$|V_{us}|$ determination from inclusive strange tau decay and lattice HVP

Peter Boyle[1], *Renwick James* Hudspith[2], *Taku* Izubuchi[3,4], *Andreas* Jüttner[5], *Christoph* Lehner[3], *Randy* Lewis[2], *Kim* Maltman[2,6], *Hiroshi* Ohki[4,7,⋆], *Antonin* Portelli[8], and *Matthew* Spraggs[8]

[1] *SUPA, School of Physics, The University of Edinburgh, Edinburgh EH9 3JZ, UK*
[2] *York University, 4700 Keele St., Toronto, ON Canada M3J IP3*
[3] *Physics Department, Brookhaven National Laboratory, Upton, NY 11973, USA*
[4] *RIKEN-BNL Research Center, Brookhaven National Laboratory, Upton, NY 11973, USA*
[5] *School of Physics and Astronomy, University of Southampton, Southampton SO17 1BJ, UK*
[6] *CSSM, University of Adelaide, Adelaide, SA 5005 Australia*
[7] *Physics Department, Nara Womens University, Nara 630-8506, Japan*
[8] *School of Physics and Astronomy, University of Edinburgh, Edinburgh EH9 3JZ, UK*

Abstract. We propose and apply a novel approach to determining $|V_{us}|$ which uses inclusive strange hadronic tau decay data and hadronic vacuum polarization functions (HVPs) computed on the lattice. The experimental and lattice data are related through dispersion relations which employ a class of weight functions having poles at space-like momentum. Implementing this approach using lattice data generated by the RBC/UKQCD collaboration, we show examples of weight functions which strongly suppress spectral integral contributions from the region where experimental data either have large uncertainties or do not exist while at the same time allowing accurate determinations of relevant lattice HVPs. Our result for $|V_{us}|$ is in good agreement with determinations from K physics and 3-family CKM unitarity. The advantages of the new approach over the conventional sum rule analysis will be discussed.

1 Introduction

The Kobayashi-Maskawa matrix element $|V_{us}|$ is an important parameter for flavor physics, one relevant to searching for new physics beyond the Standard Model (SM) of particle physics. Thus far, $|V_{us}|$ has been most precisely determined by a flavor breaking combination of $K_{\mu 2}$ and $\pi_{\mu 2}$ decays, using the lattice result $f_K/f_\pi = 1.193(3)$ [1], which yields $|V_{us}| = 0.2253(7)$. The unitarity constraint using the nuclear β decay determination, $|V_{ud}| = 0.97417(21)$ [2], similarly yields $|V_{us}| = 0.2258(9)$. There is a long-standing puzzle that the conventional version of the alternate determination employing hadronic τ decay data and flavor-breaking (FB) finite-energy sum rules (FESRs) yields very low $|V_{us}|$, recently $|V_{us}| = 0.2186(21)$ [3], which lies $\sim 3.1\sigma$ below 3-family-unitarity expectations.

The conventional FB FESR implementation relies on perturvative QCD and assumptions for unknown higher dimension OPE condensates which turn out to fail self-consistency tests [4]. Motivated

⋆ Speaker

by this observation, an alternate implementation, in which the dependence of weighted spectral integrals on the upper integration endpoint, s_0, is used to fit both $|V_{us}|$ and the $D > 4$ OPE condensates, was proposed. An updated version of this determination yields $|V_{us}| = 0.2231(22)_{exp}(4)_{th}$. The resulting error is dominated by uncertainties on the relevant weighted inclusive flavor us spectral integrals and is a factor > 2 larger than those from K-decay-based approaches. Progress can be made by improving the branching fractions (BFs) normalizing low-multiplicity exclusive us mode distributions measured by BaBar and Belle; the $\sim 25\%$ uncertainty on the contribution from "residual" modes (the higher-multiplicity-mode part of the 1999 ALEPH us distribution not re-measured at the B-factories), however, currently precludes achieving a fully competitive error [4, 5].

In this report we propose a novel dispersive approach to determining $|V_{us}|$ using inclusive strange hadronic τ decay data, hadronic vacuum polarization functions (HVPs) computed on the lattice, and weight functions,

$$\omega_N(s) = 1/\Pi_{k=1}^N (s + Q_k^2), \quad Q_k^2 > 0, \tag{1}$$

having poles at Euclidean momentum-squared Q^2. Implementing this approach using lattice data, we show examples of weight functions which allow the combination of lattice HVPs required for this analysis to be determined with good accuracy while at the same time strongly suppressing spectral integral contributions from the region of the high-multiplicity with larger-error us decay modes. In addition, the method avoids the (albeit mildly) model-dependent continuum $J = 0$ us subtraction required in the FB FESR approach, and uses precision lattice data rather than the OPE. We find results for $|V_{us}|$ from this approach in good agreement with those obtained from analyses of K physics and three-family CKM unitarity.

2 Conventional and new inclusive determinations

The conventional inclusive FB τ decay determination of $|V_{us}|$ employs experimentally determined spectral functions, $\rho(s)$, and the FESR relation

$$\int_0^{s_0} \omega(s)\rho(s)ds = -\frac{1}{2\pi i}\oint_{|s|=s_0} \omega(s)\Pi(s)ds, \tag{2}$$

where $\Pi(s)$ is the relevant HVP function, $\rho(s) = \frac{1}{\pi}\mathrm{Im}\Pi(s)$, and Eq. (2) is valid for any $s_0 > 0$ and any real $\omega(s)$ analytic inside the contour. In the SM, the differential distribution, $dR_{V/A;ij}/ds$, associated with the flavor $ij = us$ vector (V) or axial vector (A) currents, where $R_{V/A;us} = \Gamma[\tau^- \to \nu_\tau \mathrm{hadrons}_{\mathrm{V/A;us}}]/\Gamma[\tau^- \to \nu_\tau e^- \bar{\nu}_e]$ is related to the spectral function $\rho^{(J)}_{ij;V/A}(s)$, by

$$\frac{dR_{us;V/A}}{ds} = \frac{12\pi^2|V_{us}|^2 S_{EW}}{m_\tau^2}\left[\omega_\tau(s)\rho^{(0+1)}_{us;V/A}(s) - \omega_L(y_\tau)\rho^{(0)}_{us;V/A}(s)\right], \tag{3}$$

where $y_\tau = s/m_\tau^2$, $\omega_\tau(y) = (1-y)^2(1+2y)$, $\omega_L(y) = 2y(1-y)^2$, and S_{EW} is a known short-distance electroweak correction.

Here we propose a generalized dispersion relation involving the experimentally determined us V+A inclusive distribution and weights, $\omega_N(s) \equiv 1/\prod_{k=1}^N \left(s + Q_k^2\right)$, $0 < Q_1^2 < \cdots < Q_N^2$, having poles at Euclidean Q^2. From Eq. (3), $|V_{us}^2|\tilde{\rho}_{us}(s)$, with

$$\tilde{\rho}_{us}(s) \equiv \left(1 + 2\frac{s}{m_\tau^2}\right)\rho^{(1)}_{us;V+A}(s) + \rho^{(0)}_{us;V+A}(s) \tag{4}$$

is directly related to $dR_{us;V+A}/ds$. For $N \geq 3$, one has the convergent dispersion relation

$$\int_0^\infty \tilde{\rho}_{us}(s)\omega_N(s)ds = \sum_{k=1}^{N} \text{Res}\left[\tilde{\Pi}_{us}(Q_k^2)\omega_N(Q_k^2)\right] \sum_{k=1}^{N} \frac{\tilde{\Pi}_{us;V+A}(Q_k^2)}{\prod_{j\neq k}\left(Q_j^2 - Q_k^2\right)} \equiv \tilde{F}_{\omega_N}\,, \tag{5}$$

with $\tilde{\Pi}(Q^2)$ the HVP combination

$$\tilde{\Pi}_{us} \equiv \left(1 - 2\frac{Q^2}{m_\tau^2}\right)\Pi^{(1)}_{us;V+A}(Q^2) + \Pi^{(0)}_{us;V+A}(Q^2). \tag{6}$$

The idea is to evaluate the RHS of Eq. (5) using $\tilde{\Pi}_{us}(Q_k^2)$ measured on the lattice. On the LHS, $s < m_\tau^2$ contributions are determined using experimental $dR_{us;V+A}/ds$ data, and $s > m_\tau^2$ contributions approximated using pQCD. Using the experimentally determined, weighted spectral integral $\tilde{R}_{w_N}$, defined by $\tilde{R}_{w_N} \equiv \frac{m_\tau^2}{12\pi^2 S_{EW}} \int_0^{m_\tau^2} \frac{dR_{us;V+A}(s)}{ds}\omega_N(s)ds$, one finds, solving for $|V_{us}|$,

$$|V_{us}| = \sqrt{\tilde{R}_{us;w_N} / \left(\tilde{F}_{\omega_N} - \int_{m_\tau^2}^{\infty} \tilde{\rho}_{us}^{\text{pQCD}}(s)\omega_N(s)ds\right)}. \tag{7}$$

We use the following spectral distributions as input: K pole contributions from $\tau \to K\nu_\tau$ or $K_{\mu 2}$ decay+SM expectations; unit-normalized $K^-\pi^0$ and $\bar{K}^0\pi^-$ distributions from BaBar [6], and Belle [7], unit-normalized $K^-\pi^+\pi-$ and $\bar{K}^0\pi^-\pi^0$ distributions from BaBar [8] and Belle [9], the small $\bar{K}\bar{K}K$ distributions and normalization from a combination of BaBar [6] and Belle [10] results, and the remaining exclusive strange modes not remeasured by the B-factory experiments from ALEPH [11]. Unit-normalized distributions are normalized using current exclusive-mode BFs [3], with the exception of the $K\pi$ modes, where the alternate, further dispersively constrained BF results of Ref. [13] are also considered.

We employ ω_N with uniform pole spacing, Δ, and charaterize ω_N by Δ, N, and the pole-interval midpoint, $C = (Q_1^2 + Q_N^2)/2$. With large enough N and all Q_k^2 below $\sim$ 1 GeV^2, spectral integral contributions from $s > m_\tau^2$, where pQCD is used, are strongly suppressed. In Fig. 1, we plot experimental distributions, with "residual" labelling the combined distribution of exclusive us modes not remeasured at the B-factories, taken from ALEPH [11]. pQCD contributions, evaluated using the 5-loop-truncated $D = 0$ OPE form [12], and, for definiteness, $|V_{us}| = 0.2253$, are also plotted for reference.

The right panel of Fig. 1 shows a sample result for the weighted distributions, as a function of s. The $s \leq m_\tau^2$ part of the weighted spectral integral on the LHS of Eq. 5 is evaluated by summing weighted exclusive mode integral contributions. As shown in the figure, the higher-s, larger-error part of the experimental distribution can be strongly suppressed by the appropriate choice of weight function. Increasing N lowers the error of the LHS of Eq. 5, but inceases the level of cancellation in the sum of the residues on the RHS, hence increasing the error on that sum. To ensure the poles span the same interval for all N, we fix $\Delta = 0.2/(N-1)$ [GeV^2]. The error of $|V_{us}|$ can be minimized by optimizing the choice of N and C.

3 Analysis with lattice HVPs

The position-space versions of the hadronic current-current two-point functions of the flavor us V and A currents are defined by

$$C^{\mu\mu}_{V/A}(t) = \sum_{\vec{x}} \langle J^\mu_{V/A}(\vec{x}, t)(J^\mu_{V/A}(0,0))^\dagger\rangle, \tag{8}$$

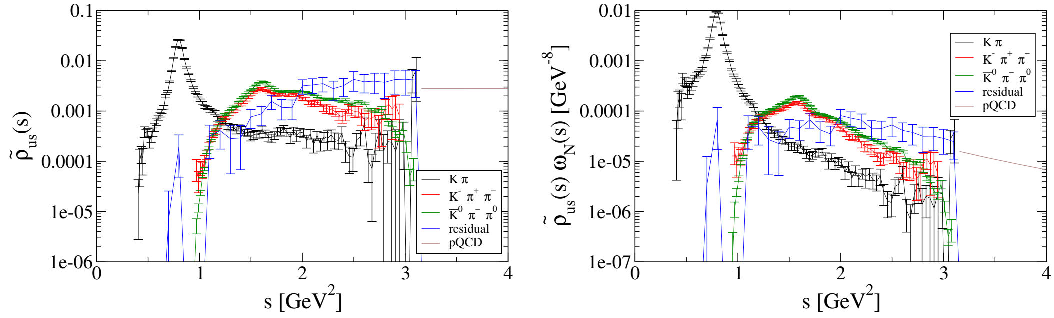

Figure 1. Experimental distribution of the $\tilde{\rho}_{us}(s)$ for each channel (left panel). Experimental weighted distributions, $\tilde{\rho}_{us}(s)\omega_N(s)$, for $N = 4$ and $C = 0.5$ GeV^2 (right panel).

with $J^{\mu}_{V/A}(\vec{x}, t)$ the flavor us V/A currents and $\mu = x, y, z, t$. Using the kernel functions of Refs. [14, 15], we compute the flavor us HVPs, $\Pi^{(J)}_{V/A}(Q^2)$, from the corresponding two-point functions. We work with the near-physical-quark-mass $n_f = 2 + 1$ Möbius domain wall fermion ensembles of the RBC and UKQCD collaborations [16], the first having size $48^3 \times 96$ and lattice cutoff $a^{-1} = 1.7295(38)$ GeV, the second size $64^3 \times 128$ and cutoff $a^{-1} = 2.3586(7)$ GeV. The all-mode-averaging (AMA) technique [18, 19] is employed to reduce cost. We correct for slight u, d, s mass mistunings by measuring the HVPs with partially quenched (PQ) physical valence quark masses [16]. AMA is also useful to reduce the number of PQ measurements required.

$\tilde{F}_{\omega_N}$ can be decomposed into four contributions, labelled by the spin $J = 0, 1$ and type, V or A, of the current involved. Due to the finite lattice temporal extent, finite time effects may exist. With the Taylor expansion of all terms on the RHS of Eq. (5) about, e.g., $Q^2 = C$, the sum of residues begins with a term proportional to $d^{N-1}\tilde{\Pi}_{us;V+A}(C)/\left(dQ^2\right)^{N-1}$. In the limit that all $Q_k^2 \to 0$, this involves the $(2N)^{th}$ time-moments of the $C^{(J)}_{V/A}$. Increasing N, with its increasing level of cancellation, thus enhances the relative weight of large-t contributions on the RHS of Eq. (5). To avoid modelling the large-t behavior, N (and C) are chosen to strongly suppress large-t contributions. The restrictions 0.1 GeV^2 $< C <$ 1 GeV^2 and $N \leq 5$ suffice for our purposes. The relative sizes of C-dependent lattice contributions from the four channels, $V^{(J)}$, $A^{(J)}$, are shown, for $N = 4$, in the left panel of Fig. 2. The right panel, similarly, shows the relative sizes of different contributions to the corresponding weighted us spectral integrals. $K\pi$ denotes the sum of $K^-\pi^0$ and $\bar{K}^0\pi^-$ contributions, *residual* the sum of contributions from $K^-\pi^0\pi^0$ and higher multiplicity us modes, and *pQCD* the contribution from $s > m_\tau^2$, evaluated using the 5-loop-truncated $D = 0$ OPE form. We have also verified that $D = 2$ OPE contributions are numerically negligible compared to $D = 0$ ones. Varying C (and N) varies the level of suppression of the pQCD and higher-multiplicity contributions, the relative size of K and $K\pi$ contributions, and hence the level of "inclusiveness" of the analysis. The systematic study of the stability of $|V_{us}|$ under such variations provides additional cross-checks.

3.1 Result

The largest contribution on the RHS of Eq. (5) is that of the $A^{(0)}$ channel. On the LHS, the K pole contribution dominates $\rho^{(0)}_{us;A}(s)$, with other, continuum contributions doubly chirally suppressed. Estimates of LHS continuum $A^{(0)}$ contributions using sum-rule results for the $K(1460)$ and $K(1830)$ decay

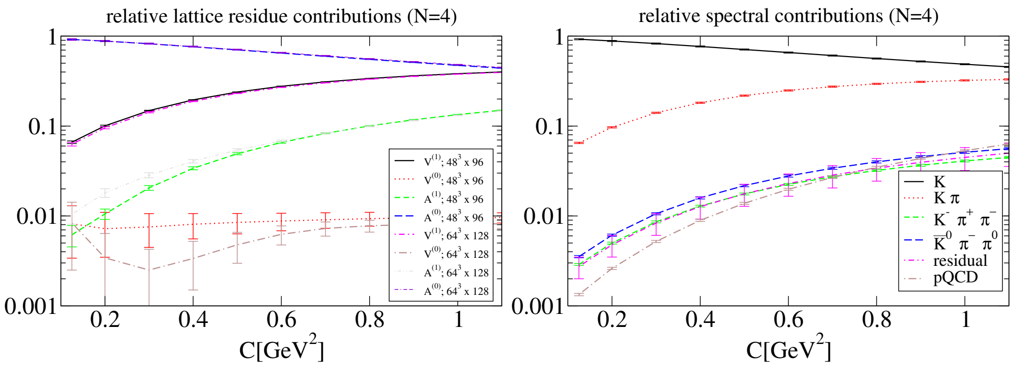

Figure 2. Relative contributions of $J = 0, 1$, V and A channels to the sum of lattice residues (left panel), and different (semi-) exclusive modes to the weighted us spectral integrals (right panel), as a function of C, for $N = 4$.

constants [20] indicate these should be numerically negligible for the ω_N of our analysis. This makes possible an "exclusive" $A^{(0)}$ analysis relating $\tilde{F}^{A^{(0)}}_{\omega_N}$ to the K-pole contribution, $\tilde{R}^K_{\omega_N} = \gamma_K \omega_N(m_K^2)$, $\gamma_K = 2|V_{us}|^2 f_K^2$. Since the simulations underlying $\tilde{F}^{A^{(0)}}_{\omega_N}$ are isospin symmetric, we need to correct γ_K for leading-order electromagnetic (EM) and strong isospin-breaking (IB) effects [13]. With PDG τ lifetime [21] and HFLAV $\tau \to K\nu_\tau$ BF [3] input, we obtain $\gamma_K[\tau_K] = 0.0012154(169)_{exp}(21)_{IB}$ GeV^2, where the second error arises from the uncertainty on the IB correction. $\gamma_K[\tau_K]$ serves as input for our main, fully τ-based analysis. The resulting exclusive $A^{(0)}$ determination is obtained via $|V_{us}^{A_0}| = \sqrt{\tilde{R}^K_{\omega_N}/\tilde{F}^{A^{(0)}}_{\omega_N}}$. Sample results for $N = 4$ are shown in Fig. 3. We find our exclusive results independent of C for $C < 1$ GeV^2 for both ensembles (confirming the absence of non-negligible continuum $A^{(0)}$ contributions), and in agreement with the results, $|V_{us}| = 0.2242(16)_{exp}(12)_{th}$ and $0.2260(3)_{exp}(12)_{th}$, obtained using $|V_{us}| = \sqrt{\gamma_K/(2f_K^2)}$, the isospin-symmetric lattice result $F_K \equiv \sqrt{2} f_K = 0.15551(83)$ [16] and $\gamma_K = \gamma_K[\tau_K]$ and $\gamma_K[K_{\mu 2}]$, respectively.

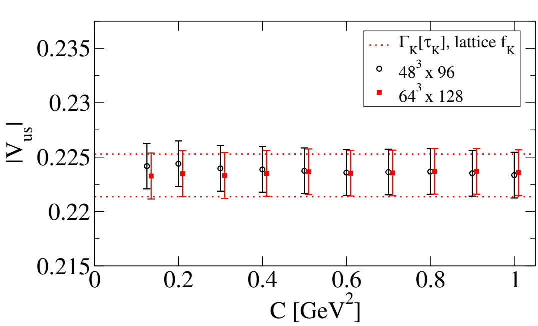

Figure 3. $|V_{us}^{A_0}|$ as a function of C for $N = 4$. The experimental input of $\gamma_K[\tau_K]$ is used. The error is statistical only. The determination obtained using $\gamma[\tau_K]$ and lattice f_K is shown for comparison.

We turn now to the fully inclusive analysis. As continuum $A^{(0)}$ contributions are negligible, statistical and systematic uncertainties can be reduced by replacing $\Pi_A^{(0)}$ with the K pole contribution, evaluated using lattice f_K. IB corrections, beyond those applied to γ_K, are relevant only for $K\pi$. We account for these using the results of Ref. [13]. A 2% uncertainty, estimated using results from a study of duality violations in the $SU(3)_F$-related flavor ud channels [17], is assigned to the pQCD contributions. Since our analysis is, by design, optimized for ω_N strongly suppressing higher-multiplicity and $s > m_\tau^2$ contributions, such an uncertainty plays a negligible role in our final error.

Several systematic uncertainties enter the lattice computation. Since we assume a continuum extrapolation linear in a^2 but have only two lattice spacings, a, $O(a^4)$ discretization uncertainties must be estimated. Based on a dimensional or ChPT analysis, uncertainties due to discretization and finite volume effects are estimated. The scale setting uncertainty is also taken into account. In Fig. 4, we show the relative errors on the total lattice residue sum, where K indicates the relative error from the K contributions and *others* the statistical error from $V^{(1)}, V^{(0)}, A^{(1)}$, and residual $A^{(0)}$ channels. For smaller C, the statistical error dominates, while for larger C, the discretization error becomes dominant.

Fig. 5 shows the result of our inclusive $|V_{us}|$ analysis. For $N = 4, 5$ we observe excellent stability with respect to C for all $C < 1$ GeV^2. For $N = 3$, $|V_{us}|$ shows signs of decreasing with increasing C for larger C. Note that this is the region where pQCD and residual mode contributions becomes larger, reaching, for example, 7% and 20% of the total spectral integral around $C \sim 1$ GeV^2. A downward shift with increasing C for $N = 3$ is exactly what one would expect to see if the older residual mode experimental results were missing higher-multiplicity strength at higher s. Since this region of the spectrum plays a larger relative role in the FB FESR determination, one would also expect to find somewhat lower $|V_{us}|$ from FB FESR analyses than from the current approach, as is indeed observed to be the case. Optimizing the values of C, we obtain the smallest error for each N. The overall best inclusive determinations,

$$|V_{us}| = \begin{cases} 0.2228(15)_{exp}(13)_{th}, & \text{for } \gamma_{\mathrm{K}}[\tau_{\mathrm{K}}] \\ 0.2245(10)_{exp}(13)_{th}, & \text{for } \gamma_{\mathrm{K}}[\mathrm{K}_{\mu 2}] \end{cases} \tag{9}$$

occur for $N = 4$ and $C = 0.7$ GeV^2, where the residual mode and pQCD contributions are highly suppressed.[1] The theoretical (lattice) error is comparable to the experimental one, and the total error less than that of the previous inclusive τ decay determinations. The result is consistent with those obtained from $K_{\ell 3}$, $\Gamma[K_{\mu 2}]/\Gamma[\pi_{\mu 2}]$, and 3-family CKM unitarity.

4 Conclusion and Discussion

We have presented a novel method to extract $|V_{us}|$ from strange inclusive τ decay distributions. There are a number of advantages over the FB FESR approach. First, our weight functions allow better suppression of contributions from the large-error, high-s region of the experimental distribution. Optimizing C and N allows dominance by K and $K\pi$ contributions to be realized, which ensures reduced experimental errors, with dominant errors coming from the lattice sub-dominant vector channel contributions. The result agrees well with $K_{\ell 3}$, $\Gamma[K_{\mu 2}]/\Gamma[\pi_{\mu 2}]$, and 3-family CKM unitarity expectations. A downward shift observed in larger C for $N = 3$ may indicate missing higher-multiplicity strength at higher s and/or uncontrolled OPE systematic errors, since this region of the spectrum plays a larger relative role in the FB FESR determination and somewhat lower $|V_{us}|$ from FB

[1]These results are obtained using the HFLAV $K\pi$ BF normalization; shifting to the alternate dispersively constrained BF normalization raises both values by 0.0013.

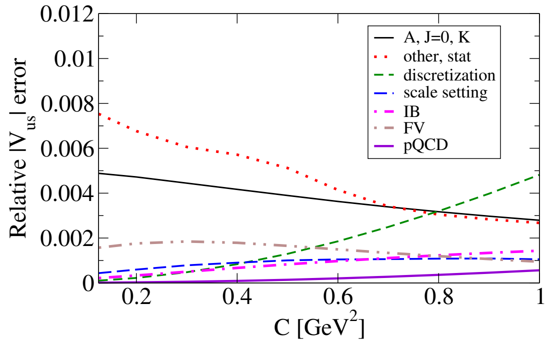

Figure 4. Lattice $|V_{us}|$ error contributions for $N = 4$.

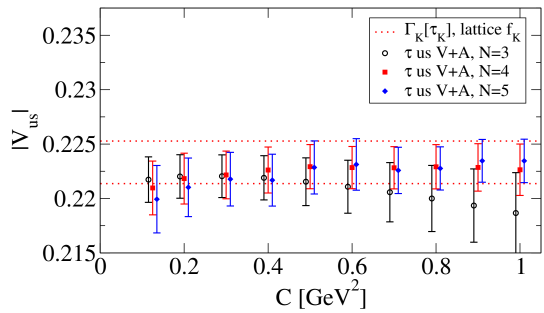

Figure 5. Result of $|V_{us}|$ as a function of C for $N = 3, 4$, and 5. The points for $N = 3$ and 5 have been shifted horizontally for clarity. Statistical and systematic errors are added in quadrature. For reference, the result $|V_{us}^{\Gamma[\tau_K]}|$ (determined only from K using f_K) is also shown.

FESR analyses is indeed observed. Key advantages of the lattice approach over the FESR approach to using the same data are better control of theory systematics thanks to non-perturbative lattice data in place of the OPE and better suppression of spectral contributions with large experimental errors. The experimental uncertainty can be further reduced through improvements to the experimental $K\pi$ branching fractions. The most dominant theoretical error is the lattice statistical error, which is improvable by straightforward lattice computational efforts.

Acknowledgments

We are grateful to members of RBC and UKQCD collaborations for fruitful discussion and support. We thank the Benasque centre for hospitality at the Benasque workshop "High Precision QCD at low energy", where this project started. This work is supported by the European Research Council under the European Unions Seventh Framework Programme (FP7/2007-2013) / ERC Grant agreement 279757, and by JSPS KAKENHI Grant Numbers 17H02906 and 17K14309. This work was supported by resources provided by the Scientific Data and Computing Center (SDCC) at Brookhaven National Laboratory (BNL), a DOE Office of Science User Facility supported by the Office of Science of the U.S. Department of Energy. The SDCC is a major component of the Computational Science Initiative (CSI) at BNL. RJH, RL and KM are supported by grants from the Natural Science and Engineering Research Council of Canada. C.L. acknowledges support through a DOE Office of Science Early

Career Award and by US DOE Contract DESC0012704(BNL). HO is supported in part by RIKEN Special Postdoctoral Researcher program and Nara Womens University Intramural Grant for Project Research. A.P. is supported in part by UK STFC grants ST/L000458/1 and ST/P000630/1.

References

[1] S. Aoki *et al.*, Eur. Phys. J. C **77**, no. 2, 112 (2017) doi:10.1140/epjc/s10052-016-4509-7 [arXiv:1607.00299 [hep-lat]].

[2] I. S. Towner and J. C. Hardy, Phys. Rev. C **91**, no. 1, 015501 (2015) doi:10.1103/PhysRevC.91.015501 [arXiv:1412.0727 [nucl-th]].

[3] The FLAV-Tau Spring 2017 report, www.slac.stanford.edu/xorg/hfag/tau/spring-2017

[4] R. J. Hudspith, R. Lewis, K. Maltman and J. M. Zanotti, arXiv:1702.01767 [hep-ph]

[5] K. Maltman, R. J. Hudspith, R. Lewis, C. E. Wolfe and J. Zanotti, PoS LATTICE **2015**, 260 (2016) [arXiv:1510.06954 [hep-ph]]; R. J. Hudspith, T. Izubuchi, R. Lewis, K. Maltman, H. Ohki and J. M. Zanotti, Mod. Phys. Lett. **A31**, 1630030 (2016) doi:10.1142/S0217732316300305

[6] B. Aubert *et al.* [BaBar Collaboration], doi:10.1103/PhysRevD.76.051104 [arXiv:0707.2922 [hep-ex]].

[7] D. Epifanov *et al.* [Belle Collaboration], Phys. Lett. B **654**, 65 (2007) doi:10.1016/j.physletb.2007.08.045 [arXiv:0706.2231 [hep-ex]].

[8] I. M. Nugent [BaBar Collaboration], Nucl. Phys. Proc. Suppl. **253-255**, 38 (2014) doi:10.1016/j.nuclphysbps.2014.09.010 [arXiv:1301.7105 [hep-ex]].

[9] S. Ryu [Belle Collaboration], Nucl. Phys. Proc. Suppl. **253-255**, 33 (2014) doi:10.1016/j.nuclphysbps.2014.09.009 [arXiv:1302.4565 [hep-ex]].

[10] M. J. Lee *et al.* [Belle Collaboration], Phys. Rev. D **81**, 113007 (2010) doi:10.1103/PhysRevD.81.113007 [arXiv:1001.0083 [hep-ex]].

[11] R. Barate *et al.* [ALEPH Collaboration], Eur. Phys. J. C **11**, 599 (1999) doi:10.1007/s100520050659 [hep-ex/9903015].

[12] P. A. Baikov, K. G. Chetyrkin and J. H. Kuhn, Phys. Rev. Lett. **101**, 012002 (2008) doi:10.1103/PhysRevLett.101.012002 [arXiv:0801.1821 [hep-ph]].

[13] M. Antonelli, V. Cirigliano, A. Lusiani and E. Passemar, JHEP **1310**, 070 (2013) doi:10.1007/JHEP10(2013)070 [arXiv:1304.8134 [hep-ph]].

[14] D. Bernecker and H. B. Meyer, Eur. Phys. J. A **47**, 148 (2011) doi:10.1140/epja/i2011-11148-6 [arXiv:1107.4388 [hep-lat]].

[15] X. Feng, S. Hashimoto, G. Hotzel, K. Jansen, M. Petschlies and D. B. Renner, Phys. Rev. D **88**, 034505 (2013) doi:10.1103/PhysRevD.88.034505 [arXiv:1305.5878 [hep-lat]].

[16] T. Blum *et al.* [RBC and UKQCD Collaborations], Phys. Rev. D **93**, no. 7, 074505 (2016) doi:10.1103/PhysRevD.93.074505 [arXiv:1411.7017 [hep-lat]].

[17] D. Boito, M. Golterman, K. Maltman and S. Peris, Phys. Rev. D**91**, 034003 (2015)

[18] T. Blum, T. Izubuchi and E. Shintani, Phys. Rev. D **88**, no. 9, 094503 (2013) doi:10.1103/PhysRevD.88.094503 [arXiv:1208.4349 [hep-lat]].

[19] E. Shintani, R. Arthur, T. Blum, T. Izubuchi, C. Jung and C. Lehner, Phys. Rev. D **91**, no. 11, 114511 (2015) doi:10.1103/PhysRevD.91.114511 [arXiv:1402.0244 [hep-lat]].

[20] K. Maltman and J. Kambor, Phys. Rev. D **65**, 074013 (2002)

[21] C. Patrignani *et al.* [Particle Data Group Collaboration], Chin. Phys. C **40**, no. 10, 100001 (2016). doi:10.1088/1674-1137/40/10/100001