Differences between short and long term risk aversion: an optimal asset allocation perspective

Jesus Gonzalo*, *Universidad Carlos III de Madrid*

Jose Olmo, *University of Southampton*

Abstract

This paper studies the long-term asset allocation problem of an investor with different risk aversion attitudes to the short and the long term. We characterize investor's preferences with a utility function exhibiting a regime shift in risk aversion at some point of the multiperiod investment horizon that is estimated using threshold nonlinearity methods. Our empirical results for a portfolio of cash, bonds and stocks suggest that long-term risk aversion is higher than short-term risk aversion and increases with the investment horizon. The exposure of the investment portfolio from stocks to bonds and cash increases with the degree of risk aversion.

Keywords: multiperiod asset allocation; GMM; parametric portfolio policies; Short and Long term risk aversion; threshold nonlinearity tests.

*Corresponding Address: Dept. Economics, Universidad Carlos III de Madrid. C/Madrid 126, Getafe, Madrid (Spain) C.P.: 28903. Jesus Gonzalo, E-mail: jesus.gonzalo@uc3m.es. We acknowledge participants at Southampton-Surrey Econometric Event and Workshop on Financial Econometrics and Structural Breaks in University of Birmingham. Financial support from the Spanish MINECO under grants ECO2013-46395, ECO2016-78652 and MDM 2014-0431, MadEco-CM grant S2015/HUM-3444 and the Bank of Spain ER program is gratefully acknowledged. The first author acknowledges the hospitality of Nuffield College during 2017.

1 Introduction

The optimal portfolio decisions of long-term investors depend on the economic and financial environment, in particular, the universe of financial assets available to the investor, their expected returns and risks, and the preferences and circumstances of investors. In a long-term optimal portfolio context, these preferences are usually modeled as the discounted sum of a stream of period utility functions characterized by a risk aversion coefficient. In this sense the degree of investor's risk aversion plays a fundamental role in determining optimal investment strategies.

This approach entertains the same parameter for describing the investor's risk aversion over different investment horizons. This assumption can be appropriate for myopic asset allocation problems involving one-period-ahead investment decisions, however, for investment decisions involving more than one period the assumption can be too simplistic and overlook the possibility of changes in individual's risk aversion between the short and the long term. Thus, for long-term asset allocation problems it may be more appropriate to consider different types of period utility functions reflecting different risk perceptions with respect to the investment horizon[1]. The main aim of this paper is to do this in an optimal asset allocation framework. More specifically, we assess the implications from an optimal portfolio theory perspective of extending the standard multiperiod optimal asset allocation problem to a setting characterized by two different types of utility function reflecting each a different risk aversion attitude to the short and the long term. For simplicity, we entertain the family of power utility functions characterized by two different relative risk aversion coefficients describing investors' preferences to the short and the long term, respectively.

Unfortunately, intertemporal asset allocation models are hard to solve in closed form unless

[1]For example, it is not difficult to construct a narrative causality theory of how unexpected electoral or referendum results can provoke different short/long term attitudes towards risk.

strong assumptions on the investor's objective function such as log preferences or a lognormal distribution for asset returns are imposed. This situation has begun to change as a result of several developments in numerical methods and continuous time finance models such as Barberis (2000) and Brennan et al. (1997, 1999). Approximate analytical solutions to the Merton (1973) model have been developed in Campbell and Viceira (1999, 2001, 2002) and Campbell et al. (2003) for models exhibiting an intertemporal elasticity of substitution close to one. Recent parametric alternatives to solving the investor optimal portfolio problem over several periods have been proposed by Aït-Sahalia and Brandt (2001) and Brandt and Santa-Clara (2006). We follow a similar methodology and develop a suitable framework in which to derive the optimal portfolio decision of investors over multiperiod investment horizons.

The distinctive and innovative feature of our framework is that investors' preferences are modelled by a power utility function that takes two different relative risk aversion coefficients depending on how close the investment horizon is to the present. To the best of our knowledge, this is the first paper to differentiate between short and long term risk aversion and analyze the consequences from a long-term optimal portfolio perspective. For the sake of generality, we also entertain the possibility of dynamics in the risk aversion coefficients. The presence of dynamics in the investor's relative risk aversion coefficient is not new in the financial literature. Brandt and Wang (2003), for example, obtain dynamics in the relative risk aversion coefficient as a consequence of entertaining utility functions that incorporate habit formation. In contrast to these authors we propose a structural model to describe the dynamics of relative risk aversion. In our model, the coefficients characterizing risk aversion to the short and long term are assumed to be parametric functions of a set of state variables used to describe the information set. By doing so, we contemplate the possibility of changes in investor's risk aversion not only driven by the structure of the investment horizon but also by time-varying economic conditions.

In the proposed model the optimal portfolio weights characterizing the optimal asset allocation are determined by a parametric linear portfolio policy rule driven by the dynamics of the set of state variables reflecting the information set. Under this assumption we obtain a set of Euler equations that can be estimated and tested using the generalized method of moments (GMM). The system of Euler equations is overidentified allowing us to test different features of the model such as the role of the state variables in driving the optimal allocation to stocks and bonds or the presence of dynamics and nonlinearities in risk aversion. Thus, the second main contribution of the paper is to propose a likelihood ratio test to formalize the existence of a regime shift in investors' risk aversion between the short and the long term. The econometric methodology to implement the test is similar in spirit to Andrews (1993), Andrews and Ploberger (1994) and Hansen (1996) that discuss how to make inference when a nuisance parameter is not identified under the null hypothesis (Davies (1977, 1987) problem). In our setting, we assume the period separating the short term from the long term to be unknown but estimated from the data. Under the null hypothesis there is a single risk aversion regime implying that the period signaling the structural break (nuisance parameter) in the risk aversion coefficient is not identified. In this scenario standard statistical inference procedures cannot be applied to statistically assess the presence of a threshold nonlinearity. Instead, we apply a p-value transformation implemented through a multiplier method to the multiperiod Euler equations defining the optimality conditions of the individual's maximization problem, see Hansen (1996) for early applications of the methodology.

We apply this methodology to analyze the optimal portfolio choice of a long-term investor that can invest in three assets - a one-month Treasury bill as riskless security, a long-term bond, and an equity portfolio. This empirical application closely follows similar studies such as Brennan et al. (1997), Brandt (1999) and Campbell et al. (2003). Our choice of state variables to proxy the dynamics of the investment opportunity set is motivated by the literature on predictive regressions for financial returns. It is defined by the detrended short-term interest rate, the U.S. credit spread, the S&P 500 trend and the one-month average of excess stock and bond returns. Our empirical findings suggest that investors exhibit two different types of risk aversion and that the threshold separating the short from the long term is around the seventh month of the investment horizon. These findings also reveal that long-term risk aversion is higher than short-term risk aversion and increases with the number of periods defining the individual's investment horizon. The analysis of the optimal portfolio weights also suggests that the exposure of the investment portfolio to cash and bonds compared to stocks increases with the degree of risk aversion (flight to quality). This phenomenon is more pronounced for very high levels of risk aversion in which investors' optimal asset allocation moves away from stocks and bonds to cash (flight to safety).

The rest of the article is structured as follows. Section 2 presents the model and derives the system of Euler equations obtained from the first order conditions of the multiperiod maximization problem of an individual exhibiting different risk aversion to the short and the long term. Section 3 discusses the implementation of GMM to estimate the optimal portfolio weights and the risk aversion coefficients and briefly discusses the corresponding asymptotic theory. Section 4 presents two types of econometric tests to assess the parametric assumptions used in the development of our model. First, we introduce in detail a threshold nonlinearity test to assess statistically the presence of two regimes in the individual's risk aversion function, and second, we discuss several specification tests to assess the suitability of the parametric policy rules proposed in the paper. Section 5 presents an empirical application to compare the optimal allocation to a portfolio of stocks, bonds and cash between investors with different attitudes towards risk with respect to the investment horizon. Section 6 concludes.

2 The Model

2.1 The investor's multiperiod objective function

We introduce first the following utility function that characterizes the preferences of individuals with different attitudes towards risk aversion in the short and the long term:

$$\sum_{j=0}^{\kappa_0} \beta^j E_t \left[\frac{W_{t+j}^{1-\gamma}}{1-\gamma} \right] + \sum_{j=\kappa_0+1}^{K} \beta^j E_t \left[\frac{W_{t+j}^{1-\gamma^*}}{1-\gamma^*} \right], \tag{1}$$

where $E_t[\cdot]$ denotes the conditional expectation with respect to the sigma-algebra generated by all the information available to the individual at time t; W_t denotes real wealth, the discount factor β measures patience, the willingness to give up wealth today for wealth tomorrow, and the coefficients γ and γ^* capture risk aversion to the short and long term, respectively. The parameter κ_0 denotes the period separating the short from the long term and is defined over K investment horizons. This function extends naturally standard formulations proposed in the literature to model the preferences of long-term investors. In this literature, investors have time-invariant period utility functions characterized by power utility functions with the same coefficient γ across investment horizons.

The above objective function can be extended to also accommodate dynamics in the coefficients γ and γ^*. By doing so, we entertain the possibility of individuals exhibiting a time-varying relative risk aversion on wealth. Brandt and Wang (2003) achieve a similar objective by introducing the presence of habit formation in the individual's utility function. In our setting, we introduce a risk aversion function $\gamma_t(j; \kappa_0)$ that, in order to guarantee the positiveness of the relative risk aversion coefficient, is defined as

$$\gamma_t(j; \kappa_0) = \exp\left((\gamma' + \eta' 1(j > \kappa_0)) Z_{t+j} \right), \tag{2}$$

where $Z_{t+j} = (1, Z_{1,t+j}, \ldots, Z_{n-1,t+j})'$ denotes a vector of $n-1$ macroeconomic and financial variables reflecting all the information available to the investor at time $t+j$; γ and η are the corresponding vectors of model parameters. This piecewise linear formulation follows the spirit of Gonzalo and Pitarakis (2012, 2017) on threshold predictive regression and Perron (1989, 1997) and Andrews (1993) on structural breaks. More compactly, the multiperiod utility function becomes

$$\sum_{j=0}^{K} \beta^j E_t \left[\frac{W_{t+j}^{1-\gamma_t(j;\kappa_0)}}{1-\gamma_t(j;\kappa_0)} \right]. \tag{3}$$

The individual begins life with an exogenous endowment of wealth $W_t = 1$. This endowment accumulates over time according to the equation

$$W_{t+1} = (1 + r_{t+1}^p) W_t. \tag{4}$$

At the beginning of period $t+1$ the individual receives income from allocating resources in an investment portfolio offering a real return r_{t+1}^p. The portfolio return is defined as

$$r_{t+1}^p(\alpha_t) = r_{f,t+1} + \alpha_t' r_{t+1}^e, \tag{5}$$

with $r_{t+1}^e = (r_{1,t+1} - r_{f,t+1}, \ldots, r_{m,t+1} - r_{f,t+1})'$ denoting the vector of excess returns on the m risky assets over the real risk-free rate $r_{f,t+1}$, and $\alpha_t = (\alpha_{1,t}, \ldots, \alpha_{m,t})'$ denoting the different allocations to risky assets. In order to be able to solve a multiperiod maximization problem that accommodates in a parsimonious way arbitrarily long investment horizons we entertain the parametric portfolio policy rule introduced in Aït -Sahalia and Brandt (2001) and used extensively in Brandt and Santa-Clara (2006) and Brandt et al. (2009). More specifically,

$$\alpha_{t+j} = \Lambda Z_{t+j}, \tag{6}$$

with Λ a $m \times n$ matrix of model parameters associated to the state variables Z_t. Time variation of the optimal asset allocation is introduced through the dynamics of the state variables. This specification of the portfolio weights has two main features. First, it allows us to study the marginal effects of the state variables on the optimal portfolio weights through the set of parameters Λ, and second, it avoids the introduction of time consuming stochastic dynamic programming methods[2].

2.2 Optimal portfolio choice under risk aversion

In this section we derive the first order conditions of the long-term optimal portfolio choice problem for a risk-averse individual with preferences described above. The investor's wealth process at time $t+j$ can be expressed in terms of the compound $j-$period gross return and the initial wealth W_t that we consider to be equal to one. More formally,

$$W_{t+j} = \prod_{i=1}^{j} (1 + r_{t+i}^{p}(\Lambda Z_{t+i-1})). \tag{7}$$

Using this characterization of the wealth process simple algebra shows that the individual's maximization problem can be written as

$$\max_{\{\Lambda\}} \left\{ \sum_{j=0}^{K} E_t \left[\beta^j \frac{\left(\prod_{i=1}^{j} (1 + r_{t+i}^{p}(\Lambda Z_{t+i-1})) \right)^{1-\gamma_t(j;\kappa_0)}}{1 - \gamma_t(j;\kappa_0)} \right] \right\}. \tag{8}$$

[2]This approach forces the individual's optimal portfolio policy rule to be linear and with the same parameter values over the long term horizon. More sophisticated models can be developed that entertain different parametric portfolio policy rules for different investment horizons $j = 1, \ldots, K$, however, this approach significantly increases the computational complexity of the methodology and is beyond the aim of this study.

The first order conditions of this optimization problem with respect to the parameter Λ provide the following system of mn Euler equations:

$$E_t\left[\sum_{j=1}^{K}\beta^j\psi_{t,j}^{h,s}(Z;\theta_0,\kappa_0)\right]=0 \tag{9}$$

with $\theta_0=(\Lambda_0,\gamma_0,\eta_0)$ and

$$\psi_{t,j}^{h,s}(Z;\theta_0,\kappa_0)=\left(\sum_{i=1}^{j}\frac{Z_{s,t+i-1}r_{h,t+i}^{e}}{1+r_{t+i}^{p}(\Lambda_0 Z_{t+i-1})}\right)\left(\prod_{i=1}^{j}(1+r_{t+i}^{p}(\Lambda_0 Z_{t+i-1}))\right)^{1-\gamma_{0t}(j;\kappa_0)}, \tag{10}$$

for $h=1,\ldots,m$ and $s=1,\ldots,n$; $\gamma_{0t}(j;\kappa_0)=\exp\left((\gamma_0'+\eta_0'1(j>\kappa_0))Z_{t+j}\right)$. The matrix Λ_0 is the solution to the first order conditions of the maximization problem for a given set of parameters γ_0 and η_0.

The set of conditional moments (9) is equivalent to the following set of conditions

$$E\left[\sum_{j=1}^{K}\beta^j\psi_{t,j}^{h,s}(Z;\theta_0,\kappa_0)U_t\right]=0,$$

for all $\Im_t-$measurable functions U_t and for all t, $1\leq t\leq T-K$, with $T>K$ the sample size. Following Giacomini and Komunjer (2005), we assume the existence of a $n\times 1$ vector of variables U_t^* that are observed at time t and that contain all of the relevant information in the sigma-algebra $\Im_t$. We refer to U_t^* as the information vector. The general requirement on $\{U_t^*\}$ is that it is a strictly stationary and mixing series. In our framework, we consider U_t^* to be the vector of state variables Z_t. Under these assumptions, the set of mn conditional moment conditions (9) becomes a set of mn^2 unconditional moment conditions given by

$$E\left[\sum_{j=1}^{K}\beta^j\psi_{t,j}^{h,s}(Z;\theta_0,\kappa_0)Z_{\widetilde{s},t}\right]=0, \tag{11}$$

indexed by $h = 1, \ldots, m$ and $s, \widetilde{s} = 1, \ldots, n$. Let $g_{it}(\theta_0, \kappa_0) = \sum_{j=1}^{K} \beta^j \psi_{t,j}^{h,s}(Z; \theta_0, \kappa_0) Z_{\widetilde{s},t}$ with $i = 1, \ldots, mn^2$ an index that accounts for all possible combinations of $h = 1, \ldots, m$ and $s, \widetilde{s} = 1, \ldots, n$ on the right hand side of the expression; and let $g_t(\theta_0, \kappa_0)$ be the mn^2 vector that stacks all these variables. Condition (11) becomes

$$E\left[g_t(\theta_0, \kappa_0)\right] = 0. \tag{12}$$

The main advantage of this approach is that the system of mn first order conditions derived from the maximization problem (8) yields an overidentified system of mn^2 unconditional moment conditions. This property is exploited in the econometric section to derive suitable estimators of the optimal portfolio weights and risk aversion coefficients at the same time as carrying out statistical tests for the parametric specifications (2) and (6).

3 Econometric methods: estimation

This section presents suitable methods to estimate the optimal portfolio weights and the parameters driving the dynamics of the risk aversion coefficient. Let $g_N(\theta_0, \kappa_0)$ be the vector that stacks the sample moment conditions $\frac{1}{N} \sum_{t=1}^{N} g_{it}(\theta_0, \kappa_0)$ with $N = T - K$ and $i = 1, \ldots, mn^2$. The idea behind GMM is to choose an estimate of θ_0, namely $\widehat{\theta}_N$, so as to make the sample moments $g_N(\widehat{\theta}_N, \kappa_0)$ as close to zero as possible.

Let $G_N(\theta, \kappa_0) = g_N(\theta, \kappa_0)' \widehat{V}_N^{-1} g_N(\theta, \kappa_0)$ with $\widehat{V}_N$ a consistent estimator of the long-run covariance matrix of $\sqrt{N} g_N(\theta_0, \kappa_0)$, defined as

$$V_0(\theta_0, \kappa) = \frac{1}{N} \sum_{t=1}^{N} \sum_{s=1}^{N} E[g_t(\theta_0, \kappa_0) g_s(\theta_0, \kappa_0)']. \tag{13}$$

This matrix captures the strong serial correlation in the sequence $g_t(\theta_0, \kappa_0)$ due to entertaining a multiperiod investment horizon in the individual's objective function. An estimator of V_0 can be obtained by applying HAC variance estimators. More specifically, let $\Gamma_N(j) = \frac{1}{N}\sum_{t=j+1}^{N} g_t(\widehat{\theta}_N, \kappa_0) g_{t-j}(\widehat{\theta}_N, \kappa_0)'$ be the sample covariance matrix between g_t and g_{t-j} constructed with the estimator $\widehat{\theta}_N$. A suitable Newey-West HAC estimator is

$$\widehat{V}_N(\widehat{\theta}_N, \kappa_0) = \Gamma_N(0) + \sum_{j=1}^{l} \frac{l-j}{j} \left(\Gamma_N(j) + \Gamma_N(j)'\right), \tag{14}$$

with l a bandwidth parameter that determines the maximum order of autocorrelation taken into account by the estimator. Using this notation, we obtain an estimator of θ_0 as the solution to the minimization problem

$$\min_{\theta \in \Theta} G_N(\theta, \kappa_0), \tag{15}$$

with Θ the parameter space for θ. In a first stage, to obtain a consistent estimator of θ_0, namely $\widetilde{\theta}_N$, we use the identity matrix as an initial candidate for $\widehat{V}_N$. In a second stage, the minimization process is repeated replacing the identity matrix by the matrix $\widehat{V}_N(\widetilde{\theta}_N, \kappa_0)$. This minimization process is iterated until a satisfactory solution is obtained with $\widehat{\theta}_N$ denoting the estimator of the model parameters obtained in the last step.

In the general case given by absence of knowledge of the true population parameter κ_0, we propose a two-step estimation procedure for estimating the model parameters.[3] For each $\kappa \in [K_{min}, K_{max}]$ with $1 < K_{min} < K_{max} < K$, we define the set of parameter estimators $\widehat{\theta}_N(\kappa)$ as

$$\widehat{\theta}_N(\kappa) = \arg\min_{\theta \in \Theta} G_N(\theta, \kappa). \tag{16}$$

[3] A similar two-step procedure for estimation of the model parameters using GMM methods is proposed by Seo and Shin (2016). These authors derive the asymptotic distribution of the model parameters including the threshold. In contrast to the conventional theory for threshold estimators derived from least squares, see, for example, the literature initiated by Chan (1993) and Hansen (1996), the threshold parameter estimator proposed by these authors is asymptotically normal irrespective of whether the regression function is continuous.

The second step of the estimation process consists of finding the strategic horizon of the investors' multiperiod objective function that minimizes $G_N(\widehat{\theta}_N(\kappa),\kappa)$ on κ. More formally,

$$\widehat{\kappa}_N = \underset{\kappa\in[K_{\min},K_{\max}]}{\arg\min}\; G_N(\widehat{\theta}_N(\kappa),\kappa). \tag{17}$$

The resulting estimator is the vector $(\widehat{\theta}_N(\widehat{\kappa}_N),\widehat{\kappa}_N)$ that will be denoted as $(\widehat{\theta}_N,\widehat{\kappa}_N)$ hereafter.

Applying standard results already derived in Chan (1993), Andrews (1993) and Hansen (2000) for least squares methods and in Seo and Shin (2016) for GMM, we state without formal proof that

$$\widehat{\kappa}_N \xrightarrow{p} \kappa_0. \tag{18}$$

We are also interested in making inference on the model parameters $\theta_0 = (\Lambda_0, \gamma_0, \eta_0)$. Abusing of notation, we interpret the quantities θ_0 and $\widehat{\theta}_N$ as vectors of dimension $(mn+2n)\times 1$. A direct application of the asymptotic theory developed in Hansen (1996, 2000), Gonzalo and Pitarakis (2002) and Gonzalo and Wolf (2005) for least squares procedures, and more specifically, theorem 1 in Seo and Shin (2016) for GMM estimation, shows that

$$\sqrt{N}\left(\widehat{\theta}_N(\widehat{\kappa}_N)-\theta_0\right) \xrightarrow{d} N\left(0, \left(D(\kappa_0)' V_0^{-1}(\theta_0,\kappa_0) D(\kappa_0)\right)^{-1}\right) \tag{19}$$

with $D(\kappa_0) \equiv E[\frac{\partial g_t(\theta,\kappa_0)}{\partial\theta}]$ a $mn^2 \times (mn+2n)$ matrix.

4 Econometric methods: hypothesis testing

This section presents a threshold nonlinearity test to statistically assess the presence of two regimes in risk aversion. Our parametric test also accommodates the presence of dynamics in both short and long-term risk aversion functions. We also exploit the overidentified system of equations (12) to propose a specification test for the parametric formulation of the risk aversion function (2) and the policy rule (6).

4.1 Threshold nonlinearity tests

Following the literature on threshold and structural break models we will distinguish two cases. One in which the timing of the break κ_0 is known, and a second case, in which κ_0 is not identified under the null hypothesis. The quantity of interest is the risk aversion function (2) that is reproduced here for the sake of clarity in the exposition:

$$\gamma_t(j;\kappa_0) = \exp\left((\gamma' + \eta' 1(j > \kappa_0))Z_{t+j}\right) \ \textit{with} \ j = 1, \ldots, K.$$

A simplified version of this function that does not entertain dynamics in risk aversion is $\gamma(j;\kappa_0) = \exp\left(\gamma_j + \eta_j 1(j > \kappa_0)\right)$, with $\gamma_j = \gamma_c$ and $\eta_j = \eta_c$ for $j = 1, \ldots, K$. For the latter case, the null hypothesis rejecting nonlinearity in risk aversion is formulated as

$$H_0 : \eta_c = 0 \text{ against } H_A : \eta_c \neq 0.$$

In the extended version that incorporates linear dynamics in risk aversion the null hypothesis is

$$H_0^{\eta} : \eta_c = \eta_1 = \ldots = \eta_{n-1} = 0 \text{ against } H_A^{\eta} : \eta_s \neq 0 \ \text{ for some } \ s = c, 1, \ldots, n-1.$$

These tests are standard for κ_0 known and appropriate test statistics can be deployed by exploiting the overidentified system of equations (12). More specifically, a suitable nonlinearity test for the above null hypotheses is the likelihood ratio test

$$L_K(\kappa_0) = N\left(G_N(\widehat{\theta}_{0N}, \kappa_0) - G_N(\widehat{\theta}_N, \kappa_0)\right), \tag{20}$$

with $\widehat{\theta}_{0N}$ denoting the model parameter estimates under the null hypothesis and $\widehat{\theta}_N$ the corresponding model parameter estimates obtained from the unrestricted model. It is important to note that the covariance matrix used for computing both statistics $G_N(\widehat{\theta}_{0N}, \kappa_0)$ and $G_N(\widehat{\theta}_N, \kappa_0)$ is obtained from the unrestricted model. Under these conditions, the asymptotic distribution of the test statistic under the null hypothesis H_0^η satisfies

$$L_K(\kappa_0) \xrightarrow{d} \chi_n^2, \tag{21}$$

with n the number of parameters defining the nonlinear segment of (2).

The most interesting case is to test for the presence of nonlinearities in the preferences of long-term investors when κ_0 is not known. In this case $\kappa_0 \in [K_{\min}, K_{\max}]$ is a nuisance parameter that cannot be identified under the null hypotheses H_0 and H_0^η of constant/linearity of the risk aversion function, respectively. Hansen (1996) shows, in very general settings, that the composite nonlinearity test is nonstandard. As proposed by this author, see also Davies (1977, 1987) or Andrews and Ploberger (1994) in different contexts, hypothesis tests for nonlinearity can be based on different functionals of the relevant test statistic computed over the domain of the nuisance parameter. In our framework, the relevant test statistic is $l_K = \sup\limits_{\kappa \in [K_{\min}, K_{\max}]} L_K(\kappa)$ with sup the supremum of the functional version of (20).

To derive the asymptotic distribution of the relevant test we define the processes $S_N(\widehat{\theta}_N, \kappa) = \sqrt{N}\ g_N(\widehat{\theta}_N, \kappa)$ and its counterpart under the null hypothesis H_0^η defined as $S_{0N}(\widehat{\theta}_{0N}, \kappa) = \sqrt{N}\ g_N(\widehat{\theta}_{0N}, \kappa)$. Both processes have the asymptotic covariance kernel defined as

$$\Sigma_0(\kappa_1, \kappa_2) = \frac{1}{N} \sum_{t=1}^{N} \sum_{s=1}^{N} E[g_t(\theta_0, \kappa_1) g_s(\theta_0, \kappa_2)'], \tag{22}$$

with $\kappa \in [K_{\min}, K_{\max}]$. The empirical counterpart of (22) is

$$\widehat{\Sigma}_N(\kappa_1, \kappa_2) = \widetilde{\Gamma}_N(0) + \sum_{j=1}^{l} \frac{l-j}{j} \left(\widetilde{\Gamma}_N(j) + \widetilde{\Gamma}_N(j)' \right), \tag{23}$$

with $\widetilde{\Gamma}_N(j) = \frac{1}{N} \sum_{t=j+1}^{N} g_t(\widehat{\theta}_N, \kappa_1) g_{t-j}(\widehat{\theta}_N, \kappa_2)'$ the functional counterpart of $\Gamma_N(j)$ defined above. The process $L_K(\kappa)$ is defined as

$$L_K(\kappa) = S'_{0N}(\widehat{\theta}_{0N}, \kappa)\widehat{\Sigma}_N(\kappa, \kappa)^{-1} S_{0N}(\widehat{\theta}_{0N}, \kappa) - S'_N(\widehat{\theta}_N, \kappa)\widehat{\Sigma}_N(\kappa, \kappa)^{-1} S_N(\widehat{\theta}_N, \kappa). \tag{24}$$

Under some suitable regularity conditions on the uniform convergence of $\widehat{\Sigma}_N(\kappa_1, \kappa_2)$ to $\Sigma_0(\kappa_1, \kappa_2)$ over its compact support, see Hansen (1996) for more technical details, the process $S_N(\widehat{\theta}_N, \kappa)$ converges weakly to a multivariate zero mean Gaussian process, $S(\theta_0, \kappa)$, defined by the covariance function $\Sigma_0(\kappa, \kappa)$. Similarly, under the null hypothesis H_0^{η}, the process $S_{0N}(\widehat{\theta}_{0N}, \kappa)$ converges to a multivariate zero-mean Gaussian process $S_0(\theta_0, \kappa)$ with covariance kernel $\Sigma_0(\kappa, \kappa)$. Under these conditions, the process $L_K(\kappa)$ converges in distribution to the following chi-square process

$$L_0(\kappa) = S'_0(\theta_0, \kappa)\Sigma_0(\kappa, \kappa)^{-1} S_0(\theta_0, \kappa) - S'(\theta_0, \kappa)\Sigma_0(\kappa, \kappa)^{-1} S(\theta_0, \kappa), \tag{25}$$

with n degrees of freedom and determined by the number of restrictions imposed by H_0^{η}. The continuous mapping theorem implies that the statistic $l_K = \sup_{\kappa \in [K_{\min}, K_{\max}]} L_K(\kappa)$ converges in distribution to $l_0 = \sup_{\kappa \in [K_{\min}, K_{\max}]} L_0(\kappa)$. Since the null distribution of (25) depends upon the covariance function $\Sigma_0(\kappa, \kappa)$, critical values cannot be tabulated. To obtain the $p-$values of the test we derive a p-value transformation similar in spirit to the work of Hansen (1996) based on a multiplier bootstrap.

Let $F_0(\cdot)$ denote the distribution function of l_0, and define $p_N = 1 - F_0(l_K)$. The above result shows that p_N converges in probability to $p_0 = 1 - F_0(l_0)$ that under the null hypothesis is uniform on $[0,1]$. Thus the asymptotic null distribution of p_N is free of nuisance parameters. The rejection rule of our test is given by $p_N < \alpha$ with α the significance level and p_N the asymptotic p-value. The random variable l_0 can be written as a continuous functional of the Gaussian processes $S(\theta_0, \kappa)$ and $S_0(\theta_0, \kappa)$ which are completely described by the covariance function $\Sigma_0(\kappa, \kappa)$. To implement the p-value transformation, we operate conditional on the sample $X_t = \{(r'_{t+1}, Z'_t)'\}_{t=1}^N$. More specifically, we define the multivariate mean-zero Gaussian processes $\widehat{S}_N(\widehat{\theta}_N, \kappa)$ and $\widehat{S}_{0N}(\widehat{\theta}_{0N}, \kappa)$ conditional on the sample X_t. These processes are vectors of dimension mn^2 that can be generated by simulating a sequence of *i.i.d.* univariate $N(0,1)$ random variables $\{v_t\}_{t=1}^N$. More specifically, each element of the multivariate process $\widehat{S}_N$ is defined as

$$\widehat{S}_{i,N}(\widehat{\theta}_N, \kappa) = \frac{1}{\sqrt{N}} \sum_{t=1}^{N} g_{it}(\widehat{\theta}_N, \kappa)\ v_t. \tag{26}$$

Similarly, each element of the multivariate process $\widehat{S}_{0N}$ is

$$\widehat{S}_{i,0N}(\widehat{\theta}_{0N}, \kappa) = \frac{1}{\sqrt{N}} \sum_{t=1}^{N} g_{it}(\widehat{\theta}_{0N}, \kappa)\ v_t. \tag{27}$$

The introduction of the zero-mean random variable v_t implies that, conditional on the sample X_t, the covariance function of the simulated process $\widehat{S}_N(\widehat{\theta}_N, \kappa)$ is equal to the sample covariance $\widehat{\Sigma}_N(\kappa, \kappa)$ of the process $S_N(\widehat{\theta}_N, \kappa)$. The corresponding conditional chi-square process is

$$\widehat{L}_K(\kappa) = \widehat{S}'_{0N}(\widehat{\theta}_{0N}, \kappa)\widehat{V}_N^{-1}(\widehat{\theta}_N, \kappa)\widehat{S}_{0N}(\widehat{\theta}_{0N}, \kappa) - \widehat{S}'_N(\widehat{\theta}_N, \kappa)\widehat{V}_N^{-1}(\widehat{\theta}_N, \kappa)\widehat{S}_N(\widehat{\theta}_N, \kappa) \tag{28}$$

and the corresponding test statistic is $\widehat{l}_K = \sup_{\kappa \in [K_{\min}, K_{\max}]} \widehat{L}_K(\kappa)$.

Let $\widehat{F}_0$ denote the conditional distribution function of $\widehat{l}_K$ and $\widehat{p}_N = 1 - \widehat{F}_0(\widehat{l}_K)$. Following similar arguments to the proof of Theorem 2 in Hansen (1996), it can be shown that the quantity $\widehat{p}_N$ is asymptotically equivalent to p_N under both the null and alternative hypotheses. The conditional distribution function $\widehat{F}_N$ is not directly observable so neither is the random variable $\widehat{p}_N$. Nevertheless, these quantities can be approximated to any desired degree of accuracy using standard simulation techniques. The following algorithm shows the implementation of this p-value transformation. Let K_b define a grid of b points over the compact set $[K_{\min}, K_{\max}]$, and let κ_i for $i = 1, \ldots, b$ be the set of equidistant points in such grid with $\kappa_1 = K_{\min}$ and $\kappa_b = K_{\max}$. For $j = 1, \ldots, J$ with J denoting the number of bootstrap replications, execute the following steps:

i) generate the sequence $\{v_{jt}\}_{t=1}^N$ *i.i.d.* random variables;

ii) conditional on the sample $X_t = \{(r'_{t+1}, Z'_t)'\}_{t=1}^N$, set the quantities $\widehat{S}_N^j(\widehat{\theta}_N, \kappa_i)$ and $\widehat{S}_{0N}^j(\widehat{\theta}_{0N}, \kappa_i)$;

iii) set $\widehat{L}_K^j(\kappa_i) = \widehat{S}_{0N}^j(\widehat{\theta}_{0N}, \kappa_i)\widehat{V}_N^{-1}(\widehat{\theta}_N, \kappa_i)\widehat{S}_{0N}^j(\widehat{\theta}_{0N}, \kappa_i) - \widehat{S}_N^j(\widehat{\theta}_N, \kappa_i)\widehat{V}_N^{-1}(\widehat{\theta}_N, \kappa_i)\widehat{S}_N^j(\widehat{\theta}_N, \kappa_i)$;

iv) set $\widehat{l}_K^j = \sup\limits_{\kappa_i \in K_b} \widehat{L}_K^j(\kappa_i)$.

This gives a random sample $(\widehat{l}_K^1, \ldots, \widehat{l}_K^J)$ from the conditional distribution $\widehat{F}_N$. The percentage of these artificial observations which exceeds the actual test statistic l_K: $\widehat{p}_N^J = \frac{1}{J}\sum\limits_{j=1}^{J} 1\left(\widehat{l}_K^j > \widehat{l}_K\right)$ is according to the Glivenko-Cantelli theorem a consistent approximation of $\widehat{p}_N$ as $J \to \infty$. In practice, the null hypothesis H_0 is rejected if $\widehat{p}_N^J < \alpha$.

4.2 Specification test

The system of equations defined in (12) entails the existence of testable restrictions of our econometric specification determined by the nonlinear risk aversion function (2) and the parametric portfolio weights (6). Estimation of $\theta_0 = (\Lambda_0, \gamma_0, \eta_0)$ sets to zero $mn + 2n$ linear combinations of the mn^2 sample orthogonality conditions $g_N(\theta_0, \kappa)$ for each $\kappa \in [K_{min}, K_{max}]$. The correct specification of the model implies that, for a fixed κ_0, there are $mn^2 - mn - 2n$ linearly independent combinations of $g_N(\widehat{\theta}_N, \kappa_0)$ that should be close to zero but are not exactly equal to zero. This hypothesis is tested using the Hansen test statistic (Hansen, 1982).

Remember that $G_N(\widehat{\theta}_N, \widehat{\kappa}_N) = g_N(\widehat{\theta}_N, \widehat{\kappa}_N)' \widehat{V}_N^{-1}(\widehat{\theta}_N, \widehat{\kappa}_N) g_N(\widehat{\theta}_N, \widehat{\kappa}_N)$. Under the null hypothesis of correct specification of the model, this statistic satisfies

$$N \; G_N(\widehat{\theta}_N, \widehat{\kappa}_N) \xrightarrow{d} \chi^2_{mn^2-mn-2n}. \tag{29}$$

The null hypothesis of correct specification of the overidentified system of equations is rejected at a significance level α if the test statistic is greater than the critical value $\chi^2_{mn^2-mn-2n,1-\alpha}$. It is worth noting that the $N-$rate consistency of the threshold parameter estimate obtained from (17) allows one to replace κ_0 by the estimator $\widehat{\kappa}_N$ without producing any change on the asymptotic distribution of the test.

5 Empirical application

We are interested in analyzing empirically the effect of considering two regimes in investor's risk aversion. To do this, we study the optimal portfolio decisions corresponding to investment horizons at four years ($K = 48$). We provide the analysis for $K = 12$ to $K = 36$ in an online appendix. For the sake of comparison, we entertain three scenarios: the benchmark case characterized by constant and static risk aversion, the case characterized by a risk aversion coefficient linearly related to our set of state variables, and the threshold type specification that extends the previous two scenarios by incorporating risk aversion to the short and the long term.

We follow seminal studies such as Brennan et al. (1997), Brandt (1999) and Campbell et al. (2003), and consider three investment assets: a one-month Treasury bill as riskless security, a long-term bond, and an equity portfolio. There are no short-selling restrictions and the discount factor to measure investor's patience is $\beta = 0.95$. Our data covers the period January 1980 to

December 2016. Monthly data are collected from Bloomberg on the S&P 500 and G0Q0 Bond Index. The G0Q0 Bond Index is a Bank of America and Merrill Lynch U.S. Treasury Index that tracks the performance of U.S. dollar denominated sovereign debt publicly issued by the U.S. government in its domestic market. The nominal yield on the U.S. one-month risk-free rate is obtained from the Fama and French database. The consumer price index (CPI) time series, used to transform nominal variables into real variables, and the yield of the Moody's Baa- and Aaa-rated corporate bonds are obtained from the U.S. Federal Reserve.

The time-variation of the optimal portfolio weights is described by a set of state variables that have been identified in the empirical literature as potential predictors of the excess stock and bond returns and the short-term ex-post real interest rates. These variables are the detrended short-term interest rate (Campbell, 1991), the U.S. credit spread (Fama and French, 1989), the S&P 500 trend (Aït-Sahalia and Brandt, 2001) and the one-month average of the excess stock and bond returns (Campbell et al., 2003). The detrended short-term interest rate detrends the short-term rate by subtracting a 12-month backwards moving average. The U.S. credit spread is defined as the yield difference between Moody's Baa- and Aaa-rated corporate bonds. The S&P 500 trend, or momentum, state variable is defined as the difference between the log of the current S&P 500 index level and the average index level over the previous 12 months. We demean and standardize all the state variables in the optimization process (Brandt et al, 2009).

[Insert Table 1 about here]

Table 1 reports summary statistics for the excess S&P 500 index return, excess G0Q0 bond index return and the short-term ex-post real interest rates. The results show higher mean and volatility for the stock index compared to the bond index and short-term real interest rate series. The skewness reveals a negative skew of the excess stock returns and a positive skew of the bond and cash series. The estimates of the kurtosis parameters also reflect the leptokurtic behavior of the three assets. Interestingly, the excess bond return has larger skewness and lower kurtosis than the S&P 500 index. This anomalous result highlights the outperformance of the G0Q0 index over the S&P 500 index mainly explained by the values of the series during the 2007-2009 period and, in particular, by the consequences of the subprime crisis on the valuation of the different risky assets.

5.1 Empirical results

The parameter estimates driving the optimal portfolio rules and dynamic risk aversion coefficients are estimated using a two-step Gauss-Newton type algorithm with numerical derivatives. The method is implemented in Matlab and code is available upon request. In a first stage we initialize the covariance matrix $\widehat{V}_N$ with the matrix $I_{mn} \otimes Z'Z$ of dimension mn^2, and in a second stage, after obtaining a first set of parameter estimates, we repeat the estimation replacing this matrix by the Newey-West estimator (14) with $l = 12$ lags. The choice of this number of lags is to obtain an estimator sufficiently robust to the presence of serial correlation in the asymptotic covariance matrix V_0. Table 2 reports estimates of the model parameters (optimal portfolio weights and risk aversion coefficients) for the three different types of investors for an investment horizon of four years ($K = 48$). The first column contains the estimates of the nonlinear process distinguishing between the short and the long term. The second column reports the parameter estimates of a simplified version of this model characterized by linear dynamics in the risk aversion coefficient. The third column contains the benchmark static model employed in the literature.

The empirical analysis presented below reveals four main features. First, the period that separates the short from the long term is found to be around the seventh month of the investment horizon. Second, the likelihood ratio tests discussed above provide strong statistical evidence of the presence of dynamics and nonlinearities in the individuals' risk aversion coefficient. In particular, our novel nonlinearity likelihood ratio test reveals the presence of nonlinearities in risk aversion when compared to the linear dynamic and constant cases. We show in the online appendix that these differences are more relevant as the investment horizon increases. Third, we observe that the allocation to bonds and stocks is negatively correlated. This finding is indicative of the existence of flight to quality effects from stocks to bonds especially during market distress episodes. Fourth, during these periods we observe a significant increase in the allocation to bonds between the constant risk aversion model and the nonlinear dynamic model. In contrast, the optimal allocation to stocks is robust to the form of the risk aversion coefficient even under market distress.

Table 2 also reveals interesting insights about risk aversion. The constant risk aversion coefficient γ_c is 3.372 and corresponds to a relative risk aversion coefficient of 29, obtained as the exponent of γ_c. The column of the bottom panel reporting the linear dynamics in the risk aversion coefficient provides overwhelming statistical evidence of the significance of the four state variables used in our analysis. These results are supported by the corresponding likelihood ratio test comparing the linear and constant risk aversion models that yields a p-value of zero. This is also observed for different investment horizons in the extended version of the empirical application reported in the online appendix. The analysis of nonlinearities in risk aversion also reveals interesting findings. In particular, there is strong evidence of nonlinearity reflected in the value of the parameter η_c. The risk aversion coefficient corresponding to the nonlinear model is $\gamma_c + \eta_c$ that is equal to 4.424, and yields a value of relative risk aversion of 83. The overall risk aversion level clearly increases from the short to the long term. The difference between the short and the long term is characterized by a threshold value κ_0 that is estimated from the data. The estimates of θ_0 in Table 2 correspond to $\widehat{\kappa}_N = 7$. The presence of nonlinearity in the risk aversion coefficient is tested using the likelihood ratio test introduced in the previous section comparing the linear model against the model with two regimes. The p-value is zero for $K = 48$. The relevance of the state variables in driving the nonlinearities in risk aversion is mixed. The results in Table 2 only show the statistical significance of the S&P 500 trend. However, unreported results obtained for non-optimal κ's also show the statistical significance of the one-month average of excess stock and bond returns.

[Insert Table 2 about here]

The analysis of the optimal portfolio allocation reveals interesting insights related to the form of risk aversion. There is clear evidence of the influence of the state variables in driving the dynamics of the optimal allocation to the bond index. In particular, the four state variables are statistically significant under the three different types of risk aversion scenarios considered in the paper. The allocation to stocks is also quite revealing of the importance of the state variables, with the results being more significant as the investment horizon increases, see also online appendix for $K = 12, 24, 36$. Overall, the results show the statistical significance of the detrended short-term interest rate, the U.S. credit spread and the one-month average of the excess bond and stock returns. These results are common across risk aversion scenarios. The analysis of the parameter values also sheds interesting findings. Thus, the state variables have a positive effect on the optimal portfolio weights reflected in positive estimates of the $\lambda_{s,\cdot}$ parameters. In contrast, the state variables have a negative effect on the optimal portfolio weights allocated to the bond index, as indicated by the negative parameter estimates of $\lambda_{b,\cdot}$. These results imply negative comovements between the optimal allocation to the S&P 500 index and the G0Q0 bond index.

We illustrate this analysis further by plotting the dynamics of the risk aversion function γ_t. The top panel of Figure 1 reports the constant and linear dynamic risk aversion coefficient (2) defined as $\gamma(j) = \exp(\widehat{\gamma}_c)$ and $\gamma_t(j) = \exp(\widehat{\gamma}' Z_{t+j})$, respectively. The bottom panel of Figure 1 plots the nonlinear version of the risk aversion function (2). For comparison purposes, we report separately the short-term dynamics $\exp(\widehat{\gamma}' Z_{t+j})$ and the long-term dynamics $\exp((\widehat{\gamma}' + \widehat{\eta}') Z_{t+j})$. The top panel of Figure 1 exhibits notable fluctuations in risk aversion during the first half of the 1980 decade due to the sharp increase in oil prices that led to a worldwide economic recession. This trend is compensated during the $2000 - 2006$ Great Moderation period. This period was characterized by economic stability, strong growth, low inflation and low and stable interest rates. During this episode the dynamic risk aversion coefficient is below the constant risk aversion coefficient $exp(\gamma_c)$. The bottom panel illustrates the additional effect of long-term risk aversion to the short-term risk aversion component. The contribution of long-term risk aversion to the overall risk aversion function is very significant during the first half of the decade of 1980 and the 2007-2009 crisis period. We also observe spikes in long-term risk aversion during the second half of the 1990 decade.

[Insert Figures 1 and 2 about here]

Figure 2 reports the dynamics of the optimal portfolio allocations to the S&P 500 index (α_{st}), the G0Q0 bond index (α_{bt}) and the one-month Treasury bill (α_{ct}) for $K = 48$ over the crisis period January 2007 to December 2011. We focus on the comparison between the constant risk aversion scenario and the case of nonlinear dynamics in risk aversion. The top panel reports the optimal allocation to stocks, the middle panel reports the optimal allocation to bonds and the bottom panel the optimal allocation to the Treasury bill (cash). The sum of the three weights is equal to one by construction. The dashed black line corresponds to the dynamic nonlinear strategy, the dotted red line to the dynamic linear strategy and the solid blue line to the constant risk aversion strategy. The allocation to stocks is very stable across time and oscillates between -0.5 and 0.5. Periods of financial distress accompanied by increases in risk aversion are corresponded by decreases in the optimal allocation to stocks. Periods of economic boom accompanied by a decrease in the overall level of risk aversion entail increases in α_{st}. It is worth noting the large drop in the optimal allocation to stocks during 2008-2009. The comparison of the optimal allocation to stocks between the constant and nonlinear risk aversion functions is not significant, though, and suggests that the presence of dynamics in individuals' risk aversion does not have a dramatic effect on the allocation to stocks. This result contrasts with the allocation to bonds reported in the middle panel of Figure 2. This allocation increases in periods of higher risk aversion providing evidence of a negative correlation between the allocation to stocks and bonds. Finally, the analysis of the optimal allocation to cash suggests that this financial instrument is used as a safety asset in periods of financial distress in which risk aversion increases considerably, see, for example, the large allocation to cash during the 2007-2009 crisis period reported in the bottom panel of Figure 2.

6 Conclusion

This paper studies the long-term asset allocation problem of an individual with different risk aversion attitudes towards the short and the long term. These different risk aversion coefficients also incorporate dynamics that are driven by variations in economic conditions and proxied by a vector of state variables. Our optimal asset allocation strategy is obtained from a parametric linear portfolio policy that accommodates an arbitrarily large number of assets in the portfolio. The parameters defining this model are estimated using GMM procedures applied to an overidentified system of Euler equations describing the first order conditions of the individual's multiperiod maximization problem.

The empirical application to a portfolio of three assets - a one-month Treasury bill as riskless security, a long-term bond, and an equity portfolio finds significant empirical evidence of the presence of dynamics in risk aversion. More importantly, we also find differences in short and long-term risk aversion. The threshold separating the short from the long term is observed to be around the seventh month of the investment horizon. The long-term risk aversion coefficient has a significant role in determining the optimal allocation to fixed income assets such as the one-month Treasury bill and the G0Q0 bond index but not to the allocation to stocks that remains very stable across risk aversion scenarios. In particular, we find a large exposure of the investment portfolio to cash and bonds as risk aversion increases and such that for very high levels of risk aversion, the allocation tilts from bonds to cash. The estimation of the parameters driving the linear portfolio policy reveals the different contributions of the state variables in determining the optimal portfolio weights. More specifically, we find that the detrended short term interest rate, the U.S. credit spread and the one-month average of the excess stock and bond returns have a positive effect on the allocation to stocks, however, the S&P 500 trend is not statistically significant. In contrast, the four state variables have a strong negative effect on the optimal allocation to bonds. These findings provide further empirical evidence of a strong negative correlation in the allocation between stocks and bonds and the allocation between stocks and cash.

References

[1] Aït-Sahalia, Y., and Brandt, M.W. (2001). Variable Selection for Portfolio Choice. Journal of Finance 56, 1297-1351.

[2] Andrews, D.W.K. (1993). Tests for parameter instability and structural change with unknown change point. Econometrica 61, 821-856.

[3] Andrews, D.W.K., and Ploberger, W. (1994). Optimal Tests When a Nuisance Parameter Is Present Only under the Alternative. Econometrica 62, 1383-1414.

[4] Barberis, N. (2000). Investing for the Long Run when Returns are Predictable, The Journal of Finance 55, 225-264.

[5] Brandt, M.W. (1999). Estimating portfolio and consumption choice: A conditional Euler equations approach. Journal of Finance 54, 1609-1646.

[6] Brandt, M.W., and Santa Clara, P. (2006). Dynamic Portfolio Selection by Augmenting the Asset Space. Journal of Finance 61, 2187-2217.

[7] Brandt, M.W., Santa Clara, P., and Valkanov, R. (2009). Parametric portfolio policies exploiting the characteristics in the cross section of equity returns. Review of Financial Studies 22, 3411-3447.

[8] Brandt, M.W., and Wang, K. Q. (2003). Time-varying risk aversion and unexpected inflation. Journal of Monetary Economics 50, 1457-1498.

[9] Brennan, M., Schwartz, E., and Lagnado, R. (1997). Strategic asset allocation. Journal of Economic Dynamics and Control 21, 1377-1403.

[10] Brennan, M., Schwartz, E., and Lagnado, R. (1999). The use of treasury bill futures in strategic asset allocation programs. In: Ziemba, W.T., Mulvey, J.M. (Eds.), World Wide Asset and Liability Modeling. Cambridge University Press, Cambridge, pp. 205-228.

[11] Campbell, J.Y. (1991). A variance decomposition for stock returns. Economic Journal 101, 157-179.

[12] Campbell, J. Y., and Viceira, L. M. (1999). Consumption and portfolio decisions when expected returns are time varying. Quarterly Journal of Economics 114, 433-495.

[13] Campbell, J. Y., and Viceira, L. M. (2001). Who should buy long-term bonds? American Economic Review 91, 99-127.

[14] Campbell, J. Y., and Viceira, L. M. (2002). Strategic Asset Allocation: Portfolio Choice for Long-Term Investors (Oxford University Press, New York).

[15] Campbell, J. Y., Chan, Y., and Viceira, L. M. (2003). A multivariate model of strategic asset allocation. Journal of Financial Economics 67, 41-80.

[16] Chan, K. S. (1993). Consistency and Limiting Distribution of the Least Squares Estimator of a Threshold Autoregressive Model. Annals of Statistics 21, 520-533.

[17] Davies, R.B. (1977). Hypothesis testing when a nuisance parameter is present only under the alternative. Biometrika 64, 247-254.

[18] Davies, R.B. (1987). Hypothesis testing when a nuisance parameter is present only under the alternative. Biometrika 74, 33-43.

[19] Fama, E. F., and French, K.R. (1989). Business conditions and expected returns on stocks and bonds. Journal of Financial Economics 25, 23-49.

[20] Giacomini, R., and Komunjer, I. (2005). Evaluation and Combination of Conditional Quantile Forecasts. Journal of Business and Economic Statistics 23, 416-431.

[21] Gonzalo, J., and Pitarakis, J.Y. (2002). Estimation and model selection based inference in single and multiple threshold models. Journal of Econometrics 110, 319-352.

[22] Gonzalo, J., and Wolf, M. (2005). Subsampling inference in threshold autoregressive models. Journal of Econometrics 127, 201-224.

[23] Gonzalo, J., and Pitarakis, J.Y. (2012). Regime specific predictability in predictive regressions. Journal of Business and Economic Statistics 30, 229-241.

[24] Gonzalo, J., and Pitarakis, J.Y. (2017). Inferring the predictability induced by a persistent regressor in a predictive threshold model. Journal of Business and Economic Statistics 35, 202-217.

[25] Hansen, B.E. (1996). Inference when a nuisance parameter is not identified under the null hypothesis. Econometrica 64, 413-430.

[26] Hansen, B.E. (2000). Sample Splitting and Threshold Estimation. Econometrica 68, 575-603.

[27] Hansen, L. (1982). Large sample properties of generalized method of moments estimators. Econometrica 50, 1029-1054.

[28] Merton, R. C. (1973). An Intertemporal Capital Asset Pricing Model. Econometrica 41, 867-887.

[29] Perron, P. (1989). The great crash, the oil price shock, and the unit root hypothesis. Econometrica 57, 1361-1401.

[30] Perron, P. (1997), Further Evidence on Breaking Trend Functions in Macroeconomic Variables. Journal of Econometrics 80, 355-385.

[31] Seo, M.H., and Shin, Y. (2016). Inference for Dynamic Panels with Threshold Effects and Endogeneity. Journal of Econometrics 195, 169-186.

asset	Mean	Std dev	10% perc.	Median	90% perc.	Skewness	Kurtosis
S&P500 index	0.0031	0.0364	-0.1166	0.0067	0.0399	-1.1773	7.9628
G0Q0 Bond index	0.0023	0.0155	-0.0161	0.0025	0.0208	0.1534	5.3247
r_f	0.0010	0.0031	-0.0027	0.0010	0.0047	0.4547	5.3857

Table 1. Summary statistics of the excess stock return, excess bond return and short-term ex-post real interest rates over the period January 1980 to December 2016 (444 observations). The return horizon is one month.

stock parameters				bond parameters			
	nonlinear	linear	constant		nonlinear	linear	constant
$\lambda_{s,c}$	0.186 [0.000]	0.231 [0.000]	0.241 [0.000]	$\lambda_{b,c}$	0.526 [0.000]	0.496 [0.000]	0.474 [0.000]
$\lambda_{s,1}$	0.089 [0.006]	0.092 [0.000]	0.066 [0.018]	$\lambda_{b,1}$	−0.162 [0.003]	−0.200 [0.000]	−0.305 [0.000]
$\lambda_{s,2}$	0.039 [0.086]	0.032 [0.013]	0.085 [0.000]	$\lambda_{b,2}$	−0.673 [0.000]	−0.762 [0.000]	−0.686 [0.000]
$\lambda_{s,3}$	−0.002 [0.930]	−0.014 [0.256]	−0.004 [0.727]	$\lambda_{b,3}$	−0.339 [0.006]	−0.345 [0.000]	−0.367 [0.000]
$\lambda_{s,4}$	0.335 [0.000]	0.400 [0.000]	0.485 [0.000]	$\lambda_{b,4}$	−0.294 [0.000]	−0.317 [0.000]	−0.402 [0.000]
short term regime				long term regime			
	nonlinear	linear	constant		nonlinear	linear	constant
γ_c	3.442 [0.000]	3.576 [0.000]	3.372 [0.000]	η_c	0.982 [0.001]	–	–
γ_1	−0.086 [0.321]	−0.188 [0.000]	–	η_1	−0.321 [0.370]	–	–
γ_2	0.425 [0.000]	0.412 [0.000]	–	η_2	0.577 [0.236]	–	–
γ_3	0.128 [0.264]	0.183 [0.000]	–	η_3	0.508 [0.111]	–	–
γ_4	0.277 [0.272]	0.125 [0.080]	–	η_4	−0.249 [0.588]		–
κ_o	7						

Table 2. Parameter estimates of the three different versions of the individual's objective function (3) for $K = 48$ and $\beta = 0.95$. The parameters λ correspond to the portfolio allocations associated to the state variables Z_t: $\lambda_{\cdot,c}$ is for the constant term, $\lambda_{\cdot,1}$ corresponds to the detrended short-term interest rate, $\lambda_{\cdot,2}$ to the U.S. credit spread, $\lambda_{\cdot,3}$ to the S&P 500 trend and $\lambda_{\cdot,4}$ to the one-month average of the excess stock and bond returns. Similarly, the vector γ describes the sensitivities of the risk aversion function (2) with respect to the state variables for the linear segment and η the corresponding sensitivities for the nonlinear segment of the function. κ_0 denotes the estimate of the threshold value corresponding to these parameter estimates. P-values are in squared brackets.

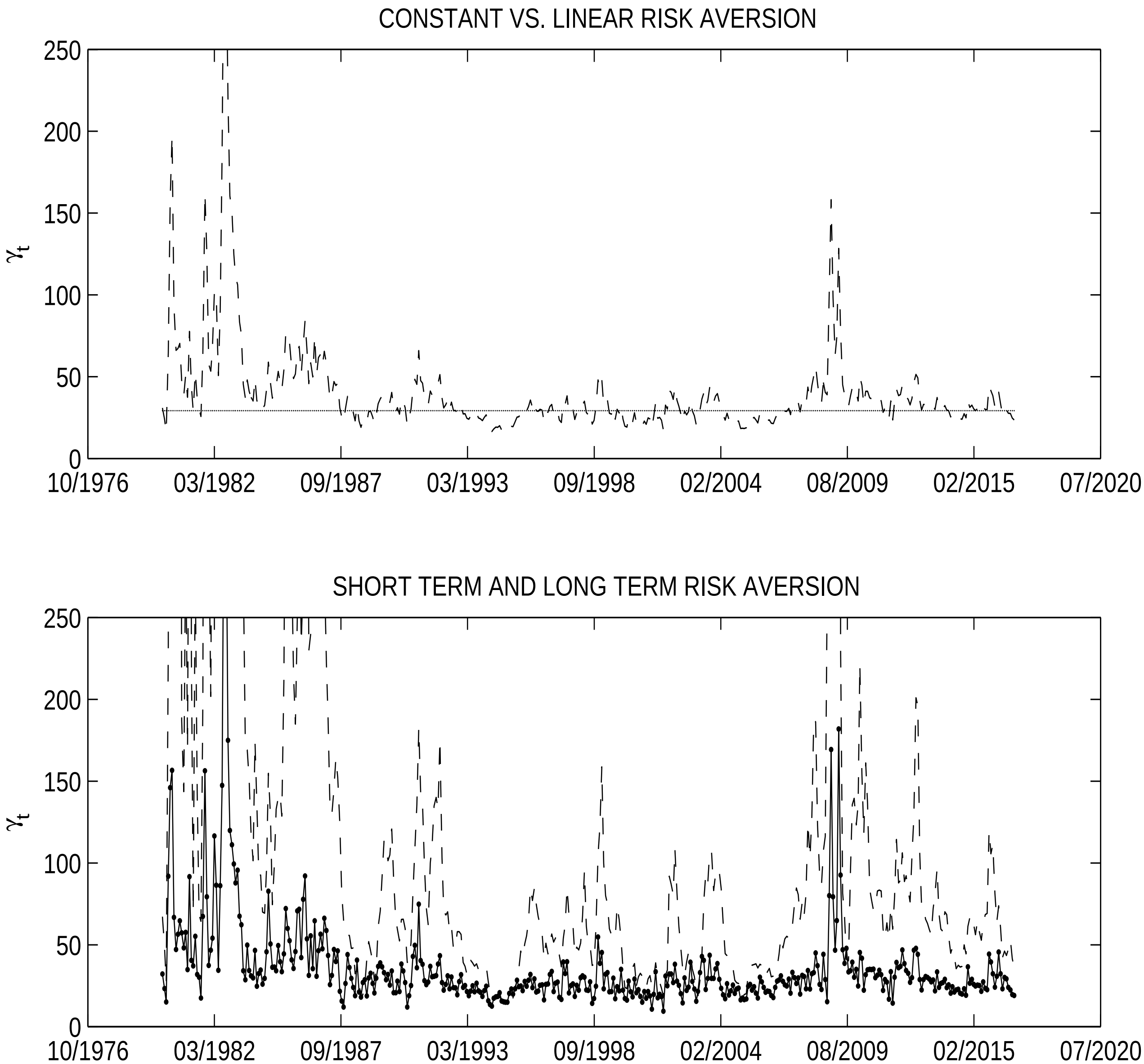

Figure 1. Dynamics of risk aversion over the period January 1980 to December 2016 for $K = 48$, $\beta = 0.95$. Top panel compares the constant and linear versions of the risk aversion function (2). Flat line for constant risk aversion and dashed line for dynamic risk aversion. Bottom panel compares the two segments defining the nonlinear version of (2). Dotted line for the short-term dynamics of risk aversion (γ parameters in (2)) and dashed for the long-term dynamics (η parameters in (2)).

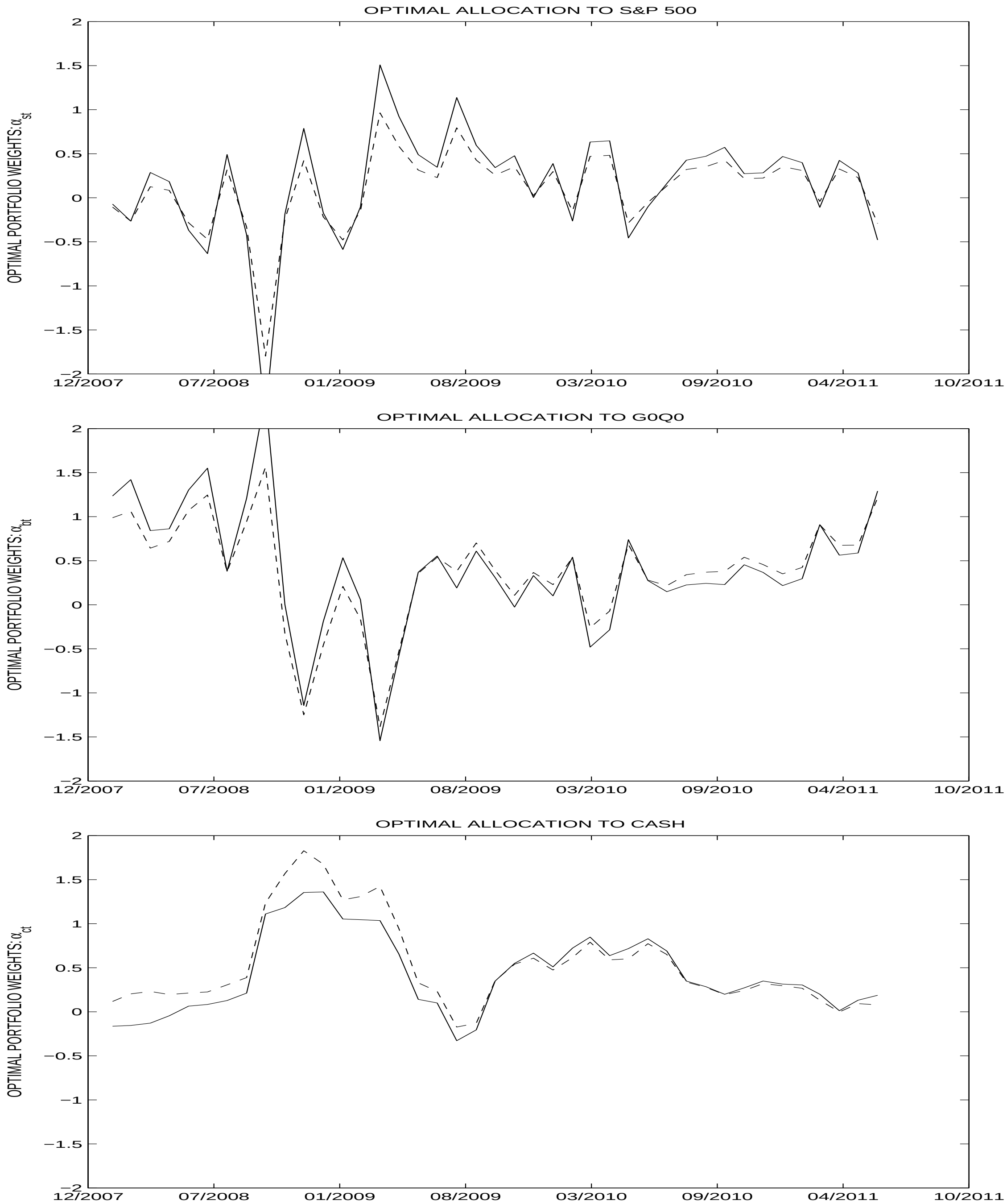

Figure 2. Dynamics of the optimal portfolio allocation to stocks, bonds and cash over the period 2007-2011 for $K = 48$, $\beta = 0.95$. Dashed black line for the dynamic nonlinear strategy and solid blue line for the constant risk aversion strategy.