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Green operators for low regularity spacetimes

Yafet Sanchez Sanchez and James Vickers

Mathematical Sciences and STAG Research Centre, University of Southampton, Southampton, SO17 1BJ.

E-mail: Y.SanchezSanchez@soton.ac.uk and J.A.Vickers@soton.ac.uk

Abstract. In this paper we define and construct advanced and retarded Green operators for the wave operator on spacetimes with low regularity. In order to do so we require that the spacetime satisfies the condition of generalised hyperbolicity which is equivalent to well-posedness of the classical inhomogeneous problem with zero initial data where weak solutions are properly supported. Moreover, we provide an explicit formula for the kernel of the Green operators in terms of an arbitrary eigenbasis of H^1 and a suitable Green matrix that solves a system of second order ODEs.

1. Introduction

The existence of Green operators for normally hyperbolic operators is well understood for globally hyperbolic spacetimes where the spacetime metric is smooth [1]. However, there are two important motivations for analysing spacetime metrics with finite differentiability. Firstly, there are several models of physical phenomena that require finite metric regularity. These include impulsive gravitational waves, stars with well-defined surfaces, general relativistic fluids and cosmic strings. Secondly, Einstein's equations, viewed as a system of hyperbolic PDEs, can be naturally formulated in function spaces with finite regularity.

When the spacetime is of finite regularity the existence of Green operators for the Klein-Gordon equation has been explored using semigroup techniques in the Hamiltonian formalism [2, 3, 4]. We complement such an analysis by providing a definition of Green operators subject only to the well posedness of the classical problem. Furthermore we show how to explicitly construct both the advanced and retarded the Green operators needed in order to formulate quantum field theory on a non-smooth background.

In the first section of the paper we give the definition of the advanced and retarded Green operators in the smooth case and define the concept of generalised hyperbolicity. In the second section of the paper we extend the definitions to the non-smooth setting and prove the existence of the Green operators for low regularity spacetimes. In the third section of the paper we give an explicit formula for the advanced Green operator in the non-smooth setting.

2. Preliminaries

2.1. The general setting

We will consider a smooth manifold M of the form $M = (0,T)\times\Sigma$ and its closure $\overline{M} = [0,T]\times\Sigma$ where Σ is a closed compact smooth manifold. Given a smooth Lorentzian metric g_{ab} of the form

$$ds^2 = N^2dt^2 - \gamma_{ij}(dx^i + \beta^i dt)(dx^j + \beta^j dt) \tag{2.1}$$

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the wave operator $\Box_g$ acting on a scalar function u is given by

$$\begin{aligned}\Box_g u = \frac{1}{N\sqrt{\gamma}}\left(\partial_t\left(N\sqrt{\gamma}\frac{1}{N^2}\partial_t u\right)\right)\\ +\frac{1}{N\sqrt{\gamma}}\left(\partial_t\left(N\sqrt{\gamma}\frac{\beta^i}{N^2}\partial_i u\right)+\partial_j\left(N\sqrt{\gamma}\frac{\beta^j}{N^2}\partial_t u\right)\right)\\ -\frac{1}{N\sqrt{\gamma}}\partial_i\left(N\sqrt{\gamma}(\gamma^{ij}-\frac{\beta^i\beta^j}{N^2})\partial_j u\right)\end{aligned} \tag{2.2}$$

where N is the lapse function, β^i the shift, γ_{ij} the metric on Σ and $\sqrt{\gamma}dx$ the induced volume form on Σ. We use a, b, c, d etc. to denote spacetime indices and i, j, k etc. to denote purely spatial indices.
The advanced zero initial data inhomogeneous problem for the wave equation on $\overline{M}$ is given by

$$\Box_g u = f \text{ on } M \tag{2.3}$$
$$u(0,x) = 0 \text{ on } \Sigma_0 = \Sigma\times\{0\} \tag{2.4}$$
$$\dot{u}(0,x) = 0 \text{ on } \Sigma_0 = \Sigma\times\{0\} \tag{2.5}$$

where $f : M \to \mathbb{R}$ is a smooth function. Similarly the retarded zero initial data inhomogeneous problem for the wave equation on $\overline{M}$ is given by

$$\Box_g u = f \text{ on } M \tag{2.6}$$
$$u(T,x) = 0 \text{ on } \Sigma_T = \Sigma\times\{T\} \tag{2.7}$$
$$\dot{u}(T,x) = 0 \text{ on } \Sigma_T = \Sigma\times\{T\} \tag{2.8}$$

where $f : M \to \mathbb{R}$ is a smooth function.
If the spacetime (M, g_{ab}) is globally hyperbolic the advanced and retarded zero initial data inhomogeneous problems are both well posed i.e. there exist unique solutions $u \in C^\infty(M)$ which depend continuously on f [1]. Moreover, in the globally hyperbolic case C^∞ well-posedness is equivalent to the existence of unique advanced and retarded Green operators [1]. For convenience we recall the definition of these.

Definition 2.1. A linear map $E^+ : D(M) \to C^\infty(M)$ satisfying

- $\Box_g E^+ = id_{D(M)}$
- $E^+\Box_g|_{D(M)} = id_{D(M)}$
- $supp(E^+\psi) \subset J^+(supp(\psi))$ for all $\psi \in D(M)$

is called an advanced Green operator for $\Box_g$. The retarded Green operators E^- are defined in a similar fashion.

For a smooth globally hyperbolic spacetime the splitting theorem says that the metric may be written in the form

$$ds^2 = +N^2dt^2 - \gamma_{ij}dx^i dx^j \tag{2.9}$$

and this remains true even in low regularity [5]. The aim of the paper is to define such advanced and retarded Green operators for the metrics of the above form where the lapse function N and the induced Riemannian metric γ_{ij} are no longer smooth.

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Notation. We denote the derivative of a function u with respect t by u_t or $\partial_t u$ and u_i or $\partial_i u$ if it is with respect to the other x^i-coordinates. The space of smooth functions of compact support will be denoted by $D(M)$. A function f on an open set $\mathcal{U}$ of $\mathbb{R}^n$ is said to be Lipschitz if there is some constant K such that for each pair of points $p, q \in \mathcal{U}$, $|f(p) - f(q)| \leq K|p-q|$, where $|p|$ denotes the usual Euclidean distance. We denote by $C^{k,1}$ those C^{k-1} functions where the k-derivative is a Lipschitz function. A function f on M is said to be Lipschitz or $C^{k,1}$ if f is Lipschitz or $C^{k,1}$ in some coordinate chart.
In the analysis below we will be working with spaces such as $L^2(\Sigma)$, $H^1(\Sigma)$ $H^{-1}(\Sigma)$ which are defined with respect to a smooth background Riemannian metric h_{ij} on Σ with ν_h the corresponding volume form. We define $L^2(\Sigma)$ to be the space of real valued functions g on Σ such that $\int_\Sigma g^2 \nu_h < \infty$ and we denote the associated inner product by $(f,g)_{L^2} = \int_\Sigma fg\nu_h$. The space $H^k(\Sigma)$ are the real valued functions g such that their first k derivatives are in $L^2(\Sigma)$ and the space $H^{-k}(\Sigma)$ are the bounded linear functionals on $H^k(\Sigma)$. Note that a different choice of background metric simply results in an equivalent norm on all these spaces.
We also define the space $L^2(M,g)$ to be the space of real valued functions ϕ on M such that $\int_M \phi^2 \nu_g < \infty$ which is defined with respect the volume form given by the metric (2.9) and $L^2(\Sigma_t, \gamma)$ to be the space of real valued functions ψ on $\Sigma_t = \{t\} \times \Sigma$ for $0 \leq t < T$ such that $\int_{\Sigma_t} \psi^2 \nu_\gamma < \infty$ which is defined with respect the volume form given by the Riemannian metric γ_{ij} that appears in (2.9) . We denote in a similar way the Sobolev space $H^k(M,g)$ and $H^k(\Sigma, \gamma)$.
We will also often think of a function $\mathbf{v}(t,x)$ as a map from $[0,T]$ to a function $\mathbf{v}(t)(\cdot)$ in some Hilbert space $X(\Sigma)$ given by $\mathbf{v}(t)(x) = \mathbf{v}(t,x)$. For example $L^2(0,T; X(\Sigma))$ is the space of functions

$$\mathbf{v} : [0,T] \to X(\Sigma) \tag{2.10}$$
$$t \mapsto \mathbf{v}(t) \tag{2.11}$$

such that $\mathbf{v}(t) \in X(\Sigma)$ and

$$\left(\int_0^T ||\mathbf{v}(t)||^2_{X(\Sigma)} dt \right)^{\frac{1}{2}} < \infty \tag{2.12}$$

When thinking of $\mathbf{v}$ in this way, we will denote the time derivative by $\dot{\mathbf{v}}$.
We will also use the space $C^k([0,T]; X(\Sigma))$ which is the space of functions

$$\mathbf{v} : [0,T] \to X(\Sigma) \tag{2.13}$$
$$t \mapsto \mathbf{v}(t) \tag{2.14}$$

such that $\mathbf{v}(t) \in X(\Sigma)$ and

$$\max_{j=0,\dots,k} \sup_{t\in[0,T]} ||\partial_{t^j} \mathbf{v}(t)||_{X(\Sigma)} < \infty \tag{2.15}$$

where $\partial_{t^l} \mathbf{v}(t)$ is the l-th derivative with respect time of $\mathbf{v}(t)$.

2.2. Generalised hyperbolicity

To prove the existence of Green operators in the non-smooth setting we will require that the spacetime satisfy the condition of *generalised hyperbolicity* (in H^1) which is essentially the requirement that solutions to the wave equation are well-posed [6], have the appropiate causal support and that the energy momentum of the solutions (see 2.23) is at least integrable.
In the low regularity setting the pointwise equation (2.3) may not make sense and therefore a notion of a weak solution is required.

Taking into account the form of the metric (2.9) and the geometric conditions that appear in Theorem 2.10 (see below) the wave operator may be written as

$$\Box_g u = \frac{\ddot{u}}{\gamma} - \frac{Lu}{\gamma} \tag{2.16}$$

where $-L$ is an elliptic operator in divergence form given by:

$$-Lu = -(\gamma^{ij}\gamma u_j)_i. \tag{2.17}$$

We can associate with the operator $-L$ the following bi-linear form given by:

$$B[u, v; t] := \int_\Sigma \gamma^{ij}(x,t)\gamma(x,t)u_i(x)v_j(x)dx^n \tag{2.18}$$

Using this bi-linear form, we make the following definition of an advanced weak solution.

Definition 2.2. We say a function:

$$u \in L^2([0,T]; H^1(\Sigma)), \text{ with } \dot{u} \in L^2([0,T]; L^2(\Sigma)), \ddot{u} \in L^2([0,T]; H^{-1}(\Sigma))$$

is an advanced weak solution of the zero initial data inhomogeneous problem (2.3), (2.4), (2.5) provided that:

(i) For each $v \in L^2([0,T]; H^1(\Sigma))$,

$$\int_0^T < \ddot{u}, v > dt + \int_0^T B[u, v; t]dt = (f, v)_{L^2(M,g)} \tag{2.19}$$

where $< \cdot, \cdot >$ denotes the dual pairing between the $H^{-1}(\Sigma)$ and $H^1(\Sigma)$ Sobolev spaces.

(ii) $u(0,x) = 0, \quad \dot{u}(0,x) = 0$

The definition of the retarded weak solution is analogous.

Remark 2.3. Notice that in our convention we have chosen to call advanced solutions those solutions that evolve from an initial time t' to a future time $t \geq t'$ and retarded if the solutions evolve in the opposite direction. We have used this convention as advanced solutions (in our sense) will be used to construct the advanced Green functions. However, there are other conventions for advanced and retarded functions in the literature with the ambiguity stemming from the fact that many authors use retarded fundamental solutions of the adjoint operator to construct the advanced Green operators. (see e.g [1] equation (3.8)).

Remark 2.4. The regularity of the weak solution implies that $u \in C([0,T], L^2(\Sigma))$ and $\dot{u} \in C([0,T], H^{-1}(\Sigma))$ (see [7]). Consequently, equation (ii) has to be understood in this sense.

Remark 2.5. The regularity needed to make sense of Equation (2.19) is that

$$u \in L^2([0,T]; H^1(\Sigma)), \text{ with }, \ddot{u} \in L^2([0,T]; H^{-1}(\Sigma)).$$

Therefore for every $u \in L^2([0,T]; H^1(\Sigma))$ with $\ddot{u} \in L^2([0,T]; H^{-1}(\Sigma))$ we can define the bounded linear functional $(\Box_g u)[\cdot]$ on $L^2([0,T]; H^1(\Sigma))$ given by

$$(\Box_g u)[v] = \int_0^T < \ddot{u}, v > dt + \int_0^T B[u, v; t]dt. \tag{2.20}$$

for all $v \in L^2([0,T]; H^1(\Sigma))$ and therefore the pairing $\langle \Box_g u, v\rangle_{L^2H^{-1}, L^2H^1}$ makes sense.

We say an advanced weak solution is regular if it satisfies a suitable energy estimate.

Definition 2.6. (**Regular Weak Solution**)
An advanced weak solution is **regular** if u satisfies the energy estimate

$$||u(t,\cdot)||_{L^2([0,T];H^1(\Sigma))} + ||\dot{u}(t,\cdot)||_{L^2([0,T];L^2(\Sigma))} + ||\ddot{u}(t,\cdot)||_{L^2([0,T];H^{-1}(\Sigma))} \\ \leq C||f||_{L^2(M,g)} \tag{2.21}$$

We will also require the weak solutions to satisfy certain support properties. In the smooth case the condition of global hyperbolicity is such that a solution u of the inhomogeneous problem satisfies $supp(u) \subset J^+(supp(f)) \cup J^-(supp(f))$. We use this observation and the definition of the support of a distribution u i.e. $supp(u) = \{x \in M|$ for all neighbourhoods U of x there exist $\varphi \in D(M)$ with $supp(\varphi) \subset U$ and $u[\varphi] \neq 0\}$ to make the following definition:

Definition 2.7. (**Causally supported Weak Solution**) We say a weak solution is **causally supported to the future** if u satisfies the support condition

$$supp(u) \subset J^+(supp(f)) \tag{2.22}$$

in the sense of distributions. The definition of causally supported to the past is analogous. We say a weak solution is **causally supported** if it is causally supported to both the past and future i.e. $supp(u) \subset J^+(supp(f)) \cup J^-(supp(f))$.

We are now in a position to extend the notion of generalised hyperbolicity to the case where the wave equation only admits weak solutions.

Definition 2.8. (**Generalised hyperbolicity**) We say a spacetime M satisfies the advanced generalised hyperbolicity condition if the wave equation is well-posed in the following sense: There exists a unique regular weak solution u of $\Box_g u = f$ in the sense of definition 2.2 and definition 2.6 with homogeneous initial conditions

(i) $u(0,\cdot) = u|_{\Sigma_0} = 0$
(ii) $\dot{u}(0,\cdot) = \frac{\partial u}{\partial t}|_{\Sigma_0} = 0$

that is causally supported to the future. With the obvious modifications we define the condition of retarded generalised hyperbolicity. If a spacetime satisfies the advanced and retarded condition of generalised hyperbolicity we say the spacetime satisfies the condition of generalised hyperbolicity.

Remark 2.9. In the case of a smooth metric, global hyperbolicity is sufficient to guarantee that the condition of generalised hyperbolicity in the above sense is also satisfied.
On the other hand even in the smooth case the notion of generalised hyperbolicity is more general. For example a spacetime with timelike boundary cannot be globally hyperbolic, but the initial value problem with reflecting timelike boundary conditions can be well-posed [8].

The following series of theorems give sufficient conditions to ensure such that a low regularity spacetime (M,g) satisfies the condition of generalised hyperbolicity. For simplicity, we only deal with the condition of advanced generalised hyperbolicity. We first show the existence and uniqueness of regular weak solutions of the initial value problem under certain regularity assumptions.

Theorem 2.10. *Let the metric g_{ab} given by Equation 2.9 on M satisfy the following conditions*

1. $\gamma^{ij} \in C^1([0,T], L^\infty(\Sigma))$.
2. *The scalar coefficient of the volume form given by $\sqrt{\gamma}$ for the induced metric γ_{ij} is bounded from below by a positive real number, i.e., $|\sqrt{\gamma}| > \eta$ for some $\eta \in \mathbb{R}^+$*
3. *The lapse function N can be chosen as $N = \sqrt{\gamma}$.*

4. There exists a constant $\theta > 0$ such that

$$\sum_{i,j=1}^{n} \gamma^{ij}\gamma\xi_i\xi_j \geq \theta|\xi|^2$$

for all $(t, x) \in M, \xi \in \mathbb{R}^n$.

Then for every $f \in L^2(M, g)$ there exists a unique weak regular solution to the initial value problem (both forwards and backwards in time).

Remark 2.11. The regularity conditions on both the lapse and the three metric required in Theorem 2.10 depend on the splitting. However the conditions required for both of these are invariant under any diffeomorphism that preserves the splitting.

Proof of Theorem 2.10 The main idea is to use Galerkin's method. We sketch the proof below. For a complete proof see [7, 9, 10].

- Introduce an orthogonal basis in $H^1(\Sigma)$ $\{w_k\}_{k=1}^{\infty}$ which is also an orthonormal basis in $L^2(\Sigma)$. An example of such a basis is given by the eigenfunctions of the Laplace-Beltrami operator with respect to the background Riemannian metric h_{ij}.
- Insert the approximate solution $u^{(m)}(t, x) = \sum_{k=1}^{m} d_k^{(m)}(t)w_k(x)$ into the projected wave equation $(u_{tt}^{(m)}, w_k)_{L^2(\Sigma)} + B[u^{(m)}, w_k] = (\gamma f, w_k)_{L^2(\Sigma)}$ for all $k = 1, ..., m$. This gives a first order system of ODEs for $d_k^{(m)}(t)$ with zero initial data on $t = 0$.
- Prove that the $u^{(m)}$ satisfy the energy estimate uniformly in m. This is the crucial step for the Galerkin's method to work as it will provide sufficient control over the approximated solutions to converge to a weak solution. The uniform control is obtained by finding a bound in terms of the norms of a truncated expansion of the source term and initial data with respect the chosen basis. While these bounds are not independent of m both terms are uniformly bounded in term of the initial data and the source. See [7] for details.
- Apply Banach-Alouglu theorem to extract a subsequence which is a solution to the weak formulation
- Use lower semi-continuity of the norm and energy estimates to prove uniqueness, stability and regularity of the weak solution.

At this point we have shown that regular weak solutions to the initial value problem exist. However, we have not shown that the support of such solutions satisfies the causal support condition to the future. We therefore prove the theorem below, which guarantees the correct support of these solutions and establishes the required generalised hyperbolicity.
Notice that Theorem 2.12 requires additional regularity conditions compared to Theorem 2.10. It does however have the advantage that the regularity condition is invariant under any diffeomorphism, not just those that respect the splitting. It may be possible to relax the regularity conditions to those of Theorem 2.10 by analysing the causal structure of spacetimes with low regularity as in [5, 12, 13] and using results on propagation of singularities for differential equations with rough coefficients as in [16].

Theorem 2.12. *Let the metric g_{ab} be a $C^{1,1}$ globally hyperbolic spacetime metric on M and the source function f be an element of $H^1(M, g)$ then (M, g) satisfies the condition of generalised hyperbolicity.*

Proof of Theorem 2.12. As described above the results of [7, 9, 10] establish well-posedness. The only thing that remains to be shown is that the weak solutions have the correct support. To do so we follow [11] and obtain an energy inequality that determines the support of the solution. Consider a weak solution u with energy-momentum tensor

$$T^{ab}[u] = \left(g^{ac}g^{bd} - \frac{1}{2}g^{ab}g^{cd}\right)\frac{\partial u}{\partial x^c}\frac{\partial u}{\partial x^d} \tag{2.23}$$

Let K be a compact subset of Σ_t. Then if we contract the energy-momentum tensor with a Lipschitz timelike vector field, Υ_a and use the fact that $J^-_{\Sigma_t}(K) \cap J^+(\Sigma_0)$ is compact by causal theory for globally hyperbolic $C^{1,1}$ metrics [12, 13, 5] (where $J^-_{\Sigma_t}(K)$ denotes the causal past of a compact region $K \subset \Sigma_t$), then provided we have sufficient regularity that one can apply the divergence theorem we obtain

$$\int_{J^-_{\Sigma_t}(K)\cap J^+(\Sigma_0)} \mathrm{div}\left(T^{ab}\Upsilon_a\right)\nu_g = \int_{\partial(J^-_{\Sigma_t}(K)\cap J^+(\Sigma_0))} T^{ab}\Upsilon_a n_b \nu_\gamma \tag{2.24}$$

In order for the above equation to be well-defined we require that $\mathrm{div}(T^{ab}\Upsilon_a)$ should be integrable with respect the volume form ν_g, and this in turn requires that the weak solutions are sufficiently regular. In fact it is enough that the weak solutions have two derivatives in $L^2(M, g)$ if the metric and the timelike vector field are in the space $C^{0,1}$ (see [14]). This is ensured by the additional regularity of the metric $g_{ab} \in C^{1,1}$ and source $f \in H^1(M, g)$ required in the conditions of Theorem 2.12 compared to Theorem 2.10. Moreover, such an improvement in regularity also guarantees that u belongs to the space $C^0([0, T], H^1(\Sigma)) \cap C^1([0, T], L^2(\Sigma))$. See [7] for a proof of these results.
The left hand side of 2.24 takes the explicit form:

$$\int_{J^-_{\Sigma_t}(K)\cap J^+(\Sigma_0)} \left(\left(g^{ab}\frac{\partial u}{\partial x^b}\Upsilon_a\right)\Box_g u + T^{ab}\nabla_b\Upsilon_a\right)\nu_g \tag{2.25}$$

while the right hand side takes the form:

$$\left(\int_K + \int_{\Sigma_0}\right) T^{ab}\Upsilon_a n_b \nu_\gamma + \int_{\mathcal{H}} T^{ab}\Upsilon_a n_b \nu_h. \tag{2.26}$$

where $\mathcal{H} = \partial(J^-_{\Sigma_t}(K) \cap J^+(\Sigma_0))\backslash(\Sigma_0 \cup K)$, ν_h is the induced volume form on $\mathcal{H}$ and n_b denotes outward pointing normal vectors on K, Σ_0 and $\mathcal{H}$.
Moreover, in the globally hyperbolic $C^{1,1}$ case we have that $\mathcal{H}$ is a null hypersurface ruled by null geodesics [13] and by the dominant energy condition [11] which follows from the specific form of 2.23 we have that

$$\int_{\mathcal{H}} T^{ab}\Upsilon_a n_b \nu_h \geq 0 \tag{2.27}$$

For the integral over K we define the energy integral:

$$E_K(t) = \int_K T^{ab}\Upsilon_a n_b \nu_\gamma \tag{2.28}$$

and we may define a similar energy integral E_{Σ_0} for the initial surface Σ_0.
We now use the energy integral $E_K(t)$ to construct a H^1 type Sobolev norm on K defined by

$$\tilde{\|}u\tilde{\|}^1_K := \left[\int_K \left(\left(\frac{\partial u}{\partial t}\right)^2 + \sum_{i=1}^{3}\left(\frac{\partial u}{\partial x^i}\right)^2\right)\nu_\gamma\right]^{\frac{1}{2}} \tag{2.29}$$

These norms are equivalent since as shown by Wilson [15]

$$C_1 E_K(t) \leq (\tilde{\|}u\tilde{\|}^1_K)^2 \leq C_2 E_K(t) \tag{2.30}$$

for constants $C_1, C_2 \geq 0$. In a similar way we may relate the initial energy integral E_{Σ_0} to an equivalent Sobolev type norm $\tilde{\|}u\tilde{\|}^1_{\Sigma_0}$ on Σ_0.
We now use the fact that in terms of this norm we have

$$\|\phi\|_{H^1(J^-_{\Sigma_{t'}}(K)\cap J^+(\Sigma_0),g)} \leq C\left(\int_0^t \left(\tilde{\|}\phi\tilde{\|}^1_K\right)^2 dt'\right)^{\frac{1}{2}} \tag{2.31}$$

where C is a constant that depends on g_{ab} and the interval $[0,t]$ [15]. Estimating all the terms in (2.24) by using the Cauchy Schwarz inequality together with the regularity of the metric and solutions gives the inequality:

$$E_K(t) \leq E_{\Sigma_0}(0) + k_0(\|\Box_g u\|_{L^2(J^-_{\Sigma_t}(K)\cap J^+(\Sigma_0),g)})^2 + k_1(\|u\|_{H^1(J^-_{\Sigma_t}(K)\cap J^+(\Sigma_0),g)})^2 \tag{2.32}$$

where k_0, k_1 are positive constants that depend on the metric g_{ab}, the vector field Υ_a and the covariant derivative $\nabla_b\Upsilon_a$.
Now rewriting (2.32) using (2.30) and (2.31) we find

$$E_K(t) \leq E_{\Sigma_0}(0) + k_0(\|\Box_g u\|_{L^2(J^-_{\Sigma_t}(K)\cap J^+(\Sigma_0),\nu_g)})^2 + k_2\int_0^t E(t')dt' \tag{2.33}$$

Using Gronwall's inequality we obtain

$$E_K(t) \leq K_4\left(E_{\Sigma_0}(0) + (\|\Box_g u\|_{L^2(J^-_{\Sigma_t}(K)\cap J^+(\Sigma_0),g)})^2\right) \text{ for all } t \leq T \tag{2.34}$$

where K_4 is a positive constant that depends on the chosen finite time t, the metric g_{ab}, the vector field Υ_a and the covariant derivative $\nabla_b\Upsilon_a$.
Rewriting everything in terms of the Sobolev type norms we obtain the expression:

$$(\tilde{\|}u\tilde{\|}^1_K)^2 \leq A\left((\tilde{\|}u\tilde{\|}^1_{\Sigma_0})^2 + (\|\Box_g u\|_{L^2(J^-_{\Sigma_t}(K)\cap J^+(\Sigma_0),g)})^2\right) \tag{2.35}$$

for some constant A.
By definition we have that the advanced weak solution has vanishing initial data, therefore $\tilde{\|}u\tilde{\|}^1_{\Sigma_0} = 0$ and we obtain the energy estimate

$$(\tilde{\|}u\tilde{\|}^1_K)^2 \leq A(\|\Box_g u\|_{L^2(J^-_{\Sigma_t}(K)\cap J^+(\Sigma_0),g)})^2 \tag{2.36}$$

This establishes the fact that the norm (2.29) of an advanced solution u of the zero initial data inhomogeneous problem $\Box_g u = f$ on $K \subset \Sigma_t$ is bounded by the value of the source f in the region $J^-_{\Sigma_t}(K) \cap J^+(\Sigma_0)$.
Suppose now that p does not belongs to $J^+(supp(f))$. Then $J^-(p)$ does not intersect $supp(f)$ and therefore we can construct a compact set $K \subset \Sigma_{t'}$ that contains p for some t' such that $J^-_{\Sigma'_t}(K) \cap J^+(\Sigma_0)$ does not intersect $supp(f)$. This implies that the right hand side of (2.36) vanishes and $u = 0$ on K. Then, taking a possible smaller region $K' \subset K$ and a small time interval ϵ we can form a neighbourhood $U = \epsilon \times K'$ that contains p and where u vanishes. In this region we have that $u = 0$ and this implies that p is not in the support of u which gives the desired result.

Remark 2.13. The norm defined by (2.29) is not a true Sobolev type norm as it is misssing the zero order terms. However, in the compact setting one has available the Poincaré inequality [7] which guarantees the vanishing of u. Alternatively one can work with a modified energy-momentum tensor S^{ab} which contains an additional $u^2 g_{ab}$ term and is equivalent to a genuine Sobolev norm (see e.g. [15]).

3. Green operators in the non-smooth setting

In this section we define the notion of advanced and retarded Green operators in a weak sense. Notice that Definition 2.1 can not be used in a setting with finite differentiability because weak solutions are in general not smooth so that one cannot expect $E^+(f) \in C^\infty(M)$ for $f \in D(M)$. In fact, even if the metric is smooth but f is not in $D(M)$ one can not expect smooth global solutions. We therefore propose the following definitions suitable for settings with finite differentiability.

Definition 3.1. A bounded linear map $E^+ : L^2(M,g) \to L^2([0,T]; H^1(\Sigma))$ satisfying

- $\langle \Box_g E^+(f), v\rangle_{L^2H^{-1},L^2H^1} = (f,v)_{L^2(M,g)}$ for all $f \in D(M), v \in L^2([0,T]; H^1(\Sigma))$
- $\langle E^+ \Box_g(f), v\rangle_{L^2H^{-1},L^2H^1} = (f,v)_{L^2(M,g)}$ for all $f \in D(M), v \in L^2([0,T]; H^1(\Sigma))$
- $supp(E^+(f)) \subset J^+(supp(f))$ for all $f \in D(M)$

is an advanced weak Green operator for $\Box_g$. A retarded weak Green operator E^- is defined similarly.

Remark 3.2. If g_{ab} is in $C^{0,1}$ and for all $f \in D(M)$ we have $E^\pm(f)$ in $H^2(M,g)$ then the first property can be restated as

$$(\Box_g E^\pm f, v)_{L^2(M,g)} = (f,v)_{L^2(M,g)} \tag{3.1}$$

for all $f \in D(M), v \in L^2([0,T]; H^1(\Sigma))$.

Remark 3.3. If for all $f \in D(M)$ we have $E^\pm(\Box_g f)$ in $L^2(M,g)$ then the second property can be restated as

$$(E^\pm \Box_g f, v)_{L^2(M,g)} = (f,v)_{L^2(M,g)} \tag{3.2}$$

for all $f \in D(M), v \in L^2([0,T]; H^1(\Sigma))$

We now show that the notion of generalised hyperbolicity is sufficient to establish the existence of weak Green operators.

Theorem 3.4. *Let (M,g) be a globally hyperbolic spacetime with $g_{ab} \in C^{1,1}$. Then on M there are unique advanced and retarded weak Green operators for $\Box_g$.*

Proof of Theorem 3.4. We will only show the existence of the advanced weak Green operator. The existence of the retarded weak Green operator follows from time reversal.
Define the linear map $E^+ : L^2(M,g) \to L^2([0,T]; H^1(\Sigma))$ which sends a source function f to the advanced regular weak solution u^+ of the advanced zero initial data inhomogeneous problem with source f . That such u^+ exist and is unique is a consequence of generalised hyperbolicity which follows from Theorem 2.12.
We first show that this map is well defined and continuous.
Let $f = g \in L^2(M,g)$, then by definition we have that $E^+(f) = u^+$ and $E^+(g) = v^+$ are regular weak solutions of the zero initial data inhomogeneous problem with source function $f = g$ causally supported to the future. By the condition of generalised hyperbolicity which guarantees uniqueness of solutions, we have $u^+ = v^+$ and therefore $E^+(f) = E^+(g)$. Therefore the map is well defined.

That the map is continuous follows directly from the regularity of solutions. We have

$$||E^+(f)||_{L^2([0,T];H^1(\Sigma))} = ||u||_{L^2([0,T];H^1(\Sigma))} \leq ||f||_{L^2(M,g)} \tag{3.3}$$

We next show that E^+ is a suitable right inverse in the sense of Definition 3.1. First notice that the regularity of the weak solutions given by the condition of generalised hyperbolicity allows us to define for every advanced weak solution u^+ the linear map $\Box_g u^+[\cdot]$. See remark 2.5. Therefore given $E^+(f) = u^+$, we have

$$\Box_g E^+(f)[v] =: \Box_g u^+[v] = \int_0^T < \ddot{u}^+, v > dt + \int_0^T B[u^+, v; t]dt = (f, v)_{L^2(M,g)} \tag{3.4}$$

which gives

$$\langle \Box_g E^+ f, v\rangle_{L^2H^{-1},L^2H^1} = (f, v)_{L^2(M,g)} \tag{3.5}$$

for all $f \in D(M), v \in L^2([0,T];H^1(\Sigma))$
Now we prove a similar equality for $E^+\Box_g f[v]$ to show that it is a left inverse. First we notice that by definition we have that $u^+ = E^+(\Box_g f)$ is an advanced weak solution of the inhomogeneous problem with source $\Box_g f$ and therefore we have that

$$\int_0^T < \ddot{u}^+, v > dt + \int_0^T B[u^+, v; t]dt = (\Box f, v)_{L^2(M,g)} \tag{3.6}$$

At the same time using integration by parts we have that

$$\int_0^T < \ddot{f}, v > dt + \int_0^T B[f, v; t]dt = (\Box f, v)_{L^2(M,g)} \tag{3.7}$$

so f is also a weak solution. Moreover, using the smoothness of f and the assumed regularity of the metric we can obtain an energy estimate which proves that f is a regular weak solution (see [7]). Then u^+ and f are weak regular solutions to the zero initial data inhomogeneous problem with the same source. So by uniqueness of the solutions we have that $u^+ = f$ which guarantees that

$$\langle E^+\Box_g f, v\rangle_{L^2H^{-1},L^2H^1} = (f, v)_{L^2(M,g)} \tag{3.8}$$

for all $f \in D(M), v \in L^2([0,T];H^1(\Sigma))$
Finally to analyse the support of $E^+(f)$ we use that the weak solutions are causally supported. We have that $supp(E^+(f)) = supp(u^+) \subset J^+(supp(f))$ for all $f \in D(M)$ which gives the desired result.

Remark 3.5. Notice that to prove the first two conditions of definition 3.1 we use Theorem 2.10 while the condition that the solution has the correct causal support is a consequence of Theorem 2.12.

Remark 3.6. It is pointed out in remark 3.4.9 in [1] that Green operators on compact spacetimes with smooth metrics cannot exist as that would imply that $\Box_g : D(M) \to D(M)$ is injective. The proof uses the fact that if $\Box_g \phi = 0$ and Green operators exist, then one has $\phi = E^\pm \Box_g \phi = E^\pm 0 = 0$ which shows injectivity. However, any constant function is in the kernel of $\Box_g$. In our case, if one tries to repeat such a result one has the following equalities $(\phi, v) = (E^\pm \Box_g \phi, v)_{L^2(M,g)} = (E^\pm 0, v)_{L^2(M,g)} = (0, v)_{L^2(M,g)} = 0$. By the density of $L^2([0,T];H^1(\Sigma))$ in $L^2(M,g)$ we can conclude that $\phi = 0$. which recovers the non-existence result in our setting also and explains the choice $M = (0,T) \times \Sigma$.

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We next show that the formal adjoint of E^+ is E^-.

Theorem 3.7. *Given $\chi \in D(M)$ and $\varphi \in D(M)$ we have that*

$$\int_M E^+(\chi)\varphi\nu_g = \int_M \chi E^-(\varphi)\nu_g \tag{3.9}$$

In order to show this result we need the following Lemma

Lemma 3.8. *Given $\chi, \varphi \in H^2(M, g)$ with $\chi(0, x) = \chi_t(0, x) = 0$ and $\varphi(T, x) = \varphi_t(T, x) = 0$ we have that*

$$\int_M \Box_g \chi\varphi\nu_g = \int_M \chi\Box_g\varphi\nu_g \tag{3.10}$$

Proof of Lemma 3.8. The proof follows straightforward from using integration by parts twice and using the boundary conditions given by the hypothesis.

Proof of Theorem 3.7.

First, notice that because the metric has regularity $C^{1,1}$ and $\chi, \varphi \in D(M)$ we have that $E^+(\chi), E^-(\varphi) \in H^2(M, g)$. [7].

Hence,

$$\begin{aligned}
\int_M E^+(\chi)\varphi\nu_g &= (E^+(\chi), \varphi)_{L^2(M,g)} && (3.11)\\
&= (E^+(\chi), \Box_g E^-(\varphi))_{L^2(M,g)} && (3.12)\\
&= (\Box_g E^+(\chi), E^-(\varphi))_{L^2(M,g)} && (3.13)\\
&= (\chi, E^-(\varphi))_{L^2(M,g)} && (3.14)\\
&= \int_M \chi E^-(\varphi)\nu_g && (3.15)
\end{aligned}$$

where the second and fourth equality follows from remark 3.2 while the third inequality used the initial data given by the advanced and retarded zero initial data inhomogeneous problem and Lemma 3.8.

In the smooth case it is often convenient to consider the Green operators as bi-distributions on the product space. We show that this result extends to the non-smooth setting.

Theorem 3.9. *The map $\mathbf{E}^+ : D(M) \otimes D(M) \to \mathbb{R}$ given by*

$$\mathbf{E}^+(fg) = (E^+(f), g)_{L^2(M,g)} \tag{3.16}$$

can be extended to a distribution on $M \times M$.

Proof of Theorem 3.9.

Given $f \otimes g \in D(M) \otimes D(M)$ we have

$$\begin{aligned}
\mathbf{E}^+(fg) &= (E^+(f), g)_{L^2(M,g)} && (3.17)\\
&\leq ||E^+(f)||_{L^2(M,g)}||g||_{L^2(M,g)} && (3.18)\\
&\leq ||E^+(f)||_{L^2([0,T],H^1(\Sigma))}||g||_{L^2(M,g)} && (3.19)\\
&\leq C_1||f||_{L^2(M,g)}||g||_{L^2(M,g)} && (3.20)\\
&\leq C_1||f \otimes g||_{L^2(M,g)} && (3.21)\\
&\leq C_2||f \otimes g||_{C^0(M)} && (3.22)
\end{aligned}$$

for some constants C_1, C_2 and where we have used Holder's inequality and that $E^+(f)$ is a regular solution. Now taking into account that $D(M) \otimes D(M)$ is dense in $D(M \times M)$ we can extend $\mathbf{E}^+(fg)$ to a unique continuous linear mapping in $D(M \times M)$. This proves the claim.

Remark 3.10. See [1] for a proof of this theorem in the smooth setting.

4. Galerkin type approximation of E^+

In this section we will provide a construction of the operator E^+ and give a formula for the kernel. To construct the map E^+ we will build a sequence of operators E_m^+ that converge in the weak operator topology to E^+. We will construct the operators E_m^+ using Galerkin's method following the methods of [7, 9, 10].

First consider a complete orthogonal basis $\{w_j(x)\}_{j\in J}$ in $H^1(\Sigma)$ which is also orthonormal in $L^2(\Sigma)$. Then inserting the m-approximate solution $u^{(m)^+}(t,x) = \sum_{k=1}^m d_k^{(m)}(t)w_k(x)$ into the equation 2.19 we find that the functions $d_k^{(m)}$ for $k = 1,...,m$ satisfies the system of second order ODEs

$$\ddot{d}_m^k(t) + \sum_{l=1}^m e^{kl}(t)d_m^l(t) = f^k(t) \tag{4.1}$$

where $e^{kl}(t) = B[w_k, w_l; t]$ are C^1 and symmetric and $f^k(t) = \int_\Sigma f(t,y)w_j(y)\gamma(y,s)dy$. (See Theorem 2.10)

We may find a Green matrix $F_m^{kj}(t,s)$ [17] for the initial value problem for this system of second order ODEs and write the solution in the form

$$d_m^k(t) = \int_0^T \sum_j F_m^{kj}(t,s)f^j(s)ds \tag{4.2}$$

So that

$$u^m(t,x) = \int_0^T \sum_{j,k} F_m^{kj}(t,s)f^j(s)w_k(x)ds \tag{4.3}$$

Recalling that $f^j(s) = \int_\Sigma f(s,y)w_j(y)\gamma(y,s)dy$ we have

$$u^m(t,x) = \int_0^T \int_\Sigma \sum_{j,k} F_m^{kj}(t,s)w_k(x)w_j(y)f(s,y)\gamma(y,s)dyds \tag{4.4}$$

We may now define the m-approximate advanced Green operator, E_m^+ on $L^2(M,g)$ by

$$E_m^+(f) := \int_0^T \int_\Sigma \sum_{j,k} F_m^{kj}(t,s)w_k(x)w_j(y)f(s,y)\gamma(y,s)dyds \tag{4.5}$$

where $F_m^{kj}(t,s)$ is the Green matrix for the (regular) ODE problem.

The next step is to show that $\{u^m(t,x)\}_m$ is bounded uniformly in $L^2([0,T];H^1(\Sigma))$. This is a consequence of the energy estimates. See [7] for a derivation of the estimates and the uniform bound. Applying the Banach-Alaoglu theorem to the bounded sequence we can obtain a subsequence $\{u^{m_l}(t,x)\}_l$ that converges weakly to u^+ in $L^2([0,T];H^1(\Sigma))$ i.e. there is a subsequence $\{u^{m_l}\}_l$ that for every linear bounded operator ϕ on $L^2([0,T];H^{-1}(\Sigma))$ we have $\lim_{l\to\infty}\langle\phi, u^{m_l}\rangle_{L^2H^{-1},L^2H^1} = \langle\phi, u^+\rangle_{L^2H^{-1},L^2H^1}$.

Notice that the weak convergence of the sequence of functions $\{u^{m_l}\}$ is equivalent to convergence of the operators $\{E_{m_l}^+(f)\}$ in the weak operator topology. By definition 4.5 we have

$$\langle\phi, u^{m_l}\rangle_{L^2H^{-1},L^2H^1} = \langle\phi, E_{m_l}^+(f)\rangle_{L^2H^{-1},L^2H^1}$$

for all $\phi \in (L^2([0,T];H^{-1}(\Sigma)))$. Now by weak convergence of u^{m_l} to u^+ we have that $\lim_{l\to\infty}\langle\phi, u^{m_l}\rangle_{L^2H^{-1},L^2H^1} = \langle\phi, u^+\rangle_{L^2H^{-1},L^2H^1}$ for all $\phi \in L^2([0,T];H^{-1}(\Sigma))$ but this is equivalent to

$$\lim_{l\to\infty}\langle\phi, E_{m_l}^+(f)\rangle_{L^2H^{-1},L^2H^1} = \langle\phi, E^+(f)\rangle_{L^2H^{-1},L^2H^1} = \langle\phi, u^+\rangle_{L^2H^{-1},L^2H^1} \tag{4.6}$$

for all $\phi \in L^2([0,T];H^{-1}(\Sigma))$ which establishes the weak convergence in the operator topology. We may therefore write for every $v \in (L^2([0,T];H^1(\Sigma)) \subset L^2([0,T];H^{-1}(\Sigma))$

$$\langle v, u^+\rangle_{L^2H^{-1},L^2H^1} = \langle v, E^+(f)\rangle_{L^2H^{-1},L^2H^1} \tag{4.7}$$

$$= \lim_{l\to\infty}\int_M E^+_{m_l}(f)(x,t)v(t,x)\gamma(x,t)dtdx \tag{4.8}$$

$$= \lim_{l\to\infty}\int_M\int_M \sum_{j,k} F^{kj}_{m_l}(t,s)w_k(x)w_j(y)f(s,y)v(t,x)\gamma(y,s)\gamma(x,t)dydsdtdx \tag{4.9}$$

where the second identity is the weak convergence of the m-approximate solutions $E^+_m(f)(x,t) = u^m(x,t)$ to the weak solution $u^+(x,t)$ in $L^2([0,T];H^1(\Sigma))$ and the third identity follows from equation 4.5. We have therefore shown that we may write the kernel of the advanced Green operator as

$$G^+(t,x;s,y) = \lim_{l\to\infty}\sum_{j,k} F^{kj}_{m_l}(t,s)w_k(x)w_j(y)\gamma(y,s)\gamma(x,t). \tag{4.10}$$

Note that in terms of the above representation the fact that the formal adjoint of E^+ is E^- is reflected in the property $F(s,t) = F^T(t,s)$ for the Green matrix of a symmetric self-adjoint system of 2nd order ODEs [17].

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