The Yagita invariant of symplectic groups of large rank

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March 26, 2018

Abstract

Fix a prime p, and let $\mathcal{O}$ denote a subring of $\mathbb{C}$ that is either integrally closed or contains a primitive pth root of 1. We determine the Yagita invariant at the prime p for the symplectic group $\mathrm{Sp}(2n, \mathcal{O})$ for all $n \geq p - 1$.

1 Introduction

The Yagita invariant $p^\circ(G)$ of a discrete group G is an invariant that generalizes the period of the p-local Tate-Farrell cohomology of G, in the following sense: it is a numerical invariant defined for any G that is equal to the period when the p-local cohomology of G is periodic. Yagita considered finite groups [6], and Thomas extended the definition to groups of finite vcd [5]. In [3] the definition was extended to arbitrary groups and $p^\circ(G)$ was computed for $G = \mathrm{GL}(n, \mathcal{O})$ for $\mathcal{O}$ any integrally closed subring of $\mathbb{C}$ and for sufficiently large n (depending on $\mathcal{O}$).

In [2], one of us computed the Yagita invariant for $\mathrm{Sp}(2(p+1), \mathbb{Z})$. Computations from [3] were used to provide an upper bound and computations with finite subgroups and with mapping class groups were used to provide a lower bound [4]. The action of the mapping class group of a surface upon the first homology of the surface gives a natural symplectic representation of the mapping class group of a genus $p+1$-surface inside $\mathrm{Sp}(2(p+1), \mathbb{Z})$. In the current paper, we compute $p^\circ(\mathrm{Sp}(2n, \mathcal{O}))$ for each $n \geq p - 1$ for each $\mathcal{O}$ for which $p^\circ(\mathrm{GL}(n, \mathcal{O}))$ was computed in [3]. By using a greater range of finite subgroups we avoid having to consider mapping class groups.

Throughout the paper, we fix a prime p. Before stating our main result we recall the definitions of the symplectic group $\mathrm{Sp}(2n, R)$ over a ring R, and of the Yagita invariant

*The first author acknowledges support from ETH Zürich, which facilitated this work

†The second author would like to thank the Isaac Newton Institute for Mathematical Sciences for support and hospitality during the programme *Non-positive Curvature, Group Actions and Cohomology*, when work on this paper was undertaken. This work was supported by EPSRC grant no. EP/K032208/1 and by a grant from the Leverhulme Trust.

$p^\circ(G)$, which depends on the prime p as well as on the group G. The group $\mathrm{Sp}(2n,R)$ is the collection of invertible $2n \times 2n$ matrices M over R such that

$$M^{\mathrm{T}}JM = J, \text{ where } J := \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}.$$

Here M^{T} denotes the transpose of the matrix M, and as usual I_n denotes the $n \times n$ identity matrix. Equivalently $M \in \mathrm{Sp}(2n,R)$ if M defines an isometry of the antisymmetric bilinear form on R^{2n} defined by $\langle x, y\rangle := x^{\mathrm{T}}Jy$. If C is cyclic of order p, then the group cohomology ring $H^*(C;\mathbb{Z})$ has the form

$$H^*(C;\mathbb{Z}) \cong \mathbb{Z}[x]/(px), \quad x \in H^2(C;\mathbb{Z}).$$

If C is a cyclic subgroup of G of order p, define $n(C)$ a positive integer or infinity to be the supremum of the integers n such that the image of $H^*(G;\mathbb{Z}) \to H^*(C;\mathbb{Z})$ is contained in the subring $\mathbb{Z}[x^n]$. Now define

$$p^\circ(G) := \text{l. c. m.}\{2n(C) \ : \ C \leq G, \ |C| = p\}.$$

It is easy to see that if $H \leq G$ then $p^\circ(H)$ divides $p^\circ(G)$ [3, Prop. 1].

In the following theorem statement and throughout the paper we let ζ_p be a primitive pth root of 1 in $\mathbb{C}$ and we let $\mathcal{O}$ denote a subring of $\mathbb{C}$ with $F \subseteq \mathbb{C}$ as its field of fractions. We assume that either $\zeta_p \in \mathcal{O}$ or that $\mathcal{O}$ is integrally closed in $\mathbb{C}$. We define $l := |F[\zeta_p] : F|$, the degree of $F[\zeta_p]$ as an extension of F. For $t \in \mathbb{R}$ with $t \geq 1$, we define $\psi(t)$ to be the largest integer power of p less than or equal to t.

Theorem 1. *With notation as above, for each $n \geq p-1$, the Yagita invariant $p^\circ(\mathrm{Sp}(2n,\mathcal{O}))$ is equal to $2(p-1)\psi(2n/l)$ for l even and equal to $2(p-1)\psi(n/l)$ for l odd.*

By the main result of [3], the above is equivalent to the statement that $p^\circ(\mathrm{Sp}(2n,\mathcal{O})) = p^\circ(\mathrm{GL}(2n,\mathcal{O}))$ when l is even and $p^\circ(\mathrm{Sp}(2n,\mathcal{O})) = p^\circ(\mathrm{GL}(n,\mathcal{O}))$ when l is odd. By definition $\mathrm{Sp}(2n,\mathcal{O})$ is a subgroup of $\mathrm{GL}(2n,\mathcal{O})$ and there is an inclusion $\mathrm{GL}(n,\mathcal{O}) \to \mathrm{Sp}(2n,\mathcal{O})$ defined by

$$A \mapsto \begin{pmatrix} A & 0 \\ 0 & (A^{\mathrm{T}})^{-1} \end{pmatrix},$$

and so for any n, $p^\circ(\mathrm{GL}(n,\mathcal{O}))$ divides $p^\circ(\mathrm{Sp}(2n,\mathcal{O}))$, which in turn divides $p^\circ(\mathrm{GL}(2n,\mathcal{O}))$.

Lemma 2. *If M is in the symplectic group, then M is conjugate to $(M^{-1})^{\mathrm{T}} = (M^{\mathrm{T}})^{-1}$, and* $\det(M) = 1$.

Proof. The matrix J defining the symplectic form satisfies $J^4 = I$, and so in particular it is invertible. The equation $M^{\mathrm{T}}JM = J$ implies the equation $JMJ^{-1} = (M^{\mathrm{T}})^{-1}$.

The usual way to show that the determinant of M is equal to 1 is via the Pfaffian. The Pfaffian is a function $A \mapsto \mathrm{pf}(A)$ on the set of skew-symmetric matrices, which is polynomial in the matrix coefficients and is a square root of the determinant, i.e., $\mathrm{pf}(A)^2 = \det(A)$ for each skew-symmetric matrix A. Given these properties, it is easy to verify that the identity $\mathrm{pf}(M^{\mathrm{T}}AM) = \det(M)\mathrm{pf}(A)$ holds for all matrices M and all skew-symmetric matrices A. Since J is invertible, $\mathrm{pf}(J) \neq 0$, and if M is symplectic, the equations

$$\mathrm{pf}(J) = \mathrm{pf}(M^{\mathrm{T}}JM) = \det(M)\mathrm{pf}(J)$$

imply that $\det(M) = 1$. □

Proposition 3. *Let $f(X)$ be a polynomial over the field $\mathbb{F}_p$, all of whose roots lie in $\mathbb{F}_p^\times$. If there is a polynomial g and an integer n so that $f(X) = g(X^n)$, then n has the form $n = mp^q$ for some m dividing $p-1$ and some positive integer q. If p is odd and for each $i \in \mathbb{F}_p^\times$, the multiplicity of i as a root of f is equal to that of $-i$, then m is even.*

Proof. The only part of this that is not contained in [3, Prop. 6] is the final statement. Since $(1-iX)(1+iX) = 1 - i^2X^2$ is a polynomial in X^2, the final statement follows. For the benefit of the reader, we sketch the rest of the proof. If $n = mp^q$ where p does not divide m, then $g(X^n) = g(X^m)^{p^q}$, so we may assume that $q = 0$. If $g(Y) = 0$ has roots y_i, then the roots of $g(X^m) = 0$ are the roots of $y_i - X^m = 0$. Since p does not divide m, these polynomials have no repeated roots; since their roots are assumed to lie in $\mathbb{F}_p$ it is now easy to show that m divides $p-1$. □

Corollary 4. *With notation as in Theorem 1, let G be a subgroup of $\mathrm{Sp}(2n, F)$. Then the Yagita invariant $p^\circ(G)$ divides the number given for $p^\circ(\mathrm{Sp}(2n, \mathcal{O}))$ in the statement of Theorem 1.*

Proof. As in [3, Cor. 7], for each $C \leq G$ of order p, we use the total Chern class to give an upper bound for the number $n(C)$ occuring in the definition of $p^\circ(G)$. If C is cyclic of order p, then C has p distinct irreducible complex representations, each 1-dimensional. If we write $H^*(C;\mathbb{Z}) = \mathbb{Z}[x]/(px)$, then the total Chern classes of these representations are $1 + ix$ for each $i \in \mathbb{F}_p$, where $i = 0$ corresponds to the trivial representation. The total Chern class of a direct sum of representations is the product of the total Chern classes, and so when viewed as a polynomial in $\mathbb{F}_p[x] = H^*(C;\mathbb{Z}) \otimes \mathbb{F}_p$, the total Chern class of any faithful representation $\rho : C \to \mathrm{GL}(2n, \mathbb{C})$ is a non-constant polynomial of degree at most $2n$ all of whose roots lie in $\mathbb{F}_p^\times$. Now let F be a subfield of $\mathbb{C}$ with $l = |F[\zeta_p] : F|$ as in the statement. The group C has $(p-1)/l$ non-trivial irreducible representations over F, each of dimension l, and the total Chern classes of these representations have the form $1 - ix^l$, where i ranges over the $(p-1)/l$ distinct lth roots of unity in $\mathbb{F}_p$. In particular, the total Chern class of any representation $\rho : C \to \mathrm{GL}(2n, F) \leq \mathrm{GL}(2n, \mathbb{C})$ is a polynomial in x^l whose x-degree is at most $2n$. If ρ has image contained in $\mathrm{Sp}(2n, \mathbb{C})$, then it factors as $\rho = \iota \circ \widetilde{\rho}$ with $\widetilde{\rho} : C \to \mathrm{Sp}(2n, \mathbb{C})$ and ι is the inclusion of $\mathrm{Sp}(2n, \mathbb{C})$ in $\mathrm{GL}(2n, \mathbb{C})$. In this case the matrix representing a generator for C is conjugate to the transpose of its own inverse; in particular it follows that the multiplicities of the irreducible complex representations of C with total Chern classes $1 + ix$ and $1 - ix$ must be equal for each i. Hence in this case, if p is odd, the total Chern class of the representation $\rho = \iota \circ \widetilde{\rho}$ is a polynomial in x^2. If $p = 2$ (which implies that $l = 1$) then the total Chern class of any representation $\rho : C \to \mathrm{GL}(2n, \mathbb{C})$ has the form $(1 + x)^i$, where i is equal to the number of non-trivial irreducible summands. Since $\mathrm{Sp}(2n,\mathbb{C}) \le \mathrm{SL}(2n,\mathbb{C})$ it follows that for symplectic representations i must be even, and so for $p = 2$, the total Chern class is a polynomial in x^2.

In summary, let $\widetilde{\rho}$ be a faithful representation of C in $\mathrm{Sp}(2n, F)$. In the case when l is odd, then the total Chern class of $\widetilde{\rho}$ is a non-constant polynomial $\tilde{f}(y) = f(x)$ in $y = x^{2l}$ such that $f(x)$ has degree at most $2n$, $\tilde{f}(y)$ has degree at most n/l, and all roots of $f, \tilde{f}$ lie in $\mathbb{F}_p^\times$. In the case when l is even, the total Chern class of ρ is a non-constant polynomial $\tilde{f}(y) = f(x)$ in $y = x^l$ such that $f(x)$ has degree at most $2n$, $\tilde{f}(y)$ has degree at most $2n/l$, and all roots of both lie in $\mathbb{F}_p^\times$. By Proposition 3, it follows that each $n(C)$ is a factor of the number given for $p^\circ(\mathrm{Sp}(2n, \mathcal{O}))$, and hence the claim. □

Lemma 5. *Let $H \le G$ with $|G : H| = m$, and let ρ be a symplectic representation of H on $V = \mathcal{O}^{2n}$. The induced representation $\mathrm{Ind}_H^G(\rho)$ is a symplectic representation of G on $W := \mathcal{O}G \otimes_{\mathcal{O}H} V \cong \mathcal{O}^{2mn}$.*

Proof. Let $e_1, \ldots, e_n, f_1, \ldots, f_n$ be the standard basis for $V = \mathcal{O}^{2n}$, so that the bilinear form $\langle v, w\rangle := v^{\mathrm{T}} J w$ on V is given by

$$\langle e_i, e_j\rangle = 0 = \langle f_i, f_j\rangle, \quad \langle e_i, f_j\rangle = -\langle f_i, e_j\rangle = \delta_{ij}.$$

The representation ρ is symplectic if and only if each $\rho(h)$ preserves this bilinear form.

Let $t_1, \ldots, t_m$ be a left transversal to H in G, so that $\mathcal{O}G = \oplus_{i=1}^m t_i \mathcal{O}H$ as right $\mathcal{O}H$-modules. Define a bilinear form $\langle\ ,\ \rangle_W$ on W by

$$\left\langle \sum_{i=1}^m t_i \otimes v^i, \sum_{i=1}^m t_i \otimes w^i \right\rangle_W := \sum_{i=1}^m \langle v^i, w^i\rangle.$$

To see that this bilinear form is preserved by the $\mathcal{O}G$-action on W, fix $g \in G$ and define a permutation π of $\{1, \ldots, m\}$ and elements $h_1, \ldots, h_m \in H$ by the equations $g t_i = t_{\pi(i)} h_i$. Now for each i, j with $1 \le i, j \le m$

$$\begin{aligned}
\langle \mathrm{Ind}(\rho(g)) t_i \otimes v, \mathrm{Ind}(\rho(g)) t_j \otimes w\rangle_W &= \left\langle t_{\pi(i)} \otimes \rho(h_i) v, t_{\pi(j)} \otimes \rho(h_j) w\right\rangle_W \\
&= \delta_{\pi(i)\pi(j)} \langle \rho(h_i) v, \rho(h_j) w\rangle \\
&= \delta_{ij} \langle \rho(h_i) v, \rho(h_i) w\rangle \\
&= \delta_{ij} \langle v, w\rangle \\
&= \langle t_i \otimes v, t_j \otimes w\rangle_W.
\end{aligned}$$

To see that $\langle\ ,\ \rangle_W$ is symplectic, define basis elements $E_1, \ldots, E_{mn}, F_1, \ldots, F_{mn}$ for W by the equations

$$E_{n(i-1)+j} := t_i \otimes e_j, \ \text{ and } \ F_{n(i-1)+j} := t_i \otimes f_j, \quad \text{for } 1 \le i \le m,\ 1 \le j \le n.$$

It is easily checked that for $1 \le i, j \le mn$

$$\langle E_i, E_j\rangle_W = 0 = \langle F_i, F_j\rangle_W, \quad \langle E_i, F_j\rangle_W = -\langle F_i, E_j\rangle_W = \delta_{ij},$$

and so with respect to this basis for W, the bilinear form $\langle\ ,\ \rangle_W$ is the standard symplectic form. □

Proposition 6. *With notation as in Theorem 1, the Yagita invariant* $p^\circ(\mathrm{Sp}(2n,\mathcal{O}))$ *is divisible by the number given in the statement of Theorem 1.*

Proof. To give lower bounds for $p^\circ(\mathrm{Sp}(2n,\mathcal{O}))$ we use finite subgroups. Firstly, consider the semidirect product $H = C_p \rtimes C_{p-1}$, where C_{p-1} acts faithfully on C_p; equivalently this is the group of affine transformations of the line over $\mathbb{F}_p$. It is well known that the image of $H^*(G;\mathbb{Z})$ inside $H^*(C_p;\mathbb{Z}) \cong \mathbb{Z}[x]/(px)$ is the subring generated by x^{p-1}. It follows that $2(p-1)$ divides $p^\circ(G)$ for any G containing H as a subgroup. The group H has a faithful permutation action on p points, and hence a faithful representation in $\mathrm{GL}(p-1,\mathbb{Z})$, where $\mathbb{Z}^{p-1}$ is identified with the kernel of the H-equivariant map $\mathbb{Z}\{1,\ldots,p\} \to \mathbb{Z}$. Since $\mathrm{GL}(p-1,\mathbb{Z})$ embeds in $\mathrm{Sp}(2(p-1),\mathbb{Z})$ we deduce that H embeds in $\mathrm{Sp}(2n,\mathcal{O})$ for each $\mathcal{O}$ and for each $n \geq p-1$.

To give a lower bound for the p-part of $p^\circ(\mathrm{Sp}(2n,\mathcal{O}))$ we use the extraspecial p-groups. For p odd, let $E(p,1)$ be the non-abelian p-group of order p^3 and exponent p, and let $E(2,1)$ be the dihedral group of order 8. (Equivalently in each case $E(p,1)$ is the Sylow p-subgroup of $\mathrm{GL}(3,\mathbb{F}_p)$.) For $m \geq 2$, let $E(p,m)$ denote the central product of m copies of $E(p,1)$, so that $E(p,m)$ is one of the two extraspecial groups of order p^{2m+1}. Yagita showed that $p^\circ(E(p,m)) = 2p^m$ for each m and p [6]. The centre and commutator subgroup of $E(p,m)$ are equal and have order p, and the abelianization of $E(p,m)$ is isomorphic to C_p^{2m}. The irreducible complex representations of $E(p,m)$ are well understood: there are p^{2m} distinct 1-dimensional irreducibles, each of which restricts to the centre as the trivial representation, and there are $p-1$ faithful representations of dimension p^m, each of which restricts to the centre as the sum of p^m copies of a single (non-trivial) irreducible representation of C_p. The group $G = E(p,m)$ contains a subgroup H isomorphic to C_p^{m+1}, and each of its faithful p^m-dimensional representations can be obtained by inducing up a 1-dimensional representation $H \to C_p \to \mathrm{GL}(1,\mathbb{C})$.

According to Bürgisser, C_p embeds in $\mathrm{Sp}(2l,\mathcal{O})$ (resp. in $\mathrm{Sp}(l,\mathcal{O})$ when l is even) provided that $\mathcal{O}$ is integrally closed in $\mathbb{C}$ [1]. Here as usual, $l := |F[\zeta_p], F|$ and F is the field of fractions of $\mathcal{O}$. If instead $\zeta_p \in \mathcal{O}$, then $l = 1$ and clearly C_p embeds in $\mathrm{GL}(1,\mathcal{O})$ and hence also in $\mathrm{Sp}(2,\mathcal{O}) = \mathrm{Sp}(2l,\mathcal{O})$. Taking this embedding of C_p and composing it with any homomorphism $H \to C_p$ we get a symplectic representation ρ of H on $\mathcal{O}^{2l}$ for any l (resp. on $\mathcal{O}^l$ for l even). For a suitable homomorphism we know that $\mathrm{Ind}_H^G(\rho)$ is a faithful representation of G on $\mathcal{O}^{2lp^m}$ (resp. on $\mathcal{O}^{lp^m}$ for l even) and by Lemma 5 we see that $\mathrm{Ind}_H^G(\rho)$ is symplectic. Hence we see that $E(m,p)$ embeds as a subgroup of $\mathrm{Sp}(2lp^m,\mathcal{O})$ for any l and as a subgroup of $\mathrm{Sp}(lp^m,\mathcal{O})$ in the case when l is even. Since $p^\circ(E(m,p)) = 2p^m$, this shows that $2p^m$ divides $p^\circ(\mathrm{Sp}(2lp^m,\mathcal{O}))$ always and that $2p^m$ divides $p^\circ(\mathrm{Sp}(lp^m,\mathcal{O}))$ in the case when l is even. □

Corollary 4 and Proposition 6 together complete the proof of Theorem 1.

We finish by pointing out that we have not computed $p^\circ(\mathrm{Sp}(2n,\mathcal{O}))$ for general $\mathcal{O}$ when $n < p-1$; to do this one would have to know which metacyclic groups $C_p \rtimes C_k$ with k coprime to p admit low-dimensional symplectic representations.

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