Proofs and refutations in school mathematics: A task design in dynamic geometry environments

Kotaro Komatsu[1] and Keith Jones[2]

[1]Shinshu University, Japan; kkomatsu@shinshu-u.ac.jp

[2]University of Southampton, UK; d.k.jones@soton.ac.uk

Although the mathematical activity of proofs and refutations is widely recognised as significant in school mathematics, much remains under-explored about ways of facilitating such activity in the classroom. In this paper, we address this issue by focusing on task design in dynamic geometry environments. In particular, we formulate three principles for the task design and use these to develop classroom tasks. We analyse a task-based interview with a triad of upper secondary school students to show how the designed tasks stimulated their activity of proofs and refutations.

Keywords: Proof, refutation, counterexample, task design, dynamic geometry environment.

Introduction

The mathematical activity of proofs and refutations described by Lakatos (1976) is significant in school mathematics because it enables students to experience authentic mathematical practice (Lampert, 1992). Although several researchers have described student behaviour with Lakatos' terminology (Komatsu, 2016; Larsen & Zandieh, 2008), few studies have examined ways of purposefully introducing such activity into classrooms (Komatsu, Tsujiyama, Sakamaki, & Koike, 2014; Komatsu, 2017; Larsen & Zandieh, 2008). Given the importance of mathematical tasks for student learning (Kieran, Doorman, & Ohtani, 2015), this study aims at developing task design principles and actual tasks for realising proofs and refutations.

To achieve this purpose, we specifically focus on dynamic geometry environments (DGEs). Research has shown the capability of dynamic geometry software (DGS) for enhancing proof-related activities such as making conjectures and subsequently constructing proofs. In particular, some studies have shown how using DGS enabled students to discover the refutations of their conjectures and proofs and cope with these refutations (Healy & Hoyles, 2001; Olivero & Robutti, 2007). The successful use of DGS in previous research was accompanied by carefully-designed tasks (Hanna, 2000). Nevertheless, how the tasks were designed was often not clarified explicitly, and task design in DGEs remains understudied (Sinclair et al., 2016).

To address these issues, the study reported in this paper focuses on the following research question: What principles can underpin the design of DGE tasks that facilitate student activity of proofs and refutations?

The meaning of proofs and refutations

Based on Lakatos (1976), we conceptualise the meaning of proofs and refutations as depicted in Figure 1. Students make conjectures (or are provided with statements), and then attempt to prove them. In this, they are confronted with refutations of the conjectures/statements or proofs, and refine them by addressing the refutations (Komatsu, 2016).

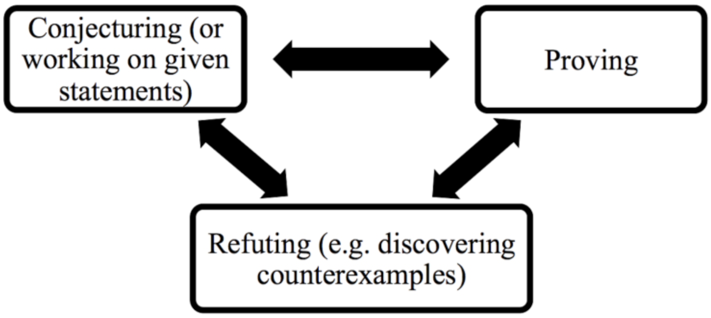


Figure 1: The conceptualisation of proofs and refutations

As there is insufficient space to explain Figure 1 fully, here we clarify only two points. First, we take the meaning of proof in a broad sense such that a deductive proof may be valid only for a subset of all cases considered in a conjecture and statement. Second, although the word *refutation* is sometimes used only for conjectures and statements, not for proofs, this study utilises *refutation* for conjectures/statements and for proofs. These two points are epistemologically consistent with Lakatos' view of mathematics. In *Proofs and Refutations* (Lakatos, 1976), he dealt with deductive proofs that were only partially valid. He also argued that proof was inextricably linked to refutations (Reid, Knipping, & Crosby, 2008) and coined the term *local counterexamples* to denote the refutations of proofs.

Task design principles

To design tasks for fostering the student activity of proofs and refutations, we develop principles for the task design from three aspects. First, Hanna (1995) pointed out that Lakatos' (1976) story rested on the topic of polyhedra, where it was relatively easy to suggest counterexamples. This confirms that it is necessary to create tasks intentionally where counterexamples can be produced. Because it is the ambiguous meaning of polyhedra that made counterexamples possible in Lakatos' research, it is essential to develop tasks whose conditions are purposefully ambiguous so that counterexamples can be proposed. In fact, we previously demonstrated that specific tasks that include hidden conditions, namely *proof problems with diagrams*, are useful for introducing proofs and refutations into secondary school geometry (Komatsu, et al., 2014; Komatsu, 2017).

Second, research indicates that students encounter difficulties in producing proper counterexamples (Hoyles & Küchemann, 2002). Thus, it is important to prepare tools that foster student production of counterexamples. DGS could play the role of such tools in geometry education because the main advantage of DGS is that students can easily transform diagrams by dragging (Arzarello, Olivero, Paola, & Robutti, 2002) and thus the students have access to various diagrams. From research on dragging modalities and measuring modalities in DGEs (Arzarello et al., 2002; Olivero & Robutti, 2007), the following are relevant to refutations of conjectures and proofs: dragging test, validation measuring, and proof measuring.

Third, several studies have reported that when students encounter counterexamples, some of them refuse to accept the counterexamples and do not try to revise their conjectures (e.g. Balacheff, 1991). For resolving this problem, we capitalise on the potential of contradictions, because if contradictions are appropriately induced, confusion generated by the contradictions can be beneficial for learning (D'Mello, Lehman, Pekrun, & Graesser, 2014). To trigger contradictions, it is likely helpful to combine tasks intentionally (rather than use a single task) where students can recognise contradictions between their solutions to tasks and their thinking in subsequent tasks (Hadas, Hershkowitz, & Schwarz, 2000; Prusak, Hershkowitz, & Schwarz, 2012).

In summary, we formulate the following principles of task design for fostering the student activity of proofs and refutations: 1) Using tasks whose conditions are purposefully ambiguous and thus allow the occurrence of counterexamples; 2) Providing tools that enhance the production of counterexamples; and 3) Increasing students' recognition of contradictions that facilitates them to revise conjectures/statements and/or proofs.

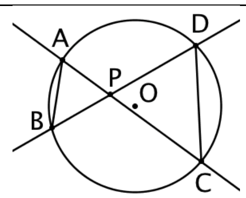
Methods

Participants

This paper analyses a task-based interview in which a triad of students, Kakeru, Sakura, and Yuka (pseudonyms), voluntarily participated. They were 11th graders (aged 16–17 years old) in an upper secondary school in Japan. According to their mathematics teacher, their mathematical capabilities were above average. The first author conducted the interview. The DGE was GeoGebra. Because the students had no experience with DGS, four hours was devoted, prior to the interview, to teaching the students the basic functions (e.g. basic construction, dragging, and measuring) of the DGE. The students had learnt geometric proofs using the conditions for congruent triangles and those for similar triangles. They were familiar with the inscribed angle theorem, the inscribed quadrilateral theorem, and the alternate segment theorem, all of which are related to tasks used in the interview.

Tasks

Q1. (1) As shown in the diagram given, there are four points A, B, C, and D on circle O. Draw lines AC and BD, and let point P be the intersection point of the lines. What relationship holds between ΔPAB and ΔPDC? Write your conjecture. (2) Prove your conjecture.



Q2. Construct the diagram shown in Q1 with DGS. Move points A, B, C, and D on circle O to examine the following questions. (1) Is your conjecture in Q1 always true? (2) Is your proof in Q1 always valid?

Figure 2: Tasks used in the interview

The tasks used in the interview are shown in Figure 2. We developed them according to the aforementioned design principles. Q1 is relevant to the first principle that involves ambiguous conditions. The condition of Q1 is vague because there is no reference to the locations of points A, B, C, and D in the problem sentences. If the locations are changed, refutations of the proof constructed in Q1 can be discovered, as shown below. The second principle corresponds to Q2, where students are invited to construct the given diagram with DGS and produce various diagrams by dragging. The third principle is related to the combination of Q1 and Q2. It is, of course, possible for Q1 to stipulate the use of DGS to produce various diagrams for making a conjecture before proving. However, we designed Q1 and Q2 in the way set out in Figure 2 because we expected that proof construction in Q1 could increase students' conviction in their conjecture and proof. This design could lead to students' recognition of a contradiction between their conviction and the subsequent refutations in Q2.

Data collection and analysis

The three students were asked to solve task Q1 collaboratively with paper and pencil and task Q2 with DGS on a desktop computer. The task-based interview lasted for approximately 35 minutes in total. It was video-recorded and the audio transcribed. We used two cameras for the recording, one placed to video the students and the other placed to record the screen of the computer. The worksheets the students completed, and the DGS file the students made, were collected. We analysed these data by focusing on what type of diagram the students produced and how they dealt with the diagrams.

Results

Conjecture, proof, and types of diagrams the students produced

Immediately after student Kakeru read the problem sentences in Q1, Sakura conjectured “similar?”. The students then wrote the following proof on their worksheet:

In ΔPAB and ΔPDC,

From the vertical angles, ∠APB = ∠DPC … (1)

From arc BC, since inscribed angles are equal, ∠PAB = ∠PDC … (2)

From (1) and (2), since two pairs of angles are equal, ΔPAB ~ ΔPDC

After that, the students worked on Q2. As they worked, they produced and examined the six types of diagrams shown in Figure 3. In Figure 3a, triangle PAB (or likewise triangle PDC) is not constructed, while both triangles are not constructed in Figures 3b and 3c. Point P is located outside circle O in Figures 3d and 3e. In the type of diagram shown in Figure 3f, the students regarded points A and C (or likewise with points B and D) to be coincident and considered line AC (or likewise BD) to be a tangent to circle O.

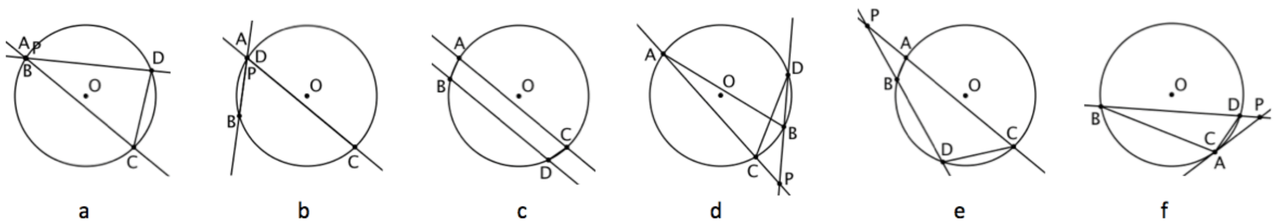


Figure 3: Types of diagrams the students produced

In the following, we report the cases regarding Figures 3e and 3f because the students devoted more efforts to these types than to the other types.

Case where point P is outside the circle

At the beginning of Q2, the students produced the type shown in Figure 3e:

116	Kakeru:	Is the conjecture in Q1, similarity, always true? [Reading the problem sentence.]
117	Sakura:	Not similar.
118	Yuka:	In this case, … impossible.
119	Kakeru:	The intersection point is outside the circle.

Here, Sakura and Yuna recognised a contradiction because although they proved their conjecture in Q1, they considered the type of Figure 3e to be a counterexample to their conjecture (lines 117 and 118). Kakeru then responded to their judgement:

132	Kakeru:	We can say that they are similar.
133	Sakura:	Why? We can't say that.
134	Kakeru:	Because.
135	Sakura:	Wait. Because.
138	Kakeru:	PAB and PDC. These are similar. This and this [angle P] are common and equal. Then, because [quadrilateral ABDC is] a quadrilateral that is inscribed to the circle.
139	Sakura:	That one.
140	Kakeru:	This [angle PAB] and this [angle PDC] are equal.

A dispute between the students can be seen in this dialogue, where Kakeru argued that their conjecture was still true (line 132), whereas Sakura objected to his argument (lines 133 and 135). To respond to Sakura's objection, particularly for showing the congruence of angles PAB and PDC, Kakeru proposed using the inscribed quadrilateral theorem (lines 138 and 140): an interior angle is equivalent to the exterior angle of the opposite angle. Sakura agreed with his thinking (line 139), and, thus, they were able to resolve the dispute by proving the similarity of the triangles in the type of Figure 3e.

Case where a line is a tangent to the circle

After producing the diagram type shown in Figure 3e, the students examined the type shown in Figure 3f (note that, strictly speaking, this type is different from the original problem where line AC cannot be drawn if points A and C coincide.) When encountering this type, Kakeru was convinced that their conjecture would be still true, and proposed using the alternate segment theorem to prove the conjecture. Nevertheless, when he started explaining his idea to Sakura and Yuka, he had a doubt as to why line AP can be considered as a tangent. The students struggled to resolve this doubt. During their struggle, as the students mentioned only once that the type of Figure 3f might be a counterexample to their conjecture, they consistently anticipated that their conjecture would be true in this type. Eventually, they judged that line AP was the tangent by measuring the degree of angle PAO and finding that it was almost 90 degrees. The subsequent student interaction was as follows:

357	Kakeru:	If we consider this as a tangent, we can use the theorem about the angle formed by a tangent and a chord.
358	Sakura:	I see.
359	Kakeru:	We can show the similarity.
360	Sakura:	This (angle DCP) and this (angle PBA) and P.

This dialogue shows that the students were able to prove their conjecture in the type of Figure 3f with the alternate segment theorem.

Examination of the initial proof

The students concluded their activity without considering Q2(2), so the interviewer questioned them as follows: "Please read again the sentences in Q2 carefully. When you say it does not hold, do you mean your conjecture is false, or your proof is invalid?" When addressing this question, the students noticed that the reasons in their initial proof were not applicable to the diagrams that they produced.

In other words, they regarded these diagrams as local counterexamples to their proof in the sense of Lakatos' (1976) terminology. For example, the following is their discussion about Figure 3e:

456	Sakura:	We wrote, "From arc BC, since inscribed angles are equal, ∠PAB = ∠PDC".
457	Kakeru:	PAB and PDC. This [the initial proof] is for this case [shown in Figure 2].
458	Sakura:	This [the last line in the proof] is valid, but the sentences [the second and third lines in the proof] are not valid, right?
461	Kakeru:	This [the initial proof] is only for this [Figure 2].
462	Yuka:	If so, this proof …
463	Sakura:	Is not always valid, right?

After that, the students pointed out that it was sufficient to revise the reasons in their initial proof by replacing the equality of vertical angles and the inscribed angle theorem with the identity of the angles and the inscribed quadrilateral theorem, respectively. They also examined and revised the initial proof in the type of Figure 3f in a similar way, with the alternate segment theorem.

Discussion and conclusion

The students in the interview were able to engage in mathematical activity of proofs and refutations depicted in Figure 1. After making and proving a conjecture, they produced diagrams to scrutinise whether their conjecture was always true. Although they initially judged the type of diagram in Figure 3e to be a counterexample to their conjecture, they modified their judgement by proving that their conjecture was still true in this type. This proof was constructed without looking back at their initial proof and revising it. However, after the interviewer's intervention asking them to consider Q2(2), the students recognised that their initial proof was not applicable to the types of diagrams in Figures 3e and 3f, and revised the proof for these types.

The three design principles and the tasks developed based on the principles were generally helpful for fostering the students' activity. Based on the first principle, we used the proof problem whose condition regarding the locations of points A, B, C, and D is ambiguous (Figure 2). This task enabled the students to produce the six types of diagrams that had the potential to refute their conjecture and proof (Figure 3).

With regard to the second principle, DGS in general and its dragging function in particular (Arzarello et al., 2002), were highly useful for producing such a variety of diagrams. In our earlier research, many students in a lower secondary school encountered difficulties in drawing diagrams that refuted their proofs in paper-and-pencil environments (Komatsu, Ishikawa, & Narazaki, 2016). Although the tasks used in that study were more difficult than those in this study, without DGS it would likely be challenging for the three students in this study to produce various diagrams different from Figure 2.

The combination of Q1 and Q2 based on the third principle played a role in stimulating the subsequent students' activity. In the case where point P was outside circle O, Sakura and Yuka felt a contradiction between the truth of their conjecture that was proved in Q1 and the refutation in Q2 where they judged the type of Figure 3e to be a counterexample to their conjecture. This contradiction triggered the dispute with Kakeru, where Sakura and Yuka's judgement was revised through Kakeru's proof showing that their conjecture was still true. In the subsequent case where a line was a tangent to circle O (Figure 3f), the students did not seem to perceive such a contradiction. This was likely related to the students' earlier experience, where they could show that the type of Figure 3e, which was initially regarded as a counterexample, did not refute their conjecture. This experience would constitute a source of their conviction in the truth of their conjecture as regards the type of Figure 3f. If the students encountered this type prior to the type of Figure 3e, they would think that it might refute their conjecture, and would perceive a contradiction between their conjecture and the refutation.

This study has limitations as it is based on a case with one set of tasks. It is necessary to develop other tasks based on the design principles of this study and conduct further empirical studies, including studies in real classroom settings, to inspect the values of the principles and tasks. Another interesting future issue is to examine whether the design principles of this study are applicable to content areas other than geometry (for example, number theory). The design principles are not conceptually restricted to geometry education; the ‘tools’ mentioned in the second design principle are not only DGS tools. This issue is worth addressing in order to extend the opportunity to introduce proofs and refutations from geometry into other topics in the mathematics curriculum.

References

Arzarello, F., Olivero, F., Paola, D., & Robutti, O. (2002). A cognitive analysis of dragging practises in Cabri environments. *ZDM – The International Journal on Mathematics Education*, *34*(3), 66–72.

Balacheff, N. (1991). Treatment of refutations: Aspects of the complexity of a constructivist approach to mathematics learning. In E. von Glasersfeld (Ed.), *Radical constructivism in mathematics education* (pp. 89–110). Dordrecht: Kluwer Academic Publishers.

D’Mello, S., Lehman, B., Pekrun, R., & Graesser, A. (2014). Confusion can be beneficial for learning. *Learning and Instruction, 29,* 153–170.

Hadas, N., Hershkowitz, R., Schwarz, B. B. (2000). The role of contradiction and uncertainty in promoting the need to prove in dynamic geometry environments. *Educational Studies in Mathematics*, *44*(1), 127–150.

Hanna, G. (1995). Challenges to the importance of proof. *For the Learning of Mathematics, 15*(3), 42–50.

Hanna, G. (2000). Proof, explanation and exploration: An overview. *Educational Studies in Mathematics, 44*(1), 5–23.

Healy, L., & Hoyles, C. (2001). Software tools for geometrical problem solving: Potentials and pitfalls. *International Journal of Computers for Mathematical Learning, 6*(3), 235–256.

Hoyles, C., & Küchemann, D. (2002). Students’ understandings of logical implication. *Educational Studies in Mathematics, 51*(3), 193–223.

Kieran, C., Doorman, M., & Ohtani, M. (2015). Frameworks and principles for task design. In A. Watson, & M. Ohtani. (Eds.), *Task design in mathematics education: An ICMI study 22* (pp. 19–81). New York: Springer.

Komatsu, K. (2016). A framework for proofs and refutations in school mathematics: Increasing content by deductive guessing. *Educational Studies in Mathematics, 92*(2), 147–162.

Komatsu, K. (2017). Fostering empirical examination after proof construction in secondary school geometry. *Educational Studies in Mathematics, 96*(2), 129–144.

Komatsu, K., Ishikawa, T., & Narazaki, A. (2016). Proof validation and modification by example generation: A classroom-based intervention in secondary school geometry. In *Proceedings of the 13th International Congress on Mathematical Education*. Hamburg, Germany.

Komatsu, K., Tsujiyama, Y., Sakamaki, A., & Koike, N. (2014). Proof problems with diagrams: An opportunity for experiencing proofs and refutations. *For the Learning of Mathematics, 34*(1), 36–42.

Lakatos, I. (1976). *Proofs and refutations: The logic of mathematical discovery*. Cambridge: Cambridge University Press.

Lampert, M. (1992). Practices and problems in teaching authentic mathematics. In F. K. Oser, A. Dick, & J. L. Patry (Eds.), *Effective and responsible teaching: The new synthesis* (pp. 295–314). San Francisco, CA: Jossey-Bass Publishers.

Larsen, S., & Zandieh, M. (2008). Proofs and refutations in the undergraduate mathematics classroom. *Educational Studies in Mathematics*, *67*(3), 205–216.

Olivero, F., & Robutti, O. (2007). Measuring in dynamic geometry environments as a tool for conjecturing and proving. *International Journal of Computers for Mathematical Learning, 12*(2), 135–156.

Prusak, N., Hershkowitz, R., & Schwarz, B. B. (2012). From visual reasoning to logical necessity through argumentative design. *Educational Studies in Mathematics*, *79*(1), 19–40.

Reid, D., Knipping, C., & Crosby, M. (2008). Refutations and the logic of practice. In O. Figueras, J. L. Cortina, S. Alatorre, T. Rojano, & A. Sepúlveda (Eds.), *Proceedings of the Joint Meeting of PME 32 and PME-NA XXX, Vol. 4* (pp. 169–176). Morelia, México: PME-NA.

Sinclair, N., Bartolini Bussi, M. G., de Villiers, M., Jones, K., Kortenkamp, U., Leung, A., & Owens, K. (2016). Recent research on geometry education: An ICME-13 survey team report. *ZDM : The International Journal on Mathematics Education, 48*(5), 691–719.