STRATIFIED LANGLANDS DUALITY IN THE A_n TOWER

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Abstract. Let $\mathbf{S}_k$ denote a maximal torus in the complex Lie group $\mathbf{G} = \mathrm{SL}_n(\mathbb{C})/C_k$ and let T_k denote a maximal torus in its compact real form $\mathrm{SU}_n(\mathbb{C})/C_k$, where k divides n. Let W denote the Weyl group of $\mathbf{G}$, namely the symmetric group $\mathfrak{S}_n$. We elucidate the structure of the extended quotient $\mathbf{S}_k/\!/W$ as an algebraic variety and of $T_k/\!/W$ as a topological space, in both cases describing them as bundles over unions of tori. Corresponding to the invariance of K-theory under Langlands duality, this calculation provides a homotopy equivalence between $T_k/\!/W$ and its dual $T_{\frac{n}{k}}/\!/W$. Hence there is an isomorphism in cohomology for the extended quotients. Moreover this is stratified as a direct sum over conjugacy classes of the Weyl group. We derive a formula for the periodic cyclic homology of the group ring of an extended affine Weyl group in terms of these extended quotients and use our formulae to compute a number of examples of homology, cohomology and K-theory.

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1. Introduction

In [7] we introduced an equivariant Poincaré duality for finite group actions on tori, giving a natural isomorphism

$$K_W^*(T) \cong K_*^W(T^\vee).$$

In the context of the action of the Weyl group on a maximal torus for a compact connected semisimple Lie group, this provides a canonical pairing between the K-theory of the group C^*-algebra of an extended affine Weyl group and of its Langlands dual. In his consideration of the K-theory of Hecke algebras, Solleveld [9, 11] determined the K-theory of the group C^*-algebra for the affine Weyl groups associated to a number of

Lie Groups including $\mathrm{SL}_n(\mathbb{C})$. In order to do so he computed the extended quotients of maximal tori up to homotopy.

In this paper we compute the extended quotient of the maximal torus by the Weyl group for all semisimple (real and complex) Lie groups of type A_n, giving a detailed construction of the extended quotients as varieties and topological spaces. In particular for $\mathrm{SU}_n(\mathbb{C})/C_k$ and its Langlands dual $\mathrm{SU}_n(\mathbb{C})/C_{\frac{n}{k}}$ with maximal tori $T_k, T_{\frac{n}{k}}$ respectively, we show that there is a homotopy equivalence between $T_k/\!/W$ and $T_{\frac{n}{k}}/\!/W$, inducing isomorphisms in cohomology, though this is not canonical. The extended quotients are stratified by the conjugacy classes of the Weyl group. The canonical pairing of $K^*_W(T_k)$ with $K^*_W(T_{\frac{n}{k}})$ provided by our Poincaré duality takes values in the representation ring $R(W)$. Taking the multiplicity of the trivial representation induces an integer valued pairing, which, via the equivariant Chern character of Baum and Connes [3], induces a faithful pairing

$$H^*(T_k/\!/W, \mathbb{C}) \times H^*(T_{\frac{n}{k}}/\!/W, \mathbb{C}) \to \mathbb{C}. \tag{1}$$

We examine the equivariant Chern character further in Section 3 and show that this is compatible with the stratification of the extended quotients by conjugacy classes. While (1) is stated here in the context of groups of type A_n as studied in this paper, we note in passing that the pairing it describes exists in general for dual tori $T, T^\vee$ corresponding to Langlands dual compact semisimple Lie groups.

In the complex case, the extended quotients are varieties which are unions of tori crossed with cyclic singularities, and we compute the structure of these varieties. We illustrate briefly the connection with p-adic groups. Let F be a local non-archimedean field such as the p-adic field $\mathbb{Q}_p$, and consider the p-adic group

$$\mathcal{G} = \mathrm{PGL}_n(F).$$

The Langlands dual of $\mathcal{G}$ is $\mathbf{G} = \mathrm{SL}_n(\mathbb{C})$. Suppose further that $p > n$. Let $\mathbf{S}$ be the standard maximal torus in $\mathrm{SL}_n(\mathbb{C})$, let T be the maximal compact subgroup of $\mathbf{S}$. According to [2, Theorem 5.2], the Iwahori-spherical block in the principal series of $\mathcal{G}$ admits the structure of the algebraic variety $\mathbf{S}/\!/W$. Furthermore, the set of tempered representations in the Iwahori-spherical block admits the structure of the extended quotient $T/\!/W$. Theorem 1.2 and Theorem 1.6, with $k = 1$ therefore provide explicit descriptions of these spaces of representations.

We now proceed to the statement of our results. Let C_k denote the cyclic group of k-th roots of unity in $\mathbb{C}$. If k divides n then, abusing notation, we will identify this group with the subgroup $C_k I_n$ in the centre of $\mathrm{SL}_n(\mathbb{C})$. We will first consider the Lie group $\mathrm{SL}_n(\mathbb{C})/C_k$ where the corresponding Weyl group is the symmetric group $\mathfrak{S}_n$. Its conjugacy classes, indexed by cycle structures, correspond to partitions of n.

Definition 1.1. For a partition μ of n given by $n = \mu_1 + \ldots + \mu_c$ the numbers $\mu_1 \ldots, \mu_c$ will be called the *parts* of μ and the parts have *multiplicities* $m_j = |\{i \mid \mu_i = j\}|$. We will denote the greatest common divisor of the parts of μ by $g(\mu)$ and the greatest common divisor of the multiplicities of the parts by $m(\mu)$. We also let $b(\mu)$ denote the number of distinct parts of μ and will write $c(\mu)$ for the total number of parts. For each i let $p_i(\mu)$

denote the number of distinct parts of μ which have multiplicity strictly greater than i. We will omit the μ decoration on these symbols where the context is clear.

Theorem 1.2. *Let $\mathbf{S}_k$ denote a maximal torus of $\mathrm{SL}_n(\mathbb{C})/C_k$ and W denote the corresponding Weyl group. The extended quotient $\mathbf{S}_k//W$ is an algebraic variety which decomposes as a disjoint union of irreducible components. Each component is the product of a complex torus with a cyclic quotient of a complex affine space. Specifically we have a disjoint union over partitions μ of n:*

$$\mathbf{S}_k//W \cong \coprod_{\mu} \coprod_{\omega \in C_{\gcd(g(\mu),k)}} (\mathbb{C}^\times)^{b(\mu)-1} \times \mathbb{A}_{\mu,\omega} \times X_{\mu,\omega}.$$

The space $X_{\mu,\omega}$ is a discrete set of cardinality $\gcd(g(\mu)/|\omega|, n/k)$. The space $\mathbb{A}_{\mu,\omega}$ is the cyclic singularity

$$\mathbb{A}^{c(\mu)-b(\mu)}/C_{\gcd(m(\mu),\frac{k}{|\omega|})}$$

where the generator η of the group $C_{\gcd(m(\mu),\frac{k}{|\omega|})}$ acts by multiplication by powers of η on the factors of $\mathbb{A}^{c(\mu)-b(\mu)}$. For each i there are $p_i(\mu)$ factors on which the generator acts by multiplication by η^i.

Corollary 1.3. *With $\mathbf{S}_k$ and W as above there is a homotopy equivalence*

$$\mathbf{S}_k//W \cong \coprod_{\mu} (\mathbb{C}^\times)^{b(\mu)-1} \times Y_\mu.$$

where Y_μ is a discrete set of cardinality $\dfrac{g(\mu)}{a}\displaystyle\sum_{s=0}^{a-1}\gcd(a,s)$ where $a = \gcd\left(g(\mu), \dfrac{n}{g(\mu)}, k, \dfrac{n}{k}\right)$.

In particular, interchanging $\mathrm{SL}_n(\mathbb{C})/C_k$ with its Langlands dual does not change the homotopy type of the extended quotient, and moreover this holds for each μ component.

Remark 1.4. While passing from $\mathrm{SL}_n(\mathbb{C})/C_k$ to the Langlands dual group $\mathrm{SL}_n(\mathbb{C})/C_{n/k}$ does not change the homotopy type of the extended quotient, the singularity structure of the cyclic quotients can and often does vary. In particular when $k = 1$ the varieties are smooth, while the dual case $k = n$, considered by Solleveld, will always give the most singular case in the tower. In that case each Y_μ has cardinality $g(\mu)$, recovering Solleveld's formula, [9].

Remark 1.5. The number $\sum_{s=0}^{a-1}\gcd(a,s)$ appearing in the formula for cardinality of the set Y_μ is the ath value of Pillai's arithmetical function [8]. The cardinality of Y_μ can alternatively be expressed as $g(\mu)$ times the sum $\sum_{d|a}\frac{\phi(d)}{d}$.

We now consider the real case.

Theorem 1.6. *Let T_k denote a maximal torus of $\mathrm{SU}_n(\mathbb{C})/C_k$ and W denote the corresponding Weyl group. The extended quotient $T_k//W$ is a disjoint union of compact orbifolds with boundary. Each component is a bundle over a compact torus with fibre a cyclic quotient of a polysimplex. Specifically we have a disjoint union over partitions μ of n:*

$$T_k//W \cong \coprod_{\mu} \coprod_{\omega \in C_{\gcd(g(\mu),k)}} E_{\mu,\omega}/C_{\gcd(m(\mu),\frac{k}{|\omega|})} \times X_{\mu,\omega}.$$

The space $X_{\mu,\omega}$ is a discrete set of cardinality $\gcd(g(\mu)/|\omega|, n/k)$. The space $E_{\mu,\omega}$ is a bundle of polysimplices over a torus of dimension $b(\mu)-1$.

The group $C_{\gcd(m(\mu),\frac{k}{|\omega|})}$ preserves the torus and acts on the polysimplicial fibres, which are products of simplices of dimensions $m_j - 1$, where j ranges over the distinct parts of the partition μ. Each simplex can be regarded as a join of $m_j/\gcd(m(\mu), \frac{k}{|\omega|})$ simplices of dimension $\gcd(m(\mu), \frac{k}{|\omega|}) - 1$ on each of which which the group acts by cyclically permuting the vertices. This action preserves the orientation on the fibres if and only if c is odd or the 2-adic norms satisfy $|c|_2 < |\gcd(m(\mu), \frac{k}{|\omega|})|_2$.

The precise construction of the bundle $E_{\mu,\omega}$ is given in section 7 below.

Corollary 1.7. *With T_k and W as above there is a homotopy equivalence:*

$$T_k//W \cong \coprod_{\mu} (S^1)^{b(\mu)-1} \times Y_\mu.$$

where Y_μ is a discrete set of cardinality $\dfrac{g(\mu)}{a}\displaystyle\sum_{s=0}^{a-1} \gcd(a,s)$ *where* $a = \gcd\left(g(\mu), \dfrac{n}{g(\mu)}, k, \dfrac{n}{k}\right)$.

In particular, interchanging $\mathrm{SU}_n(\mathbb{C})/C_k$ with its Langlands dual does not change the homotopy type of the extended quotient, and moreover this holds for each μ component.

We now consider the relationship between the homotopy equivalence provided by Corollary 1.7 and the faithful pairing in cohomology provided by (1). We begin by expanding on the construction of the pairing. There is an isomorphism $K^*_W(T) \cong K_*(C(T) \rtimes W)$ and, using our Poincaré duality $K^*_W(T^\vee) \cong K^W_*(T) \cong K^*(C(T) \rtimes W)$. The pairing between K-theory and K-homology of the algebra $C(T) \rtimes W$ thus provides a canonical integer valued pairing between $K^*_W(T)$ and $K^*_W(T^\vee)$. This pairing is obtained from the $R(W)$-valued pairing of the groups $K^*_W(T)$ and $K^*_W(T^\vee)$, by taking the multiplicity of the trivial representation as alluded to above.

Regarding $R(W)\otimes\mathbb{C}$ as the algebra of class functions on W, the characteristic function of a conjugacy class $[g]$, gives an idempotent e_g. The K-theory groups $K^*_W(T)$ and $K^*_W(T^\vee)$ are modules over $R(W)$ and hence these idempotents provide a decomposition of these two groups indexed by the conjugacy classes. The extended quotients also decompose as a coproduct indexed by conjugacy classes, for example

$$T//W \cong \coprod_{[g]} T^g/Z(g)$$

and similarly for $T^\vee$. We show in Section 3 that the equivariant Chern character of Baum and Connes [3] is compatible with these decompositions in the following sense. For each conjugacy class $[g]$ the map restricts to give:

$$e_g(K^*_W(T) \otimes \mathbb{C}) \xrightarrow{\cong} H^*(T^g/Z(g); \mathbb{C})$$
$$e_g(K^*_W(T^\vee) \otimes \mathbb{C}) \xrightarrow{\cong} H^*((T^\vee)^g/Z(g); \mathbb{C}).$$

The pairing of $K^*_W(T)$ and $K^*_W(T^\vee)$ also respects the stratification in the following sense. Given classes $\mathbf{x}$ in $K^*_W(T)$ and $\mathbf{y}$ in $K^*_W(T^\vee)$, the equivariant pairing of these is given by $(\mathbf{x} \otimes_{\mathbb{C}} \mathbf{y})\mathcal{Q} \in KK_W(\mathbb{C}, \mathbb{C}) = R(W)$ where $\mathcal{Q} \in KK_W(C(T) \otimes C(T^\vee), \mathbb{C})$ is the K-homology element giving the Poincaré duality isomorphism. The external product $K^*_W(T) \times K^*_W(T^\vee) \to K^*_W(T \times T^\vee)$ is $R(W)$-bilinear, so in particular the pairing of $e_g\mathbf{x}$ with $\mathbf{y}$ agrees with the pairing of $\mathbf{x}$ with $e_g\mathbf{y}$. As this holds for the $R(W)$-valued pairing, taking multiplicities it is also true for the integral pairing.

We conclude that the induced faithful pairing (1) respects the stratification of the extended quotients by the conjugacy classes of W, i.e. we have a direct sum of pairings indexed by the conjugacy classes of W.

Corollary 1.7 may thus be viewed as a refinement of our Poincaré duality pairing of [7], stratifying this over the conjugacy classes of the Weyl group and additionally giving an isomorphism between the homology and cohomology.

In summary Corollaries 1.3 and 1.7 allow us to compute the cohomology and K-theory groups of the extended quotients. Hence using the equivariant Chern character for K-theory, we are able to compute $K_*(C^*_r W'_a)$ in the A_n case. Analoguously, one can use the well-known identification

$$\mathrm{HP}_* \, \mathbb{C}[W'_a(G)] \simeq \bigoplus_{k \in \mathbb{Z}} \mathrm{H}^{q+2k}(\mathbf{S}^\vee /\!/ W; \mathbb{C}).$$

to compute the periodic cyclic homology $\mathrm{HP}_* \, \mathbb{C}[W'_a(G)]$. We include as appendices a number of tables of these computations.

As an application, let $\mathcal{G}$ be a split reductive p-adic group, let $\mathcal{T}$ be a maximal torus in $\mathcal{G}$. Let $\mathbf{H}, \mathbf{S}$ be the Langlands dual groups of $\mathcal{G}, \mathcal{T}$, and let G, T be the compact real forms of $\mathbf{H}, \mathbf{S}$. Let $\mathcal{I}$ denote an Iwahori subgroup of $\mathcal{G}$ and let $C^*_r(\mathcal{G}, \mathcal{I})$ denote the reduced Iwahori-spherical C^*-algebra, as in [10]. According to [10, (5.8)], we have

$$K_* \, C^*_r(\mathcal{G}, \mathcal{I}) \simeq K_* \, C^*(X \rtimes W)$$

where W is the Weyl group of $\mathcal{G}$ and X is the cocharacter group $X_*(\mathcal{T})$ of $\mathcal{T}$. By T-duality we have

$$X_*(\mathcal{T}) = X^*(\mathbf{S}) = X^*(T).$$

The Fourier transform determines the isomorphism

$$C^*(X^*(T) \rtimes W) \simeq C(T) \rtimes W$$

since the Pontryagin dual of $X^*(T)$ is T. Applying the Green-Julg theorem we infer that

$$K_* \, C^*_r(\mathcal{G}, \mathcal{I}) \simeq K^*_W(T). \tag{2}$$

We note in passing that (2) confirms, in an important special case, the conjecture on p.82 of [1].

If $\mathcal{G} = \mathrm{PGL}_n(F)$ with F a non-archimedean local field, then T is a maximal torus in $\mathrm{SU}_n(\mathbb{C})$ and W is the symmetric group $\mathfrak{S}_n$; if $\mathcal{G} = \mathrm{SL}_n(F)$, then T is a maximal torus in $\mathrm{SU}_n(\mathbb{C})/C_n$ and W is again the symmetric group $\mathfrak{S}_n$. Our results thus provide computations of $K_* \, C^*_r(\mathcal{G}, \mathcal{I})$ in these cases.

2. Examples

Example 2.1. We consider the examples of $\mathrm{SL}_6(\mathbb{C})$ and $\mathrm{PSL}_6(\mathbb{C}) = \mathrm{SL}_6(\mathbb{C})/C_6$. In the former case, since $k = 1$, the variable ω in the summation formula can only take the value 1, and we give the variety $(\mathbb{C}^\times)^{b(\mu)-1} \times \mathbb{A}_{\mu,1}$ and its multiplicity $|X_{\mu,1}|$ for each partition μ of 6. In the latter case each partition will give rise to the same number of components as for $\mathrm{SL}_6(\mathbb{C})$, but in this case they are indexed by the value of ω which ranges over certain powers of the primitive 6th root of unity ζ. We list the possible values of ω (each giving a single component) and again describe the corresponding variety $(\mathbb{C}^\times)^{b(\mu)-1} \times \mathbb{A}_{\mu,\omega}$.

	$\mathrm{SL}_6(\mathbb{C})$		$\mathrm{PSL}_6(\mathbb{C})$	
μ	$\lvert X_{\mu,1}\rvert$	$(\mathbb{C}^\times)^{b(\mu)-1} \times \mathbb{A}_{\mu,1}$	ω	$(\mathbb{C}^\times)^{b(\mu)-1} \times \mathbb{A}_{\mu,\omega}$
6	6	$\mathbb{A}^0$	1	$\mathbb{A}^0$
			ζ	$\mathbb{A}^0$
			ζ^2	$\mathbb{A}^0$
			-1	$\mathbb{A}^0$
			ζ^4	$\mathbb{A}^0$
			ζ^5	$\mathbb{A}^0$
$1+5$	1	$\mathbb{C}^\times$	1	$\mathbb{C}^\times$
2+4	2	$\mathbb{C}^\times$	1	$\mathbb{C}^\times$
			-1	$\mathbb{C}^\times$
$1+1+4$	1	$\mathbb{C}^\times \times \mathbb{A}^1$	1	$\mathbb{C}^\times \times \mathbb{A}^1$
3+3	3	$\mathbb{A}^1$	1	$\mathbb{A}^1/\langle -1\rangle$
			ζ^2	$\mathbb{A}^1/\langle -1\rangle$
			ζ^4	$\mathbb{A}^1/\langle -1\rangle$
$1+2+3$	1	$(\mathbb{C}^\times)^2$	1	$(\mathbb{C}^\times)^2$
$1+1+1+3$	1	$\mathbb{C}^\times \times \mathbb{A}^2$	1	$\mathbb{C}^\times \times \mathbb{A}^2$
2+2+2	2	$\mathbb{A}^2$	1	$\mathbb{A}^2/\langle(\zeta^2,\zeta^4)\rangle$
			-1	$\mathbb{A}^2/\langle(\zeta^2,\zeta^4)\rangle$
$1+1+2+2$	1	$\mathbb{C}^\times \times \mathbb{A}^2$	1	$\mathbb{C}^\times \times \mathbb{A}^2/\langle(-1,-1)\rangle$
$1+1+1+1+2$	1	$\mathbb{C}^\times \times \mathbb{A}^3$	1	$\mathbb{C}^\times \times \mathbb{A}^3$
$1+1+1+1+1+1$	1	$\mathbb{A}^5$	1	$\mathbb{A}^5/\langle(\zeta^1,\zeta^2,\zeta^3,\zeta^4,\zeta^5)\rangle$

Note that the values of ω allowed for $\mathrm{PSL}_6(\mathbb{C})$ are exactly the set $|X_{\mu,1}|$ for $\mathrm{SL}_6(\mathbb{C})$. This is a general property of the cases $k = 1$ and $k = n$, however for intermediate cases the picture is more complicated. For each μ, the homotopy-types of the corresponding components agree, and in many cases the components are isomorphic as varieties. As noted above this is not true in general and indeed the component varieties for SL_6 and PSL_6 are not isomorphic (or even homeomorphic) in the cases $\mu = 2+2+2$, $\mu = 1+1+2+2$ and $\mu = 1+1+1+1+1+1$. While for SL_6, the varieties are smooth and the factor $\mathbb{A}_{\mu,1}$ is simply an affine space, for PSL_6 the factor $\mathbb{A}_{\mu,\omega}$ is a cyclic singularity as classified in [5].

We remark that in the PSL_6 example the spaces $\mathbb{A}_{\mu,\omega}$ do not in fact depend on ω. This is because 6 is square-free, hence $\gcd(m, \frac{k}{|\omega|}) = \gcd(m,k)$ for all ω.

Example 2.2. We now consider the dual examples of $\mathrm{SL}_6(\mathbb{C})/C_2$ and $\mathrm{SL}_6(\mathbb{C})/C_3$. Here we will see that, while the components again do not depend on the variable ω the possible values of ω associated to a partition μ do depend on k, as does the cardinality of $X_{\mu,\omega}$. Again we also note that the singularity structure is differs between the two groups for the partitions $2+2+2$, $1+1+2+2$ and $1+1+1+1+1+1$.

	$\mathrm{SL}_6(\mathbb{C})/C_2$			$\mathrm{SL}_6(\mathbb{C})/C_3$		
μ	ω	$\lvert X_{\mu,\omega}\rvert$	$(\mathbb{C}^\times)^{b(\mu)-1}\times\mathbb{A}_{\mu,\omega}$	ω	$\lvert X_{\mu,\omega}\rvert$	$(\mathbb{C}^\times)^{b(\mu)-1}\times\mathbb{A}_{\mu,\omega}$
6	1	3	$\mathbb{A}^0$	1	2	$\mathbb{A}^0$
				ζ^2	2	$\mathbb{A}^0$
	-1	3	$\mathbb{A}^0$	ζ^4	2	$\mathbb{A}^0$
$1+5$	1	1	$\mathbb{C}^\times$	1	1	$\mathbb{C}^\times$
2+4	1	1	$\mathbb{C}^\times$	1	2	$\mathbb{C}^\times$
	-1	1	$\mathbb{C}^\times$			
$1+1+4$	1	1	$\mathbb{C}^\times\times\mathbb{A}^1$	1	1	$\mathbb{C}^\times\times\mathbb{A}^1$
3+3	1	3	$\mathbb{A}^1/\langle -1\rangle$	1	1	$\mathbb{A}^1$
				ζ^2	1	$\mathbb{A}^1$
				ζ^4	1	$\mathbb{A}^1$
$1+2+3$	1	1	$(\mathbb{C}^\times)^2$	1	1	$(\mathbb{C}^\times)^2$
$1+1+1+3$	1	1	$\mathbb{C}^\times\times\mathbb{A}^2$	1	1	$\mathbb{C}^\times\times\mathbb{A}^2$
2+2+2	1	1	$\mathbb{A}^2$	1	2	$\mathbb{A}^2/\langle(\zeta^2,\zeta^4)\rangle$
	-1	1	$\mathbb{A}^2$			
$1+1+2+2$	1	1	$\mathbb{C}^\times\times\mathbb{A}^2/\langle(-1,-1)\rangle$	1	1	$\mathbb{C}^\times\times\mathbb{A}^2$
$1+1+1+1+2$	1	1	$\mathbb{C}^\times\times\mathbb{A}^3$	1	1	$\mathbb{C}^\times\times\mathbb{A}^3$
$1+1+1+1+1+1$	1	1	$\mathbb{A}^5/\langle(-1,1,-1,1,-1)\rangle$	1	1	$\mathbb{A}^5/\langle(\zeta^2,\zeta^4,1,\zeta^2,\zeta^4)\rangle$

To illustrate the possible dependence of $X_{\mu,\omega}$ and $\mathbb{A}_{\mu,\omega}$ on ω we must consider a value of n with square factors. Specifically we will consider $n = 16$. Since there are rather a lot of partitions of 16 we shall just select a few examples to demonstrate the process by which the components of $\mathbf{S}_k//W$ are constructed.

Example 2.3 (The quotient variety for $SL_{16}(\mathbb{C})/C_2$ corresponding to $\mu = 2{+}2{+}2{+}2{+}4{+}4$). The partition has greatest common divisor $g = 2$, so the total number of components is $|Y_\mu| = \frac{2}{a}\sum_{s=0}^{a-1}\gcd(a,s) = 3$ where $a = \gcd\left(2, \frac{n}{2}, k, \frac{n}{k}\right) = 2$.

The greatest common divisor $\gcd(g,k) = 2$ hence $\omega = \pm 1$. In the case $\omega = 1$ we have $|X_{\mu,\omega}| = \gcd(\frac{g}{|\omega|}, \frac{n}{k}) = \gcd(2,8) = 2$. The multiplicities are $m_2 = 4, m_4 = 2$ so $m = 2$ so the cyclic singularity is the quotient of $\mathbb{A}^{6-2}$ by the cyclic group $C_{\gcd(m,\frac{k}{|\omega|})} = C_{\gcd(2,8)} = C_2$. This is generated by $\eta = -1$ which acts by multiplication by $(\eta^1,\eta^1,\eta^2,\eta^3) = (-1,-1,1,-1)$. The exponents of η appearing here can easily be read as the non-zero entries of the Young tableau decorated as follows:

0			
1			
0			
1			
2			
3			

The b zeros appearing in the table correspond to (the trivial action on) the torus factor, which is a codimension-1 torus in $\mathbb{C}^b$.

Turning to the case $\omega = -1$ we see that $|X_{\mu,\omega}| = \gcd(\frac{g}{|\omega|}, \frac{n}{k}) = \gcd(1, 8) = 1$. The cyclic group $C_{\gcd(m, \frac{k}{|\omega|})} = C_{\gcd(2,1)}$ is trivial, hence for $\omega = 1$ we obtain a copy of $\mathbb{C}^\times \times \mathbb{A}^4$. In conclusion this partition yields three components, two copies of the space $\mathbb{C}^\times \times \mathbb{A}^4/(-1, -1, 1, -1)$ indexed by $\omega = 1$ and one copy of $\mathbb{C}^\times \times \mathbb{A}^4$ indexed by $\omega = -1$.

We now consider the Langlands dual case $k = 8$:

Example 2.4 (The quotient variety for $SL_{16}(\mathbb{C})/C_8$ corresponding to $\mu = 2{+}2{+}2{+}2{+}4{+}4$).

Again $|Y_\mu| = 3$, since exchanging $k = 2$ with $k = 8$ interchanges k with n/k and we obtain the same value $a = 2$. As above $g = m = 2$ and $\omega = \pm 1$. In the case $\omega = 1$ we again have $|X_{\mu,\omega}| = 2$ and we obtain two copies of the variety $\mathbb{C}^\times \times \mathbb{A}^4/(-1, -1, 1, -1)$. When $\omega = -1$ we have $|X_{\mu,\omega}| = \gcd(1, 2) = 1$, and the cyclic group is $C_{\gcd(m, \frac{k}{|\omega|})} = C_{\gcd(2,4)} = C_2$. This gives the variety $\mathbb{A}^4/(-1, -1, 1, -1)$ in contrast to the case $k = 2$ where we obtained a copy of the affine space itself . Hence this partition now yields three copies of the space $\mathbb{C}^\times \times \mathbb{A}^4/(-1, -1, 1, -1)$, two indexed by $\omega = 1$ and one indexed by $\omega = -1$.

We remark that for this particular choice of partition, taking $k = 4$ (the self dual case) would yield the same quotient variety as we obtain for $k = 8$. By way of contrast consider the components arising for the partition $\mu = 4 + 4 + 4 + 4$.

Example 2.5 (The quotient variety for $SL_{16}(\mathbb{C})/C_4$ and $SL_{16}(\mathbb{C})/C_8$ corresponding to the partition $\mu = 4 + 4 + 4 + 4$). We have $g = m = 4$, and ω lies in $C_{\gcd(g,k)} = C_4$ hence $\omega \in \{\pm 1, \pm i\}$. With $\omega = 1$ we have $|X_{\mu,\omega}| = \gcd(\frac{g}{|\omega|}, \frac{n}{k}) = \gcd(4, 4) = 4$. The cyclic singularity is the quotient of $\mathbb{A}^{4-1}$ by the cyclic group $C_{\gcd(4,4)} = C_4$. This is generated by $\eta = i$ which acts by multiplication by $(\eta^1, \eta^2, \eta^3) = (i, -1, -i)$:

0			
1			
2			
3			

Now set $\omega = -1$. We have $|X_{\mu,\omega}| = \gcd(2, 4) = 2$. The cyclic singularity is the quotient of $\mathbb{A}^3$ by the cyclic group $C_{\gcd(4,2)} = C_2$. This is generated by -1 which acts by multiplication by $(-1, 1, -1)$.

For each of $\omega = \pm i$, $|X_{\mu,\omega}| = \gcd(1, 4) = 1$ and the cyclic group acting is trivial because $k/|\omega| = 1$ so these two elements yield, between them, two copies of the affine space $\mathbb{A}^3$.

Hence this partition furnishes 8 components, four of them isomorphic to the cyclic singularity $\mathbb{A}^3/(i, -1, -i)$, two of them isomorphic to $\mathbb{A}^3/(-1, 1, -1)$ and two of them isomorphic to $\mathbb{A}^3$.

Now turning to the case $k = 8$ again we have, $\omega \in \{\pm 1, \pm i\}$ since $\gcd(4, 8) = 4$. With $\omega = 1$ we have $|X_{\mu,\omega}| = \gcd(\frac{g}{|\omega|}, \frac{n}{k}) = \gcd(4, 2) = 2$. The cyclic singularity is the quotient of $\mathbb{A}^3$ by the cyclic group $C_{\gcd(4,8)} = C_4$, yielding $\mathbb{A}^3/(i, -1, -i)$. For $\omega = -1$, $|X_{\mu,\omega}| = \gcd(2, 2) = 2$. The cyclic singularity is the quotient of $\mathbb{A}^3$ by the cyclic group $C_{\gcd(4,4)} = C_4$. This is generated by i which acts by multiplication by $(i, -1, -i)$.

For each of $\omega = \pm i$, $|X_{\mu,\omega}| = \gcd(1, 2) = 1$ and the cyclic group acting is has order $\gcd(4, 2) = 2$. Hence these two elements yield, between them, two copies of the cyclic singularity $\mathbb{A}^3/(-1, 1, -1)$.

Hence this partition now furnishes only 6 components (rather than the 8 appearing in the $k = 4$ case). Four of these are isomorphic to the cyclic singularity $\mathbb{A}^3/(i, -1, -i)$ and two of them are isomorphic to $\mathbb{A}^3/(-1, 1, -1)$. In general the number of components grows with $\gcd(k, n/k)$. However if n is square free the number of components will be constant in k. See the examples in table 3.

3. Extended quotients and the equivariant Chern character

In [3] Baum and Connes introduced a version of the Chern character on equivariant K-theory. The isotropy of the action is encoded in the target cohomology groups by taking the extended quotient which they defined as follows.

Let Γ be a finite group acting on a complex affine variety X by automorphisms. The quotient variety X/Γ is obtained by collapsing each orbit to a point.

For $x \in X$, the stabilizer group of x is denoted $\Gamma_x := \{\gamma \in \Gamma : \gamma x = x\}$. Let $c(\Gamma_x)$ denote the set of conjugacy classes of Γ_x. The extended quotient is obtained by replacing the orbit of x by $c(\Gamma_x)$. This is done in the following way.

The *inertia space* of the action is defined to be

$$\widetilde{X} := \{(\gamma, x) \in \Gamma \times X : \gamma x = x\}.$$

This is an affine subvariety of $\Gamma \times X$. The group Γ acts on $\widetilde{X}$:

$$\begin{aligned} \Gamma \times \widetilde{X} &\to \widetilde{X} \\ \alpha(\gamma, x) &= (\alpha\gamma\alpha^{-1}, \alpha x), \qquad \alpha \in \Gamma, \quad (\gamma, x) \in \widetilde{X}. \end{aligned}$$

The (geometric) extended quotient $X//\Gamma$ is, by definition, the usual quotient for the action of Γ on $\widetilde{X}$:

$$X//\Gamma := \widetilde{X}/\Gamma.$$

The projection onto the second factor $(\gamma, x) \mapsto x$ is Γ-equivariant and so passes to quotient spaces to give a morphism of affine varieties

$$\rho\colon X//\Gamma \to X/\Gamma.$$

This map is referred to as the projection of the extended quotient onto the ordinary quotient. The inclusion

$$X \hookrightarrow \widetilde{X}, \quad x \mapsto (e, x)$$

where e is the identity element of Γ, is Γ-equivariant and so passes to quotient spaces to give an inclusion of affine varieties $X/\Gamma \hookrightarrow X/\!/\Gamma$. This is referred to as the inclusion of the ordinary quotient in the extended quotient.

Similarly there is a projection map from the quotient space $X/\!/\Gamma$ to the set of conjugacy classes $c(\Gamma)$ in Γ. Selecting a set C of representatives for the conjugacy classes, each element of $X/\!/\Gamma$ can be represented by a pair (γ, x) with $\gamma \in C$. The point x lies in the fixed set X^γ and is determined up to the action of the centraliser $Z(\gamma)$ of γ, thus the extended quotient may be decomposed as a disjoint union of components:

$$X/\!/\Gamma \cong \coprod_{\gamma \in C} X^\gamma / Z(\gamma).$$

If X is a topological space on which Γ acts by homeomorphisms, then the same procedure will create the topological space $X/\!/\Gamma$.

Example 3.1. Consider the action of the Coxeter group $\langle s_1, s_2, s_3 \mid s_1^2 = s_2^2 = s_3^2 = (s_1 s_2)^3 = (s_2 s_3)^3 = (s_3 s_1)^3 = e \rangle$ on the plane generated by reflections in the sides of an equilateral triangle. The triangle is a strict fundamental domain so the quotient of the plane under the action can be identified with it. We regard this triangle as a cell complex in the natural way, and note that the stabiliser of any point in the interior of the triangle is trivial, so these orbit points do not ramify in the extended quotient. Points in the interior of an edge of the triangle have stabiliser isomorphic to the cyclic group of order 2 and so these edges are doubled. The edges corresponding to the conjugacy class of the identity are attached to the triangle (which is also labelled by the identity). The vertices have stabilisers isomorphic to the dihedral group D_3 which has three conjugacy classes corresponding to the decomposition of the group into the identity element, the reflections and the rotations, so each vertex ramifies into three vertices. The vertices labelled by the identity element are attached to the triangle, those labelled by the reflection conjugacy class are attached at the endpoints to the two edges labelled by reflections lying in that conjugacy class. The remaining three points remain unattached so the extended quotient has five components, one closed triangle, one triangular loop and three additional points.

Now let X be a topological space equipped with the action of a finite group W. The equivariant Chern character for discrete groups [3] gives a map

$$\mathrm{ch}_W : K_W^j(X) \to \oplus_l H^{j+2l}(X/\!/W; \mathbb{C})$$

which becomes an isomorphism when $K_W^j(X)$ is tensored with $\mathbb{C}$. This is given by the following composition:

$$K_W^*(X) \otimes \mathbb{C} \cong \left(K^*(\widetilde{X}) \otimes \mathbb{C}\right)^W \cong \left(H^*(\widetilde{X}; \mathbb{C})\right)^W \cong H^*(X/\!/W; \mathbb{C}),$$

where the middle isomorphism is the non-equivariant Chern character.

The first term is naturally equipped with the structure of an $R(W) \otimes \mathbb{C}$-module, and we will equip each of others similarly. An element of $R(W) \otimes \mathbb{C}$ is given by a class function

χ and we lift this to a function on $\widetilde{X}$ by $\chi^*(g,x) = \chi(g)$. As χ is a class function, χ^* descends to a well defined function on the quotient $\widetilde{X}/W = X/\!/W$. This induces actions on the K-theory and cohomology groups by multiplication as required. It is obvious that the second and third isomorphisms are now $R(W)\otimes\mathbb{C}$-module isomorphisms. We will now check that the first isomorphism is also an $R(W)\otimes\mathbb{C}$-module map.

We identify $K^*_W(X)$ with the KK-group $KK^*_W(\mathbb{C}, C_0(X))$ and elements in K-theory will be represented by a (graded or ungraded) equivariant Kasparov triple $(\mathcal{E}, 1, T)$, (with the identity representation of $\mathbb{C}$). The map from $K^*_W(X)$ to $K^*(\widetilde{X})\otimes\mathbb{C}$ is defined as follows: For each $g \in W$, one restricts the KK-element to the g-fixed set X^g, to obtain an element of $K^*_{\langle g\rangle}(X^g)$. Since the action of the group $\langle g\rangle$ on X^g is trivial, the restriction can be expressed as a sum of elements of the form $(\mathcal{E}_i \otimes V_{\pi_i}, 1, T_i\otimes 1)$ where (π_i, V_{π_i}) is a representation of $\langle g\rangle$ and W acts trivially on the module $\mathcal{E}_i$ The element $(\mathcal{E}_i \otimes V_{\pi_i}, 1, T_i\otimes 1)$ is mapped to

$$(\mathcal{E}_i, 1, T_i)\otimes \chi_{\pi_i}(g) \in K^*(X^g)\otimes\mathbb{C}$$

where χ_{π_i} denotes the character of π_i. Summing over i and g gives the required W-invariant element of $K^*(\widetilde{X})\otimes\mathbb{C}$.

Given $(\mathcal{E}, 1, T) \in K^*_W(X)$, and a representation (σ, V_σ) in $R(W)$, the product is simply given by $(\mathcal{E}\otimes V_\sigma, 1, T\otimes 1)$. Applying the map has the effect of replacing restriction $\sum\limits_i(\mathcal{E}_i \otimes V_{\pi_i}, 1, T_i\otimes 1)$ by $\sum\limits_i(\mathcal{E}_i \otimes V_{\pi_i}\otimes V_\sigma, 1, T_i\otimes 1\otimes 1)$. We thus obtain

$$\sum_{g\in W}\sum_i(\mathcal{E}_i, 1, T_i)\otimes\chi_{\pi_i\otimes\sigma}(g) = \sum_{g\in W}\sum_i(\mathcal{E}_i, 1, T_i)\otimes\chi_{\pi_i}(g)\chi_\sigma(g) \in K^*(\widetilde{X})\otimes\mathbb{C}.$$

Hence the map $K^*_W(X)\otimes\mathbb{C}\to K^*(\widetilde{X})\otimes\mathbb{C}$ is an $R(W)\otimes\mathbb{C}$ module map as claimed.

Regarding $R(W)\otimes\mathbb{C}$ as the algebra of class functions on W, the characteristic function of a conjugacy class $[g]$, gives an idempotent e_g. The sum of these over all conjugacy classes yields the identity, hence $K^*_W(X)$ decomposes as a direct sum

$$K^*_W(X)\otimes\mathbb{C} = \bigoplus_{[g]\text{ a conjugacy class}} e_g(K^*_W(X)\otimes\mathbb{C})$$

The observation that the equivariant Chern character is an $R(W)\otimes\mathbb{C}$-module isomorphism implies that this takes $e_g(K^*_W(X)\otimes\mathbb{C})$ to the image of $H^*(X/\!/W;\mathbb{C})$ under the idempotent e_g, however this is precisely the cohomology of the $[g]$-component of $X/\!/W$. The equivariant Chern character thus respects the stratification by conjugacy classes, thus restricting to give isomorphisms

$$e_g(K^*_W(X)\otimes\mathbb{C}) \xrightarrow{\cong} H^*(X^g/Z(g);\mathbb{C}).$$

4. On cyclic quotients

Quotients by cyclic group actions play a key role in this paper. In particular we will be concerned with cyclic quotients of real and complex tori, and of complex affine space. The varieties that occur in the latter case were classified in [5] and we refer the reader there for details. Here we record a useful technical lemma that allows us to simplify the descriptions of the quotients arising in the former case. It allows us to change coordinates on a torus to push the action entirely onto one factor, and can be regarded as an elementary generalisation of Bezout's identity.

Lemma 4.1. *Let G be an abelian group and $\zeta \in G$. Let $T = G^n$ and $p_1, \dots, p_n \in \mathbb{Z}$ so that T is equipped with an action of the cyclic group $\langle\zeta\rangle$ defined by $\zeta \cdot (g_1, \dots, g_n) = (\zeta^{p_1} g_1, \dots, \zeta^{p_n} g_n)$. Then there is an automorphism of T which intertwines the given action with the action $\zeta \diamond (h_1, \dots, h_n) := (\zeta^d h_1, h_2, \dots, h_n)$, where $d = \gcd(p_1, \dots, p_n)$. The automorphism is algebraic in the sense that each coordinate is given by a monomial in the variables.*

Proof. For an element $A \in SL_n(\mathbb{Z})$ we define an automorphism of T as follows. For $g = (g_1, \dots, g_n) \in T$ define $Ag \in T$ by the formula

$$(Ag)_i = \prod_j g_j^{a_{ij}}.$$

It is straightforward to verify that this is an action of $SL_n(\mathbb{Z})$ by automorphisms of T. Now by the Euclidean algorithm we may choose an element $A \in SL_n(\mathbb{Z})$ such that the first column is the (transpose of the) vector $(p_1/d, \dots, p_n/d)$.

$$(A(\zeta \diamond g))_i = \prod_{j=1}^{n} (\zeta \diamond g)_j^{a_{ij}} = (\zeta^d g_1)^{a_{i1}} \prod_{j=2}^{n} g_j^{a_{ij}} = \zeta^{d a_{i1}} \prod_{j=1}^{n} g_j^{a_{ij}} = \zeta^{p_i} (Ag)_i.$$

So, $A(\zeta \diamond g) = \zeta \cdot (Ag)$ as required. □

5. Proof of Theorem 1.2

As usual we identity the conjugacy classes of $W = \mathfrak{S}_n$ with partitions of n. Let μ be a partition of n and let m_j for $j = 1, \dots n$ denote the multiplicity of j in the partition μ. We view μ as the permutation of $1, \dots, n$ with cycle type $(1)^{m_1}(2)^{m_2} \dots (n)^{m_n}$. Specifically we take μ to fix the first m_1 elements, transpose the next m_2 pairs, etc., thus selecting representatives of the conjugacy classes as required. For each μ we will compute the fixed set and identify the quotient by the centraliser $Z(\mu)$ of μ.

Let $\mathbf{S}$ be the standard maximal torus in $\mathrm{SL}_n(\mathbb{C})$ consisting of diagonal matrices such that the product of the diagonal entries is 1. Let $\mathbf{S}_k$ denote the maximal torus $\mathbf{S}/C_k$ in $\mathrm{SL}_n(\mathbb{C})/C_k$. An element of $\mathbf{S}_k$ is fixed by μ if selecting a representative $s \in \mathbf{S}$ there exists $\omega \in C_k$ such that $\mu \cdot s = \omega s$. It follows that for a cycle of μ of length j, the corresponding coordinates of s must have the form $a, \omega a, \omega^2 a, \dots \omega^{j-1} a$. Moreover $\omega^j a$ must equal a thus ω must satisfy the equation $\omega^j = 1$. So the order of ω divides the length of each cycle in μ. Since ω is an element of C_k it follows that this order, written $|\omega|$, divides $h = \gcd(g(\mu), k)$.

Elements of $\mathbf{S}_k$ fixed by μ are thus represented by elements of $\mathbf{S}$ of the form

$$s = (a_{1,1}, \dots, a_{1,m_1}, a_{2,1}, \omega a_{2,1}, \dots, a_{2,m_2}, \omega a_{2,m_2}, \dots).$$

We note that some of the multiplicities m_j will be zero, in which case the corresponding string will be empty.

We denote the C_k-orbit of s by $[a_{1,1}, \dots, a_{1,m_1}, a_{2,1} \dots, a_{2,m_2}, \dots]$, omitting terms containing a power of ω as these are determined by the others. Note that the total number of coordinates remaining in this notation is the number of parts $c(\mu)$ of μ.

The condition that s lies in $\mathbf{S}$ implies that

$$\Omega\Big(\prod_{i=1}^{m_1} a_{1,i}\Big)\Big(\prod_{i=1}^{m_2} a_{2,i}\Big)^2\Big(\prod_{i=1}^{m_3} a_{3,i}\Big)^3\cdots = 1,$$

where Ω is a power of ω. Specifically $\Omega = \omega^\alpha$ where $\alpha = \sum_{j=1}^n m_j\frac{(j-1)j}{2}$. Some of the products will be empty; for each of the non-empty products the exponent is divisible by g allowing us to write the formula as

$$\left(\Big(\prod_{i=1}^{m_1} a_{1,i}\Big)^{1/g}\Big(\prod_{i=1}^{m_2} a_{2,i}\Big)^{2/g}\Big(\prod_{i=1}^{m_3} a_{3,i}\Big)^{3/g}\cdots\right)^g = \Omega^{-1}. \tag{3}$$

The permutation μ is the product of disjoint cycles $\mu = \tau_{1,1}\dots\tau_{1,m_1}\tau_{2,1}\dots\tau_{2,m_2}\cdots$ where each $\tau_{j,i}$ is a j-cycle. The centraliser $Z(\mu)$ is the group generated by the cycles $\tau_{j,i}$ along with permutation groups $\mathfrak{S}_{m_1}, \mathfrak{S}_{m_2}, \dots$ with $\mathfrak{S}_{m_j}$ acting by permuting the m_j parts of size j. Hence $\mathfrak{S}_{m_j}$ simply permutes the coordinates $a_{j,1}, \dots, a_{j,m_j}$.

The action of $\tau_{j,i}$ on $[a_{1,1}, \dots, a_{1,m_1}, a_{2,1}\dots, a_{2,m_2}, \dots]$ has the effect of multiplying the coordinate $a_{j,i}$ by ω while leaving all other coordinates fixed.

For each j, l let $\sigma_{j,l}$ be the l-th degree symmetric polynomial in the variables $a_{j,1}^{|\omega|}, \dots, a_{j,m_j}^{|\omega|}$. By construction the actions of $\tau_{j,i}$ and $\mathfrak{S}_{m_j}$ on the coordinates leave $\sigma_{j,l}$ fixed for all j, l.

Conversely suppose that $(a_{j,i})$ and $(a'_{j,i})$ are coordinates yielding the same values of $\sigma_{j,l}$ for all j, l. It follows that the powers $a_{j,i}^{|\omega|}$ and $(a'_{j,i})^{|\omega|}$ agree up to a permutation (for each j) of the i indices. Hence multiplying each $a_{j,i}$ by some power of ω and permuting we obtain $a'_{j,i}$. So the coordinates are identified by the action of the centraliser $Z(\mu)$ precisely when they yield the same values of $\sigma_{j,l}$.

The action of the generator $\zeta = e^{2\pi i/k}$ of C_k on the coordinates has the effect of multiplying $\sigma_{j,l}$ by $\zeta^{l|\omega|}$. Note that $\zeta^{k/|\omega|}$ acts trivially on all the coordinates after factoring by the action of $Z(\mu)$, hence it is really an action of $C_{k/|\omega|}$.

To summarise we have now identified the quotient of the ω-part of the μ-fixed set by the centraliser $Z(\mu)$ as a subspace of $\mathbb{A}^{c(\mu)}/C_{k/|\omega|}$. We denote a point of the image by $[\sigma_{j,l}]$.

In order to identify the subspace we reformulate Equation 3 in terms of the variables $\sigma_{j,l}$. We have

$$\left(\prod_j \sigma_{j,m_j}^{j/g}\right)^{g/|\omega|} = \Omega^{-1} \tag{4}$$

where the index j in the product runs through the $b(\mu)$ distinct parts of μ.

The action of the generator ζ has the effect of multiplying $\prod_j \sigma_{j,m_j}^{j/g}$ by ζ^q where $q = \sum_j |\omega| j m_j/g = |\omega| n/g$. Now ζ^q generates a group of order

$$\frac{k}{\gcd(k,q)} = \frac{kg}{\gcd(kg, n|\omega|)} = \frac{g}{\gcd(g, k'|\omega|)}$$

where $k' = n/k$.

By Equation 4 the product $\prod_j \sigma_{j,m_j}^{j/g}$ can take $g/|\omega|$ values, however factoring out by the (free) action of ζ^q gives

$$\frac{g}{|\omega|} \cdot \frac{\gcd(g, k'|\omega|)}{g} = \gcd(g/|\omega|, k')$$

components. Let ξ be one of the $g/|\omega|$ roots of Ω^{-1} and consider the variety defined by the equation

$$\prod_j \sigma_{j,m_j}^{j/g} = \xi. \tag{5}$$

In these $b(\mu)$ variables this equation defines a complex torus of codimension 1. However there are a further $c(\mu) - b(\mu)$ variables $\sigma_{j,l}$ with $l < m_j$ which are unconstrained. So we obtain a variety of the form $(\mathbb{C}^\times)^{b(\mu)-1} \times \mathbb{A}^{c(\mu)-b(\mu)}$, and the component of the extended quotient corresponding to this variety is given by factoring out the action of the subgroup $C_{\gcd(k,q)} = \langle \zeta^{k/\gcd(k,q)} \rangle$. We recall that this generator acts on each coordinate $\sigma_{j,l}$ of the the variety as multiplication by $\zeta^{l|\omega|k/\gcd(k,q)}$.

First we will consider the action on the constrained coordinates σ_{j,m_j}. Here $\zeta^{k/\gcd(k,q)}$ acts by multiplication by $\zeta^{m_j|\omega|k/\gcd(k,q)}$, and we note that this is the restriction of the natural multiplication action on the larger variety $(\mathbb{C}^\times)^b$, where $b = b(\mu)$. Following the proof of Lemma 4.1 we select an element $A \in SL_b(\mathbb{Z})$ with first column equal to the transpose of the vector $\left(\frac{m_{j_1}}{m(\mu)}, \ldots, \frac{m_{j_b}}{m(\mu)}\right)$, where the indices j_i denote the distinct parts of μ. Abusing notation we will use j to denote an index in the set $J = \{j_1, \ldots, j_b\}$.

We now transform the coordinates σ_{j,m_j} by the inverse matrix A^{-1} to obtain new coordinates which we denote by ρ_i. Now (again abusing notation) the coordinates σ_{j,m_j} are recovered by the formula $\sigma_{j,m_j} = \prod_{i=1}^{b} \rho_i^{a_{ji}}$.

The subvariety defined by equation 5 is transformed by A^{-1} to the variety

$$\prod_{i=1}^{b} \prod_{j \in J} (\rho_i^{a_{ji}})^{j/g} = \prod_{i=1}^{b} \rho_i^{\sum_{j \in J} a_{ji} j/g} = \xi. \tag{6}$$

which we denote by $V_{\mu,\omega,\xi}$. Recall that there are still $c(\mu) - b(\mu)$ unconstrained coordinates $\sigma_{j,l}, j < m_j$ in this variety for which we have not changed variables.

Now let $d = \frac{m(\mu)|\omega|k}{\gcd(k,q)}$. According to the Lemma, in these coordinates the action of the generator multiplies the first coordinate ρ_1 by ζ^d and leaves $\rho_2, \ldots, \rho_b$ invariant. Let λ denote the order of ζ^d. i.e., $\lambda = \frac{k}{\gcd(d,k)}$. We will show that the quotient $\kappa := \frac{\sum_{j \in J} a_{j1} j/g}{\lambda}$ is an integer which will allow us to rewrite Equation 6 in the form

$$(\rho_1^\lambda)^\kappa \cdot \prod_{i=2}^{b} \rho_i^{\sum_{j \in J} a_{ji} j/g} = \xi. \tag{7}$$

where ρ_1^λ is left invariant by the action.

First we compute

$$\lambda = \frac{k}{\gcd\left(\frac{m(\mu)|\omega|k}{\gcd(k,q)}, k\right)} = \frac{\gcd(k,q)}{\gcd\left(m(\mu)|\omega|, \gcd(k,q)\right)} = \frac{\gcd(k,q)}{\gcd\left(m(\mu)|\omega|, k, q\right)} = \frac{\gcd(k,q)}{\gcd\left(m(\mu)|\omega|, k\right)},$$

since $m(\mu)|\omega|$ divides $q = \frac{n}{g}|\omega|$.

On the other hand, the exponent of ρ_1 is $\sum_j a_{j1} j/g = \sum_j \frac{m_j}{m(\mu)} \frac{j}{g} = \frac{n}{m(\mu)g}$. Thus

$$\kappa = \frac{n \gcd(m(\mu)|\omega|, k)}{m(\mu)g \gcd(k,q)} = \frac{\gcd(nm(\mu)|\omega|, nk)}{\gcd(m(\mu)gk, m(\mu)gq)} = \frac{\gcd(nk, nm(\mu)|\omega|)}{\gcd(m(\mu)gk, nm(\mu)|\omega|)},$$

which is an integer because $m(\mu)g$ divides n.

To recap, we have changed coordinates on the constrained variables replacing σ_{j,m_j} by ρ_i, and pushing the action there entirely onto the first coordinate ρ_1, $\rho_1 \mapsto \zeta^d \rho_1$.

We will now consider the unconstrained variables $\sigma_{j,l}$ where $l < m_j$, where the action is (still) given by $\sigma_{j,l} \mapsto \zeta^{l|\omega|k/\gcd(k,q)}\sigma_{j,l}$. Let

$$\eta = \zeta^{\lambda|\omega|k/\gcd(k,q)} = \zeta^{k|\omega|/\gcd(m(\mu)|\omega|,k)} = \zeta^{k/\gcd(m(\mu),k/|\omega|)}$$

which generates the cyclic group $C_{\gcd(m(\mu),k/|\omega|)}$. This acts on the affine space of dimension $c(\mu) - b(\mu)$ spanned by the unconstrained variables, with the generator η acting by $\sigma_{j,l} \mapsto \eta^l \sigma_{j,l}$. The cyclic singularity $\mathbb{A}_{\mu,\omega}$ is defined to be the quotient variety. It is easy to see that for each l there are $p_l(\mu)$ coordinates on which the action multiplies by η^l.

We will next define a map ϕ from the variety $V_{\mu,\omega,\xi} \subset (\mathbb{C}^\times)^b \times \mathbb{A}^{c-b}$ to $(\mathbb{C}^\times)^b \times \mathbb{A}_{\mu,\omega}$, with image a codimension 1 subvariety, where $c = c(\mu)$.

Recall that $\lambda = \frac{k}{\gcd(d,k)}$ so by Bezout's Lemma there exists an integer δ such that

$$\frac{d}{\gcd(d,k)} \cdot \delta \equiv 1 \mod \lambda. \tag{8}$$

Let $\widehat{\rho_1}$ denote a $\gcd(d,k)$-th root of ρ_1. While this is not uniquely defined, we will see that the construction that follows is independent of the choice. Set $\alpha_l = \dfrac{l|\omega|k\delta(\lambda-1)}{\gcd(k,q)}$, and define the map $\phi : V_{\mu,\omega,\xi} \subset (\mathbb{C}^\times)^b \times \mathbb{A}^{c-b} \to (\mathbb{C}^\times)^b \times \mathbb{A}_{\mu,\omega}$ by

$$\Big((\rho_1, \rho_2, \ldots, \rho_b), (\sigma_{j,l})_{j\in J, l<m_j}\Big) \mapsto \Big((\rho_1^\lambda, \rho_2, \ldots, \rho_b), (\widehat{\rho_1}^{\alpha_l}\sigma_{j,l})_{j\in J, l<m_j}\Big).$$

To see that this is well defined we need to verify that it is independent of the choice of $\widehat{\rho_1}$, but $\widehat{\rho_1}$ is defined up to multiplication by a $\gcd(d,k)$-th root of unity, all of which are powers of the primitive root ζ^λ, so it suffices to show that replacing $\widehat{\rho_1}$ by $\zeta^\lambda\widehat{\rho_1}$ does not change the image.

We have

$$\zeta^{\lambda\alpha_l} = \left(\zeta^{\frac{\lambda l|\omega|k}{\gcd(k,q)}}\right)^{\delta(\lambda-1)} = \left(\eta^l\right)^{\delta(\lambda-1)}$$

Hence as points in the quotient space $\mathbb{A}_{\mu,\omega}$,

$$\Big((\zeta^\lambda\widehat{\rho_1})^{\alpha_l}\sigma_{j,l}\Big)_{j\in J,l<m_j} = \left(\left(\eta^l\right)^{\delta(\lambda-1)}\widehat{\rho_1}^{\alpha_l}\sigma_{j,l}\right)_{j\in J,l<m_j} = \Big(\widehat{\rho_1}^{\alpha_l}\sigma_{j,l}\Big)_{j\in J,l<m_j}.$$

Now we will show that ϕ is constant on orbits of the action of $C_{\gcd(k,q)} = \langle\zeta^{k/\gcd(k,q)}\rangle$ of $V_{\mu,\omega,\xi}$. We have

$$\phi\Big((\zeta^d\rho_1,\rho_2,\dots,\rho_b),(\zeta^{l|\omega|k/\gcd(k,q)}\sigma_{j,l})\Big) = \Big((\zeta^{d\lambda}\rho_1^\lambda,\rho_2,\dots,\rho_b),((\zeta^{\frac{d}{\gcd(d,k)}}\widehat{\rho_1})^{\alpha_l}\zeta^{l|\omega|k/\gcd(k,q)}\sigma_{j,l})\Big)$$

Now $d\lambda = \frac{dk}{\gcd(d,k)}$ which is divisible by k so $\zeta^{d\lambda} = 1$. Also we compute

$$\frac{d}{\gcd(d,k)}\alpha_l + \frac{l|\omega|k}{\gcd(k,q)} = \left(\frac{d\delta(\lambda-1)}{\gcd(d,k)}+1\right)\frac{l|\omega|k}{\gcd(k,q)}.$$

Using Equation 8 we now observe that $\dfrac{d\delta(\lambda-1)}{\gcd(d,k)}+1 \equiv 0, \mod \lambda$ and set $r = \frac{1}{\lambda}\left(\frac{d\delta(\lambda-1)}{\gcd(d,k)}+1\right)$. It then follows that $\zeta^{\frac{d\alpha_l}{\gcd(d,k)}}\zeta^{l|\omega|k/\gcd(k,q)} = \eta^{rl}$, yielding

$$\begin{aligned}\phi\Big((\zeta^d\rho_1,\rho_2,\dots,\rho_b),(\zeta^{l|\omega|k/\gcd(k,q)}\sigma_{j,l})\Big) &= \Big((\rho_1^\lambda,\rho_2,\dots,\rho_b),(\eta^{rl}\widehat{\rho_1}^{\alpha_l}\sigma_{j,l})\Big)\\ &= \Big((\rho_1^\lambda,\rho_2,\dots,\rho_b),(\widehat{\rho_1}^{\alpha_l}\sigma_{j,l})\Big)\\ &= \phi\Big((\rho_1,\rho_2,\dots,\rho_b),(\sigma_{j,l})\Big)\end{aligned}$$

in $(\mathbb{C}^\times)^b\times\mathbb{A}_{\mu,\omega}$, as required.

We conclude that ϕ induces a map from the quotient of $V_{\mu,\omega,\xi}$ by the action of $C_{\gcd(k,q)}$ to the variety $(\mathbb{C}^\times)^b\times\mathbb{A}_{\mu,\omega}$. We will now show that this induced map is injective.

Suppose then that $\Big((\rho_1,\rho_2,\dots,\rho_b),(\sigma_{j,l})\Big)$ and $\Big((\rho_1',\rho_2',\dots,\rho_b'),(\sigma_{j,l}')\Big)$ have the same image under ϕ. Then ρ_1 and ρ_1' differ by a a power of ζ^d, noting that this is a primitive λ-th root of unity. The action of $C_{\gcd(k,q)}$ does not change $\rho_2,\dots\rho_b$, and it follows that up to the action of $C_{\gcd(k,q)}$ the first coordinates agree. Moreover we have noted that applying the action does not change the image under ϕ, so without loss of generality we may assume that $\rho_i' = \rho_i$ for all i. We will show that the $\sigma_{j,l}$ coordinates must now agree up to the action of the subgroup of $C_{\gcd(k,q)}$ which stabilises the ρ coordinates.

We are given that the coordinates $\widehat{\rho_1}^{\alpha_l}\sigma_{j,l}$ agree with $\widehat{\rho_1'}^{\alpha_l}\sigma_{j,l}'$ up to the action of $\langle\eta\rangle$, i.e., for some r

$$\widehat{\rho_1'}^{\alpha_l}\sigma_{j,l}' = \eta^{rl}\widehat{\rho_1}^{\alpha_l}\sigma_{j,l}$$

But since $\rho_1 = \rho_1'$, we may choose the roots $\widehat{\rho_1},\widehat{\rho_1'}$ to be equal, giving

$$\sigma_{j,l}' = \eta^{rl}\sigma_{j,l} = \zeta^{rl\lambda|\omega|k/\gcd(k,q)}\sigma_{j,l} = (\zeta^{l|\omega|k/\gcd(k,q)})^{r\lambda}\sigma_{j,l}.$$

But as noted above $\zeta^{d\lambda} = 1$ so $\zeta^{dr\lambda} = 1$ for each r. Hence

$$(\zeta^{\frac{k}{\gcd(k,q)}})^{r\lambda}\cdot\Big((\rho_1,\rho_2,\dots,\rho_b),(\sigma_{j,l})\Big) = \Big((\rho_1',\rho_2',\dots,\rho_b'),(\sigma_{j,l}')\Big)$$

so the map induced by ϕ is injective as required.

Finally we consider the image of the induced map. Given an element of $(\mathbb{C}^\times)^b \times \mathbb{A}_{\mu,\omega}$ represented by

$$\Big((\psi_1, \psi_2, \dots, \psi_b), (\tau_{j,l})\Big)$$

choose a λ-th root ρ_1 of ψ_1 and a $\gcd(d,k)$-th root $\widehat{\rho_1}$ of ρ_1. We set $\rho_i = \psi_i$ for $i \geq 2$ and $\sigma_{j,l} = \widehat{\rho_1}^{-\alpha_l}\tau_{j,l}$. Then $\Big((\rho_1, \rho_2, \dots, \rho_b), (\sigma_{j,l})\Big)$ is in the variety $V_{\mu,\omega,\xi}$ if and only if Equation 7 is satisfied by these coordinates. When this happens the image is precisely $\Big((\psi_1, \psi_2, \dots, \psi_b), (\tau_{j,l})\Big)$. Equation 7 translates in these coordinates into the equation

$$\psi_1^\kappa \cdot \prod_{i=2}^{b} \psi_i^{\sum_{j\in J} a_{ji} j/g} = \xi. \tag{9}$$

Since the variables $\tau_{j,l}$ are unconstrained the image is the product of a codimension-1 subvariety T of $(\mathbb{C}^\times)^b$ with the cyclic singularity $\mathbb{A}_{\mu,\omega}$.

Finally we will show that the exponents appearing in Equation 9 are coprime, from which it follows that the subvariety T is a single torus. Recall that g is the greatest common divisor of the parts j_i of the partition. We form a matrix B in $SL_b(\mathbb{Z})$ with first row given by $b_{1i} = j_i/g$, which is possible since the numbers j_i/g are coprime. Then the first row of the product BA has entries $(BA)_{1i} = \sum_j a_{ji} j/g$. Since the matrix BA has determinant 1 these entries are coprime. Dividing the first entry by λ to get the exponent κ of ψ_1 does not change the greatest common divisor. This completes the proof.

6. Proof of Corollary 1.3

The cyclic singularities are contractible so to prove Corollary 1.3 it suffices to show that

$$\left| \coprod_{\omega \in C_{\gcd(g(\mu),k)}} X_{\mu,\omega} \right| = \frac{g(\mu)}{a} \sum_{s=0}^{a-1} \gcd(a,s)$$

where $a = \gcd\left(g(\mu), \dfrac{n}{g(\mu)}, k, \dfrac{n}{k}\right)$.

There are $h = \gcd(g,k)$ possible values for ω. Writing ω as the pth power of the generator of C_h, the order of ω is $h/\gcd(h,p)$ so, setting $k' = n/k$, the cardinality is

$$\begin{aligned}\sum_{\omega\in C_h} \gcd(g/|\omega|, k') &= \sum_{p=0}^{h-1} \gcd\left(\frac{g\gcd(h,p)}{h}, k'\right) \\ &= \sum_{p=0}^{h-1} \gcd(\gcd(g, gp/h), k') = \sum_{p=0}^{h-1} \gcd(g, gp/h, k').\end{aligned}$$

Letting $h' = \gcd(g,k')$ and $a = hh'/g$ we observe that

$$a = \frac{\gcd(g,k)\gcd(g,k')}{g} = \frac{\gcd(g^2, gk, gk', kk')}{g} = \gcd(g,k,k',n/g).$$

The cardinality is now given by

$$\sum_{p=0}^{h-1} \gcd(h', gp/h) = \sum_{p=0}^{h-1} \gcd(h', ph'/a) = \frac{h'}{a}\sum_{p=0}^{h-1} \gcd(a, p)$$
$$= \frac{h'h}{a^2}\sum_{s=0}^{a-1} \gcd(a, s) = \frac{g}{a}\sum_{s=0}^{a-1} \gcd(a, s).$$

The third equality follows from the fact that $\gcd(a, p)$ repeats with period a, since $\gcd(a, p + a) = \gcd(a, p)$.

We remark that a is symmetric in the interchange of k with $k' = n/k$ giving the same number of components in each stratum for a group and its Langlands dual.

7. Proof of Theorem 1.6

Let T denote the standard maximal torus in $SU_n(\mathbb{C})$ consisting of diagonal matrices with entries of modulus 1 and with determinant 1. Let T_k denote the maximal torus T/C_k in $SU_n(\mathbb{C})/C_k$. The Weyl group is again the permutation group $W = \mathfrak{S}_n$ and conjugacy classes of elements correspond to partitions of n.

The fixed set in the maximal torus $T_k = T/C_k$ corresponding to a partition μ is a subspace of the fixed set for the action on $\mathbf{S}_k = \mathbf{S}/C_k$ considered in Section 5. Specifically the fixed set in T_k is the subspace of the fixed set in $\mathbf{S}_k$ where each coordinate is required to have modulus 1.

Recall that the points of the fixed set in $\mathbf{S}_k$ are parametrised by elements $\omega \in C_k$, with order dividing $h = \gcd(g, k)$, along with tuples of variables $(a_{j,i})$ where j ranges over the sizes of parts of the partition and i ranges from 1 to m_j, the multiplicity of j in the partition. These must satisfy the equation

$$\Big(\prod_{i=1}^{m_1} a_{1,i}\Big)^1 \Big(\prod_{i=1}^{m_2} a_{2,i}\Big)^2 \Big(\prod_{i=1}^{m_3} a_{3,i}\Big)^3 \cdots = \Omega^{-1}$$

where $\Omega = \omega^{\sum_{j=1}^n m_j \frac{(j-1)j}{2}}$, see Equation 3. We now require one additional constraint per variable, namely $|a_{j,i}| = 1$. The coordinates are not uniquely determined as we must factor out the action of C_k: changing each $a_{j,i}$ by a factor of $\zeta = e^{2\pi\sqrt{-1}/k}$ gives the same point of the quotient.

It is convenient to introduce polar coordinates $\theta_{j,i}$ such that $a_{j,i} = e^{2\pi\sqrt{-1}\,\theta_{j,i}}$. Since i is used as an index, we will not use it to denote the square root of -1. Taking Θ such that $\Omega = e^{2\pi\sqrt{-1}\,\Theta}$ the above equation yields

$$\sum_j \sum_{i=1}^{m_j} j\,\theta_{j,i} = -\Theta \mod \mathbb{Z} \tag{10}$$

where j ranges over the parts of μ. We denote the space of points $(\theta_{j,i})$ satisfying this equation by $V_{\mu,\omega}$.

For each μ, we must identify the quotient of this fixed set by the centraliser $Z(\mu)$ of this permutation. Recall that the centraliser is generated by cycles $\tau_{j,i}$ which act by multiplying $a_{j,i}$ by ω, and by the symmetric groups $\mathfrak{S}_{m_j}$ which act by permuting the coordinates $(a_{j,1}, \ldots, a_{j,m_j})$.

Lifting from the variables $a_{j,i}$ to $\theta_{j,i}$ has the effect of 'lifting' the centraliser as follows. The variables $\theta_{j,i}$ are determined by $a_{j,i}$, only modulo $\mathbb{Z}$. The group C_k acts by multiplying each $a_{j,i}$ by $\zeta = e^{2\pi\sqrt{-1}\frac{1}{k}}$ and hence the action lifts to an action of the infinite cyclic group $\mathbb{Z}$, where the generator, which we denote $\widetilde{\zeta}$ acts as a shift by $\frac{1}{k}$ on each $\theta_{j,i}$. It is easy to see that the cyclic group $\langle\omega\rangle$ is generated by $e^{2\pi\sqrt{-1}\frac{1}{|\omega|}}$, thus the variables $\theta_{j,i}$ can additionally be shifted by any integer multiple of $\frac{1}{|\omega|}$. Permuting the variables $a_{j,i}$ simply corresponds to permuting $\theta_{j,i}$. Hence the action of the centraliser corresponds to the action on the $\theta_{j,i}$ coordinates generated by the lattice with coordinates in $\frac{1}{|\omega|}\mathbb{Z}$, the shift $\widetilde{\zeta}$, and by these permutations. We will denote this group by $\widetilde{Z}(\mu, \omega)$.

We now introduce variables $\sigma_j = \sum_{i=1}^{m_j} \theta_{j,i}|\omega|$. These correspond to the maximal degree symmetric polynomials in the complex case. We rewrite Equation 10 in the form

$$\sum_j \frac{j}{|\omega|}\sigma_j = -\Theta \mod \mathbb{Z}$$

where the sum is taken over the distinct parts j of μ. The action of $\widetilde{\zeta}$ has the effect of shifting σ_j by $\frac{m_j|\omega|}{k}$ and hence changes $\sum_j \frac{j}{|\omega|}\sigma_j$ by

$$\sum_j \frac{j}{|\omega|}\frac{m_j|\omega|}{k} = \frac{n}{k}.$$

Note that shifting the value of a $\theta_{j,i}$ by $\frac{1}{|\omega|}$ has the effect of shifting the value σ_j by 1 and hence changing $\sum_j \frac{j}{|\omega|}\sigma_j$ by $\frac{j}{|\omega|}$.

It follows that the sum $\sum_j \frac{j}{|\omega|}\sigma_j$ can be shifted by any multiple of $\gcd(\frac{n}{k}, \frac{g}{|\omega|})$ where g is the greatest common divisor of the parts j. Thus we can assume without loss of generality that $\sum_j \frac{j}{|\omega|}\sigma_j$ lies in the finite set

$$X_{\mu,\omega} = \Big\{ -\Theta, -\Theta + 1, \ldots, -\Theta + \gcd\Big(\frac{n}{k}, \frac{g}{|\omega|}\Big) - 1\Big\}.$$

Indeed as every element of $\widetilde{Z}(\mu, \omega)$ shifts the sum by a multiple of $\gcd(\frac{n}{k}, \frac{g}{|\omega|})$, these different values give $\gcd(\frac{n}{k}, \frac{g}{|\omega|})$ distinct components of the quotient.

Now fixing the value of σ_j we consider tuples $\theta_{j,1}, \ldots, \theta_{j,m_j}$ such that the sum $\sum_{i=1}^{m_j} \theta_{j,i}|\omega|$ yields this value. We use Morton's description of symmetric products of circles, see [6]. We will consider the subgroup $\widetilde{Z}_0(\mu, \omega)$ of $\widetilde{Z}(\mu, \omega)$ generated by the sublattice

$$\{(\nu_{j,i}) : \nu_{j,i} \in \frac{1}{|\omega|}\mathbb{Z} \text{ for all } j, i \text{ and } \nu_{j,1} + \cdots + \nu_{j,m_j} = 0 \text{ for all } j\}$$

along with the permutation groups $\mathfrak{S}_{m_j}$. Note that this subgroup preserves each of the variables σ_j. By adding integer multiples of $\frac{1}{|\omega|}$ (totalling zero) to the variables $\theta_{j,i}$ we can assume that the minimum and maximum values differ by at most $\frac{1}{|\omega|}$. Additionally, the action of the symmetric group allows us to arrange the variables $\theta_{j,i}$ in ascending order, thus we may assume that

$$\theta_{j,1} \leq \theta_{j,2} \leq \cdots \leq \theta_{j,m_j} \leq \theta_{j,1} + \frac{1}{|\omega|}. \tag{11}$$

Moreover this condition yields a unique representative for each orbit of the group $\widetilde{Z}_0(\mu,\omega)$.

For $x \in X_{\mu,\omega}$ we denote by $V_{\mu,\omega,x}$ the subspace of $V_{\mu,\omega}$ defined by the equation

$$\sum_j \frac{j}{|\omega|}\sigma_j = x \tag{12}$$

along with Morton's inequalities (11). For each j, we observe that for a fixed value of σ_j, the set of points $(\theta_{j,1},\ldots,\theta_{j,m_j})$ satisfying the inequalities form a simplex of dimension $m_j - 1$. The variables σ_j can take any real values, subject to the constraint (12), hence these lie in a codimension 1 affine subspace of $\mathbb{R}^b$. It follows that the space $V_{\mu,\omega,x}$ is a product of $\mathbb{R}^{b-1}$ with a polysimplex whose component simplices have dimensions $m_j - 1$. We will think of $V_{\mu,\omega,x}$ as a bundle of polysimplices over the space $\mathbb{R}^{b-1}$.

We now consider elements of $\widetilde{Z}(\mu,\omega)$ which preserve Morton's inequalities, 11. Let W_j denote the element defined by

$$(\theta_{j,1},\ldots,\theta_{j,m_j}) \mapsto (\theta_{j,2},\ldots\theta_{j,m_j},\theta_{j,1}+\frac{1}{|\omega|}).$$

This shifts the sum $\sum_{i=1}^{m_j}\theta_{j,i}$ by $\frac{1}{|\omega|}$ and hence shifts σ_j by 1, while leaving all other $\sigma_{j'}$ invariant, and preserves Morton's inequalities. Note that $\widetilde{\zeta}$ also preserves these inequalities and that the elements W_j along with $\widetilde{\zeta}$ and $\widetilde{Z}_0(\mu,\omega)$ generate the whole of $\widetilde{Z}(\mu,\omega)$.

Each W_j changes $\sum_j \frac{j}{|\omega|}\sigma_j$ by $\frac{j}{|\omega|}$ and hence takes this value out of the set $X_{\mu,\omega}$. Similarly the shift $\widetilde{\zeta}$ changes the sum by $\frac{n}{k}$ so we must consider compositions of the form $\widetilde{\zeta}^\gamma \prod_j W_j^{\beta_j}$ where

$$\gamma\frac{n}{k} + \sum_j \frac{j}{|\omega|}\beta_j = 0.$$

Let $L_{\mu,\omega}$ denote the group of elements of this form. The quotient by $\widetilde{Z}(\mu,\omega)$ of the space $V_{\mu,\omega}$ is thus the disjoint union

$$\coprod_{x\in X_{\mu,\omega}} V_{\mu,\omega,x}/L_{\mu,\omega}$$

We remark that the elements $\widetilde{\zeta}, \{W_j\}$ are commuting elements of infinite order, so $L_{\mu,\omega}$ is an abelian group of rank $b-1$, indeed $L_{\mu,\omega}$ can be identified as a quotient of the lattice

$$\{(\gamma,(\beta_j)) \in \mathbb{Z}\times\mathbb{Z}^b \,:\, \gamma\frac{n}{k} + \sum_j \frac{j}{|\omega|}\beta_j = 0\}$$

of rank b, where the kernel is the cyclic subgroup generated by the element

$$\gamma = \frac{k}{|\omega|}, \quad \beta_j = -m_j.$$

It is easy to see that this is in the kernel and moreover for the linear part of the affine map $W_j^{\beta_j}$ to be the identity m_j must divide β_j.

We will now compute the stabiliser of the fibres of $V_{\mu,\omega,x}$, i.e. we will determine which elements $\widetilde{\zeta}^\gamma \prod_j W_j^{\beta_j}$ preserve all the σ_j variables. This will hold when

$$\frac{\gamma m_j|\omega|}{k} + \beta_j = 0$$

for each j. The elements $(\gamma, (\beta_j))$ satisfying this equation form an infinite cyclic group whose generator is given by

$$\gamma = \frac{k}{\gcd(m|\omega|, k)}, \quad \beta_j = -\frac{m_j}{\gcd(m, k/|\omega|)}.$$

The action of $\widetilde{\zeta}^\gamma \prod_j W_j^{\beta_j}$ on each simplex factor of the fibre is then the β_j power of the m_j-cycle on the vertices. This decomposes as a product of $\frac{m_j}{\gcd(m,k/|\omega|)}$ disjoint $\gcd(m, k/|\omega|)$-cycles on each simplex. The effective action is thus by a group of order $\gcd(m, k/|\omega|)$ and each simplex can be regarded as a join of $m_j/\gcd(m, \frac{k}{|\omega|})$ simplices of dimension $\gcd(m, \frac{k}{|\omega|}) - 1$ on each of which this group acts by cyclically permuting the vertices.

The action on each simplex of the join is orientable or not depending on whether or not the dimension $\gcd(m, \frac{k}{|\omega|}) - 1$ is even. It follows that the action on the join preserves orientation precisely when $(\gcd(m, \frac{k}{|\omega|}) - 1)m_j/\gcd(m, \frac{k}{|\omega|})$ is even and that the action on the polysimplex preserves the orientation when

$$\sum_j \frac{(\gcd(m, \frac{k}{|\omega|}) - 1)m_j}{\gcd(m, \frac{k}{|\omega|})} = \sum_j m_j\Big(1 - \frac{1}{\gcd(m, \frac{k}{|\omega|})}\Big) = c - \frac{c}{\gcd(m, \frac{k}{|\omega|})}$$

is even. Hence this action preserves the orientation on the fibres if and only if c is odd or the 2-adic norms satisfy $|c|_2 < |\gcd(m, \frac{k}{|\omega|})|_2$.

To complete the proof we observe that $L_{\mu,\omega}$ is the product of a free abelian group $\Gamma_{\mu,\omega}$ of rank $b-1$ with the cyclic group $C_{\gcd(m, \frac{k}{|\omega|})}$ and define $E_{\mu,\omega} = V_{\mu,\omega,x}/\Gamma_{\mu,\omega}$. Note that up to a homeomorphism induced by translation, this is independent of $x \in X_{\mu,\omega}$. The quotient $E_{\mu,\omega}$ is a bundle of polysimplices over a torus of dimension $b-1$

Thus we have

$$T//W \cong \coprod_\mu \coprod_{\omega \in C_{\gcd(g(\mu),k)}} \coprod_{x \in X_{\mu,\omega}} V_{\mu,\omega,x}/L_{\mu,\omega} \cong \coprod_\mu \coprod_{\omega \in C_{\gcd(g(\mu),k)}} \coprod_{x \in X_{\mu,\omega}} E_{\mu,\omega}/C_{\gcd(m(\mu), \frac{k}{|\omega|})}$$

where $C_{\gcd(m(\mu), \frac{k}{|\omega|})}$ acts on the fibres of the bundle as described above.

It follows that each component is a bundle over a compact torus of dimension $b-1$ with fibre a cyclic quotient of a polysimplex.

It is instructive to consider the special case when $k = 1$, for which ω can only take the value 1. In this case the components are the bundles $E_{\mu,1}$. These bundles are obtained by gluing polysimplex fibres using the compositions $\prod_j W_j^{\beta_j}$ where $\sum_j j\beta_j = 0$. In this context we have the following simple lemma in the spirit of Morton [6]. Indeed in the case where we have a partition of the form $1 + j + \cdots + j$ for some $j > 1$, the space

$E_{\mu,1}$ is simply a symmetric product of m_j circles. Morton showed that this is a simplex bundle over a circle, which is orientable or not according to whether m_j is odd or even.

Lemma 7.1. *In the case $k = 1$, given a partition μ of n with distinct parts $j_1, \ldots, j_b$ and $g = \gcd(j_1, \ldots, j_b)$, the bundle $E_{\mu,1}$ is non-orientable if and only if the vectors $(j_1/g, \ldots, j_b/g)$ and $(m_{j_1} - 1, \ldots, m_{j_b} - 1)$ are linearly independent as elements of $(\mathbb{Z}/2)^b$.*

Proof. The composition $\prod_j W_j^{\beta_j}$ preserves the orientation on the polysimplex if and only if $\sum_j (m_j - 1)\beta_j$ is even. If the vectors are linearly dependent then (noting that $(j_1/g, \ldots, j_b/g)$ is non-zero modulo 2 since its entries have greatest common divisor 1) we have either $(m_{j_1} - 1, \ldots, m_{j_b} - 1) = (0, \ldots, 0)$ modulo 2, whence each W_j preserves the fibre orientation, or $(m_{j_1} - 1, \ldots, m_{j_b} - 1) = (j_1/g, \ldots, j_b/g)$ modulo 2. In the latter case any element $(\beta_{j_1}, \ldots \beta_{j_b})$ in the lattice satisfies $\sum_j \frac{j}{g}\beta_j = 0$ hence

$$0 = \sum_j \frac{j}{g}\beta_j \cong \sum_j (m_j - 1)\beta_j \mod 2$$

and again all elements of the lattice preserve orientation on the fibres.

Conversely suppose $(j_1/g, \ldots, j_b/g)$ and $(m_{j_1} - 1, \ldots, m_{j_b} - 1)$ are linearly independent modulo 2. Then in particular there exists a pair j, j' such that $(j/g, j'/g)$ and $(m_j - 1, m_{j'} - 1)$ are linearly independent modulo 2. Set $\beta_j = j'/g, \beta_{j'} = -j/g$ and $\beta_{j''} = 0$ for all other j''. This defines a point of the lattice and the composition $W_j^{\beta_j} W_{j'}^{\beta_{j'}}$ will reverse orientation on the polysimplex fibres since the independence of $(j/g, j'/g)$ and $(m_j - 1, m_{j'} - 1)$ modulo 2 ensures that $(m_j - 1)\frac{j'}{g} - (m_{j'} - 1)\frac{j}{g}$ is odd. ☐

Example 7.2. We consider in detail the real counterpart $SU_6(\mathbb{C})$ of the example $SL_6(\mathbb{C})$ considered in section 2.

μ	$(j_1/g, \ldots, j_b/g)$	$(m_{j_1} - 1, \ldots, m_{j_b} - 1)$	$\lvert X_{\mu,1}\rvert$	$E_{\mu,1}$	orientable?
6	(1)	(0)	6	Δ^0	Yes
$1+5$	$(1,5)$	$(0,0)$	1	S^1	Yes
$2+4$	$(1,2)$	$(0,0)$	2	S^1	Yes
$1+1+4$	$(1,4)$	$(1,0)$	1	$S^1 \times \Delta^1$	Yes
$3+3$	(1)	(1)	3	Δ^1	Yes
$1+2+3$	$(1,2,3)$	$(0,0,0)$	1	$(S^1)^2$	Yes
$1+1+1+3$	$(1,3)$	$(2,0)$	1	$S^1 \tilde{\times} \Delta^2 \cong S^1 \times \Delta^2$	Yes
$2+2+2$	(1)	(2)	2	Δ^2	Yes
$1+1+2+2$	$(1,2)$	$(1,1)$	1	$S^1 \tilde{\times} (\Delta^1 \times \Delta^1)$	No
$1+1+1+1+2$	$(1,2)$	$(3,0)$	1	$S^1 \tilde{\times} \Delta^3 \cong S^1 \times \Delta^3$	Yes
$1+1+1+1+1+1$	(1)	(5)	1	Δ^5	Yes

Here the notation S^1 denotes the unit circle while Δ^p denotes the standard p-simplex. The notation $S^1 \tilde{\times} F$ denotes a twisted bundle over the circle with fibre F, which in our case is a simplex or, more generally a polysimplex. (Recall that since $k = 1$ there is no cyclic group action on the fibre.)

In the case of the $1 + 1 + 4$ partition the gluing map is $W_1^4 W_4^{-1}$ which acts as the identity on the fibre so the bundle is trivial as noted in the table. In the case when $\mu = 1 + 1 + 1 + 3$ the gluing map is $W_1^3 W_3^{-1}$ which acts as a rotation of order 3 on the simplex Δ^2. While this is non-trivial as a simplex bundle it is orientable and homeomorphic to the direct product.

For $\mu = 1+1+2+2$ the fibre is a product of two intervals, and the gluing map, $W_1^2 W_2^{-1}$ preserves the first factor and flips the second. It follows that the bundle is the product of a Möbius band with an interval and, in particular is non-orientable. We note that modulo 2 the vectors in the table reduce to $(1,0)$ and $(1,1)$ which are linearly independent.

When $\mu = 1+1+1+1+2$ the gluing map is again $W_1^2 W_2^{-1}$ acting as the square of a cyclic permutation of the vertices of the fibre which is a 3-simplex. This should be thought of as the join of two 1-simplices both of which are flipped by the gluing map, so this is a rotation of π about an axis orthogonal to both 1-simplices. The result is therefore homeomorphic to the direct product.

8. Proof of Corollary 1.7

The gluing maps defining the bundle $E_{\mu,\omega}$ and the action of the cyclic group $C_{\gcd(m(\mu),k/|\omega|)}$ are products of simplicial maps on the factors of the polysimplex. It follows that the product of the barycentres of the simplices is preserved by these actions and that the retraction onto this point defines a homotopy equivalence from the quotient $E_{\mu,\omega}/C_{\gcd(m(\mu),k/|\omega|)}$ to the $b-1$-torus over which the bundle $E_{\mu,\omega}$ is defined.

Each component is thus homotopy equivalent to $(S^1)^{b-1}$ as required. The computation of the number of components furnished by each μ is as in Section 6.

Appendix A. Computations

Our formula, together with the binomial formula for the Betti numbers of tori, allow us to compute the cohomology of the extended quotients corresponding to the extended affine Weyl groups. From the point of view of cohomology the real and complex tori $(S^1)^{b-1}, (\mathbb{C}^\times)^{b-1}$ are identical, hence the cohomology groups of the extended quotients $\mathbf{S}_k/\!/W$ and $T_k/\!/W$ are the same. Since K-theory is a compactly supported theory one must be more careful about the identification of the real and complex tori, although they do, coincidentally, have the same K-theory. For this reason we restrict to the real case when considering K-theory so that the standard Chern character argument [3] provides the ranks of the K-theory groups.

In Tables 1 and 2 below we compute the Betti numbers for small values of n in the case of $\mathrm{SL}_n(\mathbb{C})/C_k$ for $k = 1, 2$. By duality these are the same as the Betti numbers for $\mathrm{PSL}_n(\mathbb{C})$ and $\mathrm{SL}_n(\mathbb{C})/C_{n/2}$ respectively. For n_0 a triangle number there is only one partition that can contribute a top dimensional class in the cohomology of the extended quotient, that is $n_0 = 1 + 2 + \ldots + b$ so the cohomological dimension increases and the top dimensional Betti number resets to 1 at each of these. More generally we have the following.

Lemma A.1. *For each n let $b = \lfloor \frac{\sqrt{8n+1}-3}{2} \rfloor$ and let $n_0 = 1 + 2 + \ldots + b$, which is the largest triangle number less than or equal to n. Then $n = n_0 + r$ where $0 \leq r \leq b$. Let $P_2(r) := \sum_{s=0}^{r} P(s)P(r-s)$, where $P(s)$ denotes the number of partitions of s and $P_2(r)$ is the number of partitions of r into parts of two kinds. The top dimensional cohomology of $\mathbf{S}/\!/W$ appears in dimension $b-1$, and has rank $P_2(r)$, hence the generating function for the top dimensional Betti numbers $P_2(r)$ is*

$$\prod_{s=1}^{\infty} \frac{1}{(1-x^s)^2}.$$

Proof. The top dimensional cohomology is carried by tori corresponding to partitions which maximise the number of distinct parts $b(\mu)$, which increases at each triangle number since these are the minimal numbers which can be partitioned into a given number of distinct parts. So the cohomological dimension of the extended quotient is $\lfloor \frac{\sqrt{8n+1}-3}{2} \rfloor$.

Given the partition $n_0 = 1 + 2 + \ldots + b$ we obtain the partitions for $n_0 + r$ as follows. First partition r as a sum $s + (r - s)$ then choose partitions of s and $r - s$. We splice the partition of s into the partition of n_0 noting that this does not change the number of distinct parts but increases multiplicities of (some of) the first s terms. In particular it does not change the multiplicities of the last $r - s$ terms i.e. these still have multiplicity 1, Now writing the partition of $r - s = x_1 + x_2 + \ldots + x_d$ in increasing order, and noting that $d \leq r - s$, we add these terms to the last terms of the spliced partition. Each of these new terms must have multiplicity one, since the x_i terms are monotonic while the terms $b - (d - 1), \ldots, b$ are strictly increasing. Hence this produces a partition with the same number b of distinct parts.

Conversely, suppose we are given a partition of $n = n_0 + r$ with b distinct parts $j_1 < j_2 < \ldots < j_b$, which have multiplicities $m_{j_1}, \ldots, m_{j_b}$. We express the partition as the union of the partition of $j_1+j_2+\ldots+j_b$ with parts j_i together with its complement in the partition of n. Since $j_i \geq i$ it follows that $j_1+j_2+\ldots+j_b \geq n_0$, so the complementary partition is a partition of $s \leq b$, with multiplicities $m_{j_1}-1, \ldots m_{j_b}-1$. The partition $j_1+j_2+\ldots+j_b$ of $n_0+(r-s)$ is obtained from the partition $n_0 = 1+2+\ldots+b$ by adding $j_i - i$ to the ith term, where the non-zero terms $j_i - i$ define a partition of $r-s$ where $j_{i-1}-(i-1) \leq j_i - i$. Hence this produces all partitions of n into exactly b parts. □

Table 3 gives the ranks of K_0 and K_1 for n ranging from 2 to 20 and k dividing n in the case of the Lie groups $\mathrm{SU}_n(\mathbb{C})/C_k$. Note that, as remarked above, when n is square free the answers do not vary with k.

Remark A.2. We note that for $\mathrm{SU}_n(\mathbb{C})$ (and, by duality, $\mathrm{PSU}_n(\mathbb{C})$) the Euler characteristic of the extended quotient (the difference of the ranks of K_0 and K_1) is the sum of the divisors of n. This follows from considering the partitions of n with a single distinct part and summing the size of these parts.

The computations of the Betti numbers in Tables 1 and 2 can be viewed as computations of the periodic cyclic homology $\mathrm{HP}_* \, \mathbb{C}[W_a']$. Similarly the K-theory computations in Table 3 can be interpreted (identifying $K_*(C_r^* W_a')$ with $K_W^j(T_{\frac{n}{k}})$ and applying the equivariant Chern character) as computations of the ranks of $K_*(C_r^* W_a')$.

n	b_0	b_1	b_2	b_3	b_4	b_5	b_6	b_7	b_8
1	1								
2	3								
3	5	1							
4	9	2							
5	11	5							
6	20	9	1						
7	21	15	2						
8	35	25	5						
9	42	39	10						
10	61	60	18	1					
11	66	83	31	2					
12	112	132	53	5					
13	113	171	82	10					
14	168	253	129	20					
15	210	346	193	34	1				
16	279	480	290	60	2				
17	313	618	415	97	5				
18	461	882	607	157	10				
19	508	1107	841	242	20				
20	719	1533	1192	372	36				
21	852	1958	1627	551	63	1			
22	1088	2587	2248	816	105	2			
23	1277	3253	3006	1173	172	5			
24	1756	4376	4103	1685	272	10			
25	2006	5400	5387	2365	423	20			
26	2573	7031	7212	3318	642	36			
27	3106	8802	9403	4563	961	65			
28	3937	11304	12393	6277	1414	108	1		
29	4593	13895	15942	8486	2054	180	2		
30	5958	17909	20840	11480	2945	287	5		
31	6872	21787	26510	15295	4175	453	10		
32	8676	27629	34226	20394	5858	694	20		
33	10305	33853	43311	26834	8138	1055	36		
34	12655	42271	55286	35328	11213	1566	65		
35	15009	51480	69364	45962	15313	2306	110		
36	18664	64348	88029	59864	20768	3340	183	1	
37	21673	77496	109523	77103	27944	4796	295	2	
38	26559	95862	137729	99418	37385	6796	468	5	
39	31447	115954	170716	126960	49653	9560	724	10	
40	38217	142322	213011	162237	65632	13298	1107	20	
41	44623	170725	262212	205495	86178	18375	1660	36	
42	54386	209199	325553	260569	112690	25161	2461	65	
43	63303	249804	398441	327617	146468	34234	3597	110	
44	76379	303841	491402	412339	189689	46224	5203	185	
45	89696	363217	599369	515152	244298	62058	7439	298	1

TABLE 1. The Betti numbers of the extended quotient of the maximal torus of $\mathrm{SL}_n(\mathbb{C})$ by the Weyl group.

n	b_0	b_1	b_2	b_3	b_4	b_5	b_6	b_7	b_8	b_9
2	3									
4	10	2								
6	20	9	1							
8	40	27	5							
10	61	60	18	1						
12	122	139	54	5						
14	168	253	129	20						
16	306	505	295	60	2					
18	461	882	607	157	10					
20	758	1583	1210	373	36					
22	1088	2587	2248	816	105	2				
24	1848	4504	4156	1690	272	10				
26	2573	7031	7212	3318	642	36				
28	4063	11527	12518	6297	1414	108	1			
30	5958	17909	20840	11480	2945	287	5			
32	8939	28109	34516	20454	5860	694	20			
34	12655	42271	55286	35328	11213	1566	65			
36	19041	65152	88616	60021	20778	3340	183	1		
38	26559	95862	137729	99418	37385	6796	468	5		
40	38892	143835	214203	162609	65668	13298	1107	20		
42	54386	209199	325553	260569	112690	25161	2461	65		
44	77335	306262	493588	413151	189794	46226	5203	185		
46	106879	438330	734529	643953	313641	82762	10542	476	2	
48	151344	633466	1092391	995271	510521	144834	20594	1137	10	
50	206440	893139	1596122	1515435	817909	248268	38983	2555	36	
52	286682	1268240	2329944	2290931	1294065	417826	71771	5463	110	
54	390133	1771783	3355161	3420111	2020028	691336	128924	11196	300	
56	534934	2480996	4820172	5072447	3119948	1126632	226551	22131	747	2
58	719869	3424517	6843460	7441500	4763117	1809963	390279	42387	1742	10
60	979775	4745295	9701610	10856045	7205703	2870751	660346	78961	3846	36

TABLE 2. The Betti numbers of the extended quotient of the maximal torus of $\mathrm{SL}_n(\mathbb{C})/C_2$ by the Weyl group.

n, k	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
2	3 0	3 0																		
3	5 1		5 1																	
4	9 2	10 2		9 2																
5	11 5				11 5															
6	21 9	21 9	21 9			21 9														
7	23 15						23 15													
8	40 25	45 27		45 27				40 25												
9	52 39		56 41						52 39											
10	79 61	79 61			79 61					79 61										
11	97 85										97 85									
12	165 137	176 144	165 137	165 137		176 144						165 137								
13	195 181												195 181							
14	297 273	297 273					297 273							297 273						
15	404 380		404 380		404 380										404 380					
16	571 540	603 565		609 569				603 565								571 540				
17	733 715																733 715			
18	1078 1039	1078 1039	1102 1057			1102 1057			1078 1039									1078 1039		
19	1369 1349																		1369 1349	
20	1947 1905	2004 1956		1947 1905	1947 1905					2004 1956										1947 1905

Table 3. The rank of the K-theory of the extended quotient of the maximal torus of $\mathrm{SU}_n(\mathbb{C})/C_k$ by the Weyl group.

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