The Yagita invariant of symplectic groups of large rank

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Abstract

Fix a prime p, and let $\mathcal{O}$ denote a subring of $\mathbb{C}$ that is either integrally closed or contains a primitive pth root of 1. We determine the Yagita invariant at the prime p for the symplectic group $\mathrm{Sp}(2n, \mathcal{O})$ for all $n \geq p-1$.

1 Introduction

The Yagita invariant $p^\circ(G)$ of a discrete group G is an invariant that generalizes the period of the p-local Tate-Farrell cohomology of G, in the following sense: it is a numerical invariant defined for any G that is equal to the period when the p-local cohomology of G is periodic. Yagita considered finite groups [6], and Thomas extended the definition to groups of finite vcd [5]. In [3] the definition was extended to arbitrary groups and $p^\circ(G)$ was computed for $G = \mathrm{GL}(n, \mathcal{O})$ for $\mathcal{O}$ any integrally closed subring of $\mathbb{C}$ and for sufficiently large n (depending on $\mathcal{O}$).

In [2], one of us computed the Yagita invariant for $\mathrm{Sp}(2(p+1), \mathbb{Z})$. Computations from [3] were used to provide an upper bound and computations with finite subgroups and with mapping class groups were used to provide a lower bound [4]. The action of the mapping class group of a surface upon the first homology of the surface gives a natural symplectic representation of the mapping class group of a genus $p+1$ surface inside $\mathrm{Sp}(2(p+1), \mathbb{Z})$. In the current paper, we compute $p^\circ(\mathrm{Sp}(2n, \mathcal{O}))$ for each $n \geq p-1$ for each $\mathcal{O}$ for which $p^\circ(\mathrm{GL}(n, \mathcal{O}))$ was computed in [3]. By using a greater range of finite subgroups we avoid having to consider mapping class groups.

Throughout the paper, we fix a prime p. Before stating our main result we recall the definitions of the symplectic group $\mathrm{Sp}(2n, R)$ over a ring R, and of the Yagita invariant

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$p^\circ(G)$, which depends on the prime p as well as on the group G. The group $\mathrm{Sp}(2n, R)$ is the collection of invertible $2n \times 2n$ matrices M over R such that

$$M^{\mathrm{T}} J M = J, \text{ where } J := \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}.$$

Here M^{T} denotes the transpose of the matrix M, and as usual I_n denotes the $n \times n$ identity matrix. Equivalently $M \in \mathrm{Sp}(2n, R)$ if M defines an isometry of the antisymmetric bilinear form on R^{2n} defined by $\langle x, y\rangle := x^{\mathrm{T}} J y$. If C is cyclic of order p, then the group cohomology ring $H^*(C; \mathbb{Z})$ has the form

$$H^*(C; \mathbb{Z}) \cong \mathbb{Z}[x]/(px), \quad x \in H^2(C; \mathbb{Z}).$$

If C is a cyclic subgroup of G of order p, define $n(C)$ a positive integer or infinity to be the supremum of the integers n such that the image of $H^*(G; \mathbb{Z}) \to H^*(C; \mathbb{Z})$ is contained in the subring $\mathbb{Z}[x^n]$. Now define

$$p^\circ(G) := \mathrm{lcm}\{2n(C) \ : \ C \leq G, \ \ |C| = p\}.$$

It is easy to see that if $H \leq G$ then $p^\circ(H)$ divides $p^\circ(G)$ [3, Prop. 1].

2 Results

In the following theorem statement and throughout the paper we let ζ_p be a primitive pth root of 1 in $\mathbb{C}$ and we let $\mathcal{O}$ denote a subring of $\mathbb{C}$ with $F \subseteq \mathbb{C}$ as its field of fractions. We assume that either $\zeta_p \in \mathcal{O}$ or that $\mathcal{O}$ is integrally closed in $\mathbb{C}$. We define $l := |F[\zeta_p] : F|$, the degree of $F[\zeta_p]$ as an extension of F. For $t \in \mathbb{R}$ with $t \geq 1$, we define $\psi(t)$ to be the largest integer power of p less than or equal to t.

Theorem 1. *With notation as above, for each $n \geq p-1$, the Yagita invariant $p^\circ(\mathrm{Sp}(2n, \mathcal{O}))$ is equal to $2(p-1)\psi(2n/l)$ for l even and equal to $2(p-1)\psi(n/l)$ for l odd.*

By the main result of [3], the above is equivalent to the statement that $p^\circ(\mathrm{Sp}(2n, \mathcal{O})) = p^\circ(\mathrm{GL}(2n, \mathcal{O}))$ when l is even and $p^\circ(\mathrm{Sp}(2n, \mathcal{O})) = p^\circ(\mathrm{GL}(n, \mathcal{O}))$ when l is odd. By definition $\mathrm{Sp}(2n, \mathcal{O})$ is a subgroup of $\mathrm{GL}(2n, \mathcal{O})$ and there is an inclusion $\mathrm{GL}(n, \mathcal{O}) \to \mathrm{Sp}(2n, \mathcal{O})$ defined by

$$A \mapsto \begin{pmatrix} A & 0 \\ 0 & (A^{\mathrm{T}})^{-1} \end{pmatrix},$$

and so for any n, $p^\circ(\mathrm{GL}(n, \mathcal{O}))$ divides $p^\circ(\mathrm{Sp}(2n, \mathcal{O}))$, which in turn divides $p^\circ(\mathrm{GL}(2n, \mathcal{O}))$.

Before we start, we recall two standard facts concerning symplectic matrices that will be used in the proof of Corollary 3: if M is in the symplectic group then $\det(M) = 1$ and M is conjugate to the inverse of its transpose $(M^{-1})^{\mathrm{T}} = (M^{\mathrm{T}})^{-1}$. We shall use the notation $\mathbb{F}_p^\times$ to denote the multiplicative group of units in the field $\mathbb{F}_p$.

Proposition 2. *Let $f(X)$ be a polynomial over the field $\mathbb{F}_p$ and suppose that 0 is not a root of f but that f factors as a product of linear polynomials over $\mathbb{F}_p$. If there is a polynomial g and an integer n so that $f(X) = g(X^n)$, then n has the form $n = mp^q$ for some m dividing $p-1$ and some integer $q \geq 0$. If p is odd and for each $i \in \mathbb{F}_p^\times$, the multiplicity of i as a root of f is equal to that of $-i$, then m is even.*

Proof. The only part of this that is not contained in [3, Prop. 6] is the final statement. Since $(1 - iX)(1 + iX) = 1 - i^2X^2$ is a polynomial in X^2, the final statement follows. For the benefit of the reader, we sketch the rest of the proof. If $n = mp^q$ where p does not divide m, then $g(X^n) = g(X^m)^{p^q}$, so we may assume that $q = 0$. If $g(Y) = 0$ has roots y_i, then the roots of $g(X^m) = 0$ are the roots of $y_i - X^m = 0$. Since p does not divide m, these polynomials have no repeated roots; since their roots are assumed to lie in $\mathbb{F}_p$ it is now easy to show that m divides $p - 1$. □

Corollary 3. *With notation as in Theorem 1, let G be a subgroup of* $\mathrm{Sp}(2n, F)$*. Then the Yagita invariant $p^\circ(G)$ divides the number given for $p^\circ(\mathrm{Sp}(2n, \mathcal{O}))$ in the statement of Theorem 1.*

Proof. As in [3, Cor. 7], for each $C \leq G$ of order p, we use the total Chern class to give an upper bound for the number $n(C)$ occurring in the definition of $p^\circ(G)$. If C is cyclic of order p, then C has p distinct irreducible complex representations, each 1-dimensional. If we write $H^*(C;\mathbb{Z}) = \mathbb{Z}[x]/(px)$, then the total Chern classes of these representations are $1 + ix$ for each $i \in \mathbb{F}_p$, where $i = 0$ corresponds to the trivial representation. The total Chern class of a direct sum of representations is the product of the total Chern classes, and so when viewed as a polynomial in $\mathbb{F}_p[x] = H^*(C;\mathbb{Z}) \otimes \mathbb{F}_p$, the total Chern class of any faithful representation $\rho : C \to \mathrm{GL}(2n, \mathbb{C})$ is a non-constant polynomial of degree at most $2n$ all of whose roots lie in $\mathbb{F}_p^\times$. Now let F be a subfield of $\mathbb{C}$ with $l = |F[\zeta_p] : F|$ as in the statement. The group C has $(p-1)/l$ non-trivial irreducible representations over F, each of dimension l, and the total Chern classes of these representations have the form $1 - ix^l$, where i ranges over the $(p-1)/l$ distinct lth roots of unity in $\mathbb{F}_p$. In particular, the total Chern class of any representation $\rho : C \to \mathrm{GL}(2n, F) \leq \mathrm{GL}(2n, \mathbb{C})$ is a polynomial in x^l whose x-degree is at most $2n$. If ρ has image contained in $\mathrm{Sp}(2n, \mathbb{C})$, then it factors as $\rho = \iota \circ \widetilde{\rho}$ with $\widetilde{\rho} : C \to \mathrm{Sp}(2n, \mathbb{C})$ and ι is the inclusion of $\mathrm{Sp}(2n, \mathbb{C})$ in $\mathrm{GL}(2n, \mathbb{C})$. In this case the matrix representing a generator for C is conjugate to the transpose of its own inverse; in particular it follows that the multiplicities of the irreducible complex representations of C with total Chern classes $1 + ix$ and $1 - ix$ must be equal for each i. Hence in this case, if p is odd, the total Chern class of the representation $\rho = \iota \circ \widetilde{\rho}$ is a polynomial in x^2. If $p = 2$ (which implies that $l = 1$) then the total Chern class of any representation $\rho : C \to \mathrm{GL}(2n, \mathbb{C})$ has the form $(1 + x)^i$, where i is equal to the number of non-trivial irreducible summands. Since $\mathrm{Sp}(2n, \mathbb{C}) \leq \mathrm{SL}(2n, \mathbb{C})$ it follows that for symplectic representations i must be even, and so, for $p = 2$ the total Chern class is a polynomial in x^2.

In summary, let $\widetilde{\rho}$ be a faithful representation of C in $\mathrm{Sp}(2n, F)$. In the case when l is odd, then the total Chern class of $\widetilde{\rho}$ is a non-constant polynomial $\tilde{f}(y) = f(x)$ in $y = x^{2l}$ such that $f(x)$ has degree at most $2n$, $\tilde{f}(y)$ has degree at most n/l, and all roots of $f, \tilde{f}$ lie in $\mathbb{F}_p^\times$. In the case when l is even, the total Chern class of ρ is a non-constant polynomial $\tilde{f}(y) = f(x)$ in $y = x^l$ such that $f(x)$ has degree at most $2n$, $\tilde{f}(y)$ has degree at most $2n/l$, and all roots of both lie in $\mathbb{F}_p^\times$. By Proposition 2, it follows that each $n(C)$ is a factor of the number given for $p^\circ(\mathrm{Sp}(2n,\mathcal{O}))$, and hence the claim. $\square$

Lemma 4. *Let $H \le G$ with $|G : H| = m$, and let ρ be a symplectic representation of H on $V = \mathcal{O}^{2n}$. The induced representation $\mathrm{Ind}_H^G(\rho)$ is a symplectic representation of G on $W := \mathcal{O}G \otimes_{\mathcal{O}H} V \cong \mathcal{O}^{2mn}$.*

Proof. Let $e_1, \ldots, e_n, f_1, \ldots, f_n$ be the standard basis for $V = \mathcal{O}^{2n}$, so that the bilinear form $\langle v, w\rangle := v^{\mathrm{T}} J w$ on V is given by

$$\langle e_i, e_j\rangle = 0 = \langle f_i, f_j\rangle, \quad \langle e_i, f_j\rangle = -\langle f_i, e_j\rangle = \delta_{ij}.$$

The representation ρ is symplectic if and only if each $\rho(h)$ preserves this bilinear form.

Let $t_1, \ldots, t_m$ be a left transversal to H in G, so that $\mathcal{O}G = \oplus_{i=1}^m t_i \mathcal{O}H$ as right $\mathcal{O}H$-modules. Define a bilinear form $\langle\ ,\ \rangle_W$ on W by

$$\left\langle \sum_{i=1}^m t_i \otimes v^i, \sum_{i=1}^m t_i \otimes w^i \right\rangle_W := \sum_{i=1}^m \langle v^i, w^i\rangle.$$

To see that this bilinear form is preserved by the $\mathcal{O}G$-action on W, fix $g \in G$ and define a permutation π of $\{1, \ldots, m\}$ and elements $h_1, \ldots, h_m \in H$ by the equations $gt_i = t_{\pi(i)} h_i$. Now for each i, j with $1 \le i, j \le m$

$$\begin{aligned}
\langle \mathrm{Ind}(\rho(g)) t_i \otimes v, \mathrm{Ind}(\rho(g)) t_j \otimes w\rangle_W &= \left\langle t_{\pi(i)} \otimes \rho(h_i)v, t_{\pi(j)} \otimes \rho(h_j)w\right\rangle_W \\
&= \delta_{\pi(i)\pi(j)} \langle \rho(h_i)v, \rho(h_j)w\rangle \\
&= \delta_{ij} \langle \rho(h_i)v, \rho(h_i)w\rangle \\
&= \delta_{ij} \langle v, w\rangle \\
&= \langle t_i \otimes v, t_j \otimes w\rangle_W.
\end{aligned}$$

To see that $\langle\ ,\ \rangle_W$ is symplectic, define basis elements $E_1, \ldots, E_{mn}, F_1, \ldots, F_{mn}$ for W by the equations

$$E_{n(i-1)+j} := t_i \otimes e_j, \ \text{ and } \ F_{n(i-1)+j} := t_i \otimes f_j, \quad \text{for } 1 \le i \le m,\ 1 \le j \le n.$$

It is easily checked that for $1 \le i, j \le mn$

$$\langle E_i, E_j\rangle_W = 0 = \langle F_i, F_j\rangle_W, \quad \langle E_i, F_j\rangle_W = -\langle F_i, E_j\rangle_W = \delta_{ij},$$

and so with respect to this basis for W, the bilinear form $\langle\ ,\ \rangle_W$ is the standard symplectic form. $\square$

Proposition 5. *With notation as in Theorem 1, the Yagita invariant $p^\circ(\mathrm{Sp}(2n,\mathcal{O}))$ is divisible by the number given in the statement of Theorem 1.*

Proof. To give lower bounds for $p^\circ(\mathrm{Sp}(2n,\mathcal{O}))$ we use finite subgroups. Firstly, consider the semidirect product $H = C_p \rtimes C_{p-1}$, where C_{p-1} acts faithfully on C_p; equivalently this is the group of affine transformations of the line over $\mathbb{F}_p$. It is well known that the image of $H^*(G;\mathbb{Z})$ inside $H^*(C_p;\mathbb{Z}) \cong \mathbb{Z}[x]/(px)$ is the subring generated by x^{p-1}. It follows that $2(p-1)$ divides $p^\circ(G)$ for any G containing H as a subgroup. The group H has a faithful permutation action on p points, and hence a faithful representation in $\mathrm{GL}(p-1,\mathbb{Z})$, where $\mathbb{Z}^{p-1}$ is identified with the kernel of the H-equivariant map $\mathbb{Z}\{1,\ldots,p\} \to \mathbb{Z}$. Since $\mathrm{GL}(p-1,\mathbb{Z})$ embeds in $\mathrm{Sp}(2(p-1),\mathbb{Z})$ we deduce that H embeds in $\mathrm{Sp}(2n,\mathcal{O})$ for each $\mathcal{O}$ and for each $n \geq p-1$.

To give a lower bound for the p-part of $p^\circ(\mathrm{Sp}(2n,\mathcal{O}))$ we use the extraspecial p-groups. For p odd, let $E(p,1)$ be the non-abelian p-group of order p^3 and exponent p, and let $E(2,1)$ be the dihedral group of order 8. (Equivalently in each case $E(p,1)$ is the Sylow p-subgroup of $\mathrm{GL}(3,\mathbb{F}_p)$.) For $m \geq 2$, let $E(p,m)$ denote the central product of m copies of $E(p,1)$, so that $E(p,m)$ is one of the two extraspecial groups of order p^{2m+1}. Yagita showed that $p^\circ(E(p,m)) = 2p^m$ for each m and p [6]. The centre and commutator subgroup of $E(p,m)$ are equal and have order p, and the abelianization of $E(p,m)$ is isomorphic to C_p^{2m}. The irreducible complex representations of $E(p,m)$ are well understood: there are p^{2m} distinct 1-dimensional irreducibles, each of which restricts to the centre as the trivial representation, and there are $p-1$ faithful representations of dimension p^m, each of which restricts to the centre as the sum of p^m copies of a single (non-trivial) irreducible representation of C_p. The group $G = E(p,m)$ contains a subgroup H isomorphic to C_p^{m+1}, and each of its faithful p^m-dimensional representations can be obtained by inducing up a 1-dimensional representation $H \to C_p \to \mathrm{GL}(1,\mathbb{C})$.

According to Bürgisser, C_p embeds in $\mathrm{Sp}(2l,\mathcal{O})$ (resp. in $\mathrm{Sp}(l,\mathcal{O})$ when l is even) provided that $\mathcal{O}$ is integrally closed in $\mathbb{C}$ [1]. Here as usual, $l := |F[\zeta_p], F|$ and F is the field of fractions of $\mathcal{O}$. If instead $\zeta_p \in \mathcal{O}$, then $l = 1$ and clearly C_p embeds in $\mathrm{GL}(1,\mathcal{O})$ and hence also in $\mathrm{Sp}(2,\mathcal{O}) = \mathrm{Sp}(2l,\mathcal{O})$. Taking this embedding of C_p and composing it with any homomorphism $H \to C_p$ we get a symplectic representation ρ of H on $\mathcal{O}^{2l}$ for any l (resp. on $\mathcal{O}^l$ for l even). For a suitable homomorphism we know that $\mathrm{Ind}_H^G(\rho)$ is a faithful representation of G on $\mathcal{O}^{2lp^m}$ (resp. on $\mathcal{O}^{lp^m}$ for l even) and by Lemma 4 we see that $\mathrm{Ind}_H^G(\rho)$ is symplectic. Hence we see that $E(m,p)$ embeds as a subgroup of $\mathrm{Sp}(2lp^m,\mathcal{O})$ for any l and as a subgroup of $\mathrm{Sp}(lp^m,\mathcal{O})$ in the case when l is even. Since $p^\circ(E(m,p)) = 2p^m$, this shows that $2p^m$ divides $p^\circ(\mathrm{Sp}(2lp^m,\mathcal{O}))$ always and that $2p^m$ divides $p^\circ(\mathrm{Sp}(lp^m,\mathcal{O}))$ in the case when l is even. □

Corollary 3 and Proposition 5 together complete the proof of Theorem 1.

We finish by pointing out that we have not computed $p^\circ(\mathrm{Sp}(2n,\mathcal{O}))$ for general $\mathcal{O}$ when $n < p-1$; to do this one would have to know which metacyclic groups $C_p \rtimes C_k$ with k coprime to p admit low-dimensional symplectic representations.

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