u_3u_5u_7 \\
u_1u_2u_5u_6u_7 \to u_1u_2u_5u_7 & u_1u_2u_4u_5u_6 \to u_1u_2u_4u_6 & u_1u_2u_3u_6u_7 \to u_1u_3u_6u_7 \\
u_1u_2u_3u_5u_6 \to u_1u_3u_5u_6 & u_1u_2u_3u_4u_7 \to u_1u_3u_4u_7 & u_1u_2u_3u_4u_6 \to u_1u_3u_4u_6 \\
 & u_1u_2u_3u_4u_5 \to u_1u_3u_4u_5 &
\end{array}$$

Next, extend $\mathcal{M}'$ to some maximal Morse matching $\mathcal{M}$. We get

$$p(u_1u_2u_3u_4u_6) = u_1u_2u_4u_5u_6.$$

and

$$p(u_1u_2u_4u_5u_6) = 0.$$

Since $u_1u_2u_4u_5u_6 \in (u_1u_2)$, we have that $u_1u_2u_4u_5u_6$ projects to zero in

$$\operatorname{Tor}_5^S(R,k)/(u_1u_2, u_6)$$

and so the Massey product $\langle u_1u_2, u_4, u_6\rangle$ is trivial. The proof for the other two types of Massey products is similar and will be omitted. Note that adding additional edges to the chain does not change the conclusion since the corresponding generators will have multidegrees that are disjoint from the Massey product.

6.4 Representability of Massey products

By Theorem 4.1.7, we know that for any A_∞-structure $\{\mu_n\}$ obtained from the homotopy transfer theorem (Theorem 2.3.3) we have

$$\mu_n(a_1 \otimes \cdots \otimes a_n) \in \langle a_1, \ldots, a_n \rangle.$$

An interesting question is the converse, that is, if $\langle a_1, \ldots, a_n \rangle$ is a Massey product on $\mathrm{Tor}^S(R,k)$ and $a \in \langle a_1, \ldots, a_n \rangle$ does there exist an A_∞-structure $\{\mu_n\}$ such that

$$\mu_n(\alpha_1, \ldots, \alpha_n) = a?$$

The main tool in studying this question is the following proposition.

Proposition 6.4.1 ([40], Proposition 2.11)**.** Let A be a dg algebra and let $\langle \alpha_1, \ldots, \alpha_n \rangle$ be a Massey product on the homology algebra HA. Let $\alpha \in \langle \alpha_1, \ldots, \alpha_n \rangle$. If there exists a defining system $\{a_{ij}\}$ such that $\{da_{ij}\}_{j-i \geq 2}$ is a linearly independent set, then there exists an A_∞-structure $\{\mu_n\}$ on HA such that

$$\mu_n(\alpha_1 \otimes \cdots \otimes \alpha_n) = \pm\alpha.$$

The following theorem gives a partial answer to the above question.

Theorem 6.4.2. Let R be a monomial ring and let $\langle \alpha_1, \ldots, \alpha_n \rangle$ be a non-trivial Massey product on $\mathrm{Tor}^S(R,k)$. Then for all $a \in \langle \alpha_1, \ldots, \alpha_n \rangle$ there exists an A_∞-structure $\{\mu_n\}$ on $\mathrm{Tor}^S(R,k)$ such that

$$\pm\mu_n(\alpha_1, \ldots, \alpha_n) = a.$$

Proof. If $n = 2$, then there is nothing to prove since μ_2 is just the product on $\mathrm{Tor}^S(R,k)$. So assume that $n \geq 2$. Let $\langle \alpha_1, \ldots, \alpha_n \rangle$ be a non-trivial Massey product on $\mathrm{Tor}^S(R,k)$ and pick some $a \in \langle \alpha_1, \ldots, \alpha_n \rangle$. By definition there exists some defining system $\{a_{ij}\}$ for a. By Proposition 6.4.1, it is sufficient to show that the set $\{da_{ij}\}_{j-i\geq 2}$ is linearly independent. Note that

$$[da_{ij}] \in \langle \alpha_{i+1}, \ldots, \alpha_j \rangle$$

and so

$$\operatorname{mdeg} da_{ij} = \operatorname{lcm}(\alpha_{i+1}, \ldots, \alpha_j).$$

Since $\langle \alpha_1, \ldots, \alpha_n \rangle$ is non-trivial, the α_i have pairwise disjoint multidegrees. Therefore,

$$\operatorname{mdeg} da_{ij} = \operatorname{mdeg} da_{i'j'} \implies da_{ij} = da_{i'j'}.$$

Since there are no relations between components in different multidegrees, it follows that in particular there is no linear relation between the da_{ij}. $\square$

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