MOORE'S CONJECTURE FOR POLYHEDRAL PRODUCTS

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Abstract. Moore's Conjecture is shown to hold for generalized moment-angle complexes and a criterion is proved that determines when a polyhedral product is elliptic or hyperbolic.

1. Introduction

Moore's Conjecture envisions a deep relationship between the rational and torsion homotopy groups of finite CW-complexes. Let X be a finite CW-complex. The *homotopy exponent* of X at a prime p is the least power of p that annihilates the p-torsion in the homotopy groups of X. The space X is *elliptic* if it has finitely many rational homotopy groups and *hyperbolic* if it has infinitely many rational homotopy groups.

Moore's Conjecture: Let X be a finite, simply-connected CW-complex. Then the following are equivalent:

(a) X is elliptic;
(b) X has a finite homotopy exponent at every prime p;
(c) X has a finite homotopy exponent at some prime p.

The conjecture posits that the nature of the rational homotopy groups should have a profound impact on the nature of the, seemingly unrelated, torsion homotopy groups, and that torsion behaviour at one prime has a profound impact on torsion behaviour at all primes. The conjecture has been shown to hold in a number of cases. Elliptic spaces with finite exponents at all primes include spheres [13, 22], finite H-spaces [14], H-spaces with finitely generated cohomology [5], and odd primary Moore spaces [17]. Hyperbolic spaces with no exponent at any prime include wedges of simply-connected spheres, most torsion-free two-cell complexes [18], and torsion-free suspensions [20]. There are also partial results. In [16] it was shown that if X is elliptic then it has an exponent at all but finitely many primes, in [21] it was shown that if X is hyperbolic and $H_*(\Omega X;\mathbb{Z})$ is p-torsion free then, provided p is large enough, X has no exponent at p, and in [1] Moore's Conjecture was shown to hold for all but finitely many primes in the case of spaces having Lusternik-Schnirelmann category two.

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Moore's conjecture is also related to an important phenomenon in rational homotopy theory. Félix, Halperin and Thomas [8] proved the remarkable fact that a finite CW-complex is either elliptic or its total number of rational homotopy groups below dimension n grows exponentially with n. There is no hyperbolic space whose rational homotopy groups have polynomial growth.

In this paper we consider Moore's conjecture, and the notions of being elliptic or hyperbolic, in the context of polyhedral products. Let K be a simplicial complex on m vertices. For $1 \leq i \leq m$, let (X_i, A_i) be a pair of pointed CW-complexes, where A_i is a pointed subspace of X_i. Let $(\underline{X}, \underline{A}) = \{(X_i, A_i)\}_{i=1}^m$ be the sequence of CW-pairs. For each simplex (face) $\sigma \in K$, let $(\underline{X}, \underline{A})^\sigma$ be the subspace of $\prod_{i=1}^m X_i$ defined by

$$(\underline{X}, \underline{A})^\sigma = \prod_{i=1}^m \overline{X}_i \qquad \text{where} \qquad \overline{X}_i = \begin{cases} X_i & \text{if } i \in \sigma \\ A_i & \text{if } i \notin \sigma. \end{cases}$$

The *polyhedral product* determined by $(\underline{X}, \underline{A})$ and K is

$$(\underline{X}, \underline{A})^K = \bigcup_{\sigma \in K} (\underline{X}, \underline{A})^\sigma \subseteq \prod_{i=1}^m X_i.$$

The topology of polyhedral products has received a great deal of attention recently due to their central role in toric topology [2, 4, 10, 11, 12]. Important special cases include *moment-angle complexes* $\mathcal{Z}_K$, when each pair (X_i, A_i) equals (D^2, S^1), and *generalized moment-angle complexes* $\mathcal{Z}_K(D^n, S^{n-1})$, when each pair (X_i, A_i) equals (D^n, S^{n-1}) for $n \geq 2$. By [4], each generalized moment-angle complex is a finite, simply-connected CW-complex, so Moore's Conjecture may be considered.

To state our results some definitions are needed. Write $[m]$ for the vertex set $\{1, \ldots, m\}$. Let Δ^{m-1} be the standard m-simplex with vertex set $[m]$. The faces of Δ^{m-1} can be identified with sequences $(i_1, \ldots, i_k)$ for $1 \leq i_1 < \cdots < i_k \leq m$. If K is a simplicial complex on the vertex set $[m]$ then a sequence $\sigma = (i_1, \ldots, i_k)$ is a *missing face* of K if $\sigma \notin K$. It is a *minimal missing face* of K if no proper subsequence of σ is a missing face of K.

Theorem 1.1. *Let K be a simplicial complex on the vertex set $[m]$ and let $(\underline{X}, \underline{A})$ be any sequence of pairs (D^{n_i}, S^{n_i-1}) with $n_i \geq 2$ for $1 \leq i \leq m$. Then:*

(a) *$(\underline{X}, \underline{A})^K$ is elliptic if and only if the minimal missing faces of K are mutually disjoint;*

(b) *Moore's conjecture holds for $(\underline{X}, \underline{A})^K$.*

In particular, Theorem 1.1 includes generalized moment-angle complexes $\mathcal{Z}_K(D^n, S^{n-1})$ for $n \geq 2$ as a special case. Part (a) of Theorem 1.1 was proved by [2] in the special case of the moment-angle complex $\mathcal{Z}_K$, although part (b) was not. The restriction to $n \geq 2$ is made to ensure that certain retractions constructed in Theorem 4.2 involve wedges of simply-connected spheres which are hyperbolic, rather than wedge of circles which are Eilenberg-MacLane spaces.

We also give a general criterion for when a polyhedral product is elliptic or hyperbolic. This generalizes and reformulates in more combinatorial terms results obtained by Félix and Tanré [9].

Theorem 1.2. *Let K be a simplicial complex on the vertex set $[m]$ and let $(\underline{X}, \underline{A})$ be any sequence of pointed, path-connected pairs. For $1 \leq i \leq m$, let Y_i be the homotopy fibre of the inclusion $A_i \longrightarrow X_i$ and suppose that each Y_i is rationally nontrivial. Then the polyhedral product $(\underline{X}, \underline{A})^K$ is elliptic if and only if three conditions hold:*

(i) *each X_i is elliptic;*
(ii) *all the minimal missing faces of K are mutually disjoint;*
(iii) *if v is a vertex of a minimal missing face of K then Y_v is rationally homotopy equivalent to a sphere.*

For example, let K be the boundary of a pentagon. A result essentially due to MacGavran [15] shows that $\mathcal{Z}_K$ is diffeomorphic to the connected sum of 5 copies of $S^3 \times S^4$. It is well known that such a connected sum is hyperbolic. If the ingredient pairs of spaces change from (D^2, S^1) to $(C\mathbb{C}P^n, \mathbb{C}P^n)$, where $C\mathbb{C}P^n$ is the cone on $\mathbb{C}P^n$, then the polyhedral product $(\underline{C\mathbb{C}P^n}, \underline{\mathbb{C}P^n})^K$ is some analogue of a connected sum, but its homotopy type is not clear. Observe that each Y_i in this case is $\Omega\mathbb{C}P^n$, so is rationally nontrivial, and the minimal missing faces of K are not mutually disjoint. So Theorem 1.2 implies that $(\underline{C\mathbb{C}P^n}, \underline{\mathbb{C}P^n})^K$ is hyperbolic. The most pertinent point here is that this determination is made without any reference to the standard differential graded Lie algebra tools commonly used to decide ellipticity or hyperbolicity.

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2. Polyhedral product ingredients

This section contains the properties of polyhedral products that will be needed. The main results are Theorems 2.1 and 2.3, which are of independent interest.

Theorem 2.1. *Let K be a simplicial complex on the vertex set $[m]$ and let $(\underline{X}, \underline{A})$ be any sequence of pointed, path-connected CW-pairs. Then there is a homotopy fibration*

$$(\underline{CY}, \underline{Y})^K \longrightarrow (\underline{X}, \underline{A})^K \longrightarrow \prod_{i=1}^{m} X_i$$

where, for $1 \leq i \leq m$, Y_i is the homotopy fibre of the inclusion $A_i \longrightarrow X_i$.

Proof. In general, let $f\colon B \longrightarrow Z$ be a pointed, continuous map, let $I = [0, 1]$ be the unit interval with basepoint 0, and let PZ be the path space of Z. Then the homotopy fibre of f is the pullback of f and the evaluation map $ev_1\colon PZ \longrightarrow Z$, where $ev_1(\omega) = \omega(1)$. In our case, we are given m pairs of spaces (X_i, A_i). The homotopy pullback of the identity map $1_{X_i}\colon X_i \longrightarrow X_i$ is PX_i, and Y_i is defined as the homotopy pullback of the inclusion $j_i\colon A_i \longrightarrow X_i$. Note that as j_i is a subspace inclusion then Y_i is the inverse image $ev_1^{-1}(A_i) \subseteq PX_i$.

Let σ be a face in K. By the definition of a polyhedral product, $(\underline{X},\underline{A})^\sigma = \overline{X}_1 \times \cdots \times \overline{X}_m$, where $\overline{X}_i$ is X_i if $i \in \sigma$ and is A_i if $i \notin \sigma$. Consider the homotopy pullback

$$\begin{array}{ccc} Q & \longrightarrow & P(X_1 \times \cdots \times X_m) \\ \downarrow & & \downarrow{\scriptstyle ev_1} \\ (\underline{X},\underline{A})^\sigma & \xrightarrow{i^\sigma} & X_1 \times \cdots \times X_m \end{array}$$

where i^σ is the inclusion. Observe that $P(X_1 \times \cdots \times \cdots X_m)$ is homeomorphic to $PX_1 \times \cdots \times PX_m$ and under this homeomorphism ev_1 translates into a product of the m evaluation maps ev_1 on each PX_i. As the product of pullbacks is a pullback, we see that Q is homeomorphic to $\overline{Y}_1 \times \cdots \times \overline{Y}_m$, where $\overline{Y}_i$ is PX_i if $i \in \sigma$ and is Y_i if $i \notin \sigma$. That is, Q is homeomorphic to $(\underline{PX},\underline{Y})^\sigma$. Moreover, since i^σ is an inclusion, $(\underline{PX},\underline{Y})^\sigma$ is the inverse image $(ev_1 \times \cdots \times ev_1)^{-1}((\underline{X},\underline{A})^\sigma) \subseteq PX_1 \times \cdots \times PX_m$. Now $(\underline{X},\underline{A})^K$ is the union of the spaces $(\underline{X},\underline{A})^\sigma$ for all $\sigma \in K$, where intersections have been identified. Since inverse images preserve unions and intersections, we obtain that the homotopy fibre of the inclusion $(\underline{X},\underline{A})^K \longrightarrow X_1 \times \cdots \times X_m$ is homeomorphic to $(\underline{PX},\underline{Y})^K$.

Finally, since PX_i is contractible, the inclusion $Y_i \longrightarrow PX_i$ extends to a map $CY_i \longrightarrow PX_i$ which is a homotopy equivalence. Thus the induced map of pairs $(CY_i, Y_i) \longrightarrow (PX_i, Y_i)$ is a homotopy equivalence. Hence the homotopy fibre of the inclusion $(\underline{X},\underline{A})^K \longrightarrow X_1 \times \cdots \times X_m$ is homotopy equivalent to $(\underline{CY},\underline{Y})^K$. □

Next, we show that the homotopy fibration in Theorem 2.1 splits after looping. Let K be a simplicial complex on the vertex set $[m]$. If $I \subseteq [m]$ then the *full subcomplex* K_I of K is defined as the simplicial complex

$$K_I = \bigcup \{\sigma \in K \mid \text{the vertex set of } \sigma \text{ is in } I\}.$$

The definition of K_I implies that the inclusion $K_I \longrightarrow K$ is a map of simplicial complexes. This induces a map of polyhedral products $(\underline{X},\underline{A})^{K_I} \longrightarrow (\underline{X},\underline{A})^K$. There is no retraction of K_I off K as simplicial complexes, however, in [7] it was shown that there is nevertheless a retraction of $(\underline{X},\underline{A})^{K_I}$ off $(\underline{X},\underline{A})^K$.

Proposition 2.2. *Let K be a simplicial complex on the vertex set $[m]$ and let $(\underline{X},\underline{A})$ be any sequence of pointed, path-connected CW-pairs. Let $I \subseteq [m]$. Then the inclusion $(\underline{X},\underline{A})^{K_I} \longrightarrow (\underline{X},\underline{A})^K$ has a left inverse.* □

Theorem 2.3. *Let $(\underline{CY},\underline{Y})^K \longrightarrow (\underline{X},\underline{A})^K \longrightarrow \prod_{i=1}^m X_i$ be the homotopy fibration in Theorem 2.1. Then there is a homotopy equivalence*

$$\Omega(\underline{X},\underline{A})^K \simeq (\prod_{i=1}^m \Omega X_i) \times \Omega(\underline{CY},\underline{Y})^K.$$

Proof. For $1 \leq i \leq m$, let $I_i = \{i\}$. Observe that the full subcomplex K_{I_i} of K is just the vertex $\{i\}$. By the definition of the polyhedral product, $(\underline{X},\underline{A})^{K_{I_i}} = X_i$. Proposition 2.2 therefore implies that X_i retracts off $(\underline{X},\underline{A})^K$. Explicitly, the composite $X_i = (\underline{X},\underline{A})^{K_{I_i}} \longrightarrow (\underline{X},\underline{A})^K \longrightarrow \prod_{i=1}^m X_i \stackrel{\text{proj}}{\longrightarrow} X_i$ is the identity map. After looping, the loop maps $\Omega X_i \longrightarrow \Omega(\underline{X},\underline{A})^K$ may be multiplied together to obtain a map $\prod_{i=1}^m \Omega X_i \longrightarrow \Omega(\underline{X},\underline{A})^K$ which is a right homotopy inverse of the map $\Omega(\underline{X},\underline{A})^K \longrightarrow \prod_{i=1}^m \Omega X_i$. Hence, if μ is the loop multiplication on $\Omega(\underline{X},\underline{A})^K$, then the composite

$$(\prod_{i=1}^m \Omega X_i) \times \Omega(\underline{CY},\underline{Y})^K \longrightarrow \Omega(\underline{X},\underline{A})^K \times \Omega(\underline{X},\underline{A})^K \stackrel{\mu}{\longrightarrow} \Omega(\underline{X},\underline{A})^K$$

is a homotopy equivalence. □

Theorem 2.3 implies that homotopy group information about $(\underline{X},\underline{A})^K$ is determined by that of the ingredient spaces X_i and $(\underline{CY},\underline{Y})^K$. This is useful because the spaces $(\underline{CY},\underline{Y})^K$ are much better understood than the spaces $(\underline{X},\underline{A})^K$.

This section concludes with the statement of two other results which will be used later. The first, proved in [11], relates pushouts of simplicial complexes to pushouts of polyhedral products.

Proposition 2.4. *Let K be a simplicial complex on the vertex set $[m]$. Suppose that there is a pushout of simplicial complexes*

$$\begin{array}{ccc} L & \longrightarrow & K_2 \\ \downarrow & & \downarrow \\ K_1 & \longrightarrow & K \end{array}$$

Let L°, K_1° and K_1° be L, K_1 and K_2 regarded as simplicial complexes on the same vertex set as K. Then there is a pushout of polyhedral products

$$\begin{array}{ccc} (\underline{X},\underline{A})^{L^\circ} & \longrightarrow & (\underline{X},\underline{A})^{K_2^\circ} \\ \downarrow & & \downarrow \\ (\underline{X},\underline{A})^{K_1^\circ} & \longrightarrow & (\underline{X},\underline{A})^K. \end{array}$$

□

Second, we give two examples where the homotopy type of $(\underline{CY},\underline{Y})^K$ is explicitly identified. Part (a) in Lemma 2.5 is immediate from the definition of the polyhedral product, while part (b) was proved by Porter [19] when each Y_i is a loop space and more generally in [11].

Lemma 2.5. *Let $Y_1,\ldots,Y_m$ be path-connected spaces. Then the following hold:*

(a) $(\underline{CY},\underline{Y})^{\Delta^{m-1}} = \prod_{i=1}^m CY_i$;
(b) $(\underline{CY},\underline{Y})^{\partial\Delta^{m-1}} \simeq \Sigma^{m-1} Y_1 \wedge \cdots \wedge Y_m$.

□

3. Combinatorial ingredients

This section records the combinatorial information that will be needed. Let K be a simplicial complex on the index set $[m]$. For a vertex $v \in K$, the *star*, *restriction* (or *deletion*) and *link* of v are the subcomplexes

$$\begin{aligned} \mathrm{star}_K(v) &= \{\tau \in K \mid \{v\} \cup \tau \in K\}; \\ K\backslash v &= \{\tau \in K \mid \{v\} \cap \tau = \emptyset\}; \\ \mathrm{link}_K(v) &= \mathrm{star}_K(v) \cap K\backslash v. \end{aligned}$$

The *join* of two simplicial complexes K_1, K_2 on disjoint index sets is the simplicial complex

$$K_1 * K_2 = \{\sigma_1 \cup \sigma_2 \mid \sigma_i \in K_i\}.$$

From the definitions, it follows that $\mathrm{star}_K(v)$ is a join,

$$\mathrm{star}_K(v) = \{v\} * \mathrm{link}_K(v),$$

and there is a pushout

$$\begin{array}{ccc} \mathrm{link}_K(v) & \longrightarrow & \mathrm{star}_K(v) \\ \downarrow & & \downarrow \\ K\backslash v & \longrightarrow & K. \end{array}$$

A *face* of K is a simplex of K. Let Δ^{m-1} be the standard m-simplex on the vertex set $[m]$ and note that K is a subcomplex of Δ^{m-1}. Recall from the Introduction that a face $\sigma \in \Delta^{m-1}$ is a *missing face* of K if $\sigma \notin K$. It is a *minimal missing face* if any proper face of σ is a face of K. Denote the set of minimal missing faces of K by $MMF(K)$. For a simplex σ, let $\partial\sigma$ be its boundary. Observe that $\sigma \in MMF(K)$ if and only if $\sigma \notin K$ but $\partial\sigma \subseteq K$.

There is a special case which will play a crucial role in what follows. Let $\overline{K}$ be a simplicial complex on the vertex set $[m]$ with the property that it has precisely two distinct minimal missing faces and these have non-empty intersection. That is, suppose that $MMF(\overline{K}) = \{\sigma_1, \sigma_2\}$ where σ_1 and σ_2 have vertex sets I and J respectively, satisfying $I \neq J$, $I \cup J = [m]$ and $I \cap J \neq \emptyset$. Let w be a vertex in both I and J.

Consider the star-link-restriction pushout of $\overline{K}$ with respect to the vertex w:

$$\begin{array}{ccc} \mathrm{link}_{\overline{K}}(w) & \longrightarrow & \mathrm{star}_{\overline{K}}(w) \\ \downarrow & & \downarrow \\ \overline{K}\backslash w & \longrightarrow & \overline{K}. \end{array} \tag{1}$$

Let $\overline{\sigma}_1$ and $\overline{\sigma}_2$ be the proper faces of σ_1 and σ_2 on the vertex sets $\overline{I} = I\backslash\{w\}$ and $\overline{J} = J\backslash\{w\}$ respectively.

Lemma 3.1. *We have $\overline{\sigma}_1, \overline{\sigma}_2 \in MMF(star_{\overline{K}}(w))$.*

Proof. Consider $\overline{\sigma}_1$, the argument for $\overline{\sigma}_2$ being similar. First we show that $\overline{\sigma}_1$ is a missing face of $\text{star}_{\overline{K}}(w)$. For if $\overline{\sigma}_1 \in \text{star}_{\overline{K}}(w)$ then, as w is not a vertex of $\overline{\sigma}_1$, we also have $\overline{\sigma}_1 \in \overline{K}\backslash w$, implying that $\overline{\sigma}_1 \in \text{link}_{\overline{K}}(w) = \text{star}_{\overline{K}}(w) \cap \overline{K}\backslash w$. This in turn implies that $\overline{\sigma}_1 * \{w\} \in \text{star}_{\overline{K}}(w)$. But $\overline{\sigma}_1 * \{w\} = \sigma_1$, so $\sigma_1 \in \text{star}_{\overline{K}}(w)$. Therefore, by (1), $\sigma_1 \in \overline{K}$, contradicting the fact that σ_1 is a missing face of $\overline{K}$.

Next, we show that that $\overline{\sigma}_1$ is a minimal missing face of $\text{star}_{\overline{K}}(w)$. If not, then some proper face τ of $\overline{\sigma}_1$ is also a missing face of $\text{star}_{\overline{K}}(w)$. As w is not a vertex of $\overline{\sigma}_1$, it is not a vertex of τ either. Therefore $\tau * \{w\}$ is a missing face of $\text{star}_{\overline{K}}(w)$. The presence of the vertex w in $\tau * \{w\}$ implies that it is also not a face of $\overline{K}\backslash w$. On the other hand, by (1), $\overline{K}$ is the union of $\text{star}_{\overline{K}}(w)$ and $\overline{K}\backslash w$, so a face that is missing from both $\text{star}_{\overline{K}}(w)$ and $\overline{K}\backslash w$ must also be missing from $\overline{K}$. Therefore $\tau * \{w\}$ is a missing face of $\overline{K}$. But as τ is a proper face of $\overline{\sigma}_1$, $\tau * \{w\}$ is a proper face of $\overline{\sigma}_1 * \{w\} = \sigma_1$, contradicting the fact that σ_1 is a minimial missing face of $\overline{K}$. □

Corollary 3.2. *We have $\partial\overline{\sigma}_1, \partial\overline{\sigma}_2 \subseteq \text{link}_{\overline{K}}(w)$ and $\overline{\sigma}_1, \overline{\sigma}_2 \notin \text{link}_{\overline{K}}(w)$.*

Proof. Recall that a face σ of a simplicial complex K is a minimal missing face if and only if $\sigma \notin K$ but $\partial\sigma \subseteq K$. So by Lemma 3.1, $\partial\overline{\sigma}_1, \partial\overline{\sigma}_2 \subseteq \text{star}_{\overline{K}}(w)$. By definition, neither $\overline{\sigma}_1$ nor $\overline{\sigma}_2$ have w in their vertex sets, so neither do their boundaries. Therefore $\partial\overline{\sigma}_1, \partial\overline{\sigma}_2 \subseteq \overline{K}\backslash w$. Therefore, as $\text{link}_{\overline{K}}(w) = \text{star}_{\overline{K}}(w) \cap \overline{K}\backslash w$, we have $\partial\overline{\sigma}_1, \partial\overline{\sigma}_2 \subseteq \text{link}_{\overline{K}}(w)$.

Also, as $\text{link}_{\overline{K}}(w) = \text{star}_{\overline{K}}(w) \cap \overline{K}\backslash w$, it cannot be that $\overline{\sigma}_1, \overline{\sigma}_2$ are in $\text{link}_{\overline{K}}(w)$ as that would imply they are also in $\text{star}_{\overline{K}}(w)$, contradicting Lemma 3.1. □

One further observation we need regarding $\overline{K}$ is the following. Regarding w as the m^{th}-vertex of $\overline{K}$, observe that $\overline{K}\backslash w$ is a simplicial complex on the vertex set $[m-1]$.

Lemma 3.3. *There is an isomorphism of simplicial complexes $\overline{K}\backslash w \cong \Delta^{m-2}$.*

Proof. It is equivalent to show that $\overline{K}\backslash w$ has no missing faces. Suppose that $\sigma \in \Delta^{m-2}$ is a missing face of $\overline{K}\backslash w$. Then as $\overline{K}\backslash w$ is the restriction of $\overline{K}$ to the vertex set $[m-1]$, σ is also a missing face of $\overline{K}$. On the other hand, as $MMF(\overline{K}) = \{\sigma_1, \sigma_2\}$, any missing face of $\overline{K}$ must have either σ_1 or σ_2 as a subface. Thus σ must have either σ_1 or σ_2 as a subface. But this cannot happen since w is not in the vertex set of σ but it is in the vertex sets of both σ_1 and σ_2. □

4. Moore's conjecture

In this section we prove Theorems 1.1 and 1.2 as consequences of Theorem 4.2.

Proposition 4.1. *Let K be a simplicial complex on the vertex set $[m]$ and let $X_1, \ldots, X_m$ be any sequence of pointed, path-connected CW-pairs. Suppose that $\sigma_1, \sigma_2 \in MMF(K)$ and let I and J be the vertex sets of σ_1 and σ_2 respectively. If $I \neq J$, $I \cup J = [m]$ and $I \cap J \neq \emptyset$, then $(\underline{CX}, \underline{X})^{\partial\sigma_1} \vee (\underline{CX}, \underline{X})^{\partial\sigma_2}$ is a retract of $(\underline{CX}, \underline{X})^K$.*

Proof. A new simplicial complex is introduced that will act as an intermediary. In general, a simplicial complex may be characterized by listing its minimal missing faces. Let $\overline{K}$ be the simplicial complex on the vertex set $[m]$ that is characterized by the condition that $MMF(\overline{K}) = \{\sigma_1, \sigma_2\}$. Intuitively, $\overline{K}$ is obtained from K by filling in all missing faces that do not have either σ_1 or σ_2 as a subface. Rigorously, there is a map of simplicial complexes $K \longrightarrow \overline{K}$ that induces a map of polyhedral products $(\underline{CX}, \underline{X})^K \longrightarrow (\underline{CX}, \underline{X})^{\overline{K}}$. Since σ_1, σ_2 are minimal missing faces of K, we have $\sigma_1, \sigma_2 \notin K$ but $\partial\sigma_1, \partial\sigma_2 \subseteq K$. The inclusion $\partial\sigma_1 \longrightarrow K$ is a map of simplicial complexes and it induces a map of polyhedral products $(\underline{CX}, \underline{X})^{\partial\sigma_1} \longrightarrow (\underline{CX}, \underline{X})^K$. There is a similar map with respect to $\partial\sigma_2$. We will show that the composite $(\underline{CX}, \underline{X})^{\partial\sigma_1} \vee (\underline{CX}, \underline{X})^{\partial\sigma_2} \longrightarrow (\underline{CX}, \underline{X})^K \longrightarrow (\underline{CX}, \underline{X})^{\overline{K}}$ has a left homotopy inverse. Note that this composite of polyhedral products is the same as the one induced by the inclusions $\partial\sigma_1 \longrightarrow \overline{K}$ and $\partial\sigma_2 \longrightarrow \overline{K}$, so it suffices to show that the map $(\underline{CX}, \underline{X})^{\partial\sigma_1} \vee (\underline{CX}, \underline{X})^{\partial\sigma_2} \longrightarrow (\underline{CX}, \underline{X})^{\overline{K}}$ has a left homotopy inverse.

The conditions on the vertex sets I and J imply that $\overline{K}$ has the same form as in Section 3. Relabelling the spaces $X_1, \ldots, X_m$ if necessary, we may suppose that the intersection vertex w corresponds to the m^{th}-coordinate space X_m. By Proposition 2.4, the pushout of simplicial complexes in (1) implies that there is a pushout of polyhedral products

$$\begin{array}{ccc} (\underline{CX}, \underline{X})^{\mathrm{link}_{\overline{K}}(w)^\circ} & \xrightarrow{g^\circ} & (\underline{CX}, \underline{X})^{\mathrm{star}_{\overline{K}}(w)^\circ} \\ \downarrow{\scriptstyle f^\circ} & & \downarrow \\ (\underline{CX}, \underline{X})^{\overline{K}\backslash w^\circ} & \longrightarrow & (\underline{CX}, \underline{X})^{\overline{K}}. \end{array} \tag{2}$$

where $\mathrm{link}_{\overline{K}}(w)^\circ$, $\mathrm{star}_{\overline{K}}(w)^\circ$ and $\overline{K}\backslash w^\circ$ are $\mathrm{link}_{\overline{K}}(w)$, $\mathrm{star}_{\overline{K}}(w)$ and $\overline{K}\backslash w$ regarded as having vertex set $[m]$, and the maps f° and g° are induced by the inclusions $\mathrm{link}_{\overline{K}}(w)^\circ \longrightarrow \overline{K}\backslash w^\circ$ and $\mathrm{link}_{\overline{K}}(w)^\circ \longrightarrow \mathrm{star}_{\overline{K}}(w)^\circ$ respectively. The vertex sets of $\mathrm{link}_{\overline{K}}(w)$ and $\overline{K}\backslash w$ are both $[m-1]$, so by the definition of the polyhedral product,

$$(\underline{CX}, \underline{X})^{\mathrm{link}_{\overline{K}}(w)^\circ} = (\underline{CX}, \underline{X})^{\mathrm{link}_{\overline{K}}(w)} \times X_m \qquad (\underline{CX}, \underline{X})^{\overline{K}\backslash w^\circ} = (\underline{CX}, \underline{X})^{\overline{K}\backslash w} \times X_m$$

and $f^\circ = f \times 1$ where f is induced by the inclusion $\mathrm{link}_{\overline{K}}(w) \longrightarrow \overline{K}\backslash w$ and 1 is the identity map on X_m. On the other hand, the vertex set of $\mathrm{star}_{\overline{K}}(w)$ is $[m]$ so $\mathrm{star}_{\overline{K}}(w)^\circ = \mathrm{star}_{\overline{K}}(w)$. Since $\mathrm{star}_{\overline{K}}(w) = \mathrm{link}_{\overline{K}}(w) * \{w\}$, the definition of the polyhedral product implies that

$$(\underline{CX}, \underline{X})^{\mathrm{star}_{\overline{K}}(w)^\circ} = (\underline{CX}, \underline{X})^{\mathrm{link}_{\overline{K}}(w)} \times CX_m$$

and $g^\circ = 1 \times i_m$ where 1 is the identity map on $(\underline{CX}, \underline{X})^{\mathrm{link}_{\overline{K}}(w)}$ and $i_m \colon X_m \longrightarrow CX_m$ is the inclusion of the base of the cone. Putting all this together, the pushout (2) becomes the pushout

$$(3) \qquad \begin{array}{ccc} (\underline{CX}, \underline{X})^{\mathrm{link}_{\overline{K}}(w)} \times X_m & \xrightarrow{1 \times i_m} & (\underline{CX}, \underline{X})^{\mathrm{link}_{\overline{K}}(w)} \times CX_m \\ \Big\downarrow{\scriptstyle f \circ 1} & & \Big\downarrow \\ (\underline{CX}, \underline{X})^{\overline{K} \backslash w} \times X_m & \longrightarrow & (\underline{CX}, \underline{X})^{\overline{K}}. \end{array}$$

By Lemma 3.3, $\overline{K} \backslash w \cong \Delta^{m-2}$, so by Lemma 2.5 (a), $(\underline{CX}, \underline{X})^{\overline{K} \backslash w} = \prod_{i=1}^{m-1} CX_i$. Therefore, in (3), both $(\underline{CX}, \underline{X})^{\overline{K} \backslash w}$ and CX_m are contractible, implying that (3) is equivalent, up to homotopy, to the homotopy pushout

$$(4) \qquad \begin{array}{ccc} (\underline{CX}, \underline{X})^{\mathrm{link}_{\overline{K}}(w)} \times X_m & \xrightarrow{\pi_1} & (\underline{CX}, \underline{X})^{\mathrm{link}_{\overline{K}}(w)} \\ \Big\downarrow{\scriptstyle \pi_2} & & \Big\downarrow \\ X_m & \longrightarrow & (\underline{CX}, \underline{X})^{\overline{K}} \end{array}$$

where π_1 and π_2 are the projections onto the first and second factors respectively. It is well known that the pushout of the projections $A \times B \longrightarrow A$ and $A \times B \longrightarrow B$ is homotopy equivalent to the join of A and B, which in turn is homotopy equivalent to $\Sigma A \wedge B$. So (4) implies that there is a homotopy equivalence

$$(5) \qquad (\underline{CX}, \underline{X})^{\overline{K}} \simeq \Sigma (\underline{CX}, \underline{X})^{\mathrm{link}_{\overline{K}}(w)} \wedge X_m.$$

Now consider the minimal missing faces σ_1 and σ_2 of $\overline{K}$. As in Section 3, let $\overline{\sigma}_1, \overline{\sigma}_2$ be the restrictions of σ_1, σ_2 respectively to the vertex sets $\overline{I} = I \backslash \{w\}, \overline{J} = J \backslash \{w\}$. Note that as $I \neq J$ we also have $\overline{I} \neq \overline{J}$. By Corollary 3.2, $\overline{\sigma}_1, \overline{\sigma}_2 \notin \mathrm{link}_{\overline{K}}(w)$ but $\partial \overline{\sigma}_1, \partial \overline{\sigma}_2 \subseteq \mathrm{link}_{\overline{K}}(w)$. Therefore, the full subcomplex of $\mathrm{link}_{\overline{K}}(w)$ on $\overline{I}$ is $\partial \overline{\sigma}_1$, and the full subcomplex of $\mathrm{link}_{\overline{K}}(w)$ on $\overline{J}$ is $\partial \overline{\sigma}_2$. By Proposition 2.2, this implies that $(\underline{CX}, \underline{X})^{\partial \overline{\sigma}_1}$ and $(\underline{CX}, \underline{X})^{\partial \overline{\sigma}_2}$ are retracts of $(\underline{CX}, \underline{X})^{\mathrm{link}_{\overline{K}}(w)}$. By [2, Theorem 2.21], the fact that $\partial \overline{\sigma}_1$ and $\partial \overline{\sigma}_2$ are full subcomplexes of $\mathrm{link}_{\overline{K}}(w)$ on different index sets implies that that $\Sigma (\underline{CX}, \underline{X})^{\partial \overline{\sigma}_1} \vee \Sigma (\underline{CX}, \underline{X})^{\partial \overline{\sigma}_2}$ is a retract of $\Sigma (\underline{CX}, \underline{X})^{\mathrm{link}_{\overline{K}}(w)}$. Thus (5) implies that $(\Sigma (\underline{CX}, \underline{X})^{\partial \overline{\sigma}_1} \wedge X_m) \vee (\Sigma (\underline{CX}, \underline{X})^{\partial \overline{\sigma}_2} \wedge X_m)$ is a retract of $(\underline{CX}, \underline{X})^{\overline{K}}$.

We wish to choose the retraction more carefully. Restrict $\overline{K}$ to the full subcomplex on the vertex set I. Then $MMF(\overline{K}_I) = \{\sigma_1\}$, so $\overline{K}_I = \partial \sigma_1$. Therefore the star-link-restriction pushout for $\overline{K}_I$ with respect to the vertex w becomes

$$\begin{array}{ccc} \partial \overline{\sigma}_1 & \longrightarrow & \partial \overline{\sigma}_1 * \{w\} \\ \Big\downarrow & & \Big\downarrow \\ \partial \sigma_1 \backslash w & \longrightarrow & \partial \sigma_1. \end{array}$$

Note that $\partial \sigma_1 \backslash w$ is the simplex Δ^{k-1} on the vertex set $\{i_1, \ldots, i_k\}$. Now arguing as for (2) – (4) and equation (5), we obtain in place of (5) a homotopy equivalence $(\underline{CX}, \underline{X})^{\overline{K}_I} = (\underline{CX}, \underline{X})^{\partial \sigma_1} \simeq$

$\Sigma(\underline{CX},\underline{X})^{\partial\,\overline{\sigma}_1}\wedge X_m$. Thus we may choose the map $\Sigma(\underline{CX},\underline{X})^{\partial\,\overline{\sigma}_1}\wedge X_m\longrightarrow(\underline{CX},\underline{X})^{\overline{K}}$ as the composite $\Sigma(\underline{CX},\underline{X})^{\partial\,\overline{\sigma}_1}\wedge X_m\xrightarrow{\simeq}(\underline{CX},\underline{X})^{\partial\sigma_1}\longrightarrow(\underline{CX},\underline{X})^{\overline{K}}$. Doing the same for $\partial\sigma_2$ we obtain a composite $(\Sigma(\underline{CX},\underline{X})^{\partial\,\overline{\sigma}_1}\wedge X_m)\vee(\Sigma(\underline{CX},\underline{X})^{\partial\,\overline{\sigma}_2}\wedge X_m)\xrightarrow{\simeq}(\underline{CX},\underline{X})^{\partial\sigma_1}\vee(\underline{CX},\underline{X})^{\partial\sigma_2}\longrightarrow(\underline{CX},\underline{X})^{\overline{K}}$, and it is this composite that has a left homotopy inverse. In particular, we have produced a left homotopy inverse for the map $(\underline{CX},\underline{X})^{\partial\sigma_1}\vee(\underline{CX},\underline{X})^{\partial\sigma_2}\longrightarrow(\underline{CX},\underline{X})^{\overline{K}}$ induced by the inclusions $\partial\sigma_1\longrightarrow\overline{K}$ and $\partial\sigma_2\longrightarrow\overline{K}$, as required. □

Recall that for $1\leq i\leq m$, Y_i is the homotopy fibre of the inclusion $A_i\longrightarrow X_i$.

Theorem 4.2. *Let K be a simplicial complex on the vertex set $[m]$ and let $(\underline{X},\underline{A})$ be any sequence of pointed, path-connected CW-pairs. The following hold:*

(a) *if $MMF(K)=\{\sigma_1,\ldots,\sigma_n\}$ and these minimal missing faces are mutually disjoint, then there is a homotopy equivalence*

$$\Omega(\underline{X},\underline{A})^K\simeq(\prod_{i=1}^m\Omega X_i)\times(\prod_{j=1}^n\Omega(\underline{CY},\underline{Y})^{\partial\sigma_j});$$

(b) *if σ_1 and σ_2 are minimal missing faces of K with nontrivial intersection then $\Omega\left((\underline{CY},\underline{Y})^{\partial\sigma_1}\vee(\underline{CY},\underline{Y})^{\partial\sigma_2}\right)$ retracts off $\Omega(\underline{X},\underline{A})^K$.*

Proof. By Theorem 2.3, there is a homotopy equivalence

$$\Omega(\underline{X},\underline{A})^K\simeq(\prod_{i=1}^m\Omega X_i)\times\Omega(\underline{CY},\underline{Y})^K\tag{6}$$

where, for $1\leq i\leq m$, Y_i is the homotopy fibre of the inclusion $A_i\longrightarrow X_i$.

If all of the minimal missing faces of K are mutually disjoint then there is a simplicial isomorphism $K\cong K_0*K_1*\cdots*K_n$ where K_0 is a product of simplices and, for $1\leq j\leq n$, $K_j=\partial\sigma_j$ (a proof of this may be found in [3], although it may be more commonly known). In general, the definition of a polyhedral product implies that there is a homeomorphism $(\underline{X},\underline{A})^{L*M}\cong(\underline{X},\underline{A})^L\times(\underline{X},\underline{A})^M$. In our case, as K_0 is a simplex, Lemma 2.5 (a) implies that $(\underline{CY},\underline{Y})^{K_0}$ is a product of cones and so is contractible. Thus

$$(\underline{CY},\underline{Y})^K\simeq(\underline{CY},\underline{Y})^{K_1}\times\cdots\times(\underline{CY},\underline{Y})^{K_n}=(\underline{CY},\underline{Y})^{\partial\sigma_1}\times\cdots\times(\underline{CY},\underline{Y})^{\partial\sigma_n}.$$

Combining this with (6), the homotopy decomposition in part (a) follows.

Next, suppose that σ_1 and σ_2 are minimal missing faces of K that intersect nontrivially. Let I and J be the vertex sets of σ_1 and σ_2 respectively. Let $K_{I\cup J}$ be the full subcomplex of K on the index set $I\cup J$. By Proposition 2.2, $(\underline{CY},\underline{Y})^{K_{I\cup J}}$ is a retract of $(\underline{CY},\underline{Y})^K$. Further, Proposition 4.1 implies that $(\underline{CY},\underline{Y})^{\partial\sigma_1}\vee(\underline{CY},\underline{Y})^{\partial\sigma_2}$ is a retract of $(\underline{CY},\underline{Y})^{K_{I\cup J}}$. Hence $(\underline{CY},\underline{Y})^{\partial\sigma_1}\vee(\underline{CY},\underline{Y})^{\partial\sigma_2}$ is a retract of $(\underline{CY},\underline{Y})^K$. Combining this with (6), the assertion in part (b) follows. □

We now turn to Moore's Conjecture and the distinguishing of elliptic and hyperbolic spaces. For Theorem 1.1, we assume that each pair (X_i,A_i) is (D^{n_i},S^{n_i-1}) for $n_i\geq 2$. Note that the homotopy fibre Y_i of the inclusion $S^{n_i-1} \longrightarrow D^{n_i}$ is also S^{n_i-1}, so the pair (CY_i, Y_i) in Theorem 4.2 is also homotopy equivalent to (D^{n_i}, S^{n_i-1}). Note that as each X_i is D^{n_i}, the term $\prod_{i=1}^{m} \Omega X_i$ in Theorem 4.2 (a) is contractible. Also, by Lemma 2.5 (b), each term $(\underline{CY}, \underline{Y})^{\partial\sigma_i}$ in Theorem 4.2 (a) and (b) is homotopy equivalent to a simply-connected sphere.

Proof of Theorem 1.1. Theorem 4.2 (b) implies that if K has two minimal missing faces with non-trivial intersection then a wedge of two simply-connected spheres retracts off $\Omega(\underline{X}, \underline{A})^K$. The Hilton-Milnor Theorem shows that a wedge of two such spheres is hyperbolic, and Neisendorfer and Selick [18] showed that a wedge of two such spheres has no exponent at any prime p. Hence Moore's conjecture holds in this case. On the other hand, if all the minimal missing faces of K are mutually disjoint then Theorem 4.2 (a) implies that $\Omega(\underline{X}, \underline{A})^K$ is homotopy equivalent to a finite product of spheres. This is elliptic, and as each sphere has an exponent at every prime p, so does a finite product of them. Hence Moore's Conjecture holds in this case as well. □

Proof of Theorem 1.2. Recall that if Y is any space then ΣY is rationally homotopy equivalent to a wedge of spheres. In particular, if $\partial\sigma \subseteq K$ and each Y_i is rationally nontrivial then by Lemma 2.5 (b) the space $(\underline{CY}, \underline{Y})^{\partial\sigma}$ is rationally homotopy equivalent to a wedge of simply-connected spheres. Thus if v is a vertex of $\partial\sigma$ and $\mathrm{rank}(\pi_*(Y_v) \otimes \mathbb{Q}) \geq 2$ then $(\underline{CY}, \underline{Y})^{\partial\sigma}$ is rationally homotopy equivalent to a wedge of at least two simply-connected spheres.

Suppose that $(\underline{X}, \underline{A})^K$ is elliptic. The homotopy decomposition $\Omega(\underline{X}, \underline{A})^K \simeq (\prod_{i=1}^{m} \Omega X_i) \times \Omega(\underline{CY}, \underline{Y})^K$ in Theorem 2.3 then immediately implies that each X_i must be elliptic, so condition (i) holds. This homotopy decomposition also implies that $(\underline{CY}, \underline{Y})^K$ is elliptic. Let $\sigma_1, \ldots, \sigma_n$ be the minimal missing faces of K. If two of these minimal missing faces intersect, say σ_1 and σ_2, then Theorem 4.2 implies that $\Omega\left((\underline{CY}, \underline{Y})^{\partial\sigma_1} \vee (\underline{CY}, \underline{Y})^{\partial\sigma_2}\right)$ retracts off $\Omega(\underline{CY}, \underline{Y})^K$. Since each of $(\underline{CY}, \underline{Y})^{\partial\sigma_1}$ and $(\underline{CY}, \underline{Y})^{\partial\sigma_2}$ is rationally homotopy equivalent to a wedge of simply-connected spheres, the space $(\underline{CY}, \underline{Y})^{\partial\sigma_1} \vee (\underline{CY}, \underline{Y})^{\partial\sigma_2}$ is rationally homotopy equivalent to a wedge of at least two simply-connected spheres, implying that it is hyperbolic. Therefore $(\underline{CY}, \underline{Y})^K$ is hyperbolic, a contradiction. Hence the minimal missing faces of K must be mutually disjoint, implying that condition (ii) holds. Because condition (ii) holds, Theorem 4.2 implies that $\Omega(\underline{CY}, \underline{Y})^K \simeq \prod_{j=1}^{n} \Omega(\underline{CY}, \underline{Y})^{\partial\sigma_j}$. It has already been observed that if v is a vertex of $\partial\sigma_j$ and $\mathrm{rank}(\pi_*(X_v) \otimes \mathbb{Q}) \geq 2$ then $(\underline{CY}, \underline{Y})^{\partial\sigma_j}$ is rationally homotopy equivalent to a wedge of at least two simply-connected spheres, and so is hyperbolic, implying that $(\underline{CY}, \underline{Y})^K$ is hyperbolic, a contradiction. Thus condition (iii) holds.

Conversely, suppose that conditions (i) to (iii) hold. By Theorem 4.2, condition (ii) implies that $\Omega(\underline{X}, \underline{A})^K \simeq (\prod_{i=1}^{m} \Omega X_i) \times (\prod_{j=1}^{n} \Omega(\underline{CY}, \underline{Y})^{\partial\sigma_j})$, where $\sigma_1, \ldots, \sigma_n$ are the minimal missing faces of K. For each vertex v of any σ_i, condition (iii) states that Y_v is rationally homotopy equivalent to a sphere. Therefore Lemma 2.5 (b) implies that $(\underline{CY}, \underline{Y})^{\partial\sigma_i}$ is rationally homotopy equivalent to a sphere. As each X_i is elliptic by condition (i), it has finitely many rational homotopy groups.

Hence the homotopy decomposition for $\Omega(\underline{X}, \underline{A})^K$ implies that $(\underline{X}, \underline{A})^K$ has finitely many rational homotopy groups and so is elliptic. □

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