UNIVERSITY OF SOUTHAMPTON

FACULTY OF SOCIAL, HUMAN, AND MATHEMATICAL SCIENCE

Mathematical Sciences

Optimal Design of Experiments with Mixtures

by

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ABSTRACT

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Experiments involving mixtures are conducted in a variety of different areas, for example in food processing, and agricultural and chemical industrial research. The discussion and the statistical properties of such experiments were introduced in the mid fifties. In such experiments, the responses (the properties of interest) are functionally related to the proportions of the components of the mixtures and by changing these proportions the product will change as a result. This present work is concerned with the modelling and designing of such experiments when there are maximum and minimum values on the mixture components. For the modelling, fitting a models to mixture experiments is not always straightforward due to the restrictions on the proportion components, that is they must add to unity. We have contributed to mixture experiments by proposing a new class of nonlinear models, exploiting similarities of our data with compositional data, where these models outperform existing models under criteria for model comparisons. When the models have been chosen, to associate them with representation of the response surface over the region under consideration, it is necessary to make a choice of suitable designs for such regions. As the main concern of this research is the constrained mixture components, the designs under these constraints are referred to as constrained mixture designs. A set of candidate points for such restricted regions have been generated using the XVERT algorithm as illustrative in Chapter 4. However, as the proportions of the components increase, the number of design points increases considerably. To select or reduce the obtained design points in order to create mixture designs with a reasonable number of such points, the method of design optimality will be used. Different exact and near optimal designs have been constructed with respect to either standard or newly developed criteria under consideration. Besides, in Chapter 7, the comparisons between the obtained designs in terms of their efficiencies are also evaluated and demonstrated with several examples. All of the criteria and the experimental work settings have been presented throughout this research with several illustrative examples to find the characteristics and the performances of the obtained optimal designs.

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Declaration of Authorship

I, Rana Khashab, declare that the thesis entitled *"Optimal Design of Experiments with Mixtures"* and the work presented in the thesis are both my own, and have been generated by me as the result of my own original research. I confirm that:

- this work was done wholly or mainly while in candidature for a research degree at this University;
- where any part of this thesis has previously been submitted for a degree or any other qualification at this University or any other institution, this has been clearly stated;
- where I have consulted the published work of others, this is always clearly attributed;
- where I have quoted from the work of others, the source is always given. With the exception of such quotations, this thesis is entirely my own work;
- I have acknowledged all main sources of help;
- where the thesis is based on work done by myself jointly with others, I have made clear exactly what was done by others and what I have contributed myself;
- none of this work has been published before submission.

Signed:

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Chapter 1

Introduction

The design of an experiment is an essential part of the statistical methodology to improve the scientific experimentation and thus to increase the validity of the resulting conclusions. Experimental designs are required to make inferences about and carry out an analysis of the factors which affect the properties of interest (outputs) that are collected through planned experiments and to determine the relationship between them. The knowledge of the planning and interpreting the experiments is needed to decide what conditions (inputs) are to be varied (treatment factors) and what outputs (responses) are to be measured to manage the factor inputs in order to optimize the output using a statistical optimality criterion. Once the experimenter decides the treatments that are to be varied and the response that is to be observed and measured, then the experimenter hypothesizes that any measured responses are functionally related to the levels or values of factors. Such relationships can be represented analytically as

$$\text{Responses} = f(\text{the levels of factors}),$$

where $f(\text{the levels of factors})$ is assumed to be some true and unknown form of mathematical function of the factor levels.

In an experiment, such factors are also called independent variables, or explanatory variables, and each of them has two or more levels, i.e. different values of the factors. However, in some experiments, the factors under study are proportions of different components of a product or a measured yield. In these cases, the proportions of the components could be thought of as factor levels. For example, in many processes involving food processing, or in chemical industries, the product is formed by mixing several components or ingredients together in certain proportions, as is the case in:

- Animal husbandry: Purified diets consisting of mixtures of different energy supplements, such as protein, fat, and carbohydrate, are fed to several chicks, where all of these chicks initially have uniform weight and size (Cornell, 1990).
- Fruit punch: The juice may be based on a mixture of different components, watermelon, pineapple and orange for example (Cornell, 1990).
- Building construction works: The concrete is formed by mixing cement, sand, and water in certain proportions (Cornell, 1990).

In each of these cases, the main concern or properties of interest is to find the optimum proportions of the ingredients in order to maximize the product performance or to achieve the best quality of a product or the best results of a measured yield. For example, the best results or qualities of interest to the experimenter who is responsible for mixing the ingredients could be

- The weight gain for different chicks, where the weight gain is related to the components of the energy supplements.
- The fruity flavour of the punch, where this depends on the percentage of the proportions of the ingredients that are present in the punch.
- The hardness of the concrete, where it is a function of the percentages of cement, sand and water in the mixture.

Experiments with mixtures are effective tools in determining such optimum proportions, in which such experiments frequently investigate the effects of changing the proportions of different components in the blend. In the literature, the discussion on the field of mixture experiments which introduced the properties and main features of such experiments from a statistical point of view, appeared in the mid-1950s. Much of this discussion of the experimental approach involving mixtures can be found in the pioneering article by Scheffé (1958).

In such experiments and from the above cases, it is assumed that the measured characteristic of the final product, or the properties of interest, are functionally related to the proportions of the ingredients used in the product, and that, by changing the components or the ingredients'proportions, the product will change, which can be written as

$$\text{Responses} = f(\text{the proportions of the component}),$$

where the proportions of the component may be expressed in units of proportions by weight or volume. The total volume/weight of the mixture components is normally fixed in such an experiment and thus the proportions sum to 1 (or 100%). Therefore, each proportion x_r of a component can take values from zero to unity and all blends among the ingredients are possible. Thus, it is worth to mention that, the main difference between mixture experiments and other experiments, such as factorial experiments, is the selection of the factor-level combinations of those factors, whose levels are varied during the experimentation. For example, in experiments involving mixtures the total amount must be fixed, and the relative proportions of components varied. However, in the case of factorial designs, when the combinations of the levels of the factors are considered, the total amounts are varied, as well as the proportions. Therefore, the total amount is not necessarily fixed. Moreover, in factorial designs, one can change the levels of any number of factors without affecting the levels of the others. In mixture experiments, however, any change in the level or proportion of one component, will affect the proportions of one or more of the remaining components.

In the experimental mechanism studied, the main purpose of conducting experiments involving mixtures is to explore the relationship between the measured property and the components'values, to determine whether some combinations of the ingredients can be considered better in some sense than the other combinations and to model the blending surface to express the expected response as a function of the proportions of the mixture components.

There are numerous models for such experiments in the literature. For example, Scheffé (1958, 1963), who is considered a pioneer in analysing and modelling such experiments, proposed the canonical polynomial models for fitting the data collected from mixture experiments. Some researchers, such as Quenouille (1959, 1963), suggested that Scheffé's models are unqualified for describing the effect of common linear blending, where a component blends linearly when the effect of increasing its presence in the mixture with all other components keeping in fixed relative proportions to each other. Becker (1968, 1978) addressed this issue by proposing several forms of statistical models which can represent the additivity effect of one or more components on any mixture of the remaining components, and proposed a development of such models by presenting terms that can describe a wider range of linear blending effects than the models previously suggested.

However, some researchers have developed Scheffé's models by extending the terms in such models. For instance, Draper and St. John (1977) proposed the use of the inverse term for polynomial models, while Chen et al. (1985) suggested logarithmic terms for such models.

Cox (1971) suggested several models to tackle the obstacles which occur from the absence of the squared terms in Scheffé's quadratic polynomial models that make the direction and magnitude of the curvature of the response meaningless.

Moreover, some transformations in component proportions can be handled in a variety of ways. Cornell (1990) described the ratio models for such experiments, when the ratio of the component proportions are more meaningful than the proportions themselves. Cornell (1973, 1990), presented the most frequently used statistical techniques for modelling and analyzing mixture data, together with appropriate computing formulae and an illustrative example of each method.

Models with nonlinear parameters have also been applied to data from mixture experiments. For example, Focke et al. (2007) considered multi-component mixtures and assumed that only pairwise interactions between them need to be considered. Brown et al. (2015), propose a new class of models called general blending models, which integrate the models proposed by Scheffé (1958, 1963) and Becker (1968, 1978). However, their models have a large number of parameters, and it is difficult to obtain any meaningful understanding and interpretation of the experimental mechanism of the response. To tackle these complications a new class of model is proposed in this thesis for analyzing mixture experiments, based on fractional polynomial models. These models can capture the structure of the data, give a better interpretation of the experimental mechanism, and provide a better fit to the data than existing models, as will be seen in Chapter 3.

Once a decision has been made about the model, the next step is to develop suitable designs to support such models. Designing in mixture settings depends on the shape of the design region, which is defined by the constraints that are imposed on the component proportions by the experimenters. Since the proportions of components in a mixture must add to unity, the experimental region for such experiments is constrained, and will reduce to a regular (q-1)-dimensional simplex. These constraints are known as natural constraints. The traditional factorial and response surface designs are not appropriate in such cases. Thus, under such natural constraints, there are standard mixture designs for fitting standard canonical polynomial models proposed by Scheffé (1958, 1963), which are known as Simplex-Lattice designs and Simplex-Centroid designs. However, in many situations, different designs other than the standard mixture designs may be more useful to support a model in such experiments, in particular when each component in the mixture can only be varied with specific lower and upper bound constraints. Such designs are referred to as constrained mixture designs. These designs will be explained in more details in Chapter 2. In this case, once the constraints on the proportion of each component is considered, all component proportions are included in the mixture and none of these components will be 100% or 0% unlike the natural

constraints. Consequently, the restrictions on proportions of components will be of the form: $0 < L_r \leq x_r \leq U_r < 1, r = 1, 2, \cdots, q$, and the experimental region will reduce the size of the factor space that can be represented as an irregular hyperpolyhedron rather than a simplex. For such resulting regions, all component proportions that make up any mixture must be included and represented with non-zero proportions. Moreover, their position may be in some area in the middle of the simplex rather than in the vertices of the triangle where only one ingredient is represented. These additional constraints are usually imposed if the measured yield in some areas of the experimental region could not provide enough information to the experimenters, as in for example the case in the food industry, when a fruit punch juice contains either 0% or 100% pineapple. In this case such juice could not provide useful information to experimenters. Thus, the resulting constrained region with additional constraints is the area of interest in this work, as will be explained in more detail in Chapter 4.

Under statistical models that have been chosen and supported by the constrained mixture designs, the experimenter might have different aims regarding the statistical analysis for such data, where many of these objectives can be reflected in optimality criterion functions. The present research will focus on the use of such statistical functions for solving optimization problems for the region under consideration.

1.1 Research Framework

As mentioned above, the experimental design region for experiments with mixtures is defined by the allowable proportions of components that contribute to the mixture. An effective tool for illustrating the region of interest with a small number of components is a simple plot that can be used to achieve a desirable purpose. However, such a tool is not sufficient to deal with some experimental issues. For example, if the region of interest is constrained by four or five component proportions, then it is not easy to plot such a region. Since it is necessary to find the possible design points for such a restricted area, the XVERT algorithm can be used to achieve this. Such an algorithm helps to provide desirable points which can give precise estimates of the model and are sufficiently spread out over the region of interest. Since the number of the design points increases when the number of components is increased, it is necessary to select a subset of the desirable points which can be obtained in some situations using a method based on design optimality to avoid using clusters of design points in the factor space as will be seen in Chapter 5 and Chapter 6.

The main purpose of the experimental work is to make statistically precise estimates of the parameters in the statistical models, i.e. to optimize the quality of the model

parameter estimates that might be facilitated by the optimum experimental planning. Such an objective can be expressed as maximizing the precision that depends on the model matrix and error parameter estimates. This is the most common objective in relation to statistics, since the obtained estimates of the parameters will explain the quantified relationship between the response variables and the covariates.

Once the observations have been collected, nonlinear least squares (NLS) could be used to obtain an estimate of the treatment parameters in the case of fitting models with nonlinear parameters. However, in linear models, there is a specific equation which can be used to determine the single unique solution that results in the smallest error sum of squares for the model and data.

In the estimation of the treatment parameters, it is necessary to fit the assumed statistical models to the points, as the development and construction of suitable designs for the region under consideration usually depends on such models. Such designs can be obtained to locate the optimum treatment combinations. To this end, different statistical criteria will be used, either standard or newly developed, as will be seen through the discussion in Chapter 2, Section 2.4, which are referred to as the alphabetical criteria of optimality. Most of these criteria are related to the information matrix $\boldsymbol{X'X}$ which does not depend on the unknown model parameters $\boldsymbol{\beta}$ for linear models, or equivalently, the variance-covariance matrix $(\boldsymbol{X'X})^{-1}$, where $\boldsymbol{X}$ is the design matrix (or model matrix). Some of these criteria have statistically appealing properties such that they minimize the volume of the confidence region for $\boldsymbol{\beta}$, which is proportional to the determinant of the variance-covariance matrix of the parameter estimators $\hat{\boldsymbol{\beta}}$.

Unlike linear models, if the assumed models are nonlinear in their parameters then such a matrix will depend on the unknown values of the parameters. Therefore, in this case, initial values for the model parameters are needed to find the estimation for such parameters, then using the resulting estimated values as prior values for the parameters to construct the design matrix $\boldsymbol{F}$ and thus the information matrix $\boldsymbol{F'F}$.

When a relevant criterion to construct an optimal design of experiments has been chosen, sometimes the number of runs N will not be specified and we may be interested in finding a design that is independent of the sample size. This can be achieved through continuous optimal designs or approximate designs, which are usually expressed as probability measures. Such designs are considered as fundamental tools in the construction of optimal designs by choosing the design points $\boldsymbol{x_r}$ from the design space $\boldsymbol{\chi}$ with respect to the criterion under consideration. However, instead of finding the corresponding (integer-valued) optimal replications for each support point, which would result in a so-called exact design, we seek the proportions of replications or weights, which must sum to

unity. The independence of the continuous designs from the sample size is then achieved by not imposing any further constraint on the weights. To fix ideas, suppose there are q distinct support points with allocated weights to each point, then the continuous experimental design can be written as

$$\xi_1 = \left\{ \begin{array}{cccc} \boldsymbol{x_1} & \boldsymbol{x_2} & \cdots & \boldsymbol{x_q} \\ w_1 & w_2 & \cdots & w_q \end{array} \right\}, \tag{1.1}$$

where the first row represents the coordinates of each distinct point, while the second row shows the weights which are all positive real numbers such that $\sum_{r=1}^{q} w_r = 1$. However, once a specific sample size n has been selected by the experimenter, we need to round the weights in order to obtain the integer-valued replication. Thus, the above design becomes an exact experimental design, which can be represented as

$$\xi_2 = \left\{ \begin{array}{cccc} \boldsymbol{x_1} & \boldsymbol{x_2} & \cdots & \boldsymbol{x_q} \\ n_1 & n_2 & \cdots & n_q \end{array} \right\}, \tag{1.2}$$

where $n_1, n_2, \cdots, n_q$ are represent the number of replicates for the respective q support points with $n_q = nw_q$, and again they must be positive. Once the optimum solution is evaluated, then the allocated weights for each support point will be rounded to determine the exact number of replicates.

Approximate weight measures are useful in design theory, as they allow us to check the optimality of a design through the General Equivalence Theorem. However, exact optimal designs, are more applicable in a practical manner. It will be useful to make an effort to search for tailor-made exact optimal solutions under such designs.

In the case of analyzing the experimental data obtained from resulting designs after conducting an experiment, the experimenter must consider what kind of data analysis could be used to meet the objective of the statistical inferences under consideration and what the possible optimality criterion may be chosen to match that data analysis. Usually, the experimenter has more than one purpose in mind when carrying out such analysis. Thus, the experimenter can implement more than one candidate statistical criterion to the same experimental data. Then the optimum designs under different specified statistical models can be found and constructed for the design regions. To find such interesting designs it is useful to construct compound criterion functions, which is a compromise between two or more objectives while searching for designs to be more applicable in many practical situations. The notion and methodology for compound criteria were described by Atkinson et al. (2007) as a non-negatively weighted linear

combination of different design criteria. Thus, such methodology begins by defining the efficiencies over different criteria with appropriate weights in order to use them to generate compound designs. Such criteria, that need to be minimized, are weighted products of the elementary criteria, where the weights are generally arbitrary and based on the experimenter's belief.

Since our aim is to locate the optimum treatment combinations over the constrained area, the implementation of some of these statistical criteria in the exchange algorithm can be used for optimization purposes. The process for such an algorithm starts with a random initial design that is selected from the achievable points candidates sets obtained from the XVERT algorithm or otherwise. Then any exchange between the current design and the candidate list is made, and is accepted if it improves the criterion. The relevant full procedures with examples are shown in Chapter 5 and Chapter 6.

1.2 Objective and Organization of the Work

The main goals of this work are:

- Propose a new class of models to fit data from mixture experiments with restricted design regions.
- Demonstrate that this new class of models provides a good fit for many situations and compare the performance of the new models with other statistical models from the literature.
- Finding the design points for mixture experiments when the region of the interest is subject to additional constraints to the natural ones.
- Construct exact and continuous optimal designs for such experiments, using different statistical criteria and algorithms, then compare their properties.

The structure of the rest of thesis is as follows. A brief overview and outline of the main features of mixture experiments is given in Chapter 2. The new class of statistical models for such experiments is introduced in Chapter 3. The method for the XVERT Algorithm for finding design points from our restricted region which for some statistical models gives an estimates of the model parameters will be shown in Chapter 4 with some illustrative examples. Following this, we construct optimal designs for standard canonical polynomials models and our new class of models for such cases and illustrate our methods with some real examples in Chapter 5, and Chapter 6 respectively. Some continuous optimal designs that have been obtained for such experiments using other

algorithms are introduced in Chapter 7. In the final Chapter, we will draw our main conclusions and suggest some future work.

Most of the computation and calculation have been completed in the R environment on a Dell personal laptop with a 2.40 GHz Intel (R) Core (TM) i5-2430M Processor and 6 GB RAM. However, for the experimental work for constructing the determinant function in continuous designs, the Maple software on a Dell personal laptop with a 1.80 GHz Intel (R) Core (TM) CPU 2117U Processor and 2 GB RAM is used. Besides, based on academic writing style, the words "we", "us", and "our" will be used instead "I" and "me" and "my".

Chapter 2

The Main Concepts and Features for Mixture Settings

2.1 The Main Properties of Experiments Involving Mixtures

An experiment with mixtures occurs when the property of interest (response) is a function of the proportion of each component in a mixture rather than a function of the total amount of such components. Thus, the property studied, or the measured response, is assumed to depend on the proportions of the components present and not on the amount of the mixture. Moreover, the individual components which make up the mixture settings are proportions of such amounts of the mixture. Clearly, in such experiments the components making up any mixture or blend are represented as proportional amounts of the mixture by volume, weight, or fraction. Therefore, these proportions are non-negative and, if expressed as fractions of the mixture, they must add up to unity. That is, if q denotes the number of mixture components and x_r is the proportion of the r^{th} component, $r = 1, \cdots, q$, then

$$\sum_{r=1}^{q} x_r = 1 \tag{2.1}$$

and

$$0 \leq x_r \leq 1, \qquad r = 1, \cdots, q. \tag{2.2}$$

The q ingredients are also called mixture variables. These properties as mentioned before, are referred to in the literature as the natural constraints on the proportions of the mixture components.

Indeed, an individual proportion x_r could be unity, and a mixture could be a single ingredient. Such a mixture in the literature is called a pure blend or pure mixture.

The main concern that needs to be addressed is to define the experimental region within which the experimenter is going to carry out an experiment. Natural constraints will be used to determine such regions in such experiments, where the experimental region under these constraints in these experiments is known as the mixture simplex.

2.1.1 Geometrical Shapes of Mixture Regions

The feasible region of the q components that contributes to make up any blend and satisfying (2.1) and (2.2) contains the points on or inside the boundaries of a regular simplex of dimension (q-1). Some of these points can be defined as

- The q vertices of the simplex. These are obtained by putting $x_r = 1$, $r = 1, 2, \cdots, q$ and making each of the remaining proportions equal to zero, i.e. $x_s = 0$, $s = 1, 2, \cdots, q$, $s \neq r$. These q points represent the pure blends.
- The boundary points of the simplex. These are obtained by putting $x_t = 0$ for $t = 1, 2, \cdots, q$, $t \neq r$, $t \neq s$ and x_r, x_s having equal or varying values for some $r, s \in \{1, 2, \cdots, q\}$.

$\vdots$

- The internal points of the simplex. These points are obtained by putting $x_r = \frac{1}{h}$, $h \in \mathbb{N}$ for $r \in \{1, 2, \cdots, q\}$ and the remaining proportions x_s, x_t having varying or equal values for some $s, t = 1, 2, \cdots, q$, $s \neq r$, $t \neq r$.
- The centroid point of the simplex. This is the point where $x_r = \frac{1}{q}$ for each $r = 1, 2, \cdots, q$. This point represents equal proportions of volume fractions, for all the q components.

To illustrate the experimental regions, let us start by applying the natural constraints to define the 2-simplex, that is, the simplex for q= 2. The feasible region for two components is a straight line of length 1, and the extreme points are $x_r = 1$ and $x_s = 0$ for each $s \neq r$ where $r, s = 1, 2$, as can be seen in Figure 2.1. In this case, it is easy to plot 2-components to display the factor space. Here to plot such space, we defined our experimental factor space in terms of a set of the axes x and y as in Figure 2.1. Along the horizontal x_1 axis, the point in the bottom right corner can be viewed as a "mixture", where the values of x_1 and x_2 are 1 and 0 respectively. Likewise, along the vertical x_2, the point which is positioned in the top left corner could be viewed as a "mixture", in which the values of the x_1 and x_2 are 0 and 1 respectively. Moreover, if a line have been passed through these two points with considering no restrictions on the length of such

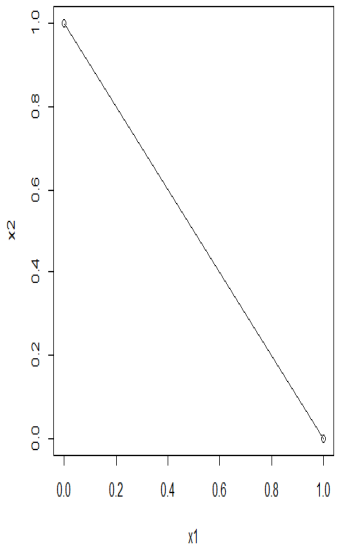

FIGURE 2.1: Two Components Line

line and applying the nonnegatively constraints, then anywhere along the line that joins these two points it will be true that $x_1 + x_2 = 1$. Thus, the two constraints will reduce the factor space to a one-dimensional simplex.

Moving to the 3-simplex and 4-simplex, that is the simplex for $q = 3$ and $q = 4$ components, the feasible regions are an equilateral triangle and tetrahedron respectively, as can be shown in Figure 2.2 and Figure 2.3 respectively. From Figure 2.2 we can see that

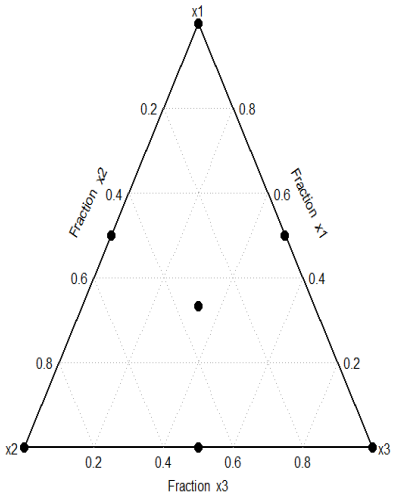

FIGURE 2.2: Three Components Triangle

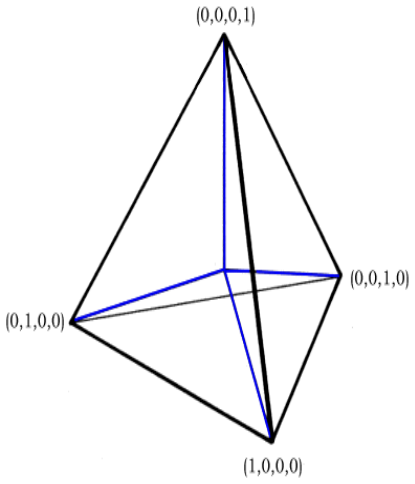

FIGURE 2.3: Four Components Tetrahedron

the vertices of the simplex or triangle represent the single-component mixtures and are denoted by $x_r = 1$ and $x_s = 0$ for $r, s = 1, 2$, and 3, $s \neq r$. Each such vertex of a simplex represents a pure component. In addition, the midpoints of the edges corresponding to the points that have been positioned on the opposite of $q - 2$ sub simplex when $x_r = 0$ and the renaming proportions having equal values. The centroid of the triangle corresponds to the mixture with equal proportions $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ for each of the components. Moreover, Figure 2.3 represents the four components whose proportions are x_1, x_2, x_3, and x_4.

It is worth highlighting the coordinates of the simplex system. Let us consider the simplex with 3-components. Each vertex represents a pure blend; such a pure blend may be indicated mathematically by 100%. In addition, 0% of a component is located on the opposite q-2 = 1-dimensional sub-simplex. For example, in Figure 2.2, 100% of the component x_1 is located at the top vertex, while, 0% of this component is located on the bottom edge of the triangle. Moreover, the proportions of the component x_1 are allocated along the right side of the triangle. Similarly, the proportions of the component x_2 are indicated along the left side of the triangle and correspond to the line sloping downward to the right, whereas, the proportions of the component x_3 correspond to the line sloping upward to the right, which are indicated along the bottom of the triangle. This kind of simplex coordinate system is often used in mixture settings.

2.2 Response Surface Methodology

In experiments involving mixtures a researcher focuses on studying the physical characteristics of the measured response surface, such as the shape or the height over the experimental region. The main motivation for studying and exploring the response surface over the region of interest is to be able to assume a proper statistical model to approximate the surface over the experimental region and test the adequacy of the model proposed to represent the response surface. Moreover, a suitable design for collecting data sets is specified, the expected optimal response is identified, the scientific interpretation of the response variation for the mechanism studied is introduced, and the adequacy of fit of the specified model is tested and checked.

We shall assume that the response is denoted by y_i for the ith run, $i = 1, 2, \cdots, n$ and varies about a mean of η_i with a common variance σ^2 for all $i = 1, 2, \cdots, n$. Then the response can be represented as

$$y_i = \eta_i + \epsilon_i, \qquad i = 1, 2, \cdots, n, \tag{2.3}$$

where the experimental errors ϵ_i are assumed to be identically distributed with the mean zero and variance σ^2. Thus the expected value of the response y_i is

$$E(y_i) = \eta_i. \tag{2.4}$$

Note that the ith observed value will also be denoted by y_i in what follows. In each experimental run, we will determine a set of the respective levels of the q continuous factors in order to approximate the functional relationship with statistical models, which is referred to as the experimental design. Generally, when finding the relationship between the response and the continuous factors, in the present case the ingredients that are present in the formulation, theoretically, the statistical functional relationship can be represented as

$$y_i = f(\boldsymbol{x_i}, \boldsymbol{\beta}) + \epsilon_i, \tag{2.5}$$

where f is assumed to be a mathematically continuous function in $\boldsymbol{x}$ and $\boldsymbol{\beta}$, which is the vector of treatment parameters that need to be estimated.

Again, the assumption for normally distributed experimental errors with mean zero will be hold. The commonly used distribution for this purpose is the normal distribution since we consider the error term as the composite of a number of minor influences due to different factors or measurement errors. As the number of these minor influences gets larger, the distribution of the error term tends to approach the normal distribution based on the Central Limit Theorem. Moreover, some methods for obtaining estimation of the

model parameters can also imply this assumption of normally distributed experimental errors, for instance, maximum likelihood estimation. Such methods uses the distribution of the experimental errors directly to obtain the model parameter estimates.

In the literature different ways are illustrated to find the estimate of the experimental error variance. For example, Draper and Smith (1998) provide the technique for such estimation, which depends on the residual mean squares, in the sense that, the residual sum of squares is divided by the corresponding number of degrees of freedom i.e. $\hat{\sigma}^2 = \frac{RSS}{(n-p)}$. Another way for the error variance estimation to be obtained is from the model-independent pure error, that can be determined from the decomposition of the residual sum of squares into the pure error and lack-of-fit components. In the sense that, $\hat{\sigma}^2 = \frac{PESS}{(n-t)}$, where t is the number of unique values of the treatment applied. Many authors have advocated the use of the pure error estimate of error variance (σ^2) instead of the one pooled with the lack-of-fit under the fitted model.

With respect to the function $f(\boldsymbol{x}, \boldsymbol{\beta})$, usually the structural form of such function is unknown, and mathematically might be approximated by a polynomial or another form of model equation. Considering the general form of the equation of an m^{th}-degree polynomial model, where for brevity of notation the η_i shall be written as η, has the general form

$$\eta = \beta_0 + \sum_{r=1}^{q} \beta_r x_r + \sum_{r=1}^{q}\sum_{s=1}^{q} \beta_{rs} x_r x_s + \sum_{r=1}^{q}\sum_{s=1}^{q}\sum_{t=1}^{q} \beta_{rst} x_r x_s x_t + \cdots, \tag{2.6}$$

where $r \leqslant s$, $r \leqslant s \leqslant t$. Then, for example, if the first degree polynomial is considered to approximate the response function, we can derive the general form of such a model by assuming $m = 1$. Then

$$\eta = \beta_0 + \sum_{r=1}^{q} \beta_r x_r. \tag{2.7}$$

Similarly, if the second degree polynomial model is considered to approximate the response, the general form of the model can be obtained by assuming $m = 2$ giving

$$\eta = \beta_0 + \sum_{r=1}^{q} \beta_r x_r + \sum_{r=1}^{q} \beta_{rr} {x_r}^2 + \sum_{r=1}^{q}\sum_{r<s}^{q} \beta_{rs} x_r x_s, \tag{2.8}$$

and so on. The above models are known as the first and second order response surface models.

Scheffé (1958), proposed the use of symmetric canonical polynomial models that were obtained from the standard models by incorporating the natural constraints in mixture components given in (2.1) and (2.2) in the standard polynomial models. From (2.7), by multiplying the β_0 term by $(x_1 + x_2 + \cdots + x_q) = 1$ and simplifying the function, the

resulting equation is

$$\eta = \sum_{r=1}^{q} \beta_r^* x_r, \tag{2.9}$$

where $\beta_r^* = \beta_r + \beta_0$, for all $r = 1, 2, \cdots, q$.

The parameter β_r^* represents the expected response to rth pure component and can be depicted as the height of the surface above the simplex at the vertex $x_r = 1$, $r = 1, 2, \cdots q$. The model obtained is known in the literature of experiments with mixtures as Scheffé's first order polynomial model; and the number of terms in (2.9) is q. Following the same methodology with the general second degree polynomial model represented in (2.8), the resulting model that we obtain after applying the identities $(x_1 + x_2 + \cdots + x_q) = 1$ and

$$x_r{}^2 = x_r \left(1 - \sum_{s=1, r \neq s}^{q} x_s \right), \tag{2.10}$$

and simplifying the model, the obtained equation is

$$\eta = \sum_{r=1}^{q} \beta_r^{**} x_r + \sum_{r=1}^{q-1} \sum_{s=r+1}^{q} \beta_{rs}^* x_r x_s, \tag{2.11}$$

where $\beta_r^{**} = \beta_0 + \beta_r + \beta_{rr}$, $r = 1, 2, \cdots, q$, and $\beta_{rs}^* = \beta_{rs} - \beta_{rr} - \beta_{ss}$, $r = 1, 2, \cdots, q-1$, $s = r+1, \cdots, q$.

This is called Scheffé's quadratic polynomial model, where a term of the form $\beta_{rs}^* x_r x_s$ represents the effect of the binary blending between the components r and s. Cornell (1990) stated that, if the amount β_{rs}^* is positive, such term is called a synergism of the binary mixture; however, it is called antagonism of the binary blend when the quantity β_{rs}^* is negative.

After carrying out the same process for $m = 3$, by multiplying some terms by the identity and then simplifying the terms, the full cubic polynomial is

$$\eta = \sum_{r=1}^{q} \beta_r^{***} x_r + \sum_{r=1}^{q-1} \sum_{s=r+1}^{q} \beta_{rs}^{**} x_r x_s + \sum_{r=1}^{q-1} \sum_{s=r+1}^{q} \delta_{rs} x_r x_s (x_r - x_s) + \sum_{r=1}^{q-2} \sum_{s=r+1}^{q-1} \sum_{t=s+1}^{q} \beta_{rst}^* x_r x_s x_t, \tag{2.12}$$

where $\beta_r^{***} = \beta_0 + \beta_r + \beta_{rr} + \beta_{rrr}$, $\beta_{rs}^{**} = (2\beta_{rs} - 2\beta_{rr} - 2\beta_{ss} - 3\beta_{rrr} - 3\beta_{sss} + \beta_{rrs} + \beta_{rss})/2$, $\beta_{rst}^* = (2\beta_{rst} + 2\beta_{rrr} + 2\beta_{sss} + 2\beta_{ttt} - \beta_{rrs} - \beta_{rrt} - \beta_{sst} - \beta_{rss} - \beta_{rtt} - \beta_{stt})/2$, and δ_{rs} $= (\beta_{rrs} - \beta_{rrr} - \beta_{rss} + \beta_{sss})$.

As can be seen from the full cubic polynomial model, the binary mixture includes the additional term $\delta_{rs} x_r x_s (x_r - x_s)$, where δ_{rs} is the cubic-coefficient of the binary mixture. In such a term, if $\delta_{rs} \neq 0$, then the term can take on positive as well as negative values, which can be detected both synergism and antagonism respectively blending between

the components r and s. Moreover, the term $\beta^*_{rst}x_rx_sx_t$ represents the ternary mixture among the components r, s, and t in the interior of the simplex.

The need of a simpler model arises, because, as the number of the component proportions q increases, the number of terms in the full cubic model that must be supported by the design increases dramatically. A commonly used special case of the full cubic polynomial that can be obtained by ignoring the terms $\delta_{rs}x_rx_s(x_r - x_s)$. These kinds of models are called special cubic polynomials and will not model as much complex curvature in a response surface as the full cubic models. The general form of the special cubic model is

$$\eta = \sum_{r=1}^{q} \beta^{***}_r x_r + \sum_{r=1}^{q-1}\sum_{s=r+1}^{q} \beta^{**}_{rs}x_rx_s + \sum_{r=1}^{q-2}\sum_{s=r+1}^{q-1}\sum_{t=s+1}^{q} \beta^{*}_{rst}x_rx_sx_t. \tag{2.13}$$

In the literature, usually the above linear polynomial models are used to model the response surface of experiments with mixtures over the entire simplex factor space. However, in some situations nonlinear models have been also applied to such data and in some cases are needed for different purposes as will be seen in Chapter 3.

Once a decision has been made about the model, to associate such polynomial models with the representation of the response surface over the region of interest, it is necessary to make a choice of suitable designs so that observations can be collected.

2.3 Designs for Mixture Settings

This section describes different designs for experiments involving mixtures according to the shape of the design region (Thompson and Myers, 1968). The shape of such a region in mixture problems is defined by the constraints on the proportions of components. For example, as mentioned before, when the components are subject to the constraints given in (2.1) and (2.2), there are standard mixture designs for fitting standard models known as Simplex-Lattice designs and Simplex-Centroid designs. On the other hand, when the mixture components are subject to additional constraints, such as a maximum and a minimum value for each component, the corresponding designs are referred to as constrained mixture designs. These designs can be described as follows.

2.3.1 Simplex-Lattice Design

This design was introduced by Scheffé (1958). In these designs, Scheffé assumed the experimental runs be placed on a uniform lattice of points to let the experimenter explore the response surface throughout the entire design simplex. To clarify this suggestion, suppose that an m-degree polynomial model is to fit the response surface of blends consisting of different q components with proportions x_r, $r = 1, 2, \cdots, q$. Then the corresponding Simplex-Lattice design can be constructed by allocating the values $0, \frac{1}{m}, \frac{2}{m}, \cdots, 1$ to each proportion x_r, where each proportion take the $(m+1)$ equally spaced values and the Simplex-Lattice design consists of all the resulting possible combinations of the above proportions. This design is denoted by the $\{q, m\}$ Simplex-Lattice design. Scheffé (1958) showed that the number of design points for the $\{q, m\}$ Simplex-Lattice design is $\binom{q+m-1}{m}$. For example, a $\{3, 2\}$ Simplex-Lattice design will consist of $\binom{3+2-1}{2}$, i.e. $\binom{4}{2} = 6$ points. Each x_r can take the value $m + 1 = 3$, where $m = 2$ in this example, then the possible values of x_r are $0, \frac{1}{2}, 1$, for $r = 1, 2, 3$ and the design points are $(1, 0, 0), (0, 1, 0), (0, 0, 1), (\frac{1}{2}, \frac{1}{2}, 0), (0, \frac{1}{2}, \frac{1}{2}), (\frac{1}{2}, 0, \frac{1}{2})$. The points, which are defined as $(1, 0, 0), (0, 1, 0), (0, 0, 1)$ represent single-component mixtures, and these points are the vertices of the triangle. The other points $(\frac{1}{2}, \frac{1}{2}, 0), (0, \frac{1}{2}, \frac{1}{2})(\frac{1}{2}, 0, \frac{1}{2})$ represent the binary blends for which the non-zero components' proportions are equal. These points are located at the midpoints of the edges of the triangle. The simplex-shaped design region for the $\{3, 2\}$ simplex-lattice is shown in Figure 2.4.

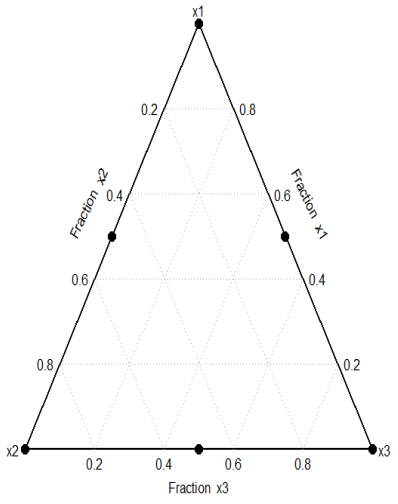

FIGURE 2.4: Simplex-Lattice Design for Three Components

2.3.2 Simplex-Centroid Design

This design was proposed by Scheffé (1963), in which runs are made using mixture experiments with q components that have equal proportions of every non-empty subset of the components. It consists of $\binom{q}{1}$ permutations of $(1, 0, 0, \cdots, 0)$, i.e. q pure blends, $\binom{q}{2}$ permutations of $(\frac{1}{2}, \frac{1}{2}, 0, \cdots, 0)$, i.e. $\binom{q}{2}$ binary blends, $\binom{q}{3}$ permutations of $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, 0, \cdots, 0)$, i.e. $\binom{q}{3}$ ternary blends,$\cdots$, $\binom{q}{q-1}$ permutations of $(\frac{1}{q-1}, \frac{1}{q-1}, \cdots, \frac{1}{q-1}, 0)$ and the overall centroid $(\frac{1}{q}, \frac{1}{q}, \cdots, \frac{1}{q})$, i.e. the q-nary blends. It can be seen that the number of points in this design is equal to $2^q - 1$. Geometrically, the Simplex-Centroid points are q vertices, $\binom{q}{2}$ centroid points in a 1-dimensional simplex, $\binom{q}{3}$ centroid points in a 2-dimensional simplices, $\cdots$, $\binom{q}{q}$ centroid point in a (q-1)-dimensional simplex. For instance, when $q = 3$, there are $2^3 - 1$=7 points. These points are the three vertices $(1, 0, 0), (0, 1, 0), (0, 0, 1)$, 3 centroid points in a 1-dimensional simplex $(\frac{1}{2}, \frac{1}{2}, 0), (0, \frac{1}{2}, \frac{1}{2})(\frac{1}{2}, 0, \frac{1}{2})$ and one centroid point in a 2-dimensional simplex $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$. The Simplex-Centroid design is shown in Figure 2.5.

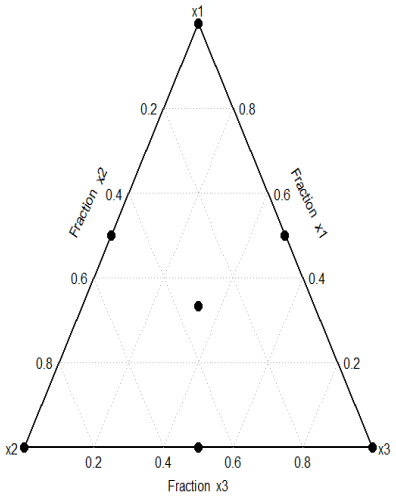

FIGURE 2.5: Simplex-Centroid Design for Three Components

2.3.3 Designs of Mixture Experiments with Lower and Upper Bound Restrictions

Restricted region designs have been developed as a technique for conducting experiments involving mixtures when some or all mixture variables have constraints placed on them. Mostly, the presence of upper and lower constraints on the components' proportions

simultaneously is created these conditions. Such constraints will be represented by L_r and U_r, $r = 1, 2, \cdots, q$, where L_r and U_r are lower and upper bounds respectively. Once the constraints of the proportion of each component is considered, all such components of the basis design are uniquely determined. Thus all component proportions are included in the blend and none of these components will be 100% or 0%. Consequently, the constrained region will reduce the size of the factor space that can be represented as a convex polyhedron, which will not be shaped like a simplex. Such a non-simplex-shaped region is considerable more complicated than the simplex because the polyhedron has more than q vertices and more than q edges. For example, referring to the animal husbandry example that was mentioned earlier in Chapter 1, the experimenter feeds the chicks with different energy supplements. The components are protein, which is represented as x_1, fats, and carbohydrates that are indicated by x_2 and x_3, respectively. It is required that between 5% and 40% of the purified diets portion should consist of protein, that the proportion of fat at least should be between 2% and 89%, and, that the proportion of carbohydrates should be bounded between 6% and 86%. These constraints on the proportions can be represented as

$$0.05 \leq x_1 \leq 0.40,\ \ 0.02 \leq x_2 \leq 0.89,\ \ 0.06 \leq x_3 \leq 0.86\,.$$

In this example, the experimenter needs to decide the constraints on the component proportions that make up the mixture because such a mixture may not contain all the ingredients or may have too much of one such component proportions which mean that the mixture will not meet the measured yield requirements.

The feasible region of interest is shown in Figure 2.6, which is a subarea that covers the region of interest within the simplex and is bounded by the •. In such designs, we need to find the coordinates of the points that are spread out adequately over the region of interest. Different algorithms from the statistical literature, such as the XVERT algorithm, can be used to find all possible points from these restricted regions. This algorithm works with the original constrained components and their upper and lower bounds to generate the coordinates of all points of such region that will be known as extreme vertices of the region of interest which adequately cover and provide maximum spread over the restricted area inside a simplex.

2.4 Design Theory

In the experimental design mechanism, the development and construction of suitable designs for the region of interest depends on the supporting statistical models, since

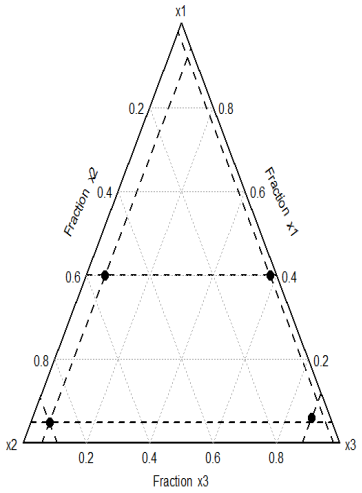

FIGURE 2.6: Non-Simplex-Shaped-Region for Constrained Area with 3-Components

the precision of the model parameters' estimation leads to the development of many statistical optimality criteria. Hence, if the model function $f(\boldsymbol{x}, \boldsymbol{\beta})$ is linear in the parameters, (2.5) can be expressed in matrix notation as

$$\boldsymbol{Y} = \boldsymbol{X}\boldsymbol{\beta} + \boldsymbol{\epsilon}, \tag{2.14}$$

where $\boldsymbol{Y} \sim \boldsymbol{\mathcal{N}}(\boldsymbol{X}\boldsymbol{\beta}, \sigma^2 \boldsymbol{I_n})$ is an $n \times 1$ vector, $\boldsymbol{\beta}$ are $p \times 1$ vectors of fixed unknown parameters and $\boldsymbol{X}$ is a design matrix (or model matrix) of order $n \times p$ and $\boldsymbol{\epsilon} \sim \boldsymbol{\mathcal{N}}(0, \sigma^2 \boldsymbol{I_n})$. Once the n observations are collected, the model parameters can be also estimated using least squares estimation (LSE) by minimizing the sum of squares

$$S(\beta) = \sum_{i=1}^{n} (y_i - f(\boldsymbol{x})\boldsymbol{\beta})^2, \tag{2.15}$$

where $f(\boldsymbol{x})$ is a $1 \times p$ vector with the elements that correspond to the elements in a row of the matrix $\boldsymbol{X}$. Then

$$\hat{\boldsymbol{\beta}} = (\boldsymbol{X'X})^{-1}\boldsymbol{X'Y} \tag{2.16}$$

with $E(\hat{\boldsymbol{\beta}}) = \boldsymbol{\beta}$ and $var(\hat{\boldsymbol{\beta}}) = \sigma^2(\boldsymbol{X'X})^{-1})$ where $E(\hat{\boldsymbol{\beta}})$ and $var(\hat{\boldsymbol{\beta}})$ are the expectation vector and the dispersion matrix of $\hat{\boldsymbol{\beta}}$ respectively.

For the individual coefficient, the variance of $\hat{\boldsymbol{\beta}}_i$ is usually proportional to the ith diagonal elements of $\boldsymbol{(X'X)^{-1}}$, and the covariance of $\hat{\boldsymbol{\beta}}_{\boldsymbol{i}}$ and $\hat{\boldsymbol{\beta}}_{\boldsymbol{j}}$ is proportional to $(i, j)th$ off-diagonal element. Thus the predicted value of the response at a point $\boldsymbol{x} = (x_1, x_2, \cdots, x_q)$

in the experimental design region $\boldsymbol{\chi}$ can be represented as

$$\hat{\boldsymbol{Y}}(\boldsymbol{x}) = f(\boldsymbol{x})\hat{\boldsymbol{\beta}}, \tag{2.17}$$

A precision measurement of the estimate $\hat{\boldsymbol{Y}}(\boldsymbol{x})$ at the point $\boldsymbol{x}$ is defined as the variance of $\hat{\boldsymbol{Y}}(\boldsymbol{x})$ that can be displayed as

$$var\hat{\boldsymbol{Y}}(\boldsymbol{x}) = \sigma^2 f(\boldsymbol{x})(\boldsymbol{X'X})^{-1} f'(\boldsymbol{x}). \tag{2.18}$$

Therefore, the standardized variance for the number of n trials is

$$d(\boldsymbol{x}, \boldsymbol{\xi}) = n\frac{var\hat{\boldsymbol{Y}}(\boldsymbol{x})}{\sigma^2} = f(\boldsymbol{x})M^{-1}(\boldsymbol{\xi}_n)f'(\boldsymbol{x}). \tag{2.19}$$

The information about the efficiency and accuracy of estimating $\boldsymbol{\beta}$ depends on the information matrix $\boldsymbol{X'X}$ that plays a central role in ordinary least squares, which is free of the parameters $\boldsymbol{\beta}$. Equivalently, such information is also contained in the variance-covariance matrix of the parameter estimates.

However, if the model function $f(\boldsymbol{x}, \boldsymbol{\beta})$ is nonlinear, the information matrix $\boldsymbol{F'F}$ will depend on the model parameters as we mentioned in Chapter 1. Moreover, the model parameters are estimated by different methods than linear models, such as nonlinear least squares (NLS) as will be seen in Chapter 3

In optimal design, the main motivation is to optimize the estimation of $\hat{\boldsymbol{\beta}}$, which is equivalent to minimizing an appropriately chosen function of $var(\hat{\boldsymbol{\beta}})$. For simplification, the optimization procedure is provided by considering the confidence interval for a single parameter or confidence region for a set of parameters. With respect to a single parameter the confidence interval can be calculated directly from the variance of the parameter estimates. However, for a set of parameters, the confidence region can be derived from the variance-covariance matrix. Hence, our motivation and for a fixed number of treatments, is to minimize the volume of this region, which is proportional to the determinant of the variance-covariance matrix. This approach results in greater precision of estimation of the whole set of parameters. Therefore, the minimization of the confidence region can be obtained by minimizing the determinant of the variance-covariance matrix, or equivalently, by maximizing the determinant of the information matrix $\boldsymbol{X'X}$. This is the aim of the D-optimality criterion and there is a lot of the theory available on this criterion (Atkinson et al., 2007). An optimality criterion, to plan an experiment should always reflect the aims of the experiment. For example, if only some of the parameters are of interest, an optimal design can be obtained by minimizing the volume of the confidence region for that subset of parameters, in the sense that, minimizing the determinant of the submatrix of the inverse of the information

matrix that corresponds to the parameters of interest. This criterion is denoted as the D_s-criterion, and can be represented as minimize $|[(\boldsymbol{X'X})^{-1}]_{22}|$, where $[(\boldsymbol{X'X})^{-1}]_{22}$ is the corresponding submatrix of the inverse of the information matrix.

While D-optimality is undoubtedly the most popular optimality criterion, another popular parameter-oriented criterion used in practice is the A- optimality criterion. Unlike the D-criterion, A-optimality ignores the covariance among the coefficient estimates and deals with only the variances of such estimates. This criterion aims to minimize the trace of $(\boldsymbol{X'X})^{-1}$ which is equivalent to minimizing the sum of variances of the individual coefficient estimates $\boldsymbol{\beta}_j$, $j = 1, 2, \cdots, q$, or the average variance of the estimators and the average squared width of corresponding confidence intervals. Such a criterion depends on the magnitude or scale of each parameter estimator. Thereby, a weighted A-criterion will be used instead, which is a special case of the L-optimality criterion that can be obtained by minimizing $tr(\boldsymbol{W}(\boldsymbol{X'X})^{-1})$, where $\boldsymbol{W}$ is a weight matrix, whose elements are chosen subjectively to reflect different levels of interest in different parameters. If the weight matrix is diagonal we will obtain the weighted A-optimality. Here, to select the weight matrix for this criterion, Gilmour and Trinca (2012a) suggest to use $a_r = \frac{1}{v_r}$ as its diagonal elements, where v_r is the reciprocals of the variances of the rth parameter estimates from $(\boldsymbol{X'X})^{-1}$ (D-optimal designs). However, if all diagonal elements are equal to each other, then the L-criterion will be defined as the A-optimality criterion. In a similar way as in D_s-optimality, the A_s-criterion is to minimize $tr[\boldsymbol{W}(\boldsymbol{X'X})^{-1}]_{22}$.

These criteria are usually known as standard optimality criteria.

For a good experimental design, several properties and characteristics have been listed by Box and Draper (1975, 1987), where many of such properties have been discussed by many authors such as Atkinson et al. (2007) and Myers et al. (2016). Some of these properties concern checks for model lack of fit, while others require enough degrees of freedom to allow an estimate of pure error. For testing lack of fit, we require a large number of different design points to achieve this purpose efficiently. However, we require replicated points to provide sufficient pure error degrees of freedom. In summary, providing degrees of freedom for pure error and lack of fit is needed. The main purpose for this requirement is that it allows a formal statistical test to be conducted that provides information about the adequacy and efficiency of the specified model, known as a lack-of-fit-test. While sufficient degrees of freedom pure errors allow precise estimation of the statistical error variance. Both issues need to be taken into account, where the standard optimality criteria do not deal with these issues.

Since the sizes of confidence intervals or regions are based on the variance-covariance matrix and the degrees of freedom for pure error due to the replications, such criteria do

not address the inference issues that the experimental data can be faced with applying the design. For example, if our motivation is obtaining unbiased confidence intervals or regions of minimal length or volume, allocating sufficient degrees of freedom for estimation pure error is vital to conduct desirable inferences. Based on the discussions given in Gilmour and Trinca (2012b), new statistical criterion functions, such as DP and AP criteria have been developed to address this issue. Thus the experimenter allows for the use of an error estimation approach which is based purely on the replicates.

The standard D-criterion has been motivated by the fact that the volume of the confidence region of the parameters is inversely proportional to the square root of $|\boldsymbol{X'X}|$ (Atkinson et al., 2007). However, the volume of the $100(1-\alpha)\%$ confidence region is actually proportional to $F_{p,d;1-\alpha}^{p/2}|\boldsymbol{X'X}|^{-1/2}$, where p is the total number of model terms, d is pure error degrees of freedom, and $F_{p,d;1-\alpha}$ is the $100(1-\alpha)$ point of the F-distribution on p and d degrees of freedom. Thus Gilmour and Trinca (2012b) propose to minimize $F_{p,d;1-\alpha}^{p}|\boldsymbol{X'X}|^{-1}$ which is defined as DP-optimality.

An alternative to the DP-optimality criterion is the AP-criterion, which is a modification of the A- or the weighted A- standard optimality criteria with respect to the weight matrix. This criterion aims to minimize $F_{1,d;1-\alpha}tr(\boldsymbol{W}(\boldsymbol{X'X})^{-1})$. Similarly, DP_s- and AP_s-optimality criteria can be represented respectively as minimizing $F_{p_2,d;1-\alpha}^{p_2}|[(\boldsymbol{X'X})^{-1}]_{22}|$, and minimizing $F_{1,d;1-\alpha}tr[\boldsymbol{W}(\boldsymbol{X'X})^{-1}]_{22}$, where p_2 is the number of parameters of interest and $|[(\boldsymbol{X'X})^{-1}]_{22}|$ is the submatrix of the inverse of the information matrix.

Being interested in the precision estimates of the parameters' fitted models or the accuracy and the goodness of that fitting, an alternative optimality criterion called the generalized optimality criterion can be applied for achieving an optimal design, which is defined as the GDP- or the GAP- optimality function. Such criteria, introduced by Egorova (2017), involve the lack-of-fit component, where the possibility under these criteria is that the model that assumed to be fitted to the data is nested within a larger model that is assumed to provide a better fit to the data collected, might not provide a good fit. This issue needs to be taken into account, where the standard criteria do not deal with this issue. These criteria seek to minimize

$$|\boldsymbol{X}_1'\boldsymbol{X}_1|^{-\frac{k_D}{p}} \times [|\boldsymbol{X}_1'\boldsymbol{X}_1|^{-1/p}F_{p,d;1-\alpha}]^{k_{DP}} \times [|L+\frac{I_g}{\tau^2}|^{-1/g}F_{g,d;1-\alpha}]^{k_{LOF}}, \tag{2.20}$$

and minimize

$$[\frac{1}{p}tr(\boldsymbol{W}(\boldsymbol{X}_1'\boldsymbol{X}_1)^{-1})]^{k_A}\times[\frac{1}{p}tr(\boldsymbol{W}(\boldsymbol{X}_1'\boldsymbol{X}_1)^{-1})F_{1,d;1-\alpha}]^{k_{AP}}\times[\frac{1}{g}tr\left(L+\frac{I_g}{\tau^2}\right)^{-1}F_{1,d;1-\alpha}]^{k_{LOF}}, \tag{2.21}$$

respectively, for the set of parameters. However, for the parameters under consideration, i.e. the model matrix does not contain the intercept, the GDP- and GAP-criteria will

minimize respectively

$$|\boldsymbol{X}_1'Q_0\boldsymbol{X}_1|^{-\frac{k_D}{(p-1)}} \times [|\boldsymbol{X}_1'Q_0\boldsymbol{X}_1|^{-1/(p-1)}F_{p-1,d;1-\alpha}]^{k_{DP}} \times [|L+\frac{I_g}{\tau^2}|^{-1/g}F_{g,d;1-\alpha}]^{k_{LOF}}, \tag{2.22}$$

and

$$[\frac{1}{(p-1)}tr(\boldsymbol{W}(\boldsymbol{X}_1'Q_0\boldsymbol{X}_1)^{-1})]^{k_A} \times [\frac{1}{(p-1)}tr(\boldsymbol{W}(\boldsymbol{X}_1'Q_0\boldsymbol{X}_1)^{-1})F_{1,d;1-\alpha}]^{k_{AP}}$$
$$\times[\frac{1}{g}tr\left(L+\frac{I_g}{\tau^2}\right)^{-1}F_{1,d;1-\alpha}]^{k_{LOF}}, \tag{2.23}$$

where $Q_0 = I_n - \frac{1}{n}11'$, L is the dispersion matrix that can be expressed as $L = \boldsymbol{X}_2'\boldsymbol{X}_2 - \boldsymbol{X}_2'\boldsymbol{X}_1(\boldsymbol{X}_1'\boldsymbol{X}_1)^{-1}\boldsymbol{X}_1'\boldsymbol{X}_2$, and $[|L+\frac{I_g}{\tau^2}|^{-1/g}F_{g,d;1-\alpha}]$, $[\frac{1}{g}tr\left(L+\frac{I_g}{\tau^2}\right)^{-1}F_{1,d;1-\alpha}]$ is a lack of fit component that aims to minimize the size of the posterior confidence region for the potential terms.

The components $F^p_{p,d;1-\alpha}|\boldsymbol{X}'\boldsymbol{X}|^{-1}$ and $F_{1,d;1-\alpha}tr(\boldsymbol{W}(\boldsymbol{X}'\boldsymbol{X})^{-1})$ represent the DP- and AP-optimality criteria respectively, where p is the number of parameters in the model, d is the number of pure error degrees of freedom, and we shall use $\alpha = 0.05$ throughought the examples in this thesis.

Here, the $\boldsymbol{X}_1$ and $\boldsymbol{X}_2$ are model matrices of the primary terms of the assumed fitted model and potential terms of an extended (full) model that could provide a better fit to the data than the assumed fitted model respectively.

For clarifying, assume that the fitted model (first- or second-degree) has the form

$$\boldsymbol{Y} = \boldsymbol{X_1}\boldsymbol{\beta_1} + \epsilon, \tag{2.24}$$

where $\boldsymbol{Y}$ is an $n \times 1$ vector of observations, $\boldsymbol{X_1}$ is an $n \times p$ $(p \geq q)$ model matrix of known settings of factors, where each column of which contains primary terms, $\boldsymbol{\beta_1}$ is a $p \times 1$ vector of unknown parameters, and ϵ is $n \times 1$ vector of independent identically distributed errors. Now suppose the extension (true) model that might provide a better fit to the data is

$$\boldsymbol{Y} = \boldsymbol{X_1}\boldsymbol{\beta_1} + \boldsymbol{X_2}\boldsymbol{\beta_2} + \epsilon, \tag{2.25}$$

where $\boldsymbol{X_2}$ is an $n \times g$ $(g \geq 0)$ an extension of the initial model matrix, which represents the extra g potential terms and $\boldsymbol{\beta_2}$ is an $g \times 1$ vector of unknown parameters. For example, if model (2.24) is of first degree in $q = 3$ components x_1, x_2, x_3, then the model (2.25) might be of the second degree in the same three components, i.e. $\boldsymbol{X_2} = [x_1x_2, x_1x_3, x_2x_3]$.

With respect to the variance scaling parameter τ^2 that describes the magnitude of each

of the potential terms, the choice of such a parameter is quite arbitrary, and some suggestions provided in the literature advocated particular values of τ^2. These suggestions are expressed in comparison to the error variance. For example, DuMouchel and Jones (1994) assumed $\tau^2 = 1$, which means that the prior variance of each potential term is the same as the error variance. On the other hand, other researchers such as Kobilinsky (1998) using $\tau^2 = 1/g$, so that in this case the total prior variance of the potential terms is the same as the error variance.

In this research, we will use the development statistical functions GDP- and GAP-criteria meanwhile advocate $\tau^2 = 1$ and $\tau^2 = 1/g$ for our optimization purposes.

During the experimental work, the developmental and standard criteria are accompanied by illustrative examples in order to construct good designs and explore the possible relationship patterns between these criteria.

The choice of the criterion for obtaining the optimal design depends on the objective of the data analysis in the experiments. For example, if the experimental analysis involves a global F-test of the parameters, then the DP-criterion is suitable for conducting the optimization output. However, if the experimental inference is related to the individual treatment parameter comparisons then the AP-criterion should be used for such purposes, while for a point estimation of such parameters, weighted A-optimality should be used.

The construction of a design which is optimum with respect to a chosen criterion is an optimization problem that can be considered as a concern for most researchers or experimenters. Most of the available algorithms can provide a method or technique that may contribute to the construction of an optimal design given a statistical model defined on a given design space. Quite a few algorithms for optimum design search have been provided in the statistical literature, some of which are presented in (Atkinson et al., 2007, Chapter 12). Many of these algorithms have been applied by many authors and researchers for searching optimal deigns such as Martin et al. (1999). They implemented the criterion under consideration into several algorithms, such as DX, KL, Fedorov, Gosset, and their adjustment algorithms for obtaining constrained mixture designs with a small number of q. In their article, they compare these algorithms and conclude that the comparison between such algorithms is not easy due to all of them having some failings. In addition, they noticed that some of them, such as DX and KL algorithms with the smallish q are usually flexible and adequate. However, for some situations, such as the parallelogram example, they investigate these algorithms are not sensible. Moreover, they notice that the Gosset is usually preferable to be applied under a bigger problem. Further, they provide a discussion and present several extensions and outcomes

for 3- and 4- component mixtures under second degree models for the results in MBS in their article Martin et al. (2001).

Many of provided algorithms begin with generating an initial random design from the candidate set of points that is composed as the set of all admissible combinations of the treatment factors, where in our case such points must be found from the restricted region. Then, each design point in the initial designs is considered for exchange with other points from the candidate point set. The combination of factor levels chosen for exchange is the one which results in the desired gain with respect to the chosen optimality criterion. During our research we will obtain our optimization results using such methodology assuming the sample size n is predetermined, thus we are not dealing with a sample size determination problem. The full procedure for this algorithm is further illustrated in Chapter 5. However, for some algorithms, there is no candidate sets of design points required. For example, the coordinate-exchange algorithm which was first introduced by Meyer and Nachtsheim (1995). This algorithm starts with generating a random initial design and going through rows and columns of the design matrix. Then, the values of each element in the design matrix are changed among all points in the candidate sets while evaluating the criterion under consideration after each change. This can be done until no further improvement can be achieved to the criterion. Thus, this algorithm has been widely used by many researchers for constructing such optimum designs.

As mentioned earlier, the data from experiments with mixtures are usually modelled using linear models, such as Scheffé polynomials and special cubic models as exhibited in the statistical literature. However, in some situations models with nonlinear parameters have been applied to such data and in some cases are needed for several statistical purposes. For example, it is important to find the best-fit values of the parameters in the model to allow us to interpret these values well. Nonlinear models in some situations can give a better interpretation of the experimental mechanism and give a good fit to the data more frequently than linear models. Thus, the improvement and approximation of such models to the unknown response surface in mixture settings are needed as will be seen through following chapter.

Chapter 3

Nonlinear Models for Experiments with Mixtures

3.1 Nonlinear Regression Models

In statistics, nonlinear regression is considered as a form of regression analysis. Unlike linear regression, under nonlinear situation, a response variable are modelled by a function which is a nonlinear combination of the model parameters, where such model depends on one or more independent variables. The primary purpose in the case of fitting regression models data analysis is to find the relationship between an input variable and the output. This kind of relationship can be expressed using some model, that has the general form of regression models displayed in (2.5). As mentioned before, if the model function $f(\boldsymbol{x}, \boldsymbol{\beta})$, for x_r, $r = 1, 2, \cdots, q$ is linear, for estimating the model parameters, there is a specific equation which can be used to obtain the single unique solution that results in the smallest error sum of squares for that model and data set. Therefore, it is just required to identify the input and output variables and the terms to include in the model. However, in some situations the observed data are modelled by a function which is nonlinear combination of the model parameters and the covariates (independent variables). In this case, the form of the model should be specified, and the parameters to be estimated must be identified. Moreover, for numerical estimation procedure, starting values of these parameters must be provided as initial values, to allow us to evaluate the model for that fit. Thus an iterative procedure is needed to fit a nonlinear models to data to converge on a solution that provides the best-fitting model to data collected.

Nonlinear regression models have many applications in engineering, physics, biology, economics and medicine, among others (Ratkowsky et al., 1983). In addition, this type

of regression is often encountered in chemical reactions and clinical trials (Begg and Kalish, 1984). There are several nonlinear models arising in the study of chemical and physical experiments. For example, one of the early experiments in the physics field was conducted by Rumford (1798). The experiment was run at the Munich Arsenal in Germany, in which water was made to boil without the use of fire and heat was generated. More details for this experiment are provided in Bates and Watts (1988). The data collected for this experiment were on the amount of heat generated by the friction of the cannon barrel, which is taken at various times. The main objective for performing this experiment was to allow the cannon barrel to cool to an ambient temperature of 60° F when it had reached after heated to a steady temperature of 130° F. Under Newton's law of cooling, which states that $\frac{df}{dx} = \theta(f - T_0)$, where T_0 is the ambient temperature, then the temperature at time x was formulated as

$$f(x;\theta) = 60 + 70e^{-\theta x}, \tag{3.1}$$

where f is predicted temperature. Since the differentiating of the function f with respect to θ gives $-70xe^{-\theta x}$, which depends on the parameter θ, the resulting model is recognized as a nonlinear model. The exponential decay for this function can be seen in Figure 3.1.

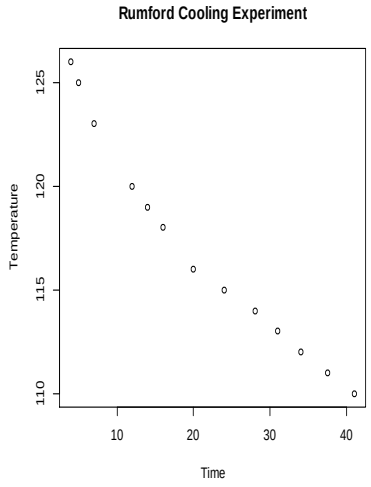

FIGURE 3.1: Rumford Cooling Experiment

Another class of models with nonlinear parameters that have been provided in the statistical literature are known as fractional polynomial models, which are an extended family of curves that was first proposed by Royston and Altman (1994). The general form of such models was derived by generalizing the common polynomial regression

models supposing that y is a response (outcome) and x is a single continuous covariate. Then the most common model relating them is a straight line model

$$E(y) = \beta_0 + \beta_1 x. \tag{3.2}$$

The straight line model is often an adequate description of the relationship between variables. However, other models must be investigated for possible improvements in fit. A simple extension of this model is a power transformation which can be represented as

$$E(y) = \beta_0 + \beta_1 x^{(\alpha)}, \tag{3.3}$$

where

$$x^{(\alpha)} = \begin{cases} x^{(\alpha)} & \text{if} \quad \alpha \neq 0 \ ; \\ log(x) & \text{if} \quad \alpha = 0 \end{cases}$$

and has been used by practitioners with different choices of α. This power transformation model can be called a first-degree fractional polynomial model or FP1 function. As in polynomial regression, the extension from the first-degree fractional polynomial FP1 to second-degree fractional polynomial FP2 can be derived straightaway (Gilmour and Trinca, 2005). There are many areas of application having several predictors or continuous factors. In this case, the form of first and second order fractional polynomial response surface models can be represented as

$$E(y) = \beta_0 + \sum_{r=1}^{q} \beta_r x_r^{(\alpha_r)}, \tag{3.4}$$

and

$$E(y) = \beta_0 + \sum_{r=1}^{q} \beta_r x_r^{(\alpha_r)} + \sum_{r=1}^{q} \beta_{rr} x_r^{2(\alpha_r)} + \sum_{r=1}^{q-1} \sum_{s=r+1}^{q} \beta_{rs} x_r^{(\alpha_r)} x_s^{(\alpha_s)}, \tag{3.5}$$

respectively, where

$$x_r^{2(\alpha_r)} = \begin{cases} x_r^{2\alpha_r} & \text{if} \quad \alpha_r \neq 0 \ ; \\ (log(x_r))^2 & \text{if} \quad \alpha_r = 0 \ . \end{cases}$$

In addition of interest is the first-order fractional polynomial response surface model with interactions, which can be written as

$$E(y) = \beta_0 + \sum_{r=1}^{q} \beta_r x_r^{(\alpha_r)} + \sum_{r=1}^{q-1} \sum_{s=r+1}^{q} \beta_{rs} x_r^{(\alpha_r)} x_s^{(\alpha_s)}, \tag{3.6}$$

which has the same number of parameters as the second-order polynomial model, but no stationary point.

With regard to the power α, the value of these parameters can be taken any real values, but there may be advantages in constraining their corresponding values to some intervals or some set of rational numbers. For example, restricting the values of the powers α to a set of integer and non-integer values such as $\{-3, -2, -1, -\frac{1}{2}, -\frac{1}{3}, 0, \frac{1}{3}, \frac{1}{2}, 1, 2, 3\}$, which has been proposed by Royston and Altman (1994). The restriction of such power terms to a small predefined set of integer and non-integer values may help to avoid having very large powers in the proposed model, whether positive or negative, as it will be difficult to obtain any meaningful understanding of them (Gilmour and Trinca, 2005). Moreover, Gilmour and Trinca (2005), provided an application of fractional polynomial models; in particular, they illustrated how these models can be fitted the presented data as well as how an appropriate and suitable model can be selected. In addition, they fitted such models and other models, such as polynomial models to the same data sets and then they investigated that, in some situations the fractional polynomial models can give a better fit to the data than polynomial models. Finally, in their article, they also gave an equally good fit for that fitting but much more reasonable interpretation.

Another application for nonlinear models that could contribute to the statistical literature can be obtained using fractional polynomial models for fitting data from experiments involving mixtures, which will be the main theme of this chapter.

3.2 Fractional Polynomial Models for Fitting Mixture Data

The models FP1 and FP2 described in (3.4) and (3.5) cannot be fitted directly to a mixture data set, due to the restrictions that have been placed on the x_r proportion components, given in (2.1) and (2.2). Thus, modification of such models is needed to fit such experimental data. Here, the expansion methodology, which is based on a binomial series expansion was the first method applied to modify these models. However, during this process, we found that the resulting functions after applying the expansion technique turn out to be unsuitable to approximate FP1 and FP2 across the model domain of the independent variables.

Thus, another approach, which is based on transformation methodology will be used to modify FP1 and FP2. Following this methodology, the continuous covariates $\boldsymbol{x_r}$ will be transformed. Referring to the statistical literature, it was found that the mixture problem is in general analogous to the compositional data problem as pointed out by

Cornell (1973). Thus, the transformation to be performed on the $\boldsymbol{x_r}$ can be handled by adopting the ideas and methods that have been applied to compositional data.

The first systematic research on such data was given by Aitchison and Shen (1980), who gave the definition and provided the main concept of such data, modelling these data sets by a logistic normal distribution, and transforming them using the log-ratio transformation. Moreover, Aitchison (1982) provided and highlighted the statistical analysis and inferences for such data when effectively has two transformation, a ratio followed by the logarithmic. The compositional data set underlying Aitchison's theory can be defined as a data set consisting of constituent vector $\boldsymbol{z}_i = (z_{i1}, z_{i2}, ..., z_{iG})$ with G components, where the constituent proportions z_{ij} represent the ith observation for the jth constituent $(i = 1, \cdots, n; j = 1, \cdots, G)$, so that $z_{ij} \geq 0$ and $\sum_{j=1}^{G} z_{ij} = 1$ for all i. As a result, z_i has dimension $g = G - 1$, i.e. $z_i \in S^g$. Based on the definition of the compositional data, the strategy used logarithmic transformations as a successful way of transforming data to normality by assuming that z_{ij} is strictly positive for all i, j and using the ratio $\boldsymbol{y}_i = (y_{i1}, y_{i2}, \ldots, y_{ig})$, $y_{ij} = \frac{z_{ij}}{z_{iG}}$, so that $\boldsymbol{y}_i$ is the $1 \times g$ vector of transformed compositional data.

Because Aitchison's method is linked to the logarithmic transformation, the achievement of the logistic-normal model when fitted to the data will depend on the extent to which the logarithm transformation is able to fit such data to normality. Moreover, this transformation is not valid with zeros in the data because the logarithm of zero is undefined (Rayens and Srinivasan, 1991). More reliable modelling for such data could be obtained by choosing the Box-Cox transformation that provides best fit (Iyengar and Dey, 1998), which can be represented as

$$y_{ij} = \begin{cases} \dfrac{\left(\frac{z_{ij}}{z_{iG}}\right)^{\lambda_{j-1}}}{\lambda_j} & \text{if} \quad \lambda_j \neq 0 \ ; \\ log\left(\frac{z_{ij}}{z_{iG}}\right) & \text{if} \quad \lambda_j = 0 \ , \end{cases}$$

for $i = 1, \cdots, n$ and $j = 1, \cdots, g$. However, for mixture data, the interesting case is not of transforming the response but the independent variables. The technique to be followed is now outlined. The mixture components are strictly positive and the sum of the proportions of all components that make up the mixture must be unity. In addition, let x_r be a proportion components of transformed compositions, by adopting Aitchison's suggestion of transformation of the proportion of components technique, the transformation for the $x_r's$ is

$$\sum_{r=1}^{q} x_r = 1, \qquad 0 < x_r \leq 1, \qquad x_r's \rightarrow \left(\frac{x_r}{x_q}\right)^{\alpha_r}, \qquad r = 1, 2, ..., q \tag{3.7}$$

As a result, the first and second order fractional polynomial models become

$$E(y) = \beta_0 + \sum_{r=1}^{q-1} \beta_r \left(\frac{x_r}{x_q}\right)^{(\alpha_r)}, \tag{3.8}$$

and

$$E(y) = \beta_0 + \sum_{r=1}^{q-1} \beta_r \left(\frac{x_r}{x_q}\right)^{(\alpha_r)} + \sum_{r=1}^{q-1} \beta_{rr} \left(\frac{x_r}{x_q}\right)^{2(\alpha_r)} + \sum_{r=1}^{q-2} \sum_{s=r+1}^{q-1} \beta_{rs} \left(\frac{x_r}{x_q}\right)^{(\alpha_r)} \left(\frac{x_s}{x_q}\right)^{(\alpha_s)} \tag{3.9}$$

respectively. The models in (3.8) and (3.9) are a new class of models for fitting mixture components called here FP1n and FP2n for first and second order modified fractional polynomial models respectively.

Before beginning to fit FP1n and FP2n to the data from mixture experiments, it is worth mentioning that there are two issues need to be considered. The first concerns the power α, while the second relates to the choice of denominator in the models. With respect to the power α, two possible cases will appear: either the model has the same value of the exponent α for each ratio of components, or the models have different values of $\alpha_{r,s}$, $r, s = 1, 2, 3, \cdots, q$. Under the parameter estimation process, if the restricted set that was proposed by Royston and Altman (1994) is considered, for finding the estimated $\alpha_{r,s}$ that appear as non-linear parameters, we could use nonlinear least squares using a partial linear algorithm and round the estimated powers. If the estimated powers are inside the range, then we can easily set the estimated value of the powers $\alpha_{r,s}$ to be as obtained value under the estimation process or equal to the nearest value from the restricted set when we decide to round them. However, if the estimated powers are outside the range, then we fix the value of some of the $\alpha_{r,s}$ in the models by choosing any value from the restricted set and then estimating the other parameters using nonlinear least squares. The main procedure for nonlinear least squares (nls) that will be used to estimate the parameters as the values that minimize the residual sum of squares is outlined as

1. Use plinear, which is a partial linear algorithm such as the Golub-Pereyra algorithm proposed by Golub and Pereyra (1973) that is extended to define a partially linear LS (least-squares) in order to identify a model containing a linear part and a nonlinear component.

2. Supply an initial guess, say $\boldsymbol{\alpha_0}$ as starting values for the parameters. The algorithm then seeks to minimize the residual sum of squares. The algorithm performs best when using starting values that are reasonably close to the minimizing values.

3. At iteration $J \geq 1$, the current guess $\boldsymbol{\alpha_J}$ is obtained by updating $\boldsymbol{\alpha_{J-1}}$. Therefore, if the residual sum of squares at $\boldsymbol{\alpha_J}$ is smaller than the residual sum of squares at $\boldsymbol{\alpha_{J-1}}$, the counter J is increased by 1 and this step is repeated. However, if no such improvement is possible, then $\boldsymbol{\alpha_{J-1}}$ is taken as estimator for the parameters.

Regarding the second issue which is the the choice of denominators in the FP1n and FP2n models, the choice of such denominators generally are not obvious. Thus, the technique that will be proposed to set the denominator will begin by trying to compare every possible denominator of x_r, $r = 1, 2, ..., q$ within the two situations (the models have the same value of the parameter α and different values of $\alpha_{r,s}$) in order to see if a further improvement in fit is possible. Then the best fitting model is chosen by their mean square residuals (MSR), i.e. the models that have the minimum (MSR). For example, in the case when fitting the FP1n model with the same value of the exponent α to the data set, we will first place x_1 in the denominator in the model, and then use a nonlinear least squares (nls) algorithm to fit the model to the data. The algorithm seeks to minimize the residual sum of squares, and at the end of the process, MSR will be evaluated. Repeating the same technique and process with x_2 and x_3 in the denominator respectively, we find the (MSR) for each of these models. In the final step, we compare between the resulting (MSR) and choose the denominator that gives the minimum (MSR) as will be explained in more detail in the following section. As the main purpose of proposed this technique is to obtain the precise estimate of the model parameters. Since the MSR can be defined statistically as a measurement to assesses the quality of an estimator, then the minimum MSR is better for conducting this purposes. In addition, such criterion tell us statistically that how much of variation in the dependent variables the proposed fitted models did not explain it. Thus the interesting is to fit the model to the data that has the minimum MSR.

This technique with the algorithm mentioned above will be used when fitting the new models to the mixtures data to evaluate the estimated model parameters. In addition, other models will be fitted to the same data set for comparison with the modified fractional polynomial model to show that, in some situations, first and second order FPn models fit the data much better than other statistical models. As the feasible interesting region is in some area in the middle of the simplex where there is a proportion of all components of the ingredients, and not in any one of the corners where only one ingredient is represented, the concern now is to fit models of the form (3.8) and (3.9) to constrained mixture data sets and to estimated the parameters in the models using the above method.

3.3 Fractional Polynomial Models with Constrained Mixtures Data: Chick Feeding Experiment

We will illustrate the new modelling approach through an example from an animal husbandry experiment. The experimental data are provided in Cornell (1990) and involve three mixture components with proportions bounded above and below. We will fit the nonlinear models FP1n and FP2n. The data were collected during an experiment on thirty groups each with different male chicks, all having similar weight and size. These chicks were force-fed purified diets consisting of energy supplements that contain protein, fat, and carbohydrates. The chicks in the experiment were fed three times a day for 10 days. At the end of this time, both the weight y_1 and the fat y_2 were measured for all chicks. The data from this experiment are listed in Table 3.1. Following the concept for minimization (MSR) technique by trying to compare every denominator of x_r, $r = 1, 2, 3$ and then selecting the denominator with the minimum (MSR), the minimum (MSR) under FP1n with considering such model have the same value of the exponent α was obtained after dividing by x_1. However with the FP1n model having different α_1 and α_2, the minimum (MSR) was obtained after dividing by x_3. Table 3.2 shows the values of MSR with different denominators in the case of fitting FP1n.

TABLE 3.2: The Value of MSR after Fitting FP1n

MSR	divided by x_1	divided by x_2	divided by x_3
MSR for model with α	282.60	286.40	432.50
MSR for model with α_1 and α_2	244.14	118.30	76.66

Here, from Table 3.2, we notice that the model with α_1 and α_2 is much better than the one with common α.

Moreover, the estimated values of the parameters for the two cases of FP1n are represented in Table 3.3.

TABLE 3.3: The Estimated Parameters for FP1n

$\hat{\alpha} = 0.5$	$\hat{\beta}_0 = 183.8$	$\hat{\beta}_1 = -18.4$	$\hat{\beta}_2 = -27.3$
$\hat{\alpha}_1 = -0.1$, $\hat{\alpha}_2 = 1$	$\hat{\beta}_0 = 594.7$	$\hat{\beta}_1 = -424.1$	$\hat{\beta}_2 = -20.9$

Following the same process and technique to obtain the estimated parameters when fitting FP2n to the data while considering the two situations (the models having α and

TABLE 3.1: Chick Feeding Experimental Data

x_1	x_2	x_3	y_1	y_2
0.05	0.89	0.06	59.00	39.00
0.05	0.73	0.22	60.00	35.00
0.05	0.09	0.86	29.00	49.00
0.05	0.25	0.70	35.00	46.00
0.05	0.57	0.38	53.00	38.00
0.05	0.41	0.54	48.00	43.00
0.12	0.82	0.06	82.00	36.00
0.12	0.66	0.22	90.00	40.00
0.12	0.02	0.86	63.00	53.00
0.12	0.18	0.70	73.00	46.00
0.12	0.50	0.38	100.00	42.00
0.12	0.34	0.54	80.00	39.00
0.19	0.75	0.06	105.00	28.00
0.19	0.59	0.22	105.00	35.00
0.19	0.43	0.38	120.00	42.00
0.19	0.27	0.54	109.00	41.00
0.19	0.11	0.70	110.00	48.00
0.26	0.68	0.06	136.00	29.00
0.26	0.04	0.70	125.00	38.00
0.26	0.20	0.54	133.00	37.00
0.26	0.36	0.38	130.00	34.00
0.26	0.52	0.22	141.00	29.00
0.33	0.61	0.06	141.00	23.00
0.33	0.45	0.22	156.00	30.00
0.33	0.29	0.38	157.00	31.00
0.33	0.13	0.54	153.00	29.00
0.40	0.54	0.06	156.00	23.00
0.40	0.38	0.22	179.00	26.00
0.40	0.22	0.38	163.00	30.00
0.40	0.06	0.54	143.00	34.00

having different values of the exponent), the optimal denominator is found to be x_2, which has the minimum MSR in comparisons to others x_r, $r = 1, 3$ in both cases. Table 3.4 represents the values of MSR with different denominators and Table 3.5 and Table 3.6 show the estimated values of the parameters with respect to the chosen denominator. Again, we notice that, the models with different different values of the power α are better than the models with common α. Moreover, the FP2n is much better than FP1n.

With respect to the other response variable fat gain y_2, the same methodology and techniques are applied. Again, we consider all possible different denominators and the

TABLE 3.4: The Value for MSR after Fitting FP2n

MSR	divided by x_1	divided by x_2	divided by x_3
MSR for FP2 having the same α	79.40	54.90	55.60
MSR for FP2 having α_1 and α_2	69.01	41.50	54.40

TABLE 3.5: The Estimated Parameters for FP2n with α

$\hat{\alpha}$ = -0.1	$\hat{\beta}_0$ = -256.4	$\hat{\beta}_1$ = -172.9	$\hat{\beta}_2$ = 278.4	$\hat{\beta}_{11}$ = 427.9	$\hat{\beta}_{22}$ = -133.3	$\hat{\beta}_{12}$ = 260.9

TABLE 3.6: The Estimated Parameters for FP2n with different α

$\hat{\alpha}_1$ = 0.5	$\hat{\alpha}_2$ = -0.1	$\hat{\beta}_0$ = -210	$\hat{\beta}_1$ = 351.9	$\hat{\beta}_2$ = 362	$\hat{\beta}_{11}$ = -65.5	$\hat{\beta}_{22}$ = -157.4	$\hat{\beta}_{12}$ = -81.8

two situations for the power α to estimate the model parameters. In the case of fitting FP1n to the data with fat gain y_2, the minimum MSR was obtained when dividing by x_2 for both scenarios, which is found to be 10.93, and 10.91 respectively. In addition, the estimated values of the parameters are shown in Table 3.7. However, in the case of fitting FP2n, the minimum MSR for the fitted model with two situations was found after dividing by x_1, which is evaluated to be respectively 7.19 and 7.06. Moreover, the output for the parameter estimation values is illustrated in Table 3.8 and Table 3.9 respectively.

TABLE 3.7: The Estimated Parameters for FP1n

$\hat{\alpha} = 0.27 \simeq 0.3$	$\hat{\beta}_0$ = 36.8	$\hat{\beta}_1$ = -24.2	$\hat{\beta}_2$ = 19.3
$\hat{\alpha}_1 = 0.26 \simeq 0.3$, $\hat{\alpha}_2 = 0.29 \simeq 0.3$	$\hat{\beta}_0$ = 38.8	$\hat{\beta}_1$ = -26.15	$\hat{\beta}_2$ = 19.05

TABLE 3.8: The Estimated Parameters for FP2n with α

$\hat{\alpha}$ = 0.5	$\hat{\beta}_0$ = 4.03	$\hat{\beta}_1$ = 10.28	$\hat{\beta}_2$ = 24.61	$\hat{\beta}_{11}$ = -0.68	$\hat{\beta}_{22}$ = -2.43	$\hat{\beta}_{12}$ = -4.57

TABLE 3.9: The Estimated Parameters for FP2n with α_1 and α_2

$\hat{\alpha}_1 = 0.65 \simeq 1$	$\hat{\alpha}_2 = 0.68 \simeq 1$	$\hat{\beta}_0$ = 17.55	$\hat{\beta}_1$ = 1.19	$\hat{\beta}_2$ = 13.68	$\hat{\beta}_{11}$ = -0.01	$\hat{\beta}_{22}$ = -1.28	$\hat{\beta}_{12}$ = -0.37

Again, FP2n is much better to fit the data than FP1n based the comparative criterion MSR.

To identify how efficient the proposed model is, one of the processes in which some of the discussions and conclusions can be made about the efficiency and the goodness of the fitted model is to make comparisons between the proposed model and existing approaches, using some statistical criteria for model comparisons.

Usually, the method that has been applied for measuring model fit is the log likelihood function. However, such function on its own does not take account of model complexity. For example, the usual requirement for making a model fit better to the data sets, which increases the value of the log likelihood function, is to add extra parameters. However, adding those parameters may result in over-fitting. To strike this balance, there are different statistical criteria that measure the trade-off between model fit and complexity of the model known as AIC and BIC which can be used under these purposes. Such criteria attempt to deal with balance by introducing a penalty term for the number of model parameters, where the minimum value of these criteria indicates the optimum model fit. Thus for given data sets, the model with lowest value of these criteria is preferred to fit such data in comparison to other models. Such criteria will be considered during our experimental work as a benchmark between our proposed models with several existing approaches.

Other criteria that can also be considered for making model comparisons through describing the goodness of the fitted model is the coefficient of determination and MSR. Such statistical criteria describe the closeness of the fitted model to evaluate and assess the performance and behavior of a proposed such model with respect to how well the model explains the data.

As mentioned before, some nonlinear models have been applied to data from mixture experiments, where the most recent class of models called general blending models (GBM) has been proposed by Brown et al. (2015). In the notation of their model, the general form of GBM can be represented as

$$E(y) = \sum_{r=1}^{q} \beta_r x_r + \sum_{r=1, r<s}^{q} \beta_{rs} \left(\frac{x_r}{x_r + x_s}\right)^{r_{rs}} \left(\frac{x_s}{x_r + x_s}\right)^{r_{sr}} (x_r + x_s)^{s_{rs}} . \tag{3.10}$$

The powers r_{rs} and r_{sr} may have possible values of $0.5, 1, 1.5, 2, 2.5$, or 3 while the values $0, 1, 2$, or 3 is the set for the power s_{rs}.

To compare the new proposed models FP1n and FP2n with the GBM model we will fit these models to the same data set to see which models are preferred to fit the data. The process followed to fit the GBM model to the data is doing a loop in R to choose the model with smallest AIC from the list of models obtained from the grid of all possible combinations of the available values of powers. However, it is worth to state

that it takes more than 2 days to find the model with the minimum AIC from the list of models. Then, the selected model is fitted to the data set for estimating the model parameters. Summary statistics compare this model with the FPn models.

Following this route, the model that has the minimum AIC compared to the other models inside the loop is

$$E(y) = x_1 + x_2 + x_3 + \left(\frac{x_1}{x_1 + x_2}\right)^{1.5} \left(\frac{x_2}{x_1 + x_2}\right)^{1.5} (x_1 + x_2)^3 + \left(\frac{x_1}{x_1 + x_3}\right)^{0.5} \left(\frac{x_3}{x_1 + x_3}\right)^{2.5} (x_1 + x_3)^3 + \left(\frac{x_2}{x_2 + x_3}\right)^{3} \left(\frac{x_3}{x_2 + x_3}\right)^{3} (x_2 + x_3)^3 . \quad (3.11)$$

Table 3.10 compares the first order FP1n model with GBM with respect to the standard criteria such as, R^2, MSR, AIC, and BIC.

TABLE 3.10: Summary Statistics for Comparing FP1n with GBM

The Models	R^2	MSR	AIC	BIC
FP1n having different α	0.9954	76.66	221.85	230.26
GBM	0.9936	107.04	232.49	242.31

This table shows that in this case and under the illustrative experimental data, even the first order FP1n model fits the data better than GBM. Note that the FP1n model with α_1 and α_2 has only 5 unknown parameters, whereas the GBM has 15 unknown parameters. As a result, a considerably simpler model provides a better fit to the data collection.

Table 3.4 and Table 3.11 show the values of MSR, AIC and BIC for the second order FP2n model respectively, where the MSR for GBM was 107 and AIC = 232.49, BIC = 242.31. This shows that with different denominators the FP2n model is a much better

TABLE 3.11: Summary Statistics for FP2

The Criteria	divided by x_1	divided by x_2	divided by x_3
AIC for FP2n having the same value of α	224.39	216.13	213.34
BIC for FP2n having the same value of α	235.61	228.74	224.55
AIC for FP2n having α_1 and α_2	220.86	214.25	205.57
BIC for FP2n having α_1 and α_2	233.47	225.74	218.18

fit to the data than GBM, with respect to the goodness and the efficiency of fitting the model in terms of prediction the data. Moreover, the discrepancy between the data

and an estimation model is less when fitting FP2n to the data set than GBM models. Furthermore, the coefficient of determination for FP2n is higher than that for GBM. That is evaluated to be 0.9988 and 0.9995 for FP2n with respect to two cases of the exponent α respectively. However, such value under GBM have been found to be 0.9936.

Apart from body weight gain y_1, the measured fat gain y_2 for each chick also can be considered for obtaining some analysis and comparisons. A similar process will be followed with fat gain in terms of fit the first and second order modified fractional polynomial models FP1n and FP2n and make comparisons between these models with the GBM. The GBM model that has the minimum AIC under this present case has the form

$$E(y) = x_1+x_2+x_3+\left(\frac{x_1}{x_1+x_2}\right)^{1.5}\left(\frac{x_2}{x_1+x_2}\right)^{2.5}(x_1+x_2)^2+\left(\frac{x_1}{x_1+x_3}\right)^{3}\left(\frac{x_3}{x_1+x_3}\right)^{0.5} + \left(\frac{x_2}{x_2+x_3}\right)^{3}\left(\frac{x_3}{x_2+x_3}\right)^{0.5}. \quad (3.12)$$

A print out of the summary statistics of the comparisons between these models is represented in Table 3.12. As it can be seen for the measured fat gain y_2 for each chick,

TABLE 3.12: Comparing FP2n with GBM

The Models	R^2	MSR	AIC	BIC
FP2n having the same value of α	0.9985	7.19	153.08	165.69
FP2n having different values of α	0.9998	7.06	152.35	163.56
GMB	0.9952	8.3	155.93	165.74

the FP2n considering two cases of the power α is better to fit the data than GBM with respect to the efficiency of the fitting model in terms of prediction the data. Here as we mentioned before the denominator for FP2n with two cases is x_1, since the minimum value of MSR in this case occurs when x_1 had been placed to the denominator in FP2n models.

Some statistical linear models also can be compared to the FP1n and FP2n. For example, some models were fitted to the chick feed data, such as first and second degree ratio models, where the researcher investigated the effect of the ratios of the different energy supplements on both weight gain and fat gain in the four chicks. The general form of these models are illustrated in (3.13) and (3.14) respectively.

$$E(y) = \beta_0 + \sum_{r=1}^{q-1}\beta_r\left(\frac{x_r}{x_q}\right) \quad (3.13)$$

and

$$E(y) = \beta_0 + \sum_{r=1}^{q-1} \beta_r \left(\frac{x_r}{x_q}\right) + \sum_{r=1}^{q-1} \beta_{rr} \left(\frac{x_r}{x_q}\right)^2 + \sum_{r=1}^{q-2} \sum_{s=r+1}^{q-1} \beta_{rs} \left(\frac{x_r}{x_q}\right)\left(\frac{x_s}{x_q}\right). \tag{3.14}$$

In these models, the researcher relies on the smallest range to place the appropriate denominator in the ratio models. In this experiment, the smallest range $U_r - L_r$ of the values of the proportion components was 0.35 for x_1, 0.87 for x_2 and 0.80 for x_3. Therefore, as suggested in Cornell (1990), the ratio of the important components are $F1 = \frac{x_2}{x_1}$ and $F2 = \frac{x_3}{x_1}$. Consequently, the first and second degree ratio models will be

$$E(y) = \beta_0 + \beta_1 \left(\frac{x_2}{x_1}\right) + \beta_2 \left(\frac{x_3}{x_1}\right), \tag{3.15}$$

and

$$E(y) = \beta_0 + \beta_1 \left(\frac{x_2}{x_1}\right) + \beta_2 \left(\frac{x_3}{x_1}\right) + \beta_{11} \left(\frac{x_2}{x_1}\right)^2 + \beta_{22} \left(\frac{x_3}{x_1}\right)^2 + \beta_{23} \left(\frac{x_2}{x_1}\right)\left(\frac{x_3}{x_1}\right) \tag{3.16}$$

respectively. Beside the results found after fitting first and second ratio models which provided in Cornell (1990), measuring some statistical criteria are evaluated for these linear models, such as AIC, and BIC. All such results are shown in Table 3.13. It can

TABLE 3.13: First and Second Degree Ratio Models with y_1

Ratio Models	R^2	MSR	AIC	BIC
First Degree Ratio Model	0.8186	341.66	302.30	307.90
Second Degree Ratio Model	0.9624	79.63	267.23	277.04

be seen that the second degree ratio model is a much better fit to the experimental data than the first degree model because the difference between the estimator and what is estimated in the second degree is less compared with first degree. Also, the goodness of fitting the second degree ratio model is much better than first degree with respect to the AIC and BIC criteria.

Now, for comparing the first degree nonlinear model FP1n to the first degree ratio model, the choice of the denominator in the FPn models whether for FP1n or FP2n in this case will be similar to the denominator in the ratio models to make the comparison more consistent. The results were compared with the first degree ratio model are shown in Table 3.14. This shows that the coefficient of determination for FP1n is higher at 86% than that from the ratio model at 82%. Moreover, the mean square residuals for FP1n at 282.60, and at 244.14 are less than that for the ratio model at 341.66. Therefore, the FP1n model fits the data well with relative quality the given set of data, and better than the first degree ratio model.

TABLE 3.14: Comparing FP1n with First Degree Ratio Model with y_1

Comparing Models	R^2	MSR	AIC	BIC
The ratio model	0.8186	341.66	302.30	307.90
FP1n having the same α	0.8555	282.60	260.16	267.17
FP1n having α_1 and α_2	0.8566	244.14	260.01	267.04

Table 3.15 shows a comparison of fitting the second degree FPn with second order ratio model. This demonstrates that how successful the fitted second degree FPn model is in explaining the variation of the data with 97% in comparison with the ratio model. With

TABLE 3.15: Comparing FP2n with Second Degree Ratio Model

Comparing Models	R^2	MSR	AIC	BIC
The ratio model	0.9624	79.63	267.23	277.04
FP2n having the same α	0.9641	79.40	224.39	235.61
FP2n having α_1 and α_2	0.9702	69.01	220.86	233.47

respect to the measured fat gain y_2 for the chicks, a similar process will be followed in terms of fitting modified FP1n and FP2n models and making comparisons between these models with the first and second degree ratio models. Summary statistics of the comparisons between these models are represented in Table 3.16. Similar performances

TABLE 3.16: Comparing FPn with Ratio Models

The Models	R^2	MSR	AIC	BIC
FP1n having the same value of α	0.7299	18.09	179.23	187.84
FP1n having different values of α	0.7341	11.34	177.71	184.71
FP2n having the same value of α	0.9985	7.19	153.08	165.69
FP2n having different values of α	0.9998	7.06	152.35	163.56
The first degree ratio model	0.4834	39.44	195.16	200.76
The second degree ratio model	0.8748	9.08	158.64	168.44

of these models can be found also when the denominator in the ratio models could be x_2 or x_3. The summary statistics of such comparisons with respect to y_1 and y_2 are represented in Table 3.17, Table 3.18 and Table 3.19, Table 3.20 respectively. In addition to ratio models, Scheffé's polynomial models have been also fitted to the data, which can also be considered for conducting such comparisons. The summary statistics for the comparisons that show the FP1n and FP2n are better to fit the data than Scheffé's models under y_1 and y_2 given in Table 3.21 and Table 3.22 respectively.

TABLE 3.17: Comparing FPn with Ratio Models under y_1 when the Denominator is x_2

The Models	R^2	MSR	AIC	BIC
FP1n having the same value of α	0.8536	286.40	260.56	267.57
FP1n having different values of α	0.9418	118.30	234.87	243.28
FP2n having the same value of α	0.9988	54.90	216.13	228.74
FP2n having different values of α	0.9995	41.50	214.25	225.74
The first degree ratio model ($\div x_2$)	0.3707	1185.10	302.30	307.91
The second degree ratio model ($\div x_2$)	0.6605	663.90	285.79	292.80

TABLE 3.18: Comparing FPn with Ratio Models under y_1 when the Denominator is x_3

The Models	R^2	MSR	AIC	BIC
FP1n having the same value of α	0.7789	432.50	272.93	279.94
FP1n having different values of α	0.9954	76.66	221.85	230.26
FP2n having the same value of α	0.9995	55.60	213.34	224.55
FP2n having different values of α	0.9997	54.40	205.57	218.18
The first degree ratio model ($\div x_3$)	0.3536	1217.40	303.11	308.72
The second degree ratio model ($\div x_3$)	0.3813	1210.00	303.79	310.80

TABLE 3.19: Comparing FPn with Ratio Models under y_2 when the Denominator is x_2

The Models	R^2	MSR	AIC	BIC
FP1n having the same value of α	0.9944	10.93	162.58	169.59
FP1n having different values of α	0.9954	10.91	161.93	162.93
FP2n having the same value of α	0.9981	7.60	161.57	162.59
FP2n having different values of α	0.9989	7.09	154.01	161.22
The first degree ratio model ($\div x_2$)	0.5059	31.87	193.82	199.43
The second degree ratio model ($\div x_2$)	0.7124	19.26	179.59	186.59

TABLE 3.20: Comparing FPn with Ratio Models under y_2 when the Denominator is x_3

The Models	R^2	MSR	AIC	BIC
FP1n having the same value of α	0.9908	18.01	177.57	184.58
FP1n having different values of α	0.9935	11.37	169.59	172.59
FP2n having the same value of α	0.9995	7.71	154.53	163.69
FP2n having different values of α	0.9997	7.30	153.49	163.10
The first degree ratio model ($\div x_3$)	0.3885	39.44	200.22	205.82
The second degree ratio model ($\div x_3$)	0.5251	31.81	194.64	201.64

TABLE 3.21: Comparing FPn with Scheffé's polynomial models under y_1

The Models	R^2	MSR	AIC	BIC
FP1n having different values of α	0.9954	76.66	221.85	230.26
FP2n having different values of α	0.9995	41.50	214.25	225.74
The first degree Scheffé model	0.9285	134.73	237.07	242.68
The second degree Scheffé model	0.9760	50.91	225.34	237.15

TABLE 3.22: Comparing FPn with Scheffé's polynomial models under y_2

The Models	R^2	MSR	AIC	BIC
FP2n having the same value of α	0.9985	7.19	153.08	165.69
FP2n having different values of α	0.9998	7.06	152.35	163.56
The second degree Scheffé model	0.8784	8.83	157.77	167.583

Overall, it can be seen that from the above comparisons, fitting FPn models to the data is better than fitting other statistical models with respect to some standard statistical criteria for model comparisons.

3.4 The Performance of Modified Fractional Polynomial Models with Manufacturing Experiments

Another example where FP1n and FP2n are much preferred to fit the data over the other existing approaches is provided in Box and Draper (2007), which includes 3 mixture components with proportions varying from 0.1 to 0.99. These components were fuel represented as x_1, and oxidizer and binder allocated as x_2 and x_3 respectively. The experimenters were interested in the effect of the combination of these variables on the response variable, which is the burning rate in the manufacturing experiments. In this

example we will fit the same models to this data set using the same method to compare these models with respect to MSR, R^2, AIC, and BIC as in the previous example. The first and second ratio models with respect to smallest range $U_r - L_r$ of the values of the proportion components will be

$$E(y) = \beta_0 + \beta_1 \left(\frac{x_1}{x_2}\right) + \beta_2 \left(\frac{x_3}{x_2}\right), \tag{3.17}$$

and

$$E(y) = \beta_0 + \beta_1 \left(\frac{x_1}{x_2}\right) + \beta_2 \left(\frac{x_3}{x_2}\right) + \beta_{11} \left(\frac{x_1}{x_2}\right)^2 + \beta_{22} \left(\frac{x_3}{x_2}\right)^2 + \beta_{23} \left(\frac{x_1}{x_2}\right)\left(\frac{x_3}{x_2}\right). \tag{3.18}$$

respectively, and the GBM derived by following the same procedure as described in the previous example can be represented as

$$E(y) = x_1 + x_2 + x_3 + \left(\frac{x_1}{x_1 + x_2}\right)\left(\frac{x_2}{x_1 + x_2}\right)^{2.5} + \left(\frac{x_1}{x_1 + x_3}\right)^3 \left(\frac{x_3}{x_1 + x_3}\right)^{0.5} + \left(\frac{x_2}{x_2 + x_3}\right)^3 \left(\frac{x_3}{x_2 + x_3}\right)^{0.5}. \tag{3.19}$$

The comparison between the proposed first degree model and the first degree linear and ratio models is illustrated in Table 3.23. It is clear that, the proposed model is much better than the linear and ratio models in the present case in terms of the coefficient of determination, that is the R^2 under FP1n is much higher at 0.69 than that from linear and ratio models which found to be at 0.61 and 0.22 respectively. Besides, it has been noticed that, the residual mean square for the first degree modified fractional polynomial models are found to be 293.58 and 274.63 with respect to the different scenarios of the exponent α, which are less than for the linear and ratio model, which found to be respectively at 597.00 and 621.45. As a result and under this situation, the first degree proposed fractional polynomial models is preferred to fit these data than the statistical linear and ratio models. With respect to make a comparison between

TABLE 3.23: Comparing different models with FP1n

The models	R^2	MSR	AIC	BIC
FP1 having the same α	0.68824	293.58	135.36	139.12
FP1 having different α	0.68995	274.63	133.45	136.67
The first degree ratio model	0.21980	621.45	143.70	146.53
The first degree Scheffé model	0.61282	597.00	143.09	145.95

second degree modified fractional polynomial models with other statistical approach, a comparison of results from fitting FP2n, the second order linear and ratio model and GBM is represented in Table 3.24. It can be seen how close a fitted FPn is to the data

with the smaller MSR 112.65 than the other models and provides best performance for all evaluation criteria. Furthermore, the efficiency of the parameterized FP2n models in terms of predicting the data and the goodness of fit is much better than other statistical models with the lowest values of AIC and BIC. On the other hand, the correlation between the response values and the predicted response values was higher for fitting FP2n than for other models. Therefore, the FP2n models is a better fit to the data than the other statistical models. Generally, as can be seen from the illustrated examples,

TABLE 3.24: Comparing different models with FP2n

The models	R^2	MSR	AIC	BIC
FP2 having different α	0.9897	112.65	120.00	125.67
GBM	0.9798	185.01	127.19	132.14
The second degree ratio model	0.8290	181.84	126.95	131.91
The second degree Scheffé model	0.9797	127.20	132.16	227.15

fitting a FPn models to the data is better than fitting other statistical models with respect to some standard statistical criteria for model comparisons.

Note that in situations where there is no constraint on the proportion of the components, i.e. when one or more proportion may be zero, our new FPn models are not suitable for fitting data from mixture experiments due to x_r occurring in the denominators. In most practical applications, however, we would expect that all components should be present. In these situations we have demonstrated that the proposed FPn models fit the data much better than other models, such as linear, Ratio and GBM models while fitting these models to the same restricted mixture components and comparing between them with respect to standard criteria. In addition, the process of fitting FPn models to each data set took a few minutes to produce results using the R software. However, fitting GBM to the same data using the same software took more than two days just to select a model from the list of models obtained from a grid of values for r_{ij}, r_{ji}, and s_{ij} that had the minimum AIC. Furthermore, FPn models have considerably fewer parameters compared with the GBM, and are thus suitable for fitting small data sets which may be impossible to fit with GBM.

3.5 Fractional Polynomial Models with More Constrained Mixtures Data

To fit FP1n and FP2n to the data from mixture experiments where the region of interest is constrained by 4-component proportions, experimental data for an illumination candle

experiment will be used, that provided in Box and Draper (2007), whose data involves four mixture components with proportions bounded above and below. The data were collected during the experiment will be used to find the treatment combination of the proportion components (x_1, x_2, x_3 x_4) that gave the maximum illumination in candles. The illumination candle experimental data are given in Table 3.25. Similar process and

TABLE 3.25: Illumination Level in Candle

The Points	x_1	x_2	x_3	x_4	y
1	0.40	0.10	0.47	0.03	75
2	0.60	0.10	0.27	0.03	195
3	0.40	0.10	0.42	0.08	180
4	0.60	0.10	0.22	0.08	300
5	0.60	0.27	0.10	0.03	220
6	0.60	0.22	0.10	0.08	350
7	0.40	0.47	0.10	0.03	145
8	0.40	0.42	0.10	0.08	230
9	0.40	0.27	0.27	0.06	190
10	0.60	0.17	0.17	0.06	310
11	0.50	0.10	0.35	0.06	220
12	0.50	0.35	0.10	0.06	260
13	0.50	0.24	0.24	0.03	260
14	0.50	0.21	0.21	0.08	410
15	0.50	0.22	0.22	0.06	425

techniques as with 3 component proportions will be used in terms of fitting the proposed models using the nonlinear least squares algorithm and the choice of the denominator in the fitted models. Two situations, the models having the same value of α and the models having different values of the power α, will be considered to illustrate the calculations required in obtaining the first and second degree FPn fitted model coefficients. In the case of fitting FP1n, the minimum MSR was obtained after dividing by x_1 in both cases of fitted model. However, with FP2n, the minimum MSR was found after dividing by x_4 when the model has the same α, whereas the minimum MSR with respect to the second situation was obtained when divided by x_2. The FP1n and FP2n fitted model coefficients are represented in Table 3.26 and Table 3.27 respectively.

TABLE 3.26: The Estimated Parameters after Fitting FP1n

$\hat{\alpha} = 2$	$\hat{\beta}_0 = 306.3$	$\hat{\beta}_1 = -132.8$	$\hat{\beta}_2 = -188.04$	$\hat{\beta}_3 = 330.03$
$\hat{\alpha_1} = 3$, $\hat{\alpha_2} = 2$, $\hat{\alpha_3} = -2$	$\hat{\beta}_0 = 381.6$	$\hat{\beta}_1 = -120.3$	$\hat{\beta}_2 = -84.9$	$\hat{\beta}_3 = -0.09$

TABLE 3.27: The Estimated Parameters after Fitting FP2n

Estimated $\hat{\alpha}$, different $\hat{\alpha}$	$\hat{\alpha}$ = -0.1	$\hat{\alpha_1}$ = -0.1, $\hat{\alpha_2}$ = -2, $\hat{\alpha_3}$ = -3
$\hat{\beta}_0$	$\hat{\beta}_0$ = 724.7	$\hat{\beta}_0$ = 204.6
$\hat{\beta}_1$	$\hat{\beta}_1$ = 848.9	$\hat{\beta}_1$ = 217
$\hat{\beta}_2$	$\hat{\beta}_2$ = -157.4	$\hat{\beta}_2$ = -117
$\hat{\beta}_3$	$\hat{\beta}_3$ = -107.2	$\hat{\beta}_3$ = -9.47
$\hat{\beta}_{11}$	$\hat{\beta}_{11}$ = -361.4	$\hat{\beta}_{11}$ = -711
$\hat{\beta}_{22}$	$\hat{\beta}_{22}$ = -113.3	$\hat{\beta}_{22}$ = -97.38
$\hat{\beta}_{33}$	$\hat{\beta}_{33}$ = -118.5	$\hat{\beta}_{33}$ = -0.008
$\hat{\beta}_{12}$	$\hat{\beta}_{12}$ = 306	$\hat{\beta}_{12}$ = 431
$\hat{\beta}_{13}$	$\hat{\beta}_{13}$ = 305.01	$\hat{\beta}_{13}$ = 19.32
$\hat{\beta}_{23}$	$\hat{\beta}_{23}$ = -174.9	$\hat{\beta}_{23}$ = 0.41

Moreover, the summary statistics of fitting and comparison between FP1n and FP2n and statistical linear models are given in Table 3.28. It can be seen that how successful the fitted second degree modified fractional polynomial models FP2n having different values of the powers α, is in explaining the variation of the data with 99% in comparison with the other fitted models. Also, FP2n with different α is a much better fit to the experimental data than other fitted models including the FP2n that have the same value of α because the mean square residual for FP2n at 816.00 is less than that for the others. Moreover, the goodness of fitting FP2n having different values of the exponents is much better than FP2n with common α and FP1n with respect to AIC and BIC criterion.

TABLE 3.28: Summary Statistics for Linear and FPn Models

The Models	R^2	MSR	AIC	BIC
FP1n having the same value of α	0.6572	4384.60	174.30	178.50
FP1n having the different values of α	0.7618	3807.70	172.80	178.40
The first degree Scheffé model	0.6544	5310.01	176.58	180.12
FP2n having the same value of α	0.9309	2206.00	162.20	170.70
FP2n having the different values of α	0.9872	816.00	140.90	150.80
The second degree Scheffé model	0.9833	3589.00	170.88	178.66

As extension of the constrained region can be defined by 4-component proportions, some experiments in the mixture field was performed with more components of the mixture variables that have constraints placed on them, as the case in the cement formulation and concrete product. The data collected during the experiments were provided in Smith (2005). The estimated coefficients of fitted such models and summary statistics for

the comparisons resulting from fitting FP1n and other statistical linear models to the 5-component proportional cement experiments are illustrated in Table 3.29 and Table 3.30 respectively. Here, the denominator in FP1n has x_5 for the model with α and x_1 with the model having different α, based on the minimum value of MSR.

TABLE 3.29: The Estimated Parameters after Fitting FP1n

Estimated $\hat{\alpha}$, different $\hat{\alpha}$	$\hat{\alpha} = 1$	$\hat{\alpha_1}$ = -1, $\hat{\alpha_2} = 1$, $\hat{\alpha_3}$ = -3, $\hat{\alpha_4}$ = -3
$\hat{\beta_0}$	$\hat{\beta_0} = 79.6$	$\hat{\beta_0} = 144.7$
$\hat{\beta_1}$	$\hat{\beta_1} = 23.7$	$\hat{\beta_1} = 0.001$
$\hat{\beta_2}$	$\hat{\beta_2} = 38.2$	$\hat{\beta_2}$ = -70.6
$\hat{\beta_3}$	$\hat{\beta_3} = 217.4$	$\hat{\beta_3}$ = -0.001
$\hat{\beta_4}$	$\hat{\beta_4} = 83.9$	$\hat{\beta_4}$ = -1833

TABLE 3.30: Summary Statistics for FP1n and Linear Models

The Models	R^2	MSR	AIC	BIC
FP1n having the same value of α	0.9862	4.18	63.70	66.70
FP1n having different values of α	0.9943	2.57	59.10	64.80
The first degree Scheffé model	0.9857	5.00	63.26	66.65

A comparison of results from fitting FP1n with considering two cases as represented in Table 3.30, shows that the efficiency of the parameterized FP1n with different α in terms of predicting the data and the goodness of fit is much better than the other models with the lowest values of AIC and BIC. On the other hand, the correlation between the response values and the predicted response values was higher for fitting FP1n with different values of the power α than for the other models. Therefore, the FP1n with different α is a better fit to the data than the model with common α and the first degree linear model. Under this present experiment the second degree modified fractional polynomial model cannot be fitted to the response surface as the number of the model parameters p exceeds the number of observations $n = 13$.

Once again, it can be conducted the same process with fitting FP1n and FP2n to the 6-component proportional concrete product experiment. In the case of FP1n, the minimum MSR was obtained after dividing by x_1, where the MSR is 10879.6 for the fitted model with $\hat{\alpha}$ = -3 and MSR is 9625.3 for the model with $\hat{\alpha_1}$ = -3, $\hat{\alpha_2}$ = -3, $\hat{\alpha_3}$ = -3, $\hat{\alpha_4}$ = 2, $\hat{\alpha_5}$ = -2. However, the denominator in FP2n having the α or different α was placed to be x_3. The value for the MSR within the two situations is 9267.6 and 9100.5 respectively.

With respect to the values of MSR, it can be conclude that FP2n fits the data much better than FP1n. Another statistical criteria can be used to determine which model is better to explain the data than the other is the coefficient of determination. In this experiment the coefficient of determination for FP2n is higher at 0.9486 and 0.9495 for two situations respectively than that from FP1n within two cases at 0.8749 and 0.9145. For the concrete product experiment and based on the summary statistics of fitted model it can be conclude that FP2n is better to fit the data than FP1n. Moreover, FP2n having different values of the exponent α is a much better fit to the data than the statistical linear models. Under the first and second degree linear models, the evaluated MSR was found to be respectively 13374.00 and 9612.01, where is considerably higher than 9625.3 and 9100.5 for FP1n and FP2n having different α respectively. Moreover, as is clear from Table 3.31, similar notification found when comparing the modification fractional polynomial models and the statistical linear models with respect to different criteria under consideration, such as AIC and BIC. Thus, we can conclude that, the proposed models are better to fit this data than other statistical approaches.

TABLE 3.31: Summary Statistics for Linear and FPn Models

The Models	*AIC*	*BIC*
FP1n having the same value of α	444.98	479.42
FP1n having different values of α	443.44	478.76
The first degree Scheffé model	451.64	472.72
FP2n having the same value of α	442.99	457.65
FP2n having different values of α	442.34	456.60
The second degree Scheffé model	444.79	479.63

As can be seen from the illustrative examples above, the resulting first and second degree FPn provide better fit to the data collected than other statistical models. Thus, the constructing an optimal designs for these models arises. Under optimization mechanism, these fitted models are consider as a main tools in optimization purposes, in the sense that, the information matrix in these models at an experimental condition depends on the unknown parameters of the models unlike linear models. Thus, the resulted estimated unknown parameters from nonlinear least squares will be used as prior values for the parameters to construct the design matrix $\boldsymbol{F}$ and thus the information matrix $\boldsymbol{F'F}$, that consider the main part in constructing optimum designs as it will be shown through examples in Chapter 6.

Chapter 4

Calculating the Coordinates for Non-Simplex-Shaped Regions

4.1 The Strategy for Creating Designs Points for Irregular Shaped Regions

For experiments involving mixtures, our interest is in experimental regions that are subareas or smaller spaces within the simplex. Such a region will not necessarily be shaped like a simplex but can be represented as an irregular shaped design region which makes the design search more challenging. One thing that all such regions have in common are vertices. In the first place, we could calculate the composition of all of the vertices of the restricted area and consider the resulting points as candidate points for a design that will be associated with such constrained regions, where these points spread out over the region under consideration.

Some pairs of such vertices will share the same one-dimensional face, so that in the second place we can evaluate the centroids of the faces and then add the obtained points to the candidate list. The composition of the centroids of the faces can be defined by averaging the composition of the vertices that have the same face. Continuing in this manner, we could calculate the centroid points of the 2-dimensional restricted planes, exhibited whenever $q \geq 4$. This can be found by grouping the vertices into groups of two or more vertices that share the same 2-dimensional constraint planes and by averaging their remaining compositions. Likewise, all remaining centroid points can be found by using all vertices that have $(q-r-1)$ proportions in common, that define a r-dimensional constraint planes where $r \leq d \leq q-2$. Generally, to evaluate r-dimensional centroids, we need first identify the vertices that define the various r-dimensional, where the minimum

number of points required to define such dimensional face is $r+1$. For example, a two-dimensional face requires at least three points, a three-dimensional face requires at least four points and so forth. The process could continue up to the point when we average all of the existing vertices, and define the composition of the $(q-1)$-dimensional overall centroid, which would in turn be added to the candidate points list. Consequently, the resulting candidate points list is composed of the vertices, the 1-dimensional centroids, the 2-dimensional centroids, $\cdots$, the $(q-1)$-dimensional centroid.

To clarify the process, consider the number of constrained ingredients to be $q = 3$. Once the composition of the vertices are determined, obtaining the centroids and overall centroid matter. Here, when $r = 1$, we have $q-r-1 = 3\text{-}1\text{-}1 = 1$ ingredient level in common i.e. the proportion components that have 1 proportion in common. Once identified such components, the compositions are grouped and averaged to get the composition of the centriods. Regarding to the overall centroid, all determined vertices will be grouped and averaged to get the composition of such centroid.

Different algorithms in the statistical literature have been evolved to create these points and to obtain the composition for any non-simplex-shaped regions, such as the XVERT algorithm. The process of such algorithm will be described in the following section.

4.2 The XVERT Algorithm

This algorithm was suggested by Snee and Marquardt (1974). They presented a procedure for generating the coordinates of all the extreme vertices of the constrained region. This procedure will be described in the five steps below:

- **Step 1**: Rank the components in terms of increasing ranges $(U_r - L_r)$, where L_r is the lower bound and U_r is the upper bound for the components and $r = 1, 2, \cdots, q$, and then list the ordered components $x_1, x_2, \cdots, x_q$, where x_1 has the smallest range and x_q has the largest range.
- **Step 2**: Form a two-level design from the upper and lower bounds of the $q-1$ components with the smallest ranges so there are 2^{q-1} combinations.
- **Step 3**: Compute the level of the q^{th} component using

$$x_q = 1 - \sum_{r=1}^{q-1} x_r. \tag{4.1}$$

- **Step 4**: A given point is an extreme vertex if $L_r \leq x_r \leq U_r$ and is called a core point. For those points which are outside of the constraint limits, x_q should be set equal to the upper or the lower limit, whichever is closer to the computed value.
- **Step 5**: From each point originally outside of the limits, generate additional points by adjusting the level of the component by an amount equal to the difference between computed values for x_q and the substituted upper and lower limits. From such points we could generate some candidate subgroups for each such point. However, additional points are generated only from those components whose adjusted levels will remain within the limits of the components.

There are 2^{q-1} original points, and each point can generate at most $q-1$ additional points. Points are not computed if the levels of at least one component fall outside the constraints. There are often replicated points which must be omitted. Replicates can occur only when the last component x_q is at its highest (U_r) or lowest (L_r) level.

The methodology for finding the boundaries of a region that is defined by placing limits on the proportions of the components x_r, is illustrated and clarified by referring to the animal husbandry example that was introduced in Chapter 2, Section 2.3.3. The experimenter in this example defines the appropriate constraints on the energy supplements to be

$$0.05 \leq x_1 \leq 0.40,\ \ 0.02 \leq x_2 \leq 0.89,\ \ 0.06 \leq x_3 \leq 0.86$$

where x_1, x_2, and x_3 represent the proportions of protein, fat, and carbohydrates, respectively, that are expected to provide the best results for the measured yield. To generate the possible component combinations of the factor space, the coordinates of all such points are found by applying the XVERT algorithm. For this the steps of the algorithm are as follows:

Step 1: Calculate the range of each component

TABLE 4.1: Rank the Components

Components	Min(L)	Max(U)	Range
x_1	0.05	0.40	0.35
x_2	0.02	0.89	0.87
x_3	0.06	0.86	0.80

Now, rank components in order of increasing ranges:

TABLE 4.2: Remark the Components

Components	Min(L)	Max(U)	Range
x_1	0.05	0.40	0.35
x_3	0.06	0.86	0.80
x_2	0.02	0.89	0.87

From this table we notice that the component x_2 has the largest range in comparison to other two components, while x_1 has the smallest range.
Step 2: Forming two level design with the components x_1 and x_3

$$L_1\ L_3 \quad L_1\ U_3 \quad U_1\ L_3 \quad U_1\ U_3$$

Step 3: Determine the levels of the omitted components (in our case x_2). Therefore, the resulting points are:

TABLE 4.3: Resulting Points

Points	x_1	x_2	x_3
A	0.05	0.89	0.06
B	0.05	0.09	0.86
C	0.40	0.54	0.06
D	0.40	-0.26	0.86

From this table we can see that the points (A,B,C) are extreme vertices, while the point (D) needs to be adjusted because the value (-0.26) is outside the limits of the component x_2 .
Step 4: Adjusting point (D), so the resulting points are given in Table 4.4. This table, representing the full set of extreme vertices, identifies 3 vertices in the core group which are indicated by (A, B, and C), and 2 vertices in the candidate subgroup which are allocated to (D1, and D2), for a total of 5 extreme vertices. Additional points that can be found are the centroids and overall centroids as exhibited in Table 4.5.

TABLE 4.4: Resulting Points

Points	x_1	x_2	x_3
A	0.05	0.89	0.06
B	0.05	0.09	0.86
C	0.40	0.54	0.06
D1	0.40	0.02	0.58
D2	0.12	0.02	0.86

TABLE 4.5: The Face Centroid and Overall Centroid

Points	x_1	x_2	x_3
A	0.05	0.89	0.06
B	0.05	0.09	0.86
C	0.40	0.54	0.06
D1	0.40	0.02	0.58
D2	0.12	0.02	0.86
F1	0.05	0.49	0.46
F2	0.085	0.055	0.86
F3	0.225	0.715	0.06
F4	0.26	0.02	0.72
F5	0.40	0.287	0.32
F6	0.204	0.312	0.484

These centroid points are obtained by grouping and averaging some vertices that have $q - r - 1 = 1$ proportion in common. For instance, from the above table, the point F1 is the centroid of the two points (A and B) that define the face, and these two points have the proportion x_1 in common that is $x_1 = 0.05$. Then, to get the composition of the points F1 this can be done by grouping these vertices (A and B) and averaging their compositions. All remaining centroid points can be found by following similar process. Thus, these faces can be represented as F1-F5 in Table 4.5. Moreover, the overall centroid's value is obtained by averaging all the factor levels of the existing vertices that are listed as point F6 in Table 4.5.

This table shows that all the desirable candidate points for the restricted region by 3-component proportions that all points are inside the limits of their components. Furthermore, the feasible region under these points are given in Figure 4.1.

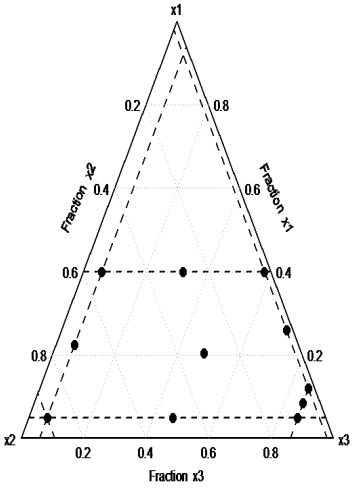

FIGURE 4.1: The Points for Non-Simplex-Shaped-Region with Constrained Area by 3-Components

As the number of the component proportions increases, the number of desirable candidate points grows rather rapidly, as in the experiment on the illumination level of candles illustrated in the following example.

4.2.1 Illustration of the XVERT Algorithm for 4-Components

An experiment considering the level of illumination in candles is an example provided in Box and Draper (2007). The chemical composition of a specific type of flare is formed from magnesium which is indicated by x_1, sodium nitrate that is represented as x_2, and strontium and binder which are represented as x_3 and x_4 respectively. Such ingredients are combined as proportions of weight. The experimenter who was responsible for conducting the experiment suggested the appropriate constraints on the weight of component proportions are

$$0.40 \leq x_1 \leq 0.60,\ \ 0.10 \leq x_2 \leq 0.50, 0.10 \leq x_3 \leq 0.50,\ \ 0.03 \leq x_4 \leq 0.08.$$

In this example it is not easy to draw a four-dimensional space in (x_1, x_2, x_3, x_4), but we can imagine that the feasible region is an irregular polyhedral space. The concern here

is to find out the coordinates of the proportions (x_1, x_2, x_3, x_4) for this space which has limitations on its boundaries. To do this, the above procedure is followed to obtain all admissible points for such a constrained area by finding all vertices, the face centroid points, mid-edge points, and the overall centroid. The procedure that we follow is

Step 1: Rank the components in terms of increasing ranges

TABLE 4.6: Rank the Components

Components	Min(L)	Max(U)	Range
x_4	0.03	0.08	0.05
x_1	0.40	0.60	0.20
x_2	0.10	0.50	0.40
x_3	0.10	0.50	0.40

From the above table, the order of the components can be represented as: $X_1 = x_4$, $X_2 = x_1$, $X_3 = x_2$, $X_4 = x_3$.

Step 2: Forming a two-level design using upper and lower bounds of the 3 components that have the smallest ranges. The number of the treatment combinations that we will obtain is $2^{q-1} = 8$ with components X_1, X_2, and X_3

$$L_4\ L_1\ L_2 \quad L_4\ L_1\ U_2 \quad L_4\ U_1\ L_2 \quad L_4\ U_1\ U_2$$

$$U_4\ L_1\ L_2 \quad U_4\ L_1\ U_2 \quad U_4\ U_1\ L_2 \quad U_4\ U_1\ U_2$$

Step 3: Determine the level of the omitted components (in our case x_3). Therefore, the resulting points are presented in Table 4.7. From this table it can be seen that

TABLE 4.7: Resulting Points

Points	x_1	x_2	x_3	x_4
1	0.40	0.10	0.47	0.03
2	0.60	0.10	0.27	0.03
3	0.40	0.10	0.42	0.08
4	0.60	0.10	0.22	0.08
5	0.40	0.50	0.07	0.03
6	0.60	0.50	-0.13	0.03
7	0.40	0.50	0.02	0.08
8	0.60	0.50	-0.18	0.08

the points (1,2,3,4) are extreme vertices that can be considered as core points of such a design, while the points (5,6,7,8) need to be adjusted because the values -0.13, -0.18 0.07, and 0.02 do not fall inside the acceptable limits. Thus, these points need to be adjusted and, from such points we could generate some candidate subgroups of four points each as follows;

Step 4: Adjusting the points (5,6,7,8), the acceptable points that we obtained are given in Table 4.8.

TABLE 4.8: Resulting Points

Points	x_1	x_2	x_3	x_4
5	0.40	0.47	0.10	0.03
6	0.60	0.27	0.10	0.03
7	0.40	0.42	0.10	0.08
8	0.60	0.22	0.10	0.08

As a result, the number of vertices over the constrained region is eight as it can be shown in Table 4.9. This table presents all admissible extreme vertices; after finding the face

TABLE 4.9: The Extreme Vertices Points

The Points	x_1	x_2	x_3	x_4
1	0.40	0.10	0.47	0.03
2	0.60	0.10	0.27	0.03
3	0.40	0.10	0.42	0.08
4	0.60	0.10	0.22	0.08
5	0.60	0.27	0.10	0.03
6	0.60	0.22	0.10	0.08
7	0.40	0.47	0.10	0.03
8	0.40	0.42	0.10	0.08

centroid, the mid-edge points and overall centroid, the points are shown in Table 4.10. For clarification, the points that are listed as 1-8 are represented as vertices, while the points 9-14 in such a table are defined as the face centroid points that have 1 ingredient level in common. In addition, the points that are listed as 15-26 are defined as the midpoints of the edges, and all these points have 2 ingredient levels in common. Finally, the last point in the above listed points is the overall centroid point.

As is clear from the present example, the number of treatment combinations increases rapidly as the number of such factors increases. Obtaining such points by applying the algorithm in the traditional manual way will become more complicated. A statistical computing package that has mixture capability could be used to obtain such points.

TABLE 4.10: Design Points for Constrained Region By 4-Components

The Points	x_1	x_2	x_3	x_4
1	0.40	0.10	0.47	0.03
2	0.60	0.10	0.27	0.03
3	0.40	0.10	0.42	0.08
4	0.60	0.10	0.22	0.08
5	0.60	0.27	0.10	0.03
6	0.60	0.22	0.10	0.08
7	0.40	0.47	0.10	0.03
8	0.40	0.42	0.10	0.08
9	0.40	0.27	0.27	0.06
10	0.60	0.17	0.17	0.06
11	0.50	0.10	0.35	0.06
12	0.50	0.35	0.10	0.06
13	0.50	0.24	0.24	0.03
14	0.50	0.21	0.21	0.08
15	0.40	0.10	0.45	0.06
16	0.60	0.10	0.25	0.06
17	0.50	0.37	0.10	0.03
18	0.50	0.32	0.10	0.08
19	0.40	0.44	0.10	0.06
20	0.40	0.29	0.29	0.03
21	0.40	0.26	0.26	0.08
22	0.60	0.25	0.10	0.06
23	0.60	0.19	0.19	0.03
24	0.60	0.16	0.16	0.08
25	0.50	0.10	0.37	0.03
26	0.50	0.10	0.32	0.08
27	0.50	0.22	0.22	0.06

Nonetheless, it is important to understand the process of the implementation of the presented algorithm. In the sense that, how such algorithm is implemented to obtain any possible points for the regions under consideration. Thus, under this specific reason, we intend to solve the mixture setting problem when $q = 3$, and 4 using traditional manual processes. An example of the statistical package could be used to find such points, where the constrained area determined by $q \geq 4$ is mixexp that is implemented in the R software (Lawson, 2016).

The XVERT method involved in this statistical package will be illustrated through the 5-component proportional cement example provided in Smith (2005). The 5-component proportions in the example are represented as a version of oxide composition that contributes to the cement formulation; this has the limitations defined by the experimenter

$$0.2098 \leq x_1 \leq 0.2743,\ \ 0.035 \leq x_2 \leq 0.08756, 0.01194 \leq x_3 \leq 0.07508,$$
$$0.02108 \leq x_4 \leq 0.04980,\ \ 0.6219 \leq x_5 \leq 0.6750.$$

The mixexp function Xvert can be used for the present example to obtain any admissible points for such constrained region by following the command below.

```
des=Xvert(5,uc=c(0.2743,0.08756,0.07508,0.0498,0.675),

lc=c(0.2098,0.035,0.01194,0.02108,0.6219),ndm=3)
```

Here, the uc and lc represent the upper and lower bounds for the proportion components respectively. Then, the resulting points that we obtain are given in Table 4.11 and Table 4.12 respectively. All points represented in Table 4.11 are the vertices that

TABLE 4.11: The Extreme Vertices Points

x_1	x_2	x_3	x_4	x_5
0.27	0.04	0.01	0.02	0.66
0.21	0.09	0.01	0.02	0.67
0.21	0.04	0.08	0.02	0.66
0.27	0.04	0.01	0.05	0.63
0.21	0.09	0.01	0.05	0.64
0.21	0.04	0.08	0.05	0.63
0.27	0.07	0.01	0.02	0.62
0.26	0.09	0.01	0.02	0.62
0.27	0.04	0.05	0.02	0.62
0.25	0.04	0.08	0.02	0.62
0.21	0.09	0.06	0.02	0.62
0.21	0.07	0.08	0.02	0.62
0.27	0.04	0.01	0.05	0.62
0.23	0.09	0.01	0.05	0.62
0.27	0.04	0.02	0.05	0.62
0.22	0.04	0.08	0.05	0.62
0.21	0.09	0.03	0.05	0.62
0.21	0.04	0.08	0.05	0.62
0.26	0.04	0.01	0.02	0.68
0.21	0.08	0.01	0.02	0.68
0.21	0.04	0.06	0.02	0.68
0.23	0.04	0.01	0.05	0.68
0.21	0.05	0.01	0.05	0.68
0.21	0.04	0.03	0.05	0.68

have 0 identical ingredients levels, while the points in Table 4.12 are defined as the face centroid points that have 1 identical ingredient levels. Furthermore, the rest of the points with the overall centroid point are listed in Table 4.13 As we can see from the above tables, the number of admissible points for restricted regions for 5-component proportions is considered too large to be involved in the experimental design.

TABLE 4.12: The Face Centroid Points

x_1	x_2	x_3	x_4	x_5
0.21	0.04	0.08	0.04	0.64
0.21	0.04	0.07	0.02	0.67
0.21	0.04	0.05	0.05	0.65
0.21	0.04	0.04	0.04	0.68
0.21	0.09	0.01	0.04	0.66
0.21	0.09	0.04	0.02	0.65
0.21	0.09	0.02	0.05	0.63
0.21	0.09	0.05	0.04	0.62
0.21	0.08	0.01	0.02	0.67
0.21	0.07	0.01	0.05	0.66
0.21	0.07	0.01	0.04	0.68
0.21	0.05	0.08	0.02	0.64
0.21	0.04	0.08	0.05	0.63
0.21	0.06	0.08	0.04	0.62
0.21	0.08	0.07	0.02	0.62
0.21	0.06	0.04	0.02	0.68
0.21	0.07	0.05	0.05	0.62
0.21	0.04	0.02	0.05	0.68
0.27	0.04	0.01	0.04	0.64
0.27	0.04	0.03	0.02	0.64
0.27	0.04	0.02	0.05	0.63
0.27	0.04	0.03	0.04	0.62
0.27	0.05	0.01	0.02	0.64
0.27	0.04	0.01	0.05	0.63
0.27	0.06	0.01	0.04	0.62
0.27	0.05	0.03	0.02	0.62
0.27	0.04	0.02	0.05	0.62
0.27	0.04	0.01	0.02	0.67
0.25	0.04	0.01	0.05	0.65
0.24	0.04	0.01	0.04	0.68
0.23	0.04	0.08	0.02	0.64
0.21	0.04	0.08	0.05	0.63
0.23	0.04	0.08	0.04	0.62
0.26	0.04	0.06	0.02	0.62
0.23	0.04	0.04	0.02	0.68
0.25	0.04	0.05	0.05	0.62
0.22	0.04	0.02	0.05	0.68
0.23	0.09	0.01	0.02	0.65
0.22	0.09	0.01	0.05	0.63
0.24	0.09	0.01	0.04	0.62
0.23	0.09	0.04	0.02	0.62
0.22	0.09	0.02	0.05	0.62
0.27	0.08	0.01	0.02	0.62
0.23	0.06	0.01	0.02	0.68
0.25	0.06	0.01	0.05	0.62
0.22	0.04	0.01	0.05	0.68
0.23	0.05	0.08	0.02	0.62
0.21	0.04	0.08	0.05	0.62

TABLE 4.13: The Centroid Points with Overall Centroid

x_1	x_2	x_3	x_4	x_5
0.21	0.04	0.06	0.04	0.66
0.21	0.09	0.03	0.04	0.64
0.21	0.08	0.01	0.04	0.67
0.21	0.05	0.08	0.04	0.63
0.21	0.07	0.05	0.02	0.65
0.21	0.06	0.04	0.05	0.64
0.21	0.07	0.06	0.04	0.62
0.21	0.05	0.03	0.04	0.68
0.27	0.04	0.02	0.04	0.63
0.27	0.05	0.01	0.04	0.63
0.27	0.05	0.02	0.02	0.63
0.27	0.04	0.01	0.05	0.62
0.27	0.05	0.02	0.04	0.62
0.26	0.04	0.01	0.04	0.66
0.22	0.04	0.08	0.04	0.63
0.25	0.04	0.05	0.02	0.65
0.24	0.04	0.04	0.05	0.64
0.25	0.04	0.05	0.04	0.62
0.23	0.04	0.03	0.04	0.68
0.23	0.09	0.01	0.04	0.64
0.23	0.09	0.03	0.02	0.64
0.22	0.09	0.02	0.05	0.63
0.23	0.09	0.03	0.04	0.62
0.25	0.07	0.01	0.02	0.65
0.24	0.06	0.01	0.05	0.64
0.26	0.07	0.01	0.04	0.62
0.23	0.05	0.01	0.04	0.68
0.22	0.05	0.08	0.02	0.63
0.21	0.04	0.08	0.05	0.62
0.22	0.05	0.08	0.04	0.62
0.25	0.06	0.05	0.02	0.62
0.23	0.05	0.03	0.02	0.68
0.24	0.06	0.04	0.05	0.62
0.22	0.04	0.02	0.05	0.68
0.21	0.06	0.04	0.04	0.65
0.27	0.04	0.02	0.04	0.63
0.24	0.04	0.04	0.04	0.65
0.22	0.09	0.02	0.04	0.63
0.24	0.06	0.01	0.04	0.65
0.22	0.04	0.08	0.04	0.63
0.24	0.06	0.04	0.02	0.65
0.23	0.05	0.03	0.05	0.64
0.24	0.06	0.04	0.04	0.62
0.22	0.05	0.02	0.04	0.68
0.23	0.05	0.04	0.04	0.64

In some situations, such a restricted region can be determined by more than 5-proportions. For example, Smith (2005) describes a concrete mixture experiment with 6-component proportions. The 6-component mixture experiment was carried out to find the optimum proportion of such components for a concrete mixture that met specified requirements for conducting the experiments. Such requirements include the fresh-concrete slump, minimum cost per cubic meter, and compressive strength after 1 day and after 28 days. The 6-components involved in concrete mixtures are: water which is represented as x_1, cement that is indicated by x_2, silica, HRWRA, coarse, and fine which are allocated to x_3, x_4, x_5, and x_6 respectively. In this experiment, the experimenter decided that suitable constraints would be

$$0.160 \leq x_1 \leq 0.185,\ \ 0.130 \leq x_2 \leq 0.150, 0.013 \leq x_3 \leq 0.027,\ \ 0.0046 \leq x_4 \leq 0.0074,$$
$$0.400 \leq x_5 \leq 0.4424,\ \ 0.250 \leq x_6 \leq 0.2924.$$

The resulting constrained region from such components has more than 6 vertices. Thus, to find such vertices with all centroids and overall centroid we will follow the same process as with 5-component proportions. The desirable points obtained following such a method for the region of interest in such cases consist of 32-vertices with 221 centroid points and an overall centroid.

As it can be seen from the designs produced by the XVERT algorithm for the constrained region by different component proportions include design points around perimeter of the region of interest augmented by the centroid. Sometimes it is needed to have additional interior points for achieving different purposes, such as checking the goodness of fitted models. Such additional interior points can obtain by averaging all possible pairs of design points of the experimental region. To obtain these points properly, a statistical computing package in the R software that has a mixture capability could be followed to evaluate such points. For example, the illustration codes for constrained region by 3-component proportions augmenting designs with interior points is

```
B=Xvert(nfac=3, lc=c(0.05, 0.02, 0.06), uc=c(0.40, 0.89, 0.86),
ndm=1)
Af=Fillv(nfac=3, B)
```

Based on the present codes, the Fillv function produces additional interior points and adding them to the design points given in Table 4.5. As a result, the obtained points for the constrained region by 3-component proportions become 66 points, and the feasible region can be displayed graphically as

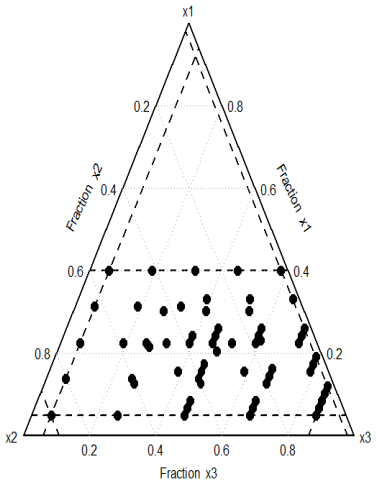

FIGURE 4.2: Additional Interior Points for Experimental Region

Following same process, we will obtain 378 design points for 4-components of illumination level in candles while for the cement and concrete experiments the obtained design points are 6903, and more than 21130 points respectively.

In summary, from the examples above, the number of potential points that had to be evaluated rapidly becomes quite large with q, especially when $q \geq 5$.

In some circumstances and for specific applications, for simplicity the researcher and experimenter do not select or use all such points for experimental purposes. They want to avoid having a very large number of the points which could lead to clusters of such points in a specific factor space occurring in such cases, and thus avoid the conglomeration of such points in a particular area within the simplex. The need arises for a method or process to reduce such a candidate list to a reasonable number of such points.

Much research was done in the statistical literature for adopting this issue during the 1960s and 1970s, by providing some methods and several algorithms that could tackle such a problem. Some of these are now incorporated into statistical computing packages.

Such methods and algorithms for designing in the mixture setting with the constrained region by upper and lower bounds for each components will be discussed in Chapter 5 and Chapter 6.

Chapter 5

Designs for Irregularly Shaped Regions

5.1 Introduction

The previous chapter was devoted to the method for generating the candidate points list by calculating all of the centroid points of any dimension from zero-dimension (the vertices) up to (q-1)-dimension (the overall centroid). After obtaining these points, we found out then that the relationship between the number of such points and the number of the component proportions is positively correlated, i.e. the number of candidate points grows rapidly with q. There is a need now for a process which selects a subset or reduces the number of achievable points by selecting some points from a candidate list to obtain a "good" design, which has a reasonable size and will support the intended models. Different methods or processes have been provided in the literature which can select the obtained points for creating a mixture design, such as a method that is based on design optimality. The search for an optimal design for such cases will be based on determining the statistical models and specifying statistical optimality criteria, since the precision of the fitted model estimates leads to the improvement of many optimality criteria.

Usually, the standard optimality theory is improved under the assumption that the assumed fitted models represent the true relationship of interest. However, the possibility that the fitted model might not provide a good fit to the data collected needs to be taken into consideration. Thus, the precision of fitted model estimates and lack-of-fit testing matter.

In the light of the reasons mentioned above for searching the optimal design for the region under consideration, this chapter is divided into two principal sections. The first

section highlights the methodology, which includes the steps for finding such optimal designs supported with different examples. The second section focuses on constructing the optimal designs for the region of interest when the precision of fitted model estimates and lack-of-fit testing are under consideration.

5.2 An Algorithmic Approach for Searching Optimal Designs

Mathematical algorithms for construction of designs; which are optimal with respect to a chosen criterion, typically include the iterative improvement of an initial N-trial design. The initial design of predetermined size (n) can be generated by selecting with replacement from the candidate design points found using the XVERT Algorithm augmented with additional interior points, such that the information matrix is non-singular. An improvement of the design in the next step is made by an exchange of points in the initial design with points selected from the candidate set, such that N remains fixed. The replacement points technique is based on a statistical optimality criterion, i.e. to determine the point from the initial design that will be replaced with the point from the candidate set, an optimality criterion is needed. For example, for D-optimality, the choice of the point $\boldsymbol{x}$ depends on the determinant of the information matrix or variance-covariance matrix. In this case, we select points at each iteration such that the determinant function is maximized or minimized respectively, i.e. we exchange a point in the initial design with each of the candidate point and evaluate the determinant in each replacement. Then we replace the initial point with the candidate point that provides the optimum criterion value. Continuing in this manner, we replace in turn each support point $\boldsymbol{x_r}$, with a new point, with the other support points fixed, where a new point satisfies the optimal condition of the criterion under consideration.

When the experimental inferences are related to point estimation for individual treatment parameters, the A-optimality criterion will be used and then can be implemented in the exchange algorithm for constructing optimal designs instead.

In the algorithmic approach we must specify a stopping criterion to tell the algorithm when to stop. This can be in terms of the maximum improvement in the criterion value over successive iterations, i.e. if any exchange improves the statistical optimality criterion by at least some small threshold value, we continue the design search. However, if no exchange improves the criterion, then there is no exchange and the process will stop.

It is worth mentioning that for some situations the main benefits of the exchange algorithm are: speed of convergence, which takes a few minutes to display the optimum

designs in our problem; the ease of use and implementation; the possibility of the application of such algorithm to different types of problems; and the possibility of applying it to different optimality criteria for constructing the optimal designs.

5.2.1 Motivating Example

Being interested in selecting a design points from candidate sets in order to construct a mixture designs, the chick feeding experimental data that involve three mixture components with proportions bounded above and below that was mentioned in Chapter 2, Section 2.3.3 will be used.

Proceeding from the candidate points for the restricted region by 3-component proportions provided in Table 4.5, augmented by the additional interior points, which become 66 points in total as described in Chapter 4, and specifying the first order polynomial model represented in (2.9) as the model to be fitted, the optimum treatment combinations (the component proportions) or (the design points) for the region of the interest can be constructed by following the illustrated process above. In this case, the first problem that we need to solve is to select N = 30 random design points from the 66 candidate points (Nc= 66) and consider the selected points as the design points for an initial design. Here and under this case, we decide the number of selected points to be as the same number of the data to be collected for the constrained region based on the three components limitations. The generated initial designs for the restricted area under standard statistically D- and A-criteria I(D) and II(A) respectively for the first degree linear model FLM is presented in Table 5.1. Then, the second issue of interest is how to obtain the treatment combination for the design space to observe the outcome in an optimal manner. The second stage of the process can handle this concern, by trying every possible exchange for each point in turn in the initial design with points from the candidate list, with calculating the optimum criterion under consideration. For example, try every possible exchange for the first component of the first design point in the starting design (0.05, 0.89, 0.06) with each point in Table 4.5 augmented by additional interior points, and for each possible exchange the objective function for either D- or A-optimality is evaluated. The process continues until all of the points in the starting design have been considered for exchange. The whole process is then repeated until a complete pass through the list of design points results in no exchange. The resulting designs in this case after conducting this process are given in Table 5.2.

The whole procedure under the present example have been repeated for 1000 "tries". Under this technique we investigated that each of the run with a different random starting design identifies the same value of the criterion. Moreover, the resulting designs

TABLE 5.1: Initial Designs for the Restricted Area by 3-Components FLM

I(D)			II(A)		
x_1	x_2	x_3	x_1	x_2	x_3
0.05	0.89	0.06	0.05	0.89	0.06
0.40	0.02	0.58	0.40	0.02	0.58
0.40	0.02	0.58	0.40	0.02	0.58
0.40	0.02	0.58	0.40	0.02	0.58
0.40	0.02	0.58	0.40	0.02	0.58
0.40	0.02	0.58	0.40	0.02	0.58
0.40	0.02	0.58	0.12	0.02	0.86
0.40	0.28	0.32	0.12	0.02	0.86
0.40	0.02	0.58	0.12	0.02	0.86
0.40	0.02	0.58	0.05	0.09	0.86
0.40	0.28	0.32	0.26	0.02	0.72
0.05	0.09	0.86	0.40	0.02	0.58
0.26	0.02	0.72	0.40	0.28	0.32
0.40	0.54	0.06	0.40	0.02	0.58
0.08	0.06	0.86	0.40	0.02	0.58
0.05	0.09	0.86	0.40	0.28	0.32
0.05	0.49	0.46	0.40	0.54	0.06
0.08	0.06	0.86	0.08	0.06	0.86
0.26	0.02	0.72	0.05	0.09	0.86
0.26	0.02	0.72	0.05	0.49	0.46
0.40	0.28	0.32	0.08	0.06	0.86
0.05	0.09	0.86	0.26	0.02	0.72
0.40	0.02	0.58	0.26	0.02	0.72
0.08	0.06	0.86	0.40	0.28	0.32
0.20	0.31	0.48	0.05	0.09	0.86
0.40	0.28	0.32	0.40	0.02	0.58
0.26	0.02	0.72	0.08	0.06	0.86
0.12	0.02	0.86	0.20	0.31	0.48
0.12	0.02	0.86	0.40	0.28	0.32
0.12	0.02	0.86	0.26	0.02	0.72

TABLE 5.2: The Optimal Designs for Constrained Region by 3-Components FLM

	I(OD)			II(OA)		
	x_1	x_2	x_3	x_1	x_2	x_3
	0.05	0.89	0.06	0.05	0.89	0.06
	0.05	0.89	0.06	0.05	0.89	0.06
	0.40	0.02	0.58	0.05	0.89	0.06
	0.05	0.89	0.06	0.05	0.09	0.86
	0.40	0.54	0.06	0.05	0.09	0.86
	0.05	0.89	0.06	0.05	0.89	0.06
	0.40	0.54	0.06	0.40	0.54	0.06
	0.40	0.02	0.58	0.05	0.09	0.86
	0.40	0.02	0.58	0.05	0.09	0.86
	0.05	0.09	0.86	0.05	0.09	0.86
	0.40	0.54	0.06	0.05	0.89	0.06
	0.05	0.89	0.06	0.05	0.09	0.86
	0.05	0.09	0.86	0.40	0.02	0.58
	0.05	0.09	0.86	0.05	0.89	0.06
	0.40	0.02	0.58	0.40	0.54	0.06
	0.05	0.09	0.86	0.40	0.02	0.58
	0.40	0.02	0.58	0.05	0.09	0.86
	0.40	0.54	0.06	0.05	0.09	0.86
	0.05	0.89	0.06	0.40	0.54	0.06
	0.05	0.09	0.86	0.05	0.09	0.86
	0.05	0.89	0.06	0.40	0.02	0.58
	0.05	0.09	0.86	0.05	0.09	0.86
	0.05	0.09	0.86	0.40	0.02	0.58
	0.40	0.02	0.58	0.40	0.54	0.06
	0.40	0.54	0.06	0.40	0.54	0.06
	0.05	0.09	0.86	0.40	0.02	0.58
	0.05	0.89	0.06	0.05	0.89	0.06
	0.40	0.54	0.06	0.05	0.09	0.86
	0.05	0.09	0.86	0.05	0.09	0.86
	0.05	0.89	0.06	0.40	0.02	0.58
D-eff	100.00			99.80		
A-eff	99.60			100.00		
df(PE; LoF)	(26;1)			(25;2)		

have a similarity in selecting design points, i.e all selecting points under this case were vertices. Furthermore, comparing the initial design ξ with the optimal design ξ^* under D-efficiency and A-efficiency, which corresponding to the comparisons of information matrices for the initial and optimal designs, the initial designs have 81% D-efficiency and 86% A-efficiency respectively, so the optimal designs have improved the initial designs considerably. As a result, in the final step the obtained designs from the numeric algorithm can be considered as near optimal designs for the constrained area by 3-component proportions.

Moreover, it can be seen from Table 5.2, although the designs under considering and specified standard statistical criteria are similar in terms of their degrees of freedom, i.e they allow only few degrees of freedom for testing lack of fit while most degrees of freedom allocated to pure error estimation, they have different properties for the inferences on estimating the treatment parameters effects. In the sense that, the obtained design (OD) is better for making inferences related to the joint treatment parameters, while the design (OA) is better for the inferences on individual parameters. Here, the number of pure error degrees of freedom are obtained by subtracting the number of the treatment combinations (design points), resulted from generating the designs from candidate points, from the unique value of treatment combinations at each exchange tried.

A first order polynomial model may not always provide a good fit to the data. To improve the fit often a higher order polynomial models can be considered. An example of such models are second order polynomial QM and special cubic models SCM given in (2.11) and (2.13) respectively. From a design point of view, the main differences between these models and first order model lies in the size of the model matrix (design matrix) X. For the quadratic model, the design matrix would be augmented with 3 additional column corresponding to the x_1x_2, x_1x_3, x_2x_3 products. Thus, the design matrix for this case would be a six-column matrix. For the special cubic model, the X matrix would be a 7-column matrix.

Applying the same methodology to the same problem but considering quadratic and special cubic models, the resulting optimal designs under standard criteria in 30 runs are given in Table 5.3 and Table 5.4 respectively. Under these models and with repetition technique we notice that there are many near equally optimal designs that have the same value of the criterion under consideration. Furthermore, the designs selected have a similarity in selecting design points. In the sense that the selected points under second degree linear models were varying between vertices, centroids, and internal points. Likewise for the special cubic model. As it can be seen from Table 5.3 and Table 5.4, the designs under considering and specified statistical criteria are extremes in that they

TABLE 5.3: The Optimal Designs Under Fitted Quadratic Model QM

	OD under QM						OA under QM					
	x_1	x_2	x_3	x_{12}	x_{13}	x_{23}	x_1	x_2	x_3	x_{12}	x_{13}	x_{23}
	0.40	0.54	0.06	0.22	0.02	0.03	0.05	0.89	0.06	0.04	0.003	0.05
	0.05	0.89	0.06	0.04	0.003	0.05	0.05	0.89	0.06	0.04	0.003	0.05
	0.40	0.02	0.58	0.01	0.23	0.01	0.05	0.09	0.86	0.004	0.04	0.08
	0.40	0.54	0.06	0.22	0.02	0.03	0.05	0.89	0.06	0.04	0.003	0.05
	0.40	0.02	0.58	0.01	0.23	0.01	0.19	0.02	0.79	0.004	0.15	0.02
	0.05	0.89	0.06	0.04	0.003	0.05	0.05	0.09	0.86	0.004	0.04	0.08
	0.23	0.38	0.39	0.09	0.09	0.15	0.05	0.09	0.86	0.004	0.04	0.08
	0.40	0.54	0.06	0.22	0.02	0.03	0.05	0.09	0.86	0.004	0.04	0.08
	0.40	0.02	0.58	0.01	0.23	0.01	0.40	0.02	0.58	0.01	0.23	0.01
	0.05	0.49	0.46	0.02	0.02	0.23	0.05	0.89	0.06	0.04	0.003	0.05
	0.05	0.89	0.06	0.04	0.003	0.05	0.05	0.49	0.46	0.02	0.02	0.23
	0.05	0.49	0.46	0.02	0.02	0.23	0.40	0.54	0.06	0.22	0.02	0.03
	0.05	0.09	0.86	0.004	0.04	0.08	0.40	0.28	0.32	0.11	0.13	0.09
	0.40	0.54	0.06	0.22	0.02	0.03	0.19	0.02	0.79	0.004	0.15	0.02
	0.05	0.49	0.46	0.02	0.02	0.23	0.40	0.54	0.06	0.22	0.02	0.03
	0.05	0.89	0.06	0.04	0.003	0.05	0.40	0.02	0.58	0.01	0.23	0.01
	0.23	0.71	0.06	0.16	0.01	0.04	0.19	0.02	0.79	0.004	0.15	0.02
	0.40	0.02	0.58	0.01	0.23	0.01	0.05	0.89	0.06	0.04	0.003	0.05
	0.05	0.09	0.86	0.004	0.04	0.08	0.23	0.38	0.39	0.09	0.09	0.15
	0.23	0.71	0.06	0.16	0.01	0.04	0.23	0.71	0.06	0.16	0.01	0.04
	0.19	0.02	0.79	0.004	0.15	0.02	0.23	0.38	0.39	0.09	0.09	0.15
	0.05	0.09	0.86	0.004	0.04	0.08	0.40	0.54	0.06	0.22	0.02	0.03
	0.05	0.49	0.46	0.02	0.02	0.23	0.05	0.49	0.46	0.02	0.02	0.23
	0.19	0.02	0.79	0.004	0.15	0.02	0.23	0.38	0.39	0.09	0.09	0.15
	0.40	0.02	0.58	0.01	0.23	0.01	0.23	0.71	0.06	0.16	0.01	0.04
	0.05	0.09	0.86	0.004	0.04	0.08	0.23	0.38	0.39	0.09	0.09	0.15
	0.23	0.38	0.39	0.09	0.09	0.15	0.23	0.38	0.39	0.09	0.09	0.15
	0.23	0.38	0.39	0.09	0.09	0.15	0.05	0.49	0.46	0.02	0.02	0.23
	0.19	0.02	0.79	0.004	0.15	0.02	0.23	0.71	0.06	0.16	0.01	0.04
	0.05	0.89	0.06	0.04	0.003	0.05	0.23	0.38	0.39	0.09	0.09	0.15
D-eff	100.00						98.20					
A-eff	93.00						100.00					
df(PE;LoF)	(22,2)						(21,3)					

allow only few degrees of freedom for testing lack of fit. Moreover, although the obtained optimal designs under the special cubic model with respect to different standard statistical criteria are equivalent in terms of their pure error degrees of freedom, they have different properties for the inferences on the treatment parameters. That is, design (OD) is being better for the experimental and inferences purposes on generalized variances of parameters, while design (OA) enhances the inferences for the individuals parameters.

As mentioned before, for the animal husbandry example, the candidate list consisted of 5-vertices, 5-face-centroids, and the overall centroid with 55 additional interior points, which produces 66 points in total. Thus, from the above optimal designs, the points that were chosen using D-criterion to support the quadratic and special cubic models are varying between vertices, face-centroids, and some points were selected from additional interior points. More precisely, the selected points under the second and the special cubic models were 4-vertices and 2 points were selected from the interior points. However, the only difference was in selecting face-centroid points, where under the second degree models the selected face-centroid points based on the D-criterion was 2-face-centroids whereas, under the third degree model it was 3-face-centroids. In contrast to the first degree model, all selected points based on the D-optimality criterion from the candidate

TABLE 5.4: The Optimal Designs for Special Cubic Model

	OD under SCM							OA under SCM						
	x_1	x_2	x_3	x_{12}	x_{13}	x_{23}	x_{123}	x_1	x_2	x_3	x_{12}	x_{13}	x_{23}	x_{123}
	0.05	0.89	0.06	0.04	0.003	0.05	0.003	0.05	0.89	0.06	0.04	0.003	0.05	0.003
	0.40	0.54	0.06	0.22	0.02	0.03	0.01	0.23	0.31	0.46	0.07	0.10	0.14	0.03
	0.05	0.89	0.06	0.04	0.003	0.05	0.003	0.40	0.28	0.32	0.11	0.13	0.09	0.04
	0.05	0.09	0.86	0.003	0.04	0.08	0.003	0.05	0.89	0.06	0.04	0.003	0.05	0.003
	0.05	0.49	0.46	0.02	0.02	0.23	0.01	0.05	0.49	0.46	0.02	0.02	0.23	0.01
	0.40	0.02	0.58	0.01	0.23	0.01	0.004	0.40	0.28	0.32	0.11	0.13	0.09	0.04
	0.40	0.54	0.06	0.22	0.02	0.03	0.01	0.05	0.89	0.06	0.04	0.003	0.05	0.003
	0.05	0.89	0.06	0.04	0.003	0.05	0.003	0.05	0.49	0.46	0.02	0.02	0.23	0.01
	0.40	0.02	0.58	0.01	0.23	0.01	0.005	0.40	0.28	0.32	0.11	0.13	0.09	0.04
	0.40	0.28	0.32	0.11	0.13	0.09	0.04	0.23	0.71	0.06	0.16	0.01	0.04	0.01
	0.05	0.09	0.86	0.004	0.04	0.08	0.004	0.05	0.89	0.06	0.04	0.003	0.05	0.003
	0.23	0.71	0.06	0.16	0.01	0.04	0.01	0.05	0.09	0.86	0.00	0.04	0.08	0.004
	0.05	0.49	0.46	0.02	0.02	0.23	0.01	0.05	0.49	0.46	0.02	0.02	0.23	0.01
	0.40	0.02	0.58	0.01	0.23	0.01	0.004	0.05	0.09	0.86	0.004	0.04	0.08	0.004
	0.05	0.09	0.86	0.004	0.04	0.08	0.004	0.23	0.31	0.46	0.07	0.10	0.14	0.03
	0.40	0.54	0.06	0.22	0.02	0.03	0.01	0.05	0.09	0.86	0.004	0.04	0.08	0.00
	0.40	0.28	0.32	0.11	0.13	0.09	0.04	0.05	0.49	0.46	0.02	0.02	0.23	0.01
	0.26	0.02	0.72	0.01	0.19	0.01	0.004	0.40	0.02	0.58	0.01	0.23	0.01	0.005
	0.05	0.89	0.06	0.04	0.003	0.05	0.003	0.23	0.71	0.06	0.16	0.01	0.04	0.01
	0.05	0.09	0.86	0.004	0.04	0.08	0.004	0.23	0.31	0.46	0.07	0.10	0.14	0.03
	0.40	0.28	0.32	0.11	0.13	0.09	0.04	0.05	0.49	0.46	0.02	0.02	0.23	0.01
	0.05	0.49	0.46	0.02	0.02	0.23	0.01	0.40	0.54	0.06	0.22	0.02	0.03	0.01
	0.23	0.71	0.06	0.16	0.01	0.04	0.01	0.26	0.02	0.72	0.01	0.19	0.01	0.004
	0.05	0.49	0.46	0.02	0.02	0.23	0.01	0.26	0.02	0.72	0.01	0.19	0.01	0.004
	0.40	0.54	0.06	0.22	0.02	0.03	0.01	0.05	0.09	0.86	0.004	0.04	0.08	0.004
	0.24	0.37	0.39	0.09	0.09	0.14	0.03	0.05	0.89	0.06	0.04	0.003	0.05	0.003
	0.24	0.37	0.39	0.09	0.09	0.14	0.03	0.23	0.38	0.39	0.09	0.09	0.15	0.03
	0.40	0.02	0.58	0.01	0.23	0.01	0.005	0.40	0.28	0.32	0.11	0.13	0.09	0.04
	0.24	0.37	0.39	0.09	0.09	0.14	0.03	0.23	0.71	0.06	0.16	0.01	0.04	0.01
	0.19	0.02	0.79	0.004	0.15	0.02	0.005	0.05	0.09	0.86	0.00	0.04	0.08	0.004
D-eff	100.00							96.00						
A-eff	91.31							100.00						
df(PE;LoF)	(21,2)							(20,3)						

set were vertices. Likewise with the A-optimality criterion, the selected points to support the quadratic and special cubic models vary between the vertices, interior, and centroids, while for the first degree the selected points were vertices.

Again, the optimum designs obtained under D- and A-criteria with quadratic and special cubic models have improved the initial designs, in the sense that the initial designs have respectively 97%, 98% D-efficiency and 95%, 96% A-efficiency.

5.2.2 Searching Optimal Designs Through the 4-Component proportional illumination Candle Constraints

To select design points from the candidate points set for the restricted region under illumination candle constraints given in Table 4.10 augmented by the interior points, again, by the same methodology and by the same manner as for 3-component constraints, we can obtain the designs, which includes the optimum treatment combinations. The designs under the first order, quadratic, and special cubic models for the constrained region by 4-component proportions in 20 runs and considering different optimality criteria are displayed in Table 5.5, Table 5.6, Table 5.7, and Table 5.8 respectively. From the above

results, we present the types of points that were chosen using the D-optimality crite-

TABLE 5.5: The Optimal Designs for Constrained Area by 4-Components Under FLM

	(OD)				(OA)			
	x_1	x_2	x_3	x_4	x_1	x_2	x_3	x_4
	0.40	0.10	0.47	0.03	0.40	0.10	0.47	0.03
	0.40	0.47	0.10	0.03	0.40	0.10	0.42	0.08
	0.40	0.10	0.42	0.08	0.40	0.10	0.47	0.03
	0.40	0.10	0.42	0.08	0.40	0.42	0.10	0.08
	0.40	0.47	0.10	0.03	0.40	0.10	0.42	0.08
	0.60	0.22	0.10	0.08	0.60	0.22	0.10	0.08
	0.60	0.22	0.10	0.08	0.60	0.27	0.10	0.03
	0.40	0.42	0.10	0.08	0.40	0.42	0.10	0.08
	0.60	0.27	0.10	0.03	0.60	0.22	0.10	0.08
	0.40	0.10	0.47	0.03	0.60	0.10	0.22	0.08
	0.60	0.10	0.27	0.03	0.40	0.47	0.10	0.03
	0.40	0.10	0.47	0.03	0.40	0.10	0.47	0.03
	0.40	0.47	0.10	0.03	0.40	0.10	0.42	0.08
	0.60	0.10	0.22	0.08	0.40	0.47	0.10	0.03
	0.60	0.10	0.27	0.03	0.40	0.47	0.10	0.03
	0.40	0.47	0.10	0.03	0.60	0.27	0.10	0.03
	0.40	0.42	0.10	0.08	0.60	0.10	0.27	0.03
	0.40	0.10	0.42	0.08	0.40	0.10	0.47	0.03
	0.60	0.10	0.27	0.03	0.60	0.10	0.27	0.03
	0.60	0.22	0.10	0.08	0.60	0.10	0.22	0.08
D-eff	100.00				95.20			
A-eff	91.80				100.00			
df(PE;LoF)	(12,4)				(12,4)			

rion for the different models in Table 5.9, where FLM, QM, and SCM as in the previous example symbolize to the first order, quadratic, and special cubic models, while the vertices, the face centroids, the midpoints of the edge, overall centroid, and additional interior points are represented as V, FC, ME, OC, and AI respectively. Similarly, with respect to the obtained designs under the A-criterion, the selected points for the first degree linear model were vertices, whereas the chosen points under higher order models were varying between vertices and centroids.

Based on the presented outcomes, we found that there is an improvement to the statistical criteria for the initial designs after conducting the process of the numerical algorithm for several "tries" of randomly starting designs for all different models. Thus, the designs resulting from an application and implementation of the technical steps of the algorithm could be accepted as near optimal designs for the region of interest that resulted from illumination candle constraints.

Apart from the improvement criteria, the designs we obtained under the second degree and special cubic linear model under D- and A-optimality criteria are more concerned for testing lack of fit by allowing the imbalance degrees of freedom in favor of that test. However, with the same criteria and the same region under consideration but for the first degree linear model, the obtained designs have different behavior, which is the imbalance tends to occur in favor of the pure error estimation.

TABLE 5.6: The Optimal Designs for Constrained Area by 4-Components Under QM

	(OD)										(OA)									
	x_1	x_2	x_3	x_4	x_{12}	x_{13}	x_{14}	x_{23}	x_{24}	x_{34}	x_1	x_2	x_3	x_4	x_{12}	x_{13}	x_{14}	x_{23}	x_{23}	x_{34}
	0.40	0.42	0.10	0.08	0.17	0.04	0.03	0.04	0.03	0.01	0.40	0.10	0.47	0.03	0.04	0.19	0.01	0.05	0.003	0.01
	0.40	0.10	0.47	0.03	0.04	0.19	0.01	0.05	0.003	0.01	0.40	0.47	0.10	0.03	0.19	0.04	0.01	0.05	0.01	0.003
	0.40	0.47	0.10	0.03	0.19	0.04	0.01	0.05	0.01	0.003	0.40	0.29	0.29	0.03	0.11	0.11	0.01	0.08	0.01	0.01
	0.60	0.22	0.10	0.08	0.13	0.06	0.05	0.02	0.02	0.01	0.50	0.10	0.37	0.03	0.05	0.19	0.01	0.04	0.003	0.01
	0.60	0.10	0.27	0.03	0.06	0.16	0.02	0.03	0.003	0.01	0.50	0.24	0.24	0.03	0.12	0.12	0.01	0.06	0.01	0.01
	0.40	0.10	0.42	0.08	0.04	0.17	0.03	0.04	0.01	0.03	0.60	0.27	0.10	0.03	0.16	0.06	0.02	0.03	0.01	0.003
	0.60	0.10	0.22	0.08	0.06	0.13	0.05	0.02	0.01	0.02	0.40	0.10	0.45	0.05	0.04	0.18	0.02	0.04	0.01	0.02
	0.40	0.29	0.29	0.03	0.11	0.11	0.01	0.08	0.01	0.01	0.50	0.10	0.32	0.08	0.05	0.16	0.04	0.03	0.01	0.03
	0.40	0.10	0.47	0.03	0.04	0.19	0.01	0.05	0.003	0.01	0.60	0.10	0.25	0.05	0.06	0.15	0.03	0.02	0.01	0.01
	0.50	0.10	0.32	0.08	0.05	0.16	0.04	0.03	0.01	0.03	0.50	0.35	0.10	0.05	0.17	0.05	0.03	0.03	0.02	0.01
	0.40	0.47	0.10	0.03	0.19	0.04	0.01	0.05	0.01	0.003	0.40	0.26	0.26	0.08	0.10	0.10	0.03	0.07	0.02	0.02
	0.40	0.42	0.10	0.08	0.17	0.04	0.03	0.04	0.03	0.01	0.40	0.44	0.10	0.05	0.18	0.04	0.02	0.04	0.02	0.01
	0.60	0.27	0.10	0.03	0.16	0.06	0.02	0.03	0.01	0.003	0.50	0.37	0.10	0.03	0.19	0.05	0.01	0.04	0.01	0.003
	0.40	0.10	0.42	0.08	0.04	0.17	0.03	0.04	0.01	0.03	0.60	0.22	0.10	0.08	0.13	0.06	0.05	0.02	0.02	0.01
	0.50	0.24	0.24	0.03	0.12	0.12	0.01	0.06	0.01	0.01	0.50	0.22	0.22	0.05	0.11	0.11	0.03	0.05	0.01	0.01
	0.60	0.10	0.25	0.05	0.06	0.15	0.03	0.02	0.01	0.01	0.50	0.32	0.10	0.08	0.16	0.05	0.04	0.03	0.03	0.01
	0.60	0.25	0.10	0.05	0.15	0.06	0.03	0.02	0.01	0.01	0.50	0.22	0.22	0.05	0.11	0.11	0.03	0.05	0.01	0.01
	0.50	0.35	0.10	0.05	0.17	0.05	0.03	0.03	0.02	0.01	0.60	0.25	0.10	0.05	0.15	0.06	0.03	0.02	0.01	0.01
	0.50	0.10	0.35	0.05	0.05	0.17	0.03	0.03	0.01	0.02	0.60	0.10	0.27	0.03	0.06	0.16	0.02	0.03	0.00	0.01
	0.40	0.27	0.27	0.05	0.11	0.11	0.02	0.07	0.01	0.01	0.50	0.22	0.22	0.05	0.11	0.11	0.03	0.05	0.01	0.01
D-eff	100.00										96.60									
A-eff	81.10										100.00									
df(PE;LoF)	(4,6)										(2,8)									

TABLE 5.7: The D-Optimal Design for Constrained Area by 4-Components Under SCM

	(OD)													
	x_1	x_2	x_3	x_4	x_{12}	x_{13}	x_{14}	x_{23}	x_{24}	x_{34}	x_{123}	x_{124}	x_{134}	x_{234}
	0.40	0.10	0.47	0.03	0.04	0.19	0.01	0.05	0.003	0.01	0.02	0.001	0.00	0.01
	0.40	0.42	0.10	0.08	0.17	0.04	0.03	0.04	0.03	0.01	0.02	0.01	0.003	0.003
	0.40	0.47	0.10	0.03	0.19	0.04	0.01	0.05	0.01	0.003	0.02	0.01	0.001	0.001
	0.50	0.10	0.37	0.03	0.05	0.19	0.01	0.04	0.003	0.01	0.02	0.001	0.001	0.01
	0.60	0.22	0.10	0.08	0.13	0.06	0.05	0.02	0.02	0.01	0.01	0.01	0.002	0.005
	0.60	0.27	0.10	0.03	0.16	0.06	0.02	0.03	0.01	0.003	0.02	0.005	0.001	0.001
	0.50	0.32	0.10	0.08	0.16	0.05	0.04	0.03	0.03	0.01	0.02	0.01	0.003	0.004
	0.40	0.10	0.42	0.08	0.04	0.17	0.03	0.04	0.01	0.03	0.02	0.003	0.008	0.01
	0.40	0.28	0.29	0.03	0.11	0.11	0.01	0.08	0.01	0.01	0.03	0.003	0.002	0.003
	0.60	0.10	0.22	0.08	0.06	0.13	0.05	0.02	0.01	0.02	0.01	0.005	0.002	0.01
	0.40	0.26	0.26	0.08	0.10	0.10	0.03	0.07	0.02	0.02	0.03	0.01	0.01	0.01
	0.60	0.19	0.19	0.03	0.11	0.11	0.02	0.03	0.01	0.01	0.02	0.003	0.001	0.003
	0.50	0.37	0.10	0.03	0.19	0.05	0.01	0.04	0.01	0.003	0.02	0.01	0.006	0.001
	0.60	0.10	0.27	0.03	0.06	0.16	0.02	0.03	0.002	0.03	0.02	0.002	0.001	0.005
	0.40	0.10	0.45	0.05	0.04	0.18	0.02	0.04	0.01	0.02	0.02	0.002	0.002	0.01
	0.50	0.10	0.32	0.08	0.05	0.16	0.04	0.03	0.01	0.03	0.02	0.004	0.003	0.01
	0.50	0.24	0.24	0.03	0.12	0.12	0.01	0.06	0.01	0.01	0.03	0.004	0.002	0.004
	0.40	0.44	0.10	0.05	0.18	0.04	0.02	0.04	0.02	0.01	0.02	0.01	0.002	0.002
	0.40	0.28	0.29	0.03	0.11	0.11	0.01	0.08	0.01	0.01	0.03	0.003	0.002	0.003
	0.50	0.22	0.22	0.05	0.11	0.11	0.03	0.05	0.01	0.01	0.02	0.01	0.003	0.01
D-eff	100.00													
A-eff	89.00													
df(PE;LoF)	(1;5)													

TABLE 5.8: The A-Optimal Design for Constrained Area by 4-Components Under SCM

	(OA)													
	x_1	x_2	x_3	x_4	x_{12}	x_{13}	x_{14}	x_{23}	x_{24}	x_{34}	x_{123}	x_{124}	x_{134}	x_{234}
	0.60	0.22	0.10	0.08	0.13	0.06	0.05	0.02	0.02	0.01	0.01	0.01	0.002	0.005
	0.40	0.47	0.10	0.03	0.19	0.04	0.01	0.05	0.01	0.006	0.02	0.01	0.001	0.001
	0.50	0.21	0.21	0.08	0.10	0.11	0.04	0.04	0.02	0.02	0.02	0.01	0.004	0.01
	0.40	0.26	0.26	0.08	0.10	0.10	0.03	0.07	0.02	0.02	0.03	0.01	0.01	0.01
	0.60	0.10	0.22	0.08	0.06	0.13	0.05	0.02	0.01	0.02	0.01	0.005	0.002	0.01
	0.60	0.27	0.10	0.03	0.16	0.06	0.02	0.03	0.01	0.00	0.02	0.001	0.001	0.005
	0.50	0.21	0.21	0.08	0.10	0.11	0.04	0.04	0.02	0.02	0.02	0.01	0.005	0.01
	0.40	0.42	0.10	0.08	0.17	0.04	0.03	0.04	0.03	0.01	0.02	0.01	0.003	0.003
	0.40	0.10	0.47	0.03	0.04	0.19	0.01	0.05	0.001	0.01	0.02	0.001	0.006	0.01
	0.50	0.37	0.10	0.03	0.19	0.05	0.01	0.04	0.01	0.006	0.02	0.01	0.001	0.002
	0.50	0.23	0.24	0.03	0.12	0.12	0.01	0.06	0.01	0.01	0.03	0.003	0.002	0.004
	0.50	0.10	0.37	0.03	0.05	0.19	0.01	0.04	0.002	0.01	0.02	0.001	0.006	0.01
	0.60	0.10	0.27	0.03	0.06	0.16	0.02	0.03	0.002	0.01	0.02	0.001	0.005	0.001
	0.50	0.10	0.32	0.08	0.05	0.16	0.04	0.03	0.01	0.03	0.02	0.004	0.003	0.01
	0.40	0.26	0.26	0.08	0.10	0.10	0.03	0.07	0.02	0.02	0.03	0.01	0.01	0.01
	0.50	0.24	0.24	0.03	0.12	0.12	0.01	0.06	0.01	0.01	0.03	0.004	0.002	0.004
	0.40	0.10	0.42	0.08	0.04	0.17	0.03	0.04	0.01	0.03	0.02	0.003	0.003	0.01
	0.45	0.29	0.24	0.03	0.13	0.11	0.01	0.07	0.01	0.01	0.03	0.004	0.002	0.003
	0.45	0.24	0.29	0.03	0.11	0.13	0.01	0.07	0.01	0.01	0.03	0.003	0.002	0.004
	0.50	0.32	0.10	0.08	0.16	0.05	0.04	0.03	0.03	0.01	0.02	0.01	0.003	0.004
A-eff	100.00													
D-eff	92.00													
df(PE;LoF)	(2;4)													

TABLE 5.9: Selected Points Based on D-optimal Designs in Illumination Candle Example

Point type	The points for LM	The points for QM	The points for SCM
V	8	8	8
FC	0	4	1
ME	0	4	9
OC	0	0	1
AI	0	0	0

5.2.3 Optimal Designs for Restricted Region by More Proportion Components

Following the same procedure by applying the process, other designs can be constructed for the region determined by larger numbers of mixture variables that have had constraints placed on them, as is the case in the cement formulation and concrete product that we mentioned earlier in Chapter 4. In order to construct optimal designs for such restricted regions resulting from the constrained proportion components, the same technique, manner, and methodology for the exchange algorithm, which are based on design optimality will be followed. In so doing, the optimum treatment combinations, proportion of the components, that were obtained after accepting any exchange that improves the criterion function under consideration were vertices for both regions of interest when the first order polynomial models had been assumed. The resulting optimal designs for the area restricted by 5-component proportions based on cement constraints and for the constrained region defined by 6-component proportions based on concrete limitations are given in Table 5.10 and Table 5.11 respectively. From Table 5.10, the obtained designs based on D- and A-criteria have improved the initial designs with respect to the criteria under consideration noticeably, which are 94% D-efficient and 93% A-efficient. Similar improvements occur to the criteria under the obtained designs for the concrete limitations example and, therefore, the resulting optimal designs under the restricted area by 5-component proportions based on cement constraints and the constrained region defined by 6-component proportions based on concrete limitations will be accepted as near optimal designs for the region of interest.

In the above examples, we constructed different optimal designs for different assumed statistical linear models and different standard criteria, such as D- and A-criteria. We have applied the process of the exchange algorithm to find such designs that are optimal and have presented the points that have been selected for such designs.

When we take into account that the assumed models may not provide a good fit to the data, the precision of estimating fitted models and testing lack-of-fit must be considered. Being interested in these purposes of the accuracy and the goodness of the fitted models,

TABLE 5.10: The Optimal Designs for Cement Formulation

	OD					OA				
	x_1	x_2	x_3	x_4	x_5	x_1	x_2	x_3	x_4	x_5
	0.21	0.09	0.06	0.02	0.62	0.21	0.04	0.08	0.02	0.66
	0.26	0.09	0.01	0.02	0.62	0.25	0.04	0.08	0.02	0.62
	0.21	0.09	0.06	0.02	0.62	0.27	0.04	0.01	0.05	0.62
	0.27	0.04	0.01	0.02	0.66	0.25	0.04	0.08	0.02	0.62
	0.21	0.05	0.01	0.05	0.68	0.21	0.08	0.01	0.02	0.68
	0.21	0.04	0.08	0.02	0.66	0.21	0.08	0.01	0.02	0.68
	0.27	0.04	0.01	0.05	0.62	0.26	0.04	0.01	0.02	0.68
	0.27	0.04	0.05	0.02	0.62	0.21	0.04	0.08	0.05	0.62
	0.23	0.09	0.01	0.05	0.62	0.27	0.04	0.01	0.05	0.62
	0.21	0.04	0.08	0.05	0.63	0.21	0.05	0.01	0.05	0.68
	0.23	0.04	0.01	0.05	0.68	0.25	0.04	0.08	0.02	0.62
	0.27	0.04	0.05	0.02	0.62	0.21	0.08	0.01	0.02	0.68
	0.21	0.04	0.06	0.02	0.68	0.27	0.04	0.01	0.05	0.62
	0.21	0.04	0.08	0.05	0.63	0.21	0.05	0.01	0.05	0.68
	0.27	0.04	0.01	0.05	0.62	0.21	0.04	0.08	0.05	0.62
	0.21	0.08	0.01	0.02	0.68	0.26	0.04	0.01	0.02	0.68
	0.26	0.04	0.01	0.02	0.68	0.21	0.04	0.08	0.05	0.62
	0.23	0.09	0.01	0.05	0.62	0.16	0.15	0.03	0.01	0.40
	0.21	0.09	0.01	0.02	0.67	0.16	0.15	0.03	0.01	0.40
	0.22	0.04	0.08	0.05	0.62	0.16	0.15	0.03	0.01	0.40
D-eff	100.00					98.20				
A-eff	97.70					100.00				
df(PE; LoF)	(5;10)					(7;8)				

TABLE 5.11: The Optimal Designs for Concrete Products

	OD						OA					
	x_1	x_2	x_3	x_4	x_5	x_6	x_1	x_2	x_3	x_4	x_5	x_6
	0.16	0.13	0.01	0.005	0.40	0.29	0.16	0.13	0.01	0.005	0.44	0.25
	0.16	0.13	0.01	0.01	0.44	0.25	0.16	0.13	0.03	0.01	0.43	0.25
	0.16	0.13	0.01	0.01	0.40	0.29	0.18	0.15	0.01	0.005	0.40	0.25
	0.18	0.15	0.01	0.005	0.40	0.25	0.16	0.13	0.01	0.005	0.40	0.29
	0.16	0.13	0.01	0.005	0.44	0.25	0.16	0.13	0.01	0.005	0.44	0.25
	0.18	0.15	0.01	0.01	0.40	0.25	0.18	0.15	0.01	0.005	0.40	0.25
	0.16	0.13	0.01	0.005	0.44	0.25	0.16	0.13	0.01	0.005	0.40	0.29
	0.16	0.13	0.03	0.005	0.40	0.28	0.16	0.13	0.01	0.005	0.44	0.25
	0.19	0.13	0.03	0.01	0.40	0.25	0.16	0.13	0.03	0.005	0.40	0.28
	0.16	0.13	0.01	0.005	0.40	0.29	0.16	0.13	0.03	0.01	0.43	0.25
	0.16	0.13	0.03	0.005	0.43	0.25	0.16	0.13	0.01	0.01	0.44	0.25
	0.16	0.15	0.01	0.01	0.40	0.27	0.16	0.13	0.01	0.01	0.40	0.29
	0.16	0.13	0.01	0.01	0.40	0.29	0.19	0.13	0.01	0.01	0.41	0.25
	0.18	0.15	0.01	0.005	0.40	0.25	0.16	0.15	0.03	0.005	0.40	0.26
	0.16	0.13	0.01	0.01	0.44	0.25	0.19	0.13	0.03	0.005	0.40	0.25
	0.19	0.13	0.03	0.01	0.40	0.25	0.16	0.13	0.01	0.005	0.44	0.25
	0.16	0.15	0.03	0.01	0.41	0.25	0.16	0.15	0.01	0.01	0.42	0.25
	0.16	0.13	0.01	0.005	0.44	0.25	0.18	0.15	0.01	0.005	0.40	0.25
	0.19	0.13	0.03	0.005	0.40	0.25	0.19	0.13	0.03	0.005	0.40	0.25
	0.16	0.15	0.03	0.005	0.40	0.26	0.16	0.15	0.01	0.01	0.40	0.27
	0.16	0.15	0.03	0.005	0.41	0.25	0.19	0.13	0.01	0.01	0.40	0.26
	0.18	0.15	0.01	0.005	0.40	0.25	0.18	0.15	0.01	0.01	0.40	0.25
	0.16	0.13	0.01	0.005	0.40	0.29	0.19	0.13	0.03	0.01	0.40	0.25
	0.16	0.13	0.01	0.01	0.44	0.25	0.16	0.15	0.03	0.01	0.40	0.26
	0.19	0.13	0.03	0.01	0.40	0.25	0.16	0.13	0.01	0.005	0.40	0.29
	0.16	0.15	0.03	0.01	0.41	0.25	0.16	0.13	0.01	0.01	0.40	0.29
	0.18	0.15	0.01	0.005	0.40	0.25	0.19	0.13	0.03	0.005	0.40	0.25
	0.19	0.13	0.01	0.01	0.40	0.26	0.16	0.15	0.03	0.005	0.40	0.26
	0.19	0.13	0.03	0.00	0.40	0.25	0.16	0.15	0.03	0.01	0.40	0.26
	0.16	0.15	0.03	0.01	0.40	0.26	0.19	0.13	0.01	0.01	0.40	0.26
D-eff	100.00						99.90					
A-eff	99.26						100.00					
df(PE; LoF)	(18;6)						(14;10)					

generalized optimality criteria can be used for constructing an optimal designs under these situations. The details are provided in the following section.

5.3 Some Considerations for the Designs

In this section, we consider the situation that the experimenter has more than one objective. In particular, the interest is not only in the parameter estimation but also in estimation the error variance precisely and interesting the lack-of-fit the assumed models. We therefore seek designs which are optimal with respective to the developed GDP- and GAP-optimality criteria introduced in (2.20) and (2.21) respectively.

Under such alternative generalized developed functions, the technique to be followed to obtain optimal designs is to begin by considering that the fitted model is nested within a larger model that is assumed to provide a better fit to the data collected to determine the number of the primary (p) and potential (g) terms in the models. Then, such optimal designs will be obtained by implementing the criteria in the exchange algorithm with considering different scenarios for the scaling parameter τ^2, that is $\tau^2 = 1$, where the prior variance of each potential term is the same as the error variance, and $\tau^2 = 1/g$, where the total prior variance of potential terms is the same as the error variance, as explained in Chapter 2. Several designs will be constructed which depend on the weights that will be allocated to the criteria components. The purpose of conducting this process is to investigate the behavior of the compound criteria as they depend on the weight allocation scheme. Moreover, we want to identify and explore any pattern and features of the relationship between the nature of the criteria and the performances of the resulting designs.

5.3.1 An application of Searching an Optimal Designs Under Generalized Determinant and Trace Based Criteria for Chick Data

Referring to the constrained region by 3-component proportions, the first degree linear models with $p = 3$ will be fitted to the data collected in 30-runs. Then, we will assume that the "true" model contains all interaction terms in the sense that $g = 4$ of them in total.

Now different weights k_r will be allocated to the components of the criteria under consideration (GDP- and GAP-criteria), where $k_r \geq 0$ and $\sum_{r=1}^{3} k_r = 1$ to investigate how efficient the designs are in terms of the single component criteria. Moreover, we will explore what the structure of such designs is in the sense of the relationship between

the weight allocation and degrees of freedom assigned to the pure error and lack-of-fit elements.

The obtained optimal designs under the exchange algorithm given in Table 5.12 and Table 5.13 was run for different 100 random initial starts, and the computational time was between 1 and 2.5 hours.

TABLE 5.12: The Summary Characteristics of GDP- and GAP- Criteria with $\tau^2 = 1$

Designs	GDP(w_i)	$df(PE; LoF)$	D-eff	DP-eff	$LoF(D)$-eff	$LoF(DP)$-eff	A-eff	AP-eff	$LoF(A)$-eff	$LoF(AP)$-eff
I	(1,0,0)	(26,1)	100.00%	99.90%	99.40%	80.9%	99.80%	99.90%	99.60%	99.40%
II	(0,1,0)	(26,1)	100.00%	100.00%	99.90%	83.90%	99.80%	99.60%	99.70%	99.20
III	(0,0,1)	(21,6)	58.20%	32.10%	59.80%	100.00%	77.30%	66.40%	99.80%	86.90%
IV	(0.5,0.5,0)	(26,1)	100.00%	99.90%	99.40%	80.9%	99.80%	99.90%	99.60%	99.40%
V	(0,0.5,0.5)	(26,1)	97.70%	97.60%	98.90%	73.10%	98.0%	97.70%	99.90%	98.90%
VI	(0.5,0,0.5)	(26,1)	100.00%	99.90%	99.40%	80.9%	99.80%	99.90%	99.60%	99.40%
VII	(1/3,1/3,1/3)	(26,1)	98.80%	98.80%	97.60%	83.40%	99.90%	99.60%	99.60%	99.60%
VIII	(0.5,0.25,0.25)	(26,1)	96.40%	99.30%	97.30%	98.60%	99.30%	89.50%	98.30%	98.60%
IX	(0.25,0.25,0.5)	(26,1)	97.90%	98.50%	95.30%	95.80%	99.40%	94.90%	99.50%	98.70%
Designs	GAP(w_i)	$df(PE; LoF)$	D-eff	DP-eff	$LoF(D)$-eff	$LoF(DP)$-eff	A-eff	AP-eff	$LoF(A)$-eff	$LoF(AP)$-eff
I	(1,0,0)	(25,2)	99.60%	99.50%	99.60%	83.90%	100.00%	99.70%	99.60%	99.30%
II	(0,1,0)	(26,1)	99.50%	99.60%	99.90%	83.90%	99.70%	100.00%	99.70%	99.20
III	(0,0,1)	(20,7)	44.40%	42.50%	99.20%	81.70%	67.90%	66.90%	64.10%	100.00%
IV	(0.5,0.5,0)	(26,1)	99.60%	99.50%	99.60%	83.90%	100.00%	99.70%	99.60%	99.30%
V	(0,0.5,0.5)	(26,1)	99.50%	99.40%	99.90%	82.90%	99.00%	99.60%	99.70%	99.20%
VI	(0.5,0,0.5)	(26,1)	99.60%	99.50%	99.60%	83.90%	100.00%	99.70%	99.60%	99.30%
VII	(1/3,1/3,1/3)	(26,1)	99.60%	99.60%	99.60%	82.90%	100.00%	99.60%	99.80%	99.10%
VIII	(0.5,0.25,0.25)	(26,1)	96.50%	99.60%	98.30%	97.60%	96.30%	98.50%	99.30%	99.60%
IX	(0.25,0.25,0.5)	(26,1)	95.90%	97.50%	97.30%	97.80%	94.40%	94.90%	99.50%	98.70%

TABLE 5.13: The Summary Characteristics of GDP- and GAP- Criteria with $\tau^2 = 1/g$

Designs	GDP(w_i)	$df(PE; LoF)$	D-eff	DP-eff	$LoF(D)$-eff	$LoF(DP)$-eff	A-eff	AP-eff	$LoF(A)$-eff	$LoF(AP)$-eff
I	(1,0,0)	(26,1)	100.00%	99.90%	99.80%	89.90%	99.80%	99.90%	99.90%	99.30%
II	(0,1,0)	(26,1)	100.00%	100.00%	99.70%	89.90%	99.8%	99.60%	99.90%	99.30
III	(0,0,1)	(4,23)	62.20%	28.80%	99.90%	100.00%	81.70%	66.70%	99.80%	82.20%
IV	(0.5,0.5,0)	(26,1)	100.00%	99.90%	99.40%	80.9%	99.80%	99.90%	99.60%	99.40%
V	(0,0.5,0.5)	(26,1)	100.00%	99.90%	99.80%	80.40%	98.90%	99.60%	99.80%	99.20%
VI	(0.5,0,0.5)	(26,1)	100.00%	99.90%	99.40%	80.9%	99.80%	99.90%	99.60%	99.40%
VII	(1/3,1/3,1/3)	(26,1)	99.80%	98.90%	97.90%	80.50%	99.90%	99.50%	99.90%	99.30%
VIII	(0.5,0.25,0.25)	(26,1)	93.40%	95.30%	97.30%	88.60%	93.30%	95.50%	96.30%	95.60%
IX	(0.25,0.25,0.5)	(26,1)	94.90%	93.50%	95.30%	89.80%	94.40%	92.90%	96.50%	95.70%
Designs	GAP(w_i)	$df(PE; LoF)$	D-eff	DP-eff	$LoF(D)$-eff	$LoF(DP)$-eff	A-eff	AP-eff	$LoF(A)$-eff	$LoF(AP)$-eff
I	(1,0,0)	(25,2)	99.60%	99.50%	99.80%	89.90%	100.00%	99.70%	99.80%	99.20%
II	(0,1,0)	(26,1)	99.50%	99.60%	99.50%	89.90%	99.70%	100.00%	99.90%	99.40
III	(0,0,1)	(8,19)	39.20%	38.30%	99.40%	76.60.70%	63.90%	63.30%	99.90%	100.00%
IV	(0.5,0.5,0)	(26,1)	99.60%	99.50%	99.60%	83.90%	100.00%	99.70%	99.60%	99.30%
V	(0,0.5,0.5)	(26,1)	97.30%	97.30%	99.70%	80.50%	98.20%	97.90%	99.90%	99.30%
VI	(0.5,0,0.5)	(26,1)	99.60%	99.50%	99.60%	83.90%	100.00%	99.70%	99.60%	99.30%
VII	(1/3,1/3,1/3)	(26,1)	99.60%	99.50%	99.60%	83.90%	100.00%	99.70%	99.60%	99.30%
VIII	(0.5,0.25,0.25)	(26,1)	94.40%	94.30%	97.30%	94.60%	95.30%	95.50%	96.30%	97.60%
IX	(0.25,0.25,0.5)	(26,1)	90.90%	93.50%	96.30%	95.80%	99.40%	94.90%	97.50%	98.70%

In all resulting optimal designs in Table 5.12, there are replicated treatment combinations and thus the designs allow pure error estimation meanwhile allowing fewer degrees of freedom for checking lack-of-fit. These degrees of freedom are subcategories resulted

from classifying the residual degrees of freedom. In almost most of the resulting optimal designs there are 26 degrees of freedom allocated to the experimental pure error (PE), and the remaining residuals degrees of freedom would be classified as 1 degree of freedom for lack-of-fit tests except for the designs obtained under A-optimality criteria and pure error components (LOF(DP) and LOF(AP)).

Generally, as it can be seen from Table 5.12 the imbalance tends to occur in favor of the pure error estimation where some weight is allocated on either of the A, AP components and D, DP elements.

It is also notable that some designs that were obtained as optimal designs with respect to different allocated weights and even with respect to different statistical criteria, for example D-optimal designs, remain optimal if some weights are distributed equally among either primary term components as in design (IV) or primary and potential components as in design (VI) in Table 5.12.

Similar performances occur under generalized-trace-based optimal criteria, i.e. the obtained A-optimal designs remain optimal if some weights are distributed equally either between two-components as in designs (IV, VI) or among all three components as design (VII) in Table 5.12.

Under exploring the relationship between optimality criteria, it can be observed from Table 5.12 that, under GDP-criteria, all designs have large D-efficiencies and DP-efficiencies (all efficiencies values are above 95%) except those designs obtained when no weight is allocated on either D- or DP-elements.

To investigate the relationship between potential and primary term components we can see that, if some weight is assigned to the DP component, the resulting designs tend to be good in terms of LOF(DP)-efficiencies. However, the opposite is not always true, in the sense that, the LOF(DP)-optimal design (design III in Table 5.12) is found to be only 58.20% D-efficient and 32.10% DP-efficient. The design for the resulted constrained region CR by 3-component proportions itself is presented in Table 5.14. Similar manner seems to be occur for the designs obtained under the GAP-criteria. However, LOF-efficiencies are large for almost all designs where is found with the values tending to be almost higher than 80%, which in this case can be evaluated.

As a result and based on the present case, we observed that there is a certain contradiction in the performances for the criterion components corresponding to the primary terms and the potential terms in the models.

It is worth mentioning that, the selected points under the LOF(DP)-optimal design from candidate points are varying between vertex (0.05,0.09,0.86), face-centroid (0.26,0.02,0.72),

TABLE 5.14: The Resulting LOF(DP)-Optimal Designs for CR by 3-components

x_1	x_2	x_3
0.24	0.30	0.46
0.05	0.09	0.86
0.15	0.38	0.46
0.26	0.02	0.72
0.05	0.09	0.86
0.15	0.38	0.46
0.24	0.04	0.72
0.24	0.04	0.72
0.24	0.04	0.72
0.31	0.50	0.19
0.24	0.30	0.46
0.26	0.02	0.72
0.24	0.30	0.46
0.24	0.30	0.46
0.26	0.02	0.72
0.31	0.50	0.19
0.30	0.30	0.40
0.24	0.30	0.46
0.24	0.30	0.46
0.05	0.09	0.86
0.05	0.69	0.26
0.15	0.38	0.46
0.24	0.04	0.72
0.24	0.30	0.46
0.05	0.69	0.26
0.30	0.30	0.40
0.05	0.09	0.86
0.26	0.02	0.72
0.15	0.38	0.46
0.31	0.50	0.19

and the remaining points are interior points that are situated in the middle of the region under consideration.

Next we are turning to the second scenario, which is advocating $\tau^2 = 1/g$. Under this assumption, the criteria under consideration depend on the inverse of the number of the potential terms. Table 5.13 provides the performances of the obtained designs with considering the same layout of the weight allocations. The obtained optimal designs seem to behave similarly to the optimal designs under $\tau^2 = 1$, in terms of the allocation of degrees of freedom to pure error and lack-of-fit components and the relationship between the nature of criteria components. As a result, in this case setting different values of the variance scaling parameter resulted in similar behavior and performances of optimal designs.

It is worth discovering the tendencies of the optimal designs and the behavior of the criteria components when higher order models are assumed to be fitted to the data. Thus, we will assume the fitted model is the second degree linear model, which contains linear and binary-interaction terms. Thus, the number of the primary terms will be $p = 6$. The "true" model that is assumed to provide a better fit to the data will be the same as in the case of fitting the first order model. Here the number of potential terms is $g = 1$. As the number of the missed terms in this case is only one, only one scenario will be under consideration. The performances of the optimal designs under GDP- and GAP-criteria together with the allocation degrees of freedom and the efficiencies with respect

to the individual component criteria are given in Table 5.15. By considering the same

TABLE 5.15: The Summary Characteristics of GDP- and GAP- Criteria with $\tau^2 = 1/g$,1

Designs	GDP(w_i)	$df(PE;LoF)$	D-eff	DP-eff	$LoF(D)$-eff	$LoF(DP)$-eff	A-eff	AP-eff	$LoF(A)$-eff	$LoF(AP)$-eff
I	(1,0,0)	(22,2)	100.00%	98.90%	99.90%	91.70%	98.20%	98.40%	99.90%	99.80%
II	(0,1,0)	(22,2)	99.80%	100.00%	99.90%	99.90%	98.20%	98.40%	99.90%	99.80
III	(0,0,1)	(22,2)	29.20%	29.20%	99.90%	100.00%	52.80%	52.90%	99.90%	99.80%
IV	(0.5,0.5,0)	(22,2)	100.00%	98.90%	99.90%	91.70%	98.20%	98.40%	99.90%	99.80%
V	(0,0.5,0.5)	(22,2)	100.00%	98.90%	99.90%	91.70%	98.20%	98.40%	99.90%	99.80%
VI	(0.5,0,0.5)	(22,2)	99.90%	99.40%	99.80%	98.51%	98.20%	98.60%	99.90%	99.30%
VII	(1/3,1/3,1/3)	(22,2)	100.00%	98.90%	99.90%	91.70%	98.20%	98.40%	99.90%	99.80%
VIII	(0.5,0.25,0.25)	(21,3)	96.40%	97.30%	97.30%	95.60%	99.30%	98.50%	97.30%	96.60%
IX	(0.25,0.25,0.5)	(22,2)	94.90%	95.50%	95.30%	97.80%	97.40%	94.90%	98.50%	98.70%
Designs	GAP(w_i)	$df(PE;LoF)$	D-eff	DP-eff	$LoF(D)$-eff	$LoF(DP)$-eff	A-eff	AP-eff	$LoF(A)$-eff	$LoF(AP)$-eff
I	(1,0,0)	(21,3)	93.00%	92.20%	99.90%	99.80%	100.00%	99.90%	99.90%	99.90%
II	(0,1,0)	(21,3)	90.40%	90.10%	99.90%	99.80%	99.90%	100.00%	99.90%	99.40
III	(0,0,1)	(20,4)	45.80%	44.90%	99.40%	99.80%	70.90%	70.30%	99.90%	100.00%
IV	(0.5,0.5,0)	(21,3)	93.00%	92.20%	99.90%	99.80%	100.00%	99.90%	99.90%	99.90%
V	(0,0.5,0.5)	(20,4)	84.20%	83.40%	99.70%	99.50%	99.20%	99.30%	99.70%	99.80%
VI	(0.5,0,0.5)	(21,3)	92.60%	91.80%	99.80%	99.70%	99.80%	99.90%	99.80%	99.90%
VII	(1/3,1/3,1/3)	(20,4)	92.10%	91.20%	99.90%	99.70%	99.90%	99.70%	99.80%	99.50%
VIII	(0.5,0.25,0.25)	(19,5)	94.40%	94.30%	97.30%	98.60%	96.30%	96.50%	98.30%	98.60%
IX	(0.25,0.25,0.5)	(20,4)	90.90%	93.50%	95.30%	97.80%	99.40%	94.90%	98.50%	96.70%

weights allocations scheme, again, similar performances can be observed as in the case of fitting the first order model, in terms of distribution of the degrees of freedom between pure error and lack-of-fit components under GDP-optimum criteria. Furthermore, the same relationship pattern holds for the primary terms and lack-of-fit components in both determinant- and trace- based criteria. However, the most notable result under this case is the large value of efficiencies of LOF(DP) and LOF(AP) elements for all optimal designs obtained either under GDP- or GAP-criteria, with all efficiencies found to be above 90%. Moreover, all optimal designs found under determinant based criteria are more extreme than those optimal designs for trace based criteria; in which they are similar as far as the experimental pure error degrees of freedom are concerned. Other than that, such optimal designs have similar quantities in terms of their efficiencies.

5.3.2 Optimal Designs Based on Generalized Compound Criteria of Illumination Candle Constraints

For the illumination candle mixture example as discussed before, the region under consideration is restricted by 4-component proportions. By assuming fitting the first degree linear model to obtain optimal designs under the GDP- and GAP-developed criteria, the same technique as for 3-component proportions will be followed. In this case, the number of the primary terms based on the assumed fitted model is $p = 4$. Second and third order terms will be taken as potential missed terms, thus there are $g = 10$ of them in total. Under the exchange algorithm, and with considering the variance scaling

parameter remains the same, i.e. $\tau^2 = 1$ and $\tau_2 = 1/g$, we ran the algorithm with 100 different random starts. In this case, as the number of the candidate points getting increases, the computation time increased to more than 15 hours. The resulting designs are presented in Table 5.16 for considering the value of scaling variance $\tau^2 = 1$, and Table 5.17 for setting $\tau^2 = 1/g$. The same relationship pattern holds for the primary

TABLE 5.16: The Summary Characteristics of GDP- and GAP- Criteria with $\tau^2 = 1$

Designs	GDP(w_i)	$df(PE; LoF)$	D-eff	DP-eff	$LoF(D)$-eff	$LoF(DP)$-eff	A-eff	AP-eff	$LoF(A)$-eff	$LoF(AP)$-eff
I	(1,0,0)	(12,4)	100.00%	95.40%	98.90%	98.90%	95.20%	95.80%	99.90%	99.80%
II	(0,1,0)	(15,1)	93.60%	100.00%	99.30%	99.00%	97.50%	98.40%	99.90%	99.40
III	(0,0,1)	(14,2)	73.60%	77.10%	98.90%	100.00%	80.80%	80.20%	99.90%	99.80%
IV	(0.5,0.5,0)	(14,2)	84.70%	87.80%	99.70%	99.30%	93.10%	93.60%	98.90%	98.70%
V	(0,0.5,0.5)	(14,2)	83.50%	86.50%	99.50%	99.40%	92.10%	92.70%	99.80%	99.50%
VI	(0.5,0,0.5)	(7,9)	64.70%	67.80%	99.50%	99.40%	78.20%	78.60%	99.90%	99.30%
VII	(1/3,1/3,1/3)	(13,3)	84.30%	86.80%	99.90%	97.50%	92.90%	93.20%	99.70%	99.30%
VIII	(0.5,0.25,0.25)	(14,2)	86.50%	89.30%	97.30%	98.60%	90.30%	93.50%	96.30%	98.60%
IX	(0.25,0.25,0.5)	(13,3)	90.90%	89.50%	96.30%	95.80%	93.40%	94.90%	99.50%	98.70%
Designs	GAP(w_i)	$df(PE; LoF)$	D-eff	DP-eff	$LoF(D)$-eff	$LoF(DP)$-eff	A-eff	AP-eff	$LoF(A)$-eff	$LoF(AP)$-eff
I	(1,0,0)	(12,4)	91.80%	94.20%	98.80%	99.20%	100.00%	99.70%	99.90%	99.80%
II	(0,1,0)	(14,2)	93.90%	98.40%	99.50%	99.40%	99.40%	100.00%	99.90%	99.70
III	(0,0,1)	(13,3)	45.90%	33.70%	99.80%	99.40%	83.10%	78.30%	99.90%	100.00%
IV	(0.5,0.5,0)	(13,3)	83.80%	86.90%	98.20%	99.20%	93.40%	93.90%	99.80%	99.90%
V	(0,0.5,0.5)	(13,3)	82.60%	85.70%	99.80%	99.80%	92.70%	93.30%	99.70%	99.60%
VI	(0.5,0,0.5)	(7,9)	63.80%	66.90%	99.20%	99.80%	93.40%	93.90%	99.80%	99.80%
VII	(1/3,1/3,1/3)	(4,12)	83.30%	86.40%	100.00%	98.90%	93.00%	93.00%	99.80%	99.70%
VIII	(0.5,0.25,0.25)	(11,5)	87.40%	88.30%	97.30%	99.60%	94.30%	93.50%	99.30%	98.60%
IX	(0.25,0.25,0.5)	(12,4)	90.90%	93.50%	95.30%	95.80%	95.40%	94.90%	99.50%	98.70%

TABLE 5.17: The Summary Characteristics of GDP- and GAP- Criteria with $\tau^2 = 1/g$

Designs	GDP(w_i)	$df(PE; LoF)$	D-eff	DP-eff	$LoF(D)$-eff	$LoF(DP)$-eff	A-eff	AP-eff	$LoF(A)$-eff	$LoF(AP)$-eff
I	(1,0,0)	(12,4)	100.00%	95.40%	99.90%	99.80%	95.20%	95.80%	99.90%	99.90%
II	(0,1,0)	(15,1)	93.60%	100.00%	99.90%	99.90%	97.50%	98.40%	99.90%	99.40
III	(0,0,1)	(14,2)	30.40%	30.20%	99.90%	100.00%	67.90%	68.80%	99.90%	99.80%
IV	(0.5,0.5,0)	(14,2)	84.70%	87.80%	99.70%	99.30%	93.10%	93.60%	98.90%	98.70%
V	(0,0.5,0.5)	(14,2)	83.50%	82.90%	99.90%	99.90%	92.60%	93.90%	99.80%	99.20%
VI	(0.5,0,0.5)	(4,12)	64.70%	64.20%	99.90%	99.91%	92.90%	94.10%	99.90%	99.30%
VII	(1/3,1/3,1/3)	(7,9)	84.70%	84.20%	99.90%	97.90%	93.00%	94.30%	99.90%	99.30%
VIII	(0.5,0.25,0.25)	(6,10)	87.40%	89.30%	97.30%	98.60%	94.30%	94.50%	98.30%	98.60%
IX	(0.25,0.25,0.5)	(14,2)	90.90%	93.50%	95.30%	95.80%	99.40%	94.90%	99.50%	98.70%
Designs	GAP(w_i)	$df(PE; LoF)$	D-eff	DP-eff	$LoF(D)$-eff	$LoF(DP)$-eff	A-eff	AP-eff	$LoF(A)$-eff	$LoF(AP)$-eff
I	(1,0,0)	(12,4)	91.80%	94.20%	99.90%	99.40%	100.00%	99.70%	99.90%	99.60%
II	(0,1,0)	(14,2)	93.90%	98.40%	99.90%	99.90%	99.40%	100.00%	99.90%	99.40
III	(0,0,1)	(14,2)	33.40%	33.10%	99.50%	99.80%	67.90%	68.90%	99.90%	100.00%
IV	(0.5,0.5,0)	(13,3)	83.80%	86.90%	99.50%	99.80%	93.40%	93.90%	99.80%	99.30%
V	(0,0.5,0.5)	(7,9)	82.90%	83.60%	99.70%	99.40%	93.00%	94.60%	99.70%	99.80%
VI	(0.5,0,0.5)	(4,12)	63.80%	63.30%	99.80%	99.70%	93.40%	94.60%	99.80%	99.90%
VII	(1/3,1/3,1/3)	(7,9)	82.90%	82.40%	99.90%	99.80%	92.90%	94.10%	99.80%	99.50%
VIII	(0.5,0.25,0.25)	(12,4)	86.40%	89.30%	97.30%	98.60%	95.30%	89.50%	98.30%	98.60%
IX	(0.25,0.25,0.5)	(7,9)	92.90%	93.50%	96.30%	98.80%	96.40%	95.90%	98.50%	99.70%

component and lack-of-fit part in the determinant and trace based criteria, in the sense that, if some weights are allocated to the DP-part, the obtained designs performs quite well in terms of LOF(DP)-efficiency. However, LOF(DP)-optimal design itself (design III in Table 5.16) is found to be 73.60% D-efficient and 77.10% DP-efficient. However,

such contradictory relationship is more evident in the determinant based criteria than in the trace optimum criteria.

Under this example, and from Table 5.16 and Table 5.17, the contradiction of the relationship between criteria that correspond to the primary terms and the criteria that correspond to the potential ones is more substantial for the trace form than the determinant components when considering $\tau^2 = 1$.

It is also notable that, although some designs were obtained when there is some weight distributed equally between D and LOF(DP) components, they seem to be compromises regarding the distribution of the degrees of freedom between pure error and lack-of-fit components. Such design (design VI in Table 5.16) has minimum values of D-efficiency as well as DP-efficiency. That is found the design to be 64.70% D-efficient and 67.80% DP-efficient. Such a design is given in Table 5.18. Same performance happens for the

TABLE 5.18: The Resulting GDP-Optimal Design for CR by 4-components

x_1	x_2	x_3	x_4
0.60	0.27	0.10	0.03
0.60	0.27	0.10	0.03
0.60	0.27	0.10	0.03
0.60	0.10	0.22	0.08
0.40	0.42	0.10	0.08
0.40	0.10	0.47	0.03
0.40	0.47	0.10	0.03
0.40	0.47	0.10	0.03
0.40	0.47	0.10	0.03
0.60	0.10	0.22	0.08
0.40	0.10	0.47	0.03
0.40	0.10	0.42	0.08
0.60	0.10	0.22	0.08
0.40	0.10	0.42	0.08
0.40	0.42	0.10	0.08
0.40	0.10	0.47	0.03
0.40	0.10	0.47	0.03
0.40	0.42	0.10	0.08
0.40	0.42	0.10	0.08
0.60	0.22	0.10	0.08

trace based criteria.

Table 5.17 summarizes the optimal designs resulting when taking $\tau^2 = 1/g$ under consideration. Most designs obtained seem to behave similarly to those design obtained under assuming the prior variance of each potential term is the same as the error variance. However, apart from this similarity, the obtained optimal designs under the trace based criteria are less extreme regarding the distribution of the degrees of freedom than the determinant form for most obtained optimal designs.

We will next assume that the second degree model is to be fitted, again with the third order terms being the potential terms. The region of interest is as before and we consider the same layout of scaling tuning τ^2 as before. Here the number of the primary terms become $p = 10$, and therefore, the third order terms will be taken as potentially missed terms, that is g =4. The performances of the resulting optimal designs under

determinant and trace criteria considering different scaling parameters are summarized in Table 5.19 and Table 5.20 respectively. For some designs, when there are no replica-

TABLE 5.19: The Summary Characteristics of GDP- and GAP- Criteria with $\tau^2 = 1$

Designs	GDP(w_i)	$df(PE; LoF)$	D-eff	DP-eff	$LoF(D)$-eff	$LoF(DP)$-eff	A-eff	AP-eff	$LoF(A)$-eff	$LoF(AP)$-eff
I	(1,0,0)	(4,6)	100.00%	93.80%	99.90%	91.90%	96.60%	93.80%	100.00%	91.90%
II	(0,1,0)	(10,0)	85.30%	100.00%	99.90%	98.80%	92.40%	96.40%	100.00%	98.80
III	(0,0,1)	(10,0)	64.12%	72.30%	99.98%	100.00%	52.20%	69.40%	99.98%	100.00%
IV	(0.5,0.5,0)	(0,10)	94.12%	0%	99.90%	0%	95.20%	0%	98.90%	0%
V	(0,0.5,0.5)	(0,10)	94.12%	0%	99.90%	0%	95.70%	0%	99.90%	0%
VI	(0.5,0,0.5)	(6,4)	96.40%	87.90%	99.70%	99.90%	94.70%	96.90%	98.90%	96.90%
VII	(1/3,1/3,1/3)	(5,5)	97.80%	82.20%	99.90%	95.90%	94.80%	96.00%	99.80%	95.90%
VIII	(0.5,0.25,0.25)	(7,3)	84.50%	82.60%	98.50%	95.60%	93.20%	94.00%	97.80%	96.60%
IX	(0.25,0.25,0.5)	(4,6)	90.50%	92.30%	98.50%	96.70%	94.60%	93.40%	96.88%	99.90%
Designs	GAP(w_i)	$df(PE; LoF)$	D-eff	DP-eff	$LoF(D)$-eff	$LoF(DP)$-eff	A-eff	AP-eff	$LoF(A)$-eff	$LoF(AP)$-eff
I	(1,0,0)	(2,8)	81.10%	94.60%	99.70%	86.60%	100.00%	91.40%	99.70%	86.60%
II	(0,1,0)	(9,1)	88.10%	98.00%	99.80%	98.50%	96.20%	100.00%	99.80%	98.50
III	(0,0,1)	(5,5)	64.13%	72.30%	99.90%	100.00%	52.20%	69.40%	99.90%	100.00%
IV	(0.5,0.5,0)	(7,3)	81.40%	88.50%	99.50%	97.60%	96.80%	99.80%	99.90%	97.60%
V	(0,0.5,0.5)	(0,10)	94.30%	0%	99.90%	0%	95.20%	0%	99.70%	0%
VI	(0.5,0,0.5)	(0,10)	94.12%	0%	99.60%	0%	95.40%	0%	99.50%	0%
VII	(1/3,1/3,1/3)	(7,3)	86.40%	89.30%	95.30%	96.60%	86.60%	89.60%	99.50%	97.60%
VIII	(0.5,0.25,0.25)	(6,4)	86.40%	89.30%	95.30%	96.60%	88.30%	88.50%	98.30%	92.60%
IX	(0.25,0.25,0.5)	(4,6)	88.90%	89.50%	95.30%	95.80%	89.40%	88.90%	98.50%	98.70%

TABLE 5.20: The Summary Characteristics of GDP- and GAP- Criteria with $\tau^2 = 1/g$

Designs	GDP(w_i)	$df(PE; LoF)$	D-eff	DP-eff	$LoF(D)$-eff	$LoF(DP)$-eff	A-eff	AP-eff	$LoF(A)$-eff	$LoF(AP)$-eff
I	(1,0,0)	(4,6)	100.00%	93.80%	97.10%	90.90%	96.60%	93.80%	89.00%	90.20%
II	(0,1,0)	(10,0)	85.30%	100.00%	95.80%	95.90%	92.40%	96.40%	94.02%	90.80
III	(0,0,1)	(8,2)	55.20%	77.40%	96.08%	100.00%	42.20%	60.40%	97.98%	100.00%
IV	(0.5,0.5,0)	(0,10)	94.12%	0%	89.80%	0%	95.20%	0%	94.90%	0%
V	(0,0.5,0.5)	(0,10)	94.12%	0%	89.80%	0%	95.70%	0%	94.90%	0%
VI	(0.5,0,0.5)	(7,3)	90.50%	90.90%	88.70%	87.09%	95.20%	96.90%	89.70%	96.70%
VII	(1/3,1/3,1/3)	(4,6)	97.80%	82.20%	99.90%	95.90%	94.80%	96.00%	99.80%	95.90%
VIII	(0.5,0.25,0.25)	(7,3)	89.50%	88.60%	96.06%	95.60%	94.20%	94.50%	96.80%	96.60%
IX	(0.25,0.25,0.5)	(3,7)	90.50%	92.30%	95.60%	95.70%	96.80%	95.50%	96.88%	96.90%
Designs	GAP(w_i)	$df(PE; LoF)$	D-eff	DP-eff	$LoF(D)$-eff	$LoF(DP)$-eff	A-eff	AP-eff	$LoF(A)$-eff	$LoF(AP)$-eff
I	(1,0,0)	(2,8)	81.10%	94.60%	96.70%	88.50%	100.00%	91.40%	98.40%	87.30%
II	(0,1,0)	(9,1)	88.10%	98.00%	95.85%	96.06%	96.20%	100.00%	95.80%	95.50
III	(0,0,1)	(4,6)	62.20%	72.50%	95.80%	100.00%	52.40%	67.20%	96.40%	100.00%
IV	(0.5,0.5,0)	(7,3)	81.40%	88.50%	99.50%	97.60%	96.80%	99.80%	99.90%	97.60%
V	(0,0.5,0.5)	(0,10)	94.30%	0%	99.90%	0%	95.20%	0%	99.70%	0%
VI	(0.5,0,0.5)	(0,10)	94.12%	0%	99.60%	0%	95.40%	0%	99.50%	0%
VII	(1/3,1/3,1/3)	(8,2)	89.50%	89.90%	94.60%	95.40%	88.90%	89.60%	95.50%	97.60%
VIII	(0.5,0.25,0.25)	(5,5)	90.40%	89.90%	96.50%	96.60%	89.90%	88.50%	97.30%	96.60%
IX	(0.25,0.25,0.5)	(4,6)	89.90%	89.80%	94.50%	96.80%	89.50%	88.80%	97.50%	97.70%

tions in the treatment combinations, some efficiencies such as D-, DP-, LOF(DP)-, and LOF(AP)-efficiency cannot be evaluated, and thus they are set to be zero.

Based on Table 5.19, we investigate that most of the performances we observed under assuming a fitted first degree linear model remain present in the present case. However, some patterns do differ. For example, for optimal designs under GDP-criteria, some designs, such as (design IV and design V) are extremes, which are more concerned

with lack-of-fit test rather than pure error estimation. In particular, those designs which obtained when some weights were distributed equally either between primary components or among primary and potential parts, perform quite well with respect to other individual primary criteria. Both designs are given in Table 5.21 and Table 5.22 respectively. Similar performances occur for the optimal designs obtained under trace-

TABLE 5.21: The Resulting GDP-Optimal Design for CR by 4-components

x_1	x_2	x_3	x_4	x_{12}	x_{13}	x_{14}	x_{23}	x_{24}	x_{34}
0.60	0.10	0.27	0.03	0.06	0.16	0.02	0.03	0.00	0.01
0.40	0.47	0.10	0.03	0.19	0.04	0.01	0.05	0.01	0.00
0.60	0.27	0.10	0.03	0.16	0.06	0.02	0.03	0.01	0.00
0.40	0.10	0.42	0.08	0.04	0.17	0.03	0.04	0.01	0.03
0.40	0.42	0.10	0.08	0.17	0.04	0.03	0.04	0.03	0.01
0.60	0.10	0.22	0.08	0.06	0.13	0.05	0.02	0.01	0.02
0.60	0.22	0.10	0.08	0.13	0.06	0.05	0.02	0.02	0.01
0.40	0.10	0.45	0.05	0.04	0.18	0.02	0.04	0.01	0.02
0.40	0.10	0.47	0.03	0.04	0.19	0.01	0.05	0.00	0.01
0.40	0.44	0.10	0.05	0.18	0.04	0.02	0.04	0.02	0.01
0.40	0.28	0.29	0.03	0.11	0.11	0.01	0.08	0.01	0.01
0.40	0.26	0.26	0.08	0.10	0.10	0.03	0.07	0.02	0.02
0.60	0.10	0.25	0.05	0.06	0.15	0.03	0.02	0.01	0.01
0.60	0.25	0.10	0.05	0.15	0.06	0.03	0.02	0.01	0.01
0.60	0.19	0.19	0.03	0.11	0.11	0.02	0.03	0.01	0.01
0.60	0.16	0.16	0.08	0.10	0.10	0.05	0.03	0.01	0.01
0.50	0.10	0.37	0.03	0.05	0.19	0.01	0.04	0.00	0.01
0.50	0.10	0.32	0.08	0.05	0.16	0.04	0.03	0.01	0.03
0.50	0.37	0.10	0.03	0.19	0.05	0.01	0.04	0.01	0.00
0.50	0.32	0.10	0.08	0.16	0.05	0.04	0.03	0.03	0.01

TABLE 5.22: The Resulting GDP-Optimal Design for CR by 4-components

x_1	x_2	x_3	x_4	x_{12}	x_{13}	x_{14}	x_{23}	x_{24}	x_{34}
0.60	0.10	0.27	0.03	0.06	0.16	0.02	0.03	0.00	0.01
0.40	0.47	0.10	0.03	0.19	0.04	0.01	0.05	0.01	0.00
0.60	0.27	0.10	0.03	0.16	0.06	0.02	0.03	0.01	0.00
0.40	0.10	0.42	0.08	0.04	0.17	0.03	0.04	0.01	0.03
0.40	0.42	0.10	0.08	0.17	0.04	0.03	0.04	0.03	0.01
0.60	0.10	0.22	0.08	0.06	0.13	0.05	0.02	0.01	0.02
0.60	0.22	0.10	0.08	0.13	0.06	0.05	0.02	0.02	0.01
0.40	0.10	0.45	0.05	0.04	0.18	0.02	0.04	0.01	0.02
0.40	0.10	0.47	0.03	0.04	0.19	0.01	0.05	0.00	0.01
0.40	0.44	0.10	0.05	0.18	0.04	0.02	0.04	0.02	0.01
0.40	0.28	0.29	0.03	0.11	0.11	0.01	0.08	0.01	0.01
0.40	0.26	0.26	0.08	0.10	0.10	0.03	0.07	0.02	0.02
0.60	0.10	0.25	0.05	0.06	0.15	0.03	0.02	0.01	0.01
0.60	0.25	0.10	0.05	0.15	0.06	0.03	0.02	0.01	0.01
0.60	0.19	0.19	0.03	0.11	0.11	0.02	0.03	0.01	0.01
0.60	0.16	0.16	0.08	0.10	0.10	0.05	0.03	0.01	0.01
0.50	0.10	0.37	0.03	0.05	0.19	0.01	0.04	0.00	0.01
0.50	0.10	0.32	0.08	0.05	0.16	0.04	0.03	0.01	0.03
0.50	0.37	0.10	0.03	0.19	0.05	0.01	0.04	0.01	0.00
0.50	0.32	0.10	0.08	0.16	0.05	0.04	0.03	0.03	0.01

based criteria. However, the extreme designs under trace criteria are obtained when equally distributing the weights allocation between primary and potential components (V, VI in Table 5.19).

Apart from the extreme behavior of some designs, the designs obtained with equal weights distributions among all three components, such as design (VII under GDP-criteria) allow a reasonable compromise between degrees of freedom for estimating pure error and testing lack-of-fit. This design, in particular, performs quite well with respect

to the other individual criteria. The illustrative design is presented in Table 5.23. Similar tendency can be found in the optimal design under the trace based criteria when unequally weights have been allocated to the criteria components, such as (design VIII in Table 5.20).

TABLE 5.23: The Resulting GDP-Optimal Design for CR by 4-components

x_1	x_2	x_3	x_4	x_{12}	x_{13}	x_{14}	x_{23}	x_{24}	x_{34}
0.40	0.47	0.10	0.03	0.19	0.04	0.01	0.05	0.01	0.00
0.40	0.10	0.47	0.03	0.04	0.19	0.01	0.05	0.00	0.01
0.40	0.47	0.10	0.03	0.19	0.04	0.01	0.05	0.01	0.00
0.40	0.42	0.10	0.08	0.17	0.04	0.03	0.04	0.03	0.01
0.60	0.27	0.10	0.03	0.16	0.06	0.02	0.03	0.01	0.00
0.40	0.29	0.29	0.03	0.11	0.11	0.01	0.08	0.01	0.01
0.40	0.42	0.10	0.08	0.17	0.04	0.03	0.04	0.03	0.01
0.40	0.10	0.45	0.05	0.04	0.18	0.02	0.04	0.01	0.02
0.45	0.10	0.37	0.08	0.05	0.17	0.04	0.04	0.01	0.03
0.60	0.27	0.10	0.03	0.16	0.06	0.02	0.03	0.01	0.00
0.55	0.27	0.10	0.08	0.15	0.06	0.04	0.03	0.02	0.01
0.40	0.29	0.29	0.03	0.11	0.11	0.01	0.08	0.01	0.01
0.60	0.10	0.27	0.03	0.06	0.16	0.02	0.03	0.00	0.01
0.40	0.10	0.42	0.08	0.04	0.17	0.03	0.04	0.01	0.03
0.60	0.10	0.22	0.08	0.06	0.13	0.05	0.02	0.01	0.02
0.40	0.10	0.47	0.03	0.04	0.19	0.01	0.05	0.00	0.01
0.60	0.10	0.25	0.05	0.06	0.15	0.03	0.02	0.01	0.01
0.50	0.35	0.10	0.05	0.17	0.05	0.03	0.03	0.02	0.01
0.60	0.22	0.10	0.08	0.13	0.06	0.05	0.02	0.02	0.01
0.50	0.10	0.37	0.03	0.05	0.19	0.01	0.04	0.00	0.01

5.3.3 Constructing an Optimal Designs for a Region Constrained by More Mixture Components

For the constrained region resulting from the cement mixture constraints, yet again the methodology for the implementation of the alternative criteria in the exchange algorithm will be used for searching and evaluating the near optimal designs over the design region, through the candidate points, which customarily involve the iterative improvement of the initial design. A 20-run design with such constraints was sought to fit the first order model with considering different values of the tuning parameter τ^2. So the number of the primary terms for the cement formulation is $p = 5$. The second order terms will be taken as potentially missed terms, and there are $g = 10$ in total of them. The performance of the optimal designs obtained under different scaling parameters are shown in Table 5.24 and Table 5.25 respectively.

In the present restricted region resulting from the cement mixture constraints, the optimal designs (I and VI) obtained under A-optimality and GAP-criteria presented in Table 5.24 are found to have relatively low DP-efficiency values in the sense that, when there is no weight allocated to the AP-component, the A-optimal design (I) has only 70.10% DP-efficiency, whereas, the GAP-optimal design (VI) is found to be 68.40% DP-efficient.

TABLE 5.24: The Summary Characteristics of GDP- and GAP- Criteria with $\tau^2 = 1$

Designs	GDP(w_i)	$df(PE; LoF)$	D-eff	DP-eff	$LoF(D)$-eff	$LoF(DP)$-eff	A-eff	AP-eff	$LoF(A)$-eff	$LoF(AP)$-eff
I	(1,0,0)	(5,10)	100.00%	97.20%	99.80%	82.90%	98.20%	93.30%	99.90%	93.00%
II	(0,1,0)	(14,1)	93.90%	100.00%	99.80%	80.20%	91.90%	86.99%	99.00%	94.50
III	(0,0,1)	(14,1)	61.12%	60.11%	99.90%	100.00%	79.10%	69.90%	99.90%	99.80%
IV	(0.5,0.5,0)	(10,5)	98.70%	97.50%	99.90%	81.30%	96.60%	97.20%	99.90%	98.50%
V	(0,0.5,0.5)	(11,4)	98.00%	99.80%	99.90%	61.80%	95.70%	96.80%	99.90%	98.90%
VI	(0.5,0,0.5)	(11,4)	98.20%	99.80%	99.80%	61.80%	97.40%	98.40%	99.90%	98.90%
VII	(1/3,1/3,1/3)	(10,5)	99.40%	97.50%	99.90%	82.50%	98.00%	98.60%	99.80%	98.50%
VIII	(0.5,0.25,0.25)	(10,5)	94.50%	92.60%	98.50%	95.60%	93.20%	94.00%	97.80%	96.60%
IX	(0.25,0.25,0.5)	(7,8)	90.50%	92.30%	98.50%	96.70%	94.60%	93.40%	96.88%	99.90%
Designs	GAP(w_i)	$df(PE; LoF)$	D-eff	DP-eff	$LoF(D)$-eff	$LoF(DP)$-eff	A-eff	AP-eff	$LoF(A)$-eff	$LoF(AP)$-eff
I	(1,0,0)	(7,8)	97.70%	70.10%	99.80%	93.90%	100.00%	98.20%	99.90%	96.20%
II	(0,1,0)	(12,3)	94.10%	98.70%	99.80%	81.80%	98.90%	100.00%	99.80%	98.90
III	(0,0,1)	(14,1)	59.60%	71.80%	99.80%	99.90%	30.20%	20.40%	99.80%	100.00%
IV	(0.5,0.5,0)	(11,4)	96.30%	94.50%	96.60%	94.90%	99.60%	99.30%	99.70%	98.70%
V	(0,0.5,0.5)	(12,3)	91.30%	93.50%	92.14%	91.20%	98.20%	99.70%	99.30%	98.70%
VI	(0.5,0,0.5)	(11,4)	90.30%	68.40%	91.20%	89.40%	98.40%	99.50%	99.20%	98.20%
VII	(1/3,1/3,1/3)	(12,3)	87.90%	92.30%	86.60%	85.80%	98.20%	99.60%	98.90%	98.30%
VIII	(0.5,0.25,0.25)	(9,6)	96.40%	97.30%	95.30%	96.60%	96.30%	98.50%	98.30%	92.60%
IX	(0.25,0.25,0.5)	(10,5)	88.90%	89.50%	95.30%	95.80%	89.40%	88.90%	98.50%	98.70%

TABLE 5.25: The Summary Characteristics of GDP- and GAP- Criteria with $\tau^2 = 1/g$

Designs	GDP(w_i)	$df(PE; LoF)$	D-eff	DP-eff	$LoF(D)$-eff	$LoF(DP)$-eff	A-eff	AP-eff	$LoF(A)$-eff	$LoF(AP)$-eff
I	(1,0,0)	(5,10)	100.00%	97.20%	95.80%	92.90%	98.20%	93.30%	96.90%	95.00%
II	(0,1,0)	(14,1)	93.90%	100.00%	98.80%	90.95%	91.90%	86.99%	99.90%	96.40
III	(0,0,1)	(14,1)	43.08%	49.17%	99.90%	100.00%	59.50%	60.50%	99.98%	99.80%
IV	(0.5,0.5,0)	(10,5)	98.70%	97.50%	99.60%	90.70%	96.60%	97.20%	100.00%	98.50%
V	(0,0.5,0.5)	(0,15)	93.90%	0%	99.90%	0%	91.70%	0%	99.90%	0%
VI	(0.5,0,0.5)	(11,4)	98.20%	99.80%	99.90%	90.70%	97.50%	97.80%	99.90%	98.90%
VII	(1/3,1/3,1/3)	(10,5)	98.20%	97.50%	99.90%	90.70%	98.10%	97.90%	99.80%	95.90%
VIII	(0.5,0.25,0.25)	(10,5)	94.50%	92.60%	89.50%	91.60%	95.20%	96.00%	97.80%	96.60%
IX	(0.25,0.25,0.5)	(11,4)	91.50%	94.30%	93.50%	92.70%	94.60%	93.40%	96.88%	99.90%
Designs	GAP(w_i)	$df(PE; LoF)$	D-eff	DP-eff	$LoF(D)$-eff	$LoF(DP)$-eff	A-eff	AP-eff	$LoF(A)$-eff	$LoF(AP)$-eff
I	(1,0,0)	(7,8)	97.70%	70.10%	99.90%	99.90%	100.00%	98.20%	99.90%	96.20%
II	(0,1,0)	(12,3)	94.10%	98.70%	99.90%	91.10%	98.90%	100.00%	98.80%	99.80
III	(0,0,1)	(14,1)	59.70%	71.80%	99.90%	99.60%	78.20%	77.40%	99.90%	100.00%
IV	(0.5,0.5,0)	(11,4)	96.30%	94.50%	95.80%	96.90%	99.60%	99.30%	97.70%	99.70%
V	(0,0.5,0.5)	(11,4)	95.00%	65.10%	99.70%	97.40%	99.80%	99.80%	99.90%	98.90%
VI	(0.5,0,0.5)	(13,2)	86.20%	69.20%	98.40%	97.50%	97.10%	98.10%	99.80%	99.60%
VII	(1/3,1/3,1/3)	(11,4)	91.90%	86.20%	99.40%	97.20%	98.30%	98.60%	99.90%	97.80%
VIII	(0.5,0.25,0.25)	(12,3)	86.40%	89.30%	93.30%	95.60%	89.30%	89.50%	98.30%	92.60%
IX	(0.25,0.25,0.5)	(10,5)	89.90%	89.50%	95.30%	95.80%	89.40%	87.90%	99.50%	98.70%

Similar patterns occur when assuming the scaling variance parameter is the inverse of the number of potential terms as given in Table 5.25. The designs obtained by allocating no weights to primary terms, such as (design V and design VI under trace-based-criteria in Table 5.25), both designs have low DP-efficiencies, which are found to be 65.10% and 69.20% respectively. On the contrary, the same designs perform quite well in terms of A- and AP- components. The optimal designs obtained under trace criteria are given in Table 5.26 and Table 5.27 respectively.

It is also notable that the designs obtained under GDP-criteria with no weights allocated to the primary terms as designs (V, VI in Table 5.24 have low LOF(DP)-efficiencies,

TABLE 5.26: The Resulting GAP-Optimal Design for CR by 5-components

x_1	x_2	x_3	x_4	x_5
0.22	0.09	0.02	0.05	0.62
0.22	0.09	0.02	0.05	0.62
0.21	0.04	0.03	0.05	0.68
0.21	0.04	0.06	0.02	0.68
0.27	0.04	0.02	0.05	0.62
0.25	0.04	0.08	0.02	0.62
0.26	0.09	0.01	0.02	0.62
0.26	0.09	0.01	0.02	0.62
0.21	0.08	0.01	0.02	0.68
0.22	0.09	0.02	0.05	0.62
0.26	0.04	0.01	0.02	0.68
0.27	0.04	0.02	0.05	0.62
0.26	0.04	0.01	0.02	0.68
0.26	0.04	0.01	0.02	0.68
0.25	0.04	0.08	0.02	0.62
0.21	0.04	0.03	0.05	0.68
0.21	0.04	0.03	0.05	0.68
0.21	0.08	0.01	0.02	0.68
0.21	0.09	0.06	0.02	0.62
0.21	0.09	0.06	0.02	0.62

TABLE 5.27: The Resulting GAP-Optimal Design for CR by 5-components

x_1	x_2	x_3	x_4	x_5
0.23	0.04	0.04	0.02	0.68
0.21	0.09	0.04	0.02	0.65
0.22	0.04	0.01	0.05	0.68
0.21	0.09	0.03	0.05	0.62
0.21	0.04	0.08	0.05	0.63
0.22	0.04	0.01	0.05	0.68
0.21	0.09	0.04	0.02	0.65
0.21	0.09	0.04	0.02	0.65
0.26	0.07	0.01	0.04	0.62
0.26	0.07	0.01	0.04	0.62
0.26	0.07	0.01	0.04	0.62
0.27	0.04	0.05	0.02	0.62
0.22	0.04	0.01	0.05	0.68
0.27	0.04	0.05	0.02	0.62
0.21	0.04	0.08	0.05	0.63
0.21	0.09	0.03	0.05	0.62
0.21	0.09	0.04	0.02	0.65
0.23	0.04	0.04	0.02	0.68
0.23	0.04	0.04	0.02	0.68
0.22	0.04	0.01	0.05	0.68

that is they only are 61.80% LOF(DP)-efficient. The designs itself are presented in Table 5.28. However, and apart from this notification, the lack-of-fit efficiencies either under determinant- or trace- based criteria for almost all resulting designs are large, where they can be evaluated and taken under consideration for optimization purposes. Moreover, the performances and relationship pattern between the primary components and lack-of-fit parts remain the same as for the previous examples.

We extend our study to an example with 6-component proportions, i.e. we use the design region based on the concrete mixture constraints. To construct an optimal designs for such design regions, assuming the first degree linear model to be fitted to the region of interest, and different statistical optimality criteria, the same process as for the cement proportion constraints will be used. However, in this case the number of the primary parts will be $p = 6$, while the number of the potential components is $g = 15$. The summary characteristics of the obtained optimal designs for the design region in this case are represented in Table 5.29 for considering the value of tuning parameter $\tau^2 = 1$,

TABLE 5.28: The Resulting GDP-Optimal Designs for CR by 5-components

(0.5,0,0.5) V					(0,0.5,0.5) VI				
x_1	x_2	x_3	x_4	x_5	x_1	x_2	x_3	x_4	x_5
0.21	0.04	0.08	0.05	0.62	0.21	0.07	0.08	0.02	0.62
0.21	0.04	0.08	0.05	0.62	0.21	0.05	0.01	0.05	0.68
0.21	0.09	0.06	0.02	0.62	0.21	0.05	0.01	0.05	0.68
0.21	0.08	0.01	0.02	0.68	0.21	0.05	0.01	0.05	0.68
0.27	0.04	0.05	0.02	0.62	0.27	0.07	0.01	0.02	0.62
0.23	0.09	0.01	0.05	0.62	0.27	0.04	0.01	0.02	0.66
0.21	0.04	0.08	0.05	0.62	0.22	0.09	0.02	0.05	0.62
0.23	0.09	0.01	0.05	0.62	0.22	0.09	0.02	0.05	0.62
0.21	0.09	0.06	0.02	0.62	0.27	0.04	0.02	0.05	0.62
0.27	0.04	0.01	0.02	0.66	0.27	0.04	0.02	0.05	0.62
0.21	0.04	0.02	0.05	0.68	0.21	0.04	0.08	0.05	0.63
0.21	0.09	0.06	0.02	0.62	0.21	0.04	0.08	0.02	0.66
0.27	0.04	0.02	0.05	0.62	0.21	0.04	0.08	0.05	0.63
0.27	0.04	0.02	0.05	0.62	0.27	0.04	0.01	0.02	0.66
0.27	0.04	0.01	0.02	0.66	0.21	0.04	0.08	0.02	0.66
0.21	0.08	0.01	0.02	0.68	0.22	0.09	0.02	0.05	0.62
0.27	0.04	0.05	0.02	0.62	0.27	0.07	0.01	0.02	0.62
0.21	0.04	0.06	0.02	0.68	0.21	0.09	0.01	0.02	0.67
0.21	0.04	0.02	0.05	0.68	0.21	0.09	0.01	0.02	0.67
0.21	0.04	0.06	0.02	0.68	0.21	0.07	0.08	0.02	0.62

while Table 5.30 shows the resulting when $\tau^2 = 1/g$.

TABLE 5.29: The Summary Characteristics of GDP- and GAP- Criteria with $\tau^2 = 1$

Designs	GDP(w_i)	$df(PE;LoF)$	D-eff	DP-eff	$LoF(D)$-eff	$LoF(DP)$-eff	A-eff	AP-eff	$LoF(A)$-eff	$LoF(AP)$-eff
I	(1,0,0)	(18,6)	100.00%	89.14%	97.90%	89.70%	99.90%	98.90%	100.00%	98.60%
II	(0,1,0)	(23,1)	96.80%	100.00%	98.80%	83.90%	98.80%	99.80%	99.90%	99.70
III	(0,0,1)	(21,3)	59.80%	62.00%	98.90%	100.00%	73.80%	74.30%	98.98%	99.80%
IV	(0.5,0.5,0)	(0,24)	92.30%	0%	89.20%	0%	97.60%	0%	85.70%	0%
V	(0,0.5,0.5)	(21,3)	96.80%	99.50%	99.90%	98.90%	99.20%	99.80%	97.90%	98.90%
VI	(0.5,0,0.5)	(19,5)	98.90%	99.50%	95.90%	96.70%	98.80%	98.90%	100.00%	98.90%
VII	(1/3,1/3,1/3)	(17,7)	98.10%	97.10%	98.90%	97.00%	98.50%	99.70%	99.80%	95.60%
VIII	(0.5,0.25,0.25)	(16,8)	94.50%	92.60%	97.50%	98.60%	98.50%	99.00%	98.80%	98.60%
IX	(0.25,0.25,0.5)	(11,13)	96.50%	94.30%	97.50%	97.70%	98.60%	98.40%	96.88%	99.90%
Designs	GAP(w_i)	$df(PE;LoF)$	D-eff	DP-eff	$LoF(D)$-eff	$LoF(DP)$-eff	A-eff	AP-eff	$LoF(A)$-eff	$LoF(AP)$-eff
I	(1,0,0)	(14,10)	99.26%	85.20%	99.90%	61.34%	100.00%	76.88%	99.90%	98.23%
II	(0,1,0)	(18,6)	97.10%	88.22%	98.90%	63.53%	99.51%	100.00%	98.90%	99.90
III	(0,0,1)	(20,4)	51.88%	53.31%	99.90%	99.80%	82.02%	82.42%	98.90%	100.00%
IV	(0.5,0.5,0)	(18,6)	99.24%	98.83%	97.90%	98.30%	99.17%	99.69%	98.70%	99.76%
V	(0,0.5,0.5)	(20,4)	92.70%	95.26%	97.90%	98.78%	98.64%	99.13%	98.89%	98.98%
VI	(0.5,0,0.5)	(18,6)	98.17%	98.52%	98.70%	98.30%	99.89%	99.86%	98.90%	97.90%
VII	(1/3,1/3,1/3)	(19,5)	93.43%	94.94%	90.40%	95.56%	98.00%	97.33%	96.90%	97.89%
VIII	(0.5,0.25,0.25)	(18,6)	96.40%	89.30%	93.30%	95.60%	93.30%	92.50%	98.30%	92.60%
IX	(0.25,0.25,0.5)	(20,4)	90.90%	93.50%	96.30%	97.80%	96.40%	97.90%	96.50%	96.70%

Table 5.30 shows that the resulted optimal design (IV) under the determinant criteria allocates all degrees of freedom to testing lack-of-fit. Thus, the efficiencies of this design with respect to some of the individual statistical optimality criteria cannot be evaluated and are set to be zero.

With respect to the designs obtained under the criteria that enhance the inferences for the individuals parameters, some optimal designs, such as (I and II in Table 5.29) are poor with respect to LOF(DP)-efficiency. These designs, in particular have minimum values of LOF(DP)-efficiency, where is found to be 61.34% and 63.53% respectively.

TABLE 5.30: The Summary Characteristics of GDP- and GAP- Criteria with $\tau^2 = 1/g$

Designs	GDP(w_i)	$df(PE; LoF)$	D-eff	DP-eff	$LoF(D)$-eff	$LoF(DP)$-eff	A-eff	AP-eff	$LoF(A)$-eff	$LoF(AP)$-eff
I	(1,0,0)	(18,6)	100.00%	89.14%	99.90%	89.90%	99.90%	98.90%	99.90%	98.30%
II	(0,1,0)	(23,1)	96.80%	100.00%	90.80%	90.90%	98.80%	99.80%	89.90%	89.70
III	(0,0,1)	(20,4)	41.60%	38.70%	100.00%	100.00%	74.70%	75.20%	100.00%	99.70%
IV	(0.5,0.5,0)	(0,24)	92.30%	0%	87.12%	0%	97.60%	0%	63.70%	0%
V	(0,0.5,0.5)	(19,5)	98.30%	90.50%	99.90%	99.80%	99.50%	99.90%	99.90%	99.60%
VI	(0.5,0,0.5)	(19,5)	98.90%	91.02%	99.90%	99.80%	99.80%	99.70%	99.90%	98.60%
VII	(1/3,1/3,1/3)	(19,5)	99.19%	91.30%	99.90%	98.80%	98.50%	98.90%	99.90%	97.90%
VIII	(0.5,0.25,0.25)	(14,10)	94.50%	94.90%	89.50%	92.60%	95.20%	96.90%	98.80%	98.60%
IX	(0.25,0.25,0.5)	(15,9)	92.50%	94.50%	93.50%	92.70%	97.60%	98.40%	98.88%	99.90%
Designs	GAP(w_i)	$df(PE; LoF)$	D-eff	DP-eff	$LoF(D)$-eff	$LoF(DP)$-eff	A-eff	AP-eff	$LoF(A)$-eff	$LoF(AP)$-eff
I	(1,0,0)	(14,10)	99.26%	85.20%	99.90%	89.68%	100.00%	76.88%	99.90%	98.29%
II	(0,1,0)	(18,6)	97.10%	88.22%	99.90%	87.57%	99.51%	100.00%	99.90%	99.49
III	(0,0,1)	(23,1)	43.17%	41.32%	99.99%	98.42%	73.45%	74.21%	99.90%	100.00%
IV	(0.5,0.5,0)	(18,6)	99.24%	98.83%	99.90%	98.90%	99.17%	99.69%	99.70%	98.49%
V	(0,0.5,0.5)	(19,5)	95.36%	87.83%	99.80%	97.77%	97.17%	98.69%	99.79%	99.61%
VI	(0.5,0,0.5)	(20,4)	97.82%	91.04%	94.89%	100.00%	99.72%	99.56%	98.80%	97.72%
VII	(1/3,1/3,1/3)	(18,6)	97.18%	88.33%	96.50%	97.51%	87.71%	98.60%	97.90%	97.80%
VIII	(0.5,0.25,0.25)	(17,7)	86.40%	89.30%	97.30%	98.60%	89.30%	89.50%	98.30%	98.60%
IX	(0.25,0.25,0.5)	(19,5)	90.90%	93.50%	95.30%	95.80%	89.40%	94.90%	99.50%	98.70%

However, LOF(DP)-optimal design turned out to be optimal when some weights are distributed equally between the standard criteria (A-optimality criteria) and lack-of-fit components when the tuning parameter is $\tau^2 = 1/g$ (design VI in Table 5.30). The design itself is shown in Table 5.31. In general, most of the patterns and performances observed

TABLE 5.31: The Resulting GAP-Optimal Design for CR by 5-components

x_1	x_2	x_3	x_4	x_5	x_6
0.16	0.13	0.01	0.005	0.44	0.25
0.16	0.13	0.03	0.005	0.40	0.28
0.16	0.13	0.01	0.005	0.40	0.29
0.19	0.13	0.03	0.005	0.40	0.25
0.19	0.15	0.01	0.005	0.40	0.25
0.16	0.13	0.01	0.005	0.44	0.25
0.19	0.13	0.03	0.005	0.40	0.25
0.16	0.13	0.01	0.005	0.40	0.29
0.19	0.13	0.03	0.005	0.40	0.25
0.19	0.13	0.01	0.01	0.40	0.26
0.16	0.15	0.03	0.01	0.41	0.25
0.16	0.13	0.01	0.01	0.44	0.25
0.16	0.13	0.01	0.01	0.40	0.29
0.16	0.15	0.03	0.01	0.41	0.25
0.16	0.15	0.03	0.01	0.41	0.25
0.16	0.15	0.01	0.005	0.40	0.27
0.19	0.13	0.01	0.01	0.40	0.26
0.19	0.13	0.01	0.01	0.40	0.26
0.16	0.13	0.01	0.01	0.44	0.25
0.19	0.15	0.01	0.005	0.40	0.25
0.16	0.13	0.03	0.005	0.40	0.28
0.16	0.15	0.01	0.005	0.40	0.27
0.16	0.13	0.01	0.01	0.40	0.29
0.16	0.15	0.01	0.005	0.40	0.27
0.16	0.13	0.03	0.005	0.40	0.28
0.16	0.13	0.01	0.005	0.44	0.25
0.19	0.15	0.01	0.005	0.40	0.25
0.19	0.13	0.01	0.01	0.40	0.26
0.16	0.13	0.01	0.005	0.44	0.25
0.16	0.15	0.03	0.01	0.41	0.25

in previous examples is remain occurring in the concrete blends example. However, some efficiencies do differ. Almost all optimal designs obtained either under criteria reinforced the experimental and inferences purposes on generalized variances of parameters or the

criteria corresponding to individual parameters have a good performances with respect to other elementary criteria components. Furthermore, and for this particular example, we notice that the performances of the optimal designs seem to be less sensitive to change in the weights allocation scheme. Also, some of the optimal designs tend to be less extreme in terms of providing degrees of freedom for pure error and lack-of-fit components.

Overall, the above examples for the generalized compound criteria have shown that, the performances of the optimal designs, for some cases, change when the scaling tuning parameter changes. Also it is illustrated that some of compound optimum designs, as shown in Table 5.12 and Table 5.13, turn out to be the same as some of the optimum designs under standard optimality criteria. In such cases, the standard and compound criteria are not far from converging to each other.

Besides that, all optimal designs for the design regions, resulted from imposing some restrictions on the component proportions, that were constructed for the alternative statistical optimality criteria, such as AP and DP criteria function, give quite extreme designs in some cases, in the sense that they allow few or no degrees of freedom for testing lack-of-fit.

Other than that, the main properties of the designs obtained can be tailored to the objective under consideration by allocating appropriate weights to the compound criteria. Generally, the experimenter should try different settings of such weights and then evaluate the designs to decide which of the resulting designs will be used in practice. Here, the compound criteria with a sensible allocation of weights informed by the evaluation reflecting the objectives of the experiments can construct compromise designs which are efficient in terms of lack-of-fit testing and pure error estimation. Such resulting designs under compound criteria in some cases can then be recommended for practical application. Thus, such compound criteria in optimum designs manner should be taken into consideration in order to carry out the statistical inferences and meet all objectives of the study.

As can be seen from the illustrative examples, several optimal designs have been found assuming different statistical linear models fitted. However, for different cases, the need for searching for optimum designs for the region under consideration for mechanistic nonlinear models arises. Thus, several optimum designs for the developed statistical criteria following the same process as for linear models in terms of constructing different optimal designs respect to the region of interest will be found. The illustrative examples with the resulting optimum designs will be presented in Chapter 6.

Chapter 6

Optimum Experimental Designs for Nonlinear Models

6.1 Designs for Nonlinear Regression Models

In some situations, the need for finding the optimum designs for mechanistic or empirical nonlinear models arises. For example, we have demonstrated in Chapter 3 that the nonlinear modified fractional polynomial models often provide a better fit to mixture data than commonly used linear models. A common feature for such models is that the optimum designs typically depend on the unknown value of $\boldsymbol{\beta}$. This leads to the concept of locally optimum designs, which are optimal for a prior chosen value of the parameter vector $\boldsymbol{\beta}$. To use the local statistical criteria in this case, the analogue of the information matrix of interest will be formulated. To derive the statistical criteria function that corresponds to nonlinear least squares or maximum likelihood estimation, the most common approach is to linearise the regression function $f(\boldsymbol{x}, \boldsymbol{\beta})$ in terms of $\boldsymbol{\beta}$ using a Taylor series expansion up to the first-order term around the selected value of $\boldsymbol{\beta}_0$ (Fedorov and Leonov, 2013). The model function becomes

$$f(\boldsymbol{x}, \boldsymbol{\beta}) \approx f(\boldsymbol{x}, \boldsymbol{\beta}_0) + \sum_{c=1}^{p} \left(\frac{\partial f(\boldsymbol{x}, \boldsymbol{\beta})}{\partial \beta_c} \mid_{\beta_c = \beta_{c0}} \right) (\boldsymbol{\beta}_c - \boldsymbol{\beta}_{c0}), \tag{6.1}$$

where the partial derivative $\frac{\partial f(\boldsymbol{x}, \boldsymbol{\beta})}{\partial \beta_c} \mid_{\beta_c = \beta_{c0}}$ is called the cth parameter sensitivity, and $\boldsymbol{\beta}$ and $\boldsymbol{\beta}_0$ are vectors of dimension $p \times 1$. The value of $\boldsymbol{\beta}_0$ is often chosen as the prior point estimate of the currently unknown $\boldsymbol{\beta}$. Thus the numerical approximation exhibited that the row elements of $\boldsymbol{F}$, which is considered as a design matrix in nonlinear regression

models, can be represented as

$$\boldsymbol{F} = \frac{\partial f(\boldsymbol{x}, \boldsymbol{\beta})}{\partial \beta_c} \Big|_{\beta_c = \beta_{c0}} . \tag{6.2}$$

Since $\boldsymbol{\beta}_0$ is given, the information matrix $\boldsymbol{F'F}$ will become identical to the Fisher information of the model, which is (asymptotically) proportional to the inverse of the variance-covariance matrix of the parameter estimator $\hat{\boldsymbol{\beta}}$. Now it becomes obvious that the locally optimum criteria function is obtained after substituting $\boldsymbol{\beta} = \boldsymbol{\beta}_0$ into the first-order linearisation of $f(\boldsymbol{x}, \boldsymbol{\beta})$ in the vicinity of $\boldsymbol{\beta}_0$.

The choice of $\boldsymbol{\beta}_0$ may depend on the experimenters' experience or a good guess about the mechanism behind the model. However, in the present case, as will be shown through the examples, the choice of prior $\boldsymbol{\beta}_0$ will be the estimated values of the model parameters from existing data sets.

Many authors have worked on the problem of constructing optimal designs for nonlinear regression models. For example, Box and Lucas (1959) carried out a study on the design of experiments for nonlinear functions. They tried to obtain a programme of trials that can be used for estimating the parameters with high accuracy. To achieve this, they assumed that a set of preliminary values of parameters were known. Then they chose the design points by maximizing the determinant of the Fisher information or, equivalently, by minimizing the asymptotic formula for generalized variance of the maximum likelihood estimates of the parameters. Finally, they illustrated their approach by using a model for chemical reactions and some standard nonlinear regression models. An extension of this work can be found in Draper and Hunter (1967) who displayed how the positioning of the design points changes based on the availability of prior information on the parameters.

Moreover, Cochran (1973) stated that we should know the values of parameters before we start finding the design points in order to estimate the model's parameters. In addition, he reviewed some works which had previously been done on experiments for nonlinear functions. White (1973) extended the general equivalence theorem to nonlinear models. She developed the information matrix and a variance function and then used the results to prove the equivalence theorem for the nonlinear model. Furthermore, under general theory of the optimum design of experiments, John and Draper (1975) reviewed some important results on the theory of design and design criteria in the case of nonlinear models. Furthermore, they discussed the algorithm for constructing the D-optimal designs. However, Silvey and Titterington (1973) outlined a slightly different algorithm than Fedorov for finding a design that are optimal under D-optimality criteria.

Abdelbasit and Plackett (1981, 1983), worked on designs for categorical data as well

as on experimental designs for binary data. In addition, they discussed the criterion of constant information for models with one or two parameters. On the other hand, Atkinson (1982), highlighted the developments in the design of experiments based on previous research. In addition, he summarized the development of designs for nonlinear models. Ford et al. (1989), highlighted some work in optimal experimental design in nonlinear issues. Besides, they described some design Methods, such as static and sequential design with their application to models in nonlinear parameters. In addition, in their articles, they conclude that, for a reliable nonlinear model a design that have been chosen can be obtained if we have a prior information about the parameters, as well as by using a sequential design technique.

Moreover, Chaudhuri and Mykland (1993), continue research on models with nonlinear parameters with the aim of searching design points that help us to estimate the parameters properly. Thus, in the first step, they used an initial design and then a fully adaptive sequential design. Then, they illustrated that, under some assumptions on the regression model, the asymptotic distribution of the maximum likelihood estimates of the parameters is normal. Following, they construct asymptotically D-optimal designs by using their approach for selecting design points successively .

In the applications for obtaining the locally optimum designs for nonlinear models, the information matrix of the model will be evaluated as the first step and then the optimum designs can be found using the algorithmic methodology process described in Chapter 5.

6.2 Numerical Results for Optimal Designs for Nonlinear Models

In this section, we will construct optimal designs for the nonlinear modified fractional polynomial models introduced in (3.8) and (3.9). Assuming these models are to be fitted to the region under consideration, we distinguish different scenarios that need to be considered for the algorithmic optimization problem: the models have the same value of the exponent α, and the models have different values of the power α. The models are nonlinear in the α or the different values of α's, respectively, and therefore the information matrices in the present case as mentioned before depend on the model parameters, so the estimated values for such parameters will be used to generate the information matrices.

From this step, we will now illustrate design search through redesigning the chick feeding experimental data. The estimated values of the parameters when considering the two situations presented previously are given in Table 3.3 for FP1n, and in Table 3.5 and Table 3.6 for FP2n. Using these estimated values with the candidate sets of design points

given in Table 4.5 augmented with interior points as explained previously in Chapter 4 for the constrained region, and substituting them into the respective matrix for nonlinear regression models given in (6.2), the candidate sets matrices for the constrained region under FPn1 (having either the same or different values of the exponent α) are presented in Table 6.1. In the given table, the second column in the obtained matrix under α and q_1, represent the derivative response function with respect to β_1, while the derivative functions with respect to β_2 and α are allocated to the third and last column as q_2 and q_3 respectively. The choice of the denominator for this case is x_1, and depends on the value of the MSR as we mentioned before in Chapter 3, Section 3.3. Moreover, the resulting matrix under different α will be obtained by applying the same derivation as for FP1n with the same value of α with considering the derivative with respect to different values of the power α that corresponding to q_3 and q_4 respectively, and the denominator in this case will be x_3. A similar method can be followed to obtain the design matrices under second degree modified fractional polynomial models while considering the two situations for the power α.

From the resulting design matrices, we can search for the optimal design points from the candidate sets under different optimality criteria, such as the developed generalized compound determinant and trace criteria, using the same methodology and by the same manner as for 3-component constraints for linear models, and can thus obtain the designs, which consist of the optimum treatment combinations. It is worth mentioning that, in the present case, the first degree nonlinear models FP1n will be assumed to fit data collected with the parameters under consideration, which means a column for the design matrix does not contain the intercept. Thus, the number of the primary terms for such models will be $p = 3$ under FP1n having common α and $p = 4$ for the model has different values of the exponent α. Moreover, the criteria functions corresponding to the subset of the model parameters given in (2.22) and (2.23) respectively will be implemented in the algorithm to find such optimum designs. Furthermore, second order terms will be considered as a potential missed terms including the intercept under this case.

The designs for FP1n models considering the two cases for the exponent and under trace- and determinant-based criteria will be evaluated with respect to other individual statistical criteria while setting different values of the variance parameter τ^2. The summaries of the characteristics of the resulting optimal designs when the model has the same value of the exponent are given in Table 6.2 and Table 6.3, while Table 6.4 and Table 6.5 show the summary characteristics of the optimum designs for the FP1n model with different α. As it can be seen from presented tables, different weights have been allocated to the elements of the criteria function under consideration. The main

TABLE 6.1: The Resulting Candidate Points for FP1n

with one value of α				with different α				
B_0	q_1	q_2	q_3	B_0	q_1	q_2	q_3	q_4
1	0.22	1.20	0.11	1	1.04	0.03	163.55	2.43
1	4.22	1.10	−228.96	1	1.02	14.83	78.75	−836.08
1	1.16	0.39	13.64	1	0.83	9.00	−665.54	−413.30
1	0.41	2.68	−130.48	1	1.22	0.02	1017.05	1.83
1	1.34	4.15	−336.61	1	1.33	0.10	1603.58	4.94
1	3.13	3.03	−315.23	1	1.25	1.07	1175.02	−1.41
1	0.84	0.89	10.94	1	0.98	0.88	−92.55	2.44
1	0.28	1.66	−33.18	1	1.11	0.03	478.29	2.08
1	1.78	0.52	−19.29	1	0.88	11.92	−491.15	−617.15
1	0.80	3.18	−194.52	1	1.26	0.06	1237.06	3.68
1	1.24	1.54	−46.00	1	1.09	0.64	399.47	5.92
1	1.42	1.19	−29.89	1	1.04	1.42	154.73	−10.46
1	0.84	0.89	10.94	1	0.98	0.88	−92.55	2.44
1	0.28	1.66	−33.18	1	1.11	0.03	478.29	2.08
1	0.49	1.79	−43.99	1	1.12	0.08	554.14	4.11
1	1.06	1.52	−37.22	1	1.09	0.49	386.33	7.30
1	0.61	1.06	7.64	1	1.01	0.33	50.54	7.65
1	0.25	1.40	−13.27	1	1.07	0.03	307.65	2.24
1	1.08	1.01	−3.89	1	1.00	1.15	10.08	−3.32
1	0.39	1.72	−37.69	1	1.11	0.05	514.59	3.22
1	0.74	1.33	−12.35	1	1.06	0.31	254.12	7.60
1	1.78	0.52	−19.29	1	0.88	11.92	−491.15	−617.15
1	2.31	2.33	−178.66	1	1.18	0.99	847.86	0.23
1	3.13	3.03	−315.23	1	1.25	1.07	1175.02	−1.41
1	3.71	2.28	−282.04	1	1.18	2.65	824.52	−54.13
1	1.61	0.92	−24.11	1	0.98	3.08	−70.50	−72.37
1	1.71	1.59	−73.91	1	1.10	1.17	429.15	−3.76
1	2.42	0.66	−63.46	1	0.92	13.37	−323.71	−724.95
1	2.65	2.61	−231.50	1	1.21	1.03	986.08	−0.58
1	2.18	1.46	−92.65	1	1.08	2.21	348.56	−36.61
1	1.04	1.33	−22.13	1	1.06	0.61	256.17	6.32
1	1.18	1.43	−35.24	1	1.07	0.68	325.77	5.42
1	1.51	1.07	−27.29	1	1.01	1.98	62.21	−28.29
1	1.01	0.69	13.55	1	0.93	2.16	−293.07	−34.69
1	0.92	1.09	−2.17	1	1.02	0.72	72.04	4.97
1	1.42	0.44	1.56	1	0.85	10.46	−593.40	−513.09
1	1.11	1.38	−28.24	1	1.07	0.65	289.47	5.89
1	1.19	0.95	−4.81	1	0.99	1.57	−43.91	−14.69
1	0.80	3.18	−194.52	1	1.26	0.06	1237.06	3.68
1	1.73	2.79	−190.93	1	1.23	0.39	1066.96	7.68
1	0.76	1.51	−26.01	1	1.09	0.25	377.20	7.28
1	0.32	2.04	−65.89	1	1.15	0.03	696.90	1.95
1	1.46	1.63	−64.04	1	1.10	0.80	458.84	3.75
1	0.60	2.90	−157.01	1	1.24	0.04	1115.91	2.85
1	1.01	2.04	−79.56	1	1.15	0.25	695.59	7.22
1	2.41	3.63	−333.82	1	1.29	0.44	1416.39	7.55
1	0.91	1.62	−39.35	1	1.10	0.31	450.22	7.60
1	0.60	2.26	−89.02	1	1.18	0.07	812.86	3.88
1	1.71	1.83	−94.11	1	1.13	0.87	577.88	2.44
1	1.04	3.57	−249.34	1	1.29	0.08	1392.01	4.36
1	1.26	2.30	−115.25	1	1.18	0.30	834.68	7.55
1	1.31	1.32	−32.70	1	1.06	0.99	246.47	0.27
1	1.28	1.95	−82.95	1	1.14	0.43	647.97	7.58
1	2.09	1.38	−80.82	1	1.07	2.32	287.95	−40.70
1	2.01	3.13	−246.24	1	1.26	0.41	1214.64	7.63
1	1.78	1.93	−106.68	1	1.14	0.85	634.86	2.89
1	0.67	1.26	−5.80	1	1.05	0.29	201.83	7.50
1	1.26	0.78	−0.20	1	0.95	2.62	−200.78	−52.68
1	0.83	1.56	−32.20	1	1.09	0.28	412.14	7.47
1	0.99	1.15	−8.64	1	1.03	0.74	124.82	4.71
1	1.23	1.27	−25.87	1	1.05	0.94	211.31	1.17
1	0.47	2.14	−75.81	1	1.16	0.05	751.39	3.02
1	0.85	1.61	−36.72	1	1.10	0.28	444.85	7.42
1	1.58	1.72	−77.54	1	1.11	0.84	514.35	3.11
1	1.55	1.13	−32.15	1	1.02	1.89	103.14	−25.07
1	1.13	2.16	−95.44	1	1.17	0.27	760.13	7.41

TABLE 6.2: The Summary Characteristics of GDP- and GAP- Criteria with $\tau^2 = 1$ and α

Designs	GDP(w_i)	$df(PE; LoF)$	D-eff	DP-eff	$LoF(D)$-eff	$LoF(DP)$-eff	A-eff	AP-eff	$LoF(A)$-eff	$LoF(AP)$-eff
I	(1,0,0)	(27,0)	100.00%	77.50%	69.85%	92.40%	87.18%	83.69%	95.02%	91.58%
II	(0,1,0)	(26,1)	91.10%	100.00%	60.99%	86.68%	82.68%	82.97%	89.81%	39.24%
III	(0,0,1)	(24,3)	48.81%	51.55%	95.20%	100.00%	94.55%	92.20%	97.53%	99.49%
IV	(0.5,0.5,0)	(27,0)	97.35%	98.34%	90.30%	91.20%	95.50%	97.30%	92.45%	90.21%
V	(0,0.5,0.5)	(26,1)	94.27%	92.38%	92.25%	95.48%	96.53%	90.61%	92.31%	94.50%
VI	(0.5,0,0.5)	(25,2)	95.43%	90.40%	94.33%	96.79%	98.10%	80.33%	90.51%	92.70%
VII	(1/3,1/3,1/3)	(25,2)	93.15%	96.10%	93.40%	96.90%	94.38%	93.35%	98.04%	99.50%
VIII	(0.5,0.25,0.25)	(24,3)	93.40%	92.30%	94.30%	91.60%	99.30%	98.50%	97.30%	96.60%
IX	(0.25,0.25,0.5)	(25,2)	94.90%	95.50%	95.30%	97.80%	97.40%	94.90%	98.50%	98.70%
Designs	GAP(w_i)	$df(PE; LoF)$	D-eff	DP-eff	$LoF(D)$-eff	$LoF(DP)$-eff	A-eff	AP-eff	$LoF(A)$-eff	$LoF(AP)$-eff
I	(1,0,0)	(26,1)	97.88%	98.80%	90.35%	91.24%	100.00%	99.77%	96.19%	92.59%
II	(0,1,0)	(26,1)	94.24%	98.74%	94.50%	92.11%	99.77%	100.00%	86.32%	92.30%
III	(0,0,1)	(25,2)	89.61%	80.99%	91.40%	96.90%	66.63%	68.97%	95.44%	100.00%
IV	(0.5,0.5,0)	(26,1)	97.64%	96.52%	94.20%	97.24%	99.20%	99.42%	91.43%	91.62%
V	(0,0.5,0.5)	(25,2)	89.20%	90.86%	95.17%	95.81%	98.16%	98.16%	99.21%	95.37%
VI	(0.5,0,0.5)	(24,3)	95.64%	96.32%	94.50%	96.37%	99.26%	96.33%	99.20%	96.77%
VII	(1/3,1/3,1/3)	(26,1)	89.77%	98.89%	93.25%	96.15%	98.57%	98.80%	97.17%	93.53%
VIII	(0.5,0.25,0.25)	(23,4)	94.40%	94.30%	97.30%	98.60%	96.30%	96.50%	98.30%	98.60%
IX	(0.25,0.25,0.5)	(24,3)	90.90%	93.50%	95.30%	87.80%	91.40%	94.90%	98.50%	96.70%

TABLE 6.3: The Summary Characteristics of GDP- and GAP- Criteria with $\tau^2 = 1/g$ and α

Designs	GDP(w_i)	$df(PE; LoF)$	D-eff	DP-eff	$LoF(D)$-eff	$LoF(DP)$-eff	A-eff	AP-eff	$LoF(A)$-eff	$LoF(AP)$-eff
I	(1,0,0)	(27,0)	100.00%	77.50%	70.80%	90.20%	87.18%	83.69%	94.50%	94.58%
II	(0,1,0)	(26,1)	91.10%	100.00%	80.75%	94.35%	82.68%	82.97%	81.61%	36.45%
III	(0,0,1)	(25,2)	97.31%	81.62%	92.49%	100.00%	94.20%	96.10%	97.53%	99.70%
IV	(0.5,0.5,0)	(27,0)	97.35%	98.34%	92.31%	93.45%	95.50%	97.30%	97.60%	99.21%
V	(0,0.5,0.5)	(24,3)	92.26%	94.56%	97.72%	95.85%	97.71%	96.69%	97.56%	96.57%
VI	(0.5,0,0.5)	(24,3)	91.69%	94.80%	93.70%	95.74%	89.25%	85.34%	89.92%	95.22%
VII	(1/3,1/3,1/3)	(25,2)	93.20%	97.97%	94.74%	95.55%	96.18%	93.14%	97.58%	99.11%
VIII	(0.5,0.25,0.25)	(25,2)	94.50%	95.30%	92.30%	95.60%	91.30%	94.50%	97.30%	96.60%
IX	(0.25,0.25,0.5)	(24,3)	94.90%	95.50%	92.30%	95.80%	89.90%	87.50%	98.50%	96.70%
Designs	GAP(w_i)	$df(PE; LoF)$	D-eff	DP-eff	$LoF(D)$-eff	$LoF(DP)$-eff	A-eff	AP-eff	$LoF(A)$-eff	$LoF(AP)$-eff
I	(1,0,0)	(26,1)	97.88%	98.80%	96.11%	94.80%	100.00%	99.77%	84.36%	95.72%
II	(0,1,0)	(26,1)	94.24%	98.74%	91.48%	96.13%	99.77%	100.00%	86.32%	95.26%
III	(0,0,1)	(24,3)	92.12%	97.98%	92.12%	97.15%	98.70%	97.55%	98.76%	100.00%
IV	(0.5,0.5,0)	(26,1)	97.64%	96.52%	95.55%	94.99%	99.20%	99.42%	91.88%	99.89%
V	(0,0.5,0.5)	(25,2)	94.49%	96.99%	92.82%	94.50%	99.94%	95.82%	89.62%	99.80%
VI	(0.5,0,0.5)	(24,3)	94.24%	91.40%	97.15%	97.34%	98.62%	96.96%	89.40%	95.20%
VII	(1/3,1/3,1/3)	(26,1)	96.50%	98.61%	97.12%	94.38%	98.70%	98.91%	92.78%	99.30%
VIII	(0.5,0.25,0.25)	(24,3)	92.30%	94.50%	96.40%	96.60%	95.30%	93.50%	88.40%	95.60%
IX	(0.25,0.25,0.5)	(23,4)	92.90%	92.50%	95.30%	97.80%	99.40%	94.90%	98.50%	96.70%

TABLE 6.4: The Summary Characteristics of GDP- and GAP- Criteria with $\tau^2 = 1$ and different α

Designs	GDP(w_i)	$df(PE; LoF)$	D-eff	DP-eff	$LoF(D)$-eff	$LoF(DP)$-eff	A-eff	AP-eff	$LoF(A)$-eff	$LoF(AP)$-eff
I	(1,0,0)	(25,1)	100.00%	100.00%	93.64%	97.12%	93.29%	96.21%	82.50%	70.30%
II	(0,1,0)	(25,1)	99.89%	100.00%	95.22%	92.14%	94.30%	96.70%	94.43%	89.11%
III	(0,0,1)	(21,5)	61.58%	63.26%	91.86%	100.00%	62.56%	61.18%	95.98%	97.42%
IV	(0.5,0.5,0)	(25,1)	88.25%	91.22%	93.37%	97.44%	98.44%	96.78%	91.81%	98.17%
V	(0,0.5,0.5)	(25,1)	99.79%	99.79%	90.12%	94.28%	98.16%	93.61%	92.96%	99.13%
VI	(0.5,0,0.5)	(25,1)	100.00%	98.00%	93.64%	97.12%	93.29%	96.21%	82.50%	70.30%
VII	(1/3,1/3,1/3)	(25,1)	99.89%	99.89%	94.32%	93.46%	96.64%	97.90%	93.90%	97.16%
VIII	(0.5,0.25,0.25)	(25,1)	95.40%	96.00%	96.30%	96.60%	92.30%	95.50%	96.10%	96.60%
IX	(0.25,0.25,0.5)	(24,2)	94.90%	95.50%	92.30%	95.20%	89.90%	87.60%	98.60%	96.70%
Designs	GAP(w_i)	$df(PE; LoF)$	D-eff	DP-eff	$LoF(D)$-eff	$LoF(DP)$-eff	A-eff	AP-eff	$LoF(A)$-eff	$LoF(AP)$-eff
I	(1,0,0)	(18,8)	95.57%	88.49%	94.17%	90.21%	100.00%	97.82%	91.46%	96.31%
II	(0,1,0)	(25,1)	97.90%	99.34%	97.76%	90.21%	99.87%	100.00%	95.57%	92.00%
III	(0,0,1)	(24,3)	92.12%	97.98%	92.12%	97.15%	98.70%	97.55%	98.76%	100.00%
IV	(0.5,0.5,0)	(25,1)	97.64%	96.52%	95.55%	94.99%	99.20%	99.42%	91.88%	99.89%
V	(0,0.5,0.5)	(25,2)	94.49%	96.99%	92.82%	94.50%	99.94%	95.82%	89.62%	99.80%
VI	(0.5,0,0.5)	(24,3)	94.24%	91.40%	97.15%	97.34%	98.62%	96.96%	89.40%	95.20%
VII	(1/3,1/3,1/3)	(26,1)	96.50%	98.61%	97.12%	94.38%	98.70%	98.91%	92.78%	99.30%
VIII	(0.5,0.25,0.25)	(24,3)	92.30%	94.50%	96.40%	96.60%	95.30%	93.50%	88.40%	95.60%
IX	(0.25,0.25,0.5)	(23,4)	92.90%	92.50%	95.30%	97.80%	99.40%	94.90%	98.50%	96.70%

TABLE 6.5: The Summary Characteristics of GDP- and GAP- Criteria with $\tau^2 = 1/g$, under different α

Designs	GDP(w_i)	$df(PE; LoF)$	D-eff	DP-eff	$LoF(D)$-eff	$LoF(DP)$-eff	A-eff	AP-eff	$LoF(A)$-eff	$LoF(AP)$-eff
I	(1,0,0)	(25,1)	100.00%	100.00%	92.11%	96.54%	93.29%	96.21%	99.28%	95.15%
II	(0,1,0)	(25,1)	99.89%	100.00%	98.58%	92.84%	94.30%	96.70%	93.14%	98.95%
III	(0,0,1)	(17,9)	97.44%	95.90%	99.94%	97.34%	99.96%	97.72%	95.11%	100.00%
IV	(0.5,0.5,0)	(25,1)	88.25%	91.22%	98.81%	97.00%	98.44%	96.78%	92.79%	96.31%
V	(0,0.5,0.5)	(24,2)	93.93%	91.92%	92.66%	95.58%	99.89%	91.61%	92.83%	95.59%
VI	(0.5,0,0.5)	(24,2)	96.74%	98.12%	95.27%	90.88%	99.96%	91.56%	92.00%	96.80%
VII	(1/3,1/3,1/3)	(25,1)	100.00%	98.00%	93.64%	97.12%	93.29%	96.21%	82.50%	70.30%
VIII	(0.5,0.25,0.25)	(24,2)	96.74%	98.12%	95.27%	90.88%	99.96%	91.56%	92.00%	96.80%
IX	(0.25,0.25,0.5)	(25,1)	92.90%	92.50%	95.30%	97.80%	99.40%	94.90%	98.50%	96.70%
Designs	GAP(w_i)	$df(PE; LoF)$	D-eff	DP-eff	$LoF(D)$-eff	$LoF(DP)$-eff	A-eff	AP-eff	$LoF(A)$-eff	$LoF(AP)$-eff
I	(1,0,0)	(18,8)	95.57%	88.49%	90.93%	98.17%	100.00%	97.82%	96.53%	95.19%
II	(0,1,0)	(25,1)	97.90%	99.34%	95.33%	98.17%	99.87%	100.00%	94.91%	95.19%
III	(0,0,1)	(17,9)	97.44%	95.90%	99.94%	98.31%	99.69%	97.72%	93.92%	100.00%
IV	(0.5,0.5,0)	(25,1)	97.64%	96.52%	93.54%	95.86%	99.20%	99.42%	91.27%	96.31%
V	(0,0.5,0.5)	(23,3)	97.86%	88.26%	96.39%	96.75%	99.96%	95.61%	92.49%	93.22%
VI	(0.5,0,0.5)	(25,1)	97.90%	95.90%	98.58%	92.85%	98.56%	94.50%	95.46%	92.85%
VII	(1/3,1/3,1/3)	(24,2)	97.62%	96.64%	94.63%	93.36%	96.37%	94.92%	96.37%	96.12%
VIII	(0.5,0.25,0.25)	(18,8)	95.57%	88.49%	90.93%	98.17%	100.00%	97.82%	96.53%	95.19%
IX	(0.25,0.25,0.5)	(23,3)	96.90%	89.50%	95.30%	95.80%	94.40%	96.90%	98.50%	96.70%

purpose for varying the allocation weights scheme between the components of the criteria is to explore how efficient the designs are with respect to the individual criteria and to investigate the relationship between the weight allocation and the distribution of the degrees of freedom between pure error and lack-of-fit components.

From Table 6.2, all resulting optimal designs under either the determinant or trace based criteria are similar as far as pure error degrees of freedom are concerned. Thus, the imbalance tends to occur in favour of the pure error with such designs.

Also we notice that, the designs that are optimal with respect to DP-criteria are poor with having lower LOF(AP)-efficiency value, and this value is even worse, in the sense that, under DP-optimal design (II) such a design is only 39.24% LOF(AP)-efficient. Similar performances occur when assuming the tuning variance value is proportional to the inverse of the potential terms as in Table 6.3. On the contrary, the optimal design with respect to LOF(AP) performs well with respect to other individual determinant criteria including DP-component, i.e. LOF(AP)-optimal design have good performances and high efficiencies in particular with respect to the DP-component, that is 80.99%. The design itself is given in Table 6.6.

TABLE 6.6: The LOF(AP)-Optimal Design for Constrained Region by 3-Components FP1n

x_1	x_2	x_3
0.13	0.40	0.47
0.13	0.40	0.47
0.40	0.54	0.06
0.22	1.20	0.11
0.13	0.40	0.47
0.40	0.02	0.58
0.13	0.40	0.47
0.13	0.40	0.47
0.13	0.40	0.47
0.13	0.40	0.47
0.05	0.09	0.86
0.13	0.40	0.47
0.40	0.54	0.06
0.13	0.40	0.47
0.40	0.54	0.06
0.40	0.54	0.06
0.05	0.09	0.86
0.40	0.54	0.06
0.13	0.40	0.47
0.40	0.54	0.06
0.40	0.54	0.06
0.13	0.40	0.47
0.13	0.40	0.47
0.13	0.40	0.47
0.13	0.40	0.47
0.13	0.40	0.47
0.40	0.54	0.06
0.05	0.09	0.86
0.05	0.89	0.06
0.13	0.40	0.47

Besides trace-based criteria, it is quite notable that, DP-optimal designs also have low efficiency with respect to other statistical criteria in particular LOF(D)-criterion. Such design is found to be 60.99% LOF(D)-efficient.

Apart from this notification, the DP-optimal designs have good performances with respect to other individual statistical determinant and trace components.

With respect to the relationship between primary and potential terms, we notice that, when all weights are allocated to the primary components either D- or DP-components, the resulting optimal designs (I and II in Table 6.2) have good performances in terms of LOF(DP)-efficiencies respectively. However, the opposite does not always occur, for example, LOF(DP)-optimal design (III) is poor with respect to other primary components, with this design having only 48.81% D-efficiency, and 51.55% DP-efficiency. The same appears true with the trace form.

We now turn to the second situation, when the models have different values of α. Table 6.4 and Table 6.5 presented the main features and performances of the resulting optimal designs when considering different values of the tuning parameter τ^2. The designs under the GDP-criteria, in particular, design (II) have good performances in comparison to the first situation when the model has the same value of the exponent α. Moreover, some compound optimal designs turned out to be the same as some of the optimum designs under standard optimality criteria. For example, the designs obtained when some weights are distributed equally between the primary and potential components (design VI in Table 6.4) or among all three components as in design (VII in Table 6.5). Such resulting designs are optimal with respect to the D-criterion. However, when these weights are unequally allocated to all three criterion components (design VIII in Table 6.5), the resulting design is optimal with respect to the trace based criterion. Such designs are given in Table 6.7.

Overall, the obtained optimal designs have good performances with respect to other statistical criteria components especially when the model has different values of α.

6.3 Optimal Designs Under Nonlinear Models with More Components

Different designs can be constructed for the regions determined by more components of mixture variables that have had constraints placed on them, as is the case in the candle constraints mixture. In order to construct optimal designs for such restricted regions resulting from the constrained proportion components, the same technique, manner, and methodology for the exchange algorithm based on design optimality with 3-component proportions provided in the previous case will be followed. Optimal designs for 20 run were found for the developed optimality criteria under the first degree modified fractional polynomial models considering different values for the scaling variance parameters. For

TABLE 6.7: The Optimal Designs Under FP1n

I(OD)			II(OA)		
x_1	x_2	x_3	x_1	x_2	x_3
0.05	0.89	0.06	0.40	0.54	0.06
0.05	0.89	0.06	0.40	0.54	0.06
0.05	0.69	0.26	0.40	0.54	0.06
0.05	0.69	0.26	0.40	0.54	0.06
0.05	0.89	0.06	0.40	0.54	0.06
0.40	0.54	0.06	0.40	0.54	0.06
0.05	0.89	0.06	0.40	0.54	0.06
0.40	0.02	0.58	0.40	0.54	0.06
0.40	0.02	0.58	0.40	0.41	0.19
0.05	0.89	0.06	0.40	0.41	0.19
0.40	0.54	0.06	0.40	0.41	0.19
0.05	0.09	0.86	0.40	0.41	0.19
0.40	0.02	0.58	0.40	0.28	0.32
0.40	0.54	0.06	0.40	0.15	0.45
0.05	0.09	0.86	0.40	0.15	0.45
0.40	0.54	0.06	0.05	0.09	0.86
0.05	0.69	0.26	0.40	0.41	0.19
0.05	0.89	0.06	0.40	0.28	0.32
0.40	0.54	0.06	0.05	0.09	0.86
0.05	0.09	0.86	0.40	0.41	0.19
0.40	0.02	0.58	0.40	0.28	0.32
0.05	0.69	0.26	0.09	0.05	0.86
0.05	0.09	0.86	0.05	0.29	0.66
0.05	0.69	0.26	0.40	0.54	0.06
0.05	0.89	0.06	0.23	0.58	0.19
0.40	0.02	0.58	0.08	0.45	0.47
0.40	0.02	0.58	0.26	0.02	0.72
0.05	0.09	0.86	0.08	0.25	0.67
0.40	0.02	0.58	1.14	0.40	0.46
0.40	0.54	0.06	1.14	0.40	0.46

this case, the number of the primary model terms is $p = 4$ when the model has the same exponent, and is $p = 6$ for the model having different α. Again, second order terms with the intercept corresponding to the full model for different situations under consideration will be assumed as a potential missed terms for this case.

The summary of the features and performances of the resulting optimal designs under different values of τ^2 are given in Table 6.8, and Table 6.9 for the model with the same exponent α and Table 6.10 and Table 6.11 for the model with different α.

Table 6.8 and Table 6.9 illustrated that, the conflict between the primary and potential parts still holds for the obtained optimal designs for the constrained region under determinant compound criteria. However, this relationship become less drastic when the fitted model has different values of the power α with $\tau^2 = 1/g$ either under determinant or trace criteria as in the resulting optimal designs under compound criteria (III in Table 6.11).

Table 6.8 and Table 6.9 present the characteristics and the performances of the optimal design under criteria for generalized determinant of model parameters which place equal weights on the primary components of the criterion (design IV). Such designs that are produced by compound GDP-criteria are less extreme, in that they allow a reasonable compromise between the degrees of freedom for pure error and lack-of-fit components. This design is given in Table 6.12. These designs perform well in terms of other deter-

TABLE 6.8: The Summary Characteristics of GDP- and GAP- Criteria with $\tau^2 = 1$ and α

Designs	GDP(w_i)	$df(PE; LoF)$	D-eff	DP-eff	$LoF(D)$-eff	$LoF(DP)$-eff	A-eff	AP-eff	$LoF(A)$-eff	$LoF(AP)$-eff
I	(1,0,0)	(14,2)	100.00%	94.76%	87.20%	88.20%	94.35%	94.05%	83.45%	86.11%
II	(0,1,0)	(15,1)	93.03%	100.00%	77.83%	87.55%	1.56%	1.55%	89.44%	86.49%
III	(0,0,1)	(13,3)	10.29%	15.03%	99.24%	100.00%	46.86%	46.54%	99.84%	94.55%
IV	(0.5,0.5,0)	(7,9)	95.00%	89.14%	91.14%	89.47%	11.56%	11.55%	83.84%	81.11%
V	(0,0.5,0.5)	(13,3)	84.20%	86.99%	92.10%	96.80%	86.86%	86.54%	82.57%	87.96%
VI	(0.5,0,0.5)	(12,4)	80.39%	59.35%	76.31%	96.36%	81.51%	81.49%	88.10%	83.41%
VII	(1/3,1/3,1/3)	(13,3)	92.50%	95.57%	86.81%	94.65%	99.96%	99.68%	82.87%	88.79%
VIII	(0.5,0.25,0.25)	(13,3)	90.20%	91.99%	89.50%	92.80%	96.86%	96.54%	84.57%	89.96%
IX	(0.25,0.25,0.5)	(13,3)	86.50%	87.57%	87.81%	90.65%	88.96%	87.68%	82.87%	87.79%
Designs	GAP(w_i)	$df(PE; LoF)$	D-eff	DP-eff	$LoF(D)$-eff	$LoF(DP)$-eff	A-eff	AP-eff	$LoF(A)$-eff	$LoF(AP)$-eff
I	(1,0,0)	(15,1)	81.39%	93.31%	88.36%	87.97%	100.00%	99.98%	61.01%	88.05%
II	(0,1,0)	(16,0)	88.53%	85.79%	79.49%	88.04%	98.70%	100.00%	58.99%	86.22%
III	(0,0,1)	(14,2)	75.40%	11.45%	92.30%	97.36%	96.80%	95.35%	94.74%	100.00%
IV	(0.5,0.5,0)	(16,0)	84.15%	86.14%	88.12%	88.72%	88.87%	88.36%	89.17%	86.39%
V	(0,0.5,0.5)	(14,2)	81.50%	81.93%	75.38%	94.32%	96.18%	96.18%	95.49%	90.47%
VI	(0.5,0,0.5)	(14,2)	85.51%	86.51%	86.71%	94.54%	94.47%	94.48%	88.81%	84.14%
VII	(1/3,1/3,1/3)	(14,2)	81.63%	85.54%	92.52%	94.92%	90.83%	90.84%	89.14%	84.36%
VIII	(0.5,0.25,0.25)	(13,3)	90.00%	89.50%	92.30%	94.60%	93.20%	94.50%	93.30%	91.60%
IX	(0.25,0.25,0.5)	(14,2)	89.90%	88.50%	95.30%	97.80%	91.40%	94.90%	95.50%	96.70%

TABLE 6.9: The Summary Characteristics of GDP- and GAP- Criteria with $\tau^2 = 1/g$, and α

Designs	GDP(w_i)	$df(PE; LoF)$	D-eff	DP-eff	$LoF(D)$-eff	$LoF(DP)$-eff	A-eff	AP-eff	$LoF(A)$-eff	$LoF(AP)$-eff
I	(1,0,0)	(14,2)	100.00%	94.76%	91.82%	96.76%	94.35%	94.05%	85.16%	88.97%
II	(0,1,0)	(15,1)	93.03%	100.00%	93.36%	97.02%	1.56%	1.55%	64.12%	88.09%
III	(0,0,1)	(14,2)	48.81%	51.55%	95.20%	100.00%	94.55%	92.20%	97.53%	99.49%
IV	(0.5,0.5,0)	(7,9)	95.00%	89.14%	94.87%	96.86%	11.56%	11.55%	88.24%	82.29%
V	(0,0.5,0.5)	(15,1)	94.27%	92.38%	92.25%	95.48%	96.53%	90.61%	92.31%	94.50%
VI	(0.5,0,0.5)	(14,2)	95.43%	90.40%	94.33%	96.79%	98.10%	80.33%	90.51%	92.70%
VII	(1/3,1/3,1/3)	(14,2)	93.15%	96.10%	93.40%	96.90%	94.38%	93.35%	98.04%	99.50%
VIII	(0.5,0.25,0.25)	(13,3)	93.40%	92.30%	94.30%	91.60%	99.30%	98.50%	97.30%	96.60%
IX	(0.25,0.25,0.5)	(14,2)	94.90%	95.50%	95.30%	97.80%	97.40%	94.90%	98.50%	98.70%
Designs	GAP(w_i)	$df(PE; LoF)$	D-eff	DP-eff	$LoF(D)$-eff	$LoF(DP)$-eff	A-eff	AP-eff	$LoF(A)$-eff	$LoF(AP)$-eff
I	(1,0,0)	(15,1)	81.39%	93.31%	93.53%	83.33%	100.00%	99.98%	72.11%	91.83%
II	(0,1,0)	(16,0)	88.53%	85.79%	86.36%	83.96%	98.70%	100.00%	82.91%	96.30%
III	(0,0,1)	(13,3)	14.84%	15.76%	80.06%	98.83%	95.40%	97.30%	84.96%	100.00%
IV	(0.5,0.5,0)	(16,0)	84.15%	86.14%	73.66%	86.56%	88.87%	88.36%	82.81%	92.48%
V	(0,0.5,0.5)	(14,2)	81.50%	91.93%	75.38%	94.32%	96.18%	96.19%	95.49%	90.47%
VI	(0.5,0,0.5)	(14,2)	85.51%	85.55%	86.71%	94.54%	94.47%	94.65%	88.81%	84.14%
VII	(1/3,1/3,1/3)	(14,2)	81.63%	85.54%	92.52%	94.92%	90.83%	90.84%	89.14%	84.36%
VIII	(0.5,0.25,0.25)	(13,3)	94.40%	92.50%	96.30%	96.60%	91.30%	93.50%	98.30%	98.60%
IX	(0.25,0.25,0.5)	(14,2)	90.90%	89.50%	95.50%	89.80%	94.40%	94.90%	94.50%	97.55%

TABLE 6.10: The Summary Characteristics of GDP- and GAP- Criteria with $\tau^2 = 1$ and different α

Designs	GDP(w_i)	$df(PE; LoF)$	D-eff	DP-eff	$LoF(D)$-eff	$LoF(DP)$-eff	A-eff	AP-eff	$LoF(A)$-eff	$LoF(AP)$-eff
I	(1,0,0)	(11,3)	100.00%	99.90%	93.03%	89.36%	93.64%	93.38%	82.13%	89.24%
II	(0,1,0)	(13,1)	94.19%	100.00%	44.40%	95.17%	93.52%	93.85%	83.06%	87.95%
III	(0,0,1)	(11,3)	62.64%	80.18%	63.21%	100.00%	62.07%	99.90%	84.52%	81.19%
IV	(0.5,0.5,0)	(11,3)	99.85%	99.86%	83.66%	86.56%	91.23%	91.03%	82.82%	92.47%
V	(0,0.5,0.5)	(10,4)	96.15%	96.16%	84.50%	82.50%	90.42%	90.23%	82.40%	95.54%
VI	(0.5,0,0.5)	(11,3)	93.66%	93.65%	80.62%	82.52%	88.20%	88.80%	80.52%	88.70%
VII	(1/3,1/3,1/3)	(12,2)	99.16%	99.17%	87.18%	91.49%	91.27%	91.18%	64.96%	92.50%
VIII	(0.5,0.25,0.25)	(11,3)	97.50%	96.30%	89.90%	91.60%	92.30%	93.50%	87.90%	95.70%
IX	(0.25,0.25,0.5)	(11,3)	93.00%	94.60%	95.03%	90.36%	94.64%	94.38%	86.50%	89.90%
Designs	GAP(w_i)	$df(PE; LoF)$	D-eff	DP-eff	$LoF(D)$-eff	$LoF(DP)$-eff	A-eff	AP-eff	$LoF(A)$-eff	$LoF(AP)$-eff
I	(1,0,0)	(11,3)	84.11%	84.12%	93.53%	83.33%	100.00%	99.72%	82.11%	91.83%
II	(0,1,0)	(13,1)	81.60%	86.62%	84.36%	83.96%	99.65%	100.00%	85.91%	96.30%
III	(0,0,1)	(13,1)	14.84%	15.76%	80.06%	98.83%	95.40%	97.30%	84.96%	100.00%
IV	(0.5,0.5,0)	(11,3)	99.86%	99.80%	73.66%	86.56%	96.45%	97.91%	82.81%	92.48%
V	(0,0.5,0.5)	(10,4)	80.94%	83.38%	76.59%	96.48%	95.45%	96.55%	84.49%	94.49%
VI	(0.5,0,0.5)	(12,2)	80.94%	83.38%	82.58%	76.64%	95.45%	97.36%	24.49%	81.49%
VII	(1/3,1/3,1/3)	(9,5)	95.82%	93.98%	83.58%	89.90%	96.94%	96.98%	82.49%	96.55%
VIII	(0.5,0.25,0.25)	(11,3)	94.40%	92.50%	96.30%	96.60%	91.30%	93.50%	98.30%	98.60%
IX	(0.25,0.25,0.5)	(10,4)	90.90%	89.50%	95.30%	89.80%	94.40%	94.90%	95.50%	96.55%

TABLE 6.11: The Summary Characteristics of GDP- and GAP- Criteria with $\tau^2 = 1/g$ and different α

Designs	GDP(w_i)	$df(PE; LoF)$	D-eff	DP-eff	$LoF(D)$-eff	$LoF(DP)$-eff	A-eff	AP-eff	$LoF(A)$-eff	$LoF(AP)$-eff
I	(1,0,0)	(11,3)	100.00%	99.90%	81.22%	83.46%	93.64%	93.38%	84.89%	96.14%
II	(0,1,0)	(13,1)	94.19%	100.00%	89.22%	88.18%	93.52%	93.85%	85.56%	95.47%
III	(0,0,1)	(11,3)	62.64%	80.18%	63.21%	100.00%	62.07%	99.90%	84.52%	81.19%
IV	(0.5,0.5,0)	(11,3)	99.85%	99.86%	83.98%	82.42%	91.23%	91.03%	88.15%	91.67%
V	(0,0.5,0.5)	(10,4)	93.58%	90.12%	87.70%	89.45%	93.65%	93.85%	93.92%	86.15%
VI	(0.5,0,0.5)	(11,3)	93.13%	93.14%	93.89%	97.11%	93.64%	89.40%	84.66%	96.48%
VII	(1/3,1/3,1/3)	(11,3)	99.64%	99.65%	83.96%	95.17%	81.40%	81.20%	93.25%	93.67%
VIII	(0.5,0.25,0.25)	(11,3)	96.00%	96.90%	84.50%	86.90%	95.60%	96.40%	86.89%	97.14%
IX	(0.25,0.25,0.5)	(11,3)	94.50%	92.11%	89.70%	89.45%	94.65%	95.85%	95.92%	89.15%
Designs	GAP(w_i)	$df(PE; LoF)$	D-eff	DP-eff	$LoF(D)$-eff	$LoF(DP)$-eff	A-eff	AP-eff	$LoF(A)$-eff	$LoF(AP)$-eff
I	(1,0,0)	(11,3)	84.11%	84.12%	92.34%	84.16%	100.00%	99.72%	89.13%	92.55%
II	(0,1,0)	(13,1)	81.60%	86.62%	89.20%	85.03%	99.65%	100.00%	89.11%	90.85%
III	(0,0,1)	(13,1)	14.84%	15.76%	80.06%	98.83%	95.40%	97.30%	84.96%	100.00%
IV	(0.5,0.5,0)	(11,3)	99.86%	99.80%	73.98%	82.42%	96.45%	97.91%	89.15%	91.67%
V	(0,0.5,0.5)	(9,5)	80.94%	83.38%	82.58%	76.64%	95.45%	97.36%	24.49%	81.49%
VI	(0.5,0,0.5)	(11,3)	93.67%	93.68%	92.35%	96.82%	96.96%	97.75%	95.57%	93.68%
VII	(1/3,1/3,1/3)	(10,4)	89.89%	93.50%	87.98%	96.59%	95.45%	97.36%	86.81%	83.98%
VIII	(0.5,0.25,0.25)	(11,3)	85.50%	84.60%	94.34%	89.16%	96.00%	95.72%	90.13%	95.55%
IX	(0.25,0.25,0.5)	(10,4)	87.90%	86.50%	90.30%	89.80%	94.50%	94.60%	94.50%	94.55%

TABLE 6.12: The GDP-Optimal Design for Constrained Region by 4-components

x_1	x_2	x_3	x_4
0.40	0.10	0.42	0.08
0.60	0.27	0.10	0.03
0.40	0.10	0.42	0.08
0.60	0.10	0.25	0.06
0.50	0.22	0.23	0.05
0.40	0.29	0.28	0.03
0.05	0.10	0.34	0.06
0.40	0.27	0.27	0.06
0.05	0.10	0.34	0.06
0.60	0.24	0.10	0.06
0.50	0.37	0.10	0.03
0.50	0.21	0.21	0.08
0.40	0.29	0.28	0.03
0.50	0.10	0.37	0.03
0.40	0.27	0.27	0.06
0.60	0.19	0.18	0.03
0.40	0.42	0.10	0.08
0.50	0.21	0.21	0.08
0.50	0.21	0.21	0.08
0.60	0.10	0.25	0.06

minant criteria components. However, the same designs have a weak performance with respect to the trace criteria, in the sense that, these designs are only 11.56% A-efficient and 11.55% AP-efficient. In a practical manner, if the experimenter aims to allow some inferences for the individual parameters, such designs are not a good choice for doing the inferences purposes in this case.

Also it is quite notable that, from Table 6.10 almost all resulting optimal designs under determinant based criteria with allocating different weights to the criterion components are extreme in terms of distribution of the degrees of freedom between pure error and lack-of-fit elements, unlike those designs that we obtained for trace based criteria. The designs that are optimal with respect to GAP-criteria, which place equal weights among all three components of the criteria seems to be less extreme in terms of allowing reasonable degrees of freedom for estimating pure error and testing lack-of-fit. Such design is illustrated in Table 6.13.

TABLE 6.13: The GDP-Optimal Design for Constrained Region by 4-components

x_1	x_2	x_3	x_4
0.60	0.10	0.25	0.06
0.40	0.26	0.26	0.08
0.40	0.10	0.47	0.03
0.60	0.10	0.25	0.06
0.50	0.10	0.32	0.08
0.60	0.10	0.25	0.06
0.40	0.26	0.26	0.08
0.50	0.34	0.10	0.06
0.60	0.24	0.10	0.06
0.60	0.24	0.10	0.06
0.50	0.37	0.10	0.03
0.60	0.10	0.27	0.03
0.50	0.37	0.10	0.03
0.40	0.26	0.26	0.08
0.40	0.47	0.10	0.03
0.40	0.42	0.10	0.08
0.60	0.24	0.10	0.06
0.40	0.10	0.47	0.03
0.60	0.10	0.27	0.03
0.50	0.10	0.34	0.06

Generally, we notice that, the obtained designs that are optimal with respect to either generalized determinant or trace criteria perform well with respect to other statistical criteria for modified fractional polynomials in particular when the models have different α. Moreover, the efficiencies of the resulting determinant and trace optimal designs for FPn having different values of the exponent have higher efficiencies with respect to the statistical criteria, such as D-, DP-, A-, and AP-optimality.

Continuing exploring the pattern and performances of the obtained optimal designs in the direction of constrained regions by 6-component proportions under nonlinear models is worthwhile.

In Table 6.14 and Table 6.15, the characteristics and the features of the designs that are optimal with respect to the determinant and trace criteria are shown for a concrete mixture constraints experiments given in Chapter 4. Due to the absence of the treatment replication in some obtained designs, the efficiencies of such designs cannot be evaluated, in the sense that the efficiencies of a certain component (DP, AP, LOF(DP), and LOF(AP)) cannot be estimated, and thus they are set to be zero. In Table 6.14,

TABLE 6.14: The Summary Characteristics of GDP- and GAP- Criteria with $\tau^2 = 1$ with α

Designs	GDP(w_i)	$df(PE; LoF)$	D-eff	DP-eff	$LoF(D)$-eff	$LoF(DP)$-eff	A-eff	AP-eff	$LoF(A)$-eff	$LoF(AP)$-eff
I	(1,0,0)	(0,24)	100.00%	0.00%	95.70%	0.00%	97.96%	0.00%	95.27%	0.00%
II	(0,1,0)	(23,1)	94.00%	100.00%	89.08%	96.67%	89.50%	95.85%	85.56%	97.47%
III	(0,0,1)	(22,2)	55.64%	60.18%	89.88%	100.00%	60.07%	70.90%	90.52%	98.19%
IV	(0.5,0.5,0)	(21,3)	95.85%	97.86%	87.26%	89.42%	91.23%	100.00%	88.15%	94.67%
V	(0,0.5,0.5)	(23,1)	94.00%	100.00%	89.08%	96.67%	89.50%	95.85%	85.56%	97.47%
VI	(0.5,0,0.5)	(23,1)	93.13%	97.93%	87.89%	97.11%	89.64%	95.40%	84.66%	96.48%
VII	(1/3,1/3,1/3)	(23,1)	94.00%	100.00%	89.08%	96.67%	89.50%	95.85%	85.56%	97.47%
VIII	(0.5,0.25,0.25)	(20,4)	95.00%	93.90%	88.50%	94.90%	91.60%	96.40%	86.89%	97.14%
IX	(0.25,0.25,0.5)	(21,3)	95.18%	95.00%	88.07%	95.45%	88.65%	94.85%	86.90%	97.50%
Designs	GAP(w_i)	$df(PE; LoF)$	D-eff	DP-eff	$LoF(D)$-eff	$LoF(DP)$-eff	A-eff	AP-eff	$LoF(A)$-eff	$LoF(AP)$-eff
I	(1,0,0)	(0,24)	96.95%	0.00%	94.50%	0.00%	100.00%	0.00%	94.60%	0.00%
II	(0,1,0)	(21,3)	96.50%	96.90%	89.43%	94.03%	92.65%	100.00%	89.14%	96.85%
III	(0,0,1)	(23,1)	59.80%	59.76%	86.06%	96.83%	73.40%	77.50%	87.96%	100.00%
IV	(0.5,0.5,0)	(19,5)	97.50%	96.88%	87.88%	93.42%	95.45%	97.95%	89.15%	92.67%
V	(0,0.5,0.5)	(23,1)	91.94%	95.88%	87.15%	96.60%	88.47%	95.98%	87.49%	94.49%
VI	(0.5,0,0.5)	(20,4)	95.67%	96.86%	89.35%	94.80%	91.96%	97.70%	89.14%	94.68%
VII	(1/3,1/3,1/3)	(21,3)	94.89%	96.50%	87.48%	95.09%	92.05%	95.45%	86.34%	95.08%
VIII	(0.5,0.25,0.25)	(21,3)	93.88%	95.43%	88.50%	96.07%	91.13%	94.59%	88.34%	96.08%
IX	(0.25,0.25,0.5)	(20,4)	95.67%	96.86%	89.35%	94.80%	91.96%	97.70%	89.14%	94.68%

almost all resulting optimal designs, either under the developed generalized determinant or trace criteria perform well with respect to the standard optimality criteria, that is they tend to have large efficiencies except those designs obtained when no weights have been allocated to D- and DP-components.

The tendency of the resulting optimal designs to the reflected relationship between the primary and potential terms is less drastic when the fitted models have different α or

TABLE 6.15: The Summary Characteristics of GDP- and GAP- Criteria with $\tau^2 = 1/g$ with α

Designs	GDP(w_i)	$df(PE; LoF)$	D-eff	DP-eff	$LoF(D)$-eff	$LoF(DP)$-eff	A-eff	AP-eff	$LoF(A)$-eff	$LoF(AP)$-eff
I	(1,0,0)	(0,24)	100.00%	0.00%	97.70%	0.00%	97.96%	0.00%	97.27%	0.00%
II	(0,1,0)	(23,1)	94.00%	100.00%	97.18%	98.23%	89.50%	95.85%	97.50%	98.66%
III	(0,0,1)	(24,0)	83.07%	89.38%	96.88%	100.00%	85.39%	89.90%	97.52%	98.90%
IV	(0.5,0.5,0)	(21,3)	95.85%	97.86%	95.60%	94.50%	91.23%	100.00%	97.11%	94.60%
V	(0,0.5,0.5)	(24,0)	92.08%	96.23%	97.40%	100.00%	83.45%	93.85%	96.92%	100.00%
VI	(0.5,0,0.5)	(23,1)	93.00%	100.00%	97.80%	97.50%	89.97%	95.40%	95.66%	96.60%
VII	(1/3,1/3,1/3)	(23,1)	93.00%	100.00%	97.80%	97.50%	89.97%	95.40%	95.66%	96.60%
VIII	(0.5,0.25,0.25)	(20,4)	94.60%	95.90%	96.50%	96.90%	89.70%	95.40%	96.89%	94.14%
IX	(0.25,0.25,0.5)	(21,3)	94.50%	92.11%	95.70%	96.45%	91.65%	91.85%	95.92%	96.15%
Designs	GAP(w_i)	$df(PE; LoF)$	D-eff	DP-eff	$LoF(D)$-eff	$LoF(DP)$-eff	A-eff	AP-eff	$LoF(A)$-eff	$LoF(AP)$-eff
I	(1,0,0)	(0,24)	96.95%	0.00%	96.60%	0.00%	100.00%	0.00%	96.60%	0.00%
II	(0,1,0)	(21,3)	96.50%	96.90%	95.40%	97.50%	92.65%	100.00%	97.44%	98.80%
III	(0,0,1)	(24,0)	89.88%	89.96%	96.06%	97.15%	90.50%	93.70%	96.96%	100.00%
IV	(0.5,0.5,0)	(19,5)	97.50%	96.88%	97.00%	89.95%	95.45%	97.95%	95.15%	88.67%
V	(0,0.5,0.5)	(23,1)	93.94%	96.38%	97.58%	97.64%	88.37%	96.50%	94.49%	89.97%
VI	(0.5,0,0.5)	(23,1)	93.67%	95.66%	96.35%	95.82%	89.43%	95.54%	96.57%	96.68%
VII	(1/3,1/3,1/3)	(21,3)	94.43%	95.50%	94.98%	92.59%	91.05%	96.36%	96.88%	94.44%
VIII	(0.5,0.25,0.25)	(19,5)	95.70%	94.65%	95.60%	89.98%	91.88%	95.72%	96.50%	89.55%
IX	(0.25,0.25,0.5)	(21,3)	93.50%	94.44%	95.89%	93.59%	90.90%	95.40%	96.88%	94.45%

TABLE 6.16: The Summary Characteristics of GDP- and GAP- Criteria with $\tau^2 = 1$ with different α

Designs	GDP(w_i)	$df(PE; LoF)$	D-eff	DP-eff	$LoF(D)$-eff	$LoF(DP)$-eff	A-eff	AP-eff	$LoF(A)$-eff	$LoF(AP)$-eff
I	(1,0,0)	(17,3)	100.00%	94.50%	94.70%	89.90%	96.96%	89.80%	95.27%	94.30%
II	(0,1,0)	(17,3)	94.19%	100.00%	90.40%	96.69%	89.90%	94.80%	89.56%	96.47%
III	(0,0,1)	(19,1)	80.64%	85.18%	89.90%	100.00%	88.70%	88.90%	90.52%	97.20%
IV	(0.5,0.5,0)	(17,3)	94.80%	95.50%	88.23%	89.42%	92.23%	100.00%	89.15%	96.67%
V	(0,0.5,0.5)	(14,6)	94.81%	96.00%	88.55%	96.60%	88.65%	95.85%	85.92%	98.15%
VI	(0.5,0,0.5)	(15,5)	93.13%	97.93%	89.90%	96.15%	89.60%	95.40%	86.66%	95.48%
VII	(1/3,1/3,1/3)	(17,3)	94.00%	96.50%	87.96%	94.70%	87.60%	95.20%	87.23%	97.67%
VIII	(0.5,0.25,0.25)	(16,4)	95.00%	93.90%	89.50%	94.90%	92.60%	96.40%	87.89%	97.14%
IX	(0.25,0.25,0.5)	(17,3)	94.19%	100.00%	90.40%	96.69%	89.90%	94.80%	89.56%	96.47%
Designs	GAP(w_i)	$df(PE; LoF)$	D-eff	DP-eff	$LoF(D)$-eff	$LoF(DP)$-eff	A-eff	AP-eff	$LoF(A)$-eff	$LoF(AP)$-eff
I	(1,0,0)	(20,0)	95.95%	94.05%	93.50%	92.90%	100.00%	89.90%	94.60%	94.50%
II	(0,1,0)	(17,3)	96.40%	96.60%	89.33%	95.07%	90.90%	100.00%	88.45%	97.80%
III	(0,0,1)	(18,2)	90.80%	92.70%	87.45%	95.80%	92.40%	94.50%	88.96%	100.00%
IV	(0.5,0.5,0)	(10,10)	97.50%	96.88%	88.88%	94.42%	95.45%	96.95%	89.90%	94.67%
V	(0,0.5,0.5)	(17,3)	92.94%	94.88%	88.50%	97.60%	88.43%	94.98%	88.49%	95.49%
VI	(0.5,0,0.5)	(17,3)	94.67%	95.86%	89.50%	93.80%	91.96%	96.70%	89.90%	95.68%
VII	(1/3,1/3,1/3)	(16,4)	94.89%	96.50%	87.48%	95.09%	92.05%	95.45%	86.34%	95.08%
VIII	(0.5,0.25,0.25)	(14,6)	92.07%	93.77%	89.45%	94.80%	91.40%	91.50%	88.96%	95.00%
IX	(0.25,0.25,0.5)	(15,5)	94.67%	95.86%	89.50%	93.80%	91.96%	96.70%	89.90%	95.68%

TABLE 6.17: The Summary Characteristics of GDP- and GAP- Criteria with $\tau^2 = 1/g$ with different α

Designs	GDP(w_i)	$df(PE; LoF)$	D-eff	DP-eff	$LoF(D)$-eff	$LoF(DP)$-eff	A-eff	AP-eff	$LoF(A)$-eff	$LoF(AP)$-eff
I	(1,0,0)	(17,3)	100.00%	94.50%	96.80%	93.90%	96.96%	89.80%	96.40%	95.50%
II	(0,1,0)	(17,3)	94.19%	100.00%	94.40%	96.90%	89.90%	94.80%	92.50%	96.90%
III	(0,0,1)	(18,2)	80.60%	84.18%	89.90%	100.00%	90.70%	91.90%	90.52%	97.20%
IV	(0.5,0.5,0)	(17,3)	94.80%	95.50%	89.50%	89.90%	92.23%	100.00%	91.09%	95.60%
V	(0,0.5,0.5)	(18,2)	89.90%	92.00%	89.90%	95.45%	88.65%	94.85%	87.92%	96.15%
VI	(0.5,0,0.5)	(17,3)	93.13%	97.93%	87.89%	97.11%	89.64%	95.40%	86.66%	96.48%
VII	(1/3,1/3,1/3)	(19,1)	93.00%	95.08%	88.90%	95.70%	86.40%	94.20%	87.25%	97.67%
VIII	(0.5,0.25,0.25)	(16,4)	94.50%	94.90%	89.50%	94.90%	92.60%	94.40%	88.89%	98.14%
IX	(0.25,0.25,0.5)	(17,3)	93.13%	97.93%	87.89%	97.11%	89.64%	95.40%	86.66%	96.48%
Designs	GAP(w_i)	$df(PE; LoF)$	D-eff	DP-eff	$LoF(D)$-eff	$LoF(DP)$-eff	A-eff	AP-eff	$LoF(A)$-eff	$LoF(AP)$-eff
I	(1,0,0)	(20,0)	95.95%	94.05%	94.90%	95.00%	100.00%	89.90%	95.60%	96.50%
II	(0,1,0)	(17,3)	96.40%	96.60%	89.90%	96.00%	90.90%	100.00%	89.90%	98.60%
III	(0,0,1)	(18,2)	90.80%	92.70%	87.45%	95.80%	92.40%	94.50%	88.96%	100.00%
IV	(0.5,0.5,0)	(10,10)	97.50%	96.88%	89.88%	96.67%	95.45%	96.95%	89.33%	97.67%
V	(0,0.5,0.5)	(17,3)	94.94%	94.90%	89.97%	96.05%	88.43%	95.00%	87.50%	96.04%
VI	(0.5,0,0.5)	(16,4)	92.07%	93.77%	89.45%	94.80%	91.40%	91.50%	88.96%	95.00%
VII	(1/3,1/3,1/3)	(16,4)	95.80%	94.90%	88.65%	96.03%	93.00%	93.40%	87.33%	94.08%
VIII	(0.5,0.25,0.25)	(10,10)	97.50%	96.88%	89.88%	96.67%	95.45%	96.95%	89.33%	97.67%
IX	(0.25,0.25,0.5)	(14,6)	92.67%	95.86%	89.50%	93.80%	91.96%	96.70%	89.90%	95.68%

even when setting the tuning variance τ^2 to be proportional to the inverse of the number of potential terms as in design III in Table 6.15.

Some obtained designs that are optimal either under statistical standard or compound criteria are not far from converging to each other, in the sense that, DP-optimal designs retain optimality when some equal weights are distributed either between primary and potential elements as in design V or even among all three components of the criteria as in design VII in Table 6.14. Such a design is given in Table 6.19. For the GDP- and GAP-optimal designs in Table 6.17 we notice that, some optimal designs are quite similar as far as the degrees of freedom for pure error are concerned. Such designs perform well with respect to the other primary criteria. Besides of this notification, although the designs (II under either GDP- or GAP-criteria) are equivalent in terms of their pure error degrees of freedom, they have different properties for the inferences on the treatment parameters. That is, design II under GDP-optimality is being better for inferences purposes on generalized variances of parameters, while design II resulting from the compound trace criterion is enhanced the inferences for the individuals parameters.

Furthermore, it is quite notable that, Designs IV in Table 6.16 and VIII in Table 6.17 under trace based criteria resulted after giving equal and different sets of weights respectively are less extreme in the sense that they offer evenly allocated degrees of freedom for estimating pure error and testing lack-of-fit. Moreover, such designs have higher efficiencies regarding to the other components of the criteria.

Overall, under searching for optimal designs assuming nonlinear models to be fitted to the data we notice that, some of the resulting optimal designs for the design regions

TABLE 6.18: The GDP-Optimal Design for Constrained Region by 6-components

x_1	x_2	x_3	x_4	x_5	x_6
0.185	0.15	0.01	0.005	0.40	0.250
0.185	0.135	0.01	0.01	0.41	0.250
0.19	0.13	0.02	0.01	0.40	0.25
0.16	0.13	0.01	0.01	0.40	0.29
0.185	0.15	0.01	0.005	0.40	0.250
0.185	0.15	0.01	0.005	0.40	0.250
0.185	0.135	0.01	0.01	0.41	0.250
0.16	0.13	0.01	0.01	0.40	0.29
0.16	0.13	0.03	0.005	0.425	0.25
0.19	0.13	0.015	0.005	0.40	0.26
0.185	0.135	0.01	0.01	0.41	0.250
0.16	0.15	0.03	0.01	0.40	0.25
0.16	0.13	0.01	0.01	0.40	0.29
0.19	0.13	0.02	0.01	0.40	0.25
0.185	0.135	0.01	0.01	0.41	0.250
0.185	0.15	0.01	0.005	0.40	0.250
0.185	0.15	0.01	0.005	0.40	0.250
0.16	0.13	0.03	0.005	0.425	0.25
0.16	0.13	0.03	0.005	0.425	0.25
0.19	0.13	0.015	0.005	0.40	0.26
0.19	0.13	0.015	0.005	0.40	0.26
0.16	0.13	0.01	0.01	0.40	0.29
0.16	0.13	0.03	0.005	0.425	0.25
0.19	0.13	0.015	0.005	0.40	0.26
0.16	0.15	0.03	0.01	0.40	0.25
0.16	0.15	0.03	0.01	0.40	0.25
0.19	0.13	0.02	0.01	0.40	0.25
0.19	0.13	0.02	0.01	0.40	0.25
0.16	0.13	0.01	0.01	0.40	0.29
0.16	0.15	0.03	0.01	0.40	0.25

resulting from restrictions that have been placed on the component proportions are less extreme regarding the distribution of the degrees of freedom. Other than that, under compound criteria, some resulting designs offer balanced degrees of freedom for pure error and lack-of-fit components.

With respect to the efficiency, the resulting optimal designs under nonlinear models perform better with respect to other individual statistical primary criteria, in particular, when the models have different values of the exponent. Furthermore, the relationship between primary and potential components becomes less considerable when nonlinear models with different values of the exponent are considered.

Since the main properties of the designs obtained can be enhanced by allocating appropriate weights to the components of the compound criteria, the experimenter should try different schemes of such weights and then obtain the designs to decide which of these resulting designs can be used in practical manner. Furthermore, the compound criteria with allocating such weights can construct designs such that they either offer compromise or evenly degrees of freedom for pure estimation and lack-of-fit testing. Such designs for some cases for the objectives under consideration can then be recommended for an application and for statistical inferences. Thus, it is worth taking these criteria under consideration for constructing an optimal designs to meet all objectives under study.

As can be seen from the above experimental work the optimal designs were obtained depending on the estimated values of the model parameters. The question arises, how

will be the performance of the designs if we consider any value of the exponent α other than the estimated values of such parameters. For example, if we consider that $\alpha = 1$ and the fitted model is the first degree modified fractional polynomial model. The summaries of the characteristics of the resulting optimal designs under different values of the parameter τ^2 for the present case can be presented in Table 6.19 and Table 6.20 respectively. From Table 6.19 we notice that some designs are more extreme than those

TABLE 6.19: The Summary Characteristics of GDP- and GAP- Criteria with $\tau^2 = 1$ and common value for the exponent α

Designs	GDP(w_i)	$df(PE;LoF)$	D-eff	DP-eff	$LoF(D)$-eff	$LoF(DP)$-eff	A-eff	AP-eff	$LoF(A)$-eff	$LoF(AP)$-eff
I	(1,0,0)	(27,0)	100.00%	82.80%	96.54%	66.60%	87.18%	85.28%	86.86%	91.58%
II	(0,1,0)	(26,1)	58.47%	100.00%	60.20%	74.72%	82.68%	82.97%	89.81%	39.24%
III	(0,0,1)	(25,2)	3.84%	5.21%	39.15%	100.00%	93.40%	89.90%	89.50%	90.40%
IV	(0.5,0.5,0)	(27,0)	72.20%	82.60%	74.32%	68.32%	91.30%	91.25%	87.45%	89.80%
V	(0,0.5,0.5)	(27,0)	92.19%	92.38%	94.25%	93.50%	95.50%	93.79%	90.31%	94.50%
VI	(0.5,0,0.5)	(26,1)	88.40%	90.50%	89.30%	85.79%	93.20%	84.55%	92.51%	92.30%
VII	(1/3,1/3,1/3)	(27,0)	85.12%	82.90%	85.85%	73.66%	89.38%	83.35%	87.20%	90.55%
VIII	(0.5,0.25,0.25)	(24,3)	90.40%	91.30%	89.30%	89.60%	92.50%	89.50%	87.30%	88.60%
IX	(0.25,0.25,0.5)	(25,2)	94.90%	93.50%	91.30%	91.80%	93.40%	93.90%	90.50%	89.70%
Designs	GAP(w_i)	$df(PE;LoF)$	D-eff	DP-eff	$LoF(D)$-eff	$LoF(DP)$-eff	A-eff	AP-eff	$LoF(A)$-eff	$LoF(AP)$-eff
I	(1,0,0)	(24,3)	83.20%	83.53%	83.44%	88.35%	100.00%	91.18%	74.84%	92.70%
II	(0,1,0)	(26,1)	72.44%	85.39%	85.98%	91.27%	81.99%	100.00%	88.81%	92.20%
III	(0,0,1)	(25,2)	89.61%	89.99%	87.55%	86.90%	81.13%	89.41%	95.44%	100.00%
IV	(0.5,0.5,0)	(27,0)	89.23%	93.82%	86.78%	89.24%	86.03%	90.42%	90.59%	98.35%
V	(0,0.5,0.5)	(26,1)	83.23%	85.60%	99.71%	93.81%	86.09%	89.90%	90.21%	92.90%
VI	(0.5,0,0.5)	(26,1)	85.64%	86.55%	99.05%	96.40%	89.90%	89.33%	90.20%	91.77%
VII	(1/3,1/3,1/3)	(27,0)	89.77%	93.72%	86.25%	90.15%	90.57%	91.80%	89.17%	96.53%
VIII	(0.5,0.25,0.25)	(27,0)	89.77%	93.72%	94.21%	84.35%	90.57%	91.80%	93.40%	83.79%
IX	(0.25,0.25,0.5)	(26,1)	80.88%	91.36%	92.27%	87.80%	96.12%	91.19%	89.50%	86.32%

TABLE 6.20: The Summary Characteristics of GDP- and GAP- Criteria with $\tau^2 = 1/g$ and common value for the exponent α

Designs	GDP(w_i)	$df(PE;LoF)$	D-eff	DP-eff	$LoF(D)$-eff	$LoF(DP)$-eff	A-eff	AP-eff	$LoF(A)$-eff	$LoF(AP)$-eff
I	(1,0,0)	(27,0)	100.00%	82.80%	96.54%	69.80%	87.18%	85.28%	95.02%	91.58%
II	(0,1,0)	(26,1)	58.47%	100.00%	79.50%	86.22%	82.68%	82.97%	89.81%	39.24%
III	(0,0,1)	(25,2)	88.16%	64.83%	82.18.15%	100.00%	94.55%	92.20%	97.53%	99.49%
IV	(0.5,0.5,0)	(27,0)	72.20%	82.60%	72.73%	86.32%	91.30%	91.25%	92.45%	90.21%
V	(0,0.5,0.5)	(26,1)	89.27%	90.38%	87.30%	88.48%	90.53%	92.61%	89.31%	90.50%
VI	(0.5,0,0.5)	(25,2)	85.43%	90.60%	94.33%	90.79%	95.10%	89.33%	90.51%	90.70%
VII	(1/3,1/3,1/3)	(27,0)	91.15%	93.10%	89.40%	87.90%	92.38%	93.35%	88.04%	89.50%
VIII	(0.5,0.25,0.25)	(24,3)	93.40%	92.30%	95.30%	92.60%	95.30%	92.50%	90.30%	90.60%
IX	(0.25,0.25,0.5)	(25,2)	89.90%	95.50%	92.30%	97.80%	97.33%	94.80%	93.50%	93.70%
Designs	GAP(w_i)	$df(PE;LoF)$	D-eff	DP-eff	$LoF(D)$-eff	$LoF(DP)$-eff	A-eff	AP-eff	$LoF(A)$-eff	$LoF(AP)$-eff
I	(1,0,0)	(24,3)	83.20%	83.53%	85.06%	88.30%	100.00%	91.18%	84.84%	97.54%
II	(0,1,0)	(26,1)	72.44%	85.39%	86.90%	89.27%	81.99%	100.00%	87.89%	86.50%
III	(0,0,1)	(25,2)	84.85%	89.90%	88.50%	89.90%	87.48%	89.90%	93.40%	100.00%
IV	(0.5,0.5,0)	(27,0)	89.23%	93.82%	87.78%	89.55%	86.03%	90.42%	92.59%	95.35%
V	(0,0.5,0.5)	(26,1)	83.23%	85.60%	99.71%	93.81%	86.09%	89.90%	90.21%	92.90%
VI	(0.5,0,0.5)	(26,1)	87.60%	88.65%	89.40%	91.40%	90.92%	88.89%	90.15%	90.60%
VII	(1/3,1/3,1/3)	(26,1)	83.23%	85.60%	99.71%	93.81%	86.09%	89.90%	90.21%	92.90%
VIII	(0.5,0.25,0.25)	(27,0)	89.77%	93.72%	92.55%	84.25%	90.57%	91.80%	92.44%	85.03%
IX	(0.25,0.25,0.5)	(26,1)	80.88%	91.36%	93.14%	87.82%	96.12%	91.19%	80.50%	80.32%

designs obtained under estimated values of the exponent α as is the case in design VII and design VIII under trace based criteria. Such designs are similar as far as the degrees

of freedom for estimating pure error are concerned and thus they do not allow any degrees of freedom for testing lack of fit. Moreover, the designs (VII and VIII under GAP-optimal criteria) under estimated value of the power α have a good performances with respect to the others much better than those designs obtained under assuming $\alpha = 1$. Similar manner occur when assuming the tuning parameter $\tau^2 = 1/g$, such as design VII under GDP-criteria and design VIII under trace form.

Generally, some obtained designs under assuming $\alpha = 1$ become more concerned with estimating pure error or even for some designs they do not provide any degrees of freedom for testing lack-of-fit. Other than that, some designs, such as DP-optimal designs have poor performance with respect to the D-criterion, unlike the obtained DP-optimal designs under estimated values of the parameters.

Chapter 7

Optimum Continuous Design Theory

7.1 Introduction

Continuous (approximate) designs are a common tool in the literature of the optimal design approach (Atkinson et al., 2007). Such designs are considered a fundamental tool in the construction of optimal designs. They choose the design points $\boldsymbol{x_r}$ from the design space $\boldsymbol{\chi}$ that satisfy the optimization conditions with respect to some statistical optimality criterion. They are used because the methodology to solve the optimization problems for finding these designs is more flexible than for exact designs in terms of ignoring the condition that the number of trials at any design point must be an integer and instead, allowing the proportions of observations in each design point $\boldsymbol{x_r}$ to be real numbers varying continuously between 0 and 1. These proportions are denoted by w_r and are called the weights. Thus, applying this methodology we can find the optimal designs independently from sample size. Moreover, we obtain a continuous convex optimization problem for optimal design criteria with respect to minor assumptions.

It is worth mentioning that, the distinction between exact and continuous designs appears in the optimization problems. In the sense that, the optimization issues in approximate designs finds the optimal measure $\boldsymbol{\xi}$ which then has to be rounded to an exact design before it can be involved in practice. However, in exact design such optimization problem finds the number of design points $x_1, x_2, \cdots, x_q$ from the design space $\boldsymbol{\chi}$ with the number of the observations $n_1, n_2, \cdots, n_q$ taken at each of these design points and subject to the constraint that they sum to n. For example, suppose a design $\boldsymbol{\xi}$ has n trials such that there are n_r replicates $r = 1, 2, \cdots, q$ at q distinct support points $x_1, x_2, \cdots, x_q$, with allocated weights w_r to each points that satisfy $\sum_{r=1}^{q} \frac{n_r}{n} = 1$, thus

$\sum_{r=1}^{q} w_r = 1$. Such a design can be represented as

$$\boldsymbol{\xi} = \left\{ \begin{array}{cccc} \boldsymbol{x_1} & \boldsymbol{x_2} & \cdots & \boldsymbol{x_q} \\ w_1 & w_2 & \cdots & w_q \end{array} \right\}. \tag{7.1}$$

When the number of design points are the same as n, then two scenarios arise. The first scenario is nw_r for all $r = 1, 2, \cdots, q$ are integers. In this case the resulting design is known as an exact design. However, in the second scenario where nw_r for some $r \in \{1, 2, \cdots, q\}$ is rational, the obtained designs are approximate designs.

For simplicity, consider the design has $n = 9$ trials with three design points $\boldsymbol{x_1}, \boldsymbol{x_2}$, and $\boldsymbol{x_3}$ such that $\boldsymbol{x_1}$ has $n_1 = 3$, $\boldsymbol{x_2}$ has $n_2 = 3$, and $\boldsymbol{x_3}$ has $n_3 = 3$, where $n = n_1 + n_2 + n_3$. Thus the optimal design can be represented as

$$\boldsymbol{\xi}^* = \left\{ \begin{array}{ccc} \boldsymbol{x_1} & \boldsymbol{x_2} & \boldsymbol{x_3} \\ 0.333 & 0.333 & 0.333 \end{array} \right\}. \tag{7.2}$$

In this case, this design is an exact design that can be implemented in practice because $n_1 = n_2 = n_3 = 0.333 \cdot 9 = 3$. On the other hand, such designs cannot be involved in a practical manner if for example n = 8, then $n_1 = n_2 = n_3 = 0.333 \cdot 8 = 2.6$ which is a fractional number. Therefore, we need to round such designs appropriately to an exact design for using in practice. Thus the exact design with n = 8 could be defined as $\boldsymbol{x_1}$ has $n_1 = 3$, $\boldsymbol{x_2}$ has $n_2 = 3$, and $\boldsymbol{x_3}$ has $n_3 = 2$, which can be represented as

$$\boldsymbol{\xi}^{**} = \left\{ \begin{array}{ccc} \boldsymbol{x_1} & \boldsymbol{x_2} & \boldsymbol{x_3} \\ \frac{3}{8} & \frac{3}{8} & \frac{2}{8} \end{array} \right\}. \tag{7.3}$$

In general, the continuous design $\boldsymbol{\xi}^{***}$, for the exact design $\boldsymbol{\xi}$ can be written as

$$\boldsymbol{\xi}^{***} = \left\{ \begin{array}{cccc} \boldsymbol{x_1} & \boldsymbol{x_2} & \cdots & \boldsymbol{x_k} \\ \frac{n_1}{n} & \frac{n_2}{n} & \cdots & \frac{n_k}{n} \end{array} \right\}. \tag{7.4}$$

Any exact design can be expressed as an equivalent continuous design; however, not every continuous design can be expressed as an exact design.

The main concept of working with the continuous design is that after formulating our objective as a convex function of the information matrix, the convex optimization problem and convex analysis theory can be applied to verify whether a design is an optimal design or not (Kiefer, 1974). Different types of statistical models can be considered in continuous optimization problems, such as statistical linear and nonlinear models. The

differences between such models coincide in the construction of the information matrix, as mentioned before. By considering such differences, the information matrix generally for a continuous design $\boldsymbol{\xi}$ can be formulated as

$$M(\boldsymbol{\xi}) = \int_{\chi} f(\boldsymbol{x}) f'(\boldsymbol{x}) \xi \, (\mathrm{d}\boldsymbol{x}) = \sum_{r=1}^{q} w_r f(\boldsymbol{x_r}) f'(\boldsymbol{x_r}), \tag{7.5}$$

where $f'(\boldsymbol{x_r})$ is the rth row of X in linear models or F in nonlinear models. The standardized variance of the predicted response is

$$d(\boldsymbol{x}, \boldsymbol{\xi}) = f(\boldsymbol{x}) M^{-1}(\xi) f'(\boldsymbol{x}), \tag{7.6}$$

where $d(\boldsymbol{x}, \boldsymbol{\xi})$ is a function of the design $\boldsymbol{\xi}$ and the point $\boldsymbol{x}$ at which the prediction is made. This function is known as the sensitivity function (Fedorov and Hackl, 2012), which is considered as the fundamental tool on which the theory of the optimum continuous designs depends for verifying and checking the optimization mechanism.

7.2 Optimization Problem and Continuous Design

Suppose that we are given a statistical model and pre-decide a design criterion, for instance D-optimality criterion. Then our motivation is to find design points $\boldsymbol{x_r}$ out of the available design space $\boldsymbol{\chi}$ at which to observe the results in an optimal manner. Different methods are provided in the literature: analytical processes using traditional mathematical derivations based on model assumptions or numerical analysis by applying algorithms that can find optimal results to be implemented in practice. Often, it is difficult to find an optimum set of the design points and their corresponding weights $\{\boldsymbol{x_r}, w_r\}_1^q$ analytically. Then a second-order numerical method, which is an algorithmic approach, can be applied to find such optimum sets. Different algorithms in the literature can provide an optimal combination of $\boldsymbol{x_r}$ and w_r for any optimization problems, such as the Nelder-Mead algorithm. The Nelder-Mead algorithm, or sometimes known as the simplex search algorithm, is designed to solve the classical optimization problems of minimizing or maximizing the objective function by a direct search method over the design space $\boldsymbol{\chi}$ and is implemented in R software.

As we are interested in the mixture problem, where the components are subject to some restrictions, whether natural or additional, as we mentioned before in Chapter 2, the general purpose of optimization of such a problem use the simplex search method in the constrained area. Our methodology for finding the optimum values of the component proportions $\boldsymbol{x_r}$ and their corresponding optimal weights w_r is applying the constrOptim

process for constrained optimization (George et al., 2016). Such a process tends to optimize a function, which is subject to inequality constraints, by supplying an initial guess, say $\boldsymbol{x_{r0}}$ as starting values of the components proportions and w_{r0} as a starting value of their allocated weights. Then the algorithm seeks to minimize the objective function which depends on the starting values of $\boldsymbol{x_{r0}}$ and w_{r0}. However, since our region of interest is constrained, the starting value must be in the feasible region. The feasible region in such a process is defined by $u_i \times \boldsymbol{\theta} - \boldsymbol{c_i} \geq 0$, where u_i is the $(k \times p)$ constraint matrix, $\boldsymbol{c_i}$ is the constraint vector of length k, and θ is the vector of the initial guessed values of $\boldsymbol{x_{r0}}$ and w_{r0}. For clarifying the process, refer to chick feeding example, where the experimenter has set suitable constraints on the proportion of the energy supplements as

$$0.05 \leq x_1 \leq 0.40,\ \ 0.02 \leq x_2 \leq 0.89,\ \ 0.06 \leq x_3 \leq 0.86\,.$$

We consider Scheffé's first order polynomial model, expressed in (2.9). First of all, we will find the constraint matrix u_i and the constraint vector $\boldsymbol{c_i}$. Since the model has three parameters, we should have at least three support points for estimating the model parameters (Atkinson et al., 2007). We will suppose that we have three support points $\boldsymbol{x_1} = (x_{11}, x_{12}, x_{13})$, $\boldsymbol{x_2} = (x_{21}, x_{22}, x_{23})$, $\boldsymbol{x_3} = (x_{31}, x_{32}, x_{33})$, with allocated weights for each support point w_1, w_2, and w_3. Considering the constraints that have been placed on each of the proportion components, we can calculate both the matrix u_i and the vector $\boldsymbol{c_i}$ for applying the process. For example, in this case and for the first degree linear model the constraint matrix can be expressed as in Table 7.1. Rows 1-6 are for the first point $\boldsymbol{x_1}$, rows 7-12 for the second point, and rows 13-18 are obtained from the third support point $\boldsymbol{x_1}$. However, the last three rows are corresponding to the weights. For simplicity, $0.05 \leq x_{11}$, $x_{11} \leq 0.40 \rightarrow -x_{11} \geq -0.40$ that resulting in the first and second row. We can use a similar method for all proportion components, all support points and for all weights. For obtaining the vector $\boldsymbol{c_i}$, all its elements are corresponding to the value for each inequality that we can find to build the above matrix. Thus the resulting constrained vector is:

$\boldsymbol{c_i} = (0.05, -0.40, 0.02, -0.89, -0.94, 0.14, 0.05, -0.40, 0.02, -0.89, -0.94, 0.14, 0.05, -0.40, 0.02, -0.89, -0.94, 0.14, 0, 0, -1)$. After we obtained the matrix u_i, and the vector $\boldsymbol{c_i}$, we guess the initial value for the $\boldsymbol{x_r}$, and w_r. Since the restrictions that are placed on them which are $\sum_r^q \boldsymbol{x_r} = 1$, and $\sum_r^q w_r = 1$, these values must be varying between 0 and 1 for both $\boldsymbol{x_r}$, and w_r. Using these constraints, we find the statistical D-optimum criterion which is our objective function, and use the constrOptim method (that coincides with the Nelder-Mead algorithm). The optimal values for weights and proportion components are represented in Table 7.2. For checking whether the above design is an

TABLE 7.1: Constraint Matrix u_i

1	2	3	4	5	6	7	8
1.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
-1.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
0.00	1.00	0.00	0.00	0.00	0.00	0.00	0.00
0.00	-1.00	0.00	0.00	0.00	0.00	0.00	0.00
-1.00	-1.00	0.00	0.00	0.00	0.00	0.00	0.00
1.00	1.00	0.00	0.00	0.00	0.00	0.00	0.00
0.00	0.00	1.00	0.00	0.00	0.00	0.00	0.00
0.00	0.00	-1.00	0.00	0.00	0.00	0.00	0.00
0.00	0.00	0.00	1.00	0.00	0.00	0.00	0.00
0.00	0.00	0.00	-1.00	0.00	0.00	0.00	0.00
0.00	0.00	-1.00	-1.00	0.00	0.00	0.00	0.00
0.00	0.00	1.00	1.00	0.00	0.00	0.00	0.00
0.00	0.00	0.00	0.00	1.00	0.00	0.00	0.00
0.00	0.00	0.00	0.00	-1.00	0.00	0.00	0.00
0.00	0.00	0.00	0.00	0.00	1.00	0.00	0.00
0.00	0.00	0.00	0.00	0.00	-1.00	0.00	0.00
0.00	0.00	0.00	0.00	-1.00	-1.00	0.00	0.00
0.00	0.00	0.00	0.00	1.00	1.00	0.00	0.00
0.00	0.00	0.00	0.00	0.00	0.00	1.00	0.00
0.00	0.00	0.00	0.00	0.00	0.00	0.00	1.00
0.00	0.00	0.00	0.00	0.00	0.00	-1.00	-1.00

TABLE 7.2: The Optimum Values of $\boldsymbol{x_r}$ and w_r for First Degree Linear Model

The Component Proportions	The weights
$\boldsymbol{x_1}$ = (0.40,0.50,0.1)	w_1= 0.333
$\boldsymbol{x_2}$ = (0.05,0.89,0.06)	w_2= 0.333
$\boldsymbol{x_3}$ = (0.05,0.09,0.86)	w_3= 0.333

optimal continuous design or not, the equivalence theorem for D-optimality is applied by letting

$$d(\boldsymbol{x}, \boldsymbol{\xi}) = f(\boldsymbol{x}) M^{-1}(\xi) f'(\boldsymbol{x}). \tag{7.7}$$

We need to check if $d(\boldsymbol{x}, \boldsymbol{\xi}) = p$ for all support points $\boldsymbol{x_r}$, where p is the number of model parameters, with $d(\boldsymbol{x}, \boldsymbol{\xi}) < p$ at suggested points from continuous region $\boldsymbol{\chi}$. In this example the resulting variance function is

$$\begin{aligned} d(\boldsymbol{x}, \boldsymbol{\xi}) \quad = \quad & (29.01x_1 - 6.58x_2 - 1.73x_3)x_1 + (-6.58x_1 + 4.47x_2 - 0.32x_3)x_2 + \\ & (-1.73x_1 - 0.32x_2 + 4.26x_3)x_3. \end{aligned} \tag{7.8}$$

By plugging the obtained value for all $\boldsymbol{x_r}$, $r = 1, 2, 3$ into $d(\boldsymbol{x}, \boldsymbol{\xi})$, we obtain

$$\begin{aligned} d(\boldsymbol{x_1}, \boldsymbol{\xi}) &= 3 \\ d(\boldsymbol{x_2}, \boldsymbol{\xi}) &= 3 \\ d(\boldsymbol{x_3}, \boldsymbol{\xi}) &= 3 \end{aligned}$$

For all support points, as we can see, the standardized variance is indeed $p = 3$, which is the number of model parameters. This continuous optimum design is represented in Figure 7.1. Besides these points some points from continuous region $\boldsymbol{\chi}$ also satisfy the conditions that $d(\boldsymbol{x}, \boldsymbol{\xi}) < p$. Here we will suppose these points to be placed in between the design points. Then, we plug these points into the sensitivity function $d(\boldsymbol{x}, \boldsymbol{\xi})$ to see if these points satisfy the conditions of such a function. Proceeding from these steps, some points, such as (0.05, 0.49, 0.46), (0.225, 0.715, 0.06) and (0.19, 0.75, 0.06) satisfy the condition that the values of $d(\boldsymbol{x}, \boldsymbol{\xi})$ at these points are less than p, as they are evaluated to be 1.44, 1.22 and 1.63 respectively. However, if the supposed points are so close to the support points, such as (0.07, 0.87, 0.06), in this case the resulting value of $d(\boldsymbol{x}, \boldsymbol{\xi})$ is approximately to be the same as the number of model parameters as it is evaluated to be 2.69.

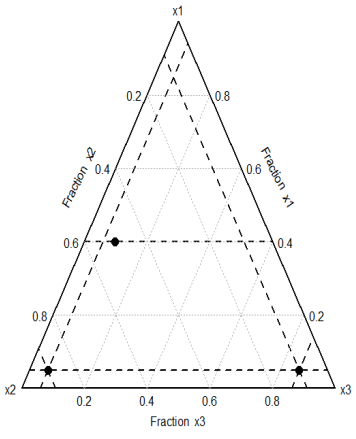

FIGURE 7.1: Optimum Set of $\boldsymbol{x_r}$ and w_r for LM with Additional Constraints

If the region of interest is the whole simplex, where the components are subject only to the natural constraints given in (2.1) and (2.2), we can apply the same method with the

same statistical model to find a continuous optimum design for this situation. The main difference between the present problem and the previous one is the difference in design regions for the proportion components. That is, in the previous case, the resulting points must be in some where in the middle of the simplex, i.e none of these components will be 100% or 0%, while for the present problem this setting can be revoked. The resulting optimum continuous design for such problem under natural constraints (NC) can be given in Table 7.3. Moreover, we can apply the equivalence theorem for D-optimum

TABLE 7.3: The Optimum Values of $\boldsymbol{x_r}$ and w_r for the First Degree Linear Model (NC)

The Component Proportions	The weights
$\boldsymbol{x_1} = (1,0,0)$	$w_1 = 0.333$
$\boldsymbol{x_2} = (0,1,0)$	$w_2 = 0.333$
$\boldsymbol{x_3} = (0,0,1)$	$w_3 = 0.333$

design where

$$d(\boldsymbol{x}, \boldsymbol{\xi}) = 3x_1^2 + 3x_2^2 + 3x_3^2. \tag{7.9}$$

Here, $d(\boldsymbol{x}, \boldsymbol{\xi}) = p = 3$, where p is the number of parameters in the first degree polynomial model. The resulting region is expressed in Figure 7.2. Following the same process as for the constrained region with respect to the search over the continuous region, we found a similar result here, in the sense that the values of the variance function given in (7.9) at the points (0.5, 0, 0.5), (0, 0.5, 0), (0.5, 0.5, 0), and (2/3, 1/3, 0) are less than the number of model parameters. This can also be seen by considering the sum $x_1^2 + x_2^2 + x_3^2$, which must be ≤ 1 if $x_1 + x_2 + x_3 = 1$ and $x_1, x_2, x_3 \geq 0$, and hence $d(\boldsymbol{x}, \boldsymbol{\xi}) \leq 3$ with $d(\boldsymbol{x}, \boldsymbol{\xi}) = 3$ at the support points across $\boldsymbol{\chi}$.

7.2.1 Continuous Design for Second Degree Linear Models

The main strategy of this part is based on two scenarios. The first scenario is by considering the same problem with the same constrained area that we have discussed above and following the process of the algorithm, while assuming the second order polynomial model given in (2.11). Here we need at least 6 support points as the number of the parameters in the second degree linear model. The resulting optimum continuous design that satisfies the variance function is displayed in Table 7.4. Here the resulting

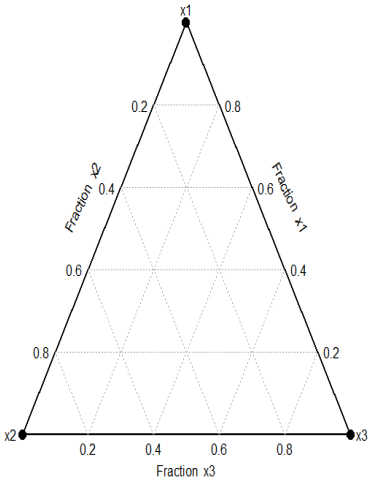

FIGURE 7.2: The Optimum Set of $\boldsymbol{x_r}$ and w_r for FLM with Natural Constraints

TABLE 7.4: The Optimum Values of $\boldsymbol{x_r}$ and w_r for the Second Degree Linear Model

The Component Proportions	The Weights
$\boldsymbol{x_1} = (0.14,0.04,0.82)$	$w_1 = 0.166667$
$\boldsymbol{x_2} = (0.20,0.09,0.71)$	$w_2 = 0.166667$
$\boldsymbol{x_3} = (0.21,0.45,0.34)$	$w_3 = 0.166667$
$\boldsymbol{x_4} = (0.40,0.54,0.06)$	$w_4 = 0.166667$
$\boldsymbol{x_5} = (0.40,0.02,0.58)$	$w_5 = 0.166667$
$\boldsymbol{x_6} = (0.10,0.04,0.86)$	$w_6 = 0.166667$

variance function for the second degree linear model in this problem is

$$\begin{aligned} d(\boldsymbol{x},\boldsymbol{\xi}) \;=\; & (3020x_1 + 102x_2 + 158x_3 - 5171x_1x_2 - 5284x_1x_3 + 182x_2x_3)x_1 + \\ & (102x_1 + 19x_2 + 6x_3 - 217x_1x_2 - 176x_1x_3 - 27x_2x_3)x_2 + \\ & (158x_1 + 6x_2 + 26x_3 - 264x_1x_2 - 331x_1x_3 - 27x_2x_3)x_3 + \\ & (-5171x_1 - 217x_2 - 264x_3 + 9103x_1x_2 + 8996x_1x_3 - 279x_2x_3)x_1x_2 + \\ & (-5284x_1 - 176x_2 - 331x_3 + 8996x_1x_2 + 9508x_1x_3 - 249x_2x_3)x_1x_3 + \\ & (182x_1 - 27x_2 - 27x_3 - 279x_1x_2 - 249x_1x_3 + 250x_2x_3)x_2x_3. \end{aligned} \tag{7.10}$$

Moreover, the value of $d(\boldsymbol{x},\boldsymbol{\xi})$ after substituting all obtained points $\boldsymbol{x_r}$, $r = 1, 2, \cdots, 6$ that are represented in Table 7.4 in presented sensitivity function is $d(\boldsymbol{x_r},\boldsymbol{\xi}) = 6$.

As can be seen the value of the variance function is the same as the number of model parameters, which occurs at all obtained support points and the restricted region for such

support points is expressed in Figure 7.3. Moreover, other points from the continuous region which are positioned in between the above points and they are shown to satisfy the equivalence theorem with $d(\boldsymbol{x}, \xi) < p$. For example, the values of the variance function for the points (0.10, 0.815, 0.085), (0.40, 0.287, 0.32), (0.40, 0.05, 0.55), and (0.17, 0.09, 0.74) are less than the number of the model parameters, which is found to be 4.92, 3.69, 5.37, and 5.28 respectively. To check further the optimality of the

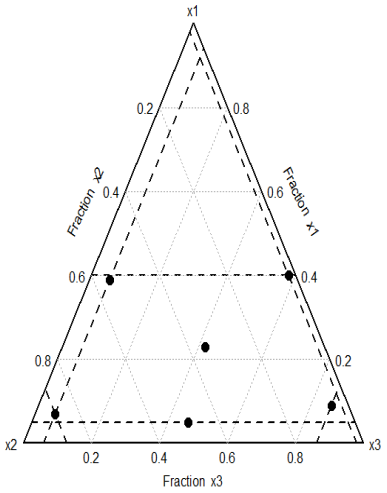

FIGURE 7.3: The Optimum Set of $\boldsymbol{x_r}$ and w_r for the Second Degree Model with Additional Constraints

6-point design, we assume that the number of support points is greater than the number of model parameters, in the sense that, we suppose there are 7 support points $\boldsymbol{x_r}$, $r = 1, 2, \cdots, 7$. The resulting optimum treatment combination for such issue following the same methodology and applying the above algorithm for obtaining the optimal points is given in Table 7.5. As we can see the weights for all resulting points are a little

TABLE 7.5: The Optimum Values of $\boldsymbol{x_r}$ and w_r with More Points

The Component Proportions	The Weights
$\boldsymbol{x_1}$ = (0.40,0.54,0.06)	$w_1 = 0.11$
$\boldsymbol{x_2}$ = (0.05,0.16,0.79)	$w_2 = 0.10$
$\boldsymbol{x_3}$ = (0.06,0.25,0.69)	$w_3 = 0.14$
$\boldsymbol{x_4}$ = (0.11,0.32,0.57)	$w_4 = 0.16$
$\boldsymbol{x_5}$ = (0.12,0.20,0.68)	$w_5 = 0.17$
$\boldsymbol{x_6}$ = (0.05,0.89,0.06)	$w_6 = 0.16$
$\boldsymbol{x_7}$ = (0.10,0.06,0.84)	$w_7 = 0.16$

different. However, the value for the sensitivity function $d(\boldsymbol{x}, \boldsymbol{\xi})$ that occurs at all such

points is 6, which is the number of the model parameters. Here, the formulation for this function can be expressed as

$$\begin{aligned} d(\boldsymbol{x},\boldsymbol{\xi}) \;=\; & (11097x_1 + 322x_2 + 476x_3 - 19456x_1x_2 - 19538x_1x_3 + 1039x_2x_3)x_1 + \\ & (322x_1 + 23x_2 + 16x_3 - 601x_1x_2 - 571x_1x_3 - 7x_2x_3)x_2 + \\ & (476x_1 + 16x_2 + 34x_3 - 832x_1x_2 - 879x_1x_3 - 4x_2x_3)x_3 + \\ & (-19456x_1 - 601x_2 - 832x_3 + 34353x_1x_2 + 34227x_1x_3 - 1798x_2x_3)x_1x_2 + \\ & (-19538x_1 - 571x_2 - 879x_3 + 34227x_1x_2 + 34625x_1x_3 - 1730x_2x_3)x_1x_3 + \\ & (1039x_1 - 7x_2 - 4x_3 - 1798x_1x_2 - 1730x_1x_3 + 476x_2x_3)x_2x_3. \end{aligned} \tag{7.11}$$

Moreover, the shape for the region defined by such points is shown in Figure 7.4. Under

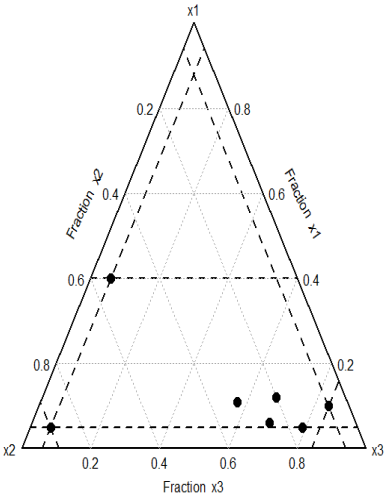

FIGURE 7.4: The Optimum Set of $\boldsymbol{x_r}$ and w_r for The Second degree Linear Model with More Support Points

this present case we found that the value of the optimality criterion under the design with equal weights for 6 points is better than for the design that having different weights for all 7 support points.

It is worth highlighting also the whole simplex, when the proportion components are subject only to the natural constraints, which is the second scenario of interest. Following the same method, the continuous optimum design in this case is presented in Table 7.6. The equivalence theorem for D-optimum design gives

TABLE 7.6: The Optimum Values of $\boldsymbol{x_r}$ and w_r for The Second Degree with Natural Constraints

The Component Proportions	The Weights
$\boldsymbol{x_1}$= (1,0,0)	w_1= 0.166667
$\boldsymbol{x_2}$= (0,1,0)	w_2= 0.166667
$\boldsymbol{x_3}$= (0,0,1)	w_3= 0.166667
$\boldsymbol{x_4}$= (0.5,0,0.5)	w_4= 0.166667
$\boldsymbol{x_5}$= (0.5,0.5,0)	w_5= 0.166667
$\boldsymbol{x_6}$= (0,0.5,0.5)	w_6= 0.166667

$$\begin{aligned} d(\boldsymbol{x},\boldsymbol{\xi}) &= (6x_1 - 12x_1x_2 - 12x_1x_3)x_1 + (6x_2 - 12x_1x_2 - 12x_2x_3)x_2 + \\ &\quad (6x_3 - 12x_1x_3 - 12x_2x_3)x_3 + (-12x_1 - 12x_2 + 144x_1x_2 + 24x_1x_3 + 24x_2x_3)x_1x_2 + \\ &\quad (-12x_1 - 12x_3 + 24x_1x_2 + 144x_1x_3 + 24x_2x_3)x_1x_3 + \\ &\quad (-12x_2 - 12x_3 + 24x_1x_2 + 24x_1x_3 + 144x_2x_3)x_2x_3, \end{aligned} \tag{7.12}$$

with $d(\boldsymbol{x},\boldsymbol{\xi}) = 6$, for all $\boldsymbol{x_r}$, $r = 1, 2, \cdots, 6$. The resulting region of interest is shown in Figure 7.5. Again, besides the obtained points from the search algorithm, other points from the continuous design region, such as (1/3, 2/3, 0), (1/3, 0, 2/3), and (0, 0.09, 0.91) also satisfy the condition of the sensitivity function that is $d(\boldsymbol{x_i},\boldsymbol{\xi}) < p$ under these points.

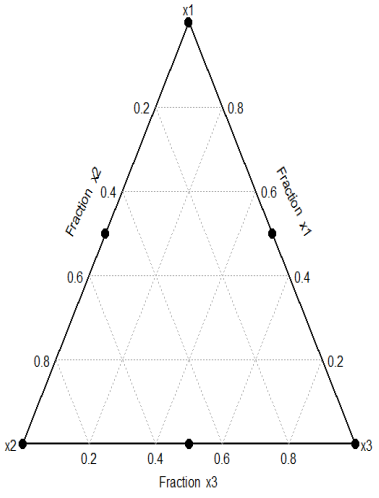

FIGURE 7.5: The Optimum Set of $\boldsymbol{x_r}$ and w_r with Natural Constraints

7.2.2 Continuous Design for the Special Cubic Models

Following the same strategy that we have discussed in the previous subsection for the second degree linear model for the continuous optimization problem, we now consider the special cubic polynomial model given in (2.13). As we explained before and under this model we require at least 7 support points. The optimum values for weights and proportion components are represented in Table 7.7. Here and under this specific situ-

TABLE 7.7: The Optimum Values of $\boldsymbol{x_r}$ and w_r for The Third Degree Linear Model

The Component Proportions	The Weights
$\boldsymbol{x_1} = (0.36,0.02,0.62)$	$w_1 = 0.1429$
$\boldsymbol{x_2} = (0.23,0.41,0.36)$	$w_2 = 0.1429$
$\boldsymbol{x_3} = (0.11,0.04,0.0.85)$	$w_3 = 0.1429$
$\boldsymbol{x_4} = (0.05,0.46,0.49)$	$w_4 = 0.1429$
$\boldsymbol{x_5} = (0.40,0.07,0.53)$	$w_5 = 0.1429$
$\boldsymbol{x_6} = (0.40,0.25,0.35)$	$w_6 = 0.1429$
$\boldsymbol{x_7} = (0.07,0.87,0.06)$	$w_7 = 0.1429$

ation the value of the variance function for all $\boldsymbol{x_r}$, $r = 1, 2, \cdots, 7$ is 7, which is indeed the same number of the model parameters. From the equivalence theorem, the obtained design above under resulted design points from the algorithmic approach is a continuous optimum design with respect to the standardized variance function, where the variance function for this case can be formulated as

$$\begin{aligned} d(\boldsymbol{x}, \boldsymbol{\xi}) \quad = \quad & (3984x_1 + 182x_2 + 190x_3 - 7166x_1x_2 - 6827x_1x_3 - 171x_2x_3 + 2559x_1x_2x_3)x_1 + \\ & (182x_1 + 29x_2 + 13x_3 - 398x_1x_2 - 327x_1x_3 - 81x_2x_3 + 380x_1x_2x_3)x_2 + \\ & (190x_1 + 13x_2 + 32x_3 - 356x_1x_2 - 406x_1x_3 - 84x_2x_3 + 394x_1x_2x_3)x_3 + \\ & (-7166x_1 - 398x_2 - 356x_3 + 13355x_1x_2 + 12354x_1x_3 + 574x_2x_3 - 6496x_1x_2x_3)x_1x_2 + \\ & (-6827x_1 - 327x_2 - 406x_3 + 12354x_1x_2 + 12130x_1x_3 + 567x_2x_3 - 6027x_1x_2x_3)x_1x_3 + \\ & (-171x_1 - 81x_2 - 84x_3 + 574x_1x_2 + 567x_1x_3 + 606x_2x_3 - 2513x_1x_2x_3)x_2x_3 + \\ & (2559x_1 + 380x_2 + 394x_3 - 6496x_1x_2 - 6027x_1x_3 - 2513x_2x_3 + 19660x_1x_2x_3)x_1x_2x_3. \end{aligned} \tag{7.13}$$

Moreover, some points from the continuous region under consideration also satisfy the condition that $d(\boldsymbol{x}, \boldsymbol{\xi}) < 7$. In the sense that, the value of the variance function the points (0.40, 0.52, 0.08) and (0.18, 0.09, 0.73) and (0.06, 0.25, 0.69) from χ are respectively 6.09, 6.44, 4.60. Moreover, the feasible constrained region for the obtained points under the algorithm is shown in Figure 7.6.

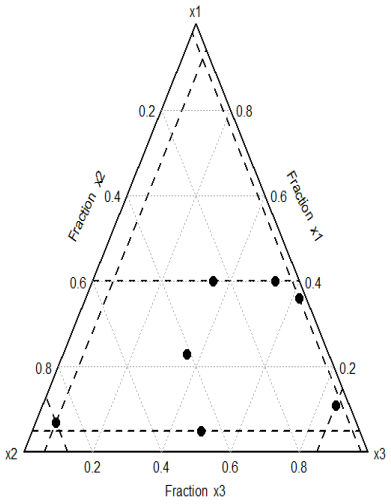

FIGURE 7.6: The Optimum Set of $\boldsymbol{x_r}$ and w_r with Additional Constraints

Furthermore, The optimum continuous design when we suggest more than 7 support points for the same restricted area is presented in Table 7.8. As we can see from this

TABLE 7.8: The Optimum Values of $\boldsymbol{x_r}$ and w_r with More than 7 Points

The Component Proportions	The Weights
$\boldsymbol{x_1} = (0.05,0.51,0.44)$	$w_1 = 0.130$
$\boldsymbol{x_2} = (0.40,0.09,0.51)$	$w_2 = 0.132$
$\boldsymbol{x_3} = (0.37,0.24,0.0.39)$	$w_3 = 0.110$
$\boldsymbol{x_4} = (0.22,0.54,0.24)$	$w_4 = 0.124$
$\boldsymbol{x_5} = (0.23,0.02,0.75)$	$w_5 = 0.121$
$\boldsymbol{x_6} = (0.07,0.07,0.86)$	$w_6 = 0.129$
$\boldsymbol{x_7} = (0.09,0.85,0.06)$	$w_7 = 0.134$
$\boldsymbol{x_8} = (0.05,0.40,0.55)$	$w_8 = 0.120$

table, the weights for obtained points are a little different, while, the value for the standardized variance function $d(\boldsymbol{x}, \boldsymbol{\xi})$ that occurs at achieved points is $p = 7$, which is the number of the parameters in the model. The formulation under this issue can be

represented as

$$
\begin{aligned}
d(\boldsymbol{x},\boldsymbol{\xi}) = (3999x_1+215x_2+272x_3-7502x_1x_2-7096x_1x_3-585x_2x_3+5413x_1x_2x_3)x_1+\\
(215x_1+33x_2+19x_3-475x_1x_2-396x_1x_3-109x_2x_3+573x_1x_2x_3)x_2+\\
(272x_1+19x_2+41x_3-525x_1x_2-565x_1x_3-115x_2x_3+646x_1x_2x_3)x_3+\\
(-7502x_1-475x_2-525x_3+14565x_1x_2+13393x_1x_3+1388x_2x_3-12352x_1x_2x_3)x_1x_2+\\
(-7096x_1-396x_2-565x_3+13393x_1x_2+12997x_1x_3+1318x_2x_3-11190x_1x_2x_3)x_1x_3+\\
(-585x_1-109x_2-115x_3+1388x_1x_2+1318x_1x_3+725x_2x_3-3499*x_1x_2x_3)x_2x_3+\\
(5413x_1+573x_2+646x_3-12352x_1x_2-11190x_1x_3-3499x_2x_3+28723x_1x_2x_3)x_1x_2x_3,
\end{aligned}
\tag{7.14}
$$

and the feasible region of such design is given in Figure 7.7. Although the design with

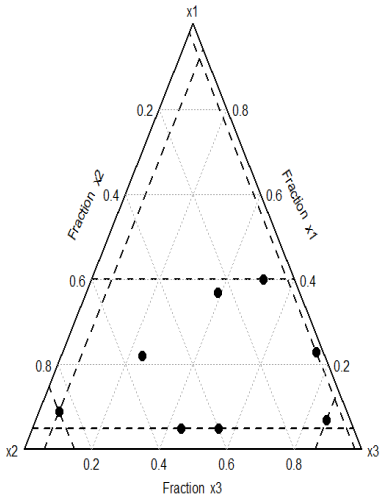

FIGURE 7.7: The Optimum Set of $\boldsymbol{x_r}$ and w_r with More Points

more support points satisfies the sensitivity function, such design do not make any improvement to the criterion under consideration in comparison to the design with 7 support points.

By applying the same methodology, with considering the same special cubic polynomial model for getting the D-optimum continuous design for the simplex shaped region, the points obtained are presented in Table 7.9. The obtained region for such points is presented in Figure 7.8. The value of the function $d(\boldsymbol{x},\boldsymbol{\xi})$ for such continuous optimum design resulted from the algorithmic is the same as the number of model parameters which satisfy the equivalence theorem in continuous optimization mechanism. Furthermore, other points from $\boldsymbol{\chi}$ satisfy also such condition for the equivalence theorem, which

TABLE 7.9: The Optimum Values of $\boldsymbol{x_r}$ and w_r with Natural Constraints

The Component Proportions	The Weights
$\boldsymbol{x_1} = (1,0,0)$	$w_1 = 0.14286$
$\boldsymbol{x_2} = (0,1,0)$	$w_2 = 0.14286$
$\boldsymbol{x_3} = (0,0,1)$	$w_3 = 0.14286$
$\boldsymbol{x_4} = (0.5,0,0.5)$	$w_4 = 0.14286$
$\boldsymbol{x_5} = (0.5,0.5,0)$	$w_5 = 0.14286$
$\boldsymbol{x_6} = (0,0.5,0.5)$	$w_6 = 0.14286$
$\boldsymbol{x_7} = (0.333,0.333,0.333)$	$w_7 = 0.14286$

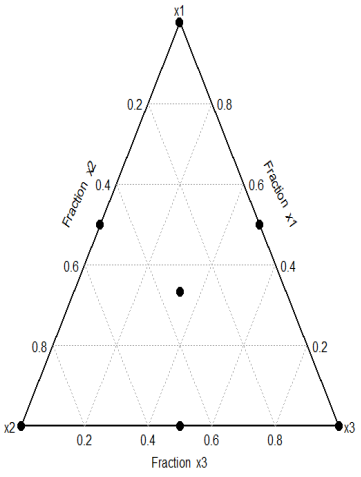

FIGURE 7.8: The Optimum Set of $\boldsymbol{x_r}$ and w_r with Natural Constraints

is $d(\boldsymbol{x}, \boldsymbol{\xi}) < p$. Such value that obtained under these points either from the algorithmic process or from $\boldsymbol{\chi}$ is

$$\begin{aligned} d(\boldsymbol{x_r}, \boldsymbol{\xi}) &= 7, r = 1, 2, \cdots, 7 \\ d(\boldsymbol{x_8} = (0, 1/3, 2/3), \boldsymbol{\xi}) &= 6 \\ d(\boldsymbol{x_9} = (1/3, 0, 2/3), \boldsymbol{\xi}) &= 6 \\ d(\boldsymbol{x_{10}} = (0.25, 0, 0.75), \boldsymbol{\xi}) &= 5 \\ d(\boldsymbol{x_{11}} = (0.25, 0.75, 0), \boldsymbol{\xi}) &= 5 \\ d(\boldsymbol{x_{12}} = (0, 0.25, 0.75), \boldsymbol{\xi}) &= 5 \\ d(\boldsymbol{x_{13}} = (1/3, 2/3, 0), \boldsymbol{\xi}) &= 6 \end{aligned} \tag{7.15}$$

Moreover, the formulation of the standardized variance function given as

$$\begin{aligned} d(\boldsymbol{x},\xi) = (7x_1 - 14x_1x_2 - 14x_1x_3 + 21x_1x_2x_3)x_1 + \\ (7x_2 - 14x_1x_2 - 14x_2x_3 + 21x_1x_2x_3)x_2 + \\ (7x_3 - 14x_1x_3 - 14x_2x_3 + 21x_1x_2x_3)x_3 + \\ (-14x_1 - 14x_2 + 168x_1x_2 + 28x_1x_3 + 28x_2x_3 - 420x_1x_2x_3)x_1x_2 + \\ (-14x_1 - 14x_3 + 28x_1x_2 + 168x_1x_3 + 28x_2x_3 - 420x_1x_2x_3)x_1x_3 + \\ (-14x_2 - 14x_3 + 28x_1x_2 + 28x_1x_3 + 168x_2x_3 - 420x_1x_2x_3)x_2x_3 + \\ (21x_1 + 21x_2 + 21x_3 - 420x_1x_2 - 420x_1x_3 - 420x_2x_3 + 8352x_1x_2x_3)x_1x_2x_3. \end{aligned} \quad (7.16)$$

7.2.3 Continuous Designs for Non-Linear Models

In this section, we will highlight the technique for finding continuous optimum designs for models with nonlinear parameters introduced in (3.8) and (3.9). The numerical optimization procedure mentioned above will be the same as for the linear models. However, the information matrix for such models for continuous optimization problems depends on the unknown model parameters unlike for linear models. Therefore, before conducting the above process, again, we will use the estimated values for the parameter models we found them before to build the information matrix $\boldsymbol{F'F}$. Different information matrices are found for the different models, which is, the models have the same value of the exponent α, or the models have different values of the power α. If the assumed model is the first degree modified fractional polynomial model FP1n and by supposing 4 points which corresponds to the number of the model parameters, the obtained points from conducting the above method are given in Table 7.10. To show this design is a

TABLE 7.10: The Optimum Values of $\boldsymbol{x_r}$ and w_r for FP1n Having α

The Component Proportions	The Weights
$\boldsymbol{x_1}$ = (0.05,0.15,0.80)	$w_1 = 0.25$
$\boldsymbol{x_2}$ = (0.06,0.88,0.06)	$w_2 = 0.25$
$\boldsymbol{x_3}$ = (0.12,0.02,0.86)	$w_3 = 0.25$
$\boldsymbol{x_4}$ = (0.40,0.49,0.11)	$w_4 = 0.25$

continuous optimum design, we again evaluate the $d(\boldsymbol{x},\boldsymbol{\xi})$ and substitute all obtained points in this function. Hence, we got

$$\begin{aligned} d(\boldsymbol{x_1}, \xi) &= 4 \\ d(\boldsymbol{x_2}, \xi) &= 4 \\ d(\boldsymbol{x_3}, \xi) &= 4 \\ d(\boldsymbol{x_4}, \xi) &= 4 \end{aligned} \tag{7.17}$$

The numerical expression for $d(\boldsymbol{x}, \boldsymbol{\xi})$ is

$$\begin{aligned} d(\boldsymbol{x}, \boldsymbol{\xi}) = {} & 71.47 - 40.91(x_2/x_1)^{0.5} - 28.44(x_3/x_1)^{0.5} + 11.54(x_2/x_1)^{0.5} log(x_2/x_1) + \\ & 7.80(x_3/x_1)^{0.5} log(x_3/x_1) + (-40.91 + 24.87(x_2/x_1)^{0.5} + 16.80(x_3/x_1)^{0.5} - \\ & 7.20(x_2/x_1)^{0.5} log(x_2/x_1) - 4.86(x_3/x_1)^{0.5} log(x_3/x_1))(x_2/x_1)^{0.5} + (-28.44 + \\ & 16.80(x_2/x_1)^{0.5} + 12.04(x_3/x_1)^{0.5} - 4.94(x_2/x_1)^{0.5} log(x_2/x_1) - 3.34(x_3/x_1)^{0.5} \\ & log(x_3/x_1))(x_3/x_1)^{0.5} + (-0.42 + 0.26(x_2/x_1)^{0.5} + 0.18(x_3/x_1)^{0.5} - 0.08(x_2/x_1)^{0.5} \\ & log(x_2/x_1) - 0.05(x_3/x_1)^{0.5} log(x_3/x_1))(-27.3(x_2/x_1)^{0.5} log(x_2/x_1) - 18.44(x_3/x_1)^{0.5} \\ & log(x_3/x_1)). \end{aligned} \tag{7.18}$$

We can see from the above sensitivity function $d(\boldsymbol{x}, \boldsymbol{\xi})$ that it attains the value 4 in all support points which is the same number of p model parameters. Moreover, for some points from continuous region, the evaluation variance function achieve the value less than p. For example,

$$\begin{aligned} d(\boldsymbol{x_5} = (0.09, 0.85, 0.06), \xi) &= 2.01 \\ d(\boldsymbol{x_6} = (0.06, 0.86, 0.08), \xi) &= 3.40 \\ d(\boldsymbol{x_7} = (0.11, 0.03, 0.86), \xi) &= 3.17 \\ d(\boldsymbol{x_8} = (0.40, 0.28, 0.32), \xi) &= 3.77 \end{aligned} \tag{7.19}$$

Finally the plot of the constrained region for such design points under simplex search algorithm is shown in Figure 7.9. To check further the optimality of the 4-support points, we suggest that there are 5 support points $\boldsymbol{x_r}$, $r = 1, 2, \cdots, 5$. The optimum set of $\boldsymbol{x_r}$ and w_r are given in Table 7.11. As we can see the weights for the resulting points

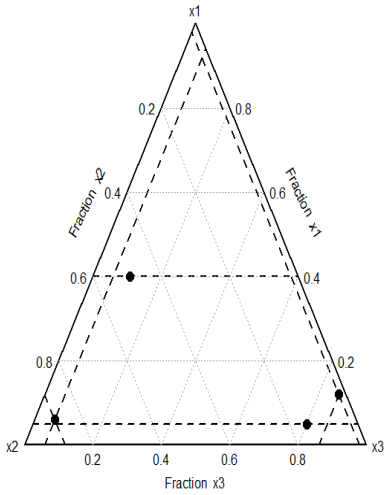

FIGURE 7.9: The Optimum Set of $\boldsymbol{x_r}$ and w_r with FP1n Having α

TABLE 7.11: The Optimum Values of $\boldsymbol{x_r}$ and w_r for FP1n Having α

The Component Proportions	The Weights
$\boldsymbol{x_1} = (0.05,0.09,0.86)$	$w_1 = 0.20$
$\boldsymbol{x_2} = (0.15,0.51,0.34)$	$w_2 = 0.25$
$\boldsymbol{x_3} = (0.40,0.02,0.58)$	$w_3 = 0.20$
$\boldsymbol{x_4} = (0.05,0.89,0.06)$	$w_4 = 0.25$
$\boldsymbol{x_5} = (0.40,0.54,0.06)$	$w_4 = 0.10$

are different. Moreover, the value of the variance function at these points is the same as the number of model parameters. Such variance function can be represented as

$$\begin{aligned} d(\boldsymbol{x},\boldsymbol{\xi}) \quad = \quad & 27.36 - 16.53(x_2/x_1)^{0.5} - 11.31(x_3/x_1)^{0.5} + 4.65(x_2/x_1)^{0.5}log(x_2/x_1) + \\ & 3.14(x_3/x_1)^{0.5}log(x_3/x_1) + (-16.53 + 11.78(x_2/x_1)^{0.5} + 7.29(x_3/x_1)^{0.5} - \\ & 3.44(x_2/x_1)^{0.5}log(x_2/x_1) - 2.32(x_3/x_1)^{0.5}log(x_3/x_1))(x_2/x_1)^{0.5} + \\ & (-11.31 + 7.29(x_2/x_1)^{0.5} + 5.20(x_3/x_1)^{0.5} - 2.18(x_2/x_1)^{0.5}log(x_2/x_1) - \\ & 1.47(x_3/x_1)^{0.5}log(x_3/x_1))(x_3/x_1)^{0.5} + (-0.17 + 0.13(x_2/x_1)^{0.5} + \\ & 0.08(x_3/x_1)^{0.5} - 0.04(x_2/x_1)^{0.5}log(x_2/x_1) - 0.03(x_3/x_1)^{0.5}log(x_3/x_1)) \\ & (-27.3(x_2/x_1)^{0.5}log(x_2/x_1) - 18.44(x_3/x_1)^{0.5}log(x_3/x_1)). \end{aligned} \tag{7.20}$$

In addition, such optimum design are represented in Figure 7.10. Again, here the obtained optimal design when the number of support points are the same as the number of model parameters is better than the other obtained design with respect to the criteria

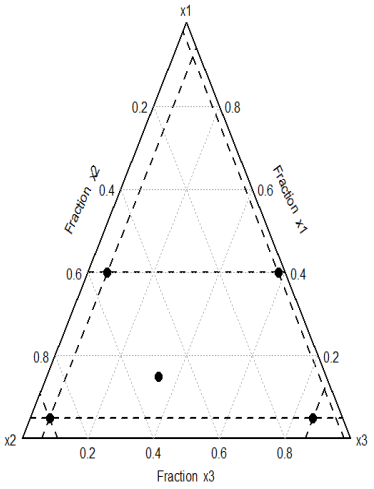

FIGURE 7.10: The Optimum Set of $\boldsymbol{x_r}$ and w_r with FP1n Having α

under consideration.

Now turning to the second situation, when the model FP1n has different values of α. Again by following the same procedure as in the first case, the optimum continuous design that satisfy the general equivalence theorem with its constrained area given in Table 7.12. Additionally points can be also considered for continuous optimization

TABLE 7.12: The Optimum Values of $\boldsymbol{x_r}$ and w_r for FP1n Having α

The Component Proportions	The Weights
$\boldsymbol{x_1} = (0.05,0.89,0.06)$	$w_1 = 0.22 \approx 0.2$
$\boldsymbol{x_2} = (0.40,0.02,0.58)$	$w_2 = 0.18 \approx 0.2$
$\boldsymbol{x_3} = (0.40,0.54,0.06)$	$w_3 = 0.18 \approx 0.2$
$\boldsymbol{x_4} = (0.06,0.88,0.06)$	$w_4 = 0.24 \approx 0.2$
$\boldsymbol{x_5} = (0.07,0.21,0.72)$	$w_5 = 0.18 \approx 0.2$

purposes. For example, (0.225, 0.715, 0.06), (0.26, 0.02, 0.72), and (0.12, 0.02, 0.86). All these points satisfy the equivalence theorem that is the obtained value of the variance function is $< p$. Furthermore, the numerical expression for $d(\boldsymbol{x}, \boldsymbol{\xi})$ under this case given

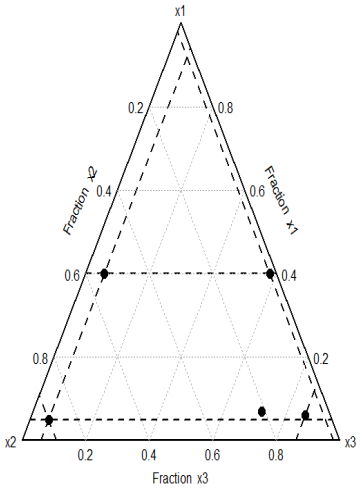

FIGURE 7.11: The Constrained Region for The Optimum Set of $\boldsymbol{x_r}$ and w_r with FP1n Having different α

as

$$
\begin{aligned}
d(\boldsymbol{x},\boldsymbol{\xi}) \;=\; & 122070.4 - 47584.9(x_2/x_1)^{-0.1} - 76386.2(x_3/x_1)^{0.1} - \\
& 4691.4log(x_2/x_1)(x_2/x_1)^{-0.1} + 7192.3(x_3/x_1)^{0.1}log(x_3/x_1) + \\
& (-47584.9 + 20043.8(x_2/x_1)^{-0.1} + 28299.1(x_3/x_1)^{0.1} + \\
& 1968.9log(x_2/x_1)(x_2/x_1)^{-0.1} - 2665.9(x_3/x_1)^{0.1}log(x_3/x_1))(x_2/x_1)^{-0.1} + \\
& +(-76386.2 + 28299.1(x_2/x_1)^{-0.1} + 49261.3(x_3/x_1)^{0.1} + 2797.2log(x_2/x_1) \\
& (x_2/x_1)^{-0.1} - 4637.3(x_3/x_1)^{0.1}log(x_3/x_1))(x_3/x_1)^{0.1} + (-4691.4 + 1968.9/(x_2/x_1)^{0.1} + \\
& 2797(x_3/x_1)^{0.1} + 193.7log(x_2/x_1)/(x_2/x_1)^{0.1} - 263.4(x_3/x_1)^{0.1}log(x_3/x_1)) \\
& log(x_2/x_1)/(x_2/x_1)^{0.1} - (-7192.3 + 2665.9/(x_2/x_1)^{0.1} + 4637.3(x_3/x_1)^{0.1} + 263.4 \\
& log(x_2/x_1)/(x_2/x_1)^{0.1} - 436.9(x_3/x_1)^{0.1}log(x_3/x1))(x_3/x_1)^{0.1}log(x_3/x_1).
\end{aligned}
\tag{7.21}
$$

Moreover, the optimum set of $\boldsymbol{x_r}$ and w_r ,that satisfy that $d(\boldsymbol{x},\boldsymbol{\xi}) = 5$, when we suppose that the number of support points are grater than the number of model parameters for the constrained area are given in Table 7.13. Furthermore, the feasible region for this design that does not make any improvement to the optimality criteria in comparison to the evaluated criteria under design with 5 support points can be represented in Figure 7.12.

TABLE 7.13: The Optimum Values of $\boldsymbol{x_r}$ and w_r for FP1n Having α

The Component Proportions	The Weights
$\boldsymbol{x_1} = (0.08,0.07,0.85)$	$w_1 = 0.18$
$\boldsymbol{x_2} = (0.40,0.02,0.58)$	$w_2 = 0.19$
$\boldsymbol{x_3} = (0.06,0.34,0.60)$	$w_3 = 0.10$
$\boldsymbol{x_4} = (0.40,0.54,0.06)$	$w_4 = 0.19$
$\boldsymbol{x_5} = (0.05,0.49,0.46)$	$w_5 = 0.15$
$\boldsymbol{x_6} = (0.05,0.89,0.06)$	$w_6 = 0.19$

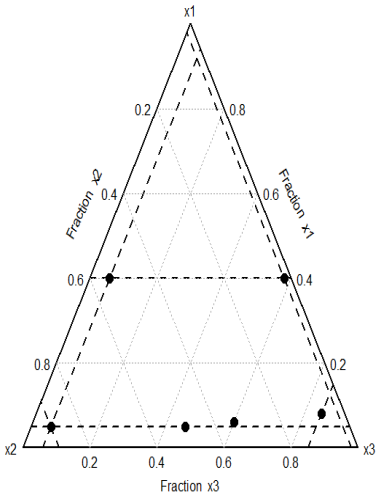

FIGURE 7.12: The Optimum Set of $\boldsymbol{x_r}$ and w_r with FP1n Having α

Next, our interesting is to find continuous optimal designs by searching the points over the design space $\boldsymbol{\chi}$ using the above process for second degree modified fractional polynomial models FP2n. Table 7.14 and Table 7.15 respectively show the results of the optimum values of the set $\{\boldsymbol{x_r}, w_r\}$ for the different cases related to the parameter α and using the estimated values for the model parameters presented in Table 3.5 for constructing the information matrices under this model. Moreover, the resulting shape of these points defining the design constrained area for both cases can be presented successively in Figure 7.13 and Figure 7.14.

TABLE 7.14: The Optimum Values of $\boldsymbol{x_r}$ and w_r for FP2n Having α

The Component Proportions	The Weights
$\boldsymbol{x_1}$ = (0.06,0.60,0.34)	$w_1 = 0.1429$
$\boldsymbol{x_2}$ = (0.28,0.26,0.46)	$w_2 = 0.1429$
$\boldsymbol{x_3}$ = (0.11,0.04,0.85)	$w_3 = 0.1429$
$\boldsymbol{x_4}$ = (0.31,0.19,0.50)	$w_4 = 0.1429$
$\boldsymbol{x_5}$ = (0.17,0.35,0.48)	$w_5 = 0.1429$
$\boldsymbol{x_6}$ = (0.38,0.04,0.58)	$w_6 = 0.1429$
$\boldsymbol{x_7}$ = (0.05,0.88,0.07)	$w_7 = 0.1429$

TABLE 7.15: The Optimum Values of $\boldsymbol{x_r}$ and w_r for FP2n Having different α

The Component Proportions	The Weights
$\boldsymbol{x_1}$ = (0.11,0.49,0.40)	$w_1 = 0.1250$
$\boldsymbol{x_2}$ = (0.19,0.20,0.61)	$w_2 = 0.1250$
$\boldsymbol{x_3}$ = (0.06,0.10,0.84)	$w_3 = 0.1250$
$\boldsymbol{x_4}$ = (0.06,0.87,0.07)	$w_4 = 0.1250$
$\boldsymbol{x_5}$ = (0.36,0.29,0.35)	$w_5 = 0.1250$
$\boldsymbol{x_6}$ = (0.30,0.03,0.67)	$w_6 = 0.1250$
$\boldsymbol{x_7}$ = (0.20,0.57,0.23)	$w_7 = 0.1250$
$\boldsymbol{x_8}$ = (0.21,0.53,0.26)	$w_8 = 0.1250$

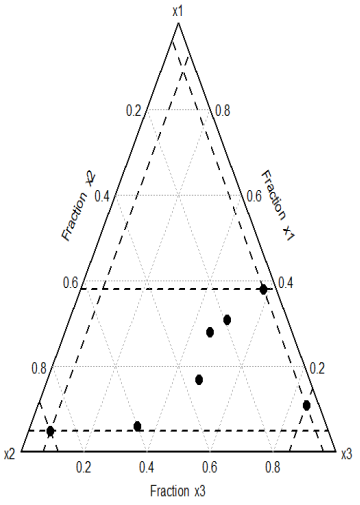

FIGURE 7.13: The Constrained Region for The Optimum Set of $\boldsymbol{x_r}$ and w_r with FP2n Having α

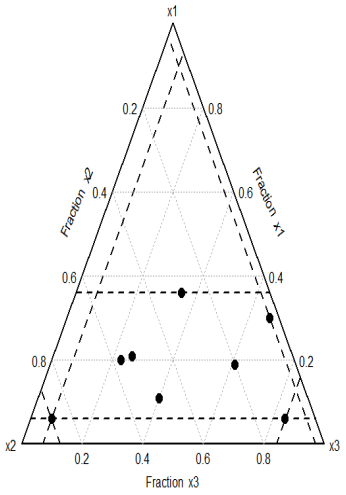

FIGURE 7.14: The Constrained Region for The Optimum Set of $\boldsymbol{x_r}$ and w_r with FP2n Having α_r

7.3 Exact Optimum Designs Based on Rounding Continuous Designs

In order to use a continuous design in a practical experiment for a specific sample size n, we need to use rounding to produce an exact design with integer replications. For example, from Table 7.2, the obtained weights for all proportion components are 0.333. If the sample size is a multiple of 3, then $n \cdot w_r$ for all $r = 1, 2, \cdots, q$ are integer. Here we have n = 30 from the previous examples, so $30 \cdot 0.333$ gives an integer number which is 10. As a result the obtained optimal design includes three distinct points each of which is replicated 10 times. From Table 5.2, the value of the determinant criterion for the optimal designs were measured to be 0.2171472, while for the rounded continuous designs found to be 0.2336816. In order to compare the rounded continuous designs, which will be symbolized as ξ_1, and the optimal design under the exchange algorithm, which will be defined as ξ_2, the relative D-efficiency will be used, which is defined as

$$D_{rel-eff} = \left\{ \frac{|M(\xi_1)|}{|M(\xi_2)|} \right\}^{\frac{1}{p}}, \tag{7.22}$$

where p is the number of parameters in the model. The differences between the D-efficiency and the relative D-efficiency lies in the resulting value, in the sense that in the latter the resulting values could be greater than one, in which case design ξ_1 is better than ξ_2 with respect to the determinant criterion. In our case the $D_{rel-eff} = 107.60\%$,

which means that the design ξ_1, the continuous optimal design, is better than design ξ_2 that represents the near optimal design obtained based on the exchange algorithm with respect to the D-optimality criterion.

With the second degree linear models, and following the same process as for the first degree linear model, the obtained optimal design includes 6 distinct points which are replicated 5 times each. Moreover, the evaluated D-optimality criterion for the rounded continuous optimal designs ξ_1 is 1.5885, while for the exact optimal design ξ_2 it is found to be 1.49712, and hence the $D_{rel-eff} = 106.10\%$. As a result, the design ξ_1 is considerably better in comparison to design ξ_2.

For both cases above, the designs constructed using a simplex search algorithm turn to be an exact design that can be implemented in practice. However, some designs cannot be involved directly in a practical manner unless we round them appropriately to exact designs as in the case of the obtained continuous optimum design for the special cubic models represented in Table 7.7. This case is consistent with the second scenario, in the sense that $n \cdot w_r$ for all $r = 1, 2, \cdots, 7$ is rational that is $4.3 \approx 4$. Thus, some points must have more replications than others. To determine which point should have more replicates, the properties of the model are considered, in the sense that, as such model has interaction terms, the points for estimating such terms are needed. In this case we will consider more replicates for the points which are associated with the face centroids, that is allocated to the points $\boldsymbol{x_4}$ and $\boldsymbol{x_6}$. Each points will be replicated 4 times which is an approximation to the value of 30×0.1428571, except the points $\boldsymbol{x_4}$ and $\boldsymbol{x_6}$ which have 5 replications for each. As a result, the $D_{rel-eff} = 109.53\%$, which shows again the design ξ_1 is better in comparison to design ξ_2, with respect to the criterion under consideration.

Similar results occur in the cases assuming the number of support points is greater than the number of model parameters in quadratic and special cubic models. The $D_{rel-eff}$ obtained after making a comparison between the design ξ_1 and ξ_2 are respectively 105.50% and 108.48%.

Moreover, with respect to the obtained designs under nonlinear models, again the same scenario that appeared for the special cubic model will occur for the proposed fractional polynomial models FP1n when this model has the same value of the exponent α and PF2n either when the model has the same or different values of the exponent. Here, $n \cdot w_r$ for all design points are rational i.e. $30 \times 0.25 = 7.5 \approx 8$ in the case of fitting FP1n, and $30 \times 0.1429 = 4.3 \approx 4$, $30 \times 0.1250 = 3.7 \approx 4$ when fitting FP2n considering the two situations of the exponents α respectively. Therefore, when the fitted model is FP1n, all obtained points are vertices augmented with the centroid in the experimental

region as presented in Figure 7.9. Thus, giving more replicates for some points than others is indeed arbitrary. We will replicate equally (8 times each) the vertices while for the centroid the given replication will be 6. Then, $D_{rel-eff} = 105.01\%$ which means that the obtained continuous design is better than the near optimal design for the region under consideration. In addition, the $D_{rel-eff}$ when the number of design points are grater than the number of the parameters in the this model is found to be 100.003%.

On the other hand, when the fitted model is FP2n, most of the obtained points are inside the feasible regions as presented in Figure 7.13 and Figure 7.14. Here, we will give the extra replication to the interior points rather than vertices as the fitted model has the cross-product terms. As a result, the $D_{rel-eff} = 102.5\%$ and $D_{rel-eff} = 106.5\%$ respectively. Again, the obtained optimal designs under continuous approach for the FP2n are better in comparison to the obtained near optimal design with respect to the criterion under consideration.

On the contrary, when the aim is to fit the FP1n model that has different values of α_r, this case will turn to be similar to the case when we want to fit the first and the second order polynomial models, i.e. all design points will have the same replications $n \cdot w_r = 6$. Then the $D_{rel-eff} = 105.10\%$, which means that the design ξ_1 is much better in comparison to design ξ_2. However, under the presented model with more support points, the tendency of the obtained design is different. In the sense that, the resulted design under simplex search algorithm will be consistent with the second scenario that is nw_r is rational. Thus, the design points will have different replication and the $D_{rel-eff}$ is found to be under this case 100.004%.

As can be seen from the obtained designs above, such designs in some cases are better in comparison to the obtained designs under the exchange algorithm. The question arises here what is the reason behind this performance. The investigation of the reason will be provided in the following section.

7.4 The Investigation of Performances of the Optimal Designs

To investigate the reason for the poor performances of the exact designs found by the exchange algorithm, we study the candidate set more closely. It is worthwhile to highlight the question that arises, namely, what are the important characteristics or performances that will happen when we augment the candidate set for the exchange algorithm by adding the obtained design points under the continuous approach, i.e. our new candidate set consists of the support points of the continuous optimal design and the points from candidate sets found previously in Chapter 4.

To proceed with the augmentation process, suppose that if the obtained design points under either first and second degree linear models and FP1n model having different values of the exponent α with their allocated weights given in Table 7.2, Table 7.4, and Table 7.12 respectively are consistent with the first scenario. That is, all design points under these models have the same replications, such as each of which is replicated 10 times, 5 times, and 6 times respectively. Then, these obtained points have been added to the generated candidate points given in Table 4.5 with additional interior points that give in total 66 design points taking into account the difference between the fitted models in terms of constructing the information matrix as mentioned before. Then, after augmentation, the exchange algorithm will run for different random starting designs and in the final step we investigate that the obtained designs make an improvement to the criterion under consideration in comparison to the near optimal designs under such algorithm found previously with respect to these assumed models. Meanwhile, such obtained designs are turn to be the exact one. For example, the design points obtained after the augmentation procedure for the first degree linear model are the same points presented in Table 7.2 with each of which replicated 10 times.

On the other hand, the performances of the obtained designs under unequally repeated points when fitting different higher order statistical models are different. For example, when the second degree linear model with the number of support points is greater than the number of the model parameters $p = 7$ assumed to be fitted. In this case, as we mentioned before, the resulting designs have unequal replications as the sample size n is not a multiple of 7. Then, we found from the obtained design under the optimization algorithm after doing the same augmentation process that the design points of such designs vary between the optimum continuous points given in Table 7.7 and the design points of the near optimum designs (OD) given in Table 5.3. Regarding the statistical optimum criterion under consideration, such obtained designs do not make an improvement to the criterion and as we mentioned before, these designs are considered as near optimal designs for the region under consideration.

Similar exploration have been found for the fitted third degree linear model and the first degree modified fractional polynomial models either have common or different values of the exponent α with assuming more support points for estimation purposes.

On the contrary, if the third degree linear model is assumed to be under consideration supposing that there are 7 optimal support points, which is the number of model parameters. Although the resulting optimum designs have unequal replication as in the case of the second degree linear model, the obtained designs under the same procedure make an improvement to the criterion for the region of interest. Again, the points in such obtained designs vary between continuous points and the points for the near optimal

designs (OD) given in Table 5.4.

Similar tendencies for the obtained design points under augmentation process occur with the first degree modified fractional polynomial having the same value of the exponent and second degree modified fractional models FP2n either have the same or different values of α. However, the obtained designs that are optimal under these models make a minor improvement to the criterion under consideration. All these results can be summarized in Table 7.16.

Generally, under augmentation process, some resulting designs that are optimal may improve the criterion under consideration significantly meanwhile they are not far from converging the exact designs. However, other designs although they satisfy the variance function properties, such designs contains a mixture of the resulted points form the optimization algorithms as well as make a minor improvement to the suggested criterion. Other than that, other designs make no improvement to the criteria under consideration.

Under these situations and for practical applications, the choice of designs depend on the characteristics and priorities that the experimenter is interested in. For example, usually the experimenter prefers to choose a design that has high efficiency in comparison to other designs. Thus in some situation it may be obvious to the experimenter which designs could be chosen to be applied in practice, in particular when the obtained designs are not far from corresponding exact designs.

Otherwise, the experimenter may also be interested in exploring the response on the whole region of interest. In this situation the designs obtained under the simplex search algorithm are recommended as in such designs the selected points are more widely spread across the feasible region, in particular if the assumed fitted model is the modified second degree fractional polynomial model.

Regarding the choice of the optimality criterion, if a criterion is selected that permits continuous designs, we recommend to construct the continuous optimal designs with respect to this criterion, and then to round the weights to construct an exact design for the chosen sample size. If, however, a criterion such as DP is of interest, where only exact designs are suitable, we need to construct the optimal designs via the methodology described in Chapter 5 and Chapter 6 with respect to the model under consideration.

TABLE 7.16: The Characteristics of The Optimal Designs

The Models	$D_{rel-eff}$	$D_{rel-eff}$ under Augmentation Process
FLM	107.60%	107.60%
SLM	106.10%| 105.50% (with more points)	106.10%| 99.998% (with more points)
SCM	109.53%|108.48% (with more points)	100.03% |99.8% (with more points)
FP1n with α	105.01% |100.003% (with more points)	100.001%|97.56% (with more points)
FP1n with different α_r	105.10%|100.004% (with more points)	105.10%|98.20% (with more points)
FP2n with α	102.5%	100.003%
FP2n with different α_r	106.5%	100.0002%

Chapter 8

Concluding the Experimental Work and Some Further Recommendations

8.1 Brief Review

The experimental work within the field of experiments with mixture for many points and aspects under consideration that has been covered through the present work is summarized in this section. Besides, some points and areas which can be considered for further research and we would like to pursue in the future are described.

In experiments with mixtures, the response depends on the proportions of the components and not on the total amount of the mixture. Thus, the proportions of the components making up any mixture must add to unity. Data from experiments with mixtures have traditionally been modelled using linear models, such as Scheffé polynomial models. More recently, models with nonlinear parameters have been applied to such data, for instance, general blending models. However, existing approaches are not flexible enough in all situations. Thus, we proposed a new class of nonlinear models for fitting the data from such experiments, by modified fractional polynomial models, as the fractional models cannot be fitted to the mixture data set straightforward due to the restriction in the proportion mixture components. Then, these models are compared with the recently proposed class of nonlinear GBM models and with several linear models. Based on these comparisons we demonstrated in our work and through different illustrative practical examples as given in Chapter 3 that our class of models is preferred and outperforms the existing approaches with respect to different criteria for comparisons.

Once a decision has been made about the model, a choice of suitable designs for the region of interest is needed so that the collection of the observations may possible.

There are different types of mixture designs according to the shape of the region under consideration, where the resulting shape is determined by the constraints that have been placed on the proportion components. Thus, the shape of the design region under mixture constraints can be divided into two categories, simplex shape when the design region results from natural constraints, and non-simplex shaped (irregular) region, on the other hand, when the component proportions are subject to further constraints.

Looking for a technique and methodology to find design points under the resulting restricted regions that can adequately cover and provide maximum spread over the region inside the simplex is needed. The XVERT algorithm is used to generate such points for the resulting restricted regions under different proportion components. Traditional methods for applying this algorithm have been shown with illustrative different examples in Chapter 4 with respect to the resulted constraints regions by 3- and 4-component mixtures. On the other hand, for the higher order constraints, using such traditional methods will become more complicated as the number of the treatment combinations becomes large with q. Thus, a statistical computing package that has mixture capability could be used to obtain all desirable candidate points for such cases. The process of such method are also provided supported by different examples. Then, we found that the candidate points produced by the XVERT algorithm include design points around the perimeter of the experimental region augmented by the centroid. Sometimes, it is necessary to find additional interior points for different purposes, such as checking the lack of fit of a model. For this, we augmented the resulting design points with the additional interior points by using a statistical package that can deal with this problem as it has been given in Chapter 4.

Under constrained regions by different proportion components, we found that the number of potential points that had to be evaluated becomes large quite rapidly with q, especially when $q \geq 4$. Thus, picking a subset or reducing the achievable points by selecting some points from candidate list to obtain a "good" designs of a reasonable size that will support the intended models in needed. The method based on design optimality is used for achieving this purpose.

Beside the standard optimality criteria, recently developed optimality, such as DP-, AP-, GDP-, and GAP- optimal criteria have been considered for obtaining optimal designs. All these criteria have been implemented in the standard exchange algorithm for constructing such designs. Different designs that are optimal under considering different statistical models are provided in Chapter 5 and Chapter 6. Based on the resulted optimal designs, we explored some features and performances of such designs and how efficient the designs are with respect to other designs under different weight allocation schemes and different values of the tuning parameter τ^2. Moreover, we showed through

different examples how the relationship between the primary and potential terms is changed under fitted nonlinear models having different values of the exponent α.

The final piece of our present research is devoted to checking whether the obtained designs are optimal designs or not. Continuous (approximate) designs are considered here, which is a common tool in the literature for the optimal design approach. Such designs are considered as a fundamental tool in the construction of the corresponding exact optimal designs by choosing the design points $\boldsymbol{x_r}$ from the design space that satisfy the optimization conditions with respect to some statistical optimality criterion. The simplex search algorithm is used for this purpose, and then we found different continuous optimum designs under different statistical models with respect to the criterion under consideration. The methodology with illustrative examples under different statistical models is displayed in Chapter 7.

8.2 Suggested Points for Future Work

For the research that has been done in the present work we considered different optimality criteria, such as determinant and trace based criteria for optimization problem. However, other statistical criteria, for instance the factorwise balance criteria (F-optimality) proposed by Ahmad (2017) could be considered and implemented in the algorithm for constructing different optimal designs under mixture components. The main concept of this criterion is making all variances of the regression coefficient estimates invariant to relabelling factor levels. Thus, it aims to minimize the sum of variation among such variances of the coefficient estimates.

Under the strategy for design optimality, several approaches have been discussed in the literature for searching optimal designs and making these designs robust by avoiding the dependence of designs on the unknown value of the model parameters. The most common approaches are standardized maximin, sequential and Bayesian approaches. Maximin D-optimum designs maximize $log|M(\xi, \beta)|$ for that value of the parameter β in which the determinant is a minimum. Many authors have considered and worked with the maximin design method for models with nonlinear parameters, such as (King and Wong, 2000). With respect to sequential designs, such designs are more important for nonlinear than for linear models as the designs in nonlinear models depend on unknown parameters. The procedure for obtaining such designs consists of several steps. The whole procedure including several examples are provided in (Atkinson et al., 2007). For the Bayesian approach, the optimum designs can be obtained by maximizing the expectation of the criterion over the assumed prior distribution, that is $E_\beta \phi(\boldsymbol{F'F}(\boldsymbol{\beta}, \xi)) = \int_\beta \phi\left(\boldsymbol{F'F}(\boldsymbol{\beta}, \xi)\right) \pi(\beta) d\beta$. More information for Bayesian optimum

designs can be found in a variety of resources, such as (Matthews and Allcock, 2004) and (Atkinson et al., 2007). Such approaches also can be considered for our further research in experiments involving mixtures.

Apart from the optimality methods, in the course of the present work we have worked with experiments involving mixtures excluding process variables. It is worthwhile considering the inclusion of process variables in mixture experiments. Process variables are factors in an experiment that do not form any proportion of the mixture but whose levels, when changed, could affect the mixture properties of the ingredients. For example, consider an experiment of making pizza with three ingredients, that is pizza sauce, flour, and cheese. Varying the proportions of ingredients will affect the taste of the pizza. In addition, varying the time or degree of baking the pizza in the oven will also affect the taste of the pizza. The time and the degree of baking are called process variables. Inclusion of process variables in experiments involving mixtures have been considered by different researchers, such as Scheffé (1963) and Czitrom (1988, 1989). The models under considering this case will be as combination models associated with the design of mixture-process experiment. Thus, such models will include terms representing component mixture properties and effects of changing the processing conditions. For example, when two process variables with two levels each and q mixture variables are include in a mixture, then the combination of the second degree linear models, for instance, that given in (2.11) with the following model in two process variables

$$\eta_p = \alpha_0 + \alpha_1 P_1 + \alpha_2 P_2 + \alpha_{12} P_1 P_2. \tag{8.1}$$

can be represented as

$$\eta(\boldsymbol{x}, P) = \sum_{r=1}^{q} \gamma_r^0 x_r + \sum_{r<s}^{q} \gamma_{rs}^0 x_r x_s + \sum_{l=1}^{2} [\sum_{r=1}^{q} \gamma_r^l x_r + \sum_{r<s}^{q} \gamma_{rs}^l x_r x_s] P_l + [\sum_{r=1}^{q} \gamma_r^{12} x_r + \sum_{r<s}^{q} \gamma_{rs}^{12} x_r x_s] P_1 P_2. \tag{8.2}$$

The combined model is obtained after rewriting the coefficient model β_r and β_{rs} as $\beta_r(P)$ and $\beta_{rs}(P)$, $r < s$, $r, s = 1, 2, \cdots, q$, to indicate that each β_r and β_{rs} is a function of the setting of $P = (P_1, P_2)$.

Similar combination technique will be followed to obtain the combination of modified fractional polynomial models that could be used to obtain any optimization and statistical references under consideration. Thus, the combination first and second degree modified fractional polynomial models with considering two process variables can be represented respectively as

$$\eta(\boldsymbol{x}, P) = \beta_0 \gamma_0^0 + \sum_{r=1}^{q-1} \gamma_r^0 Q_r^{\alpha_r} + \sum_{l=1}^{2} [\gamma_0^l + \sum_{r=1}^{q-1} \gamma_r^l Q_r^{\alpha_r}] P_l + [\gamma_0^{12} \beta_0 + \sum_{r=1}^{q-1} \gamma_r^{12} Q_r^{\alpha_r}] P_1 P_2. \tag{8.3}$$

and

$$\begin{aligned}\eta(\boldsymbol{x}, P) \quad = \quad & \beta_0\gamma_0^0 + \sum_{r=1}^{q-1}\gamma_r^0 Q_r^{\alpha_r} + \sum_{r=1}^{q-1}\gamma_{rr}^0 Q_{rr}^{2\alpha_r} + \sum_{r=1}^{q-2}\sum_{s=r+1}^{q-1}\gamma_{rs}^0 Q_r^{\alpha_r} Q_s^{\alpha_r} + \\ & \sum_{l=1}^{2}[\gamma_0^l + \sum_{r=1}^{q-1}\gamma_r^l Q_r^{\alpha_r} + \sum_{r=1}^{q-1}\gamma_{rr}^l Q_{rr}^{2\alpha_r} + \sum_{r=1}^{q-2}\sum_{s=r+1}^{q-1}\gamma_{rs}^l Q_r^{\alpha_r} Q_s^{\alpha_r}]P_l + \\ & [\gamma_0^{12}\beta_0 + \sum_{r=1}^{q-1}\gamma_{rr}^{12} Q_{rr}^{2\alpha} + \sum_{r=1}^{q-2}\sum_{s=r+1}^{q-1}\gamma_{rs}^{12} Q_r^{\alpha_r} Q_s^{\alpha_r}]P_1 P_2,\end{aligned} \tag{8.4}$$

where $Q_r = \frac{x_r}{x_q}$ and $Q_s = \frac{x_s}{x_q}$.

Furthermore, another possibility that can be taken into consideration for further research for modeling data from mixture experiments is that, when the assumption of normally distributed random errors are unlikely to be hold. Then, there is a wider class of statistical models that can be assumed to tackle this issue, and thus the methods for obtaining efficient optimum experimental designs under these classes are employed. An application of these models is generalized nonlinear models which is an extension form of nonlinear regression models. Such models allow non-normally distributed error structure. Many researchers have worked with this situation such as Wei (1997), Batchelor et al. (2007), and Biedermann and Woods (2011).

In the present work, the shape of experimental region is defined by single-component constraints. Other constraints can also be imposed to the proportion components which result in multi-component constraints. Such constraints are of the form

$$L_b \leq a_{b1}x_1 + a_{b2}x2 + \cdots + a_{bq}x_q \leq U_b, \tag{8.5}$$

where L_b and U_b are lower and upper bounds respectively for the bth two-sided constraint. Here some of a_{br} might be set as zero, where $r = 1, 2, \cdots, q$.

A special case of the these constraints, which is common in the experimental work, is Ratio constraints. To represent such constraints mathematically, consider, there are 5 component proportions present for conducting an experiment, then such constraints that have been placed on these proportions could be of the form

$$\frac{x3 + x4 + x5}{x1 + x2} \leq 1. \tag{8.6}$$

To handle the situations where the multi-component constraints occur, some algorithms that have been developed in the literature, such as CONSIM algorithm can be used, where this algorithm can be implemented by statistical computing package (Smith, 2005). Such constraints can be taken into consideration for further research in mixture settings especially when assuming modified fractional polynomial models to fit the region of interest.

The essential feature of modern experiments, in general, is the "randomization out" of the effects of factors outside the experimental structure. One of the methods that can be applied to control outside factors is allocate the responses to groups, called blocks, where each group or block has similar effects of the outside factors of the experiment. Blocking designs for experiments with mixtures under our proposed models with additional random effects to account for blocks also can be taken under consideration.

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