UNIVERSITY OF SOUTHAMPTON

FACULTY OF SOCIAL, HUMAN AND MATHEMATICAL SCIENCES

Department of Economics

Models of Network Formation and Financial Contagion

by

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ABSTRACT

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In this Thesis we study two features of production networks: their emergence and their vulnerability to idiosyncratic shocks. In order to understand the relation between economic incentives leading to the formation and the systemic properties of the network we map our economic analysis into the mathematical theory of networks.

In the first Chapter we develop a simple model of endogenous formation of input-output economies to address the theoretical nexus between trade-credit, bank credit and credit contagion. We make two contributions. First, we show that competitive markets in which heterogeneous price-taker firms compete strategically by setting trade-credit maturities have a unique symmetric equilibrium in trade-terms and the equilibrium dictates the production flow along the supply chains. Secondly, we find that the network can have a role either as shock absorber or shock amplifier and this is determined by a testable condition which holds for a general class of trade-credit networks. On these grounds, we argue that the proportional credit rationing used by banks (i.e., richer borrowers obtain larger loans) may have ambiguous effects on systemic vulnerability.

In the second Chapter we develop a model of economic networks formation which links Internal Capital Markets to the formation of Business Groups. Our model is stylized as it focuses on two interacting channels: the debt-to-equity regulations and investment profitability. In our model, a growing group of heterogeneous and financially constrained firms have a limited capability to coordinate production in a competitive market by means of pairwise credit arrangements. We show that usage of inter-firm credit is sustained by an individually rational mechanism which match several empirical features.

In the third Chapter, we look at the complex effects of financial frictions in a non-stationary market economy with heterogeneous agents. We study two classes of dynamic equilibria for which the market converges to the expression of a single type of seller.

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Declaration of Authorship

I, Andrea Giovannetti , declare that the thesis entitled *Models of Network Formation and Financial Contagion* and the work presented in the thesis are both my own, and have been generated by me as the result of my own original research. I confirm that:

- this work was done wholly or mainly while in candidature for a research degree at this University;
- where any part of this thesis has previously been submitted for a degree or any other qualification at this University or any other institution, this has been clearly stated;
- where I have consulted the published work of others, this is always clearly attributed;
- where I have quoted from the work of others, the source is always given. With the exception of such quotations, this thesis is entirely my own work;
- I have acknowledged all main sources of help;
- where the thesis is based on work done by myself jointly with others, I have made clear exactly what was done by others and what I have contributed myself;

Signed:...

Date:..

Acknowledgements

Had Nietzsche ever completed his PhD dissertation and, more important, its Acknowledgments section, hardly he would have written that the *" objective man is in truth a mirror accustomed to prostration before everything that wants to be known"* and that *"whatever personality he possesses seems to him accidental, arbitrary, or still oftener, disturbing, so much has he come to regard himself as the passage and reflection of outside forms and events."* (Nietzsche, [57]).

For it is in no other locus that the objective man hints at the humane idiosyncrasies surrounding his labor - it's a glimpse - right before dissolving his objectiveness into the magnificent stream of those passions which sustain science - gratitude and admiration.

I Myself owe human gratitude to those who steered with passion and trust my unsteady steps toward scientific rigour and sustained me by means of their personal example. First and foremost, I would like to thank my supervisor Antonella Ianni, whose dedication transcended any border of professional obligation. I am honoured of the time and effort Antonella patiently spent in routing my erratic thoughts into cogent propositions. This dissertation would not have been possible without her support along all the dimensions involved in a PhD. I also owe a profound debt of intellectual and human gratitude to Thomas Gall, Chiara Forlati, Spyros Galanis and Corrado Giulietti. Along this very line, I would like to thank the person who enlightened my path since the latter stages of my MSc, Paolo Pin, who I am honored to consider a mentor and a great source of intellectual inspiration. My time at the Department of Economics of Southampton had been so fruitful thanks to the splendid interaction with Hector Calvo Pardo, Yves Zenou, Christos Ioannou, Alex Mennuni, Tassos Magdalinos, Carmine Ornaghi, Emanuela Lotti, Zach Maniadis, Jan Podivinsky and the other scholars who fostered my personal and research development along these years. In Southampton I appreciated the true sense of collegiality by virtue of persons I am proud to call friends: those in my cohort with whom I directly shared the expectations and the pains of the PhD, with particular regards to Chi Wan Cheang, Michele Tuccio, Liu Liu and Vasiliki Koutra. Those who preceded my experience - Panos Giannarakis, Dafni Papoutsaki, Jana Sadeh, Elif Kara and Michael Kerns and those who followed it - Armine Ghazaryan, Marius Strittmatter, Burak Demirtas, Abu Siddique, Xiaocheng Hu and Anibal Neeto among the many. Friends in other departments with whom I shared also other fragments of life - Federico Vaccari and Daniele Cassese - had likewise importance. I thank *ESRC* and *Banco Popolare di Novara* for the tangible faith Society had put on my research.

Lastly, I would like to pay a homage to my mother, my father and my sister, the pillars of my life, and to Sarah, the brightest star of a firmament which has sometimes been very dim.

Chapter 1

Introduction

Tracing back to Xenophon's *Cyropaedia*, economic analysis has already put the subject of inter-connectedness in production systems under the lenses of its many sub-disciplines. Nonetheless, the importance of inter-connectedness *per se* remained relatively obscure matter until recent history vindicated a more prominent position. In this sense, after the intellectual turmoil observed in the early proceedings of the recent financial crisis (2007-2008), the ensuing Great Recession (2008-2010) triggered a timely debate in academic and wider public on the role of connectivity in economy. With the effects of the crisis rapidly unwinding beyond the financial sector, notions such as "inter-connectedness", "domino effect" and "network" became endogenous to the public debate (Carvalho, [18]). Empirical analysis related to the geometrical structure of business networks gained uplifting momentum. For instance, heated media coverage was given to a geometrical regularity uncovered by Battiston, Glattfelder and Vitali [9] in the analysis of business networks, that is that a cluster of less than 150 companies controls 40% of the *global* real economy via direct and indirect ties. Evidence related to the transformation of the US buyer-supplier architecture contained in Atalay et al. [6] had comparable far-reaching impact. On a parallel venue, the economic importance of inter-connectedness *per se* in production systems has been popularized by negative evidence provided by a wealth of papers (see Carvalho [19] for a discussion) focused on the disruptive effects of the Tōhoku tsunami, which *disconnected* the Japanese production system.

In this Thesis we study inter-connectedness in production environments by focusing on two complementary motives: the emergence of inter-firm networks in production economies and their systemic properties. A cursory outline of this work can be resumed as follows. In Chapters 2 and 3 we will look at the foundational role of firms' financing frictions for the establishment of production networks. Besides identifying a possible channel for the formation of inter-firm linkages, in Chapter 2 we will also address the issue of systemic stability in production networks. In Chapter 3, we will consider a dynamic multi-firm industry in which firms interact via a simple mechanistic protocol. We use this framework to explain the formation of Business Groups in non-stationary economies and look at their long-run properties in relation to financing and demand-driven shocks. Lastly, in Chapter 4 we focus on non-stationary economies from a more game-theoretic perspective and devise two classes of equilibria for a bargaining game in dynamic markets which may sustain a formation process which is compatible with the simple mechanism provided in Chapter 3.

Trade-Credit, Bank Credit and Systemic Vulnerability in production network

In the first part of this work, we look at inter-firm (financing) relations by grounding our theory on a general class of inter-firm contracts which goes under the name of *trade-credit*. This multi-dimensional contract raises several questions. A starting point of the literature is to try to understand why trade-credit exists. Seifert, Seifert and Protopappa-Sieke [65], show that explanations for co-existence of trade-credit and efficient financial markets fall within two broad families of motives - the demand-side and the supply-side motives - which focus on profitability and control of trade-credit from the standpoint of either buyers or suppliers, respectively. In this Chapter we entirely focus on the latter and refer the reader to [65] for an in-depth review of demand-side motives. In a seminal contribution, Biais and Gollier [10] hypothesized a *financing motive*. When banks ration credit, suppliers with better access to cheap credit are willing to *lend* working capital to liquidity constrained buyers against a premium charged over the market price, thus effectively complementing the bank in credit provision. Building upon these premises, Burkart and Ellingsen [15] explain the coexistence of bank and trade finance in competitive markets by attributing sellers a monitoring advantage over banks. While buyers can easily pocket cash loans issued by banks, the less fungible nature of inputs prevents diversion of trade-credit. Trade-credit existence is sustained by a double-coincidence of wants given by the monitor advantage and price premia on the supply-side and liquidity constraints on the demand-side. However, a growing body of papers looking at both sides of buyer-seller contracts and term structure along supply chains curtails the finance motive by showing that also credit-constrained firms offer trade-credit to their customers, thus implying that trade-credit might be a dimension of intra-sectoral competition and that differences in trade-terms may be reflective of asymmetries in firms' *market power*. Giannetti, Burkart and Ellingsen [30] reconcile the motives by showing that firms with better access to finance have higher receivables to sales, yet suppliers are more generous in trade-terms to customers that have more bargaining power. Fabbri and Klapper [24] bring evidence that trade-credit is a constituent element of supply-chains because it is used by sellers who face strong competition as a competitive gesture for locking in customers in lieu of price cuts. Van Horen [68] brings further evidence of trade-credit as a lock-in device. Klapper, Laeven and Rajan [46], note that whenever price discrimination is prohibited (e.g. the Clay Act in the U.S.), the market power hypothesis may explain why suppliers borrows at high cost in order to provide cheap financing.

In Chapter 2 we study the formation and the properties of supply chains in multi-sector economies by looking at firms' investment structure from a contract-theoretic perspective. As our focus is on the link between (formation of) trade-credit chains and balance-sheet contagion, we take a lateral approach with respect to general equilibrium models of input-output economies based upon Long and Plosser [51] discussed in the next section. Specifically, we contribute to the novel literature of trade-credit in supply chains by making two points. First, we prove that the working capital hypothesis together with the market power hypothesis discussed above provide sufficient incentives for the formation of supply chains. Supply chains may naturally emerge in contexts in which financially constrained producers compete in the non-monetary dimension of trade-terms. We make this point by nesting the bargaining hypothesis in a model of trade-credit in competitive economies which mimics the general framework proposed by Burkart and Ellingsen [15], with two major twists. First, we assume that producers in each sector *compete* for selling their output to firms belonging to the sector ahead. Secondly, as in Burkart and Ellingsen, we maintain that payment is ex-post production and that buyers can divert (part of) the output. Yet, we assume that both banks and sellers monitor buyers. More precisely, we assume that sellers' monitoring technology is costly, but costs decrease in the expected buyer's output. Our motivation is that, for instance, in the absence of formal credit-assessment tools, monitoring a buyer with large exposures in account receivables may be cheaper because of the relatively high cost of any opportunistic behavior on their part, or that sellers may trust more larger counter-parts. The core of our supply chain formation process is a simple symmetric equilibrium in trade-terms, which identifies each buyer's *strategic* supply of input given her own cash endowment, some sector-wide parameter such as the cost of bank (trade) credit and intra-sectoral competition. Because sectors are interlocked, the aggregate demand of input of some sector induces the production incentives for the sector ahead. From the perspective of the inter-sectoral flows, our interlocked economy behave similarly to a single productive cycle of well-known models (see Battiston et al. [8] discussed in the next section). Sectoral demands are satisfied sequentially. Payments are entirely on credit and are made only after a production cycle ends. The main advantage of our result is that it allows us to map individual producers' decisions directly on the "topological" properties of the feasible *class* of production architectures, with no need of pre-imposing *ad hoc* input-output structures. In this regards, our analysis complements other input-output analysis (Battiston et al. [8] and [35]) by providing a clear intuition between economic micro-incentives, macro-quantities and the input-output structure.

This brings us to the second contribution of this Chapter, that is an intuitive condition which allows us to assess the resilience to balance-sheet contagion for general classes of production architectures. The shock structure adopted in our work and the related contagion mechanics is a simple one, yet it allows us to make an inference on resilience *ex-ante* trade-credit structures realize: for all the production structures belonging to a given class, resilience is metricized as a statistical condition which disentangles the probability of contagion in the interaction between two components. These are the probability that a firm defaults upon local shocks (i.e. a shock affecting one direct counter-part) and the degree of inter-connectedness of firms populating the structure. We find that the relation between interconnectedness and individual risk is non-trivial as it depends precisely on the investment mix and the lending regime respectively selected by the producers and the bank.

In general terms, our contract-theoretic framework reasons well with the empirical consensus discussed in the next section: balance-sheet contagion is subject to the degree of heterogeneity of firms' financing capabilities, the distribution of "deep pocket" firms with respect to constrained firms and the density of the inter-firm credit architecture. Additionally, it lends us a novel testable hypothesis related to the interaction between bank credit and trade credit. We find that when the bank adopts a proportional lending regime (rich firms obtain more credit than poor firms), the response of firms' budget composition to changes in bank overdraft is non-linear in firms' wealth. Overall, this effect may be found to actually *amplify* the systemic risk for positive yet moderate levels of bank lending intensity. The intuition is that when a bank is introduced in the trade-credit architecture, only a subset of (rich) firms will fully-substitute trade credit, whereas the remaining pool of (poor) firms will leverage trade-credit supply by *adding up* trade credit and bank credit. As a result of the partial substitution, the trade-credit network will be unambiguously less inter-connected - in line with empirical findings discussed in the next section (see [62]) - yet consequential to the credit rationing policy, the bank may possibly fail to *insulate* most vulnerable firms. As a result, a greater proportion of chains will be of lower quality as stemming between poorer and more leveraged firms.

We also extend the current literature in contagion in random networks (see [23] and Hurd [38] for a reasoned review) in the following directions. First, the stochastic network which we study is micro-founded and emerges from the interaction of institutional constraints, market condition and agents' types. This implies that we do not impose artificially the network density as it depends on a variety of economic macro-factors. Secondly, we expand the classical result of Gai and Kapadia [27] by allowing our network to express *correlated* links in the fashion of Hurd [32], but the contagion dynamics which we describe are quite different from both these works. While the standard result in the literature identifies the contagion window as a generic non-monotonic relation between average network connectivity and probability of financial contagion, in our set-up such relation is mediated by the micro-founded decision taken by the various agents according to their types. Therefore, given the microeconomic foundation, the contagion (window) maps directly into a parsimonious set of economic fundamentals. Third, by making use of somewhat stronger conditions with respect to Hurd [32], we offer a tractable reformulation of his contagion threshold which highlights the straight relation between heterogeneity of agents and the geometric properties of a *vulnerable* network. This will be especially useful to highlight in analytic terms the core result of the Chapter: a risk-neutral credit rationing bank can actually *increase* the systemic risk of a trade-credit network whenever the bank-credit policy is such that firms receive credit proportionally to their endowment.

Relation to Extant Results in Inter-Firm Contagion Literature

As trade-credit is key in shaping supply chains, which are in turn the building bricks of developed economies, we may ask whether localized idiosyncratic shocks can migrate along credit-chains and cluster in substantial macroeconomic impact.

In this perspective, firms' financing behavior - and specifically the relation between trade credit and bank credit - is directly projected on aggregate production dynamics and systemic risk. Following Kiyotaki and Moore [45], several authors provided theoretical and empirical evidence that idiosyncratic stress triggered by localized defaults may funnel through trade-credit leading to system-wide failure. In an early theoretical contribution, Boissay [12] links trade-credit to contagion by showing that significant domino effects may build up along individual supply chains where a fraction of firms rely on receivables to reimburse payables. More recent contributions have tried to single out the role in contagion of the network concealed within supply chains. Battiston et al. [8] explored the effect of network interaction among producers tied via trade-credit links in the context of a dynamic multi-agent economy. They find that idiosyncratic shocks uniformly distributed among producers can induce avalanche defaults if networked firms mis-coordinate production due to delayed payments and a slowly-adaptive cost structure. In their paper, higher inter-connectivity and risk sharing is always beneficial for stability. Using complex networks theory, Henriet, Hallegatte and Tabourier [35] construct a random multi-sectoral economy by which they isolate the contribution of the network topology on systemic risk. They confirm that redundancy of connections acts as a risk-sharing device, and bring novel evidence on how clustering and initial losses concentration affect global resilience. Interestingly, by varying the degree of concentration of a fixed amount of shock, they find that the total loss of the economy is maximal when concentrated in a single firm.

Complementary to the balance-sheet credit contagion approach described above, critical developments in general equilibrium analysis have recently built upon the seminal work of Long and Plosser [51] for inspecting the micro-dependencies of aggregate fluctuations. The contributions of Gabaix [26] and Acemoglu et al. [2] in particular have reinvigorated the idea that unit-specific shocks can percolate through the wider economy conditional on the network properties of the sectoral structure. Advancing the granularity of such approach, Magerman et al. [53] address the link between domestic inter-firm trade and aggregate volatility, while Carvalho et al. [17] used the Great East Japan Earthquake shock in order to disentangle the supply-chain disruption effect on the Japanese production system. Contemporary work has addressed the critical effect of liquidity shocks over production networks. Bigio and La'O [11] extendend Acemoglu et al. model in order to study the amplifying effect of financial frictions on the aggregate economy. Furthering the characterization of firms' financial mix, Altinoglu [5] and Luo [52] endogenize a trade credit architecture within the input-output production network, proving that trade-credit strengthens the correlation of sectoral output. By introducing banks in the (trade-credit) economy, Luo shows that liquidity shocks are asymmetric in diffusing down-stream (via demand) or up-stream (via input prices) and draw policy conclusions on optimal sectoral liquidity targeting by central bank. We also address the impact of exogenous shocks on trade-credit networks in which firms adopt heterogeneous financing, yet we diverge from the literature above in two regards. First, in our work the trade-credit network architecture is endogenous. Secondly, we operationalize our contract-theoretic framework toward a complementary problem: the nexus between firms' heterogeneous investment mix and balance-sheet contagion.

In recent years, the theory of balance-sheet contagion has been strengthened by a wealth of studies. By looking at the inter-industry flows, Raddatz [62] explicitly isolated a link between the use of trade credit across sectors and the industries' output correlation. Critical for our theory, he finds that an increase in the use of bank credit relative to trade credit reduces supply-chains comovements. More recently, Boissay and Gropp found that trade-credit default chains are factual across (small) credit rationed firms, even though their impact can be moderated by *deep pocket* firms (firms that are larger and have better access to bank loans). Jacobson and Schedvin [42] bring extensive evidence that sizable portions of corporate bankruptcies can be generated by idiosyncratic shocks which diffuse along trade-credit chains and that such channel parallels the traditional demand-side amplification mechanism considered in the macroeconomic literature initiated by Long and Plosser [51]. Moreover, they validate the idea that firms react differently to their customers' default, with liquidity constrained firms being remarkably more likely to go bankrupt upon customers' failure than larger, richer firms.

The Role of financing constraint in the formation of Business Groups

In chapter 3 we study a simple model of a production economy in which an expanding population of heterogeneous producers take financing decisions. Depending on the attractiveness of the real sector and the financial constraints which determine the maximal financial exposures of lenders, a financial network may emerge as the endogenous byproduct of individual producers financing decisions. The economy in this chapter mimics a widely appreciated organizational structure generalically identified such as "Business

Groups". The morphology of Business Groups is deceptive, though. By buying and selling loans within their own group, banks belonging to a US Multibank Holding Company (MBHCs) reallocate funds by complying to a protocol similar to profit-sharing used by firms affiliated in a Korean *Chaebol* or cross-subsidization in companies partaking in an Indian Business House: they are all effectively participating into a Business Group (BG). Clusters of legally independent firms, usually operating in different sectors, may decide to resort on pre-existing relationships or even establish new ties and groups with other firms with which they share proximity under some dimension. The degree of formalization of linkages between members as well as the underlying organization ranges from loose horizontal to tight pyramidal structures [64]. Although evidence of business-groups is widespread across economies at different stages of market development, the literature fails to identify a comprehensive theoretical driver for this sort of complex organization.

In the wealth of world-wide evidences and theories collected around the many empirical realizations of BGs, Khanna and Yafeh [44] recognize that while the ubiquitous persistence of business groups is a well-documented fact, the nexus between affiliation and individual firm performance is highly sensitive to geography, institutional development and other determinants. As individual economic incentives are not univocal, also the identification of group advantages is challenging. Several drivers have been put forward in order to explain business groups' presence.

For instance, a market might be characterized by unilateral *incentives of integration*. A body of literature has focused on the one-side incentives of merges and hierarchical control. In a seminal contribution, La Porta et al. [61] found that when share-holding is concentrated, there might be an incentive for the control group to expropriate minority share-holders and *tunnel* resources from the periphery (i.e. newly acquired firms) of the group to the center of it. In this light, the incentive for group formation is a function of the majority stake-holders powers: business groups are a mechanism for resource-concentration. However, incentives for the creation of business groups and fluid internal capital market may be aligned between the business group's firms, regardless of idiosyncratic features such as the group's structure and each participant's relative power. Within-groups collusive tactics may be adopted in order to deter market-entrance (see for instance Cestone and Fumagalli, [20]) or to gain political leverage which can be used to push forward the group's special interest (Leuz and Oberholzer-Gee, [48]).

A possible unifying *structural* driver for the formation of BGs which partially encompasses the strategic incentives we touched above deals with the capability of BGs to generate Internal Capital Markets (ICMs). In fact, a majority of theories builds upon the assumption that in presence of capital market frictions, financially constrained firms may (be forced to) drift out of traditional financing venues and initiate complex inter-firm liquidity markets. While non-excluding for further explanatory layers, the idea that BGs are key for their capability to attenuate affiliates' financial constraints by means of the underlying ICM is appealing for its generality. In the context of an economy populated by conglomerates of highly independent firms (the Korean *Chaebols*) , Lee, Park and Shin [47] bring strong evidence of the causal relation between ICMs and BGs and isolate the link between the intensity of cross-subsidization - the *conglomerate* debt-to-equity ratio - and the conglomerate market efficiency. They show that the paralyzing effect on ICMs of liquidity regulations impact the profitability of the whole conglomerate. Under the assumption that ICMs are a leading *raison d'être* for BGs, the analysis of BG's efficiency maps into the study of the capital flow direction within the BG. Almeida, Kim and Kim [3] show that a low-growth to high-growth firms capital reallocation exists within ICMs and that such flow makes a dent in improving the efficiency of BGs. In particular, they find that it is generally true that BGs reallocate funds to member firms with greater investment opportunities.

In the Chapter we attempt to deliver two main contributions. First, we show that it is possible to pin down the formation of business groups to their capability of generating ICMs. By means of a simple mechanism of *pairwise* capital exchanges, we show that under certain conditions a group of independent producers with heterogeneous liquidity have the incentive to initiate a complex layer of financial transactions, and the incentive directly relates to the condition of the production market in which they operates. The resulting financial layer is flexible, in the sense that it encompasses a wide spectrum of configurations in between the two extremities of complete specialization (i.e. firms that only lend or borrow capital) and allows for the formation of multiple BGs. By varying the degree of market profitability and the underlying industry first-best investment we show how the equity-to-debt ratio modulates the BGs shapes. Our model is stylized in the sense that the formation protocol is applied to a prototypical framework endowed with the following key characteristics.

Stylized Market Regularities

- *Market is Dynamic* We will assume that in every period new (possibly liquidity constrained) firms enter in the market (i.e. the "newcomers") and seek for funding directly from firms that already settled in it (the "mature firms").
- *Investments in Business Groups flow top-to-the-bottom.* We impose that loans have a precise direction, that is mature firms can lend to newcomers but not vice-versa.
- *Exchange incentives are pairwise-determined.* In line with a substantial body of literature, we assume that firms are independent production units. Therefore, capital in exchanged via pair-wise interactions and a lender has no direct control on further transfers operated by her borrowers. While this assumption does not restrain the top-to-bottom liquidity flow, it limits the market power of any given production units to the relation with her direct counter-parties.

The main advantage stemming out from our dynamic model lies in neat relation between individual incentives, macro-variables and aggregate outcomes. In this sense we deliver testable implications of the effect of regulatory and demand shocks on the *evolution* of BGs in terms of a parsimonious set of statistics.

This leads to our second contribution. We build our market dynamics on a model of network formation. As Internal Capital Markets are observed by means of ownership participation or inferred via series of transactions (links) between distinct entities (nodes), the network approach has already been used for inspecting complementary issues. In particular Almeida et al. [4] reconstructed the expansion of Korean *chaebols* by tracking the layer of direct and indirect ownership relations and the network centrality of each production units. Our aim here is to provide a theoretical formation mechanism which matches the empirical regularities outlined above and which can be used for deriving the aggregate effect of macro-shocks in an environment of local interactions. We build our mechanism upon the class of *growing networks* models initiated in the seminal work of Jackson and Rogers [39] and Vega Redondo [69]: this class of models allows to project the effect of local interactions into the *structural* properties of a resulting aggregate system. However, with respect to the above models, we try to elaborate a formation process which is pairwise efficient and which is driven by micro-founded incentives.

The general idea behind the prototypical growing networks formation process is that the agents' population (i) is large, and (ii) it *grows* continuously in time, single agents entering sequentially each period. Concerning the interactions, it is usually assumed that (iii) agents do not revise their strategies after having taken a decision, and that (iv) agent's decisions are myopically made at their entrance conditional on the environment at time of arrival. Obviously, partner selection represents the critical feature of the (possibly multi-dimensional) individual decision set. Lastly, (v) linking decisions are taken unilaterally, so that the receiving agent is not really involved in the newcomer's decision process. In the context of *social* networks, the formation mechanism developed by Jackson and Rogers [39] and Vega-Redondo [69] under a similar set of assumptions is characterized by a mixture of global and local features. In these models, each newcomer t randomly selects m_r *friends* from the pool of available agents and then m_s further links are established with their *friends of friends* (random global search[1]). As a consequence of the latter mechanism, the composite probability for a node t' to be chosen by a newcomer is increasing in the number of her friends.

Although appealing for its tractability, the framework defined in [39] and [69] is not micro-founded and, to some extent, hinges upon dimensions that prevent a straight generalization into a proper economic setting. Namely, (i) a revenue-cost structure for linking decisions is missing, (ii) agents weight equally their friends. (iii) Agents are homogeneously characterized by discrete and exogenously posed parameters $\{m_r, m_s\}$

[1] Jackson and Rogers [39] provide an extension of their model which comprises a sketched utility preferential attachment with fixed cost and random utility of link formation

which partly determine directly the *number* of friends they have, (*iv*) these individuals are independently drawn from the available population; lastly, (*v*) among one agent's friends, only $m_r + m_s$ of them are the result of her own choice. Our aim in this Paper is to provide a model of economic network formation that accommodates an economy characterized by business groups in the following dimensions. (*i*) Linking decisions are updated via a micro-founded decision process. (*ii*) Links are weighted and ordered according to an explicit opportunity-cost structure. (*iii*) Agents retain some control over the linking process after they entered the network (i.e. links are unilateral but are required to satisfy a pairwise Incentive-Compatibility condition). (*iv*) Linking decisions depend on previously-taken decisions.

Financing Constraints in Dynamic Markets

In Chapter 4 we try to look at a possible game-theoretic foundation for the mechanism described above by studying a dynamic market in which a *time-varying* pool of heterogeneous sellers faces waves of financially constrained buyers which enter in the market at the beginning of every period. The buyers initiate series of sequential bargaining on the basis of a pre-determined demand for a generic good. We focus on two particular classes of equilibria which may sustain the dynamic market formation along a path such that a homogeneous seller-type economy does emerge in the limit. Depending on the equilibrium under consideration, buyers that satisfy their demand for the good either quit the market or settle in the market as potential sellers for next waves of buyers. In either case, sellers will leave the market upon successful trading. We show that the presence of a regulator which controls agents' financial leverage has a direct implication on the long-run characteristics of the market. Specifically, we show that financial regulations cause a tipping point in the market evolution which is critical for the existence of the two classes of equilibria

Several authors considered dynamic economies with financial constraints. In particular, Moll [56], Liu and Wang [50] and Mino [55] studied the evolution of a market economy in which firms are heterogeneous and financially constrained. Similarly to Mino, in this Chapter we consider an endogenous market process. However, our setting draws on the non-stationary dynamic bargaining framework proposed by Manea [54], which we further characterize in order to study two specific classes of equilibria which allow for the formation of a growing market economy.

More precisely, we embed our market formation process within the general *infinite horizon bargaining game played in discrete time* introduced by Manea [54]. In our model, the market formation is made of an initial stage and two intertwined processes. In the initial stage, every agent in the production market discovers her own good endowment, which is heterogeneous across the firms, and induces an individual demand for the good according to her preferences and the financial constraint currently in place. For a given *market composition* (i.e. the distribution of available units of the good among the agents who already settled in the market), the first process is the *intra-period* exchange of the good between agents entering in the market and the *current* pool of sellers. The second process is the evolution of the market composition across periods. The key dimension which links intra-period and inter-period dynamics is given by the market composition in sellers' good endowments, which is an endogenous measure that depends on the outcome of the realized exchanges and which pins downs the incentive structure of the bargaining process.

Chapter 2

Formation of Multi-Sector Economies and Trade-Credit: Can Banks Amplify Contagion Risk?

Abstract. In this paper[1] we develop a simple contract-theoretic model of Input-Output economies to address the theoretical nexus between trade-credit, bank credit and balance-sheet contagion. We make two contributions. First, we show that competitive markets in which heterogeneous price-taker firms compete strategically by setting trade-credit maturities have a unique symmetric equilibrium in trade-terms and the equilibrium dictates the production flow along the supply chains. Secondly, we find that the network can have a role either as shock absorber or shock amplifier and this is determined by a testable condition which holds for a general class of trade-credit networks. On these grounds, we argue that the proportional credit rationing used by banks (i.e., richer borrowers obtain larger loans) may have ambiguous effects on the systemic vulnerability. In fact, if for intermediate levels of bank-credit only a subset of firms substitute trade-credit in favor of bank-credit, the bank may worsen the quality of the inter-firm credit network, thus increasing the systemic vulnerability above the contagion threshold.

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2.1 Introduction

Trade-Credit, the exchange of services and goods on credit, is the most widely adopted short-term financing vehicle in inter-firm transactions[2]. The coexistence of trade-credit and traditional finance in developed economies is an empirical regularity that has received growing attention in the literature. This paper contributes to this literature by analyzing the implications of the co-existence of trade-credit networks and bank lending and suggests a novel testable hypothesis related to the propagation of shocks along supply chains.

Our contribution proceeds in three steps. We first develop a model that motivates the existence of trade credit as a strategic dimension for intra-sectoral competition in supply chains populated by firms characterized by heterogeneous access to cheaper sources of credit (i.e. bank credit). Firms in our model differ along multiple dimensions: the production stage they belong to, the mixture of own cash, trade-credit and bank credit (i.e. the financing mix) they adopt in order to carry out production and (critical for our strategic setting), the identity of trading counterparts. By interpreting this economy from the lens of a probabilistic network, we circumvent the intricacies related with the latter dimension and establish a neat relation between firms' wealth and their equilibrium financing mix, and between the latter and the network structure of the resulting inter-firm credit architecture. Third, by looking at the interaction between bank's rationing policy and firms' financing mix we offer a new testable hypothesis related to the resilience of a trade-credit architecture to localized firm bankruptcies. We show that the proportional loan rationing typically adopted by banks (i.e. to lend more to richer firms) under certain conditions can induce segmentation of borrowers which may have detrimental effects on the systemic robustness of the inter-firm credit architecture.

A starting point of the literature is to try to understand why trade-credit exists. In a seminal contribution, Biais and Gollier [10] hypothesized a *financing motive*[3]. When banks ration credit, suppliers with better access to cheap credit are willing to *lend* working capital to liquidity constrained buyers against a premium charged over the market price, thus effectively complementing the bank in credit provision. Building upon these premises, Burkart and Ellingsen [15] explain the coexistence of bank and trade finance in competitive markets by attributing sellers a monitoring advantage over banks. While buyers can easily pocket cash loans issued by banks, the less fungible nature of inputs prevents diversion of trade-credit. Trade-credit existence is sustained

[2]Seifert, Seifert and Protopappa-Sieke [65] bring extensive evidence on the magnitude of trade-credit in non-financial markets. In U.K and U.S, more than 80% of business-to-business transactions are run with some form of trade-credit arrangement and internationally, these levels are generally higher.

[3]Seifert, Seifert and Protopappa-Sieke [65], show that explanations for co-existence of trade-credit and efficient financial markets fall within two broad families of motives - the demand-side and the supply-side motives - which focus on profitability and control of trade-credit from the standpoint of either buyers or suppliers, respectively. In this paper we entirely focus on the latter and refer the reader to [65] for an in-depth review of demand-side motives.

by a double-coincidence of wants given by the monitor advantage and price premia on the supply-side and liquidity constraints on the demand-side. However, a growing body of papers looking at both sides of buyer-seller contracts and term structure along supply chains curtails the finance motive by showing that also credit-constrained firms offer trade-credit to their customers, and that the size of the customer positively relate with the credit extension. Such finding may imply that trade-credit might be a dimension of intra-sectoral competition and that differences in trade-terms may be reflective of asymmetries in firms' *market power* (Klapper, Laeven and Rajan, [46]). Giannetti, Burkart and Ellingsen [30] reconcile the motives by showing that firms with better access to finance have higher receivables to sales, yet suppliers are more generous in trade-terms to customers that have more bargaining power. Fabbri and Klapper [24] bring evidence that trade-credit is a constituent element of supply-chains because it is used by sellers who face strong competition as a competitive gesture for locking in customers in lieu of price cuts[4].

In this paper we study the formation and the properties of supply chains in multi-sector economies by looking at firms' investment structure. Specifically, we contribute to the novel literature of trade-credit in supply chains by making two points. First, we prove that the working capital hypothesis together with the market power hypothesis discussed above provide sufficient incentives for the formation of supply chains. Supply chains may naturally emerge in contexts in which producers are financially constrained and compete in the non-monetary dimension of trade-terms. We make this point by nesting the bargaining hypothesis in a model of trade-credit in competitive economies which mimics [5] the general framework proposed by Burkart and Ellingsen [15], with two major twists. First, we assume that producers in each sector *compete* for selling their output to firms belonging to the sector ahead. Secondly, as in Burkart and Ellingsen, we maintain that payment is ex-post production and that buyers can divert (part of) the output. Yet, we assume that both banks and sellers monitor buyers. Our innovation with respect to the extant literature is that we assume that sellers' monitoring technology is costly, but costs decrease *in the expected buyer's output*. Our motivation is that, for instance, in the absence of formal credit-assessment tools, monitoring a buyer with large exposures in account receivables may be cheaper because of the relatively high cost of any opportunistic behavior on their part, or that sellers may trust more larger counter-parts.

The core of our supply chain formation mechanism is a simple symmetric equilibrium in trade-terms (Theorem 2.3), which identifies each buyer's *strategic* supply of input given

[4] Van Horen [68] brings further evidence of trade-credit as a lock-in device. Klapper, Laeven and Rajan [46], note that whenever price discrimination is prohibited (e.g. the Clay Act in the U.S.), the market power hypothesis may explain why suppliers borrow at high cost in order to provide cheap financing.

[5] Some of the intersections are that buyers are heterogeneous in own endowment, both banks and lenders ration credit by means of overdrafts, sellers have a monitor advantage over banks, bank and trade-credit are perfect substitute.

her own cash endowment, some sector-wide parameter such as the cost of bank (and trade) credit, intra-sectoral competition and the bank lending policy. Because sectors are interlocked, the aggregate demand of input of a sector induces the production incentives for the sector above.

From the perspective of the inter-sectoral flows, our interlocked economy echoes a single productive cycle of Battiston et al. [8]. Sectoral demands are satisfied sequentially. Payments are entirely on credit and are made only after a production cycle ends. The main advantage of our result is that it allows us to map individual producers' decisions directly on the "topological" properties of the feasible *class* of production architectures, with no need of pre-imposing *ad hoc* input-output structures. In this regards, our analysis complements input-output analysis such as the one of Battiston et al. [8] and Henriet, Hallegatte and Tabourier [35] by providing a clear intuition between economic micro-incentives, macro-quantities and the input-output structure.

This brings us to the second contribution of this paper (Theorem 2.12), that is an intuitive condition which allows us to assess the resilience to balance-sheet contagion for general classes of production architectures. The simple trigger structure adopted in our paper (i.e. one firm going bankrupt) and the ensuing contagion mechanics (i.e. bankruptcy chains) has two advantages. First, by abstracting from market-based mechanism, our trigger is consistent with the one posed in recent natural experiments focused on the effect of exogenous shocks on the production networks (see [17] and [7] in next section) and it matches the type of direct propagation isolated in empirical studies on credit-chain contagion (see below [42] and [13]). Secondly, it allows us to make an inference on resilience *ex-ante* trade-credit structures realize: for all the production structures belonging to a given class, resilience is metricized as a statistical condition which disentangles the probability of contagion in the interaction between two components. These are the probability that a firm defaults upon local bankruptcy (i.e. a shock affecting one direct counter-part) and the *expected* degree of inter-connectedness of firms populating the structure. We find that the relation between interconnectedness and individual risk is non-trivial as it depends precisely on the investment mix and the lending regime respectively selected by the producers and the bank.

In general terms, our contract-theoretic framework reasons well with the empirical consensus cited in the next Section: balance-sheet contagion is subject to the degree of heterogeneity of firms' financing capabilities, the distribution of "deep pocket" firms with respect to constrained firms and the intensity of inter-firm connections. Additionally, it lends us a novel testable hypothesis related to the interaction between bank credit and trade credit. We find that when the bank adopts a proportional lending regime (rich firms obtain more credit than poor firms), the response of firms' budget composition to changes in bank overdraft is non-linear in firms' wealth. Overall, this effect may be found to actually *amplify* the systemic risk for positive yet moderate levels of bank lending intensity. The intuition is that when a bank is introduced in the trade-credit architecture, only a subset of (rich) firms will fully-substitute trade credit, whereas the remaining pool of (poor) firms will leverage trade-credit supply by *adding up* trade credit and bank credit. As a result of the partial substitution, the trade-credit network will be unambiguously less inter-connected - in line with empirical findings discussed in the next section (see [62]) - yet if the bank policy fails to *insulate* the poor firms, a possibly greater proportion of chains will be of lower quality as stemming between poorer and more leveraged firms.

The rest of the paper is organized as follows. In Section 2.3 we present the various elements of the model and establish the incentive structure which drives the formation of the trade-credit architecture. In Section 2.7 we present our contagion threshold and use it for isolating the channel my means of which bank lending may engender paradoxical effects over systemic resilience. We close with Section 2.8 where we validate our results via simulations. The Appendix contains an overview of our application of random network theory to the study of contagion over supply chains and collects the proofs of main results.

2.2 Relation to Extant Results in Inter-Firm Contagion Literature

As trade-credit is key in shaping supply chains, which are in turn the building bricks of developed economies, it is natural to ask whether localized idiosyncratic shocks can migrate along credit-chains and cluster in substantial macroeconomic impact.

In this perspective, firms' financing behavior - and specifically the relation between trade credit and bank credit - is directly projected on aggregate production dynamics and systemic risk. Following Kiyotaki and Moore [45], several authors provided theoretical and empirical evidence that idiosyncratic stress triggered by localized defaults may funnel through trade-credit leading to system-wide failure. In an early theoretical contribution, Boissay [12] links trade-credit to contagion by showing that significant domino effects may build up along individual supply chains where a fraction of firms rely on receivables to reimburse payables. More recent contributions have tried to single out the role in contagion of the network concealed within supply chains. Battiston et al. [8] explored the effect of network interaction among producers tied via trade-credit links in the context of a dynamic multi-agent economy. They find that idiosyncratic shocks uniformly distributed among producers can induce avalanche defaults if networked firms mis-coordinate production due to delayed payments and a slowly-adaptive cost structure. In their paper, higher inter-connectivity and risk sharing is always beneficial for stability. Using complex networks theory, Henriet, Hallegatte and Tabourier [35] construct a random multi-sectoral economy by which they isolate the contribution of the network topology on systemic risk. They confirm that redundancy of connections acts as a risk-sharing device, and bring novel evidence on how clustering and initial losses concentration affect global resilience. Interestingly, by varying the degree of concentration of a fixed amount of shock, they find that the total loss of the economy is maximal when concentrated in a single firm.

Complementary to the balance-sheet credit contagion approach described above, critical developments in general equilibrium analysis have recently built upon the seminal work of Long and Plosser [51] for inspecting the micro-dependencies of aggregate fluctuations. The contributions of Gabaix [26] and Acemoglu et al. [2] in particular have reinvigorated the idea that unit-specific shocks can percolate through the wider economy conditional on the network properties of the sectoral structure. Advancing the granularity of such approach, Magerman et al. [53] address the link between domestic inter-firm trade and aggregate volatility, while Carvalho et al. [17] used the Great East Japan Earthquake shock in order to disentangle the supply-chain disruption effect on the Japanese production system. From a similar perspective, Barrot and Sauvagnat (2016), investigated the effects of localized idiosyncratic shocks (specifically, natural disasters) on the US production architecture, showing that the drop in firms' sales caused by supply disruptions translates into value losses which propagate across the supply chain. Contemporary work has addressed the critical effect of liquidity shocks over production networks. Bigio and La'O [11] extended Acemoglu et al. model in order to study the amplifying effect of financial frictions on the aggregate economy. Furthering the characterization of firms' financial mix, Altinoglu [5] and Luo [52] endogenize a trade credit architecture within the input-output production network, proving that trade-credit strengthens the correlation of sectoral output. By introducing banks in the (trade-credit) economy, Luo shows that liquidity shocks are asymmetric in diffusing down-stream (via demand) or up-stream (via input prices) and draw policy conclusions on optimal sectoral liquidity targeting by central bank. We also address the impact of exogenous shocks on trade-credit networks in which firms adopt heterogeneous financing, yet we diverge from the literature above in two regards. First, in our work the trade-credit network architecture is endogenous. Secondly, we operationalize our contract-theoretic framework toward a complementary problem: the nexus between firms' heterogeneous investment mix and balance-sheet contagion.

In recent years, the theory of balance-sheet contagion has been validated by a wealth of empirical studies. By looking at the inter-industry flows, Raddatz [62] explicitly isolated a link between the use of trade credit across sectors and the industries' output correlation. Critical for our theory, he finds that an increase in the use of bank credit relative to trade credit reduces supply-chains co-movements. More recently, Boissay and Gropp [13] found that trade-credit default chains are factual across (small) credit rationed firms, even though their impact can be moderated by *deep pocket* firms (firms that are larger and have better access to bank loans). Jacobson and Schedvin [42] bring extensive evidence that sizable portions of corporate bankruptcies can be generated by idiosyncratic shocks which diffuse along trade-credit chains and that such channel parallels the traditional demand-side amplification mechanism considered in the macroeconomic literature initiated by Long and Plosser [51]. Moreover, they validate the idea that firms react differently to their customers' default, with liquidity constrained firms being remarkably more likely to go bankrupt upon customers' failure than larger, richer firms.

2.3 The Model

2.3.1 The Supply-Chain Economy

We consider a competitive economy made of one bank and $N = |\mathcal{N}|$ risk-neutral producers. Every firm $i \in N$ is randomly assigned a *project* R^i, that transforms a generic input I into a generic output S according to a specified production stage $t_i \in \{3, 4, ..., q\}$ and technology $A_i \subset \mathbb{R}_+$. Production stages are characterized as elements of a time sequence, capturing the idea that production is pipelined along vertical value chains (e.g both cars and fridges are produced after motors, motors are made after screws and freon gas) pointed toward a retail market which serves a fixed quantity demand D of some composite consumption good. We refer to the input quantity that is put into the project as the investment, and denote the investment of firm i by I_i. The investment is funded by means of two complementary channels, an anonymous *spot market*, and via *trade-credit* by purchase of input k_i from other firms involved in the value chain. Payments on the spot market are only accepted in cash, which is available by means of internal endowments $\omega_i \in \Omega \equiv \{0, ..., \bar{\omega}\}$, which we assume to be distributed in $\mathbb{P}_\omega$ and, if possible, through a bank overdraft $L_i \leq \bar{L}_i$. We hold that the bank conditions the overdraft $\bar{L}_i$ to the agent's internal cash ω. The other channel, trade-credit, allows firms to collect input k_i from their suppliers along the value chain after the simple promise of payment at a later stage. We refer to k_i as i's *account payables*. We call r the interest rate paid to the bank, p^s the costs for input on the spot market and by $p^{tc} \equiv p + p^s$ the price quoted by firms conceding capital k_i against trade credit, with p being the trade credit premium. In line with the literature, we assume[6] that $p^s = 0$ and $(1 + r) < p$. Therefore, given the investment I_i it holds that:

$$I_i \leq \omega_i + \bar{L}_i + k_i$$

Firms pay inputs an amount $C(I_i)$ given by:

$$C(I_i) = p \cdot k_i + (1 + r) \cdot L_i$$

[6] Alternative to the spot market, one can think of sellers offering "prompt-payment discounts" [66] (e.g. $2\% - 10$, net 30 contracts in which payment can be delayed up to thirty days after purchase, yet a rebate of 2% on purchase price is offered if payment is received before the tenth day.)

And any firm $i \in N$ produces:

$$S_i = A_i \cdot I_i$$

For simplicity[7], we assume that our is an exchange economy [8], [35] and that $A_i = A_j = 1 \ \forall i, j \in N$. The project is sold to firms belonging to the next production stage and generates an expected return of $p \cdot \mathbb{E}[s_i]$, where $s_i \leq S_i$ are *realized* sales of firm i (i.e. i's *account receivables*). We impose the following information structure:

Information structure:

1. The projects distribution and $\mathbb{P}_\omega$ are publicly known among producers and the bank.
2. Firms' endowment is private information among sellers, yet it is screened for free by the bank before issuing the loan (if any).
3. The demand of consumption goods D is common information among producers and the bank.

The production flow is characterized by means of the following regularities:

1. *Inventories are costly.* Market flow goes only one step ahead in each production stage, coherently with the popular just-in-time inventory paradigm enforced in supply-chain management (e.g. screws are sold to wheels producers, wheels are sold to car producers).
2. *Individual transactions are capped.* Firms may obtain at most one unit of input from each of the compatible producers.
3. *Producers in the bottom stage q sell to retailers the good and invest their endowment in producers in stage 1*

Formally, by defining the sub-set of firms N^m such as:

$$N^m \equiv \{i \in N : t_i = m\} \qquad for \ m = 3, 4, ..., q$$

Sectoral output (i.e. supply) of stage m is given by:

$$\bar{S}^m = \sum_{j \in N^m} S_j$$

In the above assumptions we required that inter-market exchanges are such that:

$$k_{ij} : \begin{cases} k_{ij} \in \{0, 1\}, & \forall i \in N^{m+1}, \ j \in N^m \\ k_{ij} = 0 & \text{otherwise} \end{cases} \tag{2.1}$$

[7] The assumption is made to preserve symmetry in inter-sectoral flows.

Whereby a non-negative k_{ij} implies a *link* from i to j is established[8]. Therefore, for every firm $j \in N^m$, *realized* individual sales s_j are given by:

$$s_j = \sum_{i \in N^{m+1}} k_{ij}$$

Lastly, we define $\mathbb{E}\left[\bar{S}^{m+1}\right]$ and $\mathbb{E}\left[\bar{K}^{m+1}\right]$ such as the expected aggregate supply and demand of input related to some stage $m+1$. On average, firms belonging to the stage ahead will source a quantity of capital $\bar{k}^{m+1}$ from sector N^m defined as follows:

$$\bar{k}^{m+1} = \mathbb{E}\left[\bar{K}^{m+1}\right] / N^{m+1} \tag{2.2}$$

Let us discuss the time-line and the remaining features of the model before stating the producers' problem in formal terms. The economy exists for $q+1$ fixed periods. In period $t = 1$, each firm $i \in N$ and the bank observe the firm's own endowment ω_i, and the bank sets the firms' credit allowance $\bar{L}_i$. The firm sees her allotted bank overdraft $\bar{L}_i$, the distribution of projects R and the configuration of production and exchange incentives (which we discuss below). Goods flow top-down. At the end of $t = 1$, firms anticipate the expected demand coming from the sector ahead $\mathbb{E}\left[\bar{K}^{m+1}\right]$, on the ground of which every firm i estimates her own expected sales $\mathbb{E}[s_i]$. This is equivalent to say that the distribution of admissible supply-chains $\mathcal{G}(N)$ is characterized[9] . In period $t = 2$, each firm $i \in N$ borrows L_i from the bank and purchases k_i from producers belonging to the preceding stage. Therefore, at the end of $t = 2$ supply-chains are formed (i.e. one realization $\boldsymbol{g}$ is drawn from $\mathcal{G}(N)$). In periods $t = 3, 4, ...q$ production and actual input exchanges take place. For convenience, we call stage q "retail sector". Subsequently, inter-firm debts are cleared according to the endogenous term structure obtained at $t = 2$. Lastly, firms repay the bank in period $t = q + 1$. Supply chain formation takes place according to the following set of rules:

Behavioral assumptions:

1. *The bank is a credit monopolist and firms are price takers.* In line with [25], the bank sets overdrafts to maximize its expected (possibly positive) profits. The bank transacts with each buyer separately. The firms are price takers both in the input and the output market.
2. *Purchases are on credit and firms compete on credit terms.* Firms purchase (i.e. reserve) other firms' output in period 2 (i.e. before production is initiated by firms in N^3). Input payments are processed *after* retailers in N^q have carried out production and consumption market closes. Because firms' output is non-specific,

[8] The formal definition of k_{ij} is contained in Definition **??**.

[9] In Appendix 2.6 we provide the formal definition of $\mathcal{G}(N)$.

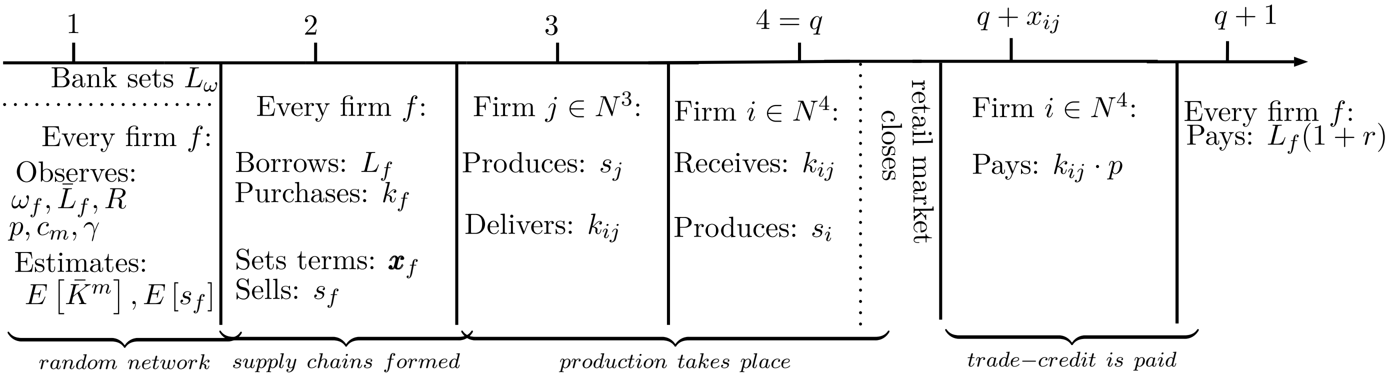

Figure 2.1: The Model Time-line for an economy characterized by only two production stages. Firm $j \in N^3$ sells on credit k_{ij} to firm $i \in N^4$ with maturity date at $q + x_{ij}$.

$j \in N^m$ competes with the other producers in N^m for selling S_j. Each seller j attracts buyers by posting a *term proposal* which dictates her customers' due date for purchase payment.

3. *Monitoring buyers is costly.* Along the lines of Jain ([43]), we assume that seller's direct oversight over purchases is costly, hence monitoring costs can be either fixed (i.e. zero with no loss of generality) or quadratic in sales (by a parameter c_m), depending on sales being below (above) a threshold $\bar{s}$. An innovation of our model, we assume that monitoring cost decreases in *buyers*' sales by a factor of γ. [10]

4. *Credit Collection is costly and Trade-Credit is capped.* In line with Burkart and Ellingsen [15] and Boissay [12], the seller may incur into an ex-post positive collection cost. In fact, the buyer i can divert a fraction $(1 - \phi) \in [0, 1]$ of $p \cdot k_i$. For this reason, purchases are rationed up to some level $k^0 \geq 0$ which we assume is exogenously set at the industry level.

5. *Retailers face a linear problem* Firms in N^q have no monitoring cost and as such they simply pass demand D upstream compatibly with k

In our framework supply-chains are equivalent to trade-credit links. Let us discuss in detail how trade credit determines the formation of the supply chains given some bank lending policy L. At the end of period 1, a generic firm $j \in N^m$ plans output volumes S_j in order to maximize profits on expected sales $\mathbb{E}[s_j]$. How do seller j estimate sales and, accordingly, set investment I_j? We begin by discussing the characterization of equilibrium expected sales focusing on a symmetric supply-chain economy populated by firms with homogeneous endowment and production incentives, which we call thereafter the benchmark supply chain. Subsequently, we study the effect on the production

[10] Overall, the cost specification has two advantages: it generates the widely observed degree of heterogeneity in suppliers' monitoring function (see Wilson and Summers([70])) and it allows us to analytically induce an equilibrium segmentation of producers' investment mix which converges to the one described by Burkart and Ellingsten [15].

structure when heterogeneity in the above dimensions is introduced. We postpone the determination of inputs demand to Section 2.4.1.

2.3.2 Formation of Supply Chains in Period 2

Following the approach in trade networks, as in Ovstrovsky [59] our discussion of market dynamics and supply chain formation will focus on the characterization of the production flow and inter-stage relations across "central" production stages $t = 3, 4, ..., q - 1$, thus abstracting from a characterization of the market interaction between retailers in stage $t = q$ and final consumers. From the behavioural assumptions above, firms belonging to the retail sector N^q have no monitoring cost, and simply pass the final demand to producers belonging to $q - 1$ compatibly with trade credit cap k^0 by solving a trivial linear problem[11]. Therefore, in the following discussion, we will focus on the behavior of firms belonging to stages $t = 3, 4, ..q - 1$.

According to the first assumption on supply chains formation in the previous section, trade-credit premium is fixed, and as such competition takes place in the dimension of payment dates. Therefore, let us define a *term proposal* such as the vector $\boldsymbol{x}_j \in \boldsymbol{X} \equiv [0, 1]^N$ such that $x = 0$ implies no credit beyond the clearing of retail market and $x = 1$ aligns the trade credit term to the bank's maturity date[12]. To introduce the matching protocol, we define the following sets. First, define ν_j^o the subset of potential buyers $i \in N^{m+1}$ which are contacted by j such as:

$$\nu_j^o = \left\{ i \in N^{m+1} : x_{ij} > 0 \right\}$$

From the above definition, the size of ν_j^o is positively related with the proposed deadline extensions. We refine this idea by imposing that for any $\nu_j^o(\boldsymbol{x}_j)$ and $\nu_j^o(\boldsymbol{x}'_j)$, $|\nu_j^o(\boldsymbol{x}'_j)| \geq \nu_j^o(\boldsymbol{x}_j)$ if $x'_{ij} \geq x_{ij}$ $\forall i \in N^{m+1}$ and strict inequality for at least one case. Our motivation is empirical and it is consistent with the bargaining power hypothesis discussed in the introduction: the size of account receivables correlates positively with length of credit deadlines. From the perspective of buyers, we may define the set of sellers with the best available term ν_i^l such as:

$$\nu_i^l = \left\{ j \in N^m : i \in \nu_j^o \; and \; x_{ij} = sup \left\{ x_{ij} \; \forall j \in N^m \right\} \right\} \qquad \forall j \in N^m$$

Therefore, for any given $j \in N^m$ and $i \in N^{m+1}$, supply chain formation takes place in Period 2 according to the following protocol:

[11] One may introduce granularity in this part of the production chain by assigning a label to each retailer and model final good demand with a system of linear equations stemming from the utility maximization of a representative agent with quadratic utility.

[12] Bank credit lines are typically renewed once a year, and renewal is regularly granted if the borrower has been able not to draw for at least 30 days during the preceding year. Trade credit, on the other hand, generally matures in 30 or 60 days [15].

Matching Protocol

1. Every seller $j \in N^m$ posts a term proposal $x_{ij} \in \boldsymbol{x}_j$ to a subset ν_j^o of potential buyers $i \in N^{m+1}$.
2. Each firm $i \in N^{m+1}$ *ranks* sellers in ν_i^l and picks the k_i sellers that are offering the best (i.e. the longest) credit terms.
3. If $|\nu_i^l| > k_i$, the potential buyer *randomly* picks k_i sellers from that set.

2.3.3 Determination of Equilibrium Expected Sales in Period 1

From the perspective of Period 1, each seller $j \in N^m$ computes the expected sales $\mathbb{E}[s_j]$ on the ground of a random demand which is a function of: (i) the expected input requirement stemming from N^{m+1}, (ii) the term proposal $\boldsymbol{x}_j$ offered by j, given the vector $\boldsymbol{x}_{-j}^m$ of proposal of all the other players in $N_{-j}^m \equiv N^m \setminus \{j\}$ and (iii) the aggregate supply $\bar{S}^m$ given the trade terms structure. The latter follows from the fact that under excess supply, the buyer's randomization mechanism defined in the matching protocol uniformly reduces the likelihood for any seller to be selected. Hence, each seller j constructs the set $\nu_j^b(x_j, \boldsymbol{x}_{-j}^m, \bar{S}^m)$ given by:

$$\nu_j^b(x_j, \boldsymbol{x}_{-j}^m, \bar{S}^m, \bar{k}^{m+1}) = \left\{ i \in \nu_i^o : j \in \nu_i^l \textit{ and } i \textit{ purchases } k_{ij} \right\}$$

Which coincides with j's expect buyers. The behavioral assumptions stated above are mapped by imposing that: $|\nu_j^b(x_j', \boldsymbol{x}_{-j}^m, \bar{S}^m)| \leq |\nu_j^b(x_j, \boldsymbol{x}_{-j}^m, \bar{S}^m)|$ for $x_j' < x_j$ and that: $|\nu_j^b(x_j, \boldsymbol{x}_{-j}^m, \bar{S}^{m\prime})| \leq |\nu_j^b(x_j, \boldsymbol{x}_{-j}^m, \bar{S}^m)|$ for $\bar{S}^m < \bar{S}^{m\prime}$. All this considered, given that in Period 1 market interaction has not taken place yet, and as such buyers' idiosyncrasies are unknown to any seller, each seller defines $\mathbb{P}(i \in \nu_j^o)$ and $\tilde{p}_{ij}$ such as:

$$\begin{aligned} \mathbb{P}(i \in \nu_j^o) &= \frac{\nu_j^o}{|N^{m+1}|} \\ \tilde{p}_{ij} \equiv P\left[i \textit{ purchases } k_{ij} | \bar{S}^m, \bar{k}^{m+1}, \boldsymbol{X} \right] &= \frac{|\nu_j^b|}{|N^{m+1}|} \end{aligned} \tag{2.3}$$

Hence, it's clear that j's expected sales $\mathbb{E}[s_j]$ in Period 1 are given by:

$$\mathbb{E}\left[s_j | \bar{S}^m, \bar{k}^{m+1}, \boldsymbol{X} \right] \equiv \sum_{i \in N^{m+1}} \tilde{p}_{ij} \tag{2.4}$$

Now, given the purchase cap in (2.1), we can restate $\tilde{p}_{ij}$ in terms of a simple binomial process which breaks down the probability of striking a deal in selling exactly one unit of capital out of the buyer's expected demand $\bar{k}^{m+1}$:

$$\begin{aligned}\tilde{p}_{ij} &= \binom{\bar{k}^{m+1}}{1} p_{ij}(x_j, \boldsymbol{x}^m_{-j}, \bar{S}^m)\left(1 - p_{ij}(x_j, \boldsymbol{x}^m_{-j}, \bar{S}^m)\right)^{\bar{k}^{m+1}-1} \\ &= \bar{k}^{m+1} \cdot p_{ij}(x_j, \boldsymbol{x}^m_{-j}, \bar{S}^m)\left(1 - p_{ij}(x_j, \boldsymbol{x}^m_{-j}, \bar{S}^m)\right)^{\bar{k}^{m+1}-1}\end{aligned} \tag{2.5}$$

Given our characterization of ν^b_j, we impose the following structure on p_{ij}. First, we require that $p_{ij} : [0,1] \times [0,1]^{N^m_{-j}} \times \mathbb{R}_+ \to [0,1]$ is continuous and strictly increasing in x_j. Second, we assume that $p_{ij}(0, \mathbf{1}, \bar{S}^m) = 0$ and $p_{ij}(1, \mathbf{0}, \bar{S}^m) = 1$ for $\bar{S}^m > 0$. Lastly, we claim that for two given expected aggregate supplied quantities $\bar{S}^m$ and S^m such that $\bar{S}^m \leq \bar{K}^{m+1} \leq S^m$, it holds that:

$$p_{ij}\left(\boldsymbol{X}, S^m\right) \leq p_{ij}\left(\boldsymbol{X}, \bar{S}^m\right) \qquad \forall i, j \in N \tag{2.6}$$

That is, $p_{ij}\left(\boldsymbol{X}, \bar{S}^m\right)$ first order stochastic dominates $p_{ij}\left(\boldsymbol{X}, S\right)$.

Now, let us introduce the implications stemming from the second regularity shaping the supply chain formation. Because sellers do operate on credit, monitoring buyers is as critical as production *per se*. In this sense, we assume that costs of monitoring buyers grows in the sales. Let us simply suppose that:

$$\mathbb{E}\left[monitoring_j | s_j\right] > 0 \qquad for\ s_j > \bar{s}$$

Where $\bar{s}$ is some threshold that captures the idea that for large volumes, the seller's direct oversight over purchases is inefficient. In absence of cheap access to credit assessment technologies which are peculiar to banks, the sellers must rely on other tools for assessing the viability and cope with monitoring burden of credit extension. We revisit the Burkart and Ellingsen's theory by refining the sellers' transaction-based information advantage. We claim that the relevant bit of information is given by the expected *architecture* of inter-firms relations, and more precisely, by a prediction on the customers' expected sales on credit. In this sense, strong empirical evidence [24], [65] suggests that in competitive input markets a seller's credit extension decision is conditioned also upon a non-monetary determinant - the buyer's *market power* [24] - which is strictly related to the structure of the market in which agents operate. The idea is that *liquidity*-constrained agents in competitive markets are likely to extend more trade-credit themselves exactly because of their limited market power. This implies that weaker sellers rely more heavily on matching trade terms (i.e. a feasible matching between payables and receivables maturities) and therefore are more exposed to negative variations in trade-credit provision from any of their creditors. Therefore, in lack of formal credit-assessment tools, monitoring a buyer characterized by large exposures in account receivables may be cheaper because of the relatively high cost of opportunistic behaviors [13]. It is then clear that such information requires the seller to have a notion of the *architecture* of the trade-credit relations in place when considering trade-credit extension. Let us consider the following specification:

$$\mathbb{E}\left[monitoring_j|s_j\right] = \left(\frac{1}{2}c_m \cdot \mathbb{E}\left[s_j\right]^2 - \gamma \cdot \sum_{i \in N^{m+1}} \mathbb{E}\left[s_i|k_{ij}=1\right]\right), \quad s_j > \bar{s} \tag{2.7}$$

In the above equation $\mathbb{E}\left[s_i|k_{ij}=1\right]$ indicates the expected sales of j's customers. The base monitoring cost (which we assume to be quadratic in s_j) incurred for monitoring j's buyers is discounted in j's customers' expected sales via γ, with $\gamma < c_m$. All that considered, in period 1 every firm $j \in N$ determines her optimal investment I_j and payment due date $\boldsymbol{x}_j$ by solving the following problem:

$$\begin{aligned}
\max_{I_j, \boldsymbol{x}_j} \; & \mathbb{E}\left[\Pi_j\right] = p \cdot \mathbb{E}\left[s_j(\bar{S}^m)|\, \boldsymbol{X}\right] - \mathbb{E}\left[monitoring_j|s_j\right] - L_j \cdot (1+r) - k_j \cdot p \\
s.t: \quad & S_j = A \cdot I_j \\
& I_j \leq \bar{L}_j + k^0 + \omega_j \\
& \Pi_j \geq 0
\end{aligned} \tag{2.8}$$

Together with the aggregate consistency condition:

$$\mathbb{E}\left[\bar{S}^m\right] = \sum_{j \in N^m} \mathbb{E}\left[S_j\right]$$

The key ingredient for the solution of (2.8) is the equilibrium structure of buyers' account receivables, which in turn depends on the trade decisions of all the agents in the economy. Apparently, in the present context every producer conditions her choices not only on the total quantities exchanged in the economy, but also by foreseeing *who* will trade with whom, as the architecture structure is critical in order to determine the monitoring costs. Evidently, the number of possible configurations stemming from this framework may potentially grow to intractable dimensions well before N reaches a size compatible with real industries. However, we may turn down a great deal of complications by claiming that in competitive markets, buyers' identity is of secondary importance for a seller. What counts is not whom exactly a potential counter-part is trading with, but the expected *size* of her positions (i.e. the number of counter-parts) within the architecture. For this reason, we exploit the high level of uncertainty to recast the determination of the trade-credit architecture in a simple N-players stochastic network formation game with complete anonymity in the vein of the approach firstly proposed by

[13] The opportunistic behaviors may encompass a variety of breaches such as the diversion of capital [15] or late payments.

Cabrales, Calvo-Armengol and Zenou [16] and re-modulated in a discrete-linking setting by Golub and Livne [34]. We proceed by steps. First, we contextualize our analysis for the benchmark case of symmetric supply chain. In such context, we solve the problem in Equation 2.8 for a symmetric equilibrium in trade-terms deadlines which we use to characterize equilibrium expected sales level. On the ground of the results obtained for the benchmark case, we explore the production structure of heterogeneous supply chains. Let us introduce a benchmark supply chain as follows.

Definition 2.1. *Consider the supply chain economy as introduced in Section 2.3.1. A benchmark supply-chain is such that within each production stage* N^m*, endowments are homogeneous:* $\omega_i = \omega_j = \omega_m \; \forall i, j \in N^m$*, for* $m = 3, 4, ..., q$*.*

We solve the game for the benchmark supply chain by means of the following symmetric equilibrium in pure strategies:

Definition 2.2. *A symmetric equilibrium in the benchmark supply chain is a vector of trade credit deadlines* $\boldsymbol{x}^*$ *such that:*

1. *Every firm* j *posts the same trade credit term structure* $x^*_{ij} = x^*, \forall j \in N^m, i \in N^{m+1}, m = 3, ...q$
2. *Every firm* j *endowed with cash* ω *invests* I^*_ω *and in expectation sells* $\mathbb{E}\left[s^*_\omega\right]$
3. *For each production stage* m*,* $\mathbb{E}\left[\bar{S}^m\right] \leq \mathbb{E}\left[\bar{K}^{m+1}\right]$

The notion of symmetric equilibrium is particularly compelling in our framework. Given that the network formation takes place in a competitive industry with no prior information about the possible counter-parties or a benchmark trade-credit architecture, the symmetric equilibrium epitomizes the lack of coordination between agents at the early stage of negotiation as well as the competitive pressure which characterizes the environment. Before presenting a general result on existence and characterization of the above equilibrium, we present an example for an intuitive setting.

Example 2.1 (**Symmetric equilibrium in simple benchmark supply-chain**). Consider an economy composed by two stages (retail and production), which we call for convenience N^r and N^p, populated by $n_r = 100$ retailers and $n_p = 5$ producers each such that $\omega_r = 1$ and $\omega_p = 2$. Suppose for simplicity that there is no bank, retail demand $D \to \infty$, trade-credit premium and monitoring cost are given by $p = 2$ and $c_m = 1$ respectively, there is no monitoring discount, such that $\gamma = 0$ and the industry credit cap is $k^0 = 1$. Given the lack of monitoring costs for retailers, each retailer $i \in N^r$ will induce a demand of input $k_i = \bar{k} = 1$. Coherently with the general structure we imposed on ν^o_j and ν^b_j , we characterize ν^o_j such that: $|\nu^o_j| = n_r \cdot x_j$ and ν^b_j as follows:

$$|\nu^b_j| \;=\; |\nu^o_j| \cdot \left(\frac{|\nu^o_j|}{\sum_{k=1}^{n_r} |\nu^o_k|} \right) \;=\; |\nu^o_j| \cdot \left(\frac{x_j}{\sum_{k=1}^{n_r} x_k} \right) \tag{2.9}$$

In Equation (2.9), the first component contains the number of retailers contacted by j, while the second component represents the share of j's proposals out of all proposals submitted by producers in N^p, and maps j's expectation of being selected by any of the contacted retailers given the term structure in $\boldsymbol{x}_{-j}$. In this example we assumed each producer takes a conservative stance on the likelihood of being selected (i.e. a "lower bound" on p_{ij}) as it abstracts from considering the size of N^r in computing the probability. In other words, it assumes that for each seller $k \in N^p$ and each buyer $i \in N^r$, $k \in \nu_i^l$, and that selection scales linearly with the offered term. Given that $k^0 = 1$, it follows from the definitions in the main text that:

$$\tilde{p}_{ij}(\boldsymbol{x}^p) = p_{ij}(\boldsymbol{x}^p) = \frac{|\nu_j^b(x_j, \boldsymbol{x}_{-j}^p)|}{n_r} = \frac{x_j^2}{\sum_{k=1}^{n_r} x_k} \tag{2.10}$$

Notice that the construction in (2.10) matches the requirements introduced in Section 2.3.3. In particular, $\tilde{p}_{ij}(0, \cdot) = 0$, $\tilde{p}_{ij}(1, \mathbf{0}) = 1$. It is also easy to check that FOSD applies. In particular, for the symmetric term structure $x_j = x_k = \bar{x}\ \forall k \in N^p$, it holds that $\tilde{p}_{ij}(\bar{x}) = \bar{x}/n_p$. Let us revisit the problem in (2.8). Given the (expected) symmetry of retailers and the other conditions specified above, each producer $j \in N^p$, faces the following problem:

$$\max_{I_j, \boldsymbol{x}_j} \mathbb{E}\left[\Pi_j\right] = p \cdot \left(\frac{n_r \cdot x_j^2}{\sum_{k=1}^{n_r} x_k} \right) - \frac{1}{2} c_m \cdot \left(\frac{n_r \cdot x_j^2}{\sum_{k=1}^{n_r} x_k} \right)^2$$

We suppressed the constraints as per the numerical parametrization we assumed in the exercise, we will show that we can safely focus on the case of unconstrained producers. By taking the first order conditions and imposing $x_j = \bar{x}\ \forall j \in N^p$, we have:

$$\frac{p}{c} \cdot \left[2n_r \bar{x} \cdot (\bar{x} n_p) - n_r \cdot (\bar{x})^2 \right] = \frac{2n_r^2 \bar{x}^3 (n_r \cdot \bar{x}) - n_p^2 \cdot (\bar{x})^4}{n_p \bar{x}}$$

In light of the result of Theorem 2.3, here it suffices to verify that the contract $\bar{x}^* = (p/c) \cdot (n_p/n_r)$ satisfies the FOC defined above. By substitution, we find that:

$$\left(\frac{p}{c}\right)^3 \left(\frac{n_p^2}{n_r}\right) \left(2n_p - \frac{1}{n_r}\right) = 2n_r^2 \cdot \left(\frac{p}{c}\right)^3 \cdot \left(\frac{n_p}{n_r}\right)^3 - \left(\frac{p}{c}\right)^3 \left(\frac{n_p^2}{n_r^3}\right) \cdot n_r^2$$

Which verifies that $\bar{x}^* = (p/c) \cdot (n_p/n_r) = 0.1$ is an extremum. By looking at second order conditions, we may confirm it is a maximum. We notice that the result weights the standard production outcome of competitive economies by the relative size of the two populations. More interestingly in terms of the resulting network, we find that the linkage probability is given by $p_{ij}(\bar{x}^*) = (p/c)/n_r = 2/100$, the set of contacted buyers is $|\nu_j^{o*}| = 10$ and, particularly, the optimal expected level of trade receivables is

$p_{ij}(\bar{x}^*) \cdot 100 = 2$, thus implying that each producer $j \in N^p$ adjusts x in order to sell $\mathbb{E}[s] = 2$ units of good. As anticipated above, $|\nu_j^{b*}| = 10$ is optimal (in Period 1) under the conservative expectations introduced above, since $\mathbb{E}[s] = |\nu_j^{o*}|/n_p = 2$. Symmetrically, one could work out an alternative scenario in which the effect of competition is assumed to be diluted in the size of the population N^r, for instance by interacting the second component of (2.9) with a parameter $\lambda \geq 0$, and consequently show that for λ large enough (i.e. $\lambda \to 2$), $|\nu_j^{b*}| \to |\nu_j^{o*}|$.

We now introduce our first main result, which gives the foundations of the trade-credit architectures. Firstly, we prove the uniqueness and existence of the symmetric equilibrium in credit extensions in the benchmark supply-chain. Subsequently, we characterize the equilibrium account receivables of any seller.

Theorem 2.3. *Given a benchmark supply-chain defined as above:*

(i) For any production stage $m \in \{3, 4, ..., q\}$ with $j \in N^m$ and expected trade-credit demand $\bar{k}^{m+1}$, there exists at most a unique symmetric equilibrium in trade terms $x^* \in [0, 1]$ which solves the problem defined in (2.8).

(ii) For $N \to \infty$, the equilibrium matching probability $\tilde{p}^*(x^*)$ of any seller $j \in N^m$, fades to 0 and trade credit terms are such that $x^* \in (0, 1)$

(iii) For every producer $j \in N^m$ endowed with $\omega \in \Omega$ and characterized by the investment mix $I_j = I_\omega$, the equilibrium expected account-receivables converges to a finite quantity $\mathbb{E}\left[s^*_{m,\omega}\right]$ defined as follows:

$$\mathbb{E}\left[s^*_{m,\omega}\right] = min\left\{S^*_\omega, \bar{S}^*_{m,\omega}\right\} \tag{2.11}$$

Where:

$$S^*_\omega = \begin{cases} min\left\{\dfrac{p}{c_m - \gamma}\ ,\ k^0 + \bar{L}_\omega + \omega\right\} & for\ \ \bar{k}^{m+1} \geq 1 \\ 0 & otherwise \end{cases}$$

And:

$$\bar{S}^*_{m,\omega} = \left(\frac{\bar{K}^{m+1}}{n \cdot S^*_{m,\omega}}\right) \cdot S^*_{m,\omega}$$

Proof of Theorem 2.3. Depending on her initial endowments, a firm must set-up her investment plan and terms offer conditional on other sellers' terms and supply. Two orders of problems are in place: first, as in the case of Cournot oligopoly with undifferentiated products [21], each producer's payoff depends on the *aggregate* production of the agents belonging to the same stage. From (2.6), we know that the probability of individual matches depends on the expected availability of excess supply. Secondly, the heterogeneity of producers' endowment will be reflected in their investment decisions.

Our strategy is to disentangle the problem by breaking it down in two respects. For the moment, we postpone the issues related with the possibility of aggregate excess supply and we begin by focusing on the problem of the unconstrained sellers, that is sellers that need no external funding. We obtain unconstrained sellers' equilibrium expected sales, thus isolating the "unconstrained" incentives that drive the formation of the trade-credit architecture. Subsequently, we tackle the constrained sellers' problem. Eventually, we construct the aggregate supply of input and devise a solution to the possibility of excess supply which is consistent with our notion of symmetric equilibrium. Therefore, we temporarily suppress the term $\bar{S}^m$ from $\mathbb{E}\left[s_j|\ \bar{S}^m, \boldsymbol{X}\right]$ and the problem for unconstrained sellers becomes:

$$\begin{aligned} &\max_{I_j,x_j}\ \mathbb{E}_{T^1}\left[\Pi_j\right] = p\cdot\mathbb{E}\left[s_j|\ \boldsymbol{X}\right] - \mathbb{E}\left[monitoring_j|s_j\right] \\ &s.t: \qquad \mathbb{E}\left[S_j\right] = A\cdot I_j \\ &\qquad\qquad\quad \Pi_j \geq 0 \end{aligned} \tag{2.12}$$

(i) For the sake of the explanation, we consider an industry made of three production stages with agents $j \in N^m$, $i \in N^{m+1}$ and $l \in N^{m+2}$. For simplicity, we assume that $N^{m+1} = N^{m+2} = n$, so that also $\bar{k}^{m+1} = \bar{k}^{m+2} = \bar{k}$. Preliminary, let us write in explicit form (2.12) such as:

$$\mathbb{E}\left[\Pi_j\right] = p\cdot\sum_i^n \tilde{p}_{ij} - \left(\frac{1}{2}\left(c_m\cdot\sum_i^n\tilde{p}_{ij}\right)^2 - \gamma\cdot\sum_i^n\sum_l^n\tilde{p}_{ij}\tilde{p}_{li}\right) \tag{2.13}$$

In which we imposed that the events in $\mathbb{E}\left[s_i|k_{jl}=1\right]$ are independent. In fact, in equilibrium, every buyer obtains her allotted equilibrium input regardless to the identity of the seller in order to produce $S_i = \mathbb{E}\left[s_i^*\right]$. Let us focus on agents in stage N^m. First, because we are looking for a symmetric equilibrium in credit terms extensions, every agent $j \in N^m$ posts the same maturity proposal to every potential customer $x_{ij} = \boldsymbol{x}_j = x_j$ and $p_{ij}(x_j, x_{-j})$ becomes $p_j(x_j, \bar{x})$ $\forall i \in N^{m+1}$. Therefore, we can rewrite the equation above such as:

$$\begin{aligned} n\bar{k}p_j(x_j,\bar{x})\left(1-p_j(x_j,\bar{x})\right)^{\bar{k}-1}\cdot\Bigg(p - \bigg(\frac{1}{2}\cdot c_m\cdot n\bar{k}p_j(x_j,\bar{x})\left(1-p_j(x_j,\bar{x})\right)^{\bar{k}-1} - \\ \gamma\cdot n\bar{k}p_i(x_i,\bar{x})\left(1-p_i(x_i,\bar{x})\right)^{\bar{k}-1}\bigg)\Bigg) \end{aligned} \tag{2.14}$$

Second, we notice that in (2.14) x_j appears only as argument of $p(x_j, \bar{x})$; because p is strictly increasing in the argument, we know that the optimal maturity x_j^* induces a unique matching probability $p(x_j, \bar{x}) = p(x_j) = p_j$. Hence, we can maximize (2.14) with respect to p_j and then obtain by bijection from p_j^* the optimal extension x_j^*. Hence, we rephrase the problem in the following:

$$\frac{d}{dp_j}\left[n\bar{k}p_j(x_j,\bar{x})\left(1-p_j(x_j,\bar{x})\right)^{\bar{k}-1}\cdot\left(p-\left(\frac{1}{2}\cdot c_m\cdot n\bar{k}p_j(x_j,\bar{x})\left(1-p_j(x_j,\bar{x})\right)^{\bar{k}-1}-\right.\right.\right.$$
$$\left.\left.\left.\gamma\cdot n\bar{k}p_i(x_i,\bar{x})\left(1-p_i(x_i,\bar{x})\right)^{\bar{k}-1}\right)\right)\right]=0$$

The differentiation leads us to:

$$\frac{1}{1-p_i}\cdot\left[\bar{k}n(1-p_j)^{\bar{k}-1}(\bar{k}p_j-1)\left(p(1-p_j)-c_m\bar{k}np_j(1-p_j)^{\bar{k}}-p_i(\bar{k}(1-p)-\right.\right.$$
$$\left.\left.\gamma\bar{k}n(1-p)\cdot(1-p_i)^{\bar{k}}+c_m\bar{k}np_j(1-p_j)^{\bar{k}}\right)=0$$

We now adopt symmetry in maturities $p_j=p_i=\bar{p}$ and obtain the condition:

$$n\bar{k}(1-\bar{p})^{\bar{k}-1}(\bar{k}\bar{p}-1)\left[c_mn\bar{k}(1-\bar{p})^{\bar{k}}\bar{p}-\gamma n\bar{k}(1-\bar{p})^{\bar{k}}\bar{p}-(1-\bar{p})p\right]\;=\;0$$

We recall that the probability of striking an agreement out of $\bar{k}$ loans is simply given by:

$$\tilde{p}=\bar{k}\cdot\bar{p}$$

The above equation becomes:

$$n\bar{k}\left(1-\frac{\tilde{p}}{\bar{k}}\right)^{\bar{k}}(1-\tilde{p})\left[\tilde{p}n\left(c_m-\gamma\right)\left(1-\frac{\tilde{p}}{\bar{k}}\right)^{\bar{k}-1}-p\right]\;=\;0 \tag{2.15}$$

Now, let us show that the above equation admits at most one internal solution in the compact $[0,1]$ for $\bar{k}>1$. First, from (2.15) it is easy to see that a trivial extremum $\tilde{p}^0=1$ always exists. We rewrite the terms in square parenthesis such as:

$$\frac{p}{(c_m-\gamma)\left(1-\frac{\tilde{p}}{\bar{k}}\right)^{\bar{k}-1}}=\tilde{p}n \tag{2.16}$$

We notice that the LhS and RhS are continuous and monotonic in the compact space $[0,1]$. Given some threshold $0<t<\infty$, depending on n being above or below t, one of the two following cases may arise:

1. Suppose $n<t$. Both for $\tilde{p}=0$ and $\tilde{p}=1$ the LhS dominates the RhS and no solution besides $\tilde{p}^0$ obtained above exists.

2. Suppose $n > t$. For $\tilde{p} = 0$ ($\tilde{p} = 1$) the RhS (LhS) dominates the LhS (RhS). Hence, the RhS and the LhS cross only once in the sub-interval $\tilde{p} \in (0,1)$, thus identifying the non-trivial extremum $\tilde{p}^* \in (0,1)$.

We now show the behaviour of the extrema in the context of our problem. By differentiating the LhS of (2.15), we obtain that:

$$\frac{n\bar{k}\left(1-\frac{\tilde{p}}{\bar{k}}\right)^{\bar{k}}}{\left(\bar{k}-\tilde{p}\right)^2}\left[n\bar{k}\left(1-\frac{\tilde{p}}{\bar{k}}\right)^{\bar{k}}(c_m-\gamma)-p\cdot\left(\tilde{p}\bar{k}(3-\tilde{p})-\tilde{p}^2+\bar{k}^2(p-2)\right)\right] \tag{2.17}$$

Now, we plug $\tilde{p}^0 = 1$ inside (2.17):

$$n\left(\frac{\bar{k}-1}{\bar{k}}\right)^{\bar{k}-1}\left(n\bar{k}\left(\frac{k-1}{\bar{k}}\right)^{\bar{k}}(c_m-\gamma)-p(k-1)\right)$$

The equation sign depends on the size of n. For n being sufficiently large, the second-order condition is positive and therefore $\tilde{p}^0$ is a local minimum. In order to inspect $\tilde{p}^* \in (0,1)$ we now turn to the case $p = 0$. The second-order condition is simply given by:

$$-n\bar{k}\left(n\left(c_m-\gamma\right)+2\cdot p\right) \tag{2.18}$$

Which is always negative for $n > 0$ given our assumption that $c_m > \gamma$. Therefore, the profit function is growing at $\tilde{p} = 0$. Recall that (2.17) is continuous in $[0,1]$. Hence, two mutually exclusive cases are possible:

1. For $n < t$, the profit function is concave in $\tilde{p}$ and the extremum $\tilde{p}^0 = 1$ is the unique maximum, a very intuitive result.

2. For $n > t$, $\tilde{p}^0$ is a local minimum. Given that the function is continuous and growing at $\tilde{p} = 0$, it must be the case that there exists a maximum in the interval $(0,1)$. Consequently, $\tilde{p}^*$ is qualified as the unique maximum in the interval.

Therefore, for $n > t$ ($n < t$), there exists a unique trade-credit term extension $\bar{x} = x^*(\bar{x} = x^0)$ such that $\tilde{p}(x^*)$ ($\tilde{p}(x^0)$) maximizes the producer's profit.

(ii) The statement follows from the observation of the first order conditions stated in (2.15). First, for $n \to \infty$, we know that $\tilde{p}^*$ is the unique maximizer of (2.15). Now, we prove by contradiction that $\tilde{p}^*(n) \to 0$ for $n \to \infty$. Suppose instead that along some

subsequence $\{n_s\}$, $\tilde{p}^*(n_s)$ converges to some limit $\bar{p}(n_s)$ and consider the equality in (2.16). For $n \to \infty$, while the LhS will converge to a fixed quantity $g(\bar{p}(n_s))$, the RhS will grow boundless, thus violating the FOC in (2.17).

(iii) We want to assess the conditions for which $\tilde{p}^*(n)n$ converges to a finite limit and characterize the equilibrium account receivables. Notice that in order for the limit to be converging, it must be that $p(n) \to 0$ fast enough. By keeping the assumption that $\bar{k} > 1$ and given the result in (i) of the current theorem, we can safely focus on the terms in square parentheses from (2.15). We elaborate two possible cases: (a) $\tilde{p}^*n \to 0$, and (b) $\tilde{p}^*n \to Q$, where Q is some finite quantity.

(a) If $\tilde{p}^*n$ converges to a finite quantity, let us rewrite (2.16) such as:

$$\frac{p}{(c_m - \gamma)\,(1)^{\bar{k}-1}} = Q$$

However, for $n \to \infty$, the probability $\tilde{p}^*$ of striking n independent deals also approximates the expected number of deals, therefore,

$$Q = \frac{p}{(c_m - \gamma)} = \mathbb{E}\,[s^*]$$

(b) If $\tilde{p}^*(n) \to 0$ fast enough,the RhS $\to 0$, thus implying that in expectations firms fail to sell any positive level of input. This might be a possibility only when the aggregate consistency condition $\bar{S}^m < \bar{K}^{m+1}$ is not met, a case which we will discuss extensively at the end of the proof.

Only the result in (a) is compatible in general terms with our incentive structure. Therefore, for $n \to \infty$, each unconstrained seller obtains an expected amount of trade-credit receivables equal to $\mathbb{E}\,[s^*]$. Now, we get back to the master problem in (2.8) and consider the constrained sellers. Let us simplify it by assuming that only agents endowed with $\omega > \omega^*$ incur in monitoring costs, where ω^* is defined such as:

$$\omega^* = \frac{p}{c_m - \gamma}$$

Hence, only firms with selling volumes potentially larger than the optimal one (as obtained from steps $(i) - (iii)$) incur in monitoring costs. For the others it must be that, compatibly with their financing constraint, they will either produce $S^* = p/(c_m - \gamma)$ or produce $S^*_\omega = \bar{L}_\omega + \omega + k^0$. This latter statement follows from the linearity of the production function, the linearity of the profit function with respect to the bank credit and the cost structure specification $(1 + r) < p$.

We now prove that the above equilibrium exists . Let us consider the effect of S^*_ω on the determination of the expected sales $\mathbb{E}\,[s_i]$. Clearly, if the aggregate supply constraint binds, $\sum_{j \in N^m} S^*_j > \bar{K}^{m+1}$. In light of the condition stated in (2.6), the probability to be selected $\tilde{p}(\bar{S}^m) \cdot n$ converges to a non-zero quantity only for $\bar{S}^m \leq \bar{K}^{m+1}$. Therefore, it must be that when the aggregate constraint binds, $\tilde{p}(\sum_{j \in N^m} S^*_j)n \to 0$. As a consequence of that, under a sufficiently large excess supply, no firm expects to be selected by buyers, so that every firm endowed with ω would expect to obtain a non-positive profit given by $\Pi_\omega = -C_\omega \leq 0$. We ask what rescale $\beta_j \in [0,1]$ - if any - would the seller be willing to accept *ex-ante* production realizes, given the expected rescales of other producers. Under symmetry, producers rescale their target by a fixed rate $\beta^*_i = \beta^*_j = \beta^* \; \forall i, j \in N^m$. It is clear that the only admissible rescale is such that:

$$\beta^*_m = \frac{\bar{K}^{m+1}}{n \cdot S^*_\omega}$$

In fact, for any of the sellers $i \in N^m$, only at rate $\beta = \beta^*_m$ there is no room for a profitable scale-up $\beta_i > \beta^*_m$. On the other hand, no seller is willing to scale below β^*_m since the expected profit is increasing for $S_i \leq S^*_\omega$. Therefore, each producer $j \in N^m$ endowed with ω, produces and sells in expectations a quantity $\bar{S}^*_\omega$ defined as:

$$\bar{S}^*_{m,\omega} = \beta^*_m \cdot S^*_\omega$$

■

The above result holds for every $\bar{k} \geq 0$ and linkage probability p_{ij} compatible with the conditions stated at the beginning of Section 2.3.3. We now validate the result by considering again the particular supply-chain economy described in Example 2.1.

Example 2.2 (**Symmetric equilibrium in the benchmark (continued)**)**.** Let us map the simple benchmark economy discussed in Example 2.1 into the general framework of Theorem 2.3. We plug the relevant parameters in the general FOC derived in Equation (2.16) with respect to p_{ij}, and obtain that:

$$\frac{p}{(c_m)\left(1 - \frac{\tilde{p}}{\bar{1}}\right)^0} = \tilde{p} n_r \;\Rightarrow\; \tilde{p}(\bar{x}^*) = \frac{p}{c_m}\frac{1}{n_r}$$

Which corresponds to the result we got by deriving the FOC for x_j and solving for the symmetric deadline $\bar{x}$ in the specific case of the exercise. in this case the optimal linkage probability $\tilde{p}(\bar{x}^*)$ is obtained first. We can then equate $(p/c_m)(1/n_r)$ with the selected functional form p_{ij} adopted in the exercise and rely on the one-to-one mapping property in order to identify $\bar{x}^*$. Notice that all the asymptotic properties (with respect to $n_r \to \infty$) discussed in the Theorem statement hold in the specific benchmark as well.

The intuition behind Theorem 2.3 is that despite the uncertainty governing the economy at time Period 1, competition in the dimension of trade-terms will bring the system in one of two distinct states. Essentially, either supply chains do not form, or firms succeed in originating positive $\bar{K}$ and $\bar{S}$, thus inducing a non-empty set $\mathcal{G}$ of admissible trade-credit architectures. As we will discuss below, supply chains will form in period 2 exactly by means of a realization $\boldsymbol{g}$ from $\mathcal{G}$. The fact that the agents can not condition their decisions upon the *identity* of their possible partners (and the partners of their partners) is not limiting as long as they can make a prediction about the *number* of them. We briefly discuss the issue of stability of the above Equilibrium. If producers expect a positive excess demand from the stage ahead, depending on financing constraints, they will produce and sell up to $S^* = \mathbb{E}[s^*] = p/(c_m - \gamma)$, that is the competitive amount, yet discounted for γ. If the opposite is true, if producers expect a positive excess supply, they will abide to a proportional supply rationing scheme similar to the one traditionally imposed in dynamic models of production in input-output economies (see for instance Henriet et al. [35]) in order to avoid a scenario in which no producer is expected to accommodate a positive supply of capital. In the proof we show in fact that for $n \to \infty$, this would be the expected unique outcome when producers induce a positive excess supply. Notice that with respect to Henriet et al. [35]), in our paper the proportional rationing scheme is part of a strategic equilibrium[14].

Therefore, depending on internal funding $\omega \in \Omega$, sellers are classified in two distinct groups, L_1 and L_2, respectively indicating constrained sellers, offering credit up to to all the available liquidity $(k^0 + \bar{L}_i + \omega_i)$ and unconstrained sellers, willing to extend credit up to $S^* = \mathbb{E}[s^*] = p/(c_m - \gamma)$. We introduce a complementary lemma before discussing the bank credit provision and analyzing the implication of the equilibrium on the demand for input.

Conjecture 2.4. *Consider a supply chain in which agents belonging to the same stage may have heterogeneous endowments.*

(i) For any production stage $m \in \{3, 4, ..., q\}$ *with* $j \in N^m$ *and expected trade-credit demand* $\bar{k}^{m+1}$*, there exists at most one vector* $\boldsymbol{x}^* \in \mathbb{R}^n$ *which solves the problem defined in* (2.8)*.*

(ii) For $N \to \infty$*, the equilibrium matching probability vector* $\boldsymbol{p}^* \in [0,1]^n$ *fades to* 0*.*

(iii) For every producer $j \in N^m$ *endowed with* $\omega \in \Omega$ *and characterized by the investment mix* $I_j = I_\omega$*, the equilibrium expected account-receivables converges to a finite quantity*

[14] Formally, the FOSD property of the linking function p_{ij} together with perfect substitutability across sellers as induced by the symmetric equilibrium allow us to invoke standard results for stability of one-to-one matchings in bipartite networks (see [1]) which guarantee the existence of stable supply chains as collections of bipartite networks (see Ostrovsky [**?**] for a discussion).

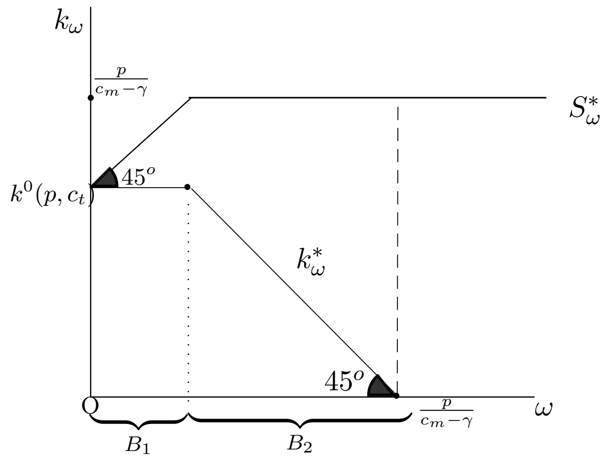

Figure 2.2: Equilibrium type segmentation induced by portfolio structure for a supply chain in which $\mathbb{E}\left[\bar{S}^m\right] \leq \mathbb{E}\left[\bar{K}^{m+1}\right]$ and no available bank-credit.

$\mathbb{E}\left[s^*_{m,\omega}\right]$ *defined as in Equation 2.11 in which* $\bar{S}^*_{m,\omega}$ *is such that:*

$$\bar{S}^*_{m,\omega} = \left(\frac{\bar{K}^{m+1}}{\sum_{\omega\in\Omega} S^*_{m,\omega}}\right) \cdot S^*_{m,\omega}$$

Sketch of Proof of Conjecture 2.4. Although intra-stage endowment heterogeneity in this context breaks the symmetry exploited in the benchmark case to obtain a unique $\bar{x}^*$ and $\tilde{p}(\bar{x})^*$, we can still rely on that results for a constructive proof of the statement. The following algorithm can be devised. First, in the current context the unconstrained credit deadline $\bar{x}^*$, the matching probability $\tilde{p}(\bar{x})$ and the equilibrium expected sales can be obtained by following the analytical steps presented in the proof of Theorem 2.3. In this setting, $\bar{x}^*$ ($\tilde{p}(\bar{x})^*$) acts as upper bound for the individual components of the vector $\boldsymbol{x}^*$ ($\boldsymbol{p}^*$). Subsequently, the elements of the credit deadline vector $\boldsymbol{x}^*$ can be individually varied in order to obtain the matching probability vector $\boldsymbol{p}^*$ which induces the equilibrium quantities as defined in Equation (2.11). ■

2.4 Proportional Bank credit Regimes

Our treatment of the bank's overdraft policy is deliberately stylized as it devised to simply match the bank rationing policy studied in Burkart and Ellingsen [15] when firms can borrow against trade-credit[15]. In order to do so, we expand Freimer's [25] classical bank lending problem to account for borrowers with heterogeneous endowments.

Coherent with Stiglitz and Weiss [67] and Smith [66], the bank faces an adverse selection problem when attempting to ration credit by price. Instead, the bank offers low interest

[15] Specifically, our finding on the linearity of the bank's credit regime refines their Proposition 3, from [15].

rates and ration loans by credit evaluation. We assume that the bank acquires liquidity at cost τ from a competitive credit market and lends at a fixed rate r. In period 1, the bank makes the arrangement of the *overdraft facility* $\bar{L}_\omega$ contingent to: (i) the buyer's internal funds ω_i and (ii) the project distribution R (and, as a consequence, the equilibrium expected sales $\mathbb{E}[s]$). Such evaluation reflects that in order to assess creditworthiness, the banks oversights at zero cost buyer's ex-ante fundamentals yet trade debits are not verifiable. As the producer can not commit *ex-ante* to its revenues, the bank evaluates the project according to an estimate over the range of possible returns. Using a simple rectangular distribution $\rho(\mathbb{E}[s])$, the bank's expected return is given by:

$$\rho(\mathbb{E}[s]) = \frac{1}{\mathbb{E}[s] - \omega}$$

and the bank investment is worth Z_ω:

$$Z_\omega = min\{\xi(\omega), (1+r)L_\omega\}$$

with Z_ω distributed according to ρ. The expression implies that conditional on the realization of $\mathbb{E}[s]$, the bank either realizes the planned profit, or gets an expected amount $\xi(\omega)$ after writing off a failed project . Typically, $\xi(\cdot)$ indicates a positive amount of capital defined in terms of a share ξ of the initial endowment, that is a recovery rate or a collateral over the buyer's internal funds:

$$\xi(\omega) = \xi \cdot \omega, \qquad \xi \in [0,1]$$

However, banks (and regulators) may enforce a tighter interpretation of loss given default over a single position: for instance, by netting the collateral from a range of standard expected penalties such as litigation costs that become apparent once the project fails, the bank may estimate a ξ within the more eventful interval $[-1,1]$. For instance, by analyzing a large sample of small business non-performing bank loans, Eales and Bosworth [22] provide strong evidence of a convex relation between severity and frequency of loss given default. Relevantly, losses given default exceeding by up to the 20% of the principal (i.e. $\xi = -0.2$) are reported twice the frequency with respect to cases with recovery rates range in $\xi \in [0.7, 0.9]$. While we hold that $\xi \in [-1,1]$, we show below that multiplicity of lending regimes characterizing our framework is non automatically imputable to our looser interpretation of ξ. Because the bank condition liquidity provision on the buyer's cash, let us re-express the overdraft facility $\bar{L}_\omega$ in terms of the implied *leverage*:

$$\alpha = \frac{\bar{L}_\omega}{\omega}$$

This formulation directly reflects that when a firm borrows against cash or its fixed assets, its borrowing capacity depends upon the collateral size. Therefore, the bank maximizes the expected profits over loan provision to ω-type entrepreneur by adjusting the leverage α given in $\bar{L}_\omega = \alpha\omega$:

$$\underset{\alpha}{max}\ \mathbb{E}\left[\Pi^B|\omega\right]$$

where $\mathbb{E}\left[\Pi^B|\omega\right]$ is equal to:

$$\int_0^{\alpha\omega(1+r)} \xi \cdot \omega\rho ds \ + \ \int_{\alpha\omega(1+r)}^{p\cdot\mathbb{E}[s]} \alpha\omega(1+r)\rho ds \ - \ (1+\tau)\alpha\omega \tag{2.19}$$

The first component reflects the expected outcome in case of firm's default (corresponding to the seized collateral in the standard interpretation of $\xi \in [0,1]$ or a penalty for $\xi < 0$), net of trade credit exposures. The second component, which is ultimately pinned down by the investment profitability $\mathbb{E}[s]$, is the expected profit conditional on the investment's success. In the following proposition we characterize the solution of (2.19) and point out that the bank's optimal investment policy may fall into two very different regimes, depending on the bank evaluation of the bad event scenario and on bank's relative cost of liquidity sourcing.

Proposition 2.5. (i) *A unique optimal overdraft facility $\bar{L}_\omega = \alpha^* \cdot \omega,\ \forall \omega \in \Omega$ exists and it is given by:*

$$\alpha^*\omega = \underbrace{p \cdot \mathbb{E}[s] \cdot \frac{(r-\tau)}{2(1+r)^2}}_{gains\ from\ trade-credit} \underbrace{+\omega \cdot \frac{(\xi(1+r)+(1+\tau))}{2(1+r)^2}}_{type-based\ rationing} \tag{2.20}$$

(ii) *The optimal credit rationing policy $\bar{L}_\omega$ is a linear function in the buyer type ω. Given the bank's relative cost of liquidity sourcing θ defined as $\theta = (1+\tau)/(1+r)$, the optimal credit rationing policy is an increasing (decreasing) function of the buyer's type if and only if $\xi > -\theta$ ($\xi < -\theta$).*

(iii) *The optimal credit rationing policy $\bar{L}_\omega$ is an increasing (decreasing) function of the bank rate r for $r < r^*$ ($r > r^*$).*

Proof of Proposition 2.5. (i) Given ω, from differentiation of the three components of (2.19) we obtain:

$$\frac{\omega^2\xi(1+r)}{p\mathbb{E}[s]-\omega} + \frac{\omega(1+r)(p\mathbb{E}[s]-\alpha\omega(1+r))}{(p\mathbb{E}[s]-\omega)} - \frac{\alpha\omega^2(1+r)^2}{(p\mathbb{E}[s]-\omega)} - (1+\tau)\omega = 0$$

The expression in (2.20) follows from algebraic manipulation. The second order condition delivers:

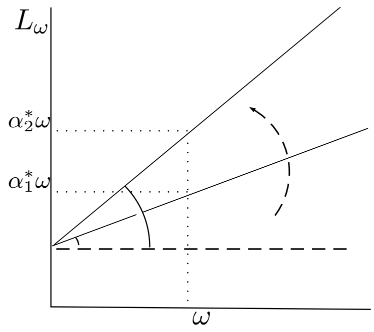

Figure 2.3: Optimal bank lending policies for ξ_1 and ξ_2, $\xi_1 < \xi_2$ in the expansive lending regime (i.e. $\theta > \bar{\theta}$)

$$-\frac{2\omega^2(1+r)^2}{(p\mathbb{E}[s]-\omega)}$$

From which we prove the second part of the statement. From the result, it stands clear that no lending is provided for $\tau > r$. (ii) Now consider $L_\omega^* = \alpha^*\omega$:

$$\alpha_\omega^* \cdot \omega = \frac{p\mathbb{E}[s](r-\tau)+\omega(\xi(1+r)+(1+\tau))}{2(1+r)^2}$$

The result in the statement follows directly by taking the derivative of the RhS with respect to ω and checking the condition for the sign reversal. (iii) By differentiating the lending policy at the optimum, we obtain the extremum r^* as follows:

$$\left.\frac{\partial^2\alpha\omega}{\partial\alpha\partial r}\right|_{\alpha=\alpha^*} > 0 \qquad \Rightarrow \qquad \frac{p\mathbb{E}[s^*](1-r+2\tau)-\omega(2+\phi(1+r)+2\tau)}{2\omega(1+r)^3} > 0$$

Which is positive if the following condition holds:

$$r^* < \frac{p\mathbb{E}[s^*](1+2\tau)-\omega(2+\phi+2\tau)}{p\mathbb{E}[s^*]+\phi\omega}$$

Which sets the bound as defined in the main text of the proof.

■

In Proposition 2.5 we obtain an explicit characterization for the optimal overdraft policy followed by the bank. The overdraft policy is made of two components. The first one is a flat-base loan which incorporates the net profitability of inter-firms trade-credit discounted for the bank credit costs. The second component is a type-specific premium which is added upon the base loan and is key for our analysis of systemic resilience. Notice in fact that only two linear bank lending *regimes* are possible, the selection of which is simply dictated by credit regulations and competitive market conditions. When

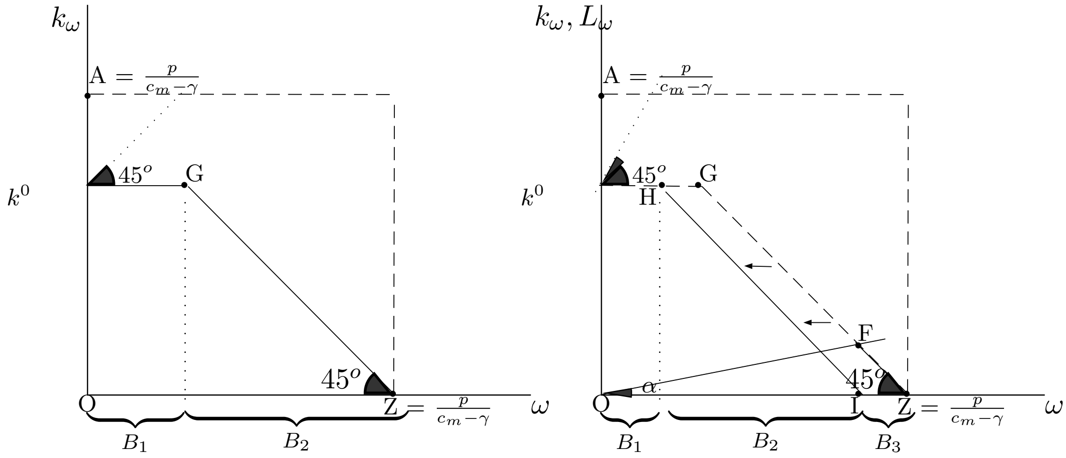

Figure 2.4: A visual representation of Proposition 2.6. *(left)* Demand for trade Credit (segment k^0GZ) with no bank credit. *(right)* Demand for Bank Loans (segment OFZ) and Trade Credit (segment k^0HIZ) under the Bank's linear *expansive* regime $\alpha^*\omega$ as obtained in Proposition 2.5.

the bank's capital sourcing is relatively cheap with respect to the cost of buyers' default (i.e. $\xi > -\theta$) the bank adopts an *expansive* overdraft constraint. The policy is abruptly reversed in a *conservative* regime if the expected penalty of the buyer's default dominates the expected net profits generated by a successful project.

How likely is one lending regime against the other? Presumably, the odds of the conservative regime are small in competitive capital markets. When r approaches τ from above, it is in fact the case that $\theta \to 1$, thus implying a smaller sensitivity of the bank's financing policy both to the economic environment and to (internal or external) changes of regulations ξ. For this reason in the following we will constrain our analysis to the expansive regime.

2.4.1 Demand for Trade-Credit: the Partial Substitution Effect

Given the bank's overdraft policy and the competitive rates r, p, for every type $\omega \in \Omega$, the solution to problem in (2.8) induces a univocal segmentation in the types' demand for bank loans and trade-credit. In the following proposition we characterize the possible segmentation arising from this general setting and show how the presence of a bank induces a *partial* substitution of trade-credit for bank credit. In order to present the broadest taxonomy, we consider a generic sector m and assume that $\bar{S}^m < \bar{K}^{m+1}$ and that $\omega^* = p/(c_m - \gamma) \in \Omega$.

Proposition 2.6. *Given the rates r, p and an expansive bank's lending policy such that $d\bar{L}_\omega/d\omega > 0$ as obtained above, each buyer i endowed with $\omega_i \in \Omega$ units of capital selects her optimal investment mix I_ω according to the following segmentation:*

- $B_1 = \left\{\omega \in \Omega : I_\omega(p, r, k^0) < \omega^*\right\}$. *For buyers in* B_1, *there exists a unique investment mix* $I_\omega(p, r, k^0)$ *given by max* $\left\{0,\ \omega_i + \bar{L}_\omega + k^0\right\}$, *which maximizes production profits* Π_ω.

- $B_2 = \left\{\omega \in \Omega : I_\omega(p, r, k^0) = \omega^* \wedge k_\omega \in \left[0, k^0\right)\right\}$ *Producers in* B_2 *saturate their bank allowance and the group's demand for trade-credit is such that* $\frac{dk}{d\omega} = -(\frac{d\bar{L}_\omega}{d\omega} + 1)$.

- $B_3 = \left\{\omega \in \Omega : I_\omega(p, r, k^0) = \omega^* \wedge \frac{dL_\omega}{d\omega} = -1\right\}$ *buyers in* B_3 *are unaffected by availability of trade-credit as they only rely on bank's credit.*

Proof of Proposition 2.6. The formal proof is omitted as it directly relies on the graphical argument we illustrated along the main text.

■

We explain the result by referring to Figure 2.4. The industry's segmentation in sets B_1, B_2 and B_3 and the relative size of each group may be obtained from the combination of technical constraints (essentially, the production technology) and economic incentives (the interests r, p and the bank's overdraft policy $\bar{L}$). Given our assumptions on the production technology, the frontier AZ identifies the demand for capital as a function of internal funds ω in absence of constraints. Clearly, given the constant returns to scale and $p > 1 + r$, the segment also identifies the demand for liquidity L_ω, were a bank available to fund the firms. Now, consider an industry which relies entirely on trade-credit for financing operations (left panel of Figure 2.4) . Demand for capital in such case is given by segment k^0GZ. Given the frontier AZ, we univocally identify a segment k^0G of tightly wealth-constrained firms that make full use of their own funds and demand k^0, as determined by the industry constraint. Due to the production frontier cap, trade-credit demand for types either declines along the 45^o degree line GZ parallel to the frontier, or, for sufficiently wealthy firms, is zero. Notice that in absence of technological constraints k^0 would coincide with A and identify the industry demand for trade-credit.

Consistently with the Burkart and Ellingsen framework [15], the introduction of a competitive financial sector offering cheaper but rationed credit (as noticeable in the shift from k^0GZ to k^0HIZ) induces a *partial* substitution of account-payables with bank loan. The bank intervention splits the buyers between firms funding the project entirely through bank's credit and firms that rely on a mix of trade-credit and bank loans. The *expansive* lending policy produces a well-defined bank-credit demand curve given by the segment OFZ in the picture. Intuitively, under some policy $\alpha > 0$, buyers are split between tightly-constrained producers that saturate the bank's allowance (sets B_1 and B_2) and a measure B_3 of non-fully constrained types whose demand for bank-credit scales proportionally with the internal funding ω and use no trade credit. Types in B_1 do not substitute account-payables with the liquidity coming from the bank because the production capacity is constrained. On the other hand, agents in groups B_2 and B_3 are

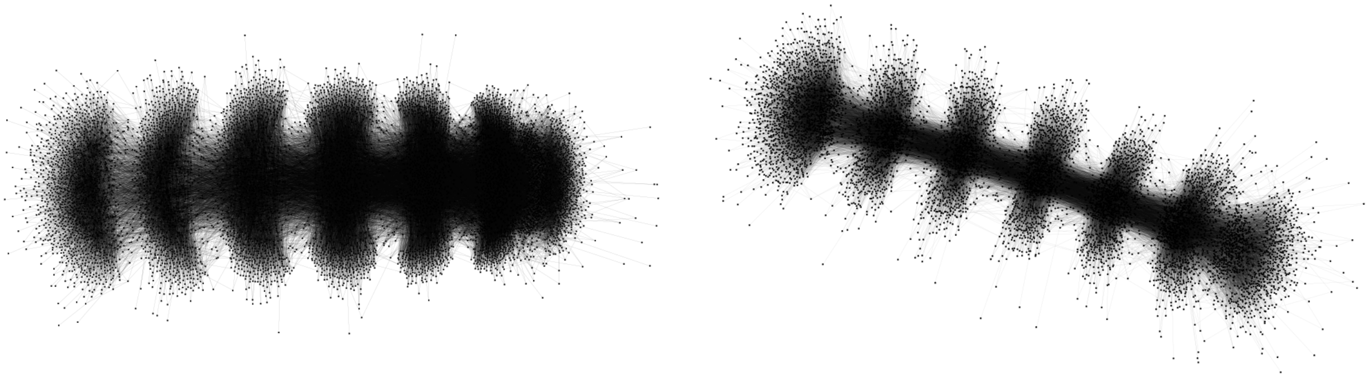

Figure 2.5: The effect of the bank's interest rate hiking from r_1 *(left panel)* to r_2 *(right panel)* for two realizations g_1 and g_2 of classes $\mathcal{G}(N)|_{r_1}$ and $\mathcal{G}(N)|_{r_2}$ respectively. The economy is made of $N = 10,000$ firms randomly assigned to stages $m \in [3,7]$. The classes are compatible with Scenario 1 as depicted in Section 2.5. Dots (arrows) represent firms (loans), and arrows point in the direction of sellers.

capable to produce $\mathbb{E}[s^*]$, yet with diverse investment mixes. The unique type $I(\alpha)$, that is the type which achieves $I = \omega^*$ by demanding exactly all of her bank's allowance and no trade-credit pivots between B_2 and B_3 and determines the relative measure of each set within Ω. In Section **??** we discuss the implication of the groups' relative size in determining the financial resilience of the trade-credit architecture.

2.5 Equilibrium Trade-Credit Architectures

The economy we just presented is coherent with the high-degree of *anonymity* governing relations in competitive markets. Since firms' trade balances are determined regardless of the *identity* of the possible counter-parts, no assumption is made on the actual layer of supply chains which may realize in period 2. The *probability* that some firm $i \in N^{m+1}$ specifically buys from some other firm $j \in N^m$ is entirely captured by the amount of credit (debit) exposure each firm is willing to take. This implies that in our model *any* inter-firm credit architecture is admissible, as long as it is drawn from a *class* of architectures $\mathcal{G}(N)$ consistent with the equilibrium quantities k^*, S^*, α^* as determined in Period 1.

For the technical definition of $\mathcal{G}(N)$ in terms of bivariate distributions and a formal connection to realized trade-credit structures, we refer the reader to Appendix 2.6. Here, we want to deliver the intuition on how incentives constrain the class of feasible architectures with respect to two types of shock which we will reconsider when discussing the ambivalent role of bank lending in inter-firm contagion. We recollect the effect of costs and prices over the class of allowed architectures $\mathcal{G}(N)$ by looking at the relative size of measures B_1, B_2, B_3 and L_1, L_2 from Proposition 2.5 and Theorem 2.3 respectively. In what follows we maintain that $\omega^* \in \Omega$.

Imagine that the bank adopts a sufficiently optimistic recovery rate estimate $\xi > -\theta$ relative to the competitive interbank and retail rates τ, r. First, suppose a scenario in which the bank's interest rate rate falls from r_1 to r_2, $r_1 > r_2$ due to a centralized monetary expansion. Given the inter-firm rate p, the monitoring cost c_m and the discount γ, the demand and supply for trade credit are well-defined functions. From Proposition 2.5, for a suitable parametrization the bank increases its credit provision across the whole industry, $\alpha_1^* < \alpha_2^*$. The effect of financial loosening on trade-credit demand is twofold. In fact, constrained firms reduce the use of account payables, implying a flow of types from B_1 to B_2 and from B_2 to B_3. On the other side of the market as sellers gain better access to liquidity, they can increase sales, with a redistribution of types from L_1 to L_2. Therefore, a loosening of overdraft policies shifts the mass of the transactions in $\mathcal{G}(N)$ toward more receivables and less payables (see Figure 2.5).

Let us briefly discuss the effect of a firm-side shock. Assume that inter-firm credit becomes costlier, pushing p well above $p^s = 0$. The effect on sales can be recovered from Theorem 2.3: the unconstrained sellers' expected sales grow by $1/(c_m - \gamma)$. The bank, on its side, will loosen the credit lines by means of $d\alpha^*/dp = 1/(c_m - \gamma) \cdot (r - \tau)/(2(1 + r)^2)$ (Proposition 2.5), less than the quantity now required by unconstrained sellers. As a consequence of that, a part of types flows from the unconstrained group into B_2 and from B_2 into B_1. As a result, we expect the class $\mathcal{G}(N)$ to shift mass toward a higher (smaller) proportion of payables (receivables).

2.6 Random Trade-Credit Network: General Definitions

The result on trade-credit contagion we stated in Theorem 2.12 - which founds the subsequent analysis on resilience - is general, in the sense that it holds for the whole *class* of trade-credit architectures induced by the parametrization of costs and competitive prices we introduced along the main text, encompassing the realized architecture in place at any moment. In this Appendix we complement the in-text intuition with a formal discussion of the technical framework behind the result.

We arrive at Theorem 2.12 by adopting the elegant methodology proposed by Hurd and Gleeson [33], [38], who draw on the theory of Random Networks[16] for studying contagion in generic classes of financial networks. In general terms, the approach circumvents the intricacies involved with tracking down possible realized chains of default along a given architecture by defining a statistical bound over the proportion of agents which would default after the default of any other firm. As we show below, the approach only requires that any non-random element of the architecture is completely determined by the architecture class $\mathcal{G}(N)$. The framework is characterized by means of three steps.

First, we switch from a deterministic analysis of specific individuals $i, j \in N$ connected via a realized loan to one of *types* of agents and links, such that each type is attached to a given proportion of agents and contracts[17] . In such framework, each firm is completely described in terms of the number of her account receivables s and payables k. Account payables and receivables are drawn from a joint distribution P_{sk}. Firms are financially linked according to a distribution Q_{ks}, which expresses the probability that among the buyers of a firm which sells to s firms, there is one that purchases from k sellers. Hence, while P_{sk} characterizes each firm's positions, Q_{ks} measures the *assortativity* [33] of an architecture class, that is the likelihood for agents of different exposures to be connected. Taken together, distributions (P_{sk}, Q_{ks}) characterize the network architecture. Any firm i is thus identified with a type τ_i which descends from a realization (s, k) of P_{sk} and, similarly, any financial contract ℓ is identified by a type τ_ℓ (s, k) identifying loans that connect a buyer with k creditors to a seller that lends to s buyers.

In the second step we inscribe any possible correlation governing inter-firms relations *inside* the random network (P_{sk}, Q_{ks}). In practical terms, it means that for every firm, any non-geometrical feature is randomly assigned conditional to the assigned label τ_i. In our case, this implies that every firm i is randomly assigned a balance sheet Δ_i from a distribution of balance sheet $\bar{\Delta}_i$ which depends only on τ_i.

Lastly, we recast the random network resilience measurement into a geometrical problem. Given the in-text definition of vulnerable firms, we identify systemic risk with the probability that the inter-firm network contains a *non-vanishing* group of vulnerable firms that take financial exposure into each other. Clearly, the default of any member of the group would trigger the default of all the others thus implying that the size of such group (if exists) gives an upper bound over the vulnerability of the whole architecture

[16]The alternative approach would require us to freeze the huge mass of highly dynamic and transient connections in one fixed realization and assume perfect information about the layer of relations, with a resulting inevitable loss of generality. For a comprehensive primer on economic random networks, we refer the reader to Jackson [40] (Ch.4-5) and references therein. In an updated literature review, Rogers and Pin [60] highlight the key feature of random network formation models, while Jackson, Rogers and Zenou [41] offer a broad discussion on the rationale for employing random network in studying diffusion processes.

[17]This is equivalent to assuming limited information and take the perspective of a probabilistic inter-firm architecture

class. The crucial problem in this last step is finding a tractable way to "navigate" the random network for assessing the size of the group of vulnerable firms.

We formalize these ideas by introducing a body of standard notations and definitions and some results brought from network theory which we use for getting to our result.

Definition 2.7 (Agents and Types)**.** *We map the basic topological features of a trade-credit architecture into a formal network structure by means of the following set of definitions:*

- *A given firm i belonging to a market of N firms which may enter into trade-credit agreements is a node. The collection of all the possible interactions between the nodes of N is given by the power set $\mathcal{G}(N)$. Interactions are in the form of lending agreements and a loan between a seller j and a buyer i is represented by the directed edge $\ell = (i, j) \in \mathcal{L}$, where $\mathcal{L} \subset N \times N$ is the set of all the directed edges. A trade-credit architecture $g \in \mathcal{G}(N)$ is therefore a pair $(N, \mathcal{L})$.*
- *The realized trade-network $\boldsymbol{g} \in \mathcal{G}(N)$ may be represented by means of the $(N \times N)$-adjacency matrix $\boldsymbol{g}$ with components:*

$$g_{ij} = \begin{cases} 1 & if\ k_{ij} = 1 \\ 0 & otherwise \end{cases}$$

- *The total number of inter-firm trade receivables and payables of a firm j is given respectively by her in-degree and out-degree, that is:*

$$\overbrace{deg^-(j) = \sum_i g_{ij} = s_j}^{Trade\ Receivables} \qquad \overbrace{deg^+(j) = \sum_i g_{ji} = k_j}^{Trade\ Payables}$$

- *A firm j has node type (s, k) if the total number of her credits is $deg^-(j) = s$ and the total number of her liabilities is $deg^+(j) = k$. In light of this characterization, the population N is understood as a collection of agents endowed with types related to their in and out degree: $N = \bigcup_{sk} N_{sk}$ such that any agent $j \in N_{sk}$ belongs to the same type (s, k)*
- *An edge $\ell = (i, j) \in \mathcal{L}$ is said to have edge type (k, s) with in-degree s and out-degree k if it is an out-edge of a node i with out-degree $k_i = k$ and an in-edge of a node j with in-degree $s_j = s$, such that $\ell \in \mathcal{L}_{ks}$. When an edge is associated to a specific seller j, we say that $\ell \in \mathcal{L}_j^-$*

The formalization allows us to take a step forward and establish a correspondence between the *class* of trade credit architectures we discussed in Section 2.5 and a probabilistic network:

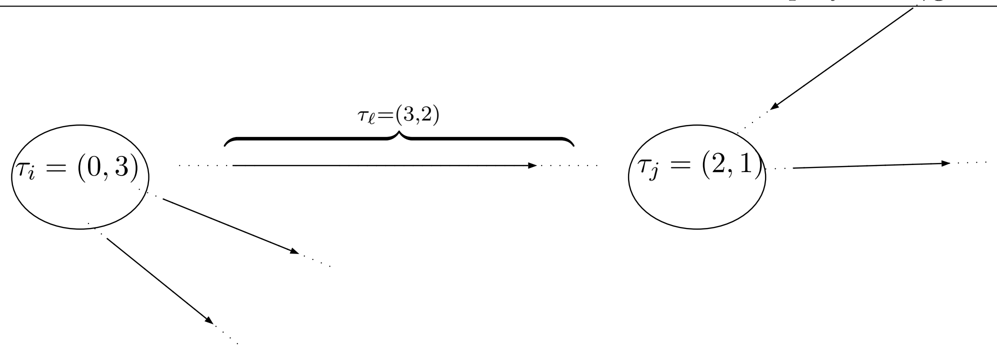

Figure 2.6: Example of two firms and one link drawn from the stochastic trade-credit configuration (P_{sk}, Q_{ks}). Given the type τ_i attached to firm i and type τ_j attached to type j both drawn from P_{sk} , the likelihood that i buys from j is encapsulated in the type τ_ℓ drawn from Q_{ks} that a buyer with k debts is linked to a seller with s assets.

Definition 2.8 (Random Network with no loops)**.**

- *A Random graph of size N is a probability distribution P on the finite set $\mathcal{G}(N)$ and it is invariant under permutation of the N node indexes.*
- *The node-types distribution has probabilities $P_{sk} = \mathbb{P}[i \in N_{sk}]$ and the edge-type distribution has probabilities $Q_{ks} = \mathbb{P}[\ell \in \mathcal{L}_{sk}]$. Therefore, P and Q are bivariate distribution with marginals $P_k^+ = \sum_s P_{sk}$, $P_s^- = \sum_k P_{sk}$ and $Q_k^+ = \sum_d Q_{ks}$, $Q_d^- = \sum_k Q_{ks}$. Edges and nodes are consistently related by the following relations:*

$$\sum_{j \in N} s_j = \sum_{i \in N} k_i$$

$$z \equiv \sum_{s,k} sP_{sk} = \sum_{s,k} kP_{sk} \tag{2.21}$$

$$Q_k^+ = \frac{kP_k^+}{z} \tag{2.22}$$

$$Q_d^- = \frac{sP_s^-}{z} \tag{2.23}$$

- *Therefore, the realized financial architecture is described the two matrices $(\hat{\boldsymbol{P}}, \hat{\boldsymbol{Q}})$ draw from (P_{sk}, Q_{ks})*
- *(no-loops) In any network g large enough and compatible with the class of trade-credit architectures defined in the main text the number of loops*[18] *tends to zero.*

Note that z in equation (2.21) is the average number of loans held (either as buyer or seller) by an agent of the network. The marginals of P_{sk} and Q_{ks} bear strictly related interpretations. For instance, while P_k^+ is the probability that a randomly selected *agent*

[18] Consider i buying from j. A loop exists between i and j if j buys from some firm k which directly or indirectly buys from i. For a formal definition, see [40]

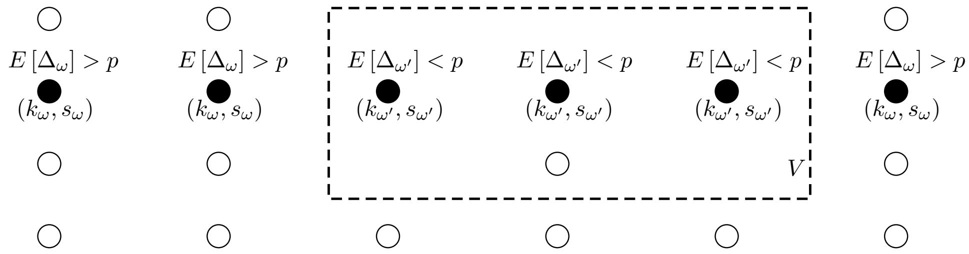

Figure 2.7: Representation of a realized distribution $\hat{\boldsymbol{P}}$ applied upon a population of two types of firms ω, ω' such that $E[\Delta_\omega] > p$ and $E[\Delta_{\omega'}] < p$. Notice that differently from $\mathcal{V}_{IN}$, the set $\mathcal{V}$ is obtained independently to the link structure.

in the network is endowed with k liabilities, the marginal Q^+_k gives the probability that a randomly selected *loan* is a liability for an agent buying from k other agents [19]. The last bullet follows from the fact that for the class of trade-credit architectures we obtained in the main text the size of account payables s is a finite quantity; then, it must be that the number of loops fades for $N \to \infty$. The latter bullet allows us to boil down the intricacies related to the exploration of network to a tractable statistical problem. The property is formalized in the following:

Definition 2.9 (Locally Tree-like Independence [33])**.** *A Random Financial Network is said to have the Locally tree-like independence (LTI) property if conditional on the frequencies in $(\hat{\boldsymbol{P}}, \hat{\boldsymbol{Q}})$, each node i or loan contract ℓ divides the graph into directed paths of receivables ("upstream flow") and payables ("downstream flow") so that the downstream flow is independent of the upstream flow and vice-versa*[20]*. Hence, the idiosyncratic shocks transmitted through different edges enter independently a node.*

Eventually, we can recast the measurement of the inter-firm architecture resilience in terms of the *geometrical* properties of the underlying random network. Let us introduce a set of definitions which characterizes the network as a collection of groups of firms, the so called *components*. As we will show below, the vulnerability of a trade-credit architecture to contagion can be understood in terms of the relative size of the component of vulnerable firms.

Definition 2.10 (Components)**.**

- *A strongly connected component $C^k \subset g$, such that $g = \bigcup_k C^k$ is a sub-network of firms in which a directed path of loans connects any firm to any other firm of the*

[19] As it is $\frac{k}{z}$ times more likely to find a buyer with k liabilities and $\frac{s}{z}$ times more likely to find a seller with s assets when we randomly pick a *loan* in the network.

[20] Suppose that a firm $i \in \tau_i = (s, k)$ buys from $j \in \tau_j = (s', k')$. LTI implies that given the probability $Q_{s'k}$ that an agent with k liabilities buys from one with s' buyers, the number of buyers s of i is independent from the number of sellers k' of j.

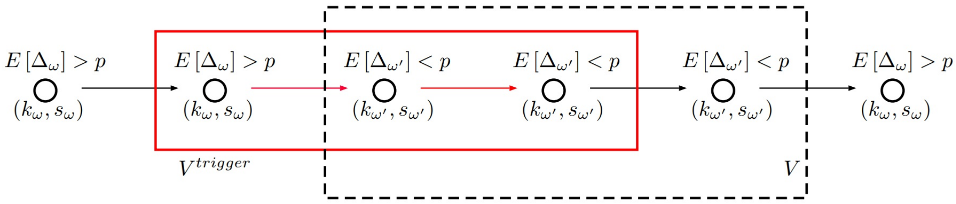

Figure 2.8: Representation of the relation taking place between the sets $\mathcal{V}_{IN}$ and $\mathcal{V}$.

component. The greatest strongly connected component C^k is the one for which $|C^k| \geq |C^m| \;\; \forall\, m \neq k$.

- *If the size of C scales linearly with the size of the network, C is called giant component*[21] *[27].*
- *The giant component of vulnerable firms $\mathcal{V}$ identifies the group of vulnerable firms that scales linearly with the size of the network, for $n \to \infty$.*

We operationalize the above notions, together with the definition of *trigger group* introduced in the main text in the following section.

2.7 Mechanics of Trade-Balance Contagion

We introduce the originating framework behind our contagion threshold. Hurd and Gleeson [33] and Hurd [38] conveyed the problem of finding the conditions which bring generic financial random networks at risk of contagion into an elegant application of the Knaster-Tarski Theorem[22]. The idea is as follows. First, one interprets the network as a collection of firms components of two possible kinds, vulnerable or safe components, and tries to determine the size of the trigger group $\mathcal{V}_{IN}$, that is the agents that may trigger (or amplify) a default cascade in the event of a shock. As the default of one agent belonging to $\mathcal{V}_{IN}$ triggers the default of the whole group of $\mathcal{V}$ firms, the proportion of firms belonging to $\mathcal{V}_{IN}$ also gives the upper bound probability for a cascade to take place. We define the probability for a buyer with k loans to be *outside* such group as:

$$b_k = \mathbb{P}\left[i \notin \mathcal{V}_{IN} | k_i = k\right] \tag{2.24}$$

And since an agent i may be defined in terms of her account receivables and payables through $\tau_i = (s, k)$, equation (2.24) is equivalent to requiring a firm of type τ_i to buy

[21] See also [40] for a definition and a proof that for $n \to \infty$ the probability to have more than one giant component vanishes

[22] Their result is a generalization of previous results [27] because it allows for assortativity of the connections.

only from safe firms or from vulnerable firms that are not buying from other vulnerable firms; that is all liabilities $\ell \in \mathcal{L}_i^+$ must belong to the set $(\mathcal{V}_{IN}^c \cap \mathcal{V}) \cup \mathcal{V}^c$. Thus, via Property 2.9, we may write:

$$b_k = (c_k)^k \tag{2.25}$$

where:

$$c_k = \sum_{s'k'} \underbrace{\mathbb{P}\left[j \in (\mathcal{V}_{IN}^c \cap \mathcal{V}) \cup \mathcal{V}^c | \; j \in N_{s'k'}, k_\ell = k\right]}_{prob.\ type\ (s',k')\ seller\ is\ outside\ trigger\ component} \; \mathbb{P}\left[j \in N_{s'k'} | k_\ell = k\right] \tag{2.26}$$

Property 2.9 allows for an explicit characterization of (2.26). By defining $P_{k'|s'}$ and $Q_{s'|k}$ such as:

$$P_{k'|s'} = \frac{P_{s'k'}}{P_{s'}^-} \qquad Q_{s'|k} = \frac{Q_{ks'}}{Q_k^+}$$

We know that a buyer i does not influence the number of buyers of any of her sellers j. Hence, the probability for an agent τ_i to buy from a firm of type τ_j is given by:

$$P\left[j \in N_{s'k'} | k_\ell = k\right] = Q_{s'|k} \cdot P_{k'|s'}$$

Which implies that c_k is easily obtained as an expected value on all the possible realizations (s', k') of τ_j by means of the following:

$$c_k = \sum_{s'k'} \left(v_{s'} b_{k'} + (1 - v_{s'})\right) Q_{s'|k} \cdot P_{k'|s'}$$

From Equations (2.24) and (2.26):

$$c_k = \sum_{s'k'} \left(v_{s'} \left(c_{k'}\right)^{k'} + (1 - v_{s'})\right) Q_{s'|k} \cdot P_{k'|s'} \tag{2.27}$$

Which implies, for all the possible types of buyers k, that:

$$\boldsymbol{c} = h\left(\boldsymbol{c}\right) \tag{2.28}$$

The mapping in equation (2.28) gives, for each type of buyer, the condition under which there may or may not be a positive probability for a cascade to take place inside the financial network.

Proposition 2.11. *(Hurd and Gleeson [33]) (i) The solution set C^* to the cascade mapping in Equation* (2.28) *is non-empty. (ii) C^* contains at most two distinct solutions c_0^* and c_∞^*.*

Proof of Proposition 2.11. (i) First, h is a mapping induced on a complete lattice with respect to the product order[23] in $[0,1]^{\mathbb{Z}^+}$. In fact, $h : [0,1]^{\mathbb{Z}^+} \to [0,1]^{\mathbb{Z}^+}$. Furthermore, h is continuous, monotone and convex as $\frac{dh(\boldsymbol{c})}{dc_k} \geq 0$ and $\frac{d^2h(\boldsymbol{c})}{dc_k{}^2} \geq 0$, $\forall k$. Therefore, by virtue of Knaster-Tarski Theorem, equation (2.28) has always at least one solution and the set of solutions is a lattice itself.

(ii) For the second part, note that $h(\mathbf{0}) > \mathbf{0}$. Because of convexity, the mapping h cuts the 45° line at most in two points. One trivial solution is given by $\boldsymbol{c}_0^* = \mathbf{1}^T = [1, 1, ..., 1]$. From (2.24) and (2.25) we see that $\boldsymbol{c}_0^*$ is equivalent to $V_{IN} = \{\emptyset\}$, that is all types of agents are safe sellers. However, because of the shape of $h(\boldsymbol{c})$, there exists also another solution $\boldsymbol{c}_\infty^* < \boldsymbol{c}_0^*$ depending on $\boldsymbol{c}_0^*$ being a non-stable fixed point. In such case, the increment of the function $h(\cdot)$ at point $\boldsymbol{c}_0^*$ would be greater than 1 , thus implying two intersections between the two curves in the interval $[0, 1]$. By defining $\boldsymbol{J}|_{\boldsymbol{c}=\boldsymbol{c}_0^*}$ as the Jacobian of the mapping given in (2.28), a well known condition[24] for stability requires that $||\boldsymbol{J}|_{\boldsymbol{c}=\boldsymbol{c}_0^*}|| < 1$, where $||\boldsymbol{J}||$ is the spectral radius of $\boldsymbol{J}$ with element $J_{kk'} = \frac{\partial h_k}{\partial c_{k'}}\big|_{\boldsymbol{c}=\boldsymbol{c}_0^*}$ given by:

$$J_{kk'} = \sum_{s'} k' v_{s'} Q_{s'|k} \cdot P_{k'|s'} \tag{2.29}$$

■

The result we will present in Theorem 2.12 is a tractable bound on the size of V_{IN} as a function of network's structure and individual vulnerability v_s. It is directly nested into Hurd and Gleeson's theoretical result, and provides a stronger yet intuitively tractable condition for assessing the network's resilience. Furthermore, the result in fact generalizes the threshold obtained by Gai and Kapadia [27] for a generic financial random network with no-assortativity and symmetric in-degree and out-degree distribution. To obtain the analytical derivation of the "contagion window" which Gai and Kapadia show in simulations, simply replace $\mathbb{E}[k|d]$ with the unconditional mean $z = \mathbb{E}[k] = \mathbb{E}[d]$.

Theorem 2.12. *Given an industry of N firms which participate to some trade-credit architecture $\boldsymbol{g}$ drawn from the class $\mathcal{G}(\Omega, N, \alpha^*, S^*)$, and given the probability $v_s(\phi) \equiv \mathbb{P}(i \in V|s_i)$ for a firm holding s receivables to be vulnerable, idiosyncratic shocks are not transmitted along the architecture if for the group of firms endowed with s receivables:*

$$v_s \cdot \mathbb{E}[k|s] < 1, \qquad s \in \{0, s^*\} \tag{2.30}$$

In which $\mathbb{E}[k|s]$ is the expected amount of payables of firms holding s receivables and s^* is as defined in Theorem 2.3.

[23] The product order in a vector space implies that for two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ in $[0,1]^{\mathbb{Z}^+}$, $\boldsymbol{a} < \boldsymbol{b}$ if $a_i \leq b_i$ $\forall i \in \mathbb{Z}^+$ with at least one strict inequality.

[24] see for example [58].

In order to prove our result on the contagion threshold, we introduce two lemmas related to matrix norms[25] which we make use in the proof. First:

Lemma 1.

(*i*) *If* $||\cdot||$ *is any norm on* $\mathbb{R}^n$, *then the quantity*

$$||\boldsymbol{J}|| = max\{||\boldsymbol{J}\cdot\boldsymbol{u}|| \mid ||\boldsymbol{u}|| = 1\}$$

defines a norm on the $n \times n$*-space of matrices known as the natural norm*

(*ii*) *The* $||\cdot||_\infty$ *matrix norm of a matrix* $\boldsymbol{J}$ *is equal to its maximal absolute row sum:*

$$||\cdot||_\infty = max\{s_1, s_2, ...\} = max\left\{\sum_d |a_{id}|, 1 \le i \le n\right\}$$

Which are used in the following:

Lemma 2. (*i*) *The spectral radius of a matrix is bounded by its matrix norm.* (*ii*) *If all the absolute row sums of* $\boldsymbol{A}$ *are strictly less than 1, then* $||\boldsymbol{J}_\infty|| < 1$ *and* $\boldsymbol{J}$ *is a convergent matrix.*

Proof of Theorem 2.12. From (2.29), define the row sum $\boldsymbol{J}_k$ as follows:

$$\boldsymbol{J}_k = \sum_{k'} J_{kk'} = \sum_{k'} \frac{\partial h_k}{\partial c_k}|_{\boldsymbol{c}=\boldsymbol{c}_0^*} = \sum_{s'k'} k' v_{s'} Q_{s'|k} \cdot P_{k'|s'} \tag{2.31}$$

From Lemma 2, a sufficient condition for stability requires that for each $\boldsymbol{J}_k$, $k \in [0, N-1]$

$$\sum_{s'k'} k' v_{s'} Q_{s'|k} \cdot P_{k'|s'} < 1$$

From the definition of $Q_{s'|k} = Q_{ks'}/Q_k^+$:

$$\sum_{s'k'} k' v_{s'} Q_{ks'} \cdot P_{k'|s'} < Q_k^+ \tag{2.32}$$

And because $Q_k^+ = \sum_d Q_{ks'}$ we can adopt the matrix notation:

$$(\boldsymbol{P}_{k'|s'} \cdot \boldsymbol{K'})^T \cdot \boldsymbol{V} \cdot \boldsymbol{Q}_{k,s'} < \boldsymbol{1}^T \cdot \boldsymbol{Q}_{k,s'} \tag{2.33}$$

Which gives:

$$\left((\boldsymbol{P}_{k'|s'} \cdot \boldsymbol{K'})^T \cdot \boldsymbol{V} - \boldsymbol{1}\right)^T \cdot \boldsymbol{Q}_{k,s'} < 0$$

[25] See [58], p. 532.

From Lemma 2, the equation is upper bounded by the condition:

$$\boldsymbol{V}\left(\boldsymbol{P}_{k'|s'} \cdot \boldsymbol{K'}\right) = \mathbf{1}$$

After rearranging:

$$\left\| \begin{matrix} 1/v_1^{**} \\ 1/v_2^{**} \\ \dots \end{matrix} \right\| = \left\| \begin{matrix} \sum_{k'} k' P_{k'|s'_1} \\ \sum_{k'} k' P_{k'|s'_2} \\ \dots \end{matrix} \right\|$$

And the result follows:

$$\left\| \begin{matrix} v_1^{**} \\ v_2^{**} \\ \dots \end{matrix} \right\| = \left\| \begin{matrix} \dfrac{1}{\sum_{k'} k' P_{k'|s'_1}} \\ \dfrac{1}{\sum_{k'} k' P_{k'|s'_2}} \\ \dots \end{matrix} \right\| = \left\| \begin{matrix} \dfrac{1}{E\left[k'|s'_1\right]} \\ \dfrac{1}{E\left[k'|s'_2\right]} \\ \dots \end{matrix} \right\|$$

■

Several observations can be derived from our result. From a methodological stand-point, the condition tells what sort of empirical regularities should be addressed for measuring $\mathcal{G}(N)$ resilience. The relation between the balance and the geometrical component is an intuitive one. Imagine that firms are split in classes according to their exposure in receivables s_i. It is clear that idiosyncratic shocks are not passed along supply chains if in expectation vulnerable firms within each class s take no trade-credit. The policy implication is strong. Rather than just cutting *tout-court* credit to vulnerable firms or pooling the risk across the whole industry, we suggest that the network may effectively *dilute* individual risks provided that each vulnerable firm's contribution to systemic risk is low when weighted over its reference class. We further explore the implications of our result in the next Section, in which we relate systemic resilience directly to the incentives driving the formation of trade-credit networks.

2.8 Discussion

In this section we explore systemic resilience as a function of the economic incentives driving the network formation process. The base-line parametrization is reported in Table 2.1. In the main exercise we consider the ambiguous effects on trade-credit network resilience of bank-rate adjustments. We conclude by testing in simulation the resilience threshold we proposed in Theorem 2.12 against the average vulnerability and the simple pooled risk indicator and show that the resilience threshold dominates the alternatives in predicting a shock outcome in Scenario 2. F For the sake of the following analysis, let us express $v_s(\phi)$ explicitly:

$$v_s(\phi) = \sum_{\omega \in \Omega} \mathbb{P}_\omega \cdot \mathbb{P}\left[\mathbb{E}\left[I_\omega\right] + p\left(\phi \cdot \mathbb{E}\left[s^*_\omega\right] - k_\omega\right) - (1+r)\, L_\omega < p\right] \tag{2.34}$$

From Theorem 2.3, we know that in equilibrium $s^*_\omega = \min\{\omega^*, \omega + k_\omega + L_\omega\}$. Therefore, we simplify the above equation in the following:

$$v_s = \sum_{\omega \in \Omega} \mathbb{P}_\omega \cdot \mathbb{I}_{(s=s^* \wedge s^*_\omega < p)} \left[\mathbb{E}\left[I_\omega\right] + p\left(\underbrace{\phi(k_\omega + \omega) - k_\omega - 1}_{\phi^{tc} > \frac{k_\omega + 1}{k_\omega + \omega}}\right) + L_\omega \left(\underbrace{p\phi - (1+r)}_{\phi^b > \frac{1+r}{p}}\right)\right] \tag{2.35}$$

Such that $\mathbb{I}_{(s=s^* \wedge s^*_\omega < p)}$ is the indicator function taking positive values for the defaults (i.e. the expression in square brackets is negative) and ϕ^{tc} (ϕ^b) indicates the lowest admissible collection costs in order to keep the trade-credit (bank credit) side of the equation above zero. In equilibrium, the probability to default against a single position depends on the magnitude of collection costs ϕ relative both to the weight of trade credit over non-bank finance $(k_\omega + 1)/(k_\omega + \omega)$, and to the relative cost of bank-credit over the trade-credit premium $(1 + r)/p$. A dynamic version of our model could easily encompass the "Indirect-Contagion" channel described by Kiyotaki and Moore [45] by looking at fluctuations in the relative cost of bank credit. Because of our cost structure, it might be that for some types $L_\omega > s^*$ as long as $s^*/L_\omega > (1 + r)/p$. This implies that when trade-credit premium and bank rates do not adjust simultaneously, a drop in premium such that $(1 + r)/p > 1$ would decrease the net-worth of all the firms relying on bank-credit.

2.8.1 Bank-Lending has ambiguous effects over network resilience.

Looser overdraft lines have an immediate impact over both the demand and the supply of trade credit. The effect over the network resilience is non-univocal. Suppose a drop in the bank rates from r_1 to r_2 such that $r^* < r_2 < r_1$. From Proposition 2.5, the bank increases credit supply by means of an upward shift of the bank loan supply. While firms characterized by intermediate levels of wealth will effectively *substitute* trade-credit with the larger bank-overdraft - thus contributing in reducing the network density - poor firms will *supplement* the original level of account payables with the new liquidity coming from the bank. As a result of this partial-substitution, the demand of trade-credit will be concentrated in firms characterized by little own funding and high bank debt. On the trade-credit supply side, the increased provision of bank liquidity will mostly benefit the constrained firms, which will be able to offer more trade-credit. In terms of Theorem 2.12, we expect an increment of v_s and that $\mathbb{E}[k|s]|_{r=r_2} \leq \mathbb{E}[k|s]|_{r=r_1}$. It is not possible to determine *a priori* whether the reduction in the density will be sufficient to cope with the lower quality of the surviving connections.

We make the point in Figure 2.9 by comparing the systemic vulnerability of a no-bank trade-credit network $\mathcal{G}^0$ against two counterfactual architectures $\mathcal{G}^1, \mathcal{G}^2$, such that in $\mathcal{G}^1$ ($\mathcal{G}^2$), the bank faces stronger (looser) loss given default regulations ξ_1 (ξ_2), such that $\xi_1 < \xi_2$. In terms of Proposition 2.5, this implies that - everything equal - bank leverage α will be looser in $\mathcal{G}^1$ than in $\mathcal{G}^2$. For both counterfactuals we simulate scenarios in which the bank faces a progressive drop of the lending rate r from $r = p = 1$ to $r = (0.10 \cdot p)$ and adjusts α accordingly. We also fix $\tau = r$ in order to focus on the effect of the type-specific component of the leverage α as derived in Proposition 2.5. As it stands clear from the graph, the proportional regime has an univocal effect over the network resilience only in $\mathcal{G}^1$ (blue line). In $\mathcal{G}^2$, the role of the bank as contagion amplifier or dampener depends entirely on the *level* of the rates r. The inverted U-shape in the graph representing the total number of defaulted firms shows that for certain rates r, the drop of density in the connectivity caused by the bank intervention is not enough to compensate for the increased individual risk. This is exactly due to the partial substitution effect.

Secondly, we fix $r = p$, corresponding to the most vulnerable configuration of the networks above and study the effects of an increase in the spread of interest rates and cost of funds $(r - \tau)$. From Proposition 2.5 we know that a positive spread will produce both a linear increment of the overall credit available in the economy and make the leverage line flatter (thus implying a lower type-discrimination by bank). As it is confirmed by the simulation, a widened spread neutralizes the partial-substitution effect by *insulating* poor firms and therefore drastically reduces the distortive effects of proportional lending for a wide range of spreads.

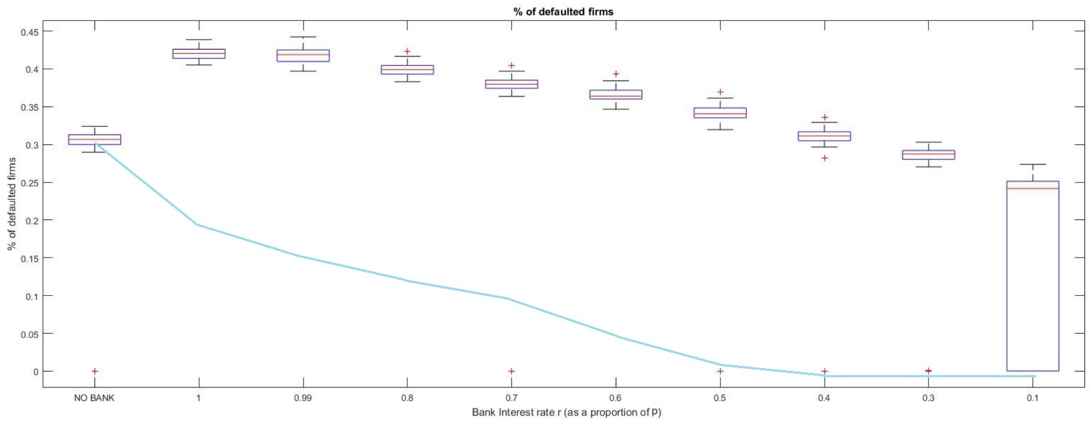

Figure 2.9: For each value of r, the box-plot traces the proportion of defaulted firms for 300 realized trade-credit architectures compatible with $\mathcal{G}^1$ as described in the text. The blue line represents the average proportion of defaulted agents in 300 realizations of architectures compatible with the network $\mathcal{G}^2$. Evidently, the interaction between trade-credit and bank-credit has an univocal effect only for $\mathcal{G}^2$. In fact, under the proportional lending regime, the growth of loan supply actually *increases* the systemic vulnerability up to a point in which most firms substitute trade credit with bank credit, thus breaking the trade-credit connectivity.

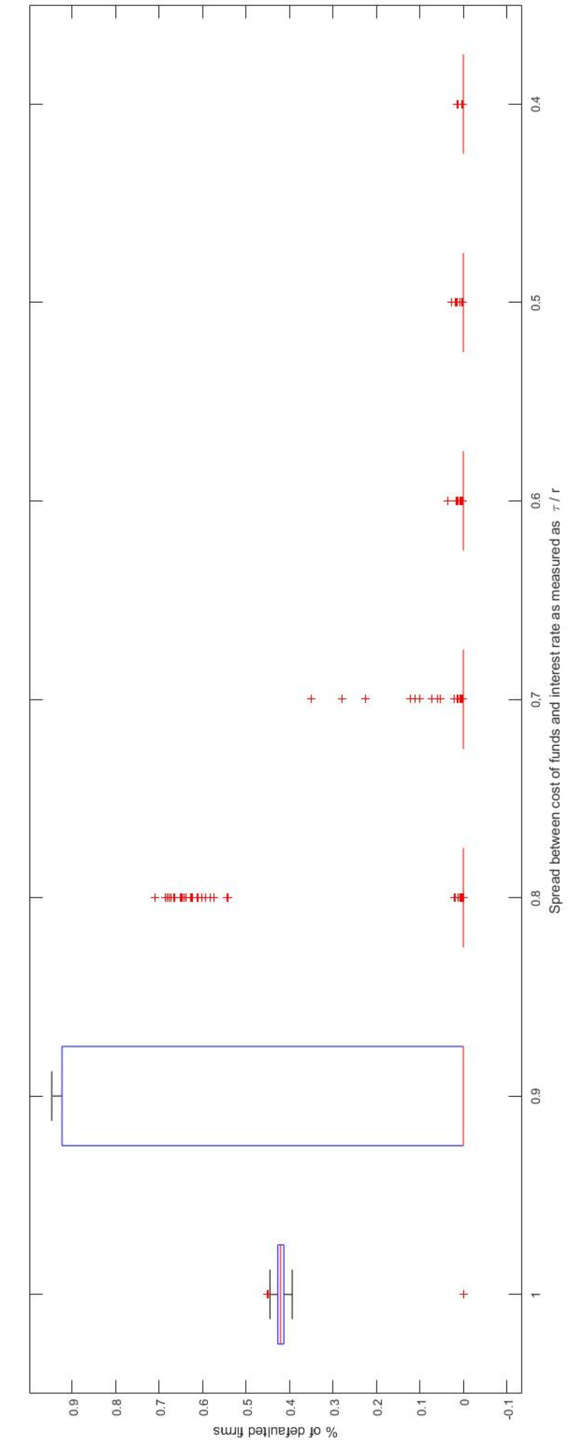

Figure 2.10: .The Inverse Relation between Bank rationing discrimination and resilience.

2.8.2 Network-weighted vs Pooled Vulnerability Indicator

Financial regulators with limited information on the inter-firm transactions layer may be tempted to control financial risk by committing to simple rules of thumbs. For example, they may discard all the network-related information and just target $\bar{v} \to 0, \bar{v} = \sum_d v_d$, for instance by enforcing a penalty to cap liabilities. However, despite the obvious effects in terms economic cool-down implied by the policy implementation, benefits are unclear. What level of $\bar{v}$ would be ”enough” to contain a domino-effect? More saliently, regulators may attempt to alter banks' and seller incentives in order to prevent vulnerable agents from entering into trade-credit agreements. In such case, regulators would target the somewhat milder condition:

$$\bar{v}\mathbb{E}[k] < 1$$

Which requires on average *across all classes of firms* that no vulnerable firms take credit. However, in our core result we take advantage of a few more topological information contained in $\mathcal{G}(N)$ and show that a more precise policy in terms of cost-benefit of risk containment lies in between the two policies. More informative than vulnerability *per se*, or than a simple pooled risk measure, it is the probability that vulnerable firms *are linked together* via chains of receivables which determines whether the shock unfolds into a domino-effect of the sort reported by Raddatz [62].

Therefore, we repeat the previous exercise and compare the actual contagion with the theoretical thresholds we proposed in Section **??**. Results are reported in Figure 2.11. The red area indicates the bank lending regimes which produce network vulnerability according to Theorem 2.12. The red line, which indicates the average vulnerability and bears no reference to the network structure, grows with the bank's liquidity expansion due to the partial substitution effect up to the point in which even poor firms reach the optimal production level and substitute trade-credit with bank-credit. Clearly, given that $r < p$, the substitution effect is beneficial over the stability as more types are left with positive buffers. From the comparison between the red area and the section in which the pooled risk indicator signals vulnerability (*i.e* when $\bar{v}\mathbb{E}[k] > 1$) the pooled risk indicator misses contagion risk for very low levels of bank credit. In fact, when no (or little) bank credit is available, *on average* the firms' exposure within the network is limited for the reason we stated in previous paragraph. However, by discarding a within-class analysis, the pooled risk indicator misses that poor firms have a higher-than-allowed exposure, thus being capable in triggering a cascade effect along the whole structure.

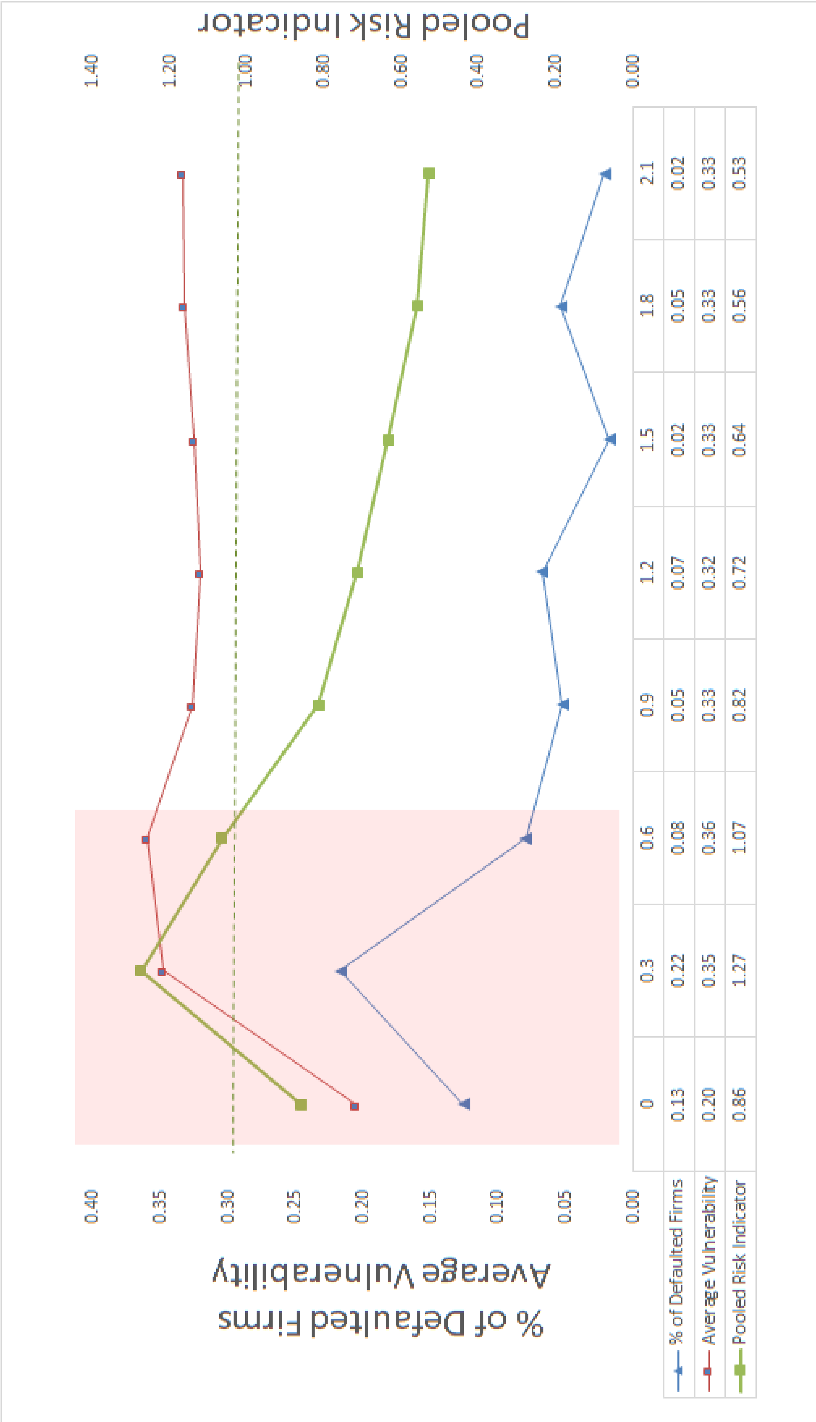

Figure 2.11: Contagion Risk indicators

2.8.3 Simulations: Set-Up

Base Parameter:	Value:
Population Size N:	$10,000$
Own funds Ω:	$\{0, 55\}$
Trade-Credit premium p	1
Bank interest rate r	variable
Trade-Credit cap. k^0:	10%
Monitoring Cost c_m:	0.7
Monitoring Discount γ:	0.69
Collection Cost $(1-\phi)$:	0.6
Cost of funds τ:	variable
Recovery rate ξ:	variable

Table 2.1: Parameter Configuration (Base-line)]

Each simulation set accounts for two distinct phases, a network formation phase and the contagion process. As discussed in Henriet et al. [35], there is no universal method for generating a uniformly randomized set of matrices with specific constraints. For this reason, we suppress production stages by noticing that for $n \to \infty$, we proved in Theorem 2.3 that the probability for any of two firms to be connected fades to zero. This, along with the *Locally Tree-like Independence* property characterizing asymptotic configuration networks (See [38] and Appendix 2.6) guarantees that firms are distributed along (random) trees which preserve the sequential production structure of our model. Each phase is articulated along a finite sequence of iterations[26] (pseudo-time) which we label $\{t_k^1\}_{k<\bar{T}^1}$ and$\{t_k^2\}_{k<\bar{T}^2}$.

1. In the first phase, at pseudo-time t_0^1, every agent i from N is randomly allocated a account payables $k_i \in [0, k^0]$. Then, account receivables $s_i \in [0, s^*]$ are assigned consistently with Proposition 2.6 and with the following rule:

$$\sum_i^N k_i = \sum_i^N s_i$$

   Conditional on (s_i, k_i), i's internal cash ω_i is randomly drawn either from B_1, B_2 or B_3 so that L_i^*, S_i^*, I_i and Δ_i are assigned.

2. At t_m^1, $m \in [3, q]$ one random agent i is selected and randomly matched. with a counter-part $j \in N$. Then, compatibly with Equation (2.1):

$$k_{ij} : \begin{cases} \mathrm{k}_{ij} = 1, & if\ s_j^{q-1} < S_j^*,\ k_i^{q-1} < k_i \\ \mathrm{k}_{ij} = 0 & otherwise \end{cases}$$

[26]For a discussion over the convergence of both the processes, the reader is referred to [38]

3. The matching process proceeds for $t_k^1, k = 2, ...k = q$ until every agent has met her own equilibrium account receivable / payable targets.

4. The resulting trade-credit network $\mathbf{g}^0 \in \mathcal{G}(N) \in [0,1]^N$ is coherent with the structure of economic incentives we discussed along the text.

In the second step we simulate the idiosyncratic shock and track down the subsequent domino-effect (if any).

1. At t_1^2, a firm j is randomly selected and defaults (i.e. $\Delta_j^0 = 0$). Because of our "hard-default" assumption, the default implies that the i-th line in $\mathbf{g}^0$ is zeroed. As a consequence of this, all j's creditor lose their position (if any) $k_{ij} \geq 0, j \in N$.

2. In the subsequent iteration t_2^2, the remaining firms' buffers are updated in order to account for j's default. Firms matching the condition $\Delta_i \leq 0$ are then removed from the network $\mathbf{g}^1$.

3. The contagion proceeds in $t_3^2, t_4^2, ..., \bar{t}^2$, that is until the buffer vector converges and $lim_{t^2 \to \infty} \boldsymbol{g}^{t^2} \to \bar{g}$.

4. General and type-specific statistics with respect to contagion rates, vulnerability and network connectivity are computed by averaging the above procedures over $m = 30$ realizations of $\mathcal{G}(N)$.

2.9 Conclusions

Widespread evidence shows that trade-credit is key in propagation of idiosyncratic shocks among non-financial businesses. Complementary, it is hold that due to firms' higher propensity toward bank loans, better access to bank-credit may effectively improve systemic resilience by reducing firms' dependence on trade-credit. In this paper we presented a two-sides equilibrium model of trade-credit chains formation in which bank lending is shown to have ambiguous effects over the systemic resilience. Depending on individual firms' investment and production menus, firms are segmented between constrained and unconstrained buyers and sellers, thus allowing us to study the contribution of each group in the formation of the credit chains as well as to systemic resilience. More specifically, our first contribution is to demonstrate how competition in the dimension of trade terms is sufficient to originate supply chains in perfectly competitive industries when firms are heterogeneous in own funds and production is constrained along a temporal dimension. When a standard risk-neutral bank is introduced in the model, we show that the credit-rationing implied by a proportional lending regime induces a partial-substitution effect on input demand in line with the classical result of Burkart and Ellingsen [15]. Our second contribution is to show for the general class of trade-credit architectures stemming from our two-sides market model how the bank-induced partial-substitution effect may disrupt the production flow by inducing a trade-balance domino-effect similar to the one hypothesized by Kiyotaki and Moore [45].

Chapter 3

Internal Capital Markets and the formation of Business Groups

Abstract. In this paper we offer a theoretical ground for a empirical regularity widely addressed as key in the formation of Business Groups. According to the related hypothesis, the worldwide ubiquity of Business groups relates to their capability of easing business groups participants' liquidity constraints by means of the creation of Internal Capital Markets. We devise a simple mechanism that founds this argument by pinning down the formation of BGs via ICMs in non-stationary economies to two testable market channels: the debt-to-equity ratio and market profitability. Our formation mechanism shows a possible route for the above channels to have macro-implications via local incentives. Specifically, we show how the two channels may affect some general properties of business group structure such as the intensity of cross-subsidization.

3.1 Introduction

By buying and selling loans within their own group, banks belonging to a US Multibank Holding Company (MBHCs) reallocate funds by using a protocol similar to the profit-sharing adopted by firms affiliated in a Korean *Chaebol* or to the cross-subsidization observed in companies partaking in an Indian Business House: they are all effectively participating into a Business Group (BG). Clusters of legally independent firms, usually operating in different sectors, may decide to resort on pre-existing relationships or even establish new ties and groups with other firms with which they share proximity under some dimension. The degree of formalization of linkages between members as well as the underlying organization ranges from loose horizontal to tight pyramidal structures [64]. Although evidence of business-groups is widespread across economies at different stages of market development, the literature fails to identify a comprehensive theoretical driver for this sort of complex organization.

In the wealth of world-wide evidence and theories collected around the many empirical realizations of BGs, Khanna and Yafeh [44] recognize that while the ubiquitous persistence of business groups is a well-documented fact, the nexus between affiliation and individual firm performance is highly sensitive to geography, institutional development and other determinants. As individual economic incentives are not univocal, also the identification of group advantages is challenging. Several drivers have been put forward in order to explain business groups' presence.

For instance, a market might be characterized by unilateral *incentives of integration.* A body of literature has focused on the one-side incentives of merges and hierarchical control. In a seminal contribution, La Porta et al. [61] found that when share-holding is concentrated, there might be an incentive for the control group to expropriate minority share-holders and *tunnel* resources from the periphery (i.e. newly acquired firms) of the group to the center of it. In this light, the incentive for group formation is a function of the stake-holder majority's powers and therefore, business groups are a mechanism for resource-concentration. However, incentives for the creation of business groups and fluid internal capital market may be aligned between the firms participating to a business group, regardless of idiosyncratic features such as the group's structure and each participant's relative power. Within-groups collusive tactics may be adopted in order to deter market-entrance (see for instance Cestone and Fumagalli, [20]) or to gain political leverage which can be used to push forward the group's special interest (Leuz and Oberholzer-Gee, [48]).

A possible unifying *structural* driver for the formation of BGs which partially encompasses the strategic incentives we touched above deals with the capability of BGs to generate Internal Capital Markets (ICMs). In fact, a majority of theories build upon the assumption that in presence of capital market frictions, financially constrained firms may (be forced to) drift out of traditional financing venues and initiate complex interfirm liquidity markets. While non-excluding for further explanatory layers, the idea that BGs are key for their capability to attenuate affiliates' financial constraints by means of the underlying ICM is appealing for its generality. In the context of an economy populated by conglomerates of highly independent firms (the Korean *Chaebols*), Lee, Park and Shin [47] bring strong evidence of the causal relation between ICMs and BGs and isolate the link between the intensity of cross-subsidization - the *conglomerate* debt-to-equity ratio - and the conglomerate market efficiency. They show that the paralyzing effect on ICMs of liquidity regulations impact the profitability of the whole conglomerate. Under the assumption that ICMs are a leading *raison d'être* for BGs, the analysis of BG's efficiency maps into the study of the capital flow direction within the BG. Almeida, Kim and Kim [3] show that a low-growth to high-growth firms capital reallocation exists within ICMs and that such flow makes a dent in improving the efficiency of BGs. In particular, they find that it is generally true that BGs reallocate funds to member firms with greater investment opportunities.

In this Paper we attempt two main contributions. First, we show that it is possible to pin down the formation of business groups to their capability of generating ICMs. By means of a simple mechanism of *pairwise* capital exchanges, we show that under certain conditions a group of independent producers with heterogeneous liquidity have the incentive to initiate a complex layer of financial transactions, and the incentive directly relates to the condition of the production market in which they operates. The resulting financial layer is flexible, in the sense that it encompasses a wide spectrum of configurations in between the two extremities of complete specialization (i.e. firms that only lend or borrow capital) and allows for the formation of multiple BGs. By varying the degree of market profitability and the underlying industry first-best investment we show how the equity-to-debt ratio modulates the BGs shapes. Our model is stylized in the sense that the formation protocol is applied to a prototypical framework endowed with the following key characteristics.

Stylized Market Regularities

- *Market is Dynamic.* We will assume that in every period new (possibly liquidity constrained) firms enter in the market (i.e. the ”newcomers”) and seek for funding directly from firms that already settled in it (the ”mature firms”).
- *Investments in Business Groups flow top-to-the-bottom.* We impose that loans have a precise direction, that is mature firms can lend to newcomers but not vice-versa.
- *Exchange incentives are pairwise-determined.* In line with a substantial body of literature (see for instance Almeida, Brian and Chang-Soo [3]), we assume that firms are independent production units. Therefore, capital in exchanged via pairwise interactions and a lender has no direct control on further transfers operated by her borrowers. While this assumption does not restrain the top-to-bottom liquidity flow, it limits the market power of any given production unit to the relation with her direct counter-parties.
- *Exchanges are pairwise efficient.* Grounded on the observation of Almeida, Kim and Kim [3], in this paper we describe an economy in which capital flows from low-growth to high-growth firms, allowing for possibly repeated exchanges along PUs. We do not factor in tunneling and other ”dark-side” distortions in capital reallocation motifs. Nonetheless, the fact that exchanges are efficient only on pairwise basis moderates the *overall* capability of Business Groups to reach efficient allocations.

The main advantage stemming out from our dynamic model lies in the neat relation between individual incentives, macro-variables and aggregate outcomes. In this sense we deliver testable implications of the effect of regulatory and demand shocks on the *evolution* of BGs in terms of a parsimonious set of statistics.

This leads to our second contribution. We build our market dynamics on a model of network formation. As Internal Capital Markets are observed by means of ownership participation or inferred via series of transactions (links) between distinct entities (nodes), the network approach has already been used for inspecting complementary issues. In particular Almeida et al. [4] reconstructed the expansion of Korean *chaebols* by tracking the layer of direct and indirect ownership relations and the network centrality of each production units. Our aim here is to provide a theoretical formation protocol which matches the empirical regularities outlined above and which can be used for deriving the aggregate effect of macro-shocks in an environment of local interactions. We build our mechanism upon the class of *growing networks* models initiated in the seminal work of Jackson and Rogers [39] and Vega Redondo [69]: this class of models allows to project the effect of local interactions into the *structural* properties of a resulting aggregate system. However, with respect to the above models, we try to elaborate a mechanism which is pairwise efficient and which is driven by micro-founded incentives.

The general idea behind the prototypical growing networks formation protocol is that the agents' population (*i*) is large, and (*ii*) it *grows* continuously in time, single agents entering sequentially each period. Concerning the interactions, it is usually assumed that (*iii*) agents do not revise their strategies after having taken a decision, and that (*iv*) agent's decisions are myopically made at their entrance conditional on the environment at time of arrival. Obviously, partner selection represents the critical feature of the (possibly multi-dimensional) individual decision set. Lastly, (*v*) linking decisions are taken unilaterally, so that the receiving agent is not really involved in the newcomer's decision process. In the context of *social* networks, the formation mechanism developed by Jackson and Rogers [39] and Vega-Redondo [69] under a similar set of assumptions is characterized by a mixture of global and local features. In these models, each newcomer t randomly selects m_r *friends* from the pool of available agents and then m_s further links are established with their *friends of friends* (random global search[1]). As a consequence of the latter mechanism, the composite probability for a node t' to be chosen by a newcomer is increasing in the number of her friends.

Although appealing for its tractability, the framework defined in [39] and [69] is not micro-founded and, to some extent, hinges upon dimensions that prevent a straight generalization into a proper economic setting. Namely, (*i*) a revenue-cost structure for linking decisions is missing, (*ii*) agents weight equally their friends. (*iii*) Agents are homogeneously characterized by discrete and exogenously posed parameters $\{m_r, m_s\}$ which partially determine the *number* of friends they have, (*iv*) these individuals are independently drawn from the available population; lastly, (*v*) among one agent's friends, only $m_r + m_s$ of them are the result of her own choice. Our aim in this Paper is to provide a model of economic network formation that accommodates an economy characterized

[1] Jackson and Rogers [39] provide an extension of their model which comprises a sketched utility preferential attachment with fixed cost and random utility of link formation

by business groups in the following dimensions: (*i*) Linking decisions are updated via a simple micro-founded decision process. (*ii*) Links are weighted and ordered according to an explicit opportunity-cost structure. (*iii*) Agents retain some control over the linking process after they entered the network (i.e. links are unilateral but are required to satisfy a pairwise Incentive-Compatibility condition). (*iv*) Linking decisions depend on previously-taken decisions.

The Paper is organized as follows. In Section 3.2 we characterize the prototypical framework for the *emergence* of Business Groups via the ICMs, we introduce the formation protocol and we derive our main theoretical result. For any given combination of macro-conditions, we identify certain statistical regularities which hold for classes of ICMs that abide with our formation protocol. In section 3.3 we study the role of debt-to-equity regulations and the industry investment profitability in shaping the geometry of the BGs compatible with our model. Lastly, we test in simulations our analytical artifact.

3.2 The Model

3.2.1 Model Set-Up

We consider an economy which extends the configuration presented in Almeida, Kim and Kim [3] by allowing for a *growing* market economy. A *large* and *expanding* group of price-taker production units (PUs) carry out production of some homogeneous good at price p and marginal cost of production c. Production implies transformation of a homogeneous and divisible input k. Production technology $f(k)$ is strictly concave and homogeneous across producers. Firms are heterogeneous in the time of entry in the pool of producers and in the initial individual capital endowment ω, which we assume distributed (with slight abuse of notation) according to some distribution p_ω over the standardized support $\Omega \equiv \{1, 2, ..., \bar{\omega}\}$.

Timing. The market is made of two distinct phases. At time $t = 0$ each firm $i \in N$ observes the market conditions which consist of the final good price p, the marginal cost of production c and the regulatory constraint α that bounds the quantity of capital every firm can borrow. In this sense, we assume the PUs are leverage-constrained. Firms set their unique production target in the final good market q^*_ω by solving a problem which we define below. In the second phase, firms sequentially enter in the market, one per period $t = 1, 2, ...$ In every period t, we label the newcomer with the time period she entered. We assume that production in the final good market takes place indefinitely later, that is, after the formation of the ICMs.

Individual firm's problem. As stated above, each firm t characterized by endowment ω_t sets her production target $q^*_{\omega_t} = q^*_\omega$ by solving the following problem:

$$\max_{I_\omega} \ \Pi_\omega = p \cdot f\left(I_\omega + \omega\right) - c \cdot \left(I_\omega + \omega\right) \qquad s.t: \quad -\beta\omega \leq I_\omega \leq \alpha\omega \tag{3.1}$$

Where I_ω is the investment, and may take negative values in case of excess of initial resources. In our model, the level of investment is determined *independently* from the cost of borrowing - which depends on pairwise interactions - and on the basis of a leverage constraint, thus reflecting the peculiar structure of ICM. Capital availability to borrowers and lenders' exposure may be restricted on the basis of legal or industry leverage requirements α and β. It is important to stress that explicit formulations of leverage α are not confined to the realm of financial business groups (see for instance [36]). There is evidence of industrial settings in which α has been institutionally targeted: [47] provides empirical evidence of the effects engendered on Korean *Cheabol* by a change in α state regulations in the aftermath of Korean 1997 financial crisis. In the following we assume $\beta = 1$ and $\alpha \in \mathbb{R}^+$. In order to construct a market segmentation, consider the first-best type $\omega = \omega^{**}$:

$$\omega^{**}(c,p): \quad I^*(\omega, \alpha, p, c) = 0, \tag{3.2}$$

Type ω^{**} agents make no investment as they are endowed with the first-best liquidity level, independently of the exogenous constraint α. Hence, ω^{**} has two roles: it rates the profitability of the industry as a function of the optimal unconstrained production level (which depends from the market-wide unitary revenue p and cost c) and gives a relative measure of the two sides of the capital market as defined by the sets $D = \{0, \omega^{**}\}$ for borrowers and $S = (\omega^{**}, \bar{\omega}\}$ for pure lenders. Depending on their type, newcomers induce the following demand for capital:

$$D_\omega = \begin{cases} I^*_\omega & if \ I^*_\omega \geq 0 \\ 0 & otherwise \end{cases} \tag{3.3}$$

In the next proposition we fully characterize the market segmentation as a function of ω^{**} and α and provide a first set of necessary conditions for the emergence of the capital market.

Proposition 3.1 (Market Segmentation). (i) *For every $\omega \in \Omega$, the constraint α and the pivotal type $\omega^{**}(p,c)$, the continuous function $D_\omega : \Omega \to \Omega$ maps each type in its own demand for capital such as:*

$$D_\omega = \begin{cases} \alpha \cdot \omega & \forall \omega \in D^* \\ (\omega^{**} - \omega) & \forall \omega \in D^{**} \\ 0 & \forall \omega \in S \end{cases} \tag{3.4}$$

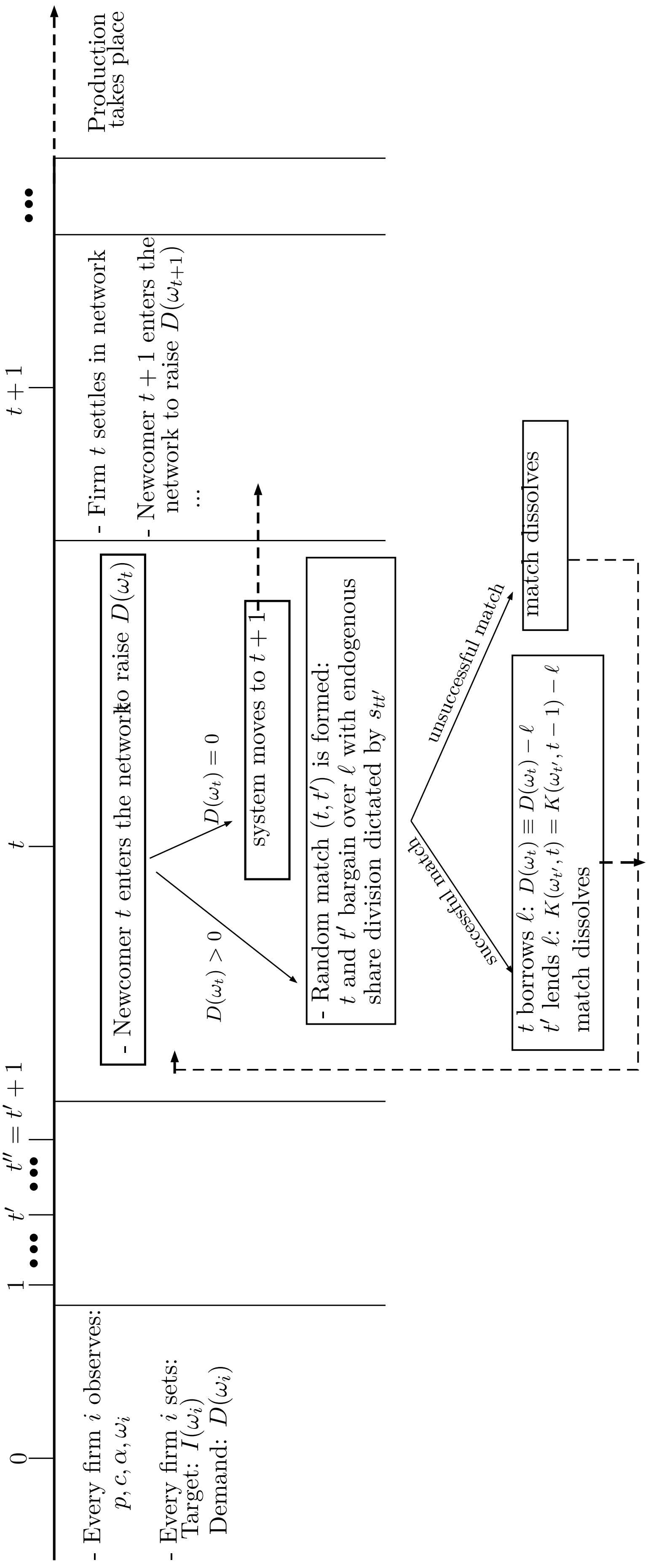

Figure 3.1: Time-Line of ICMs formation

$$\textit{with: } D^* = [0, \omega^{**}/(1+\alpha)], \quad D^{**} = [\omega^{**}/(1+\alpha), \omega^{**}], \quad S = [\omega^{**}, \bar{\omega}].$$

(ii) The average demand for capital $D(\alpha, \omega^{**}) = D^* + D^{**}$ *is given by the quantity:*

$$D(\alpha, \omega^{**}) = \frac{\alpha}{1+\alpha} \frac{(\omega^{**})^2}{2} \cdot \frac{1}{\bar{\omega}} \tag{3.5}$$

(iii) $D(\alpha, \omega^{**})$ *is concave with respect to* α *and convex with respect to* ω^{**}*.*
(iv) (balance equation) A necessary condition for market clearing in the financial sector is that $D^* + D^{**} \leq S$*, which is satisfied for every combination of* α *and* ω^{**} *such that:*

$$\frac{\alpha}{1+\alpha} \leq \left(\frac{\bar{\omega} - \omega^{**}}{\omega^{**}} \right)^2$$

Proof of Proposition 4.1. (*i*) Define $D^* \subset D$ such as the set of *fully-constrained* borrowers (borrowers for which given α and ω^{**} the constraint holds tightly) and consider $\omega^* \in D^* : f(\omega^* + \alpha\omega^*) = f(\omega^{**})$. Because $f(\cdot)$ is a bijection, this implies that $\omega^* = \omega^{**}/(1+\alpha)$. The construction of D_ω easily follows. (*ii*) The result follows from noticing that:

$$\int_0^{\omega^{**}} D_\omega p_\omega d\omega = \int_0^{\omega^*} D_\omega p_\omega d\omega + \int_{\omega^*}^{\omega^{**}} D_\omega p_\omega d\omega$$

That produces:

$$= \frac{1}{\bar{\omega}} \left(\frac{\alpha\,(\omega^*)^2}{2} + \frac{(\omega^{**})^2}{2} - \frac{(\omega^{**})^2}{1+\alpha} \right)$$

Which gives as a result the area of the triangle defined on $[0, \omega^{**}]$ with height $\alpha\omega^*$ from Figure 3.2, weighted by p_ω. (*iii*) The result is straightly assessed by checking first and second derivative with respect to the arguments.
(*iv*) By comparing the total average demand D and total average supply $[\omega^{**}, \bar{\omega}]$ we obtain that:

$$\frac{\alpha \cdot \omega^* \cdot \omega^{**}}{2} \leq \frac{(\bar{\omega} - \omega^{**})^2}{2}$$

Which after substitution gives the result in the statement. ■

Therefore, D_ω defines the aggregate quantity of capital with which every producer type aims to settle in the market and become a potential lender.

In order to introduce the payoff structure, we construct an Individual Rationality (IR) condition which restricts the exchanges, for every ordered pair (t, t'), where t is the borrower and t' is the lender and $t' < t$, to feasible *pair-wise efficient* combinations of loan sizes and interest rates $(\ell_{tt'}, \iota_{tt'})$. For any possible loan $\ell_{tt'}$, the proposition gives technical (necessary) conditions on interest rates for a bilateral trade to take place. These conditions map directly into each party's endowment, which in turn depends on the quantity of capital which had been retained for direct production.

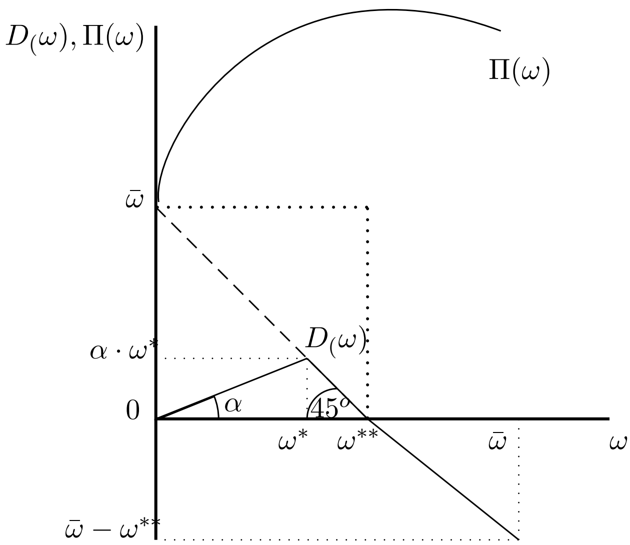

Figure 3.2: A representation of the market segmentation as induced by $I^*(\omega, \alpha, p, c)$ for a given combination of α and ω^{**}. Given the profit function Π_ω in (3.1), types are segmented via D_ω.

Definition 3.2 (Individual Rationality)**.** *Under decreasing return to scales, given a lender t' endowed with ω' and a borrower t endowed with ω, with $t' < t$, an exchange of capital is Individually Rational (IR) if:*

$$\ell_{tt'} \in \left[0, \omega'\right] \tag{3.6}$$

$$\iota_{tt'} \in \left[\bar{\iota}^{l}_{t'},\ \bar{\iota}^{u}_{t}\right] \tag{3.7}$$

Where:

$$\bar{\iota}^{l}_{t'} = \frac{p\left[f(\omega') - f(\omega' - \ell_{tt'})\right]}{\ell_{tt'}} - c, \quad \bar{\iota}^{u}_{t} = \frac{p\left[f(\omega + \ell_{tt'}) - f(\omega)\right]}{\ell_{tt'}} - c \tag{3.8}$$

We motivate the above by observing that the exchange is rational for firm t' if:

$$p \cdot f\left(\omega' - \ell\right) + \iota\ell - c\left(\omega - \ell\right) \geq p \cdot f\left(\omega\right) - c \cdot \omega$$

Which implies:

$$\iota_{t't} \geq \frac{p\left[f(\omega_{t'}) - f(\omega_{t'} - \ell_{t'})\right]}{\ell_{t'}} - c \tag{3.9}$$

Where the right hand side represents firm t's opportunity cost. Opportunity cost enters also the other way around: t borrows $\ell_{t't} \in [0, \omega_{t'}]$ to firm t at interest $\iota_{t'}$ only if:

$$\iota_{t't} \leq \frac{p\left[f(\omega_t + \ell_{t't}) - f(\omega_t)\right]}{\ell_{t't}} - c \tag{3.10}$$

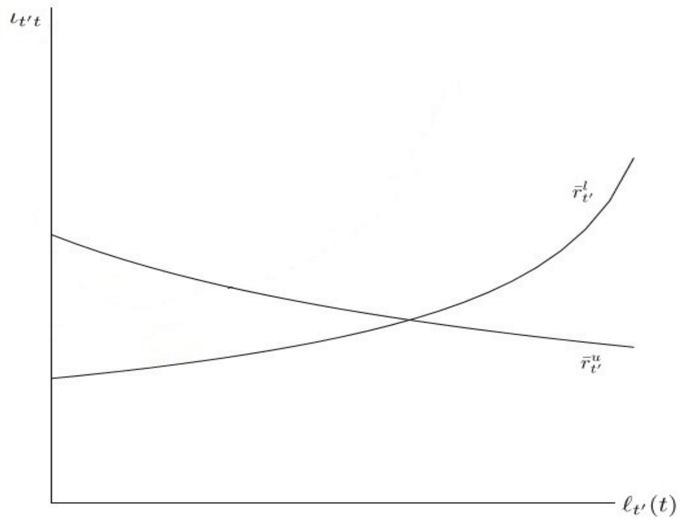

Figure 3.3: A representation of the space of feasible offer for agent 1 and 2 as obtained from the definition of Individual Rationality from Definition 3.2 with logarithmic utilities. On the x-axis: $\ell_{t'}(t)$, y-axis: $\bar{\iota}^l_{t't}(\ell_{t'}(t))$ and $\bar{\iota}^u_{t't}(\ell_{t'}(t))$.

Combining the two, we notice that exchange of resources from t' to t takes place only if:

$$\frac{p\left[f(\omega_{t'}) - f(\omega_{t'} - \ell_{t't})\right]}{\ell_{t'}} - c \leq \iota_{t'} \leq \frac{p\left[f(\omega_t + \ell_{t'}) - f(\omega_t)\right]}{\ell_{t'}} - c \tag{3.11}$$

Bargaining Surplus and Payoff Space. First, we fix $\ell_{tt'} = \ell$ small enough and assume that every matched couple bargains only over the interest rate. For every ordered pair (t, t'), where t is the borrower and t' is the lender, from Definition 3.2 we build the *spread function* $s_{tt'} : \Omega^2 \to \mathbb{R}$ such that:

$$s_{tt'} \equiv \bar{\iota}^u_t - \bar{\iota}^l_{t'} \tag{3.12}$$

The spread function is convex and increasing in the endowments of the lender and convex and decreasing in the endowment of the borrower. A visual representation of the spread function is given in Figure **??**.

3.2.2 Emergence of Internal Capital Markets

Given the simple market framework provided in the previous section, we now discuss the rationale followed by individual PUs in selecting potential lenders once they access the market. In this sense, we conceive a *probabilistic* selection protocol which matches the regularities driving the formation of ICMs we discussed in the Introduction. The probabilistic mechanism has two advantages. First, it avoids the specification of tight bargaining rules. ICMs are decentralized, unregulated markets in which PUs come to idiosyncratic pairwise arrangements, therefore the menu of available contracts can be potentially very sparse. Secondly, the probabilistic mechanism enables us to characterize the properties of general families of ICMs, thus encompassing an analysis based upon *ad hoc* ICMs. Subsequently, we address the high degree of randomness generated by the formation protocol by adopting a mean-field approximation (see for instance [14] and [69]). In the mean-field approximation, all trading actions take place deterministically at a rate proportional to the expected change [19]. On that ground, we will be able to ascertain the long-term aggregate effect of financial regulations and market profitability in shaping the structure of the emerging architectures.

Let us introduce the borrowers' selection protocol. From the previous section we know that lenders are heterogeneous and borrowers *order* lenders according to a measure of *attractiveness* which is determined by the spread between the maximum interest rate payable by the borrower and the minimum interest rate acceptable for the lender. Because the dimension of the window of jointly acceptable rates $s(\cdot)$ expands in the lender's availability of funds, we assume that the probability of matches correlates positively with $s(\cdot)$. We ground this assumption on the stylized market fact of pairwise efficiency as reported in the introduction: the larger the spread, the higher the probability the exchange is pairwise efficient and therefore, the more likely it is to take place. Hence, let us define attractiveness of a seller t' with respect to a borrower endowed with ω simply such as the ratio:

$$A_{t'}(\omega, K_{t'}(t)) = log(s(\omega, K_{t'}(t)) \equiv log\left[\frac{\bar{\iota}^u_\omega}{\bar{\iota}^l_{K_{t'}(t)}}\right] \tag{3.13}$$

In which $K_{t'}(t)$ (defined in the equation below) gives the amount of t''s residual loanable working capital available at date t. Irrespective of the surplus actual distribution, this ordering captures the fact that - for any postulated bargaining mechanism - both parts prefer to trade with a counterparty for which the measure $\bar{\iota}^u_t - \bar{\iota}^l_{K_{t'}}(t)$ is wider, as the share of surplus they may obtain is larger. The mechanism is essentially coherent with the findings of Almeida, Kim and Kim [3] which show that in *chaebol* liquidity goes from low-productivity firms to high-productivity firms and transfers are pairwise incentive compatible.

Now, we make explicit the relation between the lender's inter-period budget and the evolution of the network by assuming that the *size* of the loans is fixed independently of the borrower's features and equal for all the lenders, so that $\ell_{t'}(t) = \ell,\ \forall t, t'$ and ℓ is small enough. The law of motion of lenders' endowment can be then written as:

$$K_{t'}(t, d_{t'}) = \omega_{t'} + D_{t'} - d_{t'}(t-1) \cdot \ell \tag{3.14}$$

In which $d_{t'}(t-1)$ represents the amount of loans provided by a lender t' along the time interval $|t-t'|$. Next, let us define the *average* attractiveness of a lender such as:

$$a(K_{t'}(t)) \equiv \int_0^{\omega^{**}} \int_0^{(D_\omega - \ell)} (A_{t'}(\ x\ , \omega \mid K_{t'}(t)) \cdot p_\omega)\, dx d\omega \tag{3.15}$$

In words, $a(I_\omega, K_{t'}(t))$ evaluates the attractiveness of a lender t' from the perspective of a newcomer who is initially endowed with ω units of capital with probability p_ω and already met a fraction x of her liquidity target along the intra-period search. We imposed that the size of loan ℓ and the target D_ω do play a role on the extent of the borrower's search and t' attractiveness.

3.2.2.1 Lender Selection Protocol

How do we track the evolution of d, that is the lenders' trade-receivables along the evolution of the ICM? This depends on the probability for a borrower to be selected by a lender, in every period given the current state of the system. Because the main driver of our growing ICM is the search conducted by the borrower, the process underlying the formation of the ICM collapses in the probability for different lenders to be selected by each newcomer in the course of the market expansion. We define the probability $\xi'_{t'}(t)$ such as:

$$\xi'_{t'}(t) = \begin{cases} \text{P[t meets t' } \mid |K_{t'}(t)|] \cdot P\left[\text{t selects t}' \mid D_\omega, K_{t'}(t)\right] & for\ \mathrm{K}_{t'}(t) > 0 \\ 0 \text{ for} \quad K_{t'}(t) = 0 \end{cases}$$

Where $|K_{t'}(t)|$ is the dimension of the class of lenders endowed with t''s residual capital at t's arrival. We produce a tractable expression for $\xi'_{t'}$ by committing to three assumptions related to (i) the irrelevance of lenders' budget constraint, (ii) lender groups size and (iii) IR mechanics. First, we drop the limited liability constraint and assume instead that $\beta \to \infty$ (i.e the lender's financial exposure is allowed to grow unbounded). In Proposition 3.9 we will show that the consistency of the results we obtain via $\xi(t)$ is not distorted by the introduction of a budget constraint. Second, we assume that for large t, the probability for a borrower to be matched with a lender of any class, that is $P\left[\text{t meets t}' \mid |K_{t'}(t)|\right]$ dissolves into $1/t$. On average it must be that:

$$\xi_{t'}(t) \approx \frac{1}{t} \cdot a(D_\omega, K_{t'}, \ell, d_{t'}(t-1))$$

Third, we describe the evolution of $\bar{\iota}_t^u$ by means of a first order linear approximation. Because loans ℓ have been fixed and $d_{t'}$ in $\xi_{t'}(t)$ is unbounded, we may construct:

$$\bar{\iota}_{t'}^{l}\left(\ell, d_{t'}(t-1)\right) \approx m_{t'}(K_{t'}(t')) \cdot d_{t'}(t-1) \cdot \ell \tag{3.16}$$

In which:

$$m_{t'}(K_{t'}(t')) = \left(\frac{d\bar{\iota}^{l}(K_{t'}(t'))}{d\ell}\bigg|_{d_{t'}=0} \right)$$

Is a first order approximation in d which fixes the increment in marginal productivity caused by the reduction of residual capital $K_{t'}(t)$ while the market expands. The attractiveness of a lender t' is thus bounded by her own type, the quantity of capital she raises before settling in the market and the *number* of links she already formed in the network. Given the assumptions, $\xi_{t'}(t)$ becomes:

$$\xi_{t'}(t) = \frac{\phi(\alpha, \omega^{**})}{t \cdot m(K_{t'}(t')) \cdot d_{t'}(t-1) \cdot \ell} \tag{3.17}$$

In which $\phi(\alpha, \omega^{**})$ is the *average profitability* of the financial market and it is thus defined:

$$\phi(\alpha, \omega^{**}) = \int_0^{\omega^{**}} \int_0^{(D_t-\ell)} \left(\bar{\iota}_t^u(x,\omega) \cdot p_\omega\right) dx d\omega \tag{3.18}$$

Notice that the denominator of $\xi_{t'}(t)$ indicates the *current* productivity of t' and entirely depends on t' characteristics and her investment decisions. The numerator, $\phi(\alpha, \omega^{**})$ evaluates on average the *maximum* gains that a borrower t can make out of ℓ in the production market, given any level of capital she already raised in the network before meeting t' and any initial endowment. By providing a metric for the potential gains that come from a transaction - regardless to their assignation between the borrower and lender - $\phi(\cdot)$ measures the welfare incentives to participate to ICMs for single PUs.

3.2.2.2 Stationary Internal Capital Markets

We can now proceed to the core Theorem of this section. The Theorem is a characterization of the distribution of the average exposure in *growing* ICMs measured in terms of supplied loans for every possible type of lender ω.

Theorem 3.3. *Given the probability defined in* (3.17)*, the unconstrained stationary CDF of the number of loans agreed by a lender endowed with $\omega = \omega_{t'}$ in an ICM characterized by a lending facility given by α and a market structure encapsulated in ω^{**} is given by:*

$$P_d(d|\omega) = 1 - e^{-\frac{m(K_{t'}(t'))\ell \cdot d^2}{2\phi(\omega^{**},\alpha)}} \tag{3.19}$$

And the related conditional density is given by:

$$p_d(d|\omega) = \frac{m(K_{t'}(t'))\ell \cdot d}{\phi(\omega^{**},\alpha)} \cdot e^{-\frac{m(K_{t'}(t'))\ell \cdot d^2}{2\phi(\omega^{**},\alpha)}} \tag{3.20}$$

In which ϕ is the market profitability as defined in (3.18)*.*

Proof of Theorem 3.3. The methodology of the proof follows [39] and [69] but the resulting distribution is bivariate. We study the long-run behavior of the agents' linking dynamics by adopting a continuum approximation of the number of loans $d(t)$ issued by each lender at any time. Due to the continuum approximation, at most one update takes place almost surely at every instant. Hence, instead of "tracking" the stochastic evolution of lending for each specific lender t', we assume that agents that are identical in terms of endowments and issued loans *on average* update symmetrically, thus implying that $\xi_{t'}$ is replaced with a deterministic rule of motion given by the following simple differential equation:

$$\frac{d\ d_{t'}(t)}{dt} = \frac{\phi(\omega^{**},\alpha)}{t \cdot m(K_{t'}(t'))\ell \cdot d_{t'}(t)}$$

which is separable:

$$\int (m(K_{t'}(t'))\ell \cdot d_{t'}(t)) dd_{t'} = \int \frac{\phi(\omega^{**},\alpha)}{t} dt$$

and has initial condition $d_{t'}(t') = 0$, as we assumed that every agent (including pure lenders endowed with $\omega \geq \omega^{**}$) steps in the inter-firm financing market as a borrower. Integrating, We obtain:

$$d_{t'}(t) = \sqrt{\frac{2\phi(\cdot)}{m(K_{t'}(t'))\ell} ln(t) + constant}$$

We derive the constant from the initial conditions and produce the following law of motion:

$$d_{t'}(t) = \sqrt{\frac{2\phi(\cdot)}{m(K_{t'}(t'))\ell} ln\left(\frac{t}{t'}\right)}$$

Which shows that older lenders have more contracts in place, albeit at a decreasing growth rate in time. Therefore, the probability that at any given time period t the node entered at t' has a certain in-degree $d_{t'}$ is given by:

$$P\left[d_{t'}(t) < d\right] = P\left[\sqrt{\frac{2\phi(\cdot)}{m(K_{t'}(t'))\ell} ln\left(\frac{t}{t'}\right)} < d\right]$$

Since the only dynamical variable affecting the evolution of d is the time itself (i.e. the number of borrowers entering the market), we restate the condition in terms of time interval $|t - t'|$:

$$P\left[d_{t'}(t) < d\right] = P\left[t \cdot e^{-\frac{m(K_{t'}(t'))\ell d^2}{2\phi(\cdot)}} < t'\right]$$

In order to get rid of the time-component, rewrite the above equation as the probability for a firm to enter *after* t'. Because the probability for a firm to enter in the market after any fixed period is approximately uniform (i.e. 1/t) for $t \to \infty$, such probability is given by:

$$P\left[d_{t'}(t) < d\right] = 1 - \frac{1}{t} \cdot \left(te^{-\frac{m(K_{t'}(t'))\ell d^2}{2\phi(\cdot)}}\right)$$

Which also corresponds to the probability that at t, t' has issued d loans. The equation gives the stationary CDF in (3.19). By differentiating $P(d|\omega)$ with respect to d we derive the related density given in (3.20). ■

The unconstrained stationary distribution which we derived in Theorem 3.3 characterizes the lending side of the financing market and depends only on market conditions, endowments, regulatory constraints and average size of the loans. In Figure **??** we report two examples of the distribution. In both cases, exchange preferences are logarithmic, type space given by $\Omega = \{0, 100\}$ and first best endowment coincides with $\omega^{**} = 35$. On the left panel, financial leverage given by $\alpha = 0.14$, on the right panel, financial leverage is expanded to $\alpha = 17$.

3.3 Discussion

We are now in the position to discuss some general properties of the ICMs we developed in the previous section and map them in the extant literature. Specifically, we conduct two analytical exercises and confirm the outcomes with simulations. In the first exercise, we validate that in growing ICMs whose formation is compatible with our protocol there

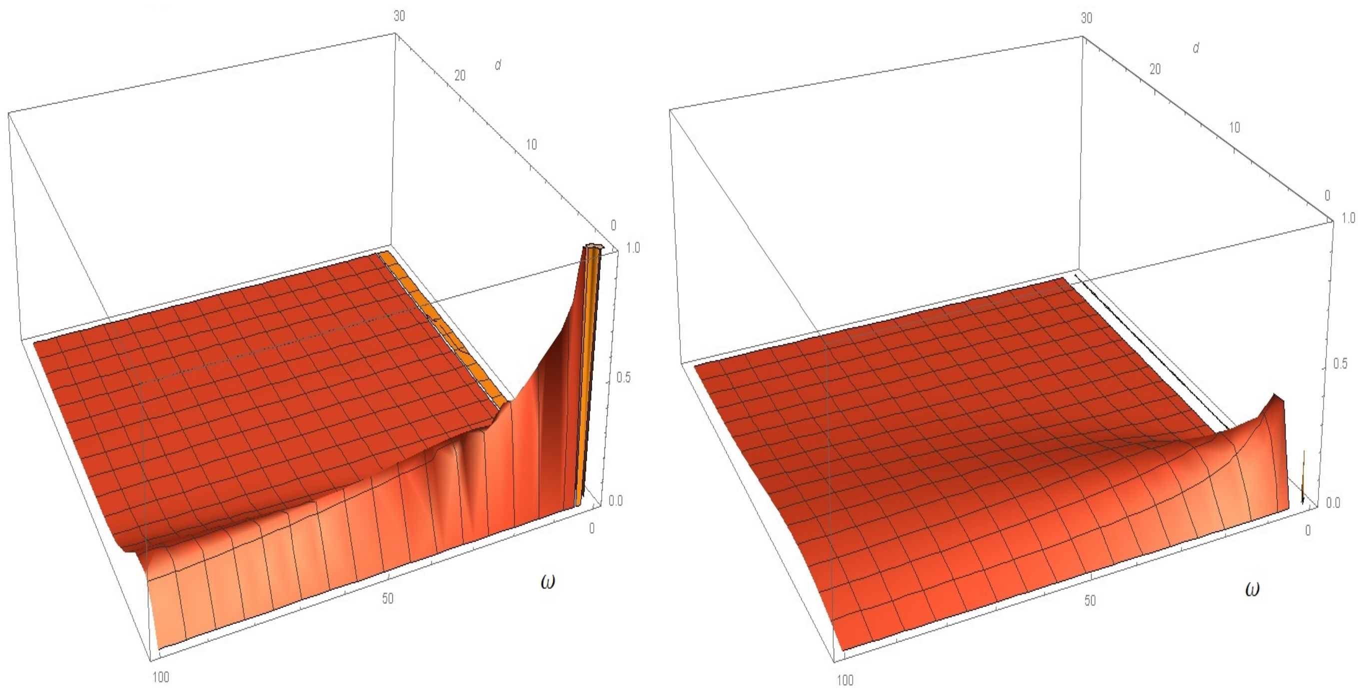

Figure 3.4: Left: Distribution of lender connections $p(d|\omega)$ for a log production function, $\Omega = \{0, 100\}$ and $\alpha = 0.14$, $\omega^{**} = 35$. Types in yellow are fully constrained lenders, red are pure lenders. Agents endowed with more capital are higher in connectivity. *(Right:)* The distribution for $\alpha = 17$ with the remaining parameters being fixed.

exists an unambiguous and positive relation between the number of loans provided by a PU and her starting wealth. This finding is coherent with the strand of empirical evidence discussed in the Introduction which highlights the importance of ICMs in the formation of BGs (see for example [3]). The second exercise is twofold. First, we want to understand whether the relation obtained in the first exercise is preserved under generic configurations of the structural parameters (i.e. the market profitability as captured by ω^{**} and the financing leverage α). Secondly, we explore the role of the structural parameters in determining the network topology and, related, the aggregate welfare. In both the exercises, we proceed to comparative statics between classes of ICMs by relying on the criterion of First Order Stochastic Dominance (FOSD)[2]. In the first exercise, the criterion is used to assess the existence of a monotonic relation between the endowments ω and loans d. In the second exercise, FOSD is adopted to provide a qualitative ordering of the networks that emerge from the different configurations of α and p, c. In Figure **??**, we visualize our results on comparative statics analysis related to the FOSD properties discussed above. Lastly, we show that the reintroduction of the limited liability constraint (which was dropped for the derivation of the mean-field distributions) does not affect our results. Eventually, we complement each step of the

[2]Given two univariate distributions $P(d)$ and $P'(d)$, $P(d)$ First Order Stochastic Dominates $P'(d)$ or, equivalently, $P(d) \succ_{FO} P'(d)$ if $P(d) \leq P'(d)$ $\forall$ $d \in \mathbb{N}$. This is to say that the mass of the distribution moves upward on the support when the distribution shifts from $P'(d)$ to $P(d)$

analysis with simulations. We begin by showing that given α and p, c, agents who are more capitalized offer more loans in the market with respect to poorer counter-parties:

Proposition 3.4. *Given the exogenous parameters α, p, c and two realizations ω', ω'' with $\omega'' < \omega'$, $P(d|\omega')$ First Order Stochastically Dominates $P(d|\omega'')$ or, equivalently, $P(d|\omega') \succ_{FO} P(d|\omega'')$*

Proof of Proposition 3.4. From the definition of FOSD, we need to show that:

$$P(d|\omega'') \geq P'(d|\omega') \quad \forall d \in \mathbb{N}$$

In order to show that the condition holds we note that it suffices to require that $\dfrac{\partial P(d|\omega)}{\partial \omega} < 0 \; \forall d \in \mathbb{N}$. From (3.19) we note that:

$$\frac{\partial P(d|\omega)}{\partial \omega} = \frac{m'(K_{t'}(t'))\ell \cdot d^2}{2\phi(\omega^{**}, \alpha)} \cdot e^{-\dfrac{m(K_{t'}(t'))\ell \cdot d^2}{2\phi(\omega^{**}, \alpha)}} < 0$$

In fact, due to decreasing returns to scale we know that $f''(\omega) < 0$ and consequently $m'(K_{t'}(t')) < 0$

■

3.3.1 Effects of Regulations and Production Determinants on ICMs

Because the inter-firm financing pool originates from a frictional problem, we expect a negative relation between financial constraints and welfare as measured in terms of density of connections. With no loss of generality, suppose that regulation becomes looser as the permitted leverage changes passing from α_1 to α_2, $\alpha_2 > \alpha_1$. First, we expect the segmentation structure of the endowments to change as $D^*_{\alpha_2} \subset D^*_{\alpha_1} \subset D$ as $\omega^{**}/(1 + \alpha_1) > \omega^{**}/(1 + \alpha_2)$. From Proposition 4.1 we know that capital demand is a concave function of α, thus the shift is leading to an aggregate increase of the quantity demanded, as shown in Figure 3.2. However, the increase in α bears a relevant effect also on the other side of the market. Albeit the quantity supplied by pure lenders is unaffected (as determined by ω^{**}), there are two orders of effects to consider: (i) the interval of full-constrained borrower types shrinks and, as a result, there is more *residual* capital available in the market that can be made free for lending at lower cost (due to DRS), resulting in a positive effect for the borrowers. (ii) From the lenders' perspective, the average attractiveness of a loan as defined in $\phi(\alpha, \omega^{**})$ changes, thus affecting the *interconnectedness* of the system. In order to understand what is the overall welfare effect of an increase of α, it suffices to check two conditions: (i) Lenders' profits are increasing in the number of loans they provide on the market (ii) The number of loans

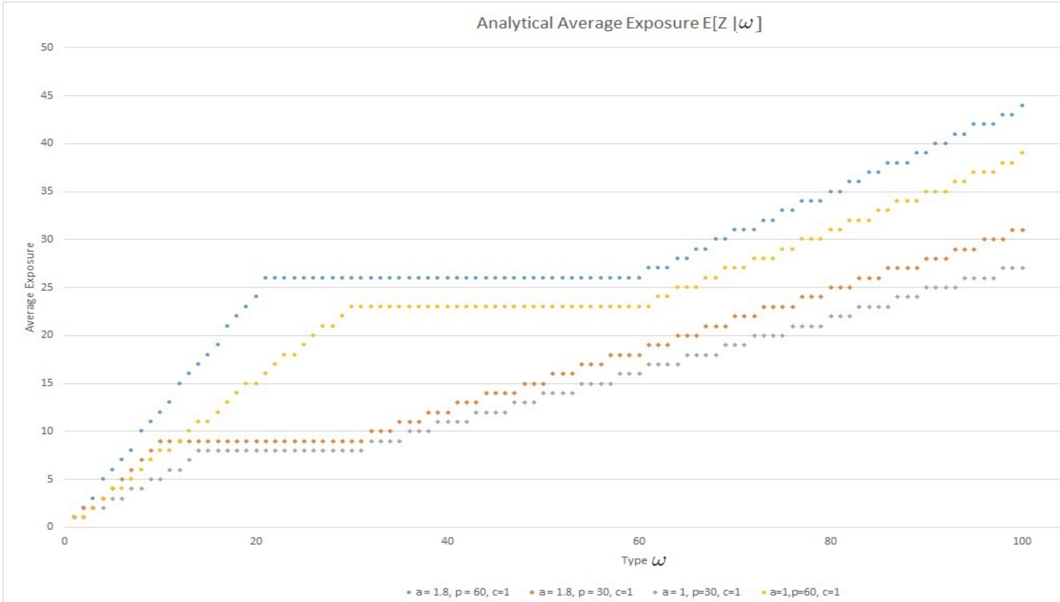

Figure 3.5: Analytical Average exposure per type for four combinations of parameters. Note FOSD applies for a positive shock in both the channels α and ω^{**}

agreed in the system increases with α raising. While the first observation follows directly from the fact that exchanges are pairwise efficient, the second one requires us to prove that there is an unambiguous and positive relation between $P(d,\omega)$ and α. We do so first by showing that there exists a monotonic and positive relation between $\phi(\cdot)$ and α (Proposition 3.5). Secondly, we use the criterion of FOSD to order the welfare of economies characterized by different leverage regulations (Proposition 3.6). Third, we apply a similar argument to discuss the role of production market on the network structure (Proposition 3.8). In Figure **??** we visualize the effect of incrementing the leverage α for a given distribution with logarithmic preferences.

Proposition 3.5. *(i) Given a parameter configuration p, c and a type space Ω, it holds that for $\alpha \to \infty$:*

$$0 \quad \leq \quad \phi(\alpha, \omega^{**}) \quad \leq \quad \frac{1}{\bar{\omega}} \left((\omega^{**} p f(\omega^{**})) - \int_0^{\omega^{**}} p f(\omega) d\omega - c \frac{(\omega^{**})^2}{2} \right) \tag{3.21}$$

Therefore, profitability is bounded between zero and the average profit that a non-fully constrained borrower can make out of her loans by direct production.

(ii) The average profitability of the financing market $\phi(\alpha, \omega^{**})$ is increasing and non-negative in $\alpha \in \mathbb{R}^+$.

Proof of Proposition 3.5. Notice that for $\ell \to 0$ it holds that:

$$\bar{\iota}^u_{t'}(\ell|x,\omega) = \frac{p\left[f(\omega_t + x + \ell) - f(\omega_t + x)\right]}{\ell} - c$$
$$= \frac{\partial f(x,\omega_t)}{\partial x} - c$$

The integrals thus become:

$$\int_0^{\omega^{**}}\int_0^{D_t}\left(p\frac{\partial f(x,\omega)}{\partial x} - c\right)\cdot p_\omega dx d\omega \;=\; \int_0^{\omega^{**}}\left[pf\left(\omega + I(\omega)\right) - pf(\omega) - cI(\omega)\right]\cdot p_\omega d\omega$$

Now, we break the problem by using the endowment segmentation that splits fully constrained from non-fully constrained types :

$$\int_0^{\omega^*}\left[pf\left(\omega+\alpha\omega\right)\right]p_\omega d\omega + \int_{\omega^*}^{\omega^{**}}\left[pf\left(\omega^{**}\right)\right]p_\omega d\omega - \int_0^{\omega^{**}}\left[pf(\omega)\cdot p_\omega\right]d\omega - \int_0^{\omega^{**}}\left[cI(\omega)\right]\cdot p_\omega d\omega$$

Where $f(\omega^{**})$ is the optimal production target as defined in (3.1). We substitute $\omega^* = \omega^{**}/(\alpha+1)$ and exploit the result in Proposition 4.1:

$$\frac{1}{\bar{\omega}}\left(\int_0^{\frac{\omega^{**}}{(\alpha+1)}}\left[pf\left(\omega+\alpha\omega\right)\right]d\omega + \frac{\alpha}{1+\alpha}\omega^{**}pf(\omega^{**}) - \int_0^{\omega^{**}} pf(\omega)d\omega - c\frac{(\omega^{**})^2}{2}\frac{\alpha}{1+\alpha}\right) \tag{3.22}$$

Which measures the average profitability of a loan for fully or partially constrained types of borrowers. The proof is simply obtained by taking the limits over α:

$$\begin{aligned} &\phi(\alpha,\omega^{**}) \to 0 && \text{for } \alpha \to 0 \\ &\phi(\alpha,\omega^{**}) \to \frac{1}{\bar{\omega}}\left(\left(\omega^{**}pf(\omega^{**})\right) - \int_0^{\omega^{**}} pf(\omega)d\omega - c\frac{(\omega^{**})^2}{2}\right) && \text{for } \alpha \to \infty \end{aligned} \tag{3.23}$$

When regulatory constraints prevent leverage of any size, there is no attractiveness in the financing system. On the contrary, regulations which potentially allow any type ω to raise in the market the related optimal capital target are such that attractiveness approaches the average profit that a non-fully constrained borrower can make out of her loans in the production sector.

(ii) In order to prove the second part of the Proposition, we recollect that $K(\omega,\alpha) = \omega(1+\alpha)$, $\forall\omega \in [0,\omega^*]$ and $K(\omega^*) = \omega^{**}$. By differentiating (3.22) with respect to α:

$$\frac{d\phi(\alpha,\omega^{**})}{d\alpha} =$$

$$\frac{1}{\bar{\omega}}\left[\frac{1}{(1+\alpha)^2}\left(\omega^{**}pf(\omega^{**})-c\frac{(\omega^{**})^2}{2}\right)+pf\left(\frac{(\omega^{**})}{(1+\alpha)}(1+\alpha)\right)\frac{-(\omega^{**})}{(1+\alpha)^2}+\int_0^{\frac{\omega^{**}}{(\alpha+1)}}\omega p\frac{df(K)}{dK}d\omega\right]$$

That gives:

$$\frac{d\phi(\alpha,\omega^{**})}{d\alpha}=\frac{1}{\bar{\omega}}\left[\int_0^{\frac{\omega^{**}}{(\alpha+1)}}\omega p\frac{df(K)}{dK}d\omega-c\frac{(\omega^{**})^2}{2(1+\alpha)^2}\right]$$

Noticing that:

$$\frac{(\omega^{**})^2}{2(1+\alpha)^2}=\int_0^{\frac{\omega^{**}}{(\alpha+1)}}\omega d\omega$$

We obtain:

$$\frac{d\phi(\alpha,\omega^{**})}{d\alpha}=\frac{1}{\bar{\omega}}\left[\int_0^{\frac{\omega^{**}}{(\alpha+1)}}\omega\left(p\frac{df(K)}{dK}-c\right)d\omega\right] \tag{3.24}$$

Now it is easy to show that (3.24) is non-negative for $\alpha \in \mathbb{R}_+$. From the solution of the producer's problem (3.1), we know that the production level for the agents whose constraint is not binding is such that $df(k)/dk = c/p$. Hence, from (3.24), non-negativity requires that:

$$\frac{df(k)}{dk} \geq \frac{c}{p} = \left.\frac{df(k)}{dk}\right|_{\omega=\omega^{**}}$$

A condition which is always fulfilled due to DRS, for all the types of borrowers $\omega \in [0, \omega^{**}]$. Thus, given the bound we found in (i), $\phi(\alpha, \omega^{**})$ is increasing in α. ■

From Proposition 3.5 it is clear that profitability is negatively affected by regulations via loss of production revenue due to the reduced capability of firms to obtain capital from the market. Because $\phi(\cdot)$ measures the potential gains of a transaction - regardless to the benefiting party - a reduction of $\phi(\cdot)$ decreases the welfare of the economy. Next, we know that the relation between profitability $\phi(\cdot)$ and the connectedness of the network stems from Theorem 3.3. In order to demonstrate that regulations affect unambiguously the geometry of the network via $\phi(\cdot)$, for all the types $\omega \in \Omega$, we use again the concept of FOSD. Differently from other works ([29],[39]) in which FOSD is assessed for a univariate distribution $P(d)$, we show that FOSD does hold also in a bivariate case such the one we presented with $P(d, \omega)$[3].

Proposition 3.6. *Given two CDF $P(d,\omega|\alpha)$ and $P'(d,\omega|\alpha')$ as defined in Theorem 3.3 with $\alpha \geq \alpha'$, $P(d,\omega|\alpha) \succ_{FO} P(d,\omega|\alpha'), \forall \omega \in \Omega$*

[3] Given two bivariate distribution $P(d,\omega)$ and $P'(d,\omega)$, we say that $P(d,\omega)$ first order stochastically dominates $P'(d,\omega)$ if $P(d,\omega) \leq P'(d,\omega)$ $\forall d \in \mathbb{N},\ \omega \in \Omega$

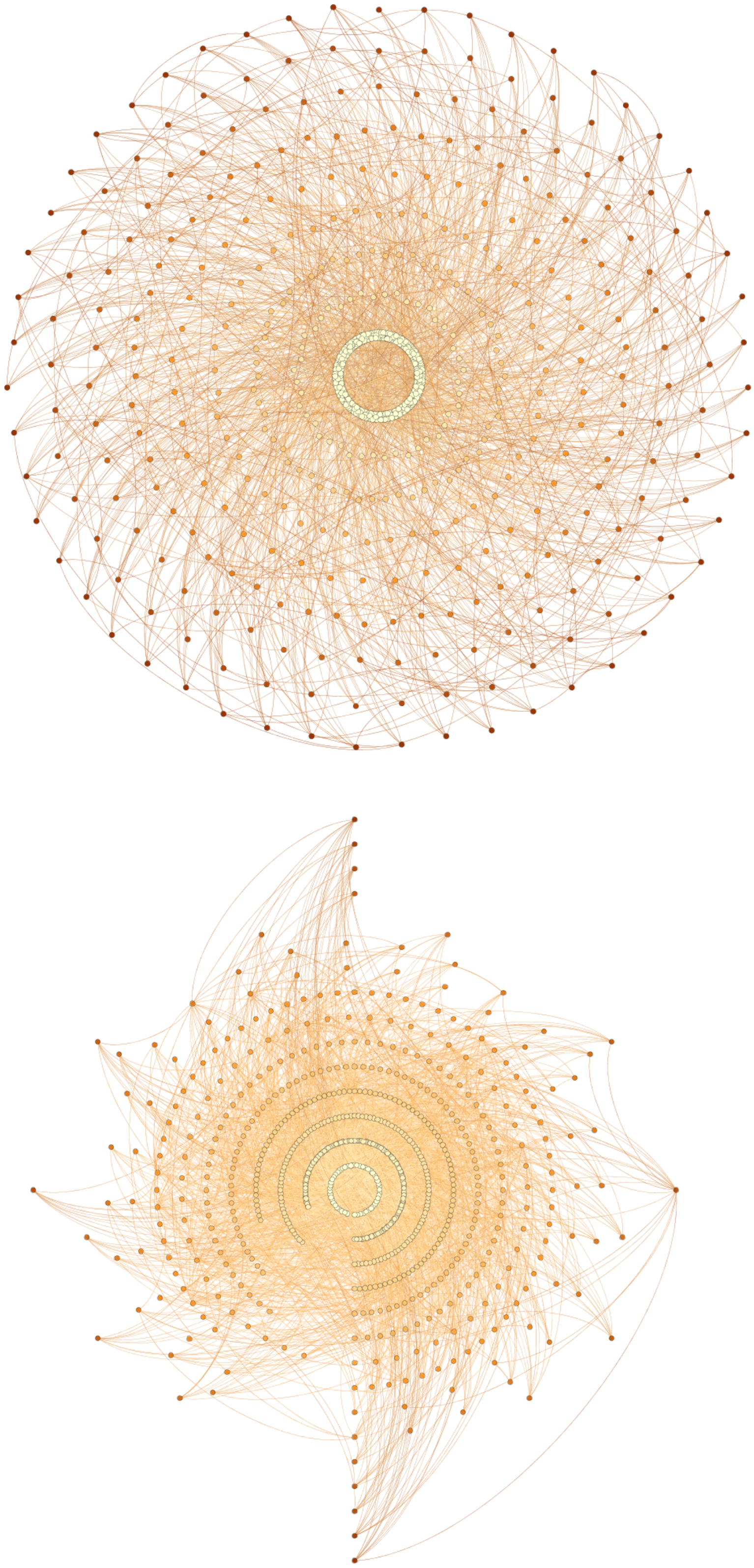

Figure 3.6: The effect of financing leverage in two realized ICMs with $t = 2500$. Agents are ordered according to the number of loans provided on the market so that the outer nodes are pure lenders (red) and central nodes are pure borrowers (white). A loosening in lending constraints allows a fraction of agents to be both borrowers and lenders in the market. (*Left:*) $\alpha = 0.1$ (left) and. (*Right:*) $\alpha = 2$.

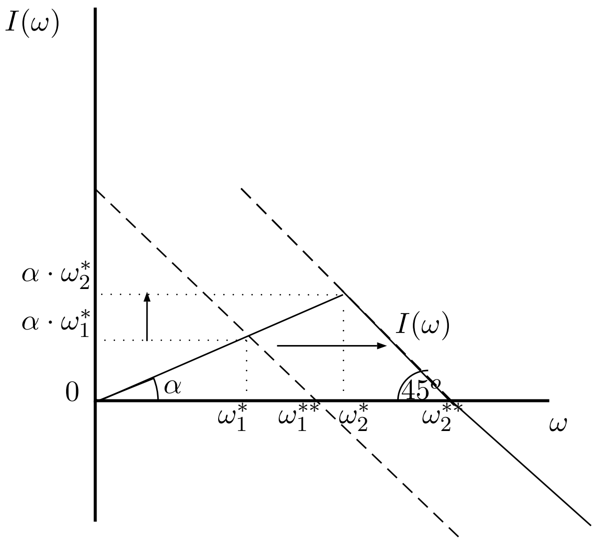

Figure 3.7: Aggregate increase of total quantity of capital demanded consequent to an exogenous increase of (p/c), for a given α. The increment is reflected in the rightward shift of both ω^{**} and ω^*.

Proof of Proposition 3.6. Adapting a result from Huang, Kira and Vertinsky (Theorem 4 in [37]) to the current context, two conditions are required to be in place to prove the proposition: (i) $P(d|\omega,\alpha) \succ_{FO} P'(d|\omega,\alpha')$. (ii) $\frac{\partial P'(d|\omega,\alpha')}{\partial \omega} < 0$ [4] . In what follows, we prove that are both satisfied. (i) From the definition of FOSD:

$$P'(d|\omega,\alpha') \geq P(d|\omega,\alpha)$$

$$e^{-\dfrac{m(K_{t'}(t'),\alpha)\ell \cdot d^2}{2\phi(\omega^{**},\alpha)}} - e^{-\dfrac{m(K_{t'}(t'),\alpha')\ell \cdot d^2}{2\phi(\omega^{**},\alpha')}} \geq 0, \qquad \forall d \in \mathbb{N}$$

First, notice that $m(K_{t'}(t')$ is decreasing in α due to DRS. Further, from Proposition 3.5 we know that $\phi(\omega^{**},\alpha)$ is increasing in α, and because $\alpha > \alpha'$ the condition is satisfied.

(ii) We proved this condition in Proposition 3.4.

■

With Proposition 3.6 we know that the effect of α on the connectedness and so on the welfare of our economy is unambiguous, but there is a second dimension which shapes the morphology of the network, that is the homogeneous good market. For example, let us assume that the ratio p/c increases, thus reflecting an increased profitability of investment in the production of the good. From (3.1) and (3.2) we know that a similar shock will channel into ω^{**} so that: $\omega_1^{**} \leq \omega_2^{**}$, thus shifting the types' mass toward borrowers. Hence - given the number of firms in the economy - the segmentation of the

[4] This implies a non-negative correlation between ω and d. Without such restriction, conditional dominance may not imply joint dominance (see [49], p.615)

population will vary according to:

$$\frac{\partial N(t)}{\partial \omega^{**}} = \frac{t}{\bar{\omega}} \cdot \left(\underbrace{\frac{1}{1+\alpha}}_{fully-constrained} + \underbrace{\frac{\alpha}{1+\alpha}}_{partially-constrained} - \underbrace{1}_{\substack{pure \\ lenders}} \right) = 0$$

The welfare implications of an increase of ω^{**} ex-ante appear to be ambiguous. A positive shock on attractiveness increases the pressure over capital supply but also shrinks the amount of pure lenders in the system. As we did for changes in regulatory policies, we establish a relation between welfare and network connectedness by ordering through FOSD the different geometries induced by changes in ω^{**}. We characterize the behavior of $\phi(\cdot)$ with respect to a change in ω^{**} and then we show how this translates in the geometric properties of the network, which we subsequently order by means of FOSD.

Proposition 3.7. *For any given admissible value of α , the average profitability of a loan $\phi(\omega^{**}, \alpha)$ is bounded below by 0 and it is strictly increasing in ω^{**}.*

Proof of Proposition 3.7. We follow the same rationale of Proposition 3.5. First, notice from (3.22) that $\phi(\omega^{**}, \alpha) \to 0$ for $\omega^{**} \to 0$. From differentiating (3.22) in Proposition 3.5 we obtain that:

$$\frac{\partial \phi(\omega^{**}, \alpha)}{\partial \omega^{**}} = \frac{\alpha}{1+\alpha}(pf(\omega^{**}) + \omega^{**}pf'(\omega^{**}) - \omega^{**}c)$$

However, by recalling how we built the types' segmentation from the producer's problem, $f'(\omega^{**}) = c/p$. Hence:

$$\frac{\partial \phi(\omega^{**}, \alpha)}{\partial \omega^{**}} = \frac{\alpha}{1+\alpha}(p \cdot f(\omega^{**}))$$

Which is always positive.

■

Eventually, we can prove the unambiguous relation between welfare and the production market condition via the following:

Proposition 3.8. *Given two CDF $P(d, \omega; \omega_1^{**})$ and $P'(d, \omega; \omega_2^{**})$ as defined in Theorem 3.3 with $\omega_1^{**} \geq \omega_2^{**}$, $P(d, \omega; \omega_1^{**}) \succ_{FO} P'(d, \omega; \omega_2^{**})$*

Proof of Proposition 3.8. The proof is omitted as it follows closely the one provided for Proposition 3.6. ■

3.3.2 Irrelevance of Individual Limited Liability over Welfare

As stated at the beginning of the previous section, the results related to the welfare effects due to a change in the model structural parameters are obtained by means of the unconstrained distribution $P(d|\omega)$. However, we can extend our insights to a model with the limited liability constraint back in place. Given a type $\omega_{t'}$, from (3.14) let us define:

$$\tilde{d}_{t'} = \frac{\omega_{t'} + D_{t'}}{\ell} \tag{3.25}$$

Where $\tilde{d}_{t'}$ is the upper bound to the number of loans that an agent endowed with $\omega_{t'}$ can offer on the market. Now, we introduce the constrained distribution $\tilde{P}(d|\omega)$ as follows:

$$\tilde{P}(d|\omega) = P\left(d|\omega, 0 \leq d \leq \tilde{d}\right) = \begin{cases} \dfrac{P(d|\omega)}{P(\tilde{d}|\omega)} & for\ d \leq \tilde{d} \\ 0 & otherwise \end{cases}$$

The idea is simple: by restricting the number of loans which t' may offer on the market, the budget constraint induces a *truncation* on $P(d|\omega = \omega_{t'})$. However, we presently show that the properties related to FOSD we depicted in Proposition 3.6 and 3.8 are preserved under *any* truncation, that is regardless to the limited liability constraint [5]. With no loss of generality, we prove the equivalence by assuming a loosening of the financial regulations from α' to α, $\alpha' < \alpha$.

Proposition 3.9. *Given two CDF $\tilde{P}(d, \omega; \alpha)$ and $\tilde{P}'(d, \omega; \alpha')$ respectively obtained as the truncation of $P(d, \omega; \alpha)$ and $P'(d, \omega; \alpha')$ as defined in* (3.25)*, with $\alpha \geq \alpha'$,*

$$P(d, \omega; \alpha) \succ_{FO} P'(d, \omega; \alpha') \;\Leftrightarrow\; \tilde{P}(d, \omega; \alpha) \succ_{FO} \tilde{P}'(d, \omega; \alpha')$$

That is the FOSD ordering holds irrespectively of the lender's budget constraint binding.

Proof. The structure of the proof closely follows the one of Proposition 3.6. However, while condition (ii) is straightforwardly assessed, condition (i) deserves some care. Given $\omega_{t'} = \omega$, we want to prove that:

$$\tilde{P}(d|\omega; \alpha) \succ_{FO} \tilde{P}'(d|\omega; \alpha') \qquad \forall d \in \left[0, \tilde{d}(\omega)\right]$$

[5] Hence, we are adopting *conditional* FOSD, which is a stronger ordering with respect to FOSD. Indeed it is not generally the case that FOSD is preserved under truncation (see for instance [63], p.249)

Let us introduce $A = \frac{m(\bar{K}_\omega,\alpha)\ell}{2\phi(\omega^{**},\alpha)}$ and $A' = \frac{m(\bar{K}_\omega,\alpha')\ell}{2\phi(\omega^{**},\alpha')}$, where $m'(\cdot) \geq m(\cdot)$ with strict inequality if $\omega \in [0, \omega^*]$ and $A' \geq A$. Hence, we rephrase the conditional FOSD above in the following:

$$\frac{1-e^{-Ad^2}}{1-e^{-A\tilde{d}^2}} \leq \frac{1-e^{-A'd^2}}{1-e^{-A'\tilde{d}^2}} \qquad \forall d \in \left[1, \tilde{d}(\omega)\right]$$

or,

$$\frac{1-e^{-Ad^2}}{1-e^{-A'\tilde{d}^2}} \leq \frac{1-e^{-A\tilde{d}^2}}{1-e^{-A'\tilde{d}^2}} \qquad \forall d \in \left[1, \tilde{d}(\omega)\right]$$

Where it is clear that both the ratios are positive and greater than unity. Hence, it suffices to show that:

$$1-e^{-Ad^2} - \left(1-e^{-A'\tilde{d}^2}\right) \leq 1-e^{-A\tilde{d}^2} - \left(1-e^{-A'\tilde{d}^2}\right)$$

$$e^{-Ad^2} - e^{-A'd^2} \geq e^{-A\tilde{d}^2} - e^{-A'\tilde{d}^2}$$

Which can be stated as:

$$g(d) \geq \tilde{B}$$

Where $\tilde{B}$ is a positive constant. Now, notice that $g(d)$ is monotonic and $g(\tilde{d}) = \tilde{B}$. In order to prove the proposition, it suffices to show that $g(d)$ is decreasing over the support. By differentiating the left side of the equation above, it is immediate to show that

$$2d\left(A'e^{-A'd^2} - Ae^{-Ad^2}\right) \leq 0$$

In fact, from the component in brakets we obtain that:

$$log(A') - log(A) \leq d^2(A' - A')$$

Which is always the case for all the d in the support.

■

3.3.3 Simulations

Since the analytical results we presented in the previous section rely on an approximation, we have to show first that the bivariate mean-field approximation is a good analogy for the actual formation protocol. To this purpose, we develop a simulation framework which replicates the market formation. The simulated market follows the time-line presented in Figure 3.1. Each simulation is conducted over $m = 100$ repetitions for

$t = 20.000$ periods. Agents draw their endowment ω from $\mathcal{U}\{1, 2, ..., 100\}$. In Figure 3.8 we plot both $P(d|\omega)$ and its empirical counter-part $\hat{P}(d|\omega)$ for a fixed set of parameters. The two distributions behave closely and the FOSD predicted by the approximation is clearly respected. However, we point out the slower decay of the empirical distribution tails. This is due to the no-constraint assumption which we implemented in order to derive the mean-field approximation.

Secondly, we are interested in comparing the mean-field approximation and the simulations precisely with respect to the FOSD properties as derived in Proposition 3.4 (type-specific FOSD), 3.6 (α-induced FOSD) and 3.8 (p/c - induced FOSD). To this purpose in Figure 3.9 we depict the empirical distribution stemming from four configurations of parameters.

Lastly, we steer our focus from distributions to actual networks. In Figure 3.6 we report the effect of a a loosening of financial constraints on lending (which maps into a shift $\alpha = 0.1 \to \alpha = 4$) for two simulated network realizations. By looking at the average exposures, we obtain a clear picture of the diverse effects implied by a change of regulations or market conditions on the network characteristics of each lender type: the structure changes according to the framework as it has been developed in Section 3.2.2.1.

3.4 Conclusion

In this paper we proposed a theoretical ground for a popular empirical conjecture, that is that Internal Capital Markets are a funding driver for the formation of Business Groups. In our model, we study the formation of Internal Capital Markets between perfectly competitive firms by focusing on two critical features. First, we show that financially constrained producers may initiate a complex network of inter-firm trade-transactions. The network formation is dictated by a simple mechanism which depends on agents' individual financial constraints, the consumer market demand and the debt-to-equity leverage regulations. On the ground of our mechanism, we propose a dynamic formation protocol which allows us to get insights on the temporal evolution of a BG whose production units can be allowed to acquire further production units which constantly flow in the market. The formation mechanism delivers neat predictions with respect to the long-term effects of macro-shocks upon the aggregate topology of the inter-firm network and it shown to be coherent with the stylized facts we reported in the Introduction.

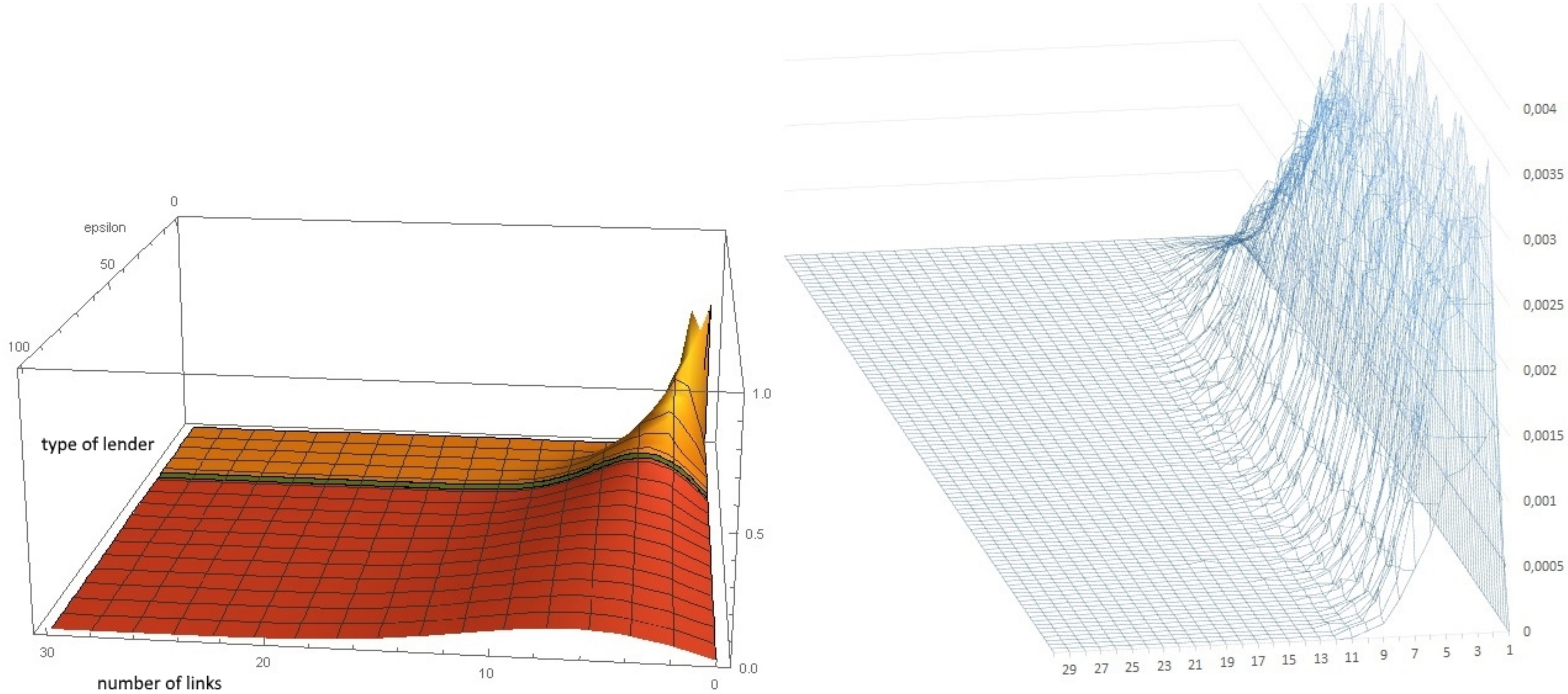

Figure 3.8: Predicted (left) and empirical (right) degree distributions $P(d|\omega)$ and $\hat{P}(d|\omega)$ for $\alpha = 0.1$, and $\omega^{**} = 30$. Coloring is used in the mean-field approximation to distinguish pure lender types (red) from fully-constrained borrowers (orange) and non-fully constrained borrowers (green). Notice the FOSD property of Proposition 3.4 as well as average type-degree distribution is preserved in the empirical distribution.

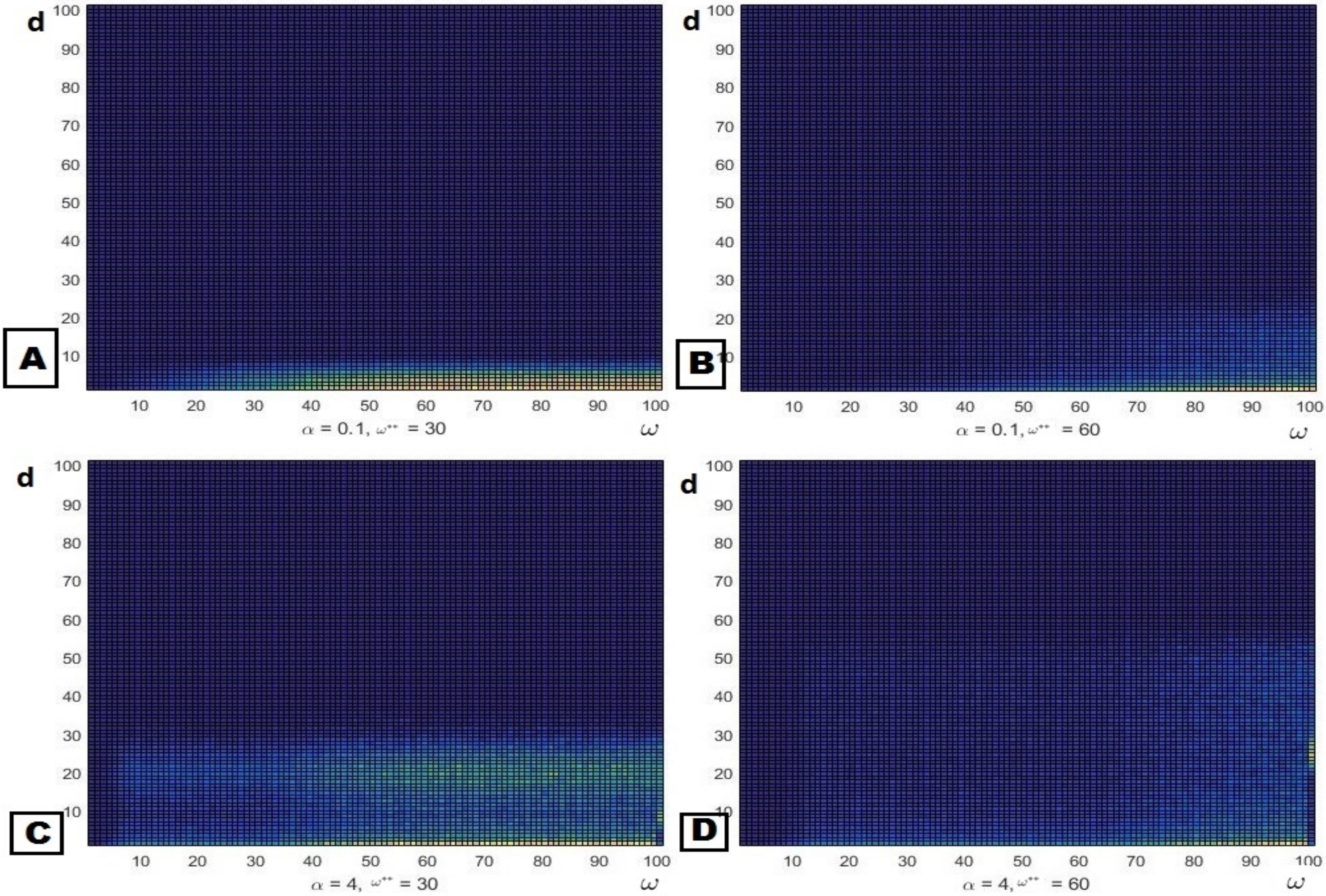

Figure 3.9: The empirical conditional distribution as obtained with four different configurations of exogenous parameters.

Chapter 4

Bargaining in Non-Stationary Markets with Frictions

Abstract. This paper specializes the dynamic market model considered in Manea [54] by introducing financial frictions in a market in which heterogeneous agents trade upon endowments of a generic good. We show that in a dynamic market with sequential waves of buyers, two classes of equilibria in which every trader clears her demand can be characterized and showed to exist. These two equilibria, which we call respectively Homogeneous equilibrium with no entry and Homogeneous equilibrium with entry, diverge only in the specification of agents' dynamics subsequent to trade. While in the former class of equilibria both sellers and buyers leave the market after successful trading, in the latter a certain type of borrower replaces sellers. For both classes of equilibria, we isolate the critical role of frictions in steering the market dynamics and we derive conditions such that the market converges to the expression of a homogeneous type of seller.

4.1 Introduction

In this paper we study a dynamic market in which a (possibly) *growing* pool of heterogeneous sellers faces *waves* of financially constrained buyers which enter in the market at the beginning of every period. The buyers initiate series of sequential bargaining on the basis of a pre-determined demand for a generic good. We show that the presence of a regulator which controls agents' financial leverage bears direct implications on the long-run characteristics of the market. Specifically, we show the existence of two classes of equilibria which may sustain the dynamic market formation along a path in which an economy with a homogeneous seller type does emerge in the limit. Why is this framework relevant and how does we contribute to the extant literature in dynamic markets? First, our paper introduces heterogeneity along a dimension which is crucial in real markets - the limited availability of exchangeable resources in a market endowed with a time dimension - and it does so by linking the evolving structure of a complex market to a systemic variable - a leverage parameter - which may be targeted. Our simple framework matches a variety of empirical environments. Internal capital markets (see for instance Almeida, Kim and Kim [3]) are decentralized environments in which complex flows of capital can take place across heterogeneous production units as a result of the heterogeneous hierarchical and control relations. Given that the efficiency of Internal capital markets in allocating resources is a highly disputed issue (see Kanna and Yafeh [44] for a review), and that internal flow of capital can be regulated by means of (either political or corporate) centralized intervention ([44]), it is fundamental to assess the relation between centralized intervention and efficiency. We can frame this question by asking whether a decentralized market between production units with heterogeneous endowments and uniform first best investment (as it is in the case for which production units are homogeneous and production features decreasing returns to scale) can autonomously converge to a situation in which every unit produces on the efficient frontier. Strikingly enough, we show that the relation between the market composition and financial constraint is non-univocal. The capability for a market to reach an efficient redistribution of inputs is subject to a tipping point which depends on the level of financial frictions in place and the initial heterogeneity of firms. When leverage is moderate, the exchange structure has a limited impact on the firms' distribution, with only a subset of firms being able to reach the first best investment level. This implies that heterogeneity does not disappear in the limit. On the other hand, when the exchange structure is characterized by a high enough financial leverage, depending on the *initial* endowment distribution, two extreme cases emerge. Either the market will attain high efficiency (with almost all firms being able to source capital from other producers) - if the initial number of rich firms is sufficiently high - or it will converge to an inefficient equilibrium in which firms fail to attain the required first best investment level. The implication is empirically relevant as it gives ontological ground to the lack of consensus over the efficiency of such widespread allocation mechanism. In terms of the contribution to the theoretical discussion, our model extends Manea by introducing rich dynamics in the evolution of the population vector. In particular, we show that individually rational exchanges can a tipping point on the state of the economy. which may induce a bifurcation in the population dynamics and study its effect over the exchange incentives.

Several authors considered dynamic economies with financial constraints. In particular, Moll [56], Liu and Wang [50] and Mino [55] studied the evolution of a market economy in which firms are heterogeneous and financially constrained. Similarly to Mino, in this paper we consider an endogenous market process. However, our setting draws on the granular non-stationary dynamic bargaining framework proposed by Manea [54], which we further characterize in order to study two complementary classes of equilibria which allows for the formation of a market economy. Other contributions have attempted to extend the framework of Manea with frictions. In particular, Lauermann, S., and

Noldeke, G. have proven existence of a steady-state equilibrium for an economy with search friction. While their main contribution is to show that a dynamic equilibrium can be achieved with non-transferable utility, they offer a limited analysis of the properties and characteristics of the steady states because the bargaining game is treated as a ”black box”.

We embed our market formation process with frictions in the general (frictionless) framework of an *infinite horizon bargaining game played in discrete time* introduced by Manea [54]. In our model, the market formation is made of an initial stage and two intertwined processes. In the initial stage, every agent in the production market discovers her own good endowment, which is heterogeneous across the firms, and induces an individual demand for the good according to her preferences and to the financial constraint currently in place. For a given *market composition*, that is the distribution of available units of the good among the agents who already settled in the market, the first process is the *intra-period* exchange of the good between agents entering in the market and the *current* pool of sellers. The second process is the evolution of the market composition across periods. The key dimension which links intra-period and inter-period dynamics is given by the market composition in sellers' good endowments, which is an endogenous measure that depends on the outcome of the realized exchanges and which pins downs the incentive structure of the bargaining process.

The two classes of equilibria respectively introduced in Definition 4.7 and Definition 4.9, diverge in the specification of the inter-period dynamics. The former equilibrium, which we call *Homogeneous Equilibrium with no Entry* expands a canonical scenario also considered in Manea [54], in which both the seller and the buyer leave the market upon successful trade, yet differently from Manea, buyers are allowed to trade multiple units of the good before leaving the market. We show that within this class of equilibria, the evolution of the market composition is subject to a tipping point which entirely depends on the level of financial frictions in place. The second equilibrium we describe, the Homogeneous Equilibrium With Entry, allows certain buyers to *settle* in the economy as potential sellers for next waves of buyers, thus replenishing the pool of sellers. With respect to this equilibrium, we show that for a compatible parametrization, the economy always converges to a expansionary path characterized by a single-seller type.

The paper is organized as follows. In Section 4.2 we define the primitives of our dynamic market and present the formation protocol. Subsequently, in Section 4.2.1 we characterize and prove existence of the two classes of equilibria which may support the market formation as the outcome of the bargaining game. In Section 4.3 we test against simulations our bargaining protocol for both classes of equilibria. Lastly, we comment the sequential entry algorithm developed in order to simulate the market with sequential waves of buyers.

4.2 The Model

We consider a market with infinite horizon which evolves in discrete periods $t = 1, 2, \dots$. In every period, a constant in-flow of heterogeneous buyers enters the market and meets sellers by means of sequences of pairwise matches. In any given period $t = 1, 2, \dots$, a buyer entered at t can be matched with any seller available in the market in period t and trade a discrete unit of a homogeneous good ℓ. Sellers and buyers are heterogeneous in the time of market entry and in the endowment of the good $\omega \in \{0, 1, 2, \dots, \bar{\omega}\} \equiv \Omega$. For simplicity we assume that types are distributed according to a rectangular distribution such that $p(\omega) = 1/\bar{\omega}, \forall \omega \in \Omega$. In order to derive a simple trading payoff structure, we will assume that preferences on possessing the good are homogeneous across the agents with decreasing utility in the units held and define $\omega^{**} \in \Omega$ as the first best endowment. For instance, ω^{**} can represent the optimal amount of working capital used by producers in some market and its determination depends on exogenous market price p, production cost c and technology.

In every period t, the *market composition* is described by the *time-invariant* measure of buyers $\boldsymbol{\mu}^B \in [0, 1]^{|\Omega|}$ entering the market at the beginning of every period and by the measure of sellers $\boldsymbol{\mu}_t^S \in [0, 1]^{|\Omega|}$ which are active at period t. In every period t, the measures $(\boldsymbol{\mu}^B,\ \boldsymbol{\mu}_t^S)$ express the proportions of active traders with respect to their endowments. In order to introduce frictions, we assume that trade is against a promise of future repayment, and that for every buyer i a heterogeneous leverage α dependent on the buyer's endowment ω regulates the buyers' budget, so that for every agent i of type ω_i the maximum purchase $\bar{q}_i$ is defined by:

$$\bar{q}_i \leq \alpha \cdot \omega_i$$

Given that we have a single first best endowment ω^{**}, similarly to the type-segmentation emerging in the framework described in Giovannetti [31], every buyer i endowed with ω_i units of capital at time of entry induces the following individual demand q_ω for the good:

$$q_{\omega_i} = max\left\{0,\ min\left\{\bar{q}_{\omega_i},\ \omega^{**} - \omega_i\right\}\right\} \tag{4.1}$$

Which allows us to produce a market segmentation that only depends on the leverage α and the exogenous preferences encapsulated in ω^{**}:

Proposition 4.1 (Market Segmentation)**.** (i) *For every $\omega \in \Omega$, the constraint α and the first best type ω^{**}, the continuous function $q : \Omega \to \Omega$ maps each type in its own* *demand for capital such as:*

$$q_\omega = \begin{cases} \alpha \cdot \omega & \forall \omega \in D^* \\ (\omega^{**} - \omega) & \forall \omega \in D^{**} \\ 0 & \forall \omega \in S \end{cases} \tag{4.2}$$

where $D^* = \{0,\ \omega^{**}/(1+\alpha)\}$ *is defined such as the group of fully-constrained buyers,* $D^{**} = \{\omega^{**}/(1+\alpha), \omega^{**}\}$ *is the segment of non-fully constrained buyers and* $S = \{\omega^{**}, \bar{\omega}\}$ *is the group of pure sellers.*

(ii) The average demand for capital $\mathcal{D}(\alpha, \omega^{**}) = D^* + D^{**}$ *is given by the quantity:*

$$D(\alpha, \omega^{**}) = \frac{\alpha}{1+\alpha} \frac{(\omega^{**})^2}{2} \cdot \frac{1}{\bar{\omega}} \tag{4.3}$$

(iii) $D(\alpha, \omega^{**})$ *is concave with respect to* α *and convex with respect to* ω^{**}*.*

Proof of Proposition 4.1. (i) Define $D^{**} \subset D$ such as the set of *fully-constrained* buyers (buyers for which given α and ω^{**} the constraint holds tightly) and consider $\omega^{**} \in D^{**}$, defined as the endowment such that due to the strictly decreasing returns in possessing the goods, the utility function is such that $u(\omega^{**} + \alpha\omega^{**}) = u(\omega^{**})$. Because $u(\cdot)$ is a bijection, this implies that $\omega^{**} = \omega^{**}/(1+\alpha)$. The construction of q_ω easily follows.

(ii) The result follows from noticing that:

$$\int_0^{\omega^{**}} q_\omega p(\omega) d\omega = \int_0^{\omega^*} q_\omega p(\omega) d\omega + \int_{\omega^*}^{\omega^{**}} q_\omega p(\omega) d\omega$$

That produces:

$$= \frac{1}{\bar{\omega}} \left(\frac{\alpha\,(\omega^{**})^2}{2} + \frac{(\omega^{**})^2}{2} - \frac{(\omega^{**})^2}{1+\alpha} \right)$$

(iii) The result is straightly assessed by checking first and second derivative with respect to the arguments.

■

Information Structure. Producer types $\omega \in \Omega$, as well as $\boldsymbol{\mu}_t^S$ and $\boldsymbol{\mu}^B$ are publicly observed by every active trader at t.

The payoff structure we use in order to characterize the bargaining mechanism at the heart of the formation protocol is a straightforward reflection of our assumption of decreasing returns to the utility of possessing the good. We assume that every matched pair bargains over the price of exchanging one unit of good according to the following surplus function.

Surplus Function and Space of Payoffs. For every ordered pair (i, j), where i is the buyer and j is the seller, the surplus function $s_{ij} : \Omega^2 \to \mathbb{R}$ is the linear mapping defined such as:

$$s_{ij} = max\{a \cdot (\omega_j - \omega_i),\ 0\} \tag{4.4}$$

With a being a positive scalar. For every possible pair (i, j), the surplus function orders the possible matches in terms of the maximum gains of pairwise trades. Albeit simple, the function captures the fact that when the double-coincidence window is determined by decreasing returns technology, the pairwise surplus is monotonically increasing (decreasing) in the type of the seller (buyer). Returns to scales are mapped into a strictly decreasing (increasing) marginal willingness to pay (accept) for obtaining further units of the good. We may think of $s(cdot)$ as stemming from an indirect utility such that the width of the window is idiosyncratic to the exchange and depends from the relative amount of good ω own by i with respect to the amount owned by j at the time in which the bargaining takes place.

For every pair of matched agents i and j at t, define $v^B_{\omega_i t}$ and $v^S_{\omega_j t}$ as elements of the space[1] of the payoffs $\mathcal{V} = \mathcal{V}^B + \mathcal{V}^S$, where $\mathcal{V}^B$ ($\mathcal{V}^S$) is the set of the seller (buyer) types' payoffs such that:

$$\mathcal{V}^S = \left\{(v^S_{\omega t})_{\omega\in\Omega, t\geq 0} | v_{\omega t} \in [0, max_{\omega\in\Omega} s_{\omega\omega'}], \forall \omega' \in \Omega, t \geq 0\right\}$$
$$\mathcal{V}^B = \left\{(v^B_{\omega t})_{\omega\in\Omega, t\geq 0} | v_{\omega' t} \in [0, max_{\omega'\in\Omega} s_{\omega\omega'}], \forall \omega \in \Omega, t \geq 0\right\}$$

Time Preferences. The players discount time according to a homogeneous discount factor $\delta \in (0, 1)$.

Market Evolution. Our market is *non-stationary* and its inter-temporal evolution is dictated by means of the following mechanism, whose elements will be described in detail later in the section.

1. *Pre-Market Stage.* Every agent $i \in N$ discovers her own endowment ω_i and induces a (possibly zero) individual demand q_{ω_i} for the good that depends on preferences and the financial leverage α.

[1] As noted in Manea [54], $\mathcal{V}$ and the other sets defined in this and in the next section can be regarded as topological vector spaces via a natural embedding in the space $\mathbb{R}^{|\Omega|}$ endowed with product topology. Because the product topology in $\mathbb{R}^{|\Omega|}$ is metrizable, the characterizations of closed sets and continuous functions in terms of convergent sequences apply for the sets defined here

2. *Entry Protocol.* Entrance is explicitly regulated by an exogenous mechanism. At the beginning of every period t, a fixed measure of *buyers* $\boldsymbol{\mu}^B$ enters in the market.

3. *Intra-Period Matching.* Every buyer (seller) is randomly matched to a counterparty of type ω with a probability π^S_ω (π^B_ω) which depends on $\boldsymbol{\mu}^S_t$ ($\boldsymbol{\mu}^B$). Each party has a probability p to submit the offer. The receiver's (sender's) outside option is endogenously determined on the basis of π^S_ω (π^B_ω). Either the traders agree on the exchange, in which case the seller transfers one unit of capital to the buyer, or they fail to. In both cases, the match dissolves. The intra-period bargaining phase terminates as soon as every buyer has cleared the (possibly zero) demand for the good.

4. *Inter-Periods Market Evolution* We separately consider two different *exit* protocols which may be put in place at the end of (3), depending on the class of equilibria we are studying. For the class of *Homogeneous equilibria with no entry* we assume that both buyers and sellers who successfully traded the good in (2) will leave the market. For the *Homogeneous equilibria with entry*, we modify the former equilibrium by allowing buyers who end up with the first best endowment ω^{**} at the end of (2) to *settle* in the market that is they become potential sellers for buyers entering at periods $t+1, t+2, ...$ In both equilibria, at the end of (3) the market (the sellers' pool) updates from μ^S_t to μ^S_{t+1}. The new measure encompasses the surviving sellers' budget - as well as for the second equilibria considered - the budget of the buyers who settle at the end of period t. Eventually, the market moves to $t+1$, with a new wave of buyers entering into the market.

Matching Technology. Agents are randomly matched in pairs. We assume that the bargaining takes place within each period t. Every producer encounters a trading partner with probability p and has a probability equal to 1/2 to submit the offer. We characterize the matching via a *soft linear search* (see for instance Gale, [28]) technology. Hence, the probability for the buyer i endowed with ω to meet a seller j with ω' at t and submit an offer is given by:

$$\pi^S_{\omega' t}(\mu^S_{\omega' t}) = \frac{p}{2} \frac{\mu^S_{\omega' t}}{\sum_{k \in \Omega} \mu^S_{kt}}$$

The (time-invariant) probability $\pi^B_\omega(\mu^B_\omega)$ for a seller to find a buyer of type ω is equivalently defined. For simplicity, in the rest of the analysis we will assume $p = 1/2$. Now, define the space $\mathcal{P}$ of the buyers' matching probabilities such as:

$$\mathcal{P} = \left\{ (\pi^S_{\omega t})_{\omega \in \Omega, t \geq 0} | \pi^S_{\omega t} \in [0,1] \, \forall \omega \in \Omega, t \geq 0 \right\}$$

Strategies. We study pure strategies. We restrict our analysis to pure strategies because the inter-period equilibrium depends on the evolution of the market composition, which in turn is determined by the rate of *actual* matches. Given a matched pair (i, j), the producer j submits to i a division $(s_{ij} - x,\ x)$ of the surplus. If the offer is accepted, j obtains $(s_{\omega_j \omega_i} - x)$ and i obtains x. Otherwise, i rejects the offer and the match dissolves. The intra-period game ends when all the buyers collect q_ω. In the intra-period bargaining game we just proposed, buyers (sellers) reservation values will depend on the composition of the sellers (buyers) available in the market at t, since matching probabilities are determined on the basis of $\boldsymbol{\mu}^B$ and $\boldsymbol{\mu}_t^S$.

Solution Concept of the Intra-Period Game. We will follow Manea [54] and restrict the solutions of the game to Subgame Perfect Equilibria (SPE) which are robust in the sense that no agent can affect the equilibrium path $\boldsymbol{\mu}_t^S$ or $\boldsymbol{\mu}^B$ by unilaterally deviating from the prescribed strategy. The intra-period game is solved by means of iterated deletion of dominated strategies. The following Theorem characterizes the strategies which are selected through iterated deletion of dominated strategies, identifies the unique payoff vector for all the types of agents and establishes the existence of the equilibrium in the intra-period game for a general class of dynamic markets which our model belongs to.

Theorem 4.2 (Manea, p. 10, [54])**.** *For every pair $\boldsymbol{\mu}_t^S$ and $\boldsymbol{\mu}^B$, there exists a unique pair of payoff vectors $\left(v_{\omega' t}^{S^*}(\boldsymbol{\mu}_t^S, \boldsymbol{\mu}^B)\right)_{\omega' \in \Omega}$ and $\left(v_{\omega t}^{B^*}(\boldsymbol{\mu}_t^S, \boldsymbol{\mu}^B)\right)_{\omega \in \Omega}$ such that:*

(i) The only date t actions which would survive iterated dominance specify that all buyers (sellers) of type ω reject any offer $x < \delta v_\omega^{B^*}(\boldsymbol{\mu}_t^S, \boldsymbol{\mu}^B)$ (respectively, $\delta v_{\omega'(t+1)}^{S^*}(\boldsymbol{\mu}_t^S, \boldsymbol{\mu}^B)$) and accept any offer $x > \delta v_\omega^{B^*}(\boldsymbol{\mu}_t^S, \boldsymbol{\mu}^B)$ (respectively, $\delta v_{\omega'(t+1)}^{S^*}(\boldsymbol{\mu}_t^S, \boldsymbol{\mu}^B)$).

(ii) In every equilibrium, the expected payoff of any active buyer (seller) of type ω at time t is given by $v_{\omega t}^{B^*}(\boldsymbol{\mu}_t^S, \boldsymbol{\mu}^B))$ (respectively, $v_{\omega' t}^{S^*}(\boldsymbol{\mu}_t^S, \boldsymbol{\mu}^B))$.

(iii) The equilibrium payoffs $v^{S^*}(\boldsymbol{\mu}_t^S, \boldsymbol{\mu}^B), v^{B^*}(\boldsymbol{\mu}_t^S, \boldsymbol{\mu}^B)$ constitute the unique bounded solution to the system of equations:

$$v_{\omega t}^{B^*}(\boldsymbol{\mu}_t^S, \boldsymbol{\mu}^B) = \sum_{\omega' \in \Omega} \pi_{\omega' t}^S(\mu_{\omega' t}^S) \left(s_{\omega\omega'} - \delta v_{\omega'(t+1)}^{S^*}\right) + \left(1 - \sum_{\omega' \in \Omega} \pi_{\omega' t}^S(\mu_{\omega' t}^S)\right) \delta v_{\omega t}^{B^*}(\boldsymbol{\mu}_t^S, \boldsymbol{\mu}^B) \tag{4.5}$$

$$v_{\omega t}^{S^*}(\boldsymbol{\mu}_t^S, \boldsymbol{\mu}^B) = \sum_{\omega' \in \Omega} \pi_{\omega'}^B(\mu_{\omega'}^B) \left(s_{\omega\omega'} - \delta v_{\omega'}^{B^*}\right) + \left(1 - \sum_{\omega' \in \Omega} \pi_{\omega'}^S(\mu_{\omega'}^B)\right) \delta v_{\omega(t+1)}^{S^*}(\boldsymbol{\mu}_t^S, \boldsymbol{\mu}^B) \tag{4.6}$$

Such that $v_{\omega t}^{B^*}$ (respectively, $v_{\omega t}^{S^*}$) represents the payoff of all the buyers (the sellers) of type $\omega \in \Omega$ that enter (that are) in the market in period $t \geq \mathbf{0}$.

(iv) An equilibrium exists.

(v) For every type $\omega \in \Omega$, the payoffs $(v^{S^*}_{\omega'}(\cdot))_{\omega' \in \Omega}$, $(v^{B^*}_{\omega}(\cdot))_{\omega \in \Omega}$ vary continuously in $\boldsymbol{\mu}^S_t, \boldsymbol{\mu}^B$

Proof of Theorem 4.2. $(i) - (v)$ is the second fundamental result of Manea (Theorem 2, [54]). We refine points (iii) and (iv). (i) is the standard outcome of the application of Subgame Perfection to bargaining models. Under Subgame Perfection, for every *type*, the expected payoff is bounded between two sequences $(m^k_{\omega t})_{\omega \in \Omega}$ and $(M^k_{\omega t})_{\omega \in \Omega}$ which converge to a common limit $(\delta v^*_{\omega t+1})_{\omega \in \Omega}$. (ii) Establishes that matched and non-matched *players* of the same type achieve the same expected payoff at the beginning of each round regardless to their matching status and it is a straight consequence of the no one-shot deviation principle. (iii) Given the sequential entry assumption we imposed upon the generic framework of Manea, we refine Manea's system of equation in (iii), Theorem 2 of [54] by adopting a stationary *within-period* payoff structure for the buyers. Buyers entering at t will trade only with the measure of sellers which is available at $t, t \geq 0$. Therefore, the matching probability $\pi^S_{\omega' t}(\mu^S_{\omega' t})$ which determines the buyers' within-period outside option *is fixed* in every period. Consequently: within every period t, $v^{B^*}_{\omega' t} = v^{B^*}_{\omega'(t+1)} \forall \omega$ and ω' in Ω. On the other hand, sellers' expected payoff $v^{S^*}_{\omega t}$ evolves *across* periods. ■

The equilibrium payoffs of both $v^{B^*}_{\omega t}$ and $v^{S^*}_{\omega t}$ obtained in Theorem 4.2 deviates from the characterization used in Manea for $v_{\omega t}$ with respect to the fact that the sole group of sellers has explicitly time-varying payoffs. For clarification, let us consider the payoff of any buyer k endowed with ω units of capital entering the market at some period t. As showed by Manea, the expected equilibrium payoff is the same for matched and unmatched agents. Therefore, the identifier k is dropped from the equation. For any buyer endowed with ω and participating to the market at t, the equilibrium element $v^{B^*}_{\omega t} \in \mathcal{V}^B$ is the period expected payoff computed on three possible matching outcomes. The left prospect identifies the expected payoff of matching with a counterpart and be the offer maker. In this case, the agent proposes the counterpart her outside option. The second prospect captures the two remaining cases, that is either the agent is matched to a counterpart as an offer receiver or no match takes place. In either case, the agent's payoff is her continuation value. We stress that buyers' equilibrium payoffs are time-invariant. Sellers equilibrium payoff can be similarly described, with the major difference that sellers' continuation value depends on next period market population vector.

As we noted above, the evolution of our market is dictated by the market composition $\boldsymbol{\mu}^S_t$. In fact, Theorem 4.2 establishes that $\boldsymbol{\mu}^S_t$ is the relevant measure which pins down the structure of the expected payoffs via the matching probability $\pi^S_{\omega' t}(\mu^S_{\omega' t}), \forall \omega \in \Omega$. The Theorem guarantees also that the economy is well defined along all the possible paths of the market composition and that sellers (buyers) of same type have equal payoff. In order to introduce an explicit law of motion for $\boldsymbol{\mu}^S_t$, we must define the structure of agreements which take place between buyers and sellers in every period t.

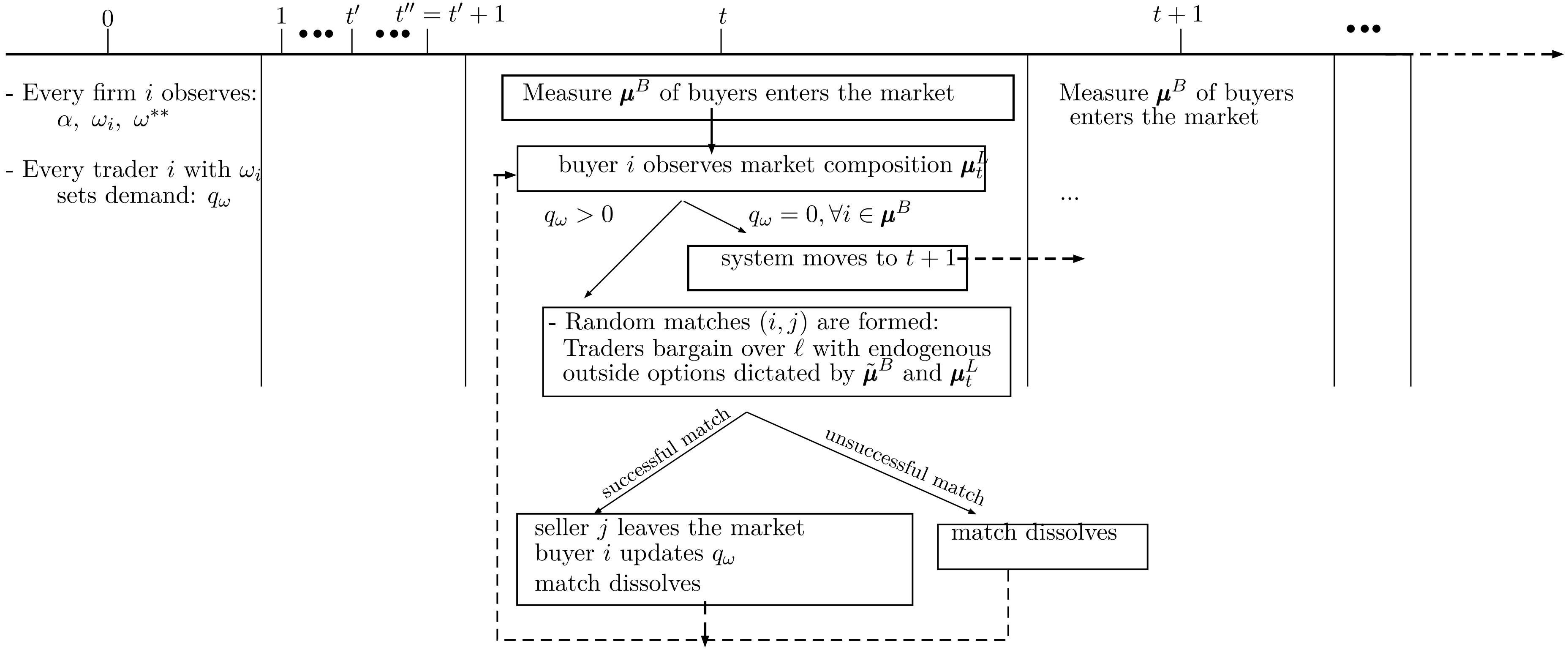

Figure 4.1: The model time-line with the inter-temporal and intra-temporal bargaining process is compatible with both the classes of homogeneous equilibria (with entry and with no entry) we introduce in the main text: the formation protocol in the figure does not specify whether at the end of every time period $t = \{1, 2, ...\}$ the measure of buyers $\boldsymbol{\mu}^B$ settles in the market.

Agreements. Since agents of the same class and endowment have equal payoff, we will identify each player i endowed with ω_i directly by means of her type ω_i. Hence, given a matched pair of types (ω, ω'), with slight abuse of notation we define $\alpha_{\omega\omega' t}$ as the fraction of types ω and ω' that reach an agreement in period t such that ω is the proposer, ω' is the receiver, with $\omega' > \omega$ being the seller. In the correspondence we do not need to specify who is the buyer and who is the seller, as from the Individual Rationality condition obtained from the characterization of the payoff function, α may take positive values only for matches in which the seller's type is higher than the buyer's one. The fraction of agreements may be 0, 1 or a value in $[0, 1]$ depending on $s_{\omega\omega'}$ and each type's continuation value. Therefore, we construct the correspondence $\alpha_{\omega\omega' t}$ such as:

$$\alpha_{\omega\omega' t} = \begin{cases} 0 & \delta v_{\omega}^{B} + \delta v_{\omega'}^{S} > s_{\omega\omega'} \\ [0,1] & \delta v_{\omega}^{B} + \delta v_{\omega'}^{S} = s_{\omega\omega'} \\ 1 & \delta v_{\omega}^{B} + \delta v_{\omega'}^{S} < s_{\omega\omega'} \end{cases} \tag{4.7}$$

Across periods, the space of path of agreement rates is defined as follows:

$$\mathcal{A} = \left\{ (\alpha_{\omega\omega' t)\omega,\omega' \in \Omega, t \geq 0} \middle| \alpha_{\omega\omega' t} \in [0,1]\,, \forall \omega, \omega' \in \Omega, t \geq 0 \right\}$$

Evolution of Market Composition. Finally, we construct the general explicit law of motion for the market composition which we will specialize further in the next section according to the equilibrium classes we will propose. First, on the ground of the definition of q_ω, let us build the measure of *settlements* $\hat{\boldsymbol{\mu}}^B$ such that, given the initial endowment $\omega' \in \Omega$ possessed by a buyer when she enters the market:

$$\hat{\mu}_{\omega'}^{B}(\alpha, \omega^{**}) = \sum_{\omega=0}^{\bar{\omega}} \mathbb{I}\left[\omega + q_{\omega}(\alpha, \omega^{**}) = \omega'\right] \cdot \mu_{\omega}^{B}$$

In which $\mathbb{I}$ is the indicator function and takes value 1 when the condition in square brackets is matched and zero otherwise. The measure of settlements encapsulates the end-period endowments of buyers and depends only on preferences and financial constraint. The measure will be used in the next section for characterizing the market's evolution for the Homogeneous equilibrium with Entry. Now, for every seller type $\omega' \in \Omega$, given an initial measure $\mu_{\omega' 0}$ of potential sellers of type ω' available in the market at period $t = 0$, the in-flow and out-flow of agents endowed with ω' is dictated by a difference equation modeled according to the specific equilibrium under consideration. In general terms, the law of motion is given by:

$$\mu^S_{\omega'(t+1)} = \phi\left(\boldsymbol{\mu}^S_t\right) \tag{4.8}$$

In which $\phi(\cdot)$ will be characterized compatibly with the equilibrium we are after. Overall, the space of the market composition is given by:

$$\mathcal{M} = \left\{(\mu^S_{\omega t})_{\omega\in\Omega, t\geq 0} | \mu^S_{\omega' t} \in [0,1]\,, \forall\omega\in\Omega\right\}$$

The characterization of the spaces $\mathcal{A}, \mathcal{M}, \mathcal{P}, \mathcal{V}$ which we defined along this Section and their relation are compatible with the general non-stationary market structure introduced by Manea [54]. Following Manea, we construct the correspondence $f : \mathcal{A} \rightrightarrows \mathcal{A}$ by composing the correspondence α and the functions v^S, π^S, μ^S, so that:

$$\mathcal{A} \overset{\mu^S}{\to} \mathcal{M} \overset{\pi^S}{\to} \mathcal{P} \overset{v}{\to} \mathcal{V} \overset{\alpha}{\rightrightarrows} \mathcal{A}$$

And use the following result:

Theorem 4.3 (Manea, p.8, Manea13)**.** *An equilibrium exists for the Bargaining Game.*

Proof of Theorem 4.3*.* See Theorem 1, [54] ∎

For every period t, Theorem 4.2 and Theorem 4.3 support the formation of the dynamic market as the equilibrium of an infinite horizon bargaining game. Moreover, the formation process is essentially pinned down to the evolution of the market composition $\boldsymbol{\mu}^S_t$, which determines the matching probabilities, the structure of payoffs and the agreements. In the next section we will restrict our focus to two specific classes of equilibria which will allow us to single out the role of the financial frictions in shaping the market.

4.2.1 No Seller Dispersion Equilibria

In the previous Section we defined the theoretical framework of our model. We may now proceed to characterize and prove existence of two classes of equilibria which sustain the market formation in presence of financial frictions. We propose two different formation mechanisms (and two related classes of equilibria). In the first class, traders will leave the market upon successful trade. In the second class, only sellers will leave the market, while certain buyers will *settle* in it.

Construction of cumulative Buyer In-Flow. The intra-period protocol we described above incurs in two sources of complexity. First, it implicitly assumes a further temporal dimension along which the sequences of bargaining take place. Furthermore, because for any of the sub-period matches the pairwise surplus is a function of both the seller's and the buyer's type at the time of match, a buyer's payoff would depend on the entire sequence of trades that may take place within the period and a seller's payoff would depend on sequences of trades within periods and across periods. By exploiting the fact that the good is traded in fixed units, we overcome this problem by decomposing each individual buyer and seller in *collections* of types which identify the various stages of good collection (for buyers) and sale (for sellers) and study the payoff for each *type*. The focus on type-matching allows us to unwind the intra-period bargaining process in *one* single match between two measures that are directly derived from the actual populations of types.

Let us focus on the demand side. We notice that given the segmentation induced by q_ω, in every period t a seller j can meet the *same* buyer i at various stages of the good collection. Precisely, a buyer i initially endowed with ω_i units of the good can be encountered under $|(q_i + \omega_i) - \omega_i + 1|$ different types. Therefore, the arrival of any buyer is equivalent to the arrival of a measure of types. We build the measure $\tilde{\boldsymbol{\mu}}^B(q)$ as follows:

$$\tilde{\mu}^B_\omega(q_\omega) = \mu^B_\omega + \sum_{\omega' \in D} \tilde{\mu}_{\omega'} \mathbb{I}\left[max\left\{0, min\left\{\omega^{**}, \omega'(1+\alpha)\right\}\right\} > \omega > \omega'\right] \tag{4.9}$$

The matching probability $\tilde{\pi}^B_\omega(\tilde{\mu}^B_\omega)$ is defined accordingly. Given (4.9), we also introduce the following result, which we will use in Theorem 4.8.

Proposition 4.4. *For every type $\omega \in \Omega$ and the cumulative buyer measure $\tilde{\mu}^B_\omega(q_\omega)$ defined above, the following properties hold:*

(i) Given a buyer's type ω and two measures $\tilde{\mu}^B_\omega(\alpha')$, $\tilde{\mu}^B_\omega(\alpha'')$ with $\alpha'' > \alpha'$, $\tilde{\mu}^B_\omega(\alpha')$ First Order Stochastically Dominates $\tilde{\mu}^B_\omega(\alpha'')$ or, equivalently, $\tilde{\mu}^B_\omega(\alpha') \succ_{FO} \tilde{\mu}^B_\omega(\alpha'')$. Moreover, it holds that:

$$\tilde{\mu}^B_\omega(q_\omega) = \frac{1}{\bar{\omega}} \cdot \left[1 + \frac{\alpha}{1+\alpha}\omega\right] \tag{4.10}$$

From which we may compute the following associated items which will be used in Theorem 4.8

(ii) $\sum_{\omega=0}^{\omega^{**}} \tilde{\pi}^B_\omega = \dfrac{(\omega^{**}+1)(2+(\alpha(2+\omega^{**})))}{2\bar{\omega}(1+\alpha)}$

(iii) $\sum_{\omega=0}^{\omega^{**}} s_{\omega\omega^{**}} \tilde{\pi}^B_{\omega^{**}} \geq \dfrac{a\omega^{**}(1+\omega^{**})(3+\alpha(2+\omega^{**}))}{6\bar{\omega}(1+\alpha)} = a \cdot \sigma$

Such that $\tilde{\pi}^B_\omega$ is the probability for a seller to be matched with a buyer and $a \cdot \sigma$ is the expected surplus generated by matches with a ω^{**}-type seller as proponent.

Proof of Proposition 4.4. (i) We prove the latter part of the statement from which it is immediate to recover the former one. For every type ω, the sum component of $\tilde{\mu}_\omega^B$ includes all the types $\omega' \in \Omega$ such that $\omega' \leq \omega$ and $\omega'(1+\alpha) \geq \omega$. By rewriting the second condition such as $\omega/(1+\alpha) - \omega' \leq 0$ we notice that if $\omega' \leq \omega$ and $\omega/(1+\alpha) - \omega' \leq 0$ hold, it must be the case that $\omega/(1+\alpha) - \omega \leq 0$ holds as well. Therefore, by using this last equation, we obtain that:

$$\tilde{\mu}_\omega^B(q_\omega) = \frac{1 + \alpha(1+\omega)}{\bar{\omega}(1+\alpha)}$$

From which the statement follows.

$(ii) - (iii)$ The results follow from a straight application of the simple series $\sum_x = 1 + 2 + ... = (n(n+1))/2$ and the definition of $s_{\omega\omega^{**}}$.

■

In order to elucidate the idea behind the measure constructed in point (i) of Proposition 4.4, we produce the following example. Let consider an environment such that $\omega \in \{1, 2, ..., 20\}$ and assume $\omega^{**} = 18$, and $\alpha = 2$. Let us assume we want to assess how many times type $\omega = 12$ will be expressed in the population of buyers across all the transactions taking place in one single period. That is we want to predict the size of the set $\tilde{\Omega}_{12} \equiv \{\omega \in \Omega : \omega \leq 12, q_\omega \geq 12\}$. According to the formula given in (i), we predict $|\tilde{\Omega}_{12}| = 9$ types will be switching at some point to type $\omega = 12$. In fact, for $\alpha = 2$ and $\omega^{**} = 18$, it is easy to see that all the types $\omega \in \{4, 5, 6, 7, 8, 9, 10, 11, 12\} \equiv \tilde{\Omega}$ will "cross" type $\omega = 12$ at some stage of the good collection. For instance, consider agents endowed with $\omega \in \{4, 10\}$. Agents with type $\omega = 4$ are allowed to collect up to 8 units of the good and will end up with $4+8 = 12$ units of the good. In the same fashion, agents with type $\omega = 10$ can collect up to 20 units of the good. However, given that $\omega^{**} = 18$, they will collect only $18 - 10 = 8$ units of the good. The same rationale applies to the remaining types in $\tilde{\Omega}$.

Now, we are interested into characterizing and proving existence for two tractable classes of equilibria which may sustain the formation of the dynamic market when a financial constraint $\alpha > 0$ is in place. Both characterization and existence for these classes of equilibria is provided. First, we define the following measure:

Definition 4.5. *Define the (time-invariant) relative measure of first best endowed buyers settling at the end of every period:*

$$\Delta = \frac{\omega^{**} - \omega^*}{\bar{\omega}} = \tilde{\mu}_\omega^B(q_{\omega^{**}}) = \frac{1}{\bar{\omega}} \cdot \left[1 + \frac{\alpha}{1+\alpha}\omega^{**}\right]$$

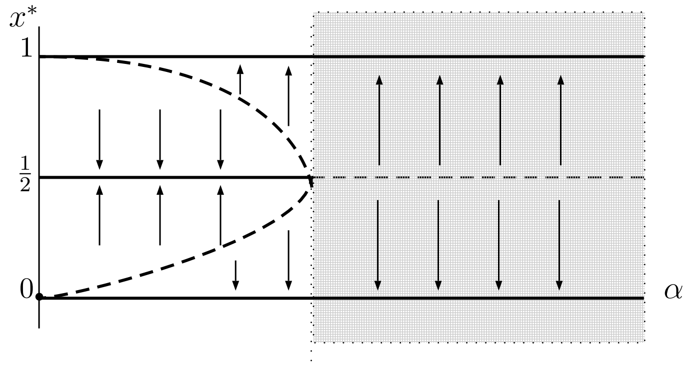

Figure 4.2: Bifurcation Diagram for the Equilibrium population index x. With no financial leverage ($\alpha = 0$), the system is at rest for any point $x \in [0, 1]$. For $\alpha \in (0, \alpha^*)$ the dotted curves identify the unstable equilibria which separate the three possible dynamical outcomes of the system. The unique unbounded stable equilibrium $x = 1/2$ is reached for all the orbits initiated within the dotted lines. Rest point $x = 0$ ($x = 1$) is a stable fixed point by construction and it is reached by orbits originated outside the dotted line between 1/2 and 1 (respectively, between 0 and 1/2). When $\alpha > \alpha^*$ the equilibrium point $x = 1/2$ loses stability and the system converges to $x = 1$ (respectively, to $x = 0$) for any x_0 above (below) 1/2. The dynamics are coherent with a pitchfork bifurcation.

In which ω^* is as defined in the proof of Proposition 4.1.

Lastly, in order to track down the evolution of ω^{**}-sellers against the aggregate of the remaining types, we also define the *market index x* as follows:

Definition 4.6. *Given any ω-type seller, $\omega \in \Omega$ and the measure $\mu^S_{\omega t}$, a market index x is defined such as:*

$$x = \frac{\mu^S_{\omega^{**}t}}{\mu^S_{\omega^{**}t} + \sum_{\omega \neq \omega^{**}} \mu^S_{\omega t}} = \mu^S_{\omega^{**}t} \tag{4.11}$$

We will use the market index x in order to construct a one-step-ahead iteration operator which will allow us to determine the market evolution. We can now provide the characterization of the equilibria we are after. We begin with the no-entry context, in which both the seller and the buyer leave the market upon successful trade.

Definition 4.7 (**Homogeneous Equilibria with no entry**). *Given a dynamic market as described by some leverage $\alpha \geq 0$, a first-best type $\omega^{**} \in \Omega$, an initial market composition $\boldsymbol{\mu}^S_0$, a buyer inflow given by $\boldsymbol{\mu}^B$, an average demand $\mathcal{D}(\omega^{**}, \alpha)$, the matching probabilities $\pi^S_t, \pi^B, t \geq 0$, the payoffs $(v^S_{\omega t})_{\omega\in\Omega}, (v^B_{\omega})_{\omega\in\Omega}$ and the laws of motion in* (4.8)*, we define Homogeneous Equilibria with no entry the class of equilibria of the bargaining game such that:*

(i) Asymptotically, every buyer type $\omega \in \mathcal{D}$ with $q_\omega > 0$ purchases the good from sellers of type ω^{**}.

(ii) There is no seller-type dispersion: $\lim_{t\to\infty} \boldsymbol{\mu}_t^S = (\bar{\mu}_\omega^S)_{\omega\in\Omega}$ *where* $\bar{\mu}_\omega^S = 0 \ \forall \omega \in \Omega \setminus \{\omega^{**}\}$ *and* $\bar{\mu}_\omega^S = 1$ *for* $\omega = \omega^{**}$.

And state our first main result result:

Theorem 4.8. *For any dynamic market with frictions fulfilling the following conditions:*

(i) Every seller j quits the market upon successful trade. Every buyer i quits the market upon satisfaction of her demand q_{ω_i}.

(ii) $\mu_{\omega^{**}0} \in \left(\frac{1}{2} \cdot \left(1 + \sqrt{1-4D^2}\right),\ 1\right]$ *for* $\alpha < \alpha^*$

$\mu_{\omega^{**}0} > \frac{1}{2}$ *for* $\alpha \geq \alpha^*$ *where* $\alpha^* = \frac{\bar{\omega}/\omega^{**}}{\omega^{**}-\bar{\omega}/\omega^{**}}$

(iii) $\delta \leq \frac{7}{4} - \frac{\sqrt{17}}{4}$

The market converges to a Homogeneous Equilibrium with no entry.

Proof of Theorem 4.8. We split the proof of the Theorem in two steps. In the first step, we focus on the dynamics involved in the market evolution in order to find the conditions that allow for the equilibria we are studying. In the second step, we discuss the conditions such that the Homogeneous equilibrium with no entry is Incentive Compatible.

Step 1. Let us study the details of the market evolution. We revisit the population law of motion for sellers of type ω^{**} in light of the equilibrium we are after. From Equation (4.8), we write:

$$\mu^S_{\omega^{**}(t+1)} = \mu^S_{\omega^{**}t} - \underbrace{\sum_{\{\omega\in\Omega|\omega\leq\omega^{**}\}} \left(\alpha_{\omega\omega^{**}t}\pi^S_{\omega^{**}t}\mu^B_\omega + \alpha_{\omega^{**}\omega t}\pi^B_\omega\mu^S_{\omega^{**}t}\right)}_{type\ \omega'\ out-flow\ (switching\ to\ type\ \omega'-1)} \tag{4.12}$$

The out-flow of ω^{**}-type sellers is due to the proportion of sellers of type ω^{**} who agreed on lending capital at t either as recipients of an offer or as proponents. Notice that due to the decreasing returns in holding the good, sellers with ω' can only exchange with buyers of lower types. From Proposition 4.1 we know that the measure of buyers ending up with ω^{**} at $t, t \geq 0$ in equilibrium is given by:

$$\hat{\mu}^B_{\omega^{**}} = \Delta = \frac{1}{\bar{\omega}} \cdot \left[1 + \frac{\alpha}{1+\alpha}\omega^{**}\right]$$

Combining the definitions of $\pi^S_{\omega^{**}}, D$ and the assumption that every agent has $p = 1/2$ probability of being active in every trade we may approximate the net flow of population of ω^{**} sellers as follows:

$$\approx - \underbrace{\frac{2}{4} \cdot \left(\frac{\mu^S_{\omega^{**}t} - D}{\mu^S_{\omega^{**}t} + \sum_{\omega \neq \omega^{**}} \mu^S_{\omega t}} \right)}_{probability\ of\ \omega^{**}-type\ sellers\ to\ trade} \times D \times \underbrace{\left(\mu^S_{\omega^{**}t} - D \right)}_{net\ mass\ of\ type\ \omega^{**}\ sellers\ at\ t+1}$$

In fact, within every period t, on average at most a measure D of sellers of every class will be matched with buyers. This leads to:

$$\mu^S_{\omega^{**}(t+1)} = \mu^S_{\omega^{**}t} - \frac{1}{2} \cdot D \cdot \left(\mu^S_{\omega^{**}t} - D \right)^2$$

In order to study the evolution of ω^{**}-type sellers as opposed to other seller-types, let us adopt the *market index* x we defined above. As $1 - x$ represents the evolution of the seller types $\omega \neq \omega^{**}$, we may keep track of the between-period change of the relative size of ω^{**}-type sellers by means of the following recursion:

$$\frac{\mu^S_{\omega^{**}(t+1)}}{\mu^S_{\omega^{**}(t+1)} + \sum_{\omega \neq \omega^{**}} \mu^S_{\omega(t+1)}} = \tau(x) = \frac{x - \frac{1}{2}(x - D)^2 \cdot D}{x - \frac{1}{2}(x - D)^2 \cdot D + \left[(1 - x) - \frac{1}{2}(1 - x - D)^2 \cdot D \right]} \tag{4.13}$$

The mapping $\tau(x)$ tracks the market evolution when exchanges are as conceived in the equilibrium definition. We proceed as follows. First, we disregard the fact that x must be bounded in the $[0, 1]$ interval and derive the (unconstrained) rest points of the system. Second, we re-introduce the bound in order to refine the properties of the system, trying to isolate the specific effect of the financial constraints on the qualitative behaviour of our economy's orbit. We impose:

$$\tau(x) = x$$

By exploiting the fact that $D = (\Delta - (1/\bar{\omega})) \cdot (\omega^{**}/2)$, after computation we obtain the index fixed points from the following expression:

$$\left(\Delta - \frac{1}{\bar{\omega}} \right) \cdot \frac{\omega^{**}}{2} \cdot (2x - 1) \cdot (D^2 + x(x - 1)) = 0$$

Which delivers the following (unconstrained) rest points:

$$x_0^* = \frac{1}{2} \qquad x_+^* = \frac{1}{2}\left(1+\sqrt{1-4D^2}\right) \qquad x_-^* = \frac{1}{2}\left(1-\sqrt{1-4D^2}\right) \tag{4.14}$$

We are interested in the stability of x_0^*. By differentiating (4.13) with respect to x at point $x = 1/2$, we find that:

$$\frac{d\tau(x)}{x}\Big|_{x=x_0^*} = \frac{2(4-D+2D^2)(2+D^2-D^3-\frac{1}{4}D)}{(D-4-2D^2+2D^3-\frac{1}{2}D)^2}$$

The stability of x_0^* requires the following two conditions to hold jointly:

$$\begin{aligned} \frac{d\tau(x)}{x}\Big|_{x=x_0^*} &< 1 \rightarrow D\cdot(D-8+28D^2+16D^4-16D^5) \leq 0 \\ \frac{d\tau(x)}{x}\Big|_{x=x_0^*} &> -1 \rightarrow 128-40D+131D^2-116D^3+48D^4-48D^5+16D^6 \geq 0 \end{aligned} \tag{4.15}$$

It is immediate to see that the first equation in (4.15) satisfies the inequality in the interval $[0,1]$. From differentiation, we find that the second equation is increasing in the interval $[0,1]$, and it breaks the inequality for $D > \bar{D}$, with $\bar{D} = 1/2$. This verifies that at $D = \bar{D}$, the equilibrium point x_0^* loses stability. Now, let us re-introduce the bound $x \in [0,1]$ and focus on the behaviour of the system in the neighborhood of $\underline{x} = 0$ and $\bar{x} = 1$. In particular, we will evaluate the iterations near $\tau(\bar{x})$, which is the relevant bound for our class of equilibria. By substitution we have that:

$$\tau(\bar{x})|_{x=1} = \frac{1-\frac{1}{2}(1-D)^2\cdot D}{1-\frac{1}{2}(1-D)^2D-\frac{1}{2}(-D)^2\dot{D}}$$

It is easy to see that for $D \in [0,\ 1/2]$ the index function is increasing at $\bar{x}$, hence $\bar{x}$ is a stable fixed point by construction. Because x_+^* for $D < 1/2$ is bounded between the two stable equilibria $x^* = 1/2$ and $\bar{x}$, it must be that x_+^* is unstable. Therefore, for $D < 1/2$, the type ω^{**} index orbit will converge to a homogeneous equilibrium whenever $\mu^S_{\omega^{**}0} \in \left(x_+^*,\ \bar{x}\right]$. On the other hand, for $D > 1/2$, the equilibrium $x = 1/2$ loses stability and the remaining two unbounded equilibria become complex. Consequently, for $D > 1/2$ the surviving stable equilibria are $\bar{x}$ and $\underline{x}$ and the ω^{**}-type index will follow the equilibrium path whenever $\mu^S_{\omega^{**}0} > 1/2$. A symmetric argument can be used to show that for an initial index $\mu^S_{\omega^{**}0} < 1/2$ the system would converge to a no ω^{**}-type equilibrium. Now, we may re-express the above argument in terms of the relation between the leverage α and the types' proportion as follows:

$$D \leq \bar{D} \ \rightarrow \ \left(\Delta - \frac{1}{\bar{\omega}}\right)\cdot\frac{(\omega^{**})^2}{2} \leq \frac{1}{2}$$

$$\frac{1}{\bar{\omega}}\left[1+\frac{\alpha}{1+\alpha}\cdot\omega^{**}\right]\leq\frac{1}{2}+\frac{1}{\bar{\omega}}$$

From which we can retrieve $\alpha = \alpha^*$ as stated in the Theorem's body.

Step 2. We now derive the conditions which allow the above dynamics to be incentive compatible. From Theorem 4.2, for every period $t \geq 0$ and buyer $\omega \in \mathcal{D}$, the system of payoffs reads:

$$\begin{aligned} v^B_\omega &= \pi^S_{\omega^{**}t}\left(s_{\omega\omega^{**}} - \delta v^S_{\omega^{**}(t+1)}\right) + \left(1-\pi^S_{\omega^{**}t}\right)\delta v^B_\omega \\ v^S_{\omega^{**}t} &= \sum_{\omega'\in\mathcal{D}}\tilde{\pi}^B_{\omega'}\left(s_{\omega\omega^{**}} - \delta v^B_{\omega'}\right) + \left(1-\sum_{\omega'\in\mathcal{D}}\tilde{\pi}^B_{\omega'}\right)\delta v^S_{\omega^{**}(t+1)} \end{aligned} \tag{4.16}$$

In which we suppressed the time notation in the buyer's payoff as we will work out an incentive-compatible condition which holds for any value of the index x along the inter-period market evolution. As we noted along the main text, within every period t, the payoff of buyers entering at t is stationary due to the entry protocol we imposed (the system moves to $t+1$ only after every buyer has cleared her own demand) and completely depends upon the market index currently in place. We rewrite the buyers' payoff by accounting for the results stated in Proposition 4.4 and Step 1 of the current Theorem:

$$v^B_\omega(x) = \pi^S_{\omega^{**}}\frac{a(\omega^{**}-\omega)-\delta v^S_{\omega^{**}(t+1)}}{1-\delta(1-\pi^S_{\omega^{**}t})} = \frac{\tau(x)^t}{4}\frac{a(\omega^{**}-\omega)-\delta v^S_{\omega^{**}(t+1)}}{1-\delta(1-\frac{\tau(x)^t}{4})}$$

By substituting v^B_ω in $v^S_{\omega^{**}t}$ we find that:

$$v^S_{\omega^{**}t}(x) = a\sigma - \delta\cdot\frac{\frac{\tau^t(x)}{4}}{1-\delta(1-\frac{\tau^t(x)}{4})}\left(\sum_{\omega\in\mathcal{D}}\tilde{\pi}^B a(\omega^{**}-\omega) - \delta v^S_{\omega^{**}(t+1)}\right) + \left(1-\left(3\sigma - \underbrace{\frac{\omega^{**}+1}{2\bar{\omega}(1+\alpha)}}_{\rho}\right)\right)\delta v^S_{\omega^{**}(}$$

Let us define $\gamma(x)$ such as:

$$\gamma(x) = \frac{\frac{\tau^t(x)}{4}}{1-\delta(1-\frac{\tau^t(x)}{4})}$$

Then, after having applied the result in point (iii) from Proposition 4.4 above, the system of payoffs reads:

$$\begin{aligned} v^B_{\omega^{**}t} &= \gamma(x)\left(a(\omega^{**}-\omega) - \delta v^S_{\omega^{**}(t+1)}\right) \\ v^S_{\omega^{**}t} &= a\sigma(1-\delta\gamma(x)) + (1+\delta\gamma(x) - 3\sigma + \rho)\delta v^S_{\omega^{**}(t+1)} \end{aligned} \tag{4.17}$$

The one-step ahead expansion of $v^S_{\omega^{**}t}$ gives us:

$$\begin{aligned} v^S_{\omega^{**}t} = a\sigma - a\sigma\delta\gamma^t(x) + \delta(a\sigma - a\sigma\delta\gamma^{t+1}(x))(1+\delta\gamma^t - 3\sigma + \rho) + \\ + \delta^2 v^S_{\omega^{**}(t+2)}(1+\delta\gamma^{t+1}(x) - 3\sigma + \rho)(1+\delta\gamma^t(x) - 3\sigma + \rho) \end{aligned}$$

By further expanding the series and discarding the decaying component, we obtain that:

$$\begin{aligned} v^S_{\omega^{**}t} = a\sigma\left(-1 + \delta\gamma^t(x) - \delta(-1+\delta\gamma^{t+1})(1+\delta\gamma^t(x) - 3\sigma + \rho) - \right. \\ - \delta^2(\delta\gamma^{t+2} - 1)(1+\delta\gamma^{t+1}(x) - 3\sigma + \rho)(1+\delta\gamma^t(x) - 3\sigma + \rho) \end{aligned}$$

Which leads to:

$$v^S_{\omega^{**}t} = a\sigma(1 - \sum_{t\geq 0}\delta^t\left(1+\delta x - 3\sigma + \rho\right)\left(1+\delta\gamma(x) - 3\sigma + \rho\right)\cdot \ldots \cdot(1+\delta\gamma^{t-1} - 3\sigma + \rho)(\delta\gamma^t - 1))$$

We rewrite the above equation:

$$v^S_{\omega^{**}t} = a\sigma(1 + \sum_{t\geq 0}\delta^t\left(1+\delta x - 3\sigma + \rho\right)\left(1+\delta\gamma(x) - 3\sigma + \rho\right)\cdot \ldots \cdot(1+\delta\gamma(x)^{t-1} - 3\sigma + \rho)(1-\delta\gamma^t))$$

From the definition given above, we know that γ can move along the following interval:

$$\bar{\gamma} \in \left[0, \frac{\frac{1}{4}}{1-\delta\left(\frac{3}{4}\right)}\right]$$

Now, we bound the series by fixing $\gamma(x)^t = \bar{\gamma}, \forall t \geq 0$ and explore the incentive-compatible index orbits. Therefore, we rewrite the above equation:

$$v^S_{\omega^{**}t} \leq a\sigma\left(1 + (1-\delta)\sum_{t\geq 0}\delta^t(1+\delta\bar{\gamma} - 3\sigma + \rho)^t\right) \tag{4.18}$$

Then, a sufficient condition for the series in (4.18) to converge is that:

$$\bar{\gamma}\delta^2 - \delta(3\sigma - \rho - 1) \leq 1$$

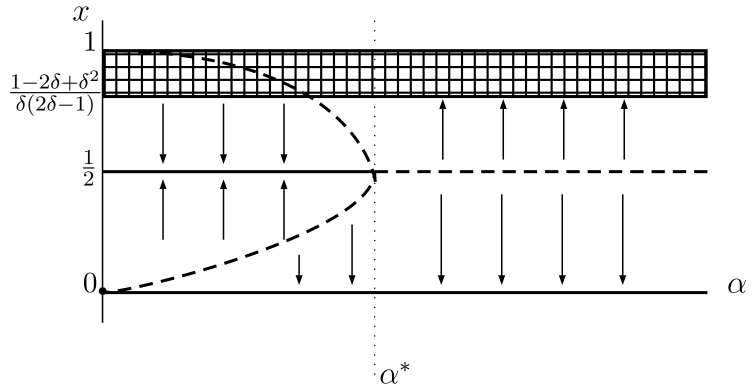

Figure 4.3: The space of incentive compatible orbits is given the area between 0 and the lower bound of the grid.

We rewrite:

$$\bar{\gamma}\delta^2 - \delta\left(\frac{(\omega^{**}+1)(\omega^{**}(3+\alpha(2+\omega^{**})-1)-1)}{2\bar{\omega}(1+\alpha)} - 1\right) - 1 \leq 0$$

For $\bar{\omega}$ large enough, we obtain that:

$$\bar{\gamma} \leq \frac{1-\delta}{\delta^2}$$

We re-express the condition in terms of index-orbit x and get a bound on δ which guarantees that all the orbits $x \in [0,1]$ are incentive compatible. From the definition of γ, the condition implies:

$$\frac{\frac{x}{4}}{1-\delta\left(1-\frac{x}{4}\right)} \leq \frac{1-\delta}{\delta^2}$$

Therefore, for $\delta \leq \bar{\delta} \equiv \frac{7}{4} - \frac{\sqrt{17}}{4}$ and $\bar{\omega}$ large enough, the series defined above converges for all the orbits $x \in [0,1]$ and we have that:

$$v^S_{\omega^{**}t} \leq a\sigma\left(1 + \frac{1-\delta}{1-\delta(1+\delta\bar{\gamma}-3\sigma+\rho)}\right) \tag{4.19}$$

We substitute (4.19) in the expression for v^B_ω:

$$v^B_{\omega t} = \bar{\gamma}\left[a(\omega^{**}-\omega) - \delta a\sigma\left(1 + \frac{1-\delta}{1-\delta(1+\delta\bar{\gamma}-3\sigma+\rho)}\right)\right]$$

And work out the matching incentive-compatible condition dictated in Equation (4.7):

$$\delta v^B_\omega + \delta v^S_{\omega^{**}} \leq s_{\omega\omega^{**}}$$
$$\Rightarrow a\delta\sigma \left[1 + \frac{1-\delta}{1-\delta(1+\delta\bar{\gamma}-3\sigma+\rho)}\right] \leq a(\omega^{**}-\omega)$$

In order to obtain an exchange which is incentive-compatible for ω^{**}-type sellers and every $\omega \in \mathcal{D}$, we set:

$$\omega^{**} - \omega = 1$$

In fact, because the surplus function is decreasing in the buyer's type, if the exchange is incentive-compatible for the buyer's type $\hat{\omega}$ such that $\hat{\omega} = \omega^{**} - 1$, it will also be for the types $\omega \leq \hat{\omega}$. We rewrite Equation (4.7) such as:

$$\delta\sigma(2 - \delta(\delta\bar{x} - 3\sigma + \rho) - 2\delta) \leq 1 - \delta(\delta\bar{\gamma} + 1 - 3\sigma + \rho)$$

From which we get:

$$\bar{\gamma} \leq \frac{1 - \delta(1 - 3\sigma + \rho) - \delta\sigma(2 - 2\delta - \delta(\rho - 3\sigma))}{\delta^2(1 - \delta\sigma)}$$

Again, by following the rationale we adopted above, we find out that for $\bar{\omega}$ large enough, the last condition stated in the Theorem's body is verified for all the values of δ such that:

$$\bar{x} = 1 \leftrightarrow \delta \leq \frac{7}{4} - \frac{\sqrt{17}}{4} \approx 0.72 \tag{4.20}$$

■

In the above result we characterized the behaviour of the economy when both buyers and sellers leave the market for a given initial pool of sellers. We demonstrated that whether the economy ends up expressing a homogeneous type of seller ω^{**} will depend jointly on the initial composition of the sellers' vector and agents' time discount δ, provided that the overall heterogeneity of types in the economy is sufficiently high.

We may now explore the dynamics of a *growing* economy by refining the market formation protocol with the introduction of a simple expansionary mechanism. Specifically, we will allow ω^{**}-type buyers to settle in the economy at the end of each trading period t. This is done in the following ancillary class of equilibria:

Definition 4.9 (**Homogeneous Equilibria with entry**). *Given a dynamic market as described by some leverage $\alpha \geq 0$, a first-best type $\omega^{**} \in \Omega$, an initial market composition $\boldsymbol{\mu}_0^S$, a buyer inflow given by $\boldsymbol{\mu}^B$, an average demand $D(\omega^{**}, \alpha)$, the matching probabilities $\pi_t^S, \pi^B, t \geq 0$, the payoffs $(v_{\omega t}^S)_{\omega \in \Omega}, (v_\omega^B)_{\omega \in \Omega}$ and the law of motions in (4.8), we define Homogeneous Equilibria with entry the class of equilibria of the bargaining game such that:*

(i) For $lim_{t \to \infty}$, every buyer type $\omega \in \mathcal{D}$ with $q_\omega > 0$ purchases the good from sellers of type ω^{**}.

(ii) Demand clears in every period: $\forall \omega \in \mathcal{D}$ s.t. $\tilde{\mu}_\omega^B > 0, \quad \delta v_{\omega^{**}}^S + \delta v_\omega^B < s_{\omega\omega^{**}}$

(iii) There is no seller-type dispersion: $\lim_{t \to \infty} \boldsymbol{\mu}_t^S = (\bar{\mu}_\omega^S)_{\omega \in \Omega}$ where $\bar{\mu}_\omega^S = 0 \ \forall \omega \in \Omega \setminus \{\omega^{**}\}$ and $\bar{\mu}_\omega^S = 1$ for $\omega = \omega^{**}$.

And state our second main result.

Lemma 3. *For any dynamic market with frictions fulfilling the following conditions:*

(i) Every buyer i such that $q_i < \omega^{**}$ and every seller j quit the market upon successful trade. Every buyer k such that $q_k = \omega^{**}$ settles in the market as potential seller.

(ii) $\omega^{**}(\omega^{**} - 2) \leq 2 \cdot \dfrac{1+\alpha}{\alpha}$, $\bar{\omega} \geq 15/2$.

(iii) $\alpha > \dfrac{\frac{\bar{\omega}}{2\omega^{**}}}{\omega^{**} - \frac{\bar{\omega}}{2\omega^{**}}}$

(iii) $\delta \leq \dfrac{7}{4} - \dfrac{\sqrt{17}}{4}$.

The market converges to a Homogeneous Equilibrium with entry.

Proof of Lemma 3. We have to expand the previous proof in the following dimensions. First, we have to add a further step which guarantees that the market formation can be sustained across periods with a positive net inflow of settled buyers. Secondly, we need to refine the market law of motions and the related rest points.

Step 1. We find a set of sufficient conditions such that the economy can clear every period according to the protocol specified in the equilibrium. In this sense, from the definitions of D and Δ (Proposition 4.1 and Definition 4.5) we know that in every period there exists a sufficient number of type ω^{**} sellers to cope with the in-flow of buyers if:

$$\Delta \geq D$$

Which can be rewritten in the following:

$$\alpha\omega^{**} + (1+\alpha) \geq \frac{\alpha(\omega^{**2})}{2} \to \alpha\omega^{**}(\omega^{**} - 2) \leq 2(1+\alpha)$$

Given that the measure of settlements is strictly increasing in the regulations α, let us consider the lower bound on $\bar{\omega}$ for $\alpha \to \infty$. In such case, from the equation above we may see that the condition $\Delta > D$ holds for all the ω^{**} such that $\omega^{**} \leq 1+\sqrt{3} \approx 3 \equiv \hat{\Omega}$. By looking at the mass of pure sellers $S = (\bar{\omega} - \omega^{**})/\bar{\omega}$ entering every period in the market, it must be that:

$$S > D$$

Taking $\omega^{**} = 3$, we obtain a bound on $\bar{\omega}$ by means of the following:

$$\frac{\bar{\omega} - 3}{\bar{\omega}} > \frac{\alpha}{1+\alpha} \cdot \frac{(3)^2}{2} \Longrightarrow \bar{\omega} \geq 15/2$$

Step 2. After having bounded the space of types, let us study the details of the market evolution for this second class of equilibria. Here, the law of motion for sellers given in Equation (4.8) becomes:

$$\mu^S_{\omega^{**}(t+1)} = \mu^S_{\omega^{**}t} + \underbrace{\hat{\mu}^B_{\omega^{**}} - \sum_{\{\omega\in\Omega|\omega\leq\omega^{**}\}} \left(\alpha_{\omega\omega^{**}t}\pi^S_{\omega^{**}t}\mu^B_{\omega} + \alpha_{\omega^{**}\omega t}\pi^B_{\omega}\mu^S_{\omega^{**}t}\right)}_{net\ equilibrium\ in-flow\ of\ \omega^{**}-type\ sellers} \tag{4.21}$$

The in-flow is given by the measure of buyers that settle in the market with ω' units of capital at the end of period t. The equation reads, in explicit form:

$$\mu^S_{\omega^{**}(t+1)} = \mu^S_{\omega^{**}t} + \Delta - \frac{1}{2}\left(\mu^S_{\omega^{**}t} + \Delta - D\right)^2 \cdot D$$

Following the proof of Theorem 4.8, we construct the index x and the one-step iterator $\tau(x)$ such that:

$$\tau(x) = \frac{x + \Delta - \frac{1}{2}(x + \Delta - D)^2 \cdot D}{x + \Delta - \frac{1}{2}(x + \Delta - D)^2 + \left(1 - x - \Delta - \frac{1}{2}(1 - x - \Delta - D)^2 \cdot D\right)}$$

Given that our goal here is to bound the system's dynamics in order to guarantee the index converges to a Homogeneous Equilibrium, we skip the intricacies related to the derivation of rest points and work directly on the conditions which enable the system to progress to $x \to 1$ for any given initial index $x^0 \in [0, 1]$. Following the proof of the previous Theorem, we differentiate $\tau(x)$ with respect to x and study the following:

$$\begin{aligned}\frac{d\tau(x)}{x} > 1 \rightarrow & 16 \cdot (-\frac{1}{2}(D(1-2x))(\Delta + x - \frac{1}{4}D(\Delta - D + x)^2) + \\ & (1 - \frac{1}{2}D(\Delta - D + x))(1 - 1/4D(\Delta - D + x)^2 - \frac{1}{4}D(+D + x - 1 - \Delta)^2)) > \\ & (-4 + D(\Delta - D + x)^2 + D(-1 - \Delta + D + x)^2)^2\end{aligned}$$

Let us study the stability of indexes $x_0 = \underline{x} = 0$

$$\underbrace{-4 + D + 4\Delta^4 D}_{A.1} + \underbrace{4\Delta^3(1-4D)D}_{A.2} \underbrace{-10D^2 + 2D^3}_{B.1} \underbrace{-4D^4 + 4D^5}_{B.2}$$
$$\underbrace{-2\Delta^2(4 - D + 6D^2 - 12D^3)}_{C.1} \underbrace{-2\Delta(4 - 9D + 2D^2 - 6D^3 + 8D^4)}_{C.2} < 0$$

Since both D and Δ are strictly less than one due to (i) of the present Lemma, it is easy to see that only $A.2$ from the equation above can be non-negative. We may easily bound the sign of $A.2$ by imposing the following sufficient condition on D and hence on α:

$$D < \frac{1}{4} \rightarrow \ \rightarrow \alpha < \frac{\frac{\bar{\omega}}{2\omega^{**}}}{\omega^{**} - \frac{\bar{\omega}}{2\omega^{**}}} \tag{4.22}$$

From which we may collect point (iii) of the Lemma's statement. We notice that the entry condition adopted in the present class of Homogeneous Equilibria implies a smaller tipping point than the one obtained in Theorem 4.8. A symmetric argument can be followed for showing that the system is always increasing at the point $x_0 = \bar{x} = 1$.

Step 3. Given that the entry condition does not affect the qualitative behavior of $\gamma(x)$ as characterized in the proof of Theorem 4.8, exchanges here follow the incentive-compatibility structure obtained for the homogeneous equilibria with no entry.

4.3 Simulations on Market Dynamics

In this section we simulate the market evolution when formation is sustained by either one or the other law of motion we characterized in the preceding section. As we focus precisely on the dynamical properties of the economy, we impose a discount factor of $\delta = 0$ thus suppressing any concern related to incentive-compatibility of exchanges. Figures 4.4 and 4.5 are referred to an environment compatible with the homogeneous equilibrium with no entry. In both simulation frameworks we assume a leverage $\alpha \rightarrow \infty$ and that $\Omega \equiv \{1, 2, ..., 10\}$. In the first scenario, we consider a type ω^{**} sellers index $x_0 \approx 0.99$. In the second scenario, everything equal, we take as starting point $x_0 = 0.49$.

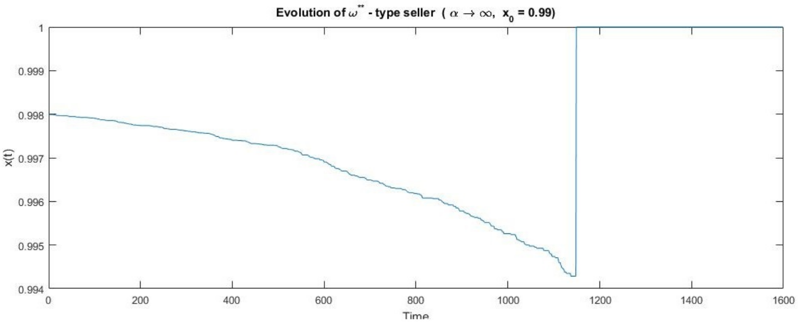

Figure 4.4: The market evolution under a Homogeneous Equilibrium with No Entry for $x_0 = 0.99$ under a very large financial leverage ($\alpha/(1+\alpha)$) $\rightarrow 1$ and $t \leq 2,000$ time periods.

In both cases, the economy is simulated for $t \leq 2,000$ periods, each period corresponding to the intra-period bargaining process which we described above. As it stands clear from the dynamics depicted in the figures, our market evolves according to the equilibrium as predicated along the main text.

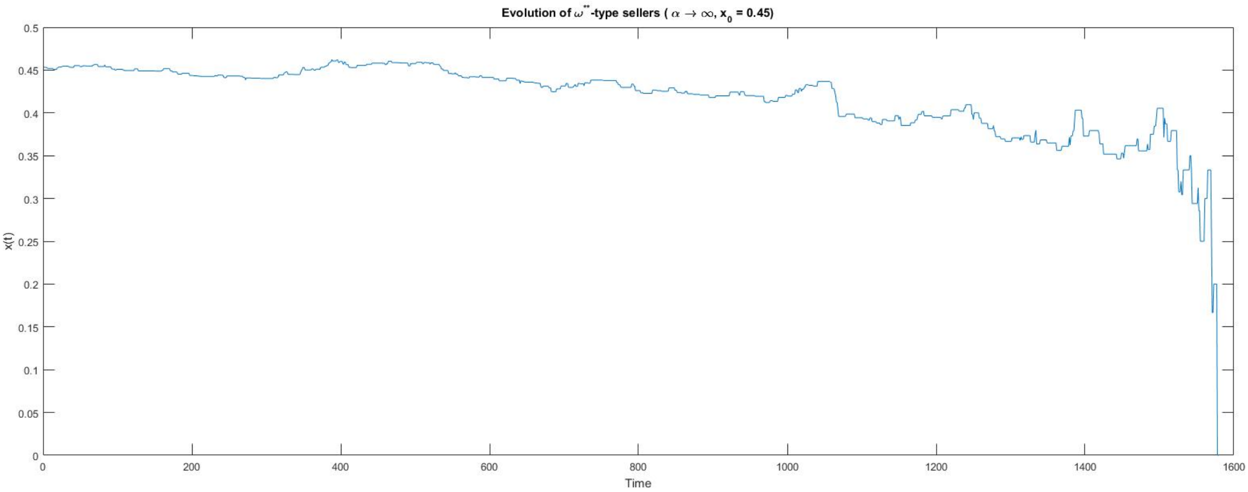

Figure 4.5: The market evolution under a Homogeneous Equilibrium with No Entry for $x_0 = 0.49$ under a very large financial leverage ($\alpha/(1+\alpha)$) $\rightarrow 1$ and $t \leq 2,000$ time periods.

We can now proceed to simulate the market evolution for the a Homogeneous Equilibrium with Entry as conceived along the main text. We keep the parametrization adopted in the previous set of simulations, yet in order to epitomize the diverse nature of this market, we assume an index initial condition given by $x_0 = 0.1$. From the resulting dynamics (captured in Figure 4.6) it is clear that no equilibrium exists in this economy besides $x^* = 1$.

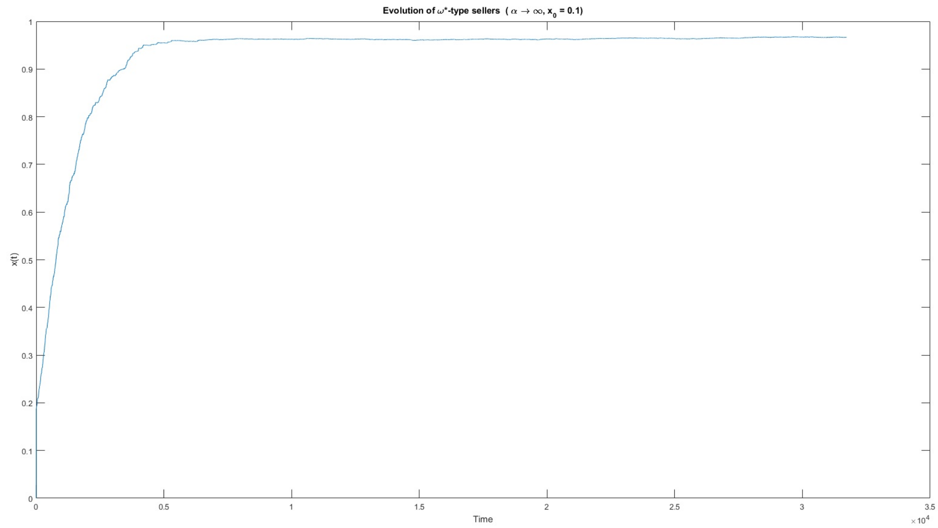

Figure 4.6: The market evolution under a Homogeneous Equilibrium with Entry for $x_0 = 0.1$ under a very large financial leverage ($\alpha/(1+\alpha)$) $\to 1$ and $t \leq 20,000$ time periods.

4.4 Conclusions

In this paper we investigated the role of financial frictions on the evolution of a dynamic market with heterogeneous agents. To this purpose, we proposed a model of dynamic market formation in which financially constrained buyers purchase multiple items from a pool of sellers whose composition evolves on the basis of realized trades. By specializing the general framework proposed in Manea [54], we pinned down the market formation to two classes of equilibria of a bargaining game, respectively defined Homogeneous Equilibria with no Entry and Homogeneous Equilibrium with Entry. These two equilibria share two critical features. First, they hold that the market evolves by means of sequential entrance of *waves* of buyers. Secondly, they postulate the emergence of a market with no-type dispersion in sellers' population. The two classes critically deviate in the specification of the end-period dynamics. While in the former setting we assumed that both sellers and buyers leave the market upon trade, in the latter, we assume a certain type of buyers will replace the exiting sellers. In both classes of equilibria, we showed that financing regulations play a critical role in steering the qualitative behavior of markets by means of a well-defined tipping point.

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