Interacting Multiple Model Estimator for Networked Control Systems: Stability, Convergence, and Performance

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***Abstract*—In this paper, we study the interacting multiple model (IMM) estimator for networked control systems with packet loss but without packet acknowledgment (ACK). The ACK is a signal sent by the actuator to inform the estimator of whether control packets are lost or not. A system with ACK is usually called a TCP-like system; otherwise, it is called a UDP-like system. We show that the stability of the IMM estimator for UDP-like systems is determined by the observation packet arrival rate (p.a.r.) and is independent of the control p.a.r. and control inputs. The IMM estimator is stable if the observation p.a.r. is greater than a critical value. We show that this critical value is the same as the critical value for the stability of the optimal estimator for its corresponding TCP-like system. If control inputs eventually tend to zero, the error covariance of the IMM estimator converges to that of the optimal estimator for TCP-like systems. We characterize the impact of the control/observation p.a.r. and the control input on estimation performance. Finally, we prove that the average estimation performance of the IMM estimator approximates that of the optimal estimator within a finite bound, and is superior to that of the linear minimum mean square error estimator.**

***Index Terms*—networked control systems; interacting multiple model estimator; stability; packet losses; packet acknowledgment**

I. Introduction

A. Background and motivation

In recent years, increasing attention has been paid to networked control systems (NCSs), in which information among sensors, controllers, and actuators is exchanged via networks [1]. The introduction of networks brings numerous advantages—including lower installation and maintenance costs, increased system flexibility, and a significant reduction in wiring—but it also causes some network-induced constraints, such as transmission delays and packet losses. For NCSs that suffer from packet losses, two fundamental communication protocols are commonly used: the transmission control protocol (TCP) and the user datagram protocol (UDP). In TCP, the lost data will be retransmitted until the sending node receives packet acknowledgment (ACK) from the receiving node. Such an ACK scheme guarantees successful data transmission but leads to several drawbacks, such as network congestion and communication delay. For unreliable networks, these drawbacks in turn prevent the ACK from being transmitted in time (without delay and loss) to implement the TCP scheme [2–4]. In UDP, no acknowledgment scheme is used and thus no retransmission for the lost data is required. The UDP scheme, at the price of less reliable delivery, avoids energy consumption for retransmitting the lost data, simplifies the protocol implementation, and allows for more timely communication, making it a preferable choice for real-time NCSs [5]. A system with the ACK scheme is usually called a TCP-like system, and the one without ACK is called a UDP-like system (see Fig. 1). This paper studies state estimation problems for UDP-like systems.

For TCP-like systems, the optimal estimator has been well known to be the time-varying Kalman filter, and its stability condition was established in the pioneering work [6]. Thereafter, the optimal estimation issues for TCP-like systems have been extensively investigated, and fruitful results have been obtained. These results mainly focus on three key aspects: stability [7–10], convergence [11, 12], and performance [13, 14]. For UDP-like systems under the special conditions "the observation matrix C is square and invertible, and there is no observation noise," the optimal estimator has been obtained in [15] and is presented in Eqs. (8) and (9). However, for general UDP-like systems, the optimal estimator is obtained only for *the system without observations lost*, and its computational load grows exponentially [16]. Hence, it cannot be used in practice. To address the computational complexity, various linear [2, 15, 17, 18] and nonlinear [19, 20] sub-optimal estimators have been proposed to approximate the optimal estimator, but they are not good approximations in terms of the estimation criterion, since the criterion for these linear estimators (see Definition 5) differs from the criterion for the optimal estimator (see Definition 1). From [19, 20], it is clear that the nonlinear estimators are not obtained according to either one of these two criteria above. As one of the most cost-efficient estimation schemes, the interacting multiple model (IMM) estimator is proven to be a good approximation for the

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optimal estimator, since it is able to obtain the estimate fairly close to the optimal one [21–23], which motivates us to apply the IMM estimator to UDP-like systems, and then explore its properties, especially the three aforementioned key aspects: stability, convergence, and performance. Moreover, we are also interested in whether the IMM estimator, performing like the optimal one, is superior to other sub-optimal estimators, such as the frequently used LMMSE estimator.

B. Literature review and contributions

The IMM estimator, a hybrid state estimation algorithm for Markov jump systems, was first proposed in [21]. Owing to its excellent estimation performance and low computational cost, it has gained broad applications in various fields [23]. However, due to the complicated structure and the nonlinear nature of the IMM estimator, it is challenging to theoretically determine its stability and performance, which are usually evaluated by the Monte Carlo method, an experimental simulation method. As reported in the recent work [24], a sharp contrast to its popularity and broad applications is that not only no condition is there to ensure the stability of the IMM estimator except [24], but also few theoretical studies are there to evaluate its performance except [22, 25]. Moreover, the convergence of the IMM estimator and the theoretical comparison of its performance with the LMMSE estimator are rarely studied. We also note that numerous variants of the IMM estimator, that is, the improved or modified IMM estimators, have been developed for different purposes—such as to accelerate computational speed [26, 27], to improve the estimation performance [28, 29], to track different types of targets [30, 31]—while, to the best of the author's knowledge, few variants are developed to achieve a performance that is theoretically superior to other sub-optimal estimators, or to facilitate the study of stability and performance.

To our best knowledge, few results are reported on the application of the IMM estimator to UDP-like systems. To characterize its stability and performance in a theoretical view still remains difficult for UDP-like systems, since

- *The aforementioned three available methods fail to apply to UDP-like systems.* 1) The stability conception for the IMM estimator and the Lyapunov approach proposed in [24] are not fit for the IMM estimator for UDP-like systems. 2) The hybrid approach proposed in [22] requires that the observation matrices for each mode are the same, which clearly does not hold for UDP-like systems. 3) The structure of the IMM estimator is too complex, so that the authors of [25] had to introduce several approximations in derivations to develop an off-line algorithm to evaluate the estimation performance. This method is efficient, but it is not a mathematically rigorous one we desire.
- *No heuristic results and methods are available from the study of the optimal estimator.* Except for the *no observation loss* case [16], few results have been reported on the optimal estimator for the general UDP-like system, that is, the UDP-like system with both control inputs and observations lost. It is clear that, unlike the case in [16], the optimal estimator will no longer remain stable for the general UDP-like system as long as the observation packet arrival rate (p.a.r.) diminishes to some extent, but the stability condition is unclear.
- Due to the complicated and nonlinear structure of the IMM estimator, how the parameters of interest—e.g. the control/observation packet loss rate and the control inputs—affect the IMM estimator's stability/convergence/performance is unknown, and the approaches developed in [22, 24, 25] cannot help to reveal the relationship among them either.

It may be concluded from the discussion above that even the IMM estimator and UDP-like systems have gained broad applications, the problems on determining the stability, convergence, and performance of the IMM estimator for UDP-like systems are still challenging and remain unsolved, which motivates our work in this paper. Our main results and contributions are summarized as follows:

1) (Stability) We show that the stability of the IMM estimator is only determined by the observation p.a.r. and is independent of the control p.a.r. and control inputs. Specifically, the IMM estimator is stable only if the observation p.a.r. is greater than a critical value. This critical value is the same as its counterpart for the optimal estimator for its corresponding TCP-like system.
2) (Convergence) The error covariance (EC) of the IMM estimator for the UDP-like system converges to the EC of the optimal estimator for the corresponding TCP-like system, if control inputs eventually tend to zero.
3) (Performance) The results on estimation performance include five points: (i) The smaller the *observation p.a.r.* is, the worse the estimation performance becomes. (ii) The graph of the relationship between the *control p.a.r.* and the conditional estimation performance is similar to a parabolic curve; (iii) As the magnitude of *control inputs* becomes larger, the expected EC remains bounded if there is no observation lost, and becomes unbounded, otherwise. (iv) The average estimation performance of the IMM estimator approximates that of the optimal estimator within a finite bound and is superior to that of the LMMSE estimator.
4) Whether all the results above obtained for the IMM estimator still hold for the optimal estimator is so far unknown. As a minor contribution, we conduct some numerical simulations and the results suggest that all the results above also hold for the optimal estimator, which may shed light on the study of the optimal estimator.

The rest of the paper is organized as follows: In Section II, the system setup and problems are formulated. In Section III, the standard and the modified IMM estimators for the UDP-like systems are proposed. The stability, convergence, and performance of the IMM estimator are studied in Sections IV, V, and VI, respectively. In Section VII, numerical examples are given to illustrate main results. Conclusions are presented in Section VIII. All the proofs of lemmas are given in Appendix.

Notations:

- $\mathcal{N}_x(\mu, P)$ denotes the Gaussian probability density function (pdf) of the random variable x with mean μ and covariance P. The subscript x sometimes is omitted for brevity when it is clear from the context.

- $x \sim \mathcal{N}_x(\mu, P)$ means that the pdf of the random variable x is $\mathcal{N}_x(\mu, P)$.
- $\mathbb{P}(\cdot)$ denotes a probability measure.
- $p(\cdot)$ and $p(\cdot|\cdot)$ denote the pdf and the conditional pdf, respectively.
- $\mathbb{E}[\cdot]$ denotes probability expectation.
- $(\cdot)'$ denotes the transpose of a matrix or vector.
- $[\cdot]_M^2$ denotes $(\cdot)'M(\cdot)$ for some Hermitian matrix M.
- $(\cdot)_I^2$ with the identity matrix I means $(\cdot)(\cdot)'$.
- $\mathrm{tr}(M)$ denotes the trace of a matrix M.
- $M^\dagger$ denotes the Moore-Penrose generalized inverse of a matrix M.
- $\sigma(M)$ and $\underline{\sigma}(M)$ denote the maximum and minimum singular values of a matrix M, respectively.
- For a vector z, $\|z\| \triangleq z'z$; For a matrix M, $\|M\| \triangleq \sigma(M)$.
- $X \succ$ (or $\succeq$) Y means $X - Y$ is a real symmetric positive definite (or semidefinite) matrix.
- $\mathsf{sat}(u, \overline{U})$ with $u = (u_1, \ldots, u_q)' \in \mathbb{R}^q$ and $\overline{U} > 0 \in \mathbb{R}$ denotes a saturation function: $\mathsf{sat}(u, \overline{U}) = (\bar{u}_1, \ldots, \bar{u}_q)$ where, for $1 \leq j \leq q$, $\bar{u}_j = u_j$ if $|u_j| \leq \overline{U}$ and $\bar{u}_j = \mathsf{sign}(u_j)\overline{U}$ otherwise. $\mathsf{sign}(x) = 1$ if $x > 0$, $\mathsf{sign}(x) = -1$ if $x < 0$, and $\mathsf{sign}(x) = 0$ if $x = 0$.

II. System setup and problem formulation

A. System setup

Consider the following system:

$$x_k = Ax_{k-1} + \theta_k B u_k + \omega_k \tag{1}$$

$$y_k = \begin{cases} Cx_k + \upsilon_k, & \text{for } \gamma_k = 1 \\ \text{no observation}, & \text{for } \gamma_k = 0 \end{cases} \tag{2}$$

where $x_k \in \mathbb{R}^n$ is the system state, $u_k \in \mathbb{R}^q$ is the control input, and $y_k \in \mathbb{R}^p$ is the observation. ω_k and υ_k are zero mean Gaussian noises with covariance $Q \succ 0$ and $R \succ 0$, respectively. θ_k and γ_k are i.i.d. Bernoulli random sequences with $\mathbb{P}(\theta_k = 1) = \theta$ and $\mathbb{P}(\gamma_k = 1) = \gamma$, and model the control and observation packet losses, respectively. That is, $\gamma_k = 1$ means that the observation y_k has been successfully received by the estimator; $\gamma_k = 0$ means that the observation is lost and thus the estimator receives nothing. $\theta_k = 1$ means that the control input u_k has been successfully delivered to the actuator, otherwise $\theta_k = 0$. The control inputs are assumed to be bounded, that is, $\|u_k\| < +\infty$ for all k. θ and γ are also known as the packet arrival rates of control and observation, respectively.

Figure 1 describes the state estimation for open-loop and closed-loop UDP-like systems. As shown later, the estimation-related results obtained in this paper are applicable to both open-loop and closed-loop UDP-like systems, since it is the value of u_k, not the way it comes from, that is required in the IMM estimator.

The estimator receives the observation from the sensor, and it thus knows the status of the observation packet losses, that is, the value of γ_k, while the value of θ_k is unknown due to the lack of ACK. Accordingly, in view of Markov jump systems, there are two unknown (or unobservable) jump modes, that is,

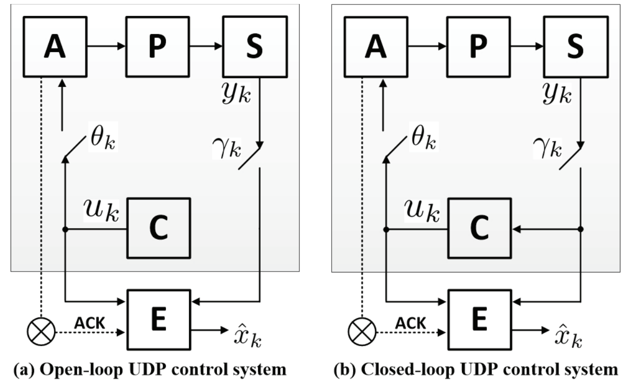

Fig. 1. State estimation for open/closed-loop UDP-like systems. The blocks P, S, E, C, and A denote the plant, sensor, estimator, controller and actuator, respectively. The symbol $\otimes$ is used to emphasize that there is no acknowledgment signal from the actuator to the estimator.

Mode 0: $\theta_k B = 0$ and Mode 1: $\theta_k B = B$, which correspond to the events $\theta_k^{[0]} \triangleq \{\theta_k = 0\}$ and $\theta_k^{[1]} \triangleq \{\theta_k = 1\}$, respectively.

Assumption 1. *The pair $(A, Q^{1/2})$ is controllable, and the pair (A, C) is observable. $x_0 \sim \mathcal{N}_{x_0}(\bar{x}_0, P_0)$, ω_k, υ_k, θ_k, and γ_k are mutually independent.*

B. Problem formulation

This section further explains why the IMM estimator is required and what problems we are devoted to solve.

Definition 1. *An estimate of x_k, denoted by $\hat{x}_k$, is said to be optimal in the minimum mean square error (MMSE) sense, if $\hat{x}_k$ minimizes $\mathbb{E}[||x_k - \hat{x}_k||^2|\mathcal{I}_k]$, where $\mathcal{I}_k \triangleq \{\gamma^k, Y^k\}$, $\gamma^k \triangleq \{\gamma_1, \ldots, \gamma_k\}$, and $Y^k \triangleq \{y_1, \ldots, y_k\}$.*

It is well known [32] that the desired optimal estimate $\hat{x}_k = \mathbb{E}[x_k|\mathcal{I}_k]$. To obtain $\mathbb{E}[x_k|\mathcal{I}_k]$, $p(x_k|\mathcal{I}_k)$ is required. However, for UDP-like systems, $p(x_k|\mathcal{I}_k)$ is a Gaussian mixture with an exponentially increasing number of terms, making its computation time-consuming and intractable [16]. The IMM estimator [21] is the right technique to deal with such kinds of pdf. By the total probability law, $p(x_k|\mathcal{I}_k) = \sum_{i=0}^{1} p(x_k|\theta_k^{[i]}, \mathcal{I}_k)p(\theta_k^{[i]}|\mathcal{I}_k)$. Define $\hat{x}_k^{[i]} \triangleq \mathbb{E}[x_k|\theta_k^{[i]}, \mathcal{I}_k]$ and $\lambda_k^{[i]} \triangleq \mathbb{P}(\theta_k^{[i]}|\mathcal{I}_k)$. The IMM estimator is able to approximately calculate $x_k^{[i]}$ and $\lambda_k^{[i]}$, and then give the desired approximate optimal estimate $\hat{x}_k = \mathbb{E}[x_k|\mathcal{I}_k] = \sum_{i=0}^{1} \lambda_k^{[i]} \hat{x}_k^{[i]}$. The IMM estimator for UDP-like systems is presented in Section III.

Definition 2. *The estimation error covariance P_k is said to be stable in the mean sense (or stable for short), if* $\sup \mathbb{E}[P_k] < +\infty$.

The IMM estimator is said to be stable/convergent, if P_k is stable/convergent, where the convergence of P_k follows the conventional definition of the convergence of matrices in a norm space. The estimation performance of the IMM estimator is evaluated by $\mathbb{E}[P_k]$ [22, 25].

The main tasks are to solve the following problems:

- **Problem 1:** How do the control/observation p.a.r. and the control input affect the stability, convergence, and estimation performance of the IMM estimator?

- **Problem 2:** As a good approximation for the optimal estimator, can the IMM estimator be proved to be near the optimal one and superior to the LMMSE estimator?

The table below illustrates where these problems are studied and where the answers are given.

Problems	Sections	Main results
1	IV/V/VI-A/VI-B	Theorems 1/2/3/4
2	VI-C/D	Theorems 5/6

III. IMM ESTIMATORS FOR UDP-LIKE SYSTEMS

By following the IMM algorithms proposed in [21, 22, 24], the standard IMM estimator for UDP-like systems is presented in Algorithm 1. Meanwhile, we propose a modified estimator to compare the estimation performance between the standard IMM and the LMMSE estimators. The modification, only involving the parameter α in (6a), is so minor that the standard and the modified IMM estimators are presented in Algorithm 1 together. By letting $\alpha = 0$ ($\alpha = 1$), Algorithm 1 yields the standard (modified) IMM estimator for UDP-like systems.

The proof of Algorithm 1 is given in Appendix.

Condition 1: C is square and invertible, and there is no observation noise, that is, $R = 0$.

For UDP-like systems under the special condition, Condition 1, when the observation y_k is successfully received, the state can be perfectly observed by $x_k = C^{-1}y_k$ and no estimation is required. Consequently, the optimal estimator can be presented in a simple form, which has been obtained in [15], as follows:

$$\text{When } \gamma_k = 1,\ \hat{x}_k = C^{-1}y_k \text{ and } P_k = 0; \tag{8}$$

$$\text{when } \gamma_k = 0,\ \begin{cases} \hat{x}_k = A\hat{x}_{k-1} + \theta B u_k \\ P_k = AP_{k-1}A' + Q + \bar{\theta}\theta B u_k u_k' B'. \end{cases} \tag{9}$$

Corollary 1. *For UDP-like systems under Condition 1, $\hat{x}_k$ and P_k calculated by the standard (or the modified) IMM estimator in Algorithm 1 reduce to (8) and (9). That is, under Condition 1, the IMM estimator takes the same form as the optimal estimator.*

Proof: Proof of (8). When $\gamma_k = 1$ and Condition 1 is satisfied, from (6b), we have $K_k = \bar{P}_k^{[\Delta]}C'(C\bar{P}_k^{[\Delta]}C' + R)^{-1} = C^{-1}$. By (6c), $\hat{x}_k^{[i]} = \bar{x}_k^{[i]} + K_k z_k^{[i]} = \bar{x}_k^{[i]} + K_k(y_k - C\bar{x}_k^{[i]}) = C^{-1}y_k$. By (7a), $\hat{x}_k = \sum_{i=0}^{1} \hat{x}_k^{[i]}\lambda_k^{[i]} = C^{-1}y_k \sum_{i=0}^{1} \lambda_k^{[i]} = C^{-1}y_k$ where $\sum_{i=0}^{1} \lambda_k^{[i]} = 1$ can be easily obtained by the definitions of $\lambda_k^{[i]}, c$, and $\theta^{[i]}$ in Algorithm 1.

By (6d) and noting that $K_k = C^{-1}$ and $R = 0$, we have $P_k^{[i]} = (I - K_kC)\bar{P}_k^{[*]}(I - K_kC)' + K_kRK_k' = 0$. Note that $\hat{x}_k^{[i]} = \hat{x}_k = C^{-1}y_k$ which has been obtained above and that $P_k^{[*]} \triangleq P_k^{[i]}$. From (7b), $P_k = P_k^{[*]} + \sum_{i=0}^{1} \lambda_k^{[i]}(\hat{x}_k - \hat{x}_k^{[i]})_I^2 = 0 + \sum_{i=0}^{1} \lambda_k^{[i]}(C^{-1}y_k - C^{-1}y_k)_I^2 = 0$. The proof of (8) is completed.

Proof of (9). When $\gamma_1 = 0$, from the mode probability updating and the measurement-update estimation steps, we have $\lambda_k^{[i]} = \theta^{[i]}$, $\hat{x}_k^{[i]} = \bar{x}_k^{[i]}$, and $P_k^{[i]} = \bar{P}_k^{[*]}$. By (3), (4), and (7a), $\hat{x}_k = \sum_{i=0}^{1} \hat{x}_k^{[i]}\lambda_k^{[i]} = \theta^{[0]}\bar{x}_k^{[0]} + \theta^{[1]}\bar{x}_k^{[1]} = \bar{\theta}A\hat{x}_{k-1} + \theta(A\hat{x}_{k-1} + Bu_k) = A\hat{x}_{k-1} + \theta Bu_k$.

Algorithm 1 The standard IMM estimator ($\alpha = 0$) and the modified IMM estimator ($\alpha = 1$) for UDP-like systems.

In the following, the superscript $i \in \{0, 1\}$ indicates that to which mode (Mode 0 or 1) the parameters correspond.

1) Mixing The mixed initial conditions $\hat{x}_{k-1}^{[0|i]}$ and $P_{k-1}^{[0|i]}$ for the mode-matched Kalman filter i are

$$\hat{x}_{k-1}^{[0|i]} = \hat{x}_{k-1},\ P_{k-1}^{[0|i]} = P_{k-1}. \tag{3}$$

2) Kalman filtering and mode probability updating

- *Time-update prediction:*

$$\bar{x}_k^{[i]} = A\hat{x}_{k-1} + iBu_k, \bar{P}_k^{[i]} = AP_{k-1}A' + Q \triangleq \bar{P}_k^{[*]}. \tag{4}$$

- *Mode probability updating:* Let $\theta^{[0]} \triangleq \bar{\theta}$ and $\theta^{[1]} \triangleq \theta$. The prior mode probability $\bar{\lambda}_k^{[i]} = \theta^{[i]}$.
If $\gamma_k = 0$, the posterior mode probability $\lambda_k^{[i]} = \theta^{[i]}$.
If $\gamma_k = 1$,

$$\text{Residual: } z_k^{[i]} \triangleq y_k - C\bar{x}_k^{[i]}$$
$$\text{Residual covariance: } P_k^Y = C\bar{P}_k^{[*]}C' + R$$
$$\text{Likelihood function: } \phi_k^{[i]} \triangleq \mathcal{N}_{y_k}(C\bar{x}_k^{[i]}, P_k^Y)$$
$$p(y_k|\mathcal{I}_{k-1}) = \bar{\theta}\phi_k^{[0]} + \theta\phi_k^{[1]} \triangleq c \tag{5}$$
$$\text{Posterior mode probability: } \lambda_k^{[i]} = \theta^{[i]}\phi_k^{[i]}/c.$$

- *Measurement-update estimation:*
If $\gamma_k = 0$, then $\hat{x}_k^{[i]} = \bar{x}_k^{[i]}$ and $P_k^{[i]} = \bar{P}_k^{[*]}$.
If $\gamma_k = 1$, then

$$\bar{P}_k^{[\Delta]} = \bar{P}_k^{[*]} + \alpha\lambda_k^{[0]}\lambda_k^{[1]}Bu_ku_k'B' \tag{6a}$$
$$K_k = \bar{P}_k^{[\Delta]}C'(C\bar{P}_k^{[\Delta]}C' + R)^{-1} \tag{6b}$$
$$\hat{x}_k^{[i]} = \bar{x}_k^{[i]} + K_kz_k^{[i]} \tag{6c}$$
$$P_k^{[i]} = \Phi(\bar{P}_k^{[*]}, K_k). \tag{6d}$$

where $\Phi(P, K) \triangleq (I - KC)P(I - KC)' + KRK'$.

3) Combining Let $P_k^{[*]} \triangleq P_k^{[i]}$.

$$\hat{x}_k = \sum_{i=0}^{1} \hat{x}_k^{[i]}\lambda_k^{[i]} \tag{7a}$$

$$P_k = P_k^{[*]} + \sum_{i=0}^{1} \lambda_k^{[i]}(\hat{x}_k - \hat{x}_k^{[i]})_I^2 \tag{7b}$$

Note that $P_k^{[*]} \triangleq P_k^{[i]} = \bar{P}_k^{[*]}$. From (7b) and (4), $P_k = P_k^{[*]} + \sum_{i=0}^{1} \lambda_k^{[i]}(\hat{x}_k - \hat{x}_k^{[i]})_I^2 = AP_{k-1}A' + Q + \bar{\theta}(A\hat{x}_{k-1} + \theta Bu_k - A\hat{x}_{k-1})_I^2 + \theta(A\hat{x}_{k-1} + \theta Bu_k - A\hat{x}_{k-1} - Bu_k)_I^2 = AP_{k-1}A' + Q + \bar{\theta}(\theta Bu_k)_I^2 + \theta(\bar{\theta}Bu_k)_I^2 = AP_{k-1}A' + Q + \bar{\theta}\theta Bu_ku_k'B'$. The proof of (9) is completed. ■

IV. STABILITY OF THE IMM ESTIMATOR

In this section, we study the stability of the IMM estimator.

Definition 3. *Let $\{\gamma_k\}$ be a sequence of i.i.d. Bernoulli random variables with $\mathbb{P}(\gamma_k = 1) = \gamma$, and let $\{T_k\}$ be a sequence of random matrices with an initial value $T_0 \succeq 0$.*

Whenever we say **the stability of $\mathbb{E}[T_k]$ is determined by γ with respect to a critical value** γ_c*, it means that there exists a real value* $\gamma_c \in [0,1)$ *such that*

$$\sup \mathbb{E}[T_k] < +\infty \text{ for } \forall T_0 \succeq 0, \text{ when } \gamma > \gamma_c;$$
$$\sup \mathbb{E}[T_k] = +\infty \text{ for some } T_0 \succeq 0, \text{ when } \gamma < \gamma_c.$$

This γ_c *is called* **the critical value for the stability of $\mathbb{E}[T_k]$**.

Theorem 1 (Stability)**.** *For the UDP-like system in (1) with bounded control inputs,*

(i) the stability of $\mathbb{E}[P_k]$ is determined by the observation p.a.r. γ with respect to a critical value γ_c, and is independent of the control p.a.r. θ and control inputs $\{u_k\}$;
(ii) the critical value γ_c for the stability of the IMM estimator for the UDP-like system is the same as that for the stability of the optimal estimator for its corresponding TCP-like system.

To prove Theorem 1, we give some preliminaries and lemmas.

Let X and Y be two random variables. Then

$$\mathrm{cov}(X) = \mathbb{E}\big[\mathrm{cov}(X|Y)\big] + \mathrm{cov}\big(\mathbb{E}[X|Y]\big), \tag{10}$$

which is an existing result established in [33].

Let $\bar{\gamma}_k \triangleq 1-\gamma_k$, $U_k \triangleq Bu_k u_k' B'$, $\mathbb{K}_k \triangleq I - K_k C$, and $\Delta_k \triangleq \gamma_k \lambda_k^{[0]}\lambda_k^{[1]}\mathbb{K}_k U_k \mathbb{K}_k' + \bar{\gamma}_k \theta\bar{\theta} U_k$. For bounded control inputs $\{u_k\}$, U_k is bounded as well. Denote the bound of U_k by U (that is, $U_k \preceq U$), and let $\Delta = \theta\bar{\theta}U$. Define

$$g(P,\gamma) \triangleq APA' - \gamma APC'(CPC'+R)^{-1}CPA' + Q \tag{11}$$

Lemma 1. *Let $P \succ 0$ and $K = PC'(CPC'+R)^{-1}$. Then (i) $\sigma(KC) < 1$; (ii) $0 < \sigma(I-KC) < 1$; (iii) $\underline{\sigma}(I-KC) > 0$.*

Lemma 2. *The following parts (i), (ii), and (iii) hold for both the standard and the modified IMM estimators.*

(i) For $\gamma_k = 0$, $P_k^{[*]} = \bar{P}_k^{[*]}$,

$$\hat{x}_k = A\hat{x}_{k-1} + \theta B u_k \tag{12a}$$
$$P_k = P_k^{[*]} + \theta\bar{\theta} B u_k u_k' B'. \tag{12b}$$

(ii) For $\gamma_k = 1$, then

$$\hat{x}_k = \mathbb{K}_k(A\hat{x}_{k-1} + \lambda_k^{[1]} B u_k) + K_k y_k \tag{13a}$$
$$P_k = P_k^{[*]} + \lambda_k^{[0]}\lambda_k^{[1]} \mathbb{K}_k B u_k u_k' B' \mathbb{K}_k'. \tag{13b}$$

(iii) $\theta\bar{\theta} \succeq \mathbb{E}[\lambda_k^{[0]}\lambda_k^{[1]}]$ and $\Delta \succeq \mathbb{E}[\Delta_k]$.
(iv) The following equations hold for the standard IMM estimator.

$$P_k = \bar{P}_k^{[*]} - \gamma_k \bar{P}_k^{[*]} C'(C\bar{P}_k^{[*]}C' + R)^{-1} C\bar{P}_k^{[*]} + \Delta_k \tag{14}$$

$$\bar{P}_{k+1}^{[*]} = g(\bar{P}_k^{[*]}, \gamma_k) + A\Delta_k A'. \tag{15}$$

Define three sequences of matrices as follows:

$$\underline{M}_{k+1} = g(\underline{M}_k, \gamma_k) \tag{16}$$
$$\overline{M}_{k+1} = g(\overline{M}_k, \gamma_k) + A\Delta A' \tag{17}$$
$$P_k^{\mathsf{tcp}} = \underline{M}_k - \gamma_k \underline{M}_k C'(C\underline{M}_k C' + R)^{-1} C\underline{M}_k \tag{18}$$

with $\overline{M}_1 = \underline{M}_1 = \bar{P}_1^{[*]} = AP_0A' + Q$. From [6], it is clear that $\underline{M}_k$ and P_k^{tcp} are the prediction and the estimation error covariances of the optimal estimator for the TCP-like system corresponding to the UDP-like system in (1), respectively.

Lemma 3. *Some existing results are given as follows:*

(i) [6, Lemma 1-c)] If $X \preceq Y$, then $g(X,\gamma) \preceq g(Y,\gamma)$.
(ii) [6, Lemma 1-a)] $\Phi(X,K) \succeq \Phi(X,K_X)$ for $\forall K$, where $K_X \triangleq XC'(CXC'+R)^{-1}$. That is, $\Phi(X,K_X) = \min_K \Phi(X,K)$.
(iii) [6, Lemma 1-b)] $g(P,\gamma) = (1-\gamma)(APA'+Q) + \gamma(A\Phi(P,K_P)A'+Q)$, when $K_P = PC'(CPC'+R)^{-1}$.
(iv) [6, Theorems 2 and 3] The stability of $\mathbb{E}[\underline{M}_k]$ is determined by the observation p.a.r. γ with a critical value γ_m.
(v) [34, Theorem 1] If R, P_0, and $Q \succ 0$, then the critical value γ_m for the stability of $\underline{M}_{k+1} = g(\underline{M}_k, \gamma_k)$ is a function of A and C, and is independent of $\{R, P_0, Q\}$.

Lemma 4.

$$\mathbb{E}[\underline{M}_k] \preceq \mathbb{E}[\bar{P}_k^{[*]}] \preceq \mathbb{E}[\overline{M}_k]. \tag{19}$$

Proof of Theorem 1: By viewing $Q + A\Delta A'$ in $g(\overline{M}_k,\gamma_k) + A\Delta A'$ as a new Q, and noting that $g(\underline{M}_k,\gamma_k)$ and $g(\overline{M}_k,\gamma_k) + A\Delta A'$ have the same A and C, it follows from Lemma 3(iv) and (v) that the critical value γ_m for the stability of $\mathbb{E}[\overline{M}_k]$ is the same as that for $\mathbb{E}[\underline{M}_k]$. By (19), the stability of $\mathbb{E}[\bar{P}_k^{[*]}]$ are the same as that of $\mathbb{E}[\overline{M}_k]$ and $\mathbb{E}[\underline{M}_k]$. Therefore, the stability of $\mathbb{E}[\overline{M}_k]$, $\mathbb{E}[\underline{M}_k]$, and $\mathbb{E}[\bar{P}_k^{[*]}]$ is determined by γ with respect to the same critical value γ_m.

From (4), it follows that if $\mathbb{E}[\bar{P}_k^{[*]}]$ is unstable then $\mathbb{E}[P_{k-1}]$ must be unstable as well. From (14) and Lemma 2(iii), we have $\mathbb{E}[P_k] \preceq \mathbb{E}[\bar{P}_k^{[*]}] + \Delta$, which implies that if $\mathbb{E}[\bar{P}_k^{[*]}]$ is stable, so is $\mathbb{E}[P_k]$. Therefore, the stability of $\mathbb{E}[P_k]$, $\mathbb{E}[\bar{P}_k^{[*]}]$, and $\mathbb{E}[\underline{M}_k]$ is equivalent. Hence, $\mathbb{E}[P_k]$ is determined by the p.a.r. γ with respect to the same critical value $\gamma_c \triangleq \gamma_m$, and is independent of the control p.a.r. θ and $\{u_k\}$, which completes the proof of part (i).

Part (ii) can be proved by noting the fact in Lemma 3(iv) that $\underline{M}_k$ is the error covariance of the optimal estimator for the corresponding TCP-like system and that $\gamma_c \triangleq \gamma_m$. ■

V. Convergence of the IMM Estimator

This section studies the convergence of the IMM estimator, that is, the convergence of P_k. Note that P_k contains $\bar{P}_k^{[*]}$, which is determined by the modified Riccati equation $\bar{P}_{k+1}^{[*]} = g(\bar{P}_k^{[*]},\gamma_k) + A\Delta_{k-1}A'$. It has been shown in [6] that $\mathbb{E}[\bar{P}_k^{[*]}]$ computed by this modified Riccati equation is not necessarily convergent. Thus, in general, P_k is not necessarily convergent either. Nevertheless, under some conditions its convergence can be characterized as in the following theorem.

Theorem 2 (Convergence)**.** *Consider the UDP-like system in (1) with $\gamma_c < \gamma < 1$, where γ_c is the critical value for the stability of the IMM estimator. If the control input u_k tends to zero, the error covariance of the IMM estimator for the UDP-like system converges to that of the optimal estimator for its* *corresponding TCP-like system. That is,*

$$\lim_{k\to\infty} P_k - P_k^{\mathsf{tcp}} = 0,$$

where P_k^{tcp} is defined in (18).

To prove Theorem 2, we present a lemma as follows.

Define

$$\bar{g}(P,\gamma,Q,R) \triangleq APA' - \gamma APC'(CPC'+R)^{-1}CPA' + Q.$$

Lemma 5. *Let $\overline{M}_{k+1}^{[*]} = \bar{g}(\overline{M}_k^{[*]}, \gamma_k, Q+\Delta_Q, R)$ with $\overline{M}_1^{[*]} = \bar{P}_1^{[*]}$ and $\Delta_Q \succeq 0$. Then $\lim_{\Delta_Q\to 0}\overline{M}_k^{[*]} = \underline{M}_k$ for all k, where $\underline{M}_k$ is defined in (16).*

Proof of Theorem 2: The prediction error covariance for TCP-like systems $\underline{M}_{k+1} = g(\underline{M}_k, \gamma_k) = \bar{g}(\underline{M}_k, \gamma_k, Q, R)$ with $\underline{M}_1 = \bar{P}_1^{[*]}$. For the IMM estimator, $\bar{P}_{k+1}^{[*]} = g(\bar{P}_k^{[*]}, \gamma_k) + A\Delta_k A' = \bar{g}(\bar{P}_k^{[*]}, \gamma_k, Q + A\Delta_k A', R)$. By the definition of Δ_k and noting that $0 \le \lambda_k^{[i]} \le 1$ and that $\sigma(\mathbb{K}_k) < 1$ by Lemma 1(ii), we have Δ_k tends to 0 when $u_k \to 0$ $(k \to +\infty)$. From Lemma 5, it follows that $\bar{P}_k^{[*]}$ converges to $\underline{M}_k$. By subtracting (18) from (14),

$$\begin{aligned} &P_k - P_k^{\mathsf{tcp}} \\ &= \big(\bar{P}_k^{[*]} - \gamma_k \bar{P}_k^{[*]} C'(C\bar{P}_k^{[*]}C'+R)^{-1}C\bar{P}_k^{[*]}\big) + \Delta_k \\ &\quad - \big(\underline{M}_k - \gamma_k \underline{M}_k C'(C\underline{M}_k C'+R)^{-1}C\underline{M}_k\big). \end{aligned}$$

Letting $\bar{P}_k^{[*]} \to \underline{M}_k$ and $\Delta_k \to 0$, we readily have $\lim_{k\to+\infty} P_k - P_k^{\mathsf{tcp}} = 0$, which completes the proof. ■

VI. Performance Analysis of the IMM Estimator

In this section, we study the impact of the observation/control p.a.r. and control inputs on the performance of the IMM estimator. Then we prove that the IMM estimator approximates the optimal estimator and outperforms the LMMSE estimator. Finally, we develop a method to improve the control performance of state feedback control systems by the IMM estimator.

A. Impact of packet losses on estimation performance

1) Impact of the loss of observations: For the loss of observation, its impact on the average estimation performance $\mathbb{E}[P_k]$ is stated in Theorem 1. As the observation p.a.r. γ diminishes, more observations are lost and the resulting estimation performance becomes worse.

2) Impact of the loss of control inputs: The control p.a.r. θ does not affect the stability of the IMM estimator, but it affects its estimation performance. However, the relationship between $\mathbb{E}[P_k]$ and θ not only appears irregular from the experimental point of view (see the graph of $\mathbb{E}[P_{50}]$ in Fig. 4), but also is difficult to be analytically characterized from the technical point of view, since

- $\mathbb{E}[P_k]$ is not convergent, since it is known from (14) that $\mathbb{E}[P_k]$ will contain $\mathbb{E}[\bar{P}_k^{[*]}]$, which is not convergent, as explained in Section V.
- The monotonic relationship between P_k and P_{k-1} is not preserved like the standard Riccati equation and is not easy to be determined either, that is, $P_{k-1}\uparrow$ does not imply $P_k\uparrow$ [1] ($P\uparrow$ means the value of P becomes larger).
- During the process of recursive computations, $\{\theta, u_j, y_j\}$ are implicitly contained in both the numerator and denominator of each $\lambda_k^{[i]}$ with $k > j$, making the relationship between $\mathbb{E}[P_k]$ and θ complicated.
- The control p.a.r. θ is assumed to be invariant at every time instant. When θ varies, it means the control p.a.r. at every time instant varies simultaneously. The variation of the control p.a.r. at every individual time instant j for $1 \le j \le k$ will impact $\mathbb{E}[P_k]$, and these cumulative impacts over 1 to $k-1$ make the relationship between $\mathbb{E}[P_k]$ and θ not easy to be determined.

The last reason stated above inspires us to study the relationship between the conditional expected error covariance and the control p.a.r. at some specific time instant. That is, we study the relationship between $\mathbb{E}[P_k|\mathcal{I}_{k-1}]$ and $\theta_{[k]}$, where $\theta_{[k]} \triangleq \mathbb{P}(\theta_k = 1)$ is the control p.a.r. at time k. Note that θ appears in (12b) and implicitly lies in $\lambda_k^{[i]}$ in (13b). From the proof of Lemma 2(i) and (ii), and the definitions of $\bar{\lambda}_k^{[i]} \triangleq p(\theta_k^{[i]}|\mathcal{I}_{k-1}) = \theta^{[i]}$ and $\lambda_k^{[i]} \triangleq p(\theta_k^{[i]}|\mathcal{I}_k)$, it is clear that *this* θ is in fact the control p.a.r at the time k and is just the $\theta_{[k]}$ defined above. In the following, we investigate how this θ in (12b) and (13b) affects $\mathbb{E}[P_k|\mathcal{I}_{k-1}]$. The subscript $[k]$ of $\theta_{[k]}$ is omitted for brevity, which will not cause confusion. Unless otherwise stated, the θ in this subsection (Sec. VI-A 2)) refers to $\theta_{[k]}$. We use the symbol $\mathbb{E}[P_k(\theta)|\mathcal{I}_{k-1}]$ to emphasize that $\mathbb{E}[P_k|\mathcal{I}_{k-1}]$ is a function of θ.

Definition 4. *A matrix function $f(\theta)$ is said to have the same monotonicity as the parabolic function $\theta(1-\theta)$ over $0 \le \theta \le 1$, denoted by $f(\theta) \propto \theta(1-\theta)$, if $f(\theta) = f(1-\theta)$ and*

$$\begin{aligned} f(\theta_1) \prec f(\theta_2), &\quad \textit{for } 0 \le \theta_1 < \theta_2 \le 1/2; \\ f(\theta_1) \succ f(\theta_2), &\quad \textit{for } 1/2 \le \theta_1 < \theta_2 \le 1. \end{aligned}$$

Theorem 3. *Consider the UDP-like system in (1) with $\gamma > \gamma_c$ under bounded control inputs. The relationship between the control p.a.r. θ and the conditional average estimation performance is*

$$\mathbb{E}[P_k(\theta)|\mathcal{I}_{k-1}] \propto \theta(1-\theta).$$

Before proving Theorem 3, we present a lemma as follows.

Lemma 6. *Define $\mathcal{L}(\theta) \triangleq \mathbb{E}[\lambda_k^{[0]}\lambda_k^{[1]}|\mathcal{I}_{k-1}]$. Then*

(i) $\mathcal{L}(\theta) = \displaystyle\int_{-\infty}^{\infty} \frac{\theta\bar{\theta}\phi_k^{[0]}\phi_k^{[1]}}{\bar{\theta}\phi_k^{[0]} + \theta\phi_k^{[1]}}\mathrm{d}y_k$

(ii) $\mathcal{L}(\theta) = \mathcal{L}(\bar{\theta})$

(iii) $\displaystyle\int_{-\infty}^{\infty} \frac{\phi_k^{[0]}}{(\phi_k^{[0]}+\phi_k^{[1]})^2}\mathrm{d}y_k = \int_{-\infty}^{\infty} \frac{\phi_k^{[1]}}{(\phi_k^{[0]}+\phi_k^{[1]})^2}\mathrm{d}y_k$

(iv) $\mathcal{L}(\theta) \propto \theta(1-\theta)$

[1] In (13b), $P_k = P_k^{[*]} + \lambda_k^{[0]}\lambda_k^{[1]}\mathbb{K}_k U_k \mathbb{K}_k'$. From (4) and (6d), it follows that $P_{k-1}\uparrow$ leads to $\bar{P}_k^{[*]}\uparrow$ and then $P_k^{[*]}\uparrow$. However, by [35, Lemma 9(2)], $\bar{P}_k^{[*]}\uparrow$ results in $\mathbb{K}_k(\mathbb{K}_k)'\downarrow$. Thus, the monotonic relationship between P_k and P_{k-1} is not easy to be determined.

(v) $\mathcal{L}(\theta) \leq \frac{\sqrt{\bar{\theta}\theta}}{2}\exp(-\frac{1}{2}[CBu_k]^2_\Lambda)$, *where* $\Lambda = (P_k^Y)^{-1}$

Proof of Theorem 3: From (14), it follows that, under the condition $\mathcal{I}_k \triangleq \{\gamma^k, Y^k\}$, $\bar{P}_k^{[*]}$ and $\mathbb{K}_k$ are deterministic. Note that U_k is assumed to be known and thus is deterministic as well. Therefore, the quantity $P_k|\mathcal{I}_{k-1}$ contains random variables y_k and γ_k, where y_k lies in $\lambda_k^{[i]}$. By the taking mathematical expectation of (14) with respect to $\{y_k, \gamma_k\}$, and noting the definition of $\mathcal{L}(\theta)$ in Lemma 6,

$$\begin{aligned}\mathbb{E}[P_k(\theta)|\mathcal{I}_{k-1}] = \bar{P}_k^{[*]} - \gamma\bar{P}_k^{[*]}C'(C\bar{P}_k^{[*]}C' + R)^{-1}C\bar{P}_k^{[*]} \\ + \gamma\mathcal{L}(\theta)\mathbb{K}_kU_k\mathbb{K}_k' + \bar{\gamma}\theta\bar{\theta}U_k. \qquad (20)\end{aligned}$$

Clearly, $\mathbb{E}[P_k(\theta)|\mathcal{I}_{k-1}]$ is determined by $\mathcal{L}(\theta)$ and $\theta\bar{\theta}$. The proof is completed by noting that $\mathcal{L}(\theta) \propto \theta(1-\theta) = \theta\bar{\theta}$ in Lemma 6(iv). ∎

Remark 1. *Note that* $\mathcal{L}(\theta) \propto \theta(1-\theta)$ *and that* $\theta(1-\theta)$ *is the entropy of* θ_k*, a notion to measure the uncertainty of a random variable. From (20), it follows that the conditional average estimation becomes worse, as the uncertainty of* θ_k *increases.*

B. Impact of control inputs on estimation performance

For the impact of control inputs, we study how the increment of control input magnitude affects the estimation performance, and the results are formulated as follows.

Theorem 4. *Consider the UDP-like system in (1).*

(i) If $\gamma_c < \gamma < 1$, $\lim_{||U_k||\to+\infty}\mathbb{E}[P_k] = +\infty$ [2];

(ii) If $\gamma = 1$ *and* C *has full column rank, then*

$$\lim_{||U_k||\to+\infty}\mathbb{E}[P_k] < +\infty.$$

Proof of (i): For the system with $\gamma_c < \gamma < 1$, from (14), $\mathbb{E}[P_k] = \mathbb{E}[P_k^{[*]}] + \bar{\gamma}\theta\bar{\theta}U_k + \gamma\mathbb{E}[\lambda_k^{[0]}\lambda_k^{[1]}\mathbb{K}_kU_k\mathbb{K}_k'] \succeq \bar{\gamma}\theta\bar{\theta}U_k$. Hence, part (i) holds.

Proof of (ii): When $\gamma = 1$, $\mathcal{I}_k = Y^k$ and, in (15), $\Delta_k = \mathbb{K}_k\lambda_k^{[0]}\lambda_k^{[1]}U_k\mathbb{K}_k'$.

$$\begin{aligned}\mathbb{E}_{Y^k}[\mathbb{K}_k\lambda_k^{[0]}\lambda_k^{[1]}U_k\mathbb{K}_k'] = \mathbb{E}_{Y^{k-1}}\Big[\mathbb{K}_k\mathbb{E}_{y_k}[\lambda_k^{[0]}\lambda_k^{[1]}|Y^{k-1}]U_k\mathbb{K}_k'\Big] \\ \preceq \mathbb{E}_{Y^{k-1}}[\mathcal{L}(\theta)U_k], \qquad (21)\end{aligned}$$

where the first equality is obtained by the independence of $\mathbb{K}_k$ and y_k, and (21) is obtained by (33) and the fact $\sigma(\mathbb{K}_k) < 1$ in Lemma 1(ii).

Note that $U_k = (Bu_k)^2_I$, and that $\Lambda \triangleq (P_k^Y)^{-1}$ is nonsingular due to $P_k^Y = C\bar{P}_k^{[*]}C' + R$. Thus, $C\Lambda C'$ is nonsingular since C has full column rank. Then, $\underline{\sigma}(C\Lambda C') > 0$. Let $l = Bu_k$. From Lemma 6(v), it follows that $\mathcal{L}(\theta)U_k \preceq \frac{\sqrt{\bar{\theta}\theta}}{2}\exp(-\frac{1}{2}[Cl]^2_\Lambda)(l)^2_I$. Define $\mathcal{H}(u_k) \triangleq \mathrm{tr}(\mathcal{L}(\theta)U_k)$. We have

$$\begin{aligned}\mathcal{H}(u_k) &\leq \frac{\sqrt{\bar{\theta}\theta}}{2}\exp(-\frac{1}{2}l'(C'\Lambda C)l)||l||^2 \\ &\leq \frac{\sqrt{\bar{\theta}\theta}}{2}\exp(-\frac{1}{2}\underline{\sigma}(C\Lambda C')||l||^2)||l||^2 \triangleq \mathcal{Q}(l).\end{aligned}$$

It is known from Algorithm 1 that both P_k^Y and $\bar{P}_k^{[*]}$ are independent of U_k. From the knowledge of calculus, it is clear when $||U_k|| \to +\infty$, $||l|| \to +\infty$ and $\lim_{||l||\to+\infty}\mathcal{Q}(l) = 0$. Thus, $\mathcal{Q}(l)$ is bounded with its bound denoted by $\overline{\mathcal{Q}}$, which implies $\mathcal{H}(u_k) \preceq \overline{\mathcal{Q}}$ for all k. Hence, $\mathbb{E}[\Delta_k] \preceq \overline{\mathcal{Q}}$. By (15) and using the established result that $\mathbb{E}[g(X,\gamma)] \preceq g(\mathbb{E}[X],\gamma)$ for a random variable X [6, Lemma 1(h)],

$$\mathbb{E}[\bar{P}_{k+1}^{[*]}] = \mathbb{E}[g(\bar{P}_k^{[*]},1) + A\Delta_kA'] \preceq g(\mathbb{E}[\bar{P}_k^{[*]}],1) + A\overline{\mathcal{Q}}A'. \qquad (22)$$

Let $T_1 = \mathbb{E}[\bar{P}_1^{[*]}]$ and define $\{T_k\}$ by the standard Riccati equation: $T_{k+1} = g(T_k,1) + A\overline{\mathcal{Q}}A'$. By using (22) and the mathematical induction method, it is easy to prove that $\mathbb{E}[\bar{P}_k^{[*]}] \preceq T_k$. It is well known that under Assumption 1 T_k is convergent and thus is bounded. Therefore, $\mathbb{E}[\bar{P}_k^{[*]}]$ is bounded. From (14), we have $\mathbb{E}[P_k] \preceq \mathbb{E}[\bar{P}_k^{[*]} + \Delta_k] \preceq \mathbb{E}[\bar{P}_k^{[*]}] + \overline{\mathcal{Q}}$. The proof is completed. ∎

C. Comparison with the LMMSE estimator

As a good approximation for the optimal estimator, the IMM estimator is expected to outperform the frequently used LMMSE estimator, but to our best knowledge, few results have been reported on comparing the LMMSE estimator with the IMM estimator or its variant. We propose a variant of it, called the modified IMM (mIMM) estimator, with its estimation performance theoretically comparable with that of the IMM and the LMMSE estimators. The modified IMM (mIMM) estimator has been given in Algorithm 1 in Section III by letting the $\alpha = 1$ in (6a).

Definition 5 (Optimal linear estimation)**.** *An estimate of* x_k*, denoted by* $\hat{x}_k^L$*, is said to be an optimal linear estimation in the MMSE sense, if* $\hat{x}_k^L$ *is a linear function of* $\{\gamma_1y_1,\ldots,\gamma_ky_k\}$ *and* $\hat{x}_k^L$ *minimizes* $\mathbb{E}[||x_k - \hat{x}_k^L||^2]$*, where* $\gamma_ky_k = \emptyset$ *(an empty set) when* $\gamma_k = 0$*.*

Denote the prediction and the estimation error covariances of the LMMSE estimator for UDP-like systems by $\bar{P}_k^L$ and P_k^L, respectively. They have been obtained in [15, 17] and are shown in Algorithm 2.

Algorithm 2 The LMMSE estimator for UDP-like systems

Time-update prediction: Let $U_k \triangleq Bu_ku_k'B'$.

$$\bar{P}_{k+1}^L = AP_k^LA' + Q + \theta\bar{\theta}U_{k+1}, \text{ with } P_0^L = P_0.$$

Measurement-update estimation:
If $\gamma_{k+1} = 0$, then $P_{k+1}^L = \bar{P}_{k+1}^L$.
If $\gamma_{k+1} = 1$, then

$$\begin{aligned}&K_{k+1}^L = \bar{P}_{k+1}^LC'(C\bar{P}_{k+1}^LC' + R)^{-1} \\ &P_{k+1}^L = (I - K_{k+1}^LC)\bar{P}_{k+1}^L = \Phi(\bar{P}_{k+1}^L, K_{k+1}^L) \\ &\bar{P}_{k+1}^L = A\bar{P}_k^LA' - \gamma_kA\bar{P}_{k+1}^LC'(C\bar{P}_{k+1}^LC' + R)^{-1}C\bar{P}_{k+1}^LA' \\ &\qquad + Q + \theta\bar{\theta}U_{k+1} = \bar{g}(\bar{P}_k^L, \gamma_k, Q + \theta\bar{\theta}U_{k+1}, R).\end{aligned}$$

[2] This result accounts for the need of the bounded input assumption.

Theorem 5. *For UDP-like systems, the estimation performance of the standard IMM estimator is superior to that of the LMMSE estimator in the mean sense, that is,*

$$\mathbb{E}[P_k] \preceq \mathbb{E}[P_k^L].$$

Before proving Theorem 5, we present a lemma.

Denote the estimation error covariance of the mIMM estimator (actually, a sub-optimal IMM estimator) by P_k^{sub}.

Lemma 7. $P_k \preceq P_k^{sub}$

Proof of Theorem 5: P_k^{sub} can be calculated via Algorithm 1 by letting $\alpha = 1$ and $P_k^{sub} = P_k$. From (6d), (12b), and (13b), it is easy to obtain that

$$P_k^{sub} = \bar{\gamma}_k(\bar{P}_k^{[*]} + \theta\bar{\theta}U_k) + \gamma_k(\Phi(\bar{P}_k^{[\Delta]}, K_k). \tag{23}$$

From Algorithm 2, we have

$$P_k^L = \bar{\gamma}_k\bar{P}_k^L + \gamma_k\Phi(\bar{P}_k^L, K_k^L). \tag{24}$$

From Algorithms 1 and 2, it is clear that $P_0^L = P_0$, and thus $\mathbb{E}[P_0^{sub}] \preceq \mathbb{E}[P_0^L]$ naturally holds. Suppose that it also holds for $1, \ldots, k-1$. We check the case k as follows.

By subtracting (23) from (24), $P_k^L - P_k^{sub} = \bar{\gamma}_k\bar{P}_k^L + \gamma_k\Phi(\bar{P}_k^L, K_k^L) - \bar{\gamma}_k(\bar{P}_k^{[*]} + \theta\bar{\theta}U_{k-1}) - \gamma_k\Phi(\bar{P}_k^{[\Delta]}, K_k) = \bar{\gamma}_k A(P_{k-1}^L - P_{k-1}^{sub})A' + \gamma_k(\Phi(\bar{P}_k^L, K_k^L) - \Phi(\bar{P}_k^{[\Delta]}, K_k))$. For the second term in the equation above, $\Phi(\bar{P}_k^L, K_k^L) - \Phi(\bar{P}_k^{[\Delta]}, K_k) \succeq \Phi(\bar{P}_k^L, K_k^L) - \Phi(\bar{P}_k^{[\Delta]}, K_k^L) = (I - K_k^L C)(\bar{P}_k^L - \bar{P}_k^{[\Delta]})(I - K_k^L C)' \succeq \underline{\sigma}_{\mathbb{K}_k^L}^2(\bar{P}_k^L - \bar{P}_k^{[\Delta]})$, where the first and the last inequalities are obtained by Lemma 3(ii) and Lemma 1(iii), respectively. By the hypothesis $\mathbb{E}[P_{k-1}^{sub}] \preceq \mathbb{E}[P_{k-1}^L]$, we have

$$\begin{aligned}\mathbb{E}[P_k^L - P_k^{sub}] &\succeq \bar{\gamma}A\mathbb{E}[P_{k-1}^L - P_{k-1}^{sub}]A' + \gamma\underline{\sigma}_{\mathbb{K}_k^L}^2\mathbb{E}[\bar{P}_k^L - \bar{P}_k^{[\Delta]}] \\ &\succeq \gamma\underline{\sigma}_{\mathbb{K}_k^L}^2\mathbb{E}[\bar{P}_k^L - \bar{P}_k^{[\Delta]}]. \end{aligned}\tag{25}$$

By noting that $\bar{P}_k^{[*]} = AP_{k-1}^{sub}A' + Q$ and using (24) and (6a),

$$\begin{aligned}&\bar{P}_k^L - \bar{P}_k^{[\Delta]} \\ &= AP_{k-1}^L A' + Q + \theta\bar{\theta}U_k - (AP_{k-1}^{sub}A' + Q + \lambda_k^{[0]}\lambda_k^{[1]}U_k) \\ &= A(P_{k-1}^L - P_{k-1}^{sub})A' + (\theta\bar{\theta} - \lambda_k^{[0]}\lambda_k^{[1]})U_k.\end{aligned}$$

From the hypothesis above and Lemma 2(iii), it follows $\mathbb{E}[\bar{P}_k^L - \bar{P}_k^{[\Delta]}] \succeq A\mathbb{E}[P_{k-1}^L - P_{k-1}^{sub}]A' + \mathbb{E}[(\theta\bar{\theta} - \lambda_k^{[0]}\lambda_k^{[1]})]U_k \succeq 0$. By (25), $\mathbb{E}[P_k^L] \succeq \mathbb{E}[P_k^{sub}]$ is proved. Then, by Lemma 7, $\mathbb{E}[P_k^L] \succeq \mathbb{E}[P_k]$. The proof is completed. ■

D. Comparison with the optimal estimator

In most literature, the fact that the IMM estimator is an good approximation for the optimal estimator is usually illustrated by numerical simulations. However, from a theoretical point of view, it is still unsure that whether the approximation error will be accumulated to a certain extent that the IMM estimator diverges from the optimal one. The following theorem gives an answer to this question, and it gives the condition under which the distance between them is bounded in the mean sense.

Denote the optimal state estimate $\hat{x}_k^{opt}$ and its estimation covariance by P_k^{opt}.

Theorem 6. *Consider the UDP-like system in (1) and (2) with bounded control inputs. Let γ_c be the critical value for the stability of the IMM estimator. If the observation p.a.r. $\gamma > \gamma_c$, then the distance $D_k \triangleq P_k^{opt} - P_k$ is bounded in the mean sense, that is,*

$$-d \preceq \mathbb{E}[D_k] \preceq d,$$

where d is a positive definite constant matrix and only depends on the initial value P_0.

Before proving Theorem 6, we give a lemma as follows, whose proof is given in Appendix.

Lemma 8. *For UDP-like systems with bounded control inputs and $\gamma > \gamma_c$, the average LMMSE error covariance $\mathbb{E}[P_k^L]$ is bounded, that is, $\mathbb{E}[P_k^L] \preceq d$, where d is a positive definite constant matrix and only depends on the initial value P_0.*

Proof of Theorem 6*:* For the LMMSE estimate $\hat{x}_k^L$, according to Definition 1, it is clear that $P_k^{opt} = \mathbb{E}[(x_k - \hat{x}_k^{opt})_I^2|\mathcal{I}_k] \preceq \mathbb{E}[(x_k - \hat{x}_k^L)_I^2|\mathcal{I}_k]$. By Definition 5 and taking the mathematical expectation with respect to $\mathcal{I}_k$, $\mathbb{E}[P_k^{opt}] \preceq \mathbb{E}[(x_k - \hat{x}_k^L)_I^2] = P_k^L$. Obviously, $\mathbb{E}[P_k^{opt}] \preceq \mathbb{E}[P_k^L]$. From Theorem 5 and Lemma 8, it follows that

$$\mathbb{E}[P_k] \preceq \mathbb{E}[P_k^L] \preceq d \text{ and } \mathbb{E}[P_k^{opt}] \preceq \mathbb{E}[P_k^L] \preceq d, \tag{26}$$

where d is given in Lemma 8. (26) means that both $\mathbb{E}[P_k]$ and $\mathbb{E}[P_k^{opt}]$ lie between 0 and the upper bound d, which in fact implies that the distance between $\mathbb{E}[P_k]$ and $\mathbb{E}[P_k^{opt}]$ is bounded. $\mathbb{E}[P_k^{opt}] \preceq d \Rightarrow \mathbb{E}[P_k^{opt}] - \mathbb{E}[P_k] \preceq d - \mathbb{E}[P_k] \preceq d \Rightarrow \mathbb{E}[P_k^{opt} - P_k] \preceq d$. Similarly, it is easy to obtain $-d \preceq \mathbb{E}[P_k^{opt} - P_k]$. The proof is completed. ■

Remark 2. *Theorem 6 suggests that P_k approximates P_k^{opt} within a finite bound in the mean sense, but P_k is not necessarily an upper bound of P_k^{opt}. This result also implies the average distance between $\hat{x}_k$ and $\hat{x}_k^{opt}$ is bounded. That is, $\mathbb{E}[\|\hat{x}_k - \hat{x}_k^{opt}\|^2] = \mathbb{E}[\|(\hat{x}_k - x_k) + (x_k - \hat{x}_k^{opt})\|^2] \leq 2(\mathbb{E}[\|\hat{x}_k - x_k\|^2] + \mathbb{E}[\|x_k - \hat{x}_k^{opt}\|^2]) = 2\mathrm{tr}(\mathbb{E}[P_k] + \mathbb{E}[P_k^{opt}]) \leq 4\mathrm{tr}(d)$.*

VII. Numerical Examples

In this section, numerical examples are presented to illustrate the obtained results for the IMM estimator and the properties of the optimal estimator.

Consider the system in (1) with following parameters:

$$A = \begin{bmatrix}\sigma & 0 \\ 0 & 0.5\end{bmatrix}, \sigma > 1, B = \begin{bmatrix}-1 \\ 1\end{bmatrix},$$
$$C = \begin{bmatrix}-1 & 1 \\ 0 & 1\end{bmatrix}, Q = R = \begin{bmatrix}20 & 0 \\ 0 & 20\end{bmatrix}.$$

Some specifications for the simulations are given as follows: 1) For the convenience of computing the critical value, we choose the form above for A, since, for such a structure of A, the critical value for the TCP-like system is $\gamma_c = 1 - 1/\sigma^2$ [6]. 2) It is known from [16] that the computation of the optimal estimator for UDP-like systems is in fact NP-hard. Thus, its simulation time is chosen to be 30. 3) Unless otherwise stated, the simulations for the optimal estimator are performed using the same system parameters and control inputs as the corresponding simulations for the IMM estimator.

A. *Numerical examples for the IMM estimator*

Stability: For σ with different values $\{1.0541,\ 1.1952,\ 1.4142,\ 1.8257\}$, the corresponding critical values γ_c are $\{0.1, 0.3, 0.5, 0.7\}$, respectively. Under bounded control inputs ($\|u_k\| < 5$) and different control p.a.r.s, Fig. 2 shows that $\mathbb{E}[P_k]$ is stable as long as γ is greater than the critical value γ_c, and this critical value γ_c is the same as that of the optimal estimator for the corresponding TCP-like system, as stated in Theorem 1.

Convergence: Under $u_k = 10\exp(-k/5)$ and different pairs of $\{\theta, \gamma\}$, Fig. 3 shows that P_k, the EC of the IMM estimator for the UDP-like system, converges to S_k, the EC of the optimal estimator for the corresponding TCP-like system, as claimed in Theorem 2.

Estimator performance: **(i)** The impact of *the observation p.a.r.* on estimator performance is illustrated in Fig. 2. **(ii)** As shown in Fig. 4, the relationship between $\mathbb{E}[P_k]$ with $k = 50$ and *the control p.a.r.* θ appears irregular, but the graphs of $\mathbb{E}[P_k|\mathcal{I}_{k-1}]$ with $k = 30$ and 100 are similar to the parabolas with the symmetry axis at $\theta = 0.5$, as formulated in Theorem 3. **(iii)** How the increment of *control input magnitude* affects the performance is illustrated in Fig. 5, in which it can be seen that the expected EC of the IMM estimator remains bounded when C has full column rank and observations are successfully transmitted (that is, $\gamma = 1$), and tends to infinity when observations are randomly lost (that is, $\gamma < 1$), like the phenomenon described in Theorem 4. **(iv)** Fig. 6 shows that under different circumstances $\mathbb{E}[P_k^L - P_k]$ is positive, which implies that the IMM estimator is superior to the LMMSE one in the mean sense, as described in Theorem 5.

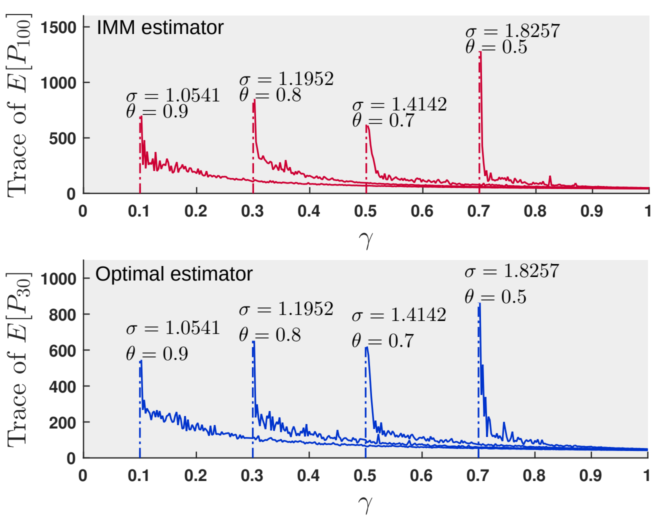

Fig. 2. Critical values for the IMM and the optimal estimators.

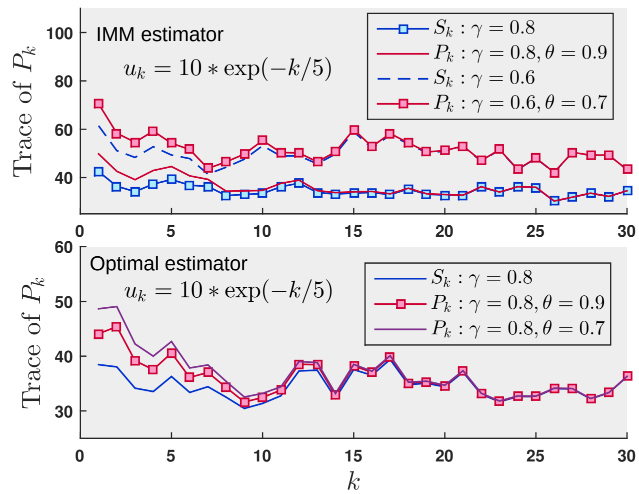

Fig. 3. Convergence of the IMM and the optimal estimators.

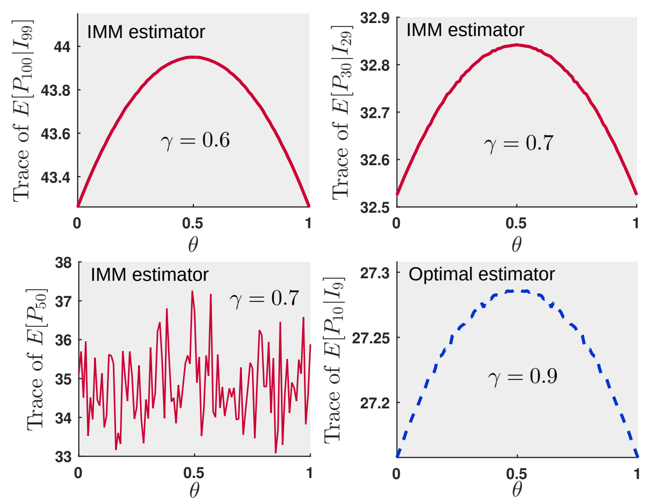

Fig. 4. The impact of the control p.a.r. on the estimation performance of the IMM and the optimal estimators.

B. *Numerical examples for the optimal estimator*

The IMM estimator is a good approximation for the optimal estimator, and we naturally wonder what are the results on these properties of the optimal estimator. Thus, the numerical simulations on the key properties are performed in parallel using the same system parameters for both the IMM and the optimal estimators, except for the comparison with the LMMSE one, as the optimal estimator no doubt outperforms the LMMSE one. As shown above, the optimal estimator has the same properties as that of the IMM estimator. The properties of the optimal estimator are shown experimentally, but, to our best knowledge, they are not presented in the existing literature, and might shed light on the further theoretical investigation of the optimal estimator.

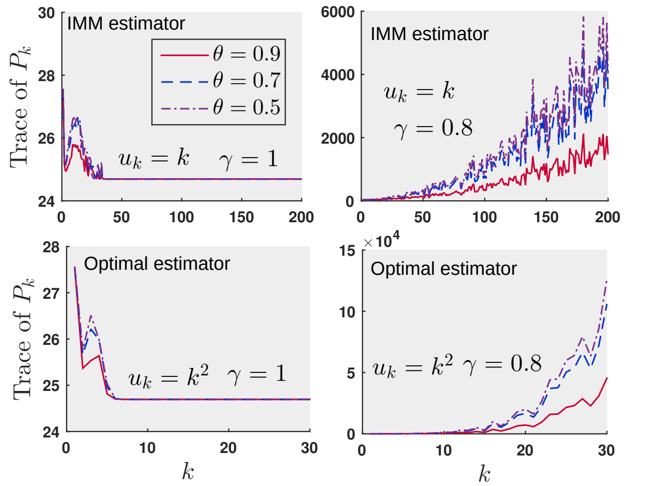

Fig. 5. The impact of the input magnitude on the estimation performance of the IMM and the optimal estimators.

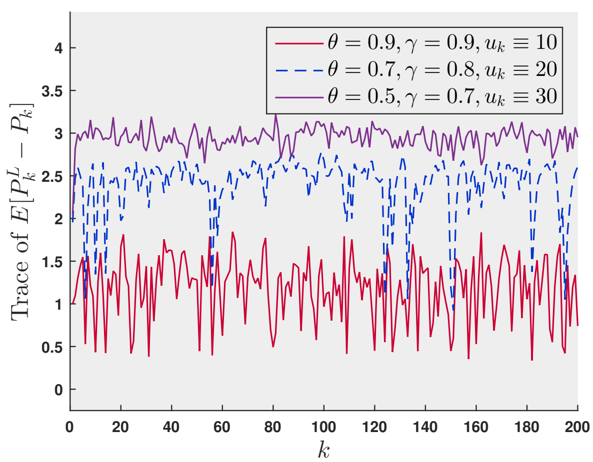

Fig. 6. Comparison of the estimation performance between the IMM and the LMMSE estimators.

C. Motivating examples of using IMM estimator to improve LQG control performance

In the previous sections, we have addressed some estimation problems on the IMM estimator for UDP-like systems. In this section, we give some simulation examples to show its application in control problems. We first propose a motivating idea, a controller re-design method, on improving control performance of a linear quadratic Gaussian (LQG) controller by the IMM estimator, then show the effectiveness of the proposed method by simulation examples. Finally, for future investigation, we provide some suggestions on how to theoretically analyze this method and design controllers by the IMM estimator.

1) A motivating idea: The LQG controller design for TCP-like systems has been extensively studied in [15, 17, 36, 37], while for general UDP-like systems, whether an optimal LQG controller exists or not remains unknown. Some sub-optimal LQG controllers, such as [2, 18], are developed with an objective to minimize the LQG cost J_N, which is also called the LQG performance and is defined as follows:

$$J_N = \mathbb{E}[x_N' W x_N] + \sum_{j=1}^{N} \mathbb{E}[x_{j-1}' W x_{j-1} + \theta_j u_j' \Lambda u_j].$$

The IMM estimator yields excellent estimation performance in various applications [24], which motivates us to consider that whether a better state estimator can improve in some way the control performance of the LQG controller for UDP-like systems. We propose an implementation of this idea as follows.

2) A controller-redesign method by using the IMM estimator: For general UDP-like systems, we consider two scenarios:

- There is no bounded input constraint on the controller considered. Denote the controller by

$$u_k = L_k \hat{x}_{k-1}. \tag{27}$$

- The controller considered has a bounded input constraint, which is denoted by

$$u_k = \mathsf{sat}(L_k \hat{x}_{k-1}, \overline{U}), \tag{28}$$

  where the saturation function $\mathsf{sat}()$ is defined in the notation list in Sec. I, and $\overline{U}$ is an upper of this controller.

By the IMM estimator, we propose a re-designed control law to improve the LQG performance of the controllers (27) and (28) as follows:

$$u_k = \mathsf{sat}(L_k \hat{x}_{k-1}^{imm}, \overline{U}^*), \tag{29}$$

where $\hat{x}_k^{imm}$ denotes the IMM estimate, and the upper bound $\overline{U}^*$ is determined as follows:

- For the controller (28) with a bounded input constraint, let $\overline{U}^* = \overline{U}$ in (29). It would be unreasonable to let $\overline{U}^* < \overline{U}$ in (29), as it will impose more constraints on the re-designed controller (29).
- For the controller (27) without a bounded input constraint, we first need to determine an upper bound for the controller (27)—as the IMM estimate $\hat{x}_k^{imm}$ can be stably calculated when control inputs are bounded—but, it is easy to determine an upper bound $\overline{U}^*$ such that imposing this upper bound onto the controller (27) has little, even no, impact on its LQG performance. The desired upper bound $\overline{U}^*$ can be determined in the following way: for a given upper bound $\overline{U}$, run numerical simulations with the control input $u_k = \mathsf{sat}(L_k \hat{x}_{k-1}, \overline{U})$, and then compute the corresponding LQG cost $\overline{J}$. Enlarge $\overline{U}$ until the desired $\overline{U}^*$ is found, such that J and $\overline{J}$ are close within a satisfied small gap, e.g., $|J - \bar{J}| \leq 10^{-8}$.

Remark 3. *For the controller (27) without a bounded input constraint, the performance J, in fact, can be viewed the limit of $\overline{J}$ as $\overline{U} \to +\infty$. From the knowledge of limit theory, it follows that $|J - \bar{J}|$ can be arbitrarily small by enlarging $\overline{U}$. Therefore, the way above to determine $\overline{U}^*$ is feasible, which will be verified in Fig. 7.*

3) Simulation results and feasibility of the proposed method: We apply the proposed controller-redesign method to two LQG controllers, which are developed in [2] and [18], respectively, and illustrate its effectiveness in improving the LQG performance of these two controllers by numerical simulations. Some specifications on simulations are given as follows:

On controllers with a bounded input constraint: It needs pointing out that, in this section, for a controller with a bounded input constraint, what we are interested in is to improve its LQG performance *when this controller exists*, not to design a controller with a bounded input constraint. We discovered that the LQG controllers developed in [2] and [18] are still able to stabilize the system when the upper bound is not too small. Therefore, to simulate an LQG controller with a bounded input constraint, we first design the LQG controller by the methods proposed in [2] or [18], and then determine an upper bound $\overline{U}$, like $\overline{U}$ given in the top two figures in Fig. 8 for the scalar/MIMO system, such that the resulting closed-loop system is still stable.

On system models used in simulations: We first use two systems: one is a scalar system with the following parameters: $A = 1.1$, $B = C = Q = R = W = \Lambda = 1$; and the other is a $4th$ order MIMO system[3]. They are frequently used to evaluate LQG controllers for UDP-like systems. Then, we randomly generate some systems for simulations. It is impossible to test all kinds of systems. We consider the randomly-generated systems with the following form: $A = a_1$, $B = b_1$, and $C = c_1$ for single variable systems; and $A = \mathsf{diag}(a_1, \ldots, a_n)$, $B = (b_1, \ldots, b_n)'$, $C = \left[\begin{smallmatrix} c_{11} & \ldots & c_{1n} \\ c_{21} & \ldots & c_{2n} \end{smallmatrix}\right]$ for multi-variable systems, $Q = 0.01 \cdot I_n$, and $R = 0.01 \cdot I_q$, where a_i, b_i, c_1, c_{ij} are mutually independent random variables uniformly distributed on $[-1.5, 1.5]$. The reasons to adopt the form above are the following: 1) most of the systems can be diagonalized; 2) the form above may cover a class of systems close to the frequently used scalar and $4th$ order systems above (when the $4th$ order system is diagonalized); 3) the condition for the stability of the IMM estimator (Theorem 1) requires that $\gamma > \lambda_c$ and a lower bound of λ_c is known in [6] to be $1 - \rho_A^{-2}$, where $\rho_A = \max|\rho_i|$ and ρ_i are the eigenvalues of A. For the case $\gamma = 0.7$ considered in the following Table I, due to $0.7 > \lambda_c > 1 - \rho_A^{-2}$ and $\rho_A = \max(|a_i|)$, we have $|a_i| < 1.8257$. We set $a_i \in [-1.5, 1.5]$, leaving some margin for the stability.

Define some performance-related symbols for the LQG controller $u_k = L_k \hat{x}_{k-1}$ developed in [18] as follows:

$$J^{JFI} \qquad u_k = L_k \hat{x}_{k-1} \tag{30a}$$

$$\overline{J}^{JFI} \qquad u_k = \mathsf{sat}(L_k \hat{x}_{k-1}, \overline{U}^*) \tag{30b}$$

$$\overline{J}^{JFI}_{IMM} \qquad u_k = \mathsf{sat}(L_k \hat{x}^{imm}_{k-1}, \overline{U}^*) \tag{30c}$$

$$J^{JFI}_{IMM} \qquad u_k = L_k \hat{x}^{imm}_{k-1} \tag{30d}$$

where the symbol involving J denotes the LQG performance of the controller on its right-hand side. The symbols $\{J^{AJC}, \overline{J}^{AJC}, \overline{J}^{AJC}_{IMM}, J^{AJC}_{IMM}\}$ for the LQG controller developed in [2] can be defined by replacing "JFI" in the equations above with "AJC".

Fig. 7 shows a comparison of the LQG performances of the two controllers and the proposed method for both the scalar and the MIMO systems, when there is no bounded input constraint. In Fig. 7, $\overline{J}^{AJC \text{ or } JFI}_{IMM} \leq J^{AJC \text{ or } JFI}$ means that the proposed method (29) yield a better LQG performance than the two LQG controllers. In the following, for brevity, denote "AJC or JFI" by $\star$. It is also shown in Fig. 7 that $J^\star$ and $\overline{J}^\star$ overlap. Actually, the numerical simulations return that $\overline{J}^\star - J^\star = 0$. It suggests that imposing the desired upper bound $\overline{U}^* = 10^5$ to these two LQG controllers has no impact on their LQG performance, and thus the way to determine $\overline{U}^*$ is feasible.

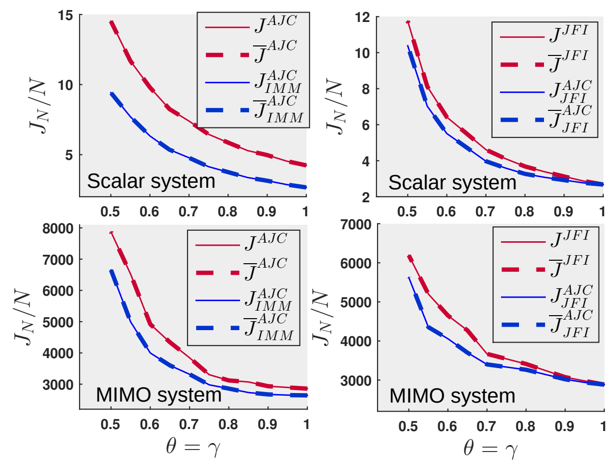

Fig. 7. Comparisons of the LQG performances of the controllers in [2, 18] and the proposed method, without a bounded input constraint. ($N = 20000$)

Fig. 8 presents the simulation results on the controller developed in [2] ([18]) for the scalar (MIMO) system with a bounded input constraint, and the upper bound $\overline{U}$ is shown in the top two figures. The graphs of other cases are similar to Fig. 8, and are not presented here for brevity. $\overline{J}^\star_{IMM} \leq \overline{J}^\star$ means that the proposed method improves the performance of the controller developed in [2, 18] with bounded inputs.

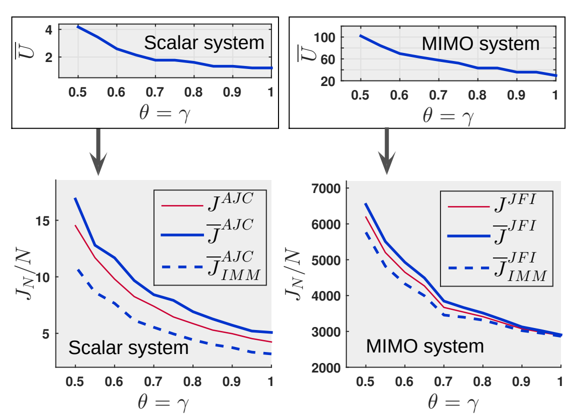

Fig. 8. Comparisons of the LQG performances of the controller in [2, 18] and the proposed method, with a bounded input constraint. ($N = 20000$)

[3]Please refer to [2, 15] for the parameters of this MIMO system.

TABLE I
AVERAGE IMPROVEMENT OF THE LQG PERFORMANCE

	AJC: $\theta=\gamma$			JFI: $\theta=\gamma$		
	0.7	0.8	0.9	0.7	0.8	0.9
$n=1$	10.14%	16.03%	14.70%	9.85%	9.48%	8.49%
$n=2$	14.72%	17.55%	15.35%	7.61%	8.83%	7.14%
$n=3$	12.97%	16.33%	14.74%	10.92%	9.19%	7.82%
$n=4$	13.61%	14.91%	15.62%	8.44%	9.07%	8.32%

Table I shows the average LQG performance improvement of the proposed method with respect to the controllers developed in [2, 18]. Take the top-left number 10.14% in Table I for example. To calculate this number, we randomly generate 10^3 systems with the aforementioned form and the system order $n=1$, then compute the LQG performance J^{AJC} of the controller developed in [2] and that $\overline{J}_{IMM}^{AJC}$ of the proposed method for each randomly-generated system under the control/observation p.a.r. $\theta=\gamma=0.7$. (For each randomly-generated system, we have checked that it is feasible to design an LQG controller by the method developed in [2, 18].) Finally, we calculate the average LQG performance improvement (that is, 10.14%) of all these $(J^{AJC}-\overline{J}_{IMM}^{AJC})/J^{AJC}$. For each case in Table I, we randomly generate 10^3 systems. The simulation results show that the proposed method indeed is able to improve the LQG performance of the controllers developed in [2, 18] for all these randomly-generated systems with or without a bounded input constraint, That is, $J^\star \geq \overline{J}_{IMM}^\star$ ($\overline{J}^\star \geq \overline{J}_{IMM}^\star$) for the controller without (with) a bounded input constraint. Table I shows the case that there is no bounded input constraint, but a table for the bounded input case is not given here, since for a fixed upper bound, some randomly-generated systems are stable, but others may be not. Thus, we cannot obtain a table like Table I, under some fixed upper bound. These simulation results suggest that the feasibility and effectiveness of the proposed method in improving the LQG performance of the LQG controllers developed in [2, 18].

4) Some suggestions: The simulation results above show the effectiveness of the proposed method. However, a theoretical proof of $\overline{J}_{IMM}^\star \leq J^\star$ (or $\overline{J}^\star$) for systems without (or with) a bounded input constraint will involve the details of controller design, which is challenging—especially when control inputs are bounded—and beyond the scope of this paper. Thus, some suggestions on controller design and LQG performance analysis are given as follows.

On the upper bound of control inputs: By the methodologies on designing LQG controllers for systems with bounded inputs, such as the quadratic programming approach [38] and the stochastic linearization method [39], one may obtain the impact of the upper bound $\overline{U}^*$ on the LQG performance. After knowing the impact of upper bounds, it is possible to consider the control law $u_k = L_k\hat{x}_{k-1}^{imm}$ in (30d) for the case of no bounded input constraint. We also present the simulation results of this control law in Fig. 7[4], which shows that $J_{IMM}^\star$ and $\overline{J}_{IMM}^\star$ also overlap. It means the LQG performances of $u_k = L_k\hat{x}_{k-1}^{imm}$ and $u_k = \mathsf{sat}(L_k\hat{x}_{k-1}^{imm}, \overline{U}^*)$ are quite close.

[4] $J_{IMM}^\star$ is not shown in Fig. 8, as the control law in (30d) is equal to that in (30c), that is, $J_{IMM}^\star = \overline{J}_{IMM}^\star$, for the bounded input constraint case.

Controller design: It is known that for general UDP-like systems, finding the optimal control u_k is quite involved and computing the optimal estimate $\hat{x}_k^{opt}$ is impractical, while it would be interesting to study the existence of the optimal control gain L_k^* such that $u_k = L_k^*\hat{x}_k^{opt}$ minimizes the LQG cost, with $\hat{x}_k^{opt}$ approximately computed by the IMM estimator. If such control strategy exists, it may provide a better LQG performance than some LQG controllers, such as [2] and [18], with the state estimated by linear estimators.

LQG performance analysis: For a given LQG controller $u_k = L_k\hat{x}_{k-1}$, e.g., the one proposed in [2] or [18], one may consider finding the optimal quantity $\breve{x}_k$, for the given L_k, such that $u_k = L_k\breve{x}_{k-1}$ minimizes the LQG cost. It would be possible to obtain 1) a new control law $u_k = L_k\breve{x}_{k-1}$ with a better LQG performance than $u_k = L_k\hat{x}_{k-1}$, and 2) the relationship between $\breve{x}_k$ and $\hat{x}_k^{opt}$, which is a possible way to analyze and explain how the IMM estimator affects the LQG performance by noting that $\hat{x}_k^{opt} \simeq \hat{x}_k^{imm}$.

VIII. CONCLUSION

In this paper, we have studied the stability, convergence, and performance of the IMM estimator for UDP-like systems, and revealed the impact of the control/observation packet arrival rate and the control input on the estimation performance of the IMM estimator, and proved that the IMM estimator outperforms the LMMSE estimator in the mean sense. Finally, we have proposed a controller re-design method to improve the LQG performance of LQG controllers developed in [2, 18] for general UDP-like systems, and illustrated the effectiveness by numerical simulations.

APPENDIX

Proof of Algorithm 1 *:* **(Mixing)** According to the standard IMM algorithm, the mixed initial conditions $\hat{x}_{k-1}^{[0|i]}$ and $P_{k-1}^{[0|i]}$ for the mode-matched filter i are defined and computed as follows:

$$\begin{aligned}\hat{x}_{k-1}^{[0|i]} &\triangleq \mathbb{E}[x_{k-1}|\theta_k^{[i]}, \mathcal{I}_{k-1}] \\ &= \mathbb{E}[x_{k-1}|\mathcal{I}_{k-1}] = \hat{x}_{k-1} \qquad (31)\\ P_{k-1}^{[0|i]} &\triangleq \mathbb{E}[(\hat{x}_{k-1}^{[0|i]} - x_{k-1})_I^2|\theta_k^{[i]}, \mathcal{I}_{k-1}] \\ &= \mathbb{E}[(\hat{x}_{k-1} - x_{k-1})_I^2|\mathcal{I}_{k-1}] = P_{k-1}, \qquad (32)\end{aligned}$$

where (31) and (32) are obtained by definitions of $\hat{x}_{k-1}$ and P_{k-1}, and by noting that $\theta_k^{[i]} = \{\theta_k = i\}$ is independent of $\mathcal{I}_{k-1}$. Thus, (3) holds.

(Time-update prediction) Based on the initial conditions $\hat{x}_{k-1}^{[0|i]}$ and $P_{k-1}^{[0|i]}$, for Mode 0 (that is, $\theta_k = 0$, $x_k = Ax_{k-1} + \omega_k$), by using Kalman filter, we have $\bar{x}_k^{[0]} = A\hat{x}_{k-1}$ and $\bar{P}_k^{[0]} = AP_{k-1}A' + Q$, which proves that (4) holds for Mode 0. Similarly, it is easy to check that it also holds for Mode 1.

(Mode probability updating) A key assumption in the IMM estimator is that the conditional pdf of $x_{k-1}^{[0|i]}$ is Gaussian, that is, $p(x_{k-1}|\theta_k^{[i]}, \mathcal{I}_{k-1}) = \mathcal{N}_{x_{k-1}}(\hat{x}_{k-1}^{[0|i]}, P_{k-1}^{[0|i]}) = \mathcal{N}_{x_{k-1}}(\hat{x}_{k-1}, P_{k-1})$. From (1) and (2), we have

$$p(x_k|\theta_k^{[i]}, \mathcal{I}_{k-1}) = \mathcal{N}_{x_k}(\bar{x}_k^{[i]}, \bar{P}_k^{[*]})$$

$$p(y_k|\theta_k^{[i]},\mathcal{I}_{k-1})=\mathcal{N}_{y_k}(C\bar{x}_k^{[i]},P_k^Y),$$

where $P_k^Y=C\bar{P}_k^{[*]}C'+R$. Hence, $p(y_k|\theta_k^{[i]},\mathcal{I}_{k-1})=\phi_k^{[i]}$. By the total probability law and the fact that $\theta_k^{[i]}$ is independent of $\mathcal{I}_{k-1}$, $p(y_k|\mathcal{I}_{k-1})=\sum_{i=0}^{1}p(y_k|\theta_k^{[i]},\mathcal{I}_{k-1})p(\theta_k^{[i]}|\mathcal{I}_{k-1})=\bar{\theta}\phi_k^{[0]}+\theta\phi_k^{[1]}$, which proves (5).

The prior mode probability $\bar{\lambda}_k^{[i]}\triangleq p(\theta_k^{[i]}|\mathcal{I}_{k-1})$. Since $\theta_k^{[i]}$ is independent of $\mathcal{I}_{k-1}$, $p(\theta_k^{[i]}|\mathcal{I}_{k-1})=p(\theta_k^{[i]})=\theta^{[i]}$. If $\gamma_k=0$, there is no useful observation available for updating (or correcting) the mode probability, then the posterior mode probability $\lambda_k^{[i]}\triangleq p(\theta_k^{[i]}|\mathcal{I}_k)=p(\theta_k^{[i]}|\mathcal{I}_{k-1})=\bar{\lambda}_k^{[i]}=\theta^{[i]}$. If $\gamma_k=1$, y_k is used to update the posterior mode probability. By Bayesian formula, $\lambda_k^{[i]}\triangleq p(\theta_k^{[i]}|\mathcal{I}_k)=p(\theta_k^{[i]}|\mathcal{I}_{k-1})p(y_k|\theta_k^{[i]},\mathcal{I}_{k-1})/p(y_k|\mathcal{I}_{k-1})=\bar{\lambda}_k^{[i]}\phi_k^{[i]}/c=\theta^{[i]}\phi_k^{[i]}/c$. Thus, the mode probability updating step holds.

(Measurement-update estimation) When $\gamma_k=0$, no useful observation is available for updating the prediction. Thus, $\hat{x}_k^{[i]}=\bar{x}_k^{[i]}$ and $P_k^{[i]}=\bar{P}_k^{[*]}$ hold for both the standard and the modified IMM estimators.

When $\gamma_k=1$, for the standard IMM estimator, $\alpha=0$ and $\bar{P}_k=\bar{P}_k^{[*]}$. It is clear that the equations (6b)–(6d) are the standard measurement-update steps in Kalman filtering. Hence, (6) holds for the standard IMM estimator. When $\gamma_k=1$, for the modified IMM estimator, the filtering gain K_k is designed by (6a) and (6b) with $\alpha=1$. By (6c), $x_k-\hat{x}_k^{[i]}=(I-K_kC)(x_k-\bar{x}_k^{[i]})-K_k\upsilon_k$. Thus, it is clear that the estimation error covariance takes the form as in (6d).[5] Therefore, (6) holds for both the standard and the modified IMM estimators.

(Combination) Eq. (7) is known as the combination step in the IMM estimator. Clearly, $\hat{x}_k=\sum_{i=0}^{1}\hat{x}_k^{[i]}\lambda_k^{[i]}$ holds.

$$\begin{aligned}P_k&=\int_{-\infty}^{\infty}(x_k-\hat{x}_k)_I^2p(x_k|\mathcal{I}_k)\mathrm{d}x_k\\&=\sum_{i=0}^{1}\int_{-\infty}^{\infty}(x_k-\hat{x}_k)_I^2p(x_k|\theta_k^{[i]},\mathcal{I}_k)p(\theta_k^{[i]}|\mathcal{I}_k)\mathrm{d}x_k\\&=\lambda_k^{[0]}\int_{-\infty}^{\infty}\big((x_k-\hat{x}_k^{[0]})-(\hat{x}_k-\hat{x}_k^{[0]})\big)_I^2\\&\quad\times p(x_k|\theta_k^{[0]},\mathcal{I}_k)\mathrm{d}x_k+\lambda_k^{[1]}\int_{-\infty}^{\infty}((x_k-\hat{x}_k^{[1]})-\\&\quad(\hat{x}_k-\hat{x}_k^{[1]}))_I^2p(x_k|\theta_k^{[1]},\mathcal{I}_k)\mathrm{d}x_k\\&=P_k^{[*]}+\sum_{i=0}^{1}\lambda_k^{[i]}(\hat{x}_k-\hat{x}_k^{[i]})_I^2.\end{aligned}$$

The proof of Algorithm 1 is completed. ■

Proof of Lemma 1: Using the matrix inverse lemma $XC'(CXC'+Y)^{-1}=(X^{-1}+C'Y^{-1}C)^{-1}C'Y^{-1}$ [32], we have $KC=(P^{-1}+C'R^{-1}C)^{-1}C'R^{-1}C$. It has been proved in [40, Theorem 7.7.3 and Corollary 7.7.4] that if $X\succ 0$ and $Y\succeq 0$, then $\sigma(X^{-1}Y)<1$ is equivalent to $X\succ Y$. By viewing $P^{-1}+C'R^{-1}C$ and $C'R^{-1}C$ as X and Y, part (i) is proved. From part (i), it is clear that $0<\sigma(I-KC)<1$ holds. Part (iii) is a straightforward result of Part (ii). ■

Proof of Lemma 2: *Proof of (i):* For $\gamma_k=0$, by (7) and (4), we have $\hat{x}_k=\sum_{i=0}^{1}\hat{x}_k^{[i]}\lambda_k^{[i]}=\sum_{i=0}^{1}\bar{x}_k^{[i]}\theta^{[i]}=\bar{\theta}A\hat{x}_{k-1}+\theta(A\hat{x}_{k-1}+Bu_k)=A\hat{x}_{k-1}+\theta Bu_k$. Since $e_k=x_k-\hat{x}_k=A(x_{k-1}-\hat{x}_{k-1})+(\theta_k-\theta)Bu_k+\omega_k$, it is easy to obtain that $P_k=AP_{k-1}A'+\bar{\theta}\theta Bu_ku_k'B'+Q=\bar{P}_k^{[*]}+\bar{\theta}\theta Bu_ku_k'B'$, which proves (12).

Proof of (ii): For $\gamma_k=1$, from (7), (6c), and (4),

$$\begin{aligned}\hat{x}_k&=\sum_{i=0}^{1}\hat{x}_k^{[i]}\lambda_k^{[i]}\\&=\lambda_k^{[0]}((I-K_kC)A\hat{x}_{k-1}+K_ky_k)\\&\quad+\lambda_k^{[1]}((I-K_kC)(A\hat{x}_{k-1}+Bu_k)+K_ky_k)\\&=\mathbb{K}_k(A\hat{x}_{k-1}+\lambda_k^{[1]}Bu_k)+K_ky_k,\end{aligned}$$

which proves (13a). From (7b), it follows that to prove (13b) is to prove $\sum_{i=0}^{1}\lambda_k^{[i]}(\hat{x}_k-\hat{x}_k^{[i]})_I^2=\lambda_k^{[0]}\lambda_k^{[1]}\mathbb{K}_kU_k\mathbb{K}_k'$. From (6c) and (13a), $\hat{x}_k-\hat{x}_k^{[0]}=\mathbb{K}_k(A\hat{x}_{k-1}+\lambda_k^{[1]}Bu_k)+K_ky_k-(\mathbb{K}_kA\hat{x}_{k-1}+K_ky_k)=\mathbb{K}_k\lambda_k^{[1]}Bu_k$. Similarly, $\hat{x}_k-\hat{x}_k^{[1]}=-\mathbb{K}_k\lambda_k^{[0]}Bu_k$. Then $\sum_{i=0}^{1}\lambda_k^{[i]}(\hat{x}_k-\hat{x}_k^{[i]})_I^2=\lambda_k^{[0]}(\mathbb{K}_k\lambda_k^{[1]}Bu_k)_I^2+\lambda_k^{[1]}(\mathbb{K}_k\lambda_k^{[0]}Bu_k)_I^2=\lambda_k^{[0]}\lambda_k^{[1]}\mathbb{K}_kU_k\mathbb{K}_k'$, where $\lambda_k^{[0]}+\lambda_k^{[1]}=1$ is used.

Proof of (iii): Note that θ_k is a Bernoulli random variable. It is evident that $\mathrm{cov}(\theta_k)=\theta\bar{\theta}$. In the sequel, we compute $\mathrm{cov}(\theta_k|Y^k)$. By the definition that $\lambda_k^{[i]}\triangleq\mathbb{P}(\theta_k^{[i]}|Y^k)$, the conditional expectation $\mathbb{E}[\theta_k|Y^k]=1\cdot\mathbb{P}(\{\theta_k=1\}|Y^k)+0\cdot\mathbb{P}(\{\theta_k=0\}|Y^k)=\mathbb{P}(\theta_k^{[1]}|Y^k)=\lambda_k^{[1]}$ and the conditional covariance $\mathrm{cov}(\theta_k|Y^k)=\sum_{\theta_k=0}^{1}(\theta_k-\mathbb{E}[\theta_k|Y^k])^2\mathbb{P}(\theta_k|Y^k)=(0-\lambda_k^{[1]})^2\lambda_k^{[0]}+(1-\lambda_k^{[1]})^2\lambda_k^{[1]}=\lambda_k^{[0]}\lambda_k^{[1]}$. Observe that in (10), $\mathrm{cov}\big(\mathbb{E}[X|Y]\big)\succeq 0$. From (10), it follows $\theta\bar{\theta}=\mathrm{cov}(\theta_k)\succeq\mathbb{E}[\mathrm{cov}(\theta_k|Y_k)]=\mathbb{E}[\lambda_k^{[0]}\lambda_k^{[1]}|Y^k]$, which implies $\theta\bar{\theta}\succeq\mathbb{E}[\lambda_k^{[0]}\lambda_k^{[1]}]$. By this result and Lemma 1(ii), it is easy to obtain $\Delta\succeq\mathbb{E}[\Delta_k]$. Part (iii) is proved.

Proof of (iv): By computing $(1-\gamma_k)*(12b)+\gamma_k*(13b)$ and using the equality in the Riccati equation that $\Phi(\bar{P}_k^{[*]},K_k)=(I-K_kC)\bar{P}_k^{[*]}$, (14) can be readily obtained. From (4) and the definition of $g(\cdot)$, it follows that (15) holds. ■

Proof of Lemma 4: We prove this lemma by the mathematical induction. Since $\overline{M}_1=\underline{M}_1=\bar{P}_1^{[*]}$, (19) holds for $k=1$.

Suppose that (19) holds for $1,\ldots,n$. By Lemma 3(iii),

$$\begin{aligned}\underline{M}_{n+1}&=\bar{\gamma}_n(A\underline{M}_nA'+Q)+\gamma_n(A\Phi(\underline{M}_n,K_{\underline{M}_n})A'+Q)\\\bar{P}_{n+1}^{[*]}&=\bar{\gamma}_n(A\bar{P}_n^{[*]}A'+Q)+\gamma_n(A\Phi(\bar{P}_n^{[*]},K_{\bar{P}_n^{[*]}})A'+Q).\end{aligned}$$

Subtracting $\underline{M}_{n+1}$ from $\bar{P}_{n+1}^{[*]}$ in (15) and using Lemma 3(ii) yield $\bar{P}_{n+1}^{[*]}-\underline{M}_{n+1}\succeq\bar{\gamma}_nA(\bar{P}_n^{[*]}-\underline{M}_n)A'+A\Delta_nA'+\gamma_nA(\Phi(\bar{P}_n^{[*]},K_{\underline{M}_n})-\Phi(\underline{M}_n,K_{\underline{M}_n}))A'$, where $\Phi(\bar{P}_n^{[*]},K_{\underline{M}_n})-\Phi(\underline{M}_n,K_{\underline{M}_n})=(I-K_{\underline{M}_n}C)(\bar{P}_n^{[*]}-\underline{M}_n)(I-K_{\underline{M}_n}C)'\succeq\underline{\sigma}(I-K_{\underline{M}_n}C)^2(\bar{P}_n^{[*]}-\underline{M}_n)$. Noting that $\mathbb{E}[\bar{P}_n^{[*]}-\underline{M}_n]\succeq 0$ (by hypothesis), $\Delta_n\succeq 0$, and the minimum

[5] Actually, (6d) can also be readily proved by the fact in [32] that for a given filtering gain K_k in (6c) with appropriate dimension, the estimation error covariance $P_k^{[i]}$ always takes the form as in (6d), no matter what value K_k takes. Only when K_k is determined by (6b) with $\alpha=0$, the resulting estimation $\hat{x}_k^{[i]}$ is optimal in the MMSE sense; otherwise, $\hat{x}_k^{[i]}$ is a sub-optimal estimation.

singular value $\underline{\sigma}(I-K_{\underline{M}_n}C) > 0$ (by Lemma 1(iii)), we have $\mathbb{E}[\bar{P}_{n+1}^{[*]} - \underline{M}_{n+1}] \succeq 0$, which proves $\mathbb{E}[\bar{P}_k^{[*]}] \succeq \mathbb{E}[\underline{M}_k]$.

Similarly, by following the same line of arguments above, $\overline{M}_{n+1} - \bar{P}_{n+1}^{[*]} \succeq \bar{\gamma}_n A(\overline{M}_n - \bar{P}_n^{[*]})A' + A(\Delta - \Delta_n)A' + \gamma_n A(\Phi(\overline{M}_n, K_{\overline{M}_n}) - \Phi(\bar{P}_n^{[*]}, K_{\overline{M}_n}))A'$, where $\Phi(\overline{M}_n, K_{\overline{M}_n}) - \Phi(\bar{P}_n^{[*]}, K_{\overline{M}_n}) = (I - K_{\overline{M}_n}C)(\overline{M}_n - \bar{P}_n^{[*]})(I - K_{\overline{M}_n}C)' \succeq \underline{\sigma}(I - K_{\overline{M}_n}C)^2(\bar{P}_n^{[*]} - \overline{M}_n)$. Noting that $\mathbb{E}[\overline{M}_n - \bar{P}_n^{[*]}] \succeq 0$, $\mathbb{E}[\Delta - \Delta_n] \succeq 0$ (by Lemma 2(iii)), and $\underline{\sigma}(I - K_{\overline{M}_n}C) > 0$ (by Lemma 1(iii)), we have $\mathbb{E}[\overline{M}_{n+1} - \bar{P}_{n+1}^{[*]}] \succeq 0$, which proves $\mathbb{E}[\overline{M}_k] \succeq \mathbb{E}[\bar{P}_k^{[*]}]$. ■

Proof of Lemma 5: It is pointed out in the proof of Theorem 2 in [34] that $\bar{g}(P,\gamma,Q,R)$ is a monotonically increasing function with respect to P, Q, and R. Given a real number $\alpha > 1$, we define two sequences

$$\underline{Z}_{k+1} = \bar{g}(\underline{Z}_k, \gamma_k, Q/\alpha, R/\alpha), \qquad \underline{Z}_1 = \bar{P}_1^{[*]}/\alpha;$$
$$\overline{Z}_{k+1} = \bar{g}(\overline{Z}_k, \gamma_k, \alpha Q, \alpha R), \qquad \overline{Z}_1 = \alpha \bar{P}_1^{[*]}.$$

Note that $\overline{Z}_1 = \alpha\bar{P}_1^{[*]} = \alpha\underline{M}_1$. Suppose that $\overline{Z}_k = \alpha\underline{M}_k$.

$$\begin{aligned}\overline{Z}_{k+1} &= \bar{g}(\overline{Z}_k, \gamma_k, \alpha Q, \alpha R) = \bar{g}(\alpha\underline{M}_k, \gamma_k, \alpha Q, \alpha R)\\ &= A\alpha\underline{M}_k A' - \gamma_k \alpha A\underline{M}_k C'(C\underline{M}_k C' + R)^{-1}C\underline{M}_k A' + \alpha Q\\ &= \alpha\underline{M}_{k+1}.\end{aligned}$$

According to the mathematical induction method, we have $\overline{Z}_k = \alpha\underline{M}_k$. Similarly, it is easy to verify that $\underline{Z}_k = \underline{M}_k/\alpha$ holds.

By noting that $Q + \Delta_Q$ with $\Delta_Q \to 0$ is equivalent to αQ with $\alpha \to 1^+$ (that is, $\alpha \to 1$ and $\alpha > 1$), $\overline{M}_k^{[*]}$ can be rewritten as $\overline{M}_{k+1}^{[*]} = \bar{g}(\overline{M}_k^{[*]}, \gamma_k, \alpha Q, R)$.

Observe that $\underline{Z}_1 \preceq \overline{M}_1^{[*]}$ holds for $\alpha > 1$. We suppose that $\underline{Z}_k \preceq \overline{M}_k^{[*]}$. Because of the monotonicity of $\bar{g}(P,\gamma,Q,R)$, $\overline{M}_{k+1}^{[*]} = \bar{g}(\overline{M}_k^{[*]}, \gamma_k, \alpha Q, R) \succeq \bar{g}(\underline{Z}_k, \gamma_k, Q/\alpha, R/\alpha) = \underline{Z}_{k+1}$. By the mathematical induction method, it follows that $\underline{Z}_k \preceq \overline{M}_k^{[*]}$. By following the same line of arguments, we have $\overline{M}_k^{[*]} \preceq \overline{Z}_k$. Consequently, $\frac{1}{\alpha}\underline{M}_k \preceq \underline{Z}_k \preceq \overline{M}_k^{[*]} \preceq \overline{Z}_k = \alpha\underline{M}_k$. Letting $\alpha \to 1^+$ and noting the aforementioned equivalence between $\alpha \to 1^+$ and $\Delta_Q \to 0$, we have $\lim_{\Delta_Q\to 0}\overline{M}_k^{[*]} = \underline{M}_k$ for all k. The proof is completed. ■

Proof of Lemma 6: *Proof of (i):* According to $\phi_k^{[i]}$ and (5) in Algorithm 1,

$$\begin{aligned}\mathcal{L}(\theta) &\triangleq \mathbb{E}[\lambda_k^{[0]}\lambda_k^{[1]}|\mathcal{I}_{k-1}]\\ &= \int_{-\infty}^{\infty}\lambda_k^{[0]}\lambda_k^{[1]}p(y_k|\mathcal{I}_{k-1})\mathrm{d}y_k = \int_{-\infty}^{\infty}\frac{\theta\bar{\theta}\phi_k^{[0]}\phi_k^{[1]}}{\bar{\theta}\phi_k^{[0]} + \theta\phi_k^{[1]}}\mathrm{d}y_k,\end{aligned}$$

which proves part (i).

For notational brevity, let $t \triangleq y_k - C(A\hat{x}_{k-1} + \frac{1}{2}Bu_k)$, $\delta_u \triangleq CBu_k/2$, $\Lambda \triangleq (P_k^Y)^{-1}$, $\eta \triangleq \sqrt{2\pi P_k^Y}$. Let $\phi^{[0]}(t) \triangleq \frac{1}{\eta}\exp(-\frac{1}{2}[t+\delta_u]_\Lambda^2)$ and $\phi^{[1]}(t) \triangleq \frac{1}{\eta}\exp(-\frac{1}{2}[t-\delta_u]_\Lambda^2)$.

By the notations defined above and some simple algebraic computations, it is easy to check the following three equalities hold.

$$\mathcal{L}(\theta) = \int_{-\infty}^{\infty}\frac{\theta\bar{\theta}\phi^{[0]}(t)\phi^{[1]}(t)}{\bar{\theta}\phi^{[0]}(t) + \theta\phi^{[1]}(t)}\mathrm{d}t \triangleq \mathcal{L}_0 \quad (33)$$

$$\int_{-\infty}^{\infty}\frac{\phi_k^{[0]}}{(\phi_k^{[0]} + \phi_k^{[1]})^2}\mathrm{d}y_k = \int_{-\infty}^{\infty}\frac{\phi^{[0]}(t)}{(\phi^{[0]}(t) + \phi^{[1]}(t))^2}\mathrm{d}t \triangleq \mathcal{L}_1$$
$$\int_{-\infty}^{\infty}\frac{\phi_k^{[1]}}{(\phi_k^{[0]} + \phi_k^{[1]})^2}\mathrm{d}y_k = \int_{-\infty}^{\infty}\frac{\phi^{[1]}(t)}{(\phi^{[0]}(t) + \phi^{[1]}(t))^2}\mathrm{d}t \triangleq \mathcal{L}_2.$$

Proof of (ii): By noting that $\phi^{[0]}(-t) = \phi^{[1]}(t)$ and letting $z = -t$,

$$\begin{aligned}\mathcal{L}(\theta) = \mathcal{L}_0 &= \int_{+\infty}^{-\infty}\frac{\theta\bar{\theta}\phi^{[0]}(-z)\phi^{[1]}(-z)}{\bar{\theta}\phi^{[0]}(-z) + \theta\phi^{[1]}(-z)}\mathrm{d}(-z)\\ &= \int_{-\infty}^{+\infty}\frac{\theta\bar{\theta}\phi^{[1]}(z)\phi^{[0]}(z)}{\bar{\theta}\phi^{[1]}(z) + \theta\phi^{[0]}(z)}\mathrm{d}(z) = \mathcal{L}(\bar{\theta}),\end{aligned}$$

which proves part (ii).

Proof of (iii): Similarly, by letting $z = -t$ and using $\phi^{[0]}(-z) = \phi^{[1]}(z)$,

$$\begin{aligned}\mathcal{L}_1 &= \int_{+\infty}^{-\infty}\frac{\phi^{[0]}(-z)}{(\phi^{[0]}(-z) + \phi^{[1]}(-z))^2}\mathrm{d}(-z)\\ &= \int_{-\infty}^{\infty}\frac{\phi^{[1]}(z)}{(\phi^{[1]}(z) + \phi^{[0]}(z))^2}\mathrm{d}(z) = \mathcal{L}_2,\end{aligned}$$

which proves part (iii).

Proof of (iv): To study the monotonicity of $\mathcal{L}(\theta)$, we calculate the derivative of $\mathcal{L}(\theta)$ in (33) as follows:

$$\frac{\mathrm{d}\mathcal{L}(\theta)}{\mathrm{d}\theta} = \int_{-\infty}^{\infty}\frac{\bar{\theta}^2\phi^{[0]} - \theta^2\phi^{[1]}}{(\bar{\theta}\phi^{[0]} + \theta\phi^{[1]})^2}\mathrm{d}t.$$

When $0 < \theta < 1/2$, $\bar{\theta} = 1 - \theta > \theta$. We have

$$\begin{aligned}\frac{\mathrm{d}\mathcal{L}(\theta)}{\mathrm{d}\theta} &> \frac{1}{(1-\theta)^2}\int_{-\infty}^{\infty}\frac{\bar{\theta}^2\phi^{[0]} - \theta^2\phi^{[1]}}{(\phi^{[0]} + \phi^{[1]})^2}\mathrm{d}t\\ &= \frac{\bar{\theta}^2 - \theta^2}{(1-\theta)^2}\int_{-\infty}^{\infty}\frac{\phi^{[0]}}{(\phi^{[0]} + \phi^{[1]})^2}\mathrm{d}t > 0,\end{aligned}$$

where the last equality is obtained by using part (iii). Similarly, for $1/2 < \theta < 1$, we have

$$\frac{\mathrm{d}\mathcal{L}(\theta)}{\mathrm{d}\theta} < \frac{\bar{\theta}^2 - \theta^2}{(1-\theta)^2}\int_{-\infty}^{\infty}\frac{\phi^{[0]}}{(\phi^{[0]} + \phi^{[1]})^2}\mathrm{d}t < 0.$$

Note that $\mathcal{L}(\theta)$ in part (i) is a continuous function with respect to θ. From the monotonicity of $\mathcal{L}(\theta)$ established above and the fact $\mathcal{L}(0) = \mathcal{L}(1)$ (by part (ii)), it is clear that $\mathcal{L}(\theta)$ takes the minimum value at $\theta = 0, 1$ and the maximum value at $\theta = 1/2$. Part (iv) is proved.

Proof of (v): Let $\varpi = t\Lambda\delta_u' + \delta_u\Lambda t'$.

$$\phi^{[0]}(t) = \eta^{-1}\exp(-0.5([t]_\Lambda^2 + \varpi + [\delta_u]_\Lambda^2))$$
$$\phi^{[1]}(t) = \eta^{-1}\exp(-0.5([t]_\Lambda^2 - \varpi + [\delta_u]_\Lambda^2)).$$

Then

$$\phi^{[0]}(t)\phi^{[1]}(t) = \eta^{-2}\exp(-([t]_\Lambda^2 + [\delta_u]_\Lambda^2)) \quad (34)$$

$$\begin{aligned}\bar{\theta}\phi^{[0]}(t) + \theta\phi^{[1]}(t) &= \eta^{-1}\exp(-0.5([t]_\Lambda^2 + [\delta_u]_\Lambda^2))\\ &\times \big(\bar{\theta}\exp(-0.5\varpi) + \theta\exp(0.5\varpi)\big).\end{aligned} \quad (35)$$

Observe that $\bar{\theta}\exp(-0.5\varpi) + \theta\exp(0.5\varpi) \geq 2\sqrt{\bar{\theta}\theta\exp(-0.5\varpi + 0.5\varpi)} \geq 2\sqrt{\bar{\theta}\theta}$. By (34) and (35),

$$\frac{\bar{\theta}\theta\phi^{[0]}(t)\phi^{[1]}(t)}{\bar{\theta}\phi^{[0]}(t) + \theta\phi^{[1]}(t)} \leq \frac{\sqrt{\bar{\theta}\theta}}{2\eta}\exp(-\frac{1}{2}([t]_\Lambda^2 + [\delta_u]_\Lambda^2)).$$

Observe that $\int_{-\infty}^{\infty}\frac{1}{\eta}\exp(-\frac{1}{2}[t]_{\Lambda}^2) = 1$, we have $\mathcal{L}(\theta) \leq \frac{\sqrt{\theta\bar{\theta}}}{2}\exp(-\frac{1}{2}[\delta_u]_{\Lambda}^2)$, which proves part (v). ■

Proof of Lemma 7: According to Definition 1, it is well known [32] that the desired optimal estimate

$$\hat{x}_k = \mathbb{E}[x_k|\mathcal{I}_k] = \int_{-\infty}^{\infty} x_k p(x_k|\mathcal{I}_k)\mathrm{d}x_k$$
$$= \int_{-\infty}^{\infty} x_k \sum_{i=0}^{1} p(x_k|\theta_k^{[i]},\mathcal{I}_k)p(\theta_k^{[i]}|\mathcal{I}_k)\mathrm{d}x_k = \sum_{i=0}^{1}\hat{x}_k^{[i]}\lambda_k^{[i]}$$

where $\hat{x}_k^{[i]}$ and $\lambda_k^{[i]}$ are defined in Section II. The standard assumption adopted in the IMM estimator is that $p(x_{k-1}|\theta_k^{[i]},\mathcal{I}_{k-1})$ is a Gaussian pdf. Under this assumption, $\hat{x}_k^{[i]}$ and $\lambda_k^{[i]}$ can be calculated via Kalman filter as obtained in Algorithm 1 by the standard IMM estimator, and thus the standard IMM estimator yields $\hat{x}_k$ that minimizes $P_k = \mathbb{E}[(x_k-\hat{x}_k)_I^2|\mathcal{I}_k]$. In other words, the modified IMM estimator provides another estimate, denoted by $\hat{x}_k^s$, whose estimation EC $P_k^{sub} = \mathbb{E}[(x_k-\hat{x}_k^s)_I^2|\mathcal{I}_k] = \mathbb{E}[(x_k-\hat{x}_k+\hat{x}_k-\hat{x}_k^s)_I^2|\mathcal{I}_k] = \mathbb{E}[(x_k-\hat{x}_k)_I^2+(\hat{x}_k-\hat{x}_k^s)_I^2+2(x_k-\hat{x}_k)(\hat{x}_k-\hat{x}_k^s)|\mathcal{I}_k]$. Since $\hat{x}_k = \mathbb{E}[x_k|\mathcal{I}_k]$, $P_k^{sub} = P_k + \mathbb{E}[(\hat{x}_k-\hat{x}_k^s)_I^2|\mathcal{I}_k] \succeq P_k$, which proves this lemma. ■

Proof of Lemma 8*:* The recursive equation for the prediction error covariance $\bar{P}_k^L$ for UDP-like systems has been obtained in [15] and is given in Algorithm 2. That is,

$$\bar{P}_{k+1}^L = \bar{g}(\bar{P}_k^L,\gamma_k,Q+\theta\bar{\theta}Bu_{k+1}u_{k+1}'B',R).$$

Define $Q_u \triangleq Q+\theta\bar{\theta}B\bar{U}B'$ where $\bar{U}$ is an upper bound of u_ku_k', and then construct a matrix sequence

$$\bar{M}_{k+1}^L = \bar{g}(\bar{M}_k^L,\gamma_k,Q_u,R), \quad \text{with } \bar{M}_0^L = \bar{P}_0^L. \tag{36}$$

By using the mathematical induction method and following the same line of arguments as those in the proof of Lemma 4, it is easy to prove that $\bar{P}_k^L \preceq \bar{M}_k^L$. $\bar{M}_{k+1}^L = \bar{g}(\bar{M}_k^L,\gamma_k,Q_u,R)$ is a modified Riccati equation, whose stability has been established in [6]. It is shown in [6] the stability of $\bar{M}_k^L$ is determined by a critical value denoted by γ_M. By comparing $\bar{M}_k^L$ with the prediction error covariance $\underline{M}_k$ for TCP-like systems in (16), the modified Riccati equation $g(\underline{M}_k,\gamma_k)$ in (16) and $\bar{g}(\bar{M}_k^L,\gamma_k,Q_u,R)$ in (36) have the same A and C. By Lemma 3(iv)(v) and Theorem 1(ii), we have $\gamma_M = \gamma_c$, and $\mathbb{E}[\bar{M}_k^L] \preceq d$ for some constant d if $\gamma > \gamma_c$, where d depends on the initial value $\bar{P}_0$. Therefore, when $\gamma > \gamma_c$, $\mathbb{E}[\bar{P}_k^L] \preceq \mathbb{E}[\bar{M}_k^L] \preceq d$. From Algorithm 2, $P_k^L = \bar{P}_k^L - \gamma_k K_k^L C\bar{P}_k^L \preceq \bar{P}_k^L$. Therefore, $\mathbb{E}[P_k^L] \preceq \mathbb{E}[\bar{P}_k^L] \preceq d$. The proof is completed. ■

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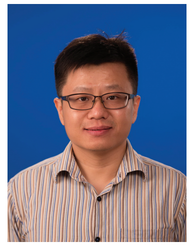

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