Cluster Adjacency Properties of Scattering Amplitudes in $\mathcal{N}=4$ Supersymmetric Yang-Mills Theory

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We conjecture a new set of analytic relations for scattering amplitudes in planar $\mathcal{N}=4$ super Yang-Mills theory. They generalize the Steinmann relations and are expressed in terms of the cluster algebras associated to $\mathrm{Gr}(4,n)$. In terms of the symbol, they dictate which letters can appear consecutively. We study heptagon amplitudes and integrals in detail and present symbols for previously unknown integrals at two and three loops which support our conjecture.

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Introduction.—Scattering amplitudes in perturbative quantum field theory have an intricate analytic structure. In certain cases the analytic structure becomes tractable enough to submit to a general mathematical description. The case of planar maximally supersymmetric Yang-Mills theory is particularly striking. Here there is evidence that the branch cut singularities of certain perturbative loop amplitudes are described by cluster algebras associated to the Grassmannians $\mathrm{Gr}(4,n)$ [1]. More precisely, the locations of branch point singularities are given by cluster $\mathcal{A}$ coordinates of the associated cluster algebra.

The $\mathcal{A}$ coordinates are polynomials in the Grassmannian Plücker coordinates $(ijkl)$ which are totally antisymmetric sl_4-invariant combinations of momentum twistors Z_i [2]. Such variables neatly describe the geometry of a polygonal lightlike Wilson loop dual to the planar scattering amplitude [3–5]. The sides of the polygon are given by the massless particle momenta p_i and the corners, denoted by x_i, are then related to lines in momentum twistor space $\mathbb{P}^3$ after picking a preferred bitwistor I, corresponding to the null cone at infinity,

$$p_i = x_{i+1} - x_i, \qquad x_i = \frac{Z_{i-1}\wedge Z_i}{(i-1iI)}. \tag{1}$$

Steinmann relations [6,7] are the requirement that a scattering amplitude does not have consecutive discontinuities in overlapping channels. In the context of amplitudes in planar $\mathcal{N}=4$ super yang-Mills theory, their importance was emphasized in Ref. [8] and they have been usefully employed to construct amplitudes in Refs. [9,10].

In order to see the appearance of Steinmann relations in massless amplitudes it is useful to define an appropriate infrared finite quantity [9]. In planar $\mathcal{N}=4$ this quantity is the *BDS-like subtracted amplitude* [11] which exists for n-point amplitudes with $n\geq 6$ and $n\neq 0$ mod 4. These amplitudes do not have consecutive branch cuts in overlapping three-particle or higher Mandelstam invariants. For example, a discontinuity around $s_{123}=0$ cannot itself have a discontinuity around $s_{234}=0$.

Adjacency rules from $\mathrm{Gr}(4,n)$ *clusters.*—Polylogarithms and symbols: The amplitudes that we discuss here are believed to be expressible in terms of polylogarithms, i.e., iterated integrals over a set of logarithmic singularities. A polylogarithm of weight k, denoted as $f^{(k)}$, satisfies

$$df^{(k)} = \sum_r f_r^{(k-1)} d\log\phi_r, \tag{2}$$

where ϕ_r (the *letters*) are algebraic functions of some set of coordinates. Functions of weight one are rational linear combinations of logarithms of the letters.

The symbol map, defined following Eq. (2), maps a weight-k function to a k-fold tensor,

$$\mathcal{S}[f^{(k)}] = \sum_r \mathcal{S}[f_r^{(k-1)}]\otimes\phi_r = \sum_{\vec{\phi}} c_{\vec{\phi}}\phi_{r_1}\otimes\cdots\otimes\phi_{r_k} \tag{3}$$

with rational coefficients $c_{\vec{\phi}}$. The symbol encodes both the differential and the branch-cut structure of $f^{(k)}$ and obeys the integrability conditions,

$$\sum_{\vec{\phi}} c_{\vec{\phi}}\phi_{r_1}\otimes\cdots\phi_{r_{i-1}}\otimes\phi_{r_{i+2}}\ldots\phi_{r_k}\frac{d\phi_{r_i}\wedge d\phi_{r_{i+1}}}{\phi_{r_i}\phi_{r_{i+1}}} = 0. \tag{4}$$

The symbol was used in Ref. [12] to deduce a beautifully simple formula for the two-loop hexagon remainder [13–15].

Terms of the form $[\phi_{r_1}\otimes\phi_{r_2}\otimes\cdots]$ in the symbol of a function of weight k indicate a logarithmic branch point where ϕ_{r_1} vanishes. The discontinuity across the associated branch cut is a function of weight $(k-1)$ whose symbol is the sum of the terms $[\phi_{r_2}\otimes\cdots]$. The Steinmann relations of Ref. [9] have implications on the initial pairs in the symbol, which we will refer to as *initial* Steinmann relations. Stronger, *extended* Steinmann conditions have been observed to hold on *all* neighboring pairs in the symbols of 6-point amplitudes and integrals [16,17].

Differential operators also act simply on the symbol,

$$d[\phi_1 \otimes \cdots \otimes \phi_k] = d\log\phi_k[\phi_1 \otimes \cdots \otimes \phi_{k-1}]. \quad (5)$$

This allows differential equations for integrals to be set up and solved at the level of the symbol.

Cluster algebras: There is evidence [1] that cluster algebras [18–20] address the question of which set of letters ϕ_i, called the *alphabet*, appear in the symbol of certain scattering amplitudes in planar $\mathcal{N}=4$ super Yang-Mills theory. A $\mathrm{Gr}(4,n)$ cluster algebra consists of clusters that are represented by quiver diagrams of *nodes* connected by arrows and carry the $\mathcal{A}$ coordinates, which are the letters potentially appearing in the symbol of the n-particle scattering amplitude.

A cluster is mapped to a different one under an operation called *mutation* which replaces the value of a node and changes the connectivity of the quiver according to the rules explained in Ref. [1] in detail. Mutations define a connectivity structure between different clusters, such that each cluster can be thought of living on a vertex of a polytope. For $n \geq 8$ the mutations do not close on finitely many clusters and therefore the polytope has infinitely many vertices.

The $\mathrm{Gr}(4,7)$ cluster algebra generates 49 distinct $\mathcal{A}$ coordinates, of which seven are *frozen* (i.e., appear in every cluster) and the remainder (*unfrozen*) are distributed in sextets across 833 clusters. These 49 coordinates admit 42 homogeneous ratios which can be chosen as [21]

$$\begin{aligned} a_{11} &= \frac{\langle 1234\rangle\langle 1567\rangle\langle 2367\rangle}{\langle 1237\rangle\langle 1267\rangle\langle 3456\rangle} & a_{21} &= \frac{\langle 1234\rangle\langle 2567\rangle}{\langle 1267\rangle\langle 2345\rangle} \\ a_{31} &= \frac{\langle 1567\rangle\langle 2347\rangle}{\langle 1237\rangle\langle 4567\rangle} & a_{41} &= \frac{\langle 2457\rangle\langle 3456\rangle}{\langle 2345\rangle\langle 4567\rangle} \\ a_{51} &= \frac{\langle 1(23)(45)(67)\rangle}{\langle 1234\rangle\langle 1567\rangle} & a_{61} &= \frac{\langle 1(34)(56)(72)\rangle}{\langle 1234\rangle\langle 1567\rangle} \end{aligned} \quad (6)$$

and cyclically related a_{ij} for $j > 1$. Here we write $\langle a(bc)(de)(fg)\rangle = \langle acde\rangle\langle acfg\rangle - \langle acde\rangle\langle abfg\rangle$. For $\mathrm{Gr}(4,6)$ there are 14 clusters generating 15 $\mathcal{A}$ coordinates admitting 9 homogeneous ratios. Note that for any odd n one may choose the alphabet to be given by the unfrozen $\mathcal{A}$ coordinates, rendered homogeneous by combinations of the frozen coordinates. Therefore, we can assign one of the a_{ij} for each unfrozen node of a $\mathrm{Gr}(4,7)$ cluster so that the initial cluster effectively becomes

$$\begin{array}{ccc} a_{24} & \longrightarrow & a_{37} \\ \downarrow & \nwarrow & \downarrow \\ a_{13} & \longrightarrow & a_{17} \\ \downarrow & \nwarrow & \downarrow \\ a_{32} & \longrightarrow & a_{27} \end{array}$$

It is possible to mutate the initial cluster on nodes a_{17}, a_{27}, and a_{32} to reach a cluster that has the topology of an E_6 Dynkin diagram. Finally, mutating a_{37} to a_{26}, a_{53} to a_{41} and a_{26} to a_{33} only changes the direction of some arrows and yields

$$\begin{array}{ccccccccc} & & & & a_{13} & & & & \\ & & & & \uparrow & & & & \\ a_{24} & \longrightarrow & a_{51} & \longrightarrow & a_{62} & \longleftarrow & a_{41} & \longleftarrow & a_{33} \end{array}$$

Cluster adjacency: While mutations of clusters generate the letters of the symbol alphabet, the alphabet itself contains no information on either the connectivity of the clusters or the distribution of the $\mathcal{A}$ coordinates across them. However, a survey of all known Maximally Helicity Violating (MHV) and NMHV (next-to MHV) BDS-like subtracted heptagon amplitudes reveals that only certain pairs of letters appear in neighboring slots of the symbol. This leads us to conjecture a much more general set of adjacency relations for BDS-like subtracted amplitudes:

Two distinct $\mathcal{A}$ coordinates can appear consecutively in a symbol only if there is a cluster containing both.

We believe that the above conjecture will apply to any BDS-like subtracted amplitude which is expressed in terms of cluster polylogarithms. It has been conjectured that all MHV and NMHV amplitudes in planar $\mathcal{N}=4$ super Yang-Mills theory will have a polylogarithmic form [22], though it has not yet been tested whether the alphabets are dictated by the $\mathrm{Gr}(4,n)$ cluster structures for $n \geq 8$ beyond two-loop MHV amplitudes.

For eight-point amplitudes, a BDS-like subtracted amplitude in the sense of Ref. [11], which uses only two-particle Mandelstam invariants to provide a solution to the conformal Ward identity of [23], does not exist. Proceeding to nine points, one again has a canonical BDS-like subtracted amplitude, constructible from the two-loop results in Ref. [24] and we have verified directly that it does obey the above conjecture. That is, for each neighboring pair appearing in the symbol we can find a cluster containing that pair.

It is important to emphasise that the above adjacency conjecture introduces a much more detailed role for the cluster structure over and above the fact that the alphabet can be obtained from the union over all cluster $\mathcal{A}$ coordinates. It is the structure of the individual clusters which constrains both sequences of discontinuities (reading the symbol from the left) and successive derivatives (reading from the right).

TABLE I. The neighborhood and connectivity relations of the coordinates a_{i1} with the 42-letter alphabet. Other relations can be inferred by cyclic symmetry. The relations in the dashed box imply the extended Steinmann conditions. ◆: There are clusters where the coordinates appear together connected by an arrow. ●: There are clusters where the coordinates appear together but they are never connected. ○: The coordinates never appear in the same cluster but there is a mutation that links them. ◇: The coordinates do not appear in the same cluster nor there is a mutation that links them.

	a_{1i}	a_{2i}	a_{3i}	a_{4i}	a_{5i}	a_{6i}
a_{11}	● ○ ○ ◆ ◆ ○ ○	◆ ◆ ○ ● ◆ ● ○	◆ ○ ● ◆ ● ○ ◆	● ○ ◆ ○ ○ ◆ ○	● ○ ◆ ○ ○ ◆ ○	◇ ◆ ○ ○ ○ ○ ◆
a_{21}	◆ ○ ● ◆ ● ○ ◆	● ○ ● ◆ ◆ ● ○	◆ ○ ◆ ◆ ○ ◆ ●	◆ ○ ◆ ○ ● ◆ ○	○ ◆ ● ○ ◆ ○ ◆	○ ◆ ○ ● ○ ◆ ○
a_{31}	◆ ◆ ○ ● ◆ ● ○	◆ ● ◆ ○ ◆ ◆ ○	● ○ ● ◆ ◆ ● ○	◆ ○ ◆ ● ○ ◆ ○	○ ◆ ○ ◆ ○ ● ◆	○ ○ ◆ ○ ● ○ ◆
a_{41}	● ○ ◆ ○ ○ ◆ ○	◆ ○ ◆ ● ○ ◆ ○	◆ ○ ◆ ○ ● ◆ ○	● ◇ ◆ ○ ○ ◆ ◇	● ○ ○ ○ ○ ○ ○	◇ ◆ ◇ ○ ○ ◇ ◆
a_{51}	● ○ ◆ ○ ○ ◆ ○	○ ◆ ○ ◆ ○ ● ◆	○ ◆ ● ○ ◆ ○ ◆	● ○ ○ ○ ○ ○ ○	● ◇ ◆ ○ ○ ◆ ◇	◇ ◆ ◇ ○ ○ ◇ ◆
a_{61}	◇ ◆ ○ ○ ○ ○ ◆	○ ○ ◆ ○ ● ○ ◆	○ ◆ ○ ● ○ ◆ ○	◇ ◆ ◇ ○ ○ ◇ ◆	◇ ◆ ◇ ○ ○ ◇ ◆	● ◇ ○ ◇ ◇ ○ ◇

In the case of planar $\mathcal{N} = 4$ heptagon amplitudes, only 840 of the 1764 possible ordered pairs of 42 letters are contained in the same cluster. We summarize this information in Table I, also distinguishing whether such pairs are connected by a quiver arrow as well as whether two letters that never appear together mutate into each other. If one letter mutates into another, this pair never appears together in a third cluster.

The (extended) Steinmann constraints on heptagon symbols are a subset of the cluster adjacency conditions: the letters a_{12}, a_{13}, a_{16}, or a_{17} never share a cluster with a_{11}. In fact, four dihedral copies of the initial cluster mutate to another cluster where a_{11} is replaced by one of these four letters which cannot appear after an initial a_{11}.

Only 784 of the 840 allowed adjacencies actually occur in the known 7-particle amplitudes, while the pairs

$$[a_{11} \otimes a_{41}] \quad \& \text{ cyclic} + \text{parity} \tag{7a}$$

$$[a_{21} \otimes a_{64}] \quad \& \text{ cyclic} + \text{reflection} \tag{7b}$$

and their reverses do not appear even though they are permitted by our conjecture. Nevertheless, in the following section we compute the symbol of a three-loop integral and find that its symbol has adjacent pairs of the form Eq. (7a).

Heptagon integrals.—Let us consider the following finite double pentaladder integral (drawn in Fig. 1)

$$I^{(3)} = \int d\tilde{Z} \frac{1}{\prod_{i=1}^{4}(ABi-1i)\prod_{i=4}^{7}(EFi-1i)} \times \frac{(AB13)(EF46)N}{(CD34)(ABCD)(CDEF)(CD67)}. \tag{8}$$

The measure is $d\tilde{Z} = (d^4 Z_{AB}/i\pi^2)(d^4 Z_{CD}/i\pi^2)(d^4 Z_{EF}/i\pi^2)$ and the numerator $N = (2345)(3467)[7(12)(34)(56)]$ ensures that the integral has unit leading singularity [25]. There exists a set of four second-order differential operators which relate $I^{(3)}$ to two-loop integrals [26]. Using the notation $O_{ij} = Z_i \cdot (\partial/\partial Z_j)$ they are

$$(4567) N O_{45} O_{34} N^{-1} I^{(3)} = -(3467) I^{(2)}, \tag{9}$$

$$(3456) N O_{65} O_{76} N^{-1} I^{(3)} = -(3467) I^{(2)}, \tag{10}$$

$$(1237) N O_{32} O_{43} N^{-1} I^{(3)} = +(1347) \tilde{I}^{(2)}, \tag{11}$$

$$(1234) N O_{12} O_{71} N^{-1} I^{(3)} = -(1347) \tilde{I}^{(2)}. \tag{12}$$

The two-loop integrals $I^{(2)}$ and $\tilde{I}^{(2)}$ are shown in Fig. 2.

The second-order operators above reduce the weight by two. Therefore they must annihilate the final entries in the symbol of $I^{(3)}$. Using this condition we can construct a set of ten multiplicative combinations out the 42 possible heptagon letters for the final entries.

The integral $I^{(2)}$ obeys a similar set of differential equations, except that the integrals on the rhs are the one-loop hexagons $I^{(1)}$ and $\tilde{I}^{(1)}$ depicted in Fig. 3. Therefore $I^{(2)}$ can also only have the same ten possible final entries as $I^{(3)}$. Of the 322 weight-4 (initial) Steinmann heptagon symbols constructed in Ref. [10], there is a unique linear combination with the correct final entries for $I^{(2)}$ which we conclude must be the result up to a scale. The details of the rhs differential equations were not

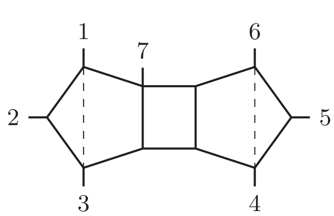

FIG. 1. Seven-point, three-loop, massless integral.

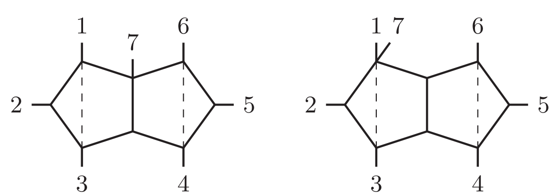

FIG. 2. The seven-point, two-loop integrals $I^{(2)}$ and $\tilde{I}^{(2)}$.

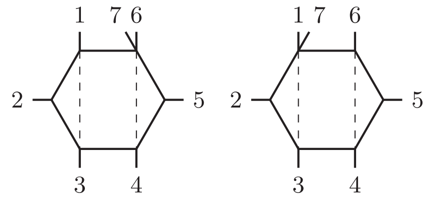

FIG. 3. The one-loop hexagon integrals $I^{(1)}$ and $\tilde{I}^{(1)}$.

required to obtain it, and indeed they can be used to derive formulas for the one-loop hexagons in Fig. 3.

Returning to $I^{(3)}$, we find that of the 3192 weight-6 (initial) Steinmann heptagon symbols constructed in Ref. [10], seven of them have good final entries. However, only one of these produces our result for $I^{(2)}$ on the rhs of Eqs. (9) and (10) and we conclude this is the symbol of $I^{(3)}$. Either of the Eqs. (11) or (12) can then be used to derive a result for $\tilde{I}^{(2)}$. We quote all symbols in ancillary files [27].

When analyzing the symbol of the three-loop integral $I^{(3)}$, pairs not present in the MHV and NMHV heptagon data were found, namely, $[a_{11} \otimes a_{41}]$ and $[a_{41} \otimes a_{11}]$. Their cyclic and parity copies will therefore be found in the associated cyclic and parity copies of the integral, completing the set Eq. (7). This evidence supports our conjecture that it is the cluster structure which controls the appearance of consecutive letters.

Consequences of the adjacency rules.—Neighbor sets: To describe the sets of allowed neighbors of a given letter, the E_6 topology cluster described above is very useful. It contains one of each of the six cyclic classes of letters given in Eq. (6). If we freeze the node a_{13} the cluster algebra reduces to an A_5 algebra which generates 20 letters all of which are in a cluster with a_{13}. Including a_{13} itself we find 21 possible neighbors for a_{13}. The analysis applies similarly to all a_{1i} type letters and is in accordance with the first line of Table I. If we freeze either of the nodes a_{24} or a_{37} the cluster algebra reduces to a D_5 algebra, generating 25 allowed neighbors in addition to the letter itself. Likewise freezing either a_{41} or a_{51} leads to an $A_4 \times A_1$ algebra which generates $14 + 2 = 16$ neighboring letters in addition to the letter itself. Finally, if we freeze the node a_{62} we obtain an $A_2 \times A_2 \times A_1$ subalgebra which generates $5 + 5 + 2 = 12$ allowed neighbors in addition to a_{62} itself. Each subalgebra responsible for generating the allowed neighbour set of a given letter corresponds to a subpolytope in the whole E_6 polytope.

Integrability: Let us now analyze the restrictions that integrability places on symbols. We may construct all integrable weight-two symbols which obey the cluster adjacency conditions. In the heptagon case we find 573 possible weight two integrable words (with no additional conditions on the initial entries). Of these 441 are symmetric in the exchange of the initial and final entries (and hence are trivially integrable) and 132 are antisymmetric. The symmetric ones correspond to products of logarithms and the antisymmetric ones to combinations of dilogarithms.

The number of antisymmetric symbols is the same as without the cluster adjacency conditions. Indeed, inspecting the pairs which appear in the antisymmetric integrable weight-two words one finds that they not only automatically obey the adjacency condition but in fact they obey the stronger condition of always being *connected* pairs; i.e., they appear together in some cluster connected by an arrow. Thus the Poisson structure (i.e., the connectivity) encodes the integrability conditions.

The cluster adjacency criterion is therefore really a constraint on the *symmetric* part. It implies that, even though any symmetric pair $[a \otimes b] + [b \otimes a]$ is integrable, to be admissible a and b must appear together in some cluster. Moreover admissible pairs which are never connected by an arrow in any cluster must appear symmetrically.

Triplets: We can go further and construct all weight three integrable words (again no restriction on initial entries) of which we find 4906. Examining the triplets therein we find two cases. Either there is a cluster in which all three letters of the triplet appear, or the triplet is of the form $[a \otimes b \otimes a']$ where a' is the result of applying a mutation to a and where b is in the intersection of the connected sets of a and a'. This implies that some pairs of letters, such as a_{11} and a_{61}, which neither share a cluster nor mutate into each other, never appear together in a triplet—there must be at least two letters appearing between them in an admissible symbol.

Cluster heptagon functions: Now let us impose the initial entry conditions appropriate for scattering amplitudes. In the heptagon case this corresponds to allowing only a_{1i} letters in the initial entries. In Table II we record the dimensions of the spaces of cluster adjacent symbols. Up to weight three all initial Steinmann symbols are cluster adjacent. At weight four there are 14 initial Steinmann symbols which fail to be cluster adjacent. They are of the form $[u \otimes (1-u) \otimes u \otimes u]$ for $u = (a_{11}a_{12}/a_{15})$ or $u = a_{11}a_{13}$ and cyclic copies. The failure of cluster adjacency comes in the last two slots as in either case the letter u may not appear next to itself. Interestingly, if we impose *extended* Steinmann conditions, then up to weight seven the dimensions come out to be the same as with the cluster adjacency condition; i.e., the extended Steinmann condition and initial entry condition

TABLE II. Dimensions of various spaces constructed from the $\mathcal{A}$ coordinates of the Gr(4, 7) cluster algebra. The dimensions that are yet unknown are indicated by ?.

Function space	1	2	3	4	5	6	7	8
7gon	7	42	237	1288	6763	?	?	?
Initial Steinmann 7gon	7	28	97	322	1030	3192	9570	?
Cluster 7gon	7	28	97	308	911	2555	6826	?

together imply cluster adjacency. We do not yet know if this pattern continues.

Hexagon case: The discussion for the $\mathrm{Gr}(4,6)$ hexagon case is analogous to that for the heptagon. In terms of the inhomogeneous $\mathcal{A}$ coordinates the cluster adjacency constraints in this case are as follows. First, the Plückers $(ii+1i+3i+4)$ and $(i+1i+2i+4i+5)$ may not appear consecutively, which is the hexagon extended Steinmann condition [16,17]. Second, the Plückers $(ii+2i+3i+4)$ may not appear next to $(i-1i+1i+2i+3)$, $(i+1i+3i+4i+5)$, or $(i-2i-1i+1i+2)$. These conditions allow for 40 integrable weight-two words. With the hexagon initial entry condition assumed, the dimensions of the spaces of integrable words again coincide with those obtained just assuming the extended Steinmann conditions of Refs. [16,17]. As for the heptagon case, it would be interesting to prove whether this is always the case.

NMHV final entries: In Ref. [10] a set of final entry conditions for NMHV heptagon amplitudes due to Caron-Huot were given. Here we note that our adjacency condition also applies to these combinations in the sense that the final letters of the NMHV symbols in fact lie in the same clusters as the poles of the accompanying rational functions or *R invariants*. A similar structure is exhibited in the NMHV hexagon amplitudes, as can been seen by inspecting the coproduct relations found in Ref. [28], and conjecturally will be applicable to multipoint NMHV amplitudes.

A_2 and A_3 functions: In Ref. [29] special weight-four functions which have simple $B_3 \otimes \mathbb{C}^*$ and $B_2 \wedge B_2$ coproduct components were used to describe the two-loop MHV amplitudes. Those relations are best described in terms of cluster $\mathcal{X}$ coordinates while our adjacency pattern relates to $\mathcal{A}$ coordinates. It would be fascinating to explore the connections between the two.

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