

Virtually torsion-free covers of minimax groups

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Abstract

We prove that every finitely generated, virtually solvable minimax group can be expressed as a homomorphic image of a virtually torsion-free, virtually solvable minimax group. This result enables us to generalize a theorem of Ch. Pittet and L. Saloff-Coste about random walks on finitely generated, virtually solvable minimax groups. Moreover, the paper identifies properties, such as the derived length and the nilpotency class of the Fitting subgroup, that are preserved in the covering process. Finally, we determine exactly which infinitely generated, virtually solvable minimax groups also possess this type of cover.

Couvertures virtuellement sans torsion de groupes minimax

Résumé

Nous prouvons que tout groupe de type fini minimax et virtuellement résoluble peut être exprimé comme l'image homomorphe d'un groupe minimax virtuellement résoluble et virtuellement sans torsion. Ce résultat permet de généraliser un théorème de Ch. Pittet et L. Saloff-Coste concernant les marches aléatoires sur les groupes de type fini minimax et virtuellement résolubles. En outre, l'article identifie des propriétés conservées dans le processus de couverture, telles que la classe de résolubilité et la classe de nilpotence du sous-groupe de Fitting. Enfin, nous déterminons exactement quels groupes de type infini minimax et virtuellement résolubles admettent également ce type de couverture.

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1 Introduction

In this paper, we study *virtually solvable minimax groups*; these are groups G that possess a series

$$1 = G_0 \trianglelefteq G_1 \trianglelefteq \cdots \trianglelefteq G_r = G$$

such that each factor G_i/G_{i-1} is either finite, infinite cyclic, or quasicyclic. (Recall that a group is *quasicyclic* if, for some prime p , it is isomorphic to $\mathbb{Z}(p^\infty) := \mathbb{Z}[1/p]/\mathbb{Z}$.) Denoting the class of virtually solvable minimax groups by \mathfrak{M} , we determine which \mathfrak{M} -groups can be realized as quotients of virtually torsion-free \mathfrak{M} -groups. Moreover, our results on quotients allow us to settle a longstanding question about a lower bound for the return probability of a random walk on the Cayley graph of a finitely generated \mathfrak{M} -group (see §1.3). We are indebted to Lison Jacoboni for pointing out the relevance of our work to this question.

The importance of \mathfrak{M} -groups arises primarily from the special status among virtually solvable groups that is enjoyed by finitely generated \mathfrak{M} -groups. As shown by the first author in [14], the latter comprise all the finitely generated, virtually solvable groups without any sections isomorphic to a wreath product of a finite cyclic group with an infinite cyclic one. In particular, any finitely generated, virtually solvable group of finite abelian section rank is minimax (a property first established by D. J. S. Robinson [23]). For background on these and other properties of \mathfrak{M} -groups, we refer the reader to J. C. Lennox and Robinson's treatise [16] on infinite solvable groups.

Within the class \mathfrak{M} , we distinguish two subclasses: first, the subclass \mathfrak{M}_1 consisting of all the \mathfrak{M} -groups that are virtually torsion-free; second, the complement of \mathfrak{M}_1 in \mathfrak{M} , which we denote \mathfrak{M}_∞ . It has long been apparent that the groups in \mathfrak{M}_1 possess a far more transparent structure than those in \mathfrak{M}_∞ . For example, an \mathfrak{M} -group belongs to \mathfrak{M}_1 if and only if it is residually finite. As a result, every finitely generated \mathfrak{M}_∞ -group fails to be linear over any field. In contrast, \mathfrak{M}_1 -groups are all linear over \mathbb{Q} and hence can be studied with the aid of the entire arsenal of the theory of linear groups over \mathbb{R} , including via embeddings into Lie groups. The latter approach is particularly fruitful when tackling problems of an analytic nature, such as those that arise in the investigation of random walks (see [21]).

A further consequence of the \mathbb{Q} -linearity of \mathfrak{M}_1 -groups is that the finitely generated ones fall into merely countably many isomorphism classes. On the other hand, there are uncountably many nonisomorphic, finitely generated \mathfrak{M}_∞ -groups (see either [16, p. 104] or Proposition 7.12 below). Other differences between \mathfrak{M}_1 -groups and \mathfrak{M}_∞ -groups are evident in their respective algorithmic properties. The word problem, for instance, is solvable for every finitely generated group in \mathfrak{M}_1 (see [2]). However, since the number of possible algorithms is countable, there exist uncountably many nonisomorphic, finitely generated \mathfrak{M}_∞ -groups with unsolvable word problem.

Our goal here is to explore the following two questions concerning the relationship between \mathfrak{M}_1 -groups and \mathfrak{M}_∞ -groups. In phrasing the questions, we employ a parlance that will be used throughout the paper, and that also occurs in its title: if a group G can be expressed as a homomorphic image of a group G^* , we say that G is *covered by* G^* . In this case, G^* is referred to as a *cover* of G and any epimorphism $G^* \rightarrow G$ as a *covering*. A cover of G that belongs to the class \mathfrak{M}_1 is called an \mathfrak{M}_1 -*cover* of G , and the corresponding covering is designated an \mathfrak{M}_1 -*covering*.

Question 1.1. *Under what conditions is it possible to cover an \mathfrak{M}_∞ -group by an \mathfrak{M}_1 -group?*

Question 1.2. *If an \mathfrak{M}_∞ -group G can be covered by an \mathfrak{M}_1 -group G^* , how can we choose G^* so that it retains many of the properties enjoyed by G ?*

Answering the above questions should enable us to reduce certain problems about \mathfrak{M} -groups to the more tractable case where the group belongs to \mathfrak{M}_1 . A current problem inviting such an approach arises in Ch. Pittet and L. Saloff-Coste's study [21] of random walks on the Cayley graphs of finitely generated \mathfrak{M} -groups. Their paper establishes a lower bound on the probability of return for this sort of random walk when the group is virtually torsion-free, but their methods fail to apply to \mathfrak{M}_∞ -groups.* One way to extend their bound to the latter case is to prove that any finitely generated member of the class \mathfrak{M}_∞ can be expressed as a homomorphic image of an \mathfrak{M}_1 -group. In Theorem 1.5 below, we establish that the

*The hypothesis that the group is virtually torsion-free should be included in the statement of [21, Theorem 1.1], for the proof requires that assumption. We thank Lison Jacoboni for bringing this mistake to our attention.

condition that the group be finitely generated is indeed one possible answer to Question 1.1, thus yielding the desired generalization of Pittet and Saloff-Coste's result (Corollary 1.8).

1.1 Structure of \mathfrak{M} -groups

Before stating our main results, we summarize the structural properties of \mathfrak{M} -groups in Proposition 1.3 below; proofs of these may be found in [16]. In the statement of the proposition, as well as throughout the rest of the paper, we write $\text{Fitt}(G)$ for the *Fitting subgroup* of a group G , namely, the subgroup generated by all the nilpotent normal subgroups. In addition, we define $R(G)$ to be the *finite residual* of G , by which we mean the intersection of all the subgroups of finite index.

Proposition 1.3. *If G is an \mathfrak{M} -group, then the following four statements hold.*

- (i) $\text{Fitt}(G)$ is nilpotent and $G/\text{Fitt}(G)$ is virtually abelian.
- (ii) If G belongs to \mathfrak{M}_1 , then $G/\text{Fitt}(G)$ is finitely generated.
- (iii) $R(G)$ is a direct product of finitely many quasicyclic groups.
- (iv) G is a member of \mathfrak{M}_1 if and only if $R(G) = 1$. □

1.2 Covers of finitely generated \mathfrak{M} -groups

Our main result, Theorem 1.13, characterizes all the \mathfrak{M} -groups that can be covered by an \mathfrak{M}_1 -group. We will postpone describing this theorem until §1.4, focusing first on its implications for finitely generated \mathfrak{M} -groups. We begin with this special case because of its immediate relevance to random walks, as well as its potential to find further applications.

While discussing our results, we will refer to the following example; it is the simplest instance of a finitely generated \mathfrak{M}_∞ -group, originally due to P. Hall [7]. More sophisticated examples of such groups are described in the final section of the paper.

Example 1.4. Take p to be a prime, and let G^* be the group of upper triangular 3×3 matrices (a_{ij}) with entries in the ring $\mathbb{Z}[1/p]$ such that $a_{11} = a_{33} = 1$ and a_{22} is a power of p . Let A be the central subgroup consisting of all the matrices $(a_{ij}) \in G^*$ with $a_{22} = 1$, $a_{12} = a_{23} = 0$, and $a_{13} \in \mathbb{Z}$. Set $G = G^*/A$. Then G is a finitely generated solvable minimax group with a quasicyclic center.

In Example 1.4, there is an epimorphism from the finitely generated, torsion-free solvable minimax group G^* to G . The kernel of this epimorphism is a central cyclic subgroup of G^* . Moreover, the groups G and G^* are structurally very similar. For instance, they have the same derived length, and the nilpotency classes of $\text{Fitt}(G)$ and $\text{Fitt}(G^*)$ coincide.

It turns out that the group G in Example 1.4 is, in certain respects, quite typical for a finitely generated \mathfrak{M}_∞ -group. In Theorem 1.5 below, we prove that such groups always admit an \mathfrak{M}_1 -covering exhibiting most of the properties manifested by the covering in Example 1.4. In the statement of the theorem, the *spectrum* of a group G , written $\text{spec}(G)$, is the set of primes for which G has a quasicyclic section. Also, $\text{solv}(G)$ denotes the *solvable radical* of G , that is, the subgroup generated by all the solvable normal subgroups. Note that, if G is an \mathfrak{M} -group, $\text{solv}(G)$ is a solvable normal subgroup of finite index. Another piece of notation refers to the derived length of G if G is solvable, written $\text{der}(G)$. Finally, if N is a nilpotent group, then its nilpotency class is denoted $\text{nil } N$.

Theorem 1.5. *Let G be a finitely generated \mathfrak{M} -group, and write $N = \text{Fitt}(G)$ and $S = \text{solv}(G)$. Then there is a finitely generated \mathfrak{M}_1 -group G^* and an epimorphism $\phi : G^* \rightarrow G$ satisfying the following four properties, where $N^* = \text{Fitt}(G^*)$ and $S^* = \text{solv}(G^*)$.*

- (i) $\text{spec}(G^*) = \text{spec}(G)$.
- (ii) $N^* = \phi^{-1}(N)$; hence $S^* = \phi^{-1}(S)$.
- (iii) $\text{der}(S^*) = \text{der}(S)$.
- (iv) $\text{nil } N^* = \text{nil } N$ and $\text{der}(N^*) = \text{der}(N)$.

The first part of statement (ii) in Theorem 1.5 implies that $\text{Ker } \phi$ is nilpotent, but the theorem says little more about the kernel's properties. In particular, Theorem 1.5 makes no claims about the kernel of the covering being central or polycyclic, although both properties hold for the kernel in Example 1.4. Indeed, it is not possible in general to form a covering whose kernel has either of these two attributes. We illustrate this phenomenon in §7 with two examples, one where the spectrum contains two primes and another where it consists of a single prime.

The reader will notice that, in all of the examples appearing in the paper, the kernel of the covering is abelian. Nevertheless, as will become clear in the proof of Theorem 1.5, our construction may conceivably produce a kernel with nilpotency class larger than one (see Remark 6.2). Whether this is merely an artifact of the techniques employed here, or whether certain groups will only admit coverings with nonabelian kernels, remains a mystery.

Open Question 1.6. *Can the group G^* and epimorphism $\phi : G^* \rightarrow G$ in Theorem 1.5 always be chosen so that $\text{Ker } \phi$ is abelian?*

We conjecture that this question has a negative answer.

Acknowledgement. We are grateful to an anonymous referee for posing Open Question 1.6.

1.3 Application to random walks

We now describe in detail the application of Theorem 1.5 to random walks alluded to above. For the statement of the result, we require some notation specific to this matter. First, if f and g are functions from the positive integers to the nonnegative real numbers, we write $f(m) \lesssim g(m)$ whenever there are positive constants a , b , and c such that $f(m) \geq ag(bm)$ for $m > c$. Also, if $f(m) \lesssim g(m)$ and $g(m) \lesssim f(m)$, then we write $f(m) \sim g(m)$.

Suppose that G is a group with a finite symmetric generating set S ; by *symmetric*, we mean that it is closed under inversion. Consider the simple random walk on the Cayley graph of G with respect to S . For any positive integer m , let $P_{(G,S)}(2m)$ be the probability of returning to one's starting position after $2m$ steps. It is shown in [20] that, for any other finite symmetric generating set T , $P_{(G,T)}(2m) \sim P_{(G,S)}(2m)$.

In [21] the following lower bound is established for the probability of return for a random walk on the Cayley graph of a finitely generated \mathfrak{M}_1 -group.

Theorem 1.7. (Pittet and Saloff-Coste [21, Theorem 1.1]) *Let G be an \mathfrak{M}_1 -group with a finite symmetric generating set S . Then*

$$P_{(G,S)}(2m) \gtrsim \exp(-m^{\frac{1}{3}}). \quad \square$$

With the aid of Theorem 1.5, we generalize Theorem 1.7 to all finitely generated \mathfrak{M} -groups. As related at the end of the paper in the acknowledgements, this corollary and its proof were communicated to the authors by Lison Jacoboni.

Corollary 1.8. *Let G be an \mathfrak{M} -group with a finite symmetric generating set S . Then*

$$P_{(G,S)}(2m) \gtrsim \exp(-m^{\frac{1}{3}}).$$

Proof. According to Theorem 1.5, there is an \mathfrak{M}_1 -group G^* and an epimorphism $\phi : G^* \rightarrow G$. Also, we can select G^* so that it has a finite symmetric generating set S^* with $\phi(S^*) = S$. Consider random walks on the Cayley graphs of both groups with respect to these generating sets, with both walks commencing at the identity element. Then $P_{(G,S)}(2m)$ is equal to the probability that the random walk on G^* will take us to an element of $\text{Ker } \phi$ after $2m$ steps. As a consequence, $P_{(G,S)}(2m) \geq P_{(G^*,S^*)}(2m)$. Moreover, by Theorem 1.7, we have $P_{(G^*,S^*)}(2m) \gtrsim \exp(-m^{\frac{1}{3}})$. Hence $P_{(G,S)}(2m) \gtrsim \exp(-m^{\frac{1}{3}})$. \square

It is proved in [20] that the probability of return for groups of exponential growth is always bounded above by the function $\exp(-m^{\frac{1}{3}})$. Hence we have the

Corollary 1.9. *Let G be an \mathfrak{M} -group with a finite symmetric generating set S . If G has exponential growth, then*

$$P_{(G,S)}(2m) \sim \exp(-m^{\frac{1}{3}}). \quad \square$$

Recall that a finitely generated solvable group has either exponential or polynomial growth, with the latter property holding if and only if the group is virtually nilpotent (see [19] and [31]). Furthermore, N. Varopoulos shows in [29] that any finitely generated group of polynomial growth of degree d has probability of return $m^{-d/2}$.

1.4 Covers of non-finitely-generated \mathfrak{M} -groups

Having discussed finitely generated \mathfrak{M} -groups, we turn next to the question of what other \mathfrak{M} -groups can be realized as quotients of \mathfrak{M}_1 -groups. As we shall see below, not all \mathfrak{M} -groups admit such a description; moreover, even if there is an \mathfrak{M}_1 -cover, it may not be possible to construct one with the same spectrum as the group. To distinguish \mathfrak{M} -groups with different spectra, we will employ the symbols \mathfrak{M}^π and \mathfrak{M}_1^π to denote the subclasses of \mathfrak{M} and \mathfrak{M}_1 , respectively, consisting of all those members with a spectrum contained in a set of primes π . Alternatively, an \mathfrak{M}^π -group will be referred to as a π -*minimax group*. In Proposition 1.11 below (proved in §3.4), we identify three properties that must be satisfied by an \mathfrak{M} -group in order for it to have an \mathfrak{M}_1^π -cover. One of these properties requires the following notion, which will play a prominent role in the paper.

Definition 1.10. Let π be a set of primes. A π -*number* is a nonzero integer whose prime divisors all belong to π . If G is a group and A a $\mathbb{Z}G$ -module, then we say that the action of G on A is π -*integral* if, for each $g \in G$, there are integers $\alpha_0, \alpha_1, \dots, \alpha_m$ such that α_m is a π -number and $(\alpha_0 + \alpha_1 g + \dots + \alpha_m g^m) \in \text{Ann}_{\mathbb{Z}G}(A)$.

Considering inverses of elements of G , we immediately see that we can replace the condition that α_m is a π -number in Definition 1.10 with the condition that α_0 is a π -number. Indeed, as the reader may easily verify, this is also equivalent to requiring that both α_0 and α_m are π -numbers.

Proposition 1.11. *Let π be a set of primes and G an \mathfrak{M} -group with Fitting subgroup N . Set $Q = G/N$. If G is a homomorphic image of an \mathfrak{M}_1^π -group, then G satisfies the following three conditions.*

- (a) Q is finitely generated.
- (b) $\text{spec}(N) \subseteq \pi$.
- (c) Q acts π -integrally on N_{ab} .

We point out that statement (b) in Proposition 1.11 is obvious and (a) is an immediate consequence of Proposition 1.3(ii). The proof of the third statement is also quite straightforward and is based on the fact that, for any $\mathbb{Z}G$ -module A whose underlying abelian group is in \mathfrak{M}_1^π , the action of Q on A is π -integral (Lemma 3.21).

In Example 1.12, we describe some situations where either statement (a) or (c) in Proposition 1.11 is false while the other two assertions hold. These examples will serve two purposes: first, to exhibit \mathfrak{M}_∞ -groups that cannot be expressed as a homomorphic image of any \mathfrak{M}_1 -group; second, to demonstrate that there are \mathfrak{M}_∞ -groups G that cannot be realized as quotients of \mathfrak{M}_1 -groups with the same spectrum as G , but nevertheless occur as quotients of \mathfrak{M}_1 -groups with larger spectra. The descriptions involve the ring \mathbb{Z}_p of p -adic integers and its multiplicative group of units, denoted \mathbb{Z}_p^* .

Example 1.12. Let p be a prime. The groups that we discuss are denoted G_1 , G_2 , and G_3 . In each case, G_i is defined to be a semidirect product $\mathbb{Z}(p^\infty) \rtimes \Lambda_i$, where Λ_i is a certain minimax subgroup of \mathbb{Z}_p^* . In this semidirect product, we assume that the action of Λ_i on $\mathbb{Z}(p^\infty)$ arises from the natural \mathbb{Z}_p -module structure on $\mathbb{Z}(p^\infty)$. Note that, with this definition, we necessarily have $\text{Fitt}(G_i) \cong \mathbb{Z}(p^\infty)$.

- For $i = 1$, we take q to be a prime such that $p \mid q - 1$. The subgroup of \mathbb{Z}_p^* consisting of all the p -adic integers congruent to 1 modulo p is isomorphic to \mathbb{Z}_p if p is odd and $\mathbb{Z}_2 \oplus (\mathbb{Z}/2)$ if $p = 2$. Thus \mathbb{Z}_p^* has a subgroup isomorphic to $\mathbb{Z}[1/q]$ that contains q . Choose Λ_1 to be such a subgroup. Hence G_1 satisfies conditions (b) and (c) in Proposition 1.11 for $\pi = \{p, q\}$. However, G_1 fails to fulfill (a) and so cannot be covered by an \mathfrak{M}_1 -group. (This example also appears in [16, p. 92].)
- For $i = 2$, let Λ_2 be a cyclic subgroup generated by an element of \mathbb{Z}_p^* that is transcendental over \mathbb{Q} . For the group G_2 , assertion (a) from Proposition 1.11 is true, and (b) holds for $\pi = \{p\}$. However, condition (c) in Proposition 1.11 is not satisfied for any set of primes π . Hence there is no \mathfrak{M}_1 -group that has a quotient isomorphic to G_2 .
- For $i = 3$, let q be a prime distinct from p , and take Λ_3 to be the cyclic subgroup of \mathbb{Z}_p^* generated by q . For the group G_3 , statement (a) in Proposition 1.11 is true, and (b) holds for $\pi = \{p\}$. Condition (c), on the other hand, is not satisfied for $\pi = \{p\}$. Hence Proposition 1.11 implies that G_3 cannot be realized as a quotient of a p -minimax group. However, (b) and (c) are both fulfilled for $\pi = \{p, q\}$. In fact, G_3 is a quotient of the torsion-free $\{p, q\}$ -minimax group $\mathbb{Z}[1/pq] \rtimes C_\infty$, where the generator of C_∞ acts on $\mathbb{Z}[1/pq]$ by multiplication by q .

The main theorem of the paper, Theorem 1.13, encompasses Proposition 1.11 and its converse, thus providing a complete characterization of all the \mathfrak{M} -groups that occur as quotients of \mathfrak{M}_1^π -groups. At the same time, the theorem identifies another set of conditions that is equivalent to properties (a), (b), and (c) from Proposition 1.11.

Theorem 1.13. *Let π be a set of primes and G an \mathfrak{M} -group. Write $N = \text{Fitt}(G)$ and $Q = G/N$. Then the following three statements are equivalent.*

- I. *G can be expressed as a homomorphic image of an \mathfrak{M}_1^π -group.*
- II. *Q is finitely generated, $\text{spec}(N) \subseteq \pi$, and Q acts π -integrally on N_{ab} .*
- III. *Q is finitely generated, $\text{spec}(N) \subseteq \pi$, and Q acts π -integrally on the image of $R(G)$ in N_{ab} .*

Moreover, if the above conditions are fulfilled, then we can select an \mathfrak{M}_1^π -minimax group G^ and epimorphism $\phi : G^* \rightarrow G$ so that properties (ii)-(iv) in Theorem 1.5 hold.*

Remark 1.14. It follows from Theorem 1.13 that the solvable minimax groups G that are quotients of \mathfrak{M}_1^π -groups for $\pi = \text{spec}(G)$ are precisely those solvable minimax groups that belong to the class \mathcal{U} from [18]. In that paper, the second author shows that the groups in this class enjoy two cohomological properties that are not manifested by all solvable minimax groups.

Our next lemma shows that Theorem 1.5 is a special case of the implication (III) \implies (I) in Theorem 1.13.

Lemma 1.15. *If G is a finitely generated \mathfrak{M} -group, then G satisfies statement (III) in Theorem 1.13 for $\pi = \text{spec}(G)$.*

Proof. Since Q is finitely presented, N_{ab} must be finitely generated as a $\mathbb{Z}Q$ -module. Because $\mathbb{Z}Q$ is a Noetherian ring, it follows that N_{ab} is a Noetherian module. Therefore N_{ab} must be virtually torsion-free, implying that the image of $R(G)$ in N_{ab} is trivial. Hence (III) holds. \square

After laying the foundations for our argument in §§3-5, we prove Theorem 1.13 in §6. In that section, we also discuss how the theorem can be extended to yield a π -minimax cover that is entirely torsion-free, rather than just virtually torsion-free (Corollaries 6.3 and 6.4). With a torsion-free cover, however, we are unable to achieve quite the same degree of resemblance between the cover and the original group.

2 Strategy and terminology

2.1 Strategy for proving Theorem 1.13

In proving Theorem 1.13, we introduce a new method for studying \mathfrak{M} -groups, one that we hope will give rise to further advances in the theory of such groups. Our approach involves embedding an \mathfrak{M} -group G densely in a locally compact, totally disconnected topological group and taking advantage of certain features of the structure of this new group. In this section, we describe the main aspects of the argument for (II) \implies (I), highlighting the roles played by four pivotal propositions, **Propositions 4.15, 3.12, 4.20, and 3.28**. For our discussion here, we will assume that G has torsion at merely a single prime p . This means that the finite residual P of G is a direct product of finitely many quasicyclic p -groups.

The first step is to densely embed N in a nilpotent, locally compact topological group N_p such that the compact subgroups of N_p are all polycyclic pro- p groups. The group N_p is constructed by forming the direct limit of the pro- p completions of the finitely generated

subgroups of N . In the case where N is abelian and written additively, this is equivalent to tensoring N with \mathbb{Z}_p . The precise definition and properties of N_p , called the *tensor p -completion* of N , are discussed in §5. Moreover, in §4, we investigate the class \mathfrak{N}_p of topological groups to which these tensor p -completions belong.

Drawing on a technique originally due to Peter Hilton, we build a topological group extension $1 \rightarrow N_p \rightarrow G_{(N,p)} \rightarrow Q \rightarrow 1$ that fits into a commutative diagram of the form

$$\begin{array}{ccccccc} 1 & \longrightarrow & N & \longrightarrow & G & \longrightarrow & Q \longrightarrow 1 \\ & & \downarrow & & \downarrow & & \parallel \\ 1 & \longrightarrow & N_p & \longrightarrow & G_{(N,p)} & \longrightarrow & Q \longrightarrow 1, \end{array}$$

and in which N_p is an open normal subgroup. We now employ the first of our four key propositions, **Proposition 4.15**, which allows us to obtain two closed subgroups R_0 and X of $G_{(N,p)}$ satisfying the following properties.

- (a) R_0 is a radicable normal subgroup of $G_{(N,p)}$ contained in N_p . (A group R is said to be *radicable* if the map $r \mapsto r^m$ is a surjection $R \rightarrow R$ for any $m \in \mathbb{Z}$.)
- (b) $G_{(N,p)} = R_0 X$.
- (c) $X \cap N_p$ is a polycyclic pro- p -group.

Notice that, since Q is virtually polycyclic, (c) implies that X must be virtually torsion-free.

At this stage, we will invoke the most important proposition for the proof, **Proposition 3.12**. This result will enable us to construct a sequence

$$\cdots \xrightarrow{\psi_{i+1}} R_i \xrightarrow{\psi_i} R_{i-1} \xrightarrow{\psi_{i-1}} \cdots \xrightarrow{\psi_2} R_1 \xrightarrow{\phi_1} R_0 \quad (2.1)$$

of group epimorphisms such that every R_i is a radicable nilpotent group upon which X acts as a group of automorphisms and each ψ_i commutes with the action of X . In addition, each R_i will contain an isomorphic copy of P , and the preimage under ψ_i of the copy of P in R_{i-1} will be the copy of P in R_i . Finally, the map $P \rightarrow P$ induced by ψ_i will be the epimorphism $u \mapsto u^p$.

In the case that R_0 is abelian, such a sequence (2.1) can be formed very easily merely by making each $R_i = R_0$ and ψ_i the p -powering map. Extending this construction to nilpotent groups, however, constitutes one of the most subtle aspects of the paper. Our method is based on the observation that $H^2(R, A) \cong H^2(P, A)^R$ whenever R is a radicable nilpotent group with torsion subgroup P and A is a finite $\mathbb{Z}R$ -module (Lemma 3.13). Gleaned from the Lyndon-Hochschild-Serre (LHS) spectral sequence, this property results from the fact that most of the cohomology of a torsion-free radicable nilpotent group vanishes when the coefficient module is finite. It is precisely this feature of radicable nilpotent groups that makes passing into N_p desirable.

Our next step is to consider the inverse limit Ω of (2.1), letting ψ be the canonical epimorphism $\Omega \rightarrow R_0$. The preimage Ψ of P in Ω is the inverse limit of the sequence $\cdots \rightarrow P \rightarrow P \rightarrow P$, where each homomorphism $P \rightarrow P$ is the p -power map. This means that Ψ is isomorphic to the direct sum of finitely many copies of \mathbb{Q}_p , so that Ω is torsion-free. We define a topology on Ω that makes it a locally compact, totally disconnected group and the action of X on Ω continuous. Thus the semidirect product $\Gamma := \Omega \rtimes X$ is a virtually torsion-free, locally compact group that covers $G_{(N,p)} = R_0 X$ via the continuous

epimorphism $(r, x) \mapsto \psi(r)x$, where $r \in \Omega$ and $x \in X$. Moreover, it will turn out that, like G , the group Γ acts π -integrally on the abelianization of its Fitting subgroup.

Inside the group Γ , we assemble a π -minimax cover for G , making use of **Propositions 4.20 and 3.28**. The former permits us to find a π -minimax nilpotent cover for N within Γ . From this cover of N , we stitch together a cover for G ; to show that the latter cover is π -minimax, we exploit the π -integrality of the action of Γ on the abelianization of its Fitting subgroup, applying **Proposition 3.28**.

2.2 Notation and terminology

Rings and modules

Let p be a prime. Then \mathbb{Z}_p is the ring of p -adic integers, \mathbb{Z}_p^* is the multiplicative group of units in \mathbb{Z}_p , \mathbb{Q}_p is the field of p -adic rational numbers, and \mathbb{F}_p is the field with p elements.

Let π be a set of primes. A π -number is a nonzero integer whose prime divisors all belong to π . The ring $\mathbb{Z}[\pi^{-1}]$ is the subring of \mathbb{Q} consisting of all rational numbers of the form m/n where $m \in \mathbb{Z}$ and n is a π -number.

The term *module* will always refer to a left module. Moreover, if R is a ring and A an R -module, we will write the operation $R \times A \rightarrow A$ as $(r, a) \mapsto r \cdot a$.

A *section* of a module is a quotient of a submodule.

If G is a group and R a ring, then RG denotes the group ring of G over R .

Abstract groups

If p is a prime, then $\mathbb{Z}(p^\infty)$ is the *quasicyclic p -group*, that is, the inductive limit of the cyclic p -groups \mathbb{Z}/p^k for $k \geq 1$.

Let G be a group. The center of G is written $Z(G)$. The *Fitting subgroup* of G , denoted $\text{Fitt}(G)$, is the subgroup generated by all the nilpotent normal subgroups of G . We use $R(G)$ to represent the *finite residual* of G , which is the intersection of all the subgroups of finite index.

A *section* of a group is a quotient of a subgroup.

The *spectrum* of a group G , denoted $\text{spec}(G)$, is the set of primes for which G has a quasicyclic section.

A *series* in a group G will always mean a series of the form

$$1 = G_0 \trianglelefteq G_1 \trianglelefteq \cdots \trianglelefteq G_r = G.$$

If $\phi : G \rightarrow Q$ and $\psi : H \rightarrow Q$ are group homomorphisms, then

$$G \times_Q H := \{(g, h) \mid g \in G, h \in H, \text{ and } \phi(g) = \psi(h)\}.$$

If H is a subgroup of a group G and $g \in G$, then $H^g := g^{-1}Hg$.

If g and h are elements of a group G , then $[g, h] := g^{-1}h^{-1}gh$. If H and K are subgroups of G , then $[H, K] := \langle [h, k] \mid h \in H, k \in K \rangle$. Moreover, if H_1, \dots, H_r are subgroups of G , then $[H_1, \dots, H_r]$ is defined recursively as follows: $[H_1, \dots, H_r] = [[H_1, \dots, H_{r-1}], H_r]$. For $H, K \leq G$ and $i \geq 1$, we abbreviate $[H, \underbrace{K, \dots, K}_i]$ to $[H, {}_iK]$.

Let G be a group. The derived series of G is written

$$\cdots \leq G^{(1)} \leq G^{(0)} = G.$$

The subgroups $G^{(1)}$ and $G^{(2)}$ will often be represented by G' and G'' , respectively. Also, the lower and upper central series are written

$$\cdots \leq \gamma_2 G \leq \gamma_1 G = G \quad \text{and} \quad 1 = Z_0(G) \leq Z_1(G) \leq \cdots,$$

respectively.

Let π be a set of primes. A π -torsion group, also referred to as a π -group, is one in which the order of every element is a π -number. A group G is said to be π -radicable if, for any π -number n and $g \in G$, there is an $x \in G$ such that $x^n = g$. A group is *radicable* if it is π -radicable for π the set of all primes. Throughout the paper, we will often write abelian groups additively, in which case we will use the term *divisible* rather than *radicable*.

A *Černikov group* is a group that is a finite extension of a direct product of finitely many quasicyclic groups.

A virtually solvable group G is said to have *finite torsion-free rank* if it possesses a subgroup series of finite length whose factors are either infinite cyclic or torsion. Because of the Schreier refinement theorem, the number of infinite cyclic factors in any such a series is an invariant of G , called its *Hirsch length* and denoted $h(G)$.

A virtually solvable group is said to have *finite abelian section rank* if its elementary abelian sections are all finite.

If G is an \mathfrak{M} -group and p a prime, then $m_p(G)$ represents the number of factors isomorphic to $\mathbb{Z}(p^\infty)$ in any series in which each factor is finite, cyclic, or quasicyclic. By Schreier's refinement theorem, this number does not depend on the particular series selected.

Let N be a nilpotent group and $H \leq N$. If π is a set of primes, then the π -isolator of H is the subgroup of N consisting of all the elements x such that $x^m \in H$ for some π -number m .

Let G and K be groups. If G acts upon K on the left, then we call K a G -group. In this case, we write the operation $G \times K \rightarrow K$ as $(g, k) \mapsto g \cdot k$. A homomorphism between G -groups that commutes with the action of G is a G -group homomorphism.

Topological groups

If G is an abstract group, then the *profinite completion* of G , denoted \hat{G} , is the inverse limit of all the finite quotients of G . We regard \hat{G} as a topological group with respect to the topology on this inverse limit induced by the product topology. This topology makes \hat{G} both compact and totally disconnected. The canonical homomorphism from G to \hat{G} is called the *profinite completion map* and denoted $c^G : G \rightarrow \hat{G}$.

If G is an abstract group and p a prime, then the *pro- p completion* of G , denoted \hat{G}_p , is the inverse limit of all the quotients of G that are finite p -groups. As with the profinite completion, \hat{G}_p is a compact, totally disconnected topological group. The canonical homomorphism from G to \hat{G}_p is called the *pro- p completion map* and denoted $c_p^G : G \rightarrow \hat{G}_p$.

Let G be a topological group. If $H \leq G$, then \bar{H} denotes the closure of H in G . We say that G is *topologically finitely generated* if there are elements g_1, \dots, g_r of G such that $G = \langle g_1, \dots, g_r \rangle$. If G happens to be generated as an abstract group by finitely many elements, then we say that G is *abstractly finitely generated*.

Let p be a prime. A pro- p group G is called *cyclic* if there is an element g of G such that $G = \langle g \rangle$. Hence any cyclic pro- p group must be either a finite cyclic p -group or a copy of \mathbb{Z}_p . A *polycyclic pro- p group* is a pro- p group with a series of finite length in which each factor is a cyclic pro- p group. Since we will also be applying the adjectives *cyclic* and *polycyclic* to

abstract groups, we will adhere to the convention that, when used in their pro- p senses, the terms *cyclic* and *polycyclic* will always be immediately followed by the adjective *pro- p* .

A topological space is *σ -compact* if it is the union of countably many compact subspaces.

A topological group is *locally elliptic* if every compact subset is contained in a compact open subgroup.

Acknowledgement. We thank an anonymous referee for pointing out the relevance of local ellipticity to our discussion in §4.

3 Preliminary results about abstract groups

In this section, we establish an array of results concerning abstract groups that we require for the proof of Theorem 1.13. Foremost among these is Proposition 3.12, which is proved in §3.2. For the parts of Theorem 1.13 addressing nilpotency class and derived length, we need to investigate $\text{nil } G$ and $\text{der}(G)$ for a group G that can be written as a product of two subgroups, one of which is normal. This is accomplished in §3.1, whose principal result is Proposition 3.8. In §3.3, we discuss coverings of groups, and finally, in §3.4, we study π -integral actions.

3.1 Nilpotent and solvable actions

In order to examine the nilpotency class of a product, we require the well-known concept of a nilpotent action.

Definition 3.1. Let G be a group and K a G -group. We define the *lower G -central series*

$$\cdots \leq \gamma_3^G K \leq \gamma_2^G K \leq \gamma_1^G K$$

of K as follows: $\gamma_1^G K = K$; $\gamma_i^G K = \langle k(g \cdot l)k^{-1}l^{-1} \mid k \in K, l \in \gamma_{i-1}^G K, g \in G \rangle$ for $i > 1$.

We say that the action of G on K is *nilpotent* if there is a nonnegative integer c such that $\gamma_{c+1}^G K = 1$. The smallest such integer c is called the *nilpotency class* of the action, written $\text{nil}_G K$.

Dual to the lower G -central series is the upper G -central series.

Definition 3.2. Let G be a group and K a G -group. Define $Z^G(K)$ to be the subgroup of $Z(K)$ consisting of all the elements that are centralized by G . The *upper G -central series*

$$Z_0^G(K) \leq Z_1^G(K) \leq Z_2^G(K) \leq \cdots$$

is defined as follows: $Z_0^G(K) = 1$; $Z_i^G(K) = Z^G(K/Z_{i-1}^G(K))$ for $i \geq 1$.

As stated in the following well-known lemma, the lengths of the upper and lower G -central series are the same.

Lemma 3.3. *Let K be a G -group. If $i \geq 0$, then $\gamma_{i+1}^G K = 1$ if and only if $Z_i^G(K) = K$. \square*

Analogous to the notion of a nilpotent action is that of a solvable action, defined below. Although we are unaware of any specific reference, it is likely that this concept has been studied before.

Definition 3.4. Let G be a group and K a G -group. We define the G -derived series

$$\cdots \leq \delta_2^G K \leq \delta_1^G K \leq \delta_0^G K$$

of K as follows: $\delta_0^G K = K$; $\delta_i^G K = \gamma_2^{G^{(i-1)}}(\delta_{i-1}^G K)$ for $i \geq 1$.

The action of G on K is *solvable* if there is a nonnegative integer d such that $\delta_d^G K = 1$. The smallest such integer d is called the *derived length* of the action, denoted $\text{der}_G(K)$.

We wish to use the lower G -central series and the G -derived series to describe the lower central series and derived series of a group that can be written as a product of two subgroups, one of which is normal. First, however, we mention an elementary property of commutator subgroups, Lemma 3.5. This can be proved with the aid of Hall and L. Kalužnin's "three subgroup lemma" [16, 1.2.3]; we leave the details to the reader.

Lemma 3.5. Let G be a group and K a normal subgroup of G . For each $i > 0$,

$$[K, \gamma_i G] \leq [K, {}_i G]. \quad \square$$

In the next lemma, we investigate the lower central series and derived series of a product.

Lemma 3.6. Let G be a group such that $G = KX$, where $K \trianglelefteq G$ and $X \leq G$. For each $i > 0$,

$$\gamma_i G = [K, {}_{i-1} G] \gamma_i X \quad \text{and} \quad G^{(i)} = [K, G, G', \dots, G^{(i-1)}] X^{(i)}.$$

Proof. We content ourselves with deriving the first equation; the second may be deduced by a similar argument. We proceed by induction on i . For $i = 1$, the equation clearly holds. Suppose $i > 1$. Let $g \in \gamma_{i-1} G$ and $h \in G$. By the inductive hypothesis, $g = ax$, where $a \in [K, {}_{i-2} G]$ and $x \in \gamma_{i-1} X$. Also, write $h = by$ with $b \in K$ and $y \in X$. We have

$$[g, h] = [ax, by] = [a, y]^x [x, y] [a, b]^{xy} [x, b]^y.$$

In this product, the first and third factors plainly belong to $[K, {}_{i-1} G]$, and the second to $\gamma_i X$. The fourth, meanwhile, is an element of $[K, \gamma_{i-1} G]$. But, by Lemma 3.5, this latter subgroup is contained in $[K, {}_{i-1} G]$. As a result, $[g, h] \in [K, {}_{i-1} G] \gamma_i X$. We have thus shown $\gamma_i G = [K, {}_{i-1} G] \gamma_i X$. \square

The first factors in the representations of $\gamma_i G$ and $G^{(i)}$ provided in Lemma 3.6 permit an alternative description, namely, as terms in the lower X -central series and X -derived series of K .

Lemma 3.7. Let G be a group such that $G = KX$, where $K \trianglelefteq G$ and $X \leq G$. Then, for all $i \geq 1$,

$$\gamma_i^X K = [K, {}_{i-1} G] \quad \text{and} \quad \delta_i^X K = [K, G, G', \dots, G^{(i-1)}].$$

Proof. The first equation follows immediately from the definition of $\gamma_i^X K$. For the second equation, we proceed by induction on i . We have $\delta_1^X K = \gamma_2^X K$, and, by the first equation, $\gamma_2^X K = [K, G]$. This establishes the case $i = 1$. Assume next that $i > 1$. Then

$$\delta_i^X K = \gamma_2^{X^{(i-1)}}(\delta_{i-1}^X K) = \gamma_2^{X^{(i-1)}}([K, G, G', \dots, G^{(i-2)}]).$$

Moreover, from the first equation, we obtain

$$\gamma_2^{X^{(i-1)}}([K, G, G', \dots, G^{(i-2)}]) = [K, G, G', \dots, G^{(i-2)}, H],$$

where $H = [K, G, G', \dots, G^{(i-2)}]X^{(i-1)}$. But, by Lemma 3.6, $H = G^{(i-1)}$, yielding the second equation. \square

Combining the preceding two lemmas gives rise to

Proposition 3.8. *Let G be a group such that $G = KX$, where $K \trianglelefteq G$ and $X \leq G$. Then statements (i) and (ii) below hold.*

(i) *G is nilpotent if and only if X is nilpotent and acts nilpotently on K . In this case,*

$$\text{nil } G = \max\{\text{nil } X, \text{nil}_X K\}.$$

(ii) *G is solvable if and only if X and K are solvable. In this case,*

$$\text{der}(G) = \max\{\text{der}(X), \text{der}_X(K)\}.$$

\square

3.2 Radicable nilpotent groups

In this subsection, we examine the attributes of radicable nilpotent groups that underpin the proof of Theorem 1.13. We start with three very basic, and undoubtedly well-known, properties.

Lemma 3.9. *The three statements below hold for any nilpotent group N .*

(i) *N is radicable if and only if N_{ab} is divisible.*

(ii) *If N is radicable, then $\gamma_i N$ is radicable for all $i \geq 1$.*

(iii) *If $K \trianglelefteq N$ such that K and N/K are radicable, then N is radicable.*

Proof. Assertions (i) and (ii) can be deduced using the epimorphisms from the tensor powers of N_{ab} to the factors in the lower central series of N induced by the iterated commutator maps. In addition, one requires the obvious fact that the property of radicableity is preserved by central extensions.

To prove (iii), we set $Q = N/K$ and consider the exact sequence $K_{\text{ab}} \rightarrow N_{\text{ab}} \rightarrow Q_{\text{ab}} \rightarrow 0$ of abelian groups. Since K_{ab} and Q_{ab} are divisible, so is N_{ab} . It follows, then, from (i) that N is radicable. \square

Next we show that the terms in the lower central G -series and derived G -series of a nilpotent radicable G -group are radicable.

Lemma 3.10. *Let G be a group and R a radicable nilpotent G -group. Then, for all $i \geq 1$, $\gamma_i^G R$ and $\delta_i^G R$ are radicable.*

Proof. We need only show that $\gamma_2^G R$ is radicable; the entire conclusion will then follow by induction. To begin with, observe $\gamma_2^G R/R' = (IG)(R_{\text{ab}})$, where IG denotes the augmentation ideal in $\mathbb{Z}G$. Since R_{ab} is divisible, $(IG)(R_{\text{ab}})$ is too. Moreover, Lemma 3.9(ii) implies that R' must be radicable. Therefore, by Lemma 3.9(iii), $\gamma_2^G R$ is radicable. \square

Lemma 3.10 gives rise to the following two observations about extensions.

Lemma 3.11. *Let G be a group and $1 \rightarrow F \rightarrow \bar{R} \rightarrow R \rightarrow 1$ an extension of G -groups such that F is finite and \bar{R} is nilpotent and radicable.*

- (i) *If G acts nilpotently on R , then G also acts nilpotently on \bar{R} and $\text{nil}_G \bar{R} = \text{nil}_G R$.*
- (ii) *If G acts solvably on R , then G also acts solvably on \bar{R} and $\text{der}_G(\bar{R}) = \text{der}_G(R)$.*

Proof. We just prove (i), the proof of (ii) being similar. Let $c = \text{nil}_G R$. Then $\gamma_{c+1}^G \bar{R}$ is finite. But $\gamma_{c+1}^G \bar{R}$ is radicable by Lemma 3.10. Therefore $\gamma_{c+1}^G \bar{R} = 1$. Hence $\text{nil}_G \bar{R} = c$. \square

The most important property of radicable nilpotent groups for our purposes is described in Proposition 3.12 below.

Proposition 3.12. *Let X be a group that acts upon a nilpotent radicable group R . Assume further that the torsion subgroup P of R is p -minimax for some prime p . Then there exist a nilpotent radicable X -group \bar{R} and an X -group epimorphism $\psi : \bar{R} \rightarrow R$ such that there is an X -group isomorphism $\nu : P \rightarrow \psi^{-1}(P)$ with $\psi\nu(u) = u^p$ for all $u \in P$. Furthermore, it follows that $\text{Ann}_{\mathbb{Z}X}(\bar{R}_{\text{ab}}) = \text{Ann}_{\mathbb{Z}X}(R_{\text{ab}})$.*

We prove Proposition 3.12 using the cohomological classification of group extensions. Our argument hinges on the following lemma concerning the cohomology of radicable nilpotent groups.

Lemma 3.13. *Let R be a radicable nilpotent group with torsion subgroup P . For any finite $\mathbb{Z}R$ -module A , $H^2(R, A)$ is mapped isomorphically onto $H^2(P, A)^R$ by the restriction homomorphism.*

Lemma 3.13 will be proved using the following fact derived from the LHS spectral sequence, which is a special case of [12, Theorem 2].

Lemma 3.14. *Let $1 \rightarrow K \rightarrow G \rightarrow Q \rightarrow 1$ be a group extension and A a $\mathbb{Z}G$ -module. Suppose that the cohomology groups $H^1(K, A)$, $H^2(Q, A^K)$, and $H^3(Q, A^K)$ are all trivial. Then $H^2(G, A)$ is mapped isomorphically onto $H^2(K, A)^G$ by the restriction homomorphism.* \square

Proof of Lemma 3.13. Notice first that R must act trivially on A . Hence, by Lemma 3.14, the conclusion will follow if we show $H^n(R/P, A) = 0$ for $n = 2, 3$. To verify this, we employ the universal coefficient exact sequence

$$0 \longrightarrow \text{Ext}_{\mathbb{Z}}^1(H_{n-1}(R/P, \mathbb{Z}), A) \longrightarrow H^n(R/P, A) \longrightarrow \text{Hom}_{\mathbb{Z}}(H_n(R/P, \mathbb{Z}), A) \longrightarrow 0 \quad (3.1)$$

for $n \geq 1$. Since R/P is torsion-free, nilpotent, and radicable, $H_n(R/P, \mathbb{Z})$ is torsion-free and divisible for $n \geq 1$ (see, for instance, [10, Proposition 4.8]). As a result, the second and fourth groups in sequence (3.1) are both trivial. Thus $H^n(R/P, A) = 0$ for $n \geq 1$. \square

In proving Proposition 3.12, we will also avail ourselves of the following simple cohomological property of radicable abelian groups.

Lemma 3.15. *Let p be a prime and R a radicable abelian group whose p -torsion subgroup is minimax. In addition, define $\rho : R \rightarrow R$ by $\rho(r) = r^p$ for every $r \in R$. Finally, set $A = \text{Ker } \rho$. Then the cohomology class of the group extension $1 \rightarrow A \rightarrow R \xrightarrow{\rho} R \rightarrow 1$ is fixed by the canonical action of $\text{Aut}(R)$ on $H^2(R, A)$.*

Proof. Let $\xi \in H^2(R, A)$ be the cohomology class of the extension $1 \rightarrow A \rightarrow R \xrightarrow{\rho} R \rightarrow 1$. Take $\alpha \in \text{Aut}(R)$. Then the diagram

$$\begin{array}{ccccccc} 1 & \longrightarrow & A & \longrightarrow & R & \xrightarrow{\rho} & R \longrightarrow 1 \\ & & \downarrow \alpha & & \downarrow \alpha & & \downarrow \alpha \\ 1 & \longrightarrow & A & \longrightarrow & R & \xrightarrow{\rho} & R \longrightarrow 1 \end{array}$$

commutes. As a result, we have $\alpha \cdot \xi = \xi$. \square

Finally, we require the group-theoretic lemma below.

Lemma 3.16. *Let G be a group with a finite normal subgroup F . Let $\phi : R \rightarrow G$ and $\psi : R \rightarrow G$ be homomorphisms where R is a radicable group. If $\phi(r)F = \psi(r)F$ for all $r \in R$, then $\phi = \psi$.*

Proof. Since $\psi(R)$ is radicable, it must centralize F . As a result, the map $r \mapsto \phi(r)\psi(r^{-1})$ defines a homomorphism $R \rightarrow F$. But all such homomorphisms are trivial, so that $\phi = \psi$. \square

Armed with the preceding lemmas, we embark on the proof of Proposition 3.12.

Proof of Proposition 3.12. Take A to be the subgroup of P consisting of all its elements of order dividing p . The group \bar{R} will be obtained by employing cohomology with coefficients in the $\mathbb{Z}R$ -module A . By Lemma 3.13, we know that the restriction map induces a natural isomorphism $\theta : H^2(R, A) \rightarrow H^2(P, A)^R$. Consider now the short exact sequence $1 \rightarrow A \rightarrow P \xrightarrow{\rho} P \rightarrow 1$, where $\rho(u) = u^p$ for all $u \in P$. Let $\xi \in H^2(P, A)$ be the cohomology class of this extension. Then Lemma 3.15 implies $\xi \in H^2(P, A)^R$. Thus there is a unique element ζ of $H^2(R, A)$ such that $\theta(\zeta) = \xi$. Regarding $H^2(P, A)$ as a $\mathbb{Z}X$ -module and invoking Lemma 3.15 again, we have $x \cdot \xi = \xi$ for all $x \in X$. It follows, then, from the naturality of θ that $x \cdot \zeta = \zeta$ for all $x \in X$.

Form a group extension $1 \rightarrow A \xrightarrow{\iota} \bar{R} \xrightarrow{\psi} R \rightarrow 1$ corresponding to $\zeta \in H^2(R, A)$. Then \bar{R} is plainly nilpotent. Moreover, because $\theta(\zeta) = \xi$, there is an isomorphism $\nu : P \rightarrow \psi^{-1}(P)$ such that the diagram

$$\begin{array}{ccccccc} 1 & \longrightarrow & A & \longrightarrow & P & \xrightarrow{\rho} & P \longrightarrow 1 \\ & & \parallel & & \downarrow \nu & & \parallel \\ 1 & \longrightarrow & A & \xrightarrow{\iota} & \psi^{-1}(P) & \xrightarrow{\psi} & P \longrightarrow 1 \end{array}$$

commutes. Thus $\psi\nu(u) = u^p$ for all $u \in P$. Also, we see that \bar{R} is an extension of P by R/P . Hence \bar{R} is radicable by Lemma 3.9(iii).

For each $x \in X$, let α_x and β_x be the automorphisms of A and R , respectively, that are induced by x . Since $x \cdot \zeta = \zeta$ for all $x \in X$, we can find, for each $x \in X$, an automorphism $\gamma_x : \bar{R} \rightarrow \bar{R}$ that renders the diagram

$$\begin{array}{ccccccc} 1 & \longrightarrow & A & \xrightarrow{\iota} & \bar{R} & \xrightarrow{\psi} & R \longrightarrow 1 \\ & & \downarrow \alpha_x & & \downarrow \gamma_x & & \downarrow \beta_x \\ 1 & \longrightarrow & A & \xrightarrow{\iota} & \bar{R} & \xrightarrow{\psi} & R \longrightarrow 1 \end{array}$$

commutative. Furthermore, in view of Lemma 3.16, γ_x must be the only automorphism of \bar{R} that makes this diagram commute. As a result, the assignment $x \mapsto \gamma_x$ defines a homomorphism from X to $\text{Aut}(\bar{R})$, thus equipping \bar{R} with an action of X . This action plainly makes ψ into an X -group homomorphism. We claim that the same holds for ν . To see this, notice that ν induces an X -group homomorphism $P/A \rightarrow \psi^{-1}(P)/\iota(A)$. Hence it is easy to deduce from Lemma 3.16 that ν is an X -group homomorphism.

To verify the assertion about the annihilators, let $\lambda \in \text{Ann}_{\mathbb{Z}X}(R_{\text{ab}})$. Consider now the $\mathbb{Z}X$ -module epimorphism $\bar{R}_{\text{ab}} \rightarrow R_{\text{ab}}$ induced by ψ . In light of Lemma 3.16, the fact that λ annihilates R_{ab} implies that λ must have the same effect on \bar{R}_{ab} . \square

3.3 Elementary properties of coverings

In this subsection, we collect several fundamental facts concerning coverings that are required for the proofs of our main results. We begin by making some elementary observations about extending a covering of a quotient to the entire group; the proofs of these are left to the reader.

Lemma 3.17. *Let $1 \rightarrow K \xrightarrow{\iota} G \xrightarrow{\epsilon} Q \rightarrow 1$ be a group extension. Suppose that there is a group Q^* equipped with an epimorphism $\psi : Q^* \rightarrow Q$. Let $G^* = G \times_Q Q^*$ and define the homomorphisms $\phi : G^* \rightarrow G$, $\iota^* : K \rightarrow G^*$, and $\epsilon^* : G^* \rightarrow Q^*$ as follows:*

$$\begin{aligned} \phi(g, q) &= g \text{ for all } (g, q) \in G^*; \\ \iota^*(k) &= (\iota(k), 1) \text{ for all } k \in K; \\ \epsilon^*(g, q) &= q \text{ for all } (g, q) \in G^*. \end{aligned}$$

Then the diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & K & \xrightarrow{\iota^*} & G^* & \xrightarrow{\epsilon^*} & Q^* & \longrightarrow & 1 \\ & & \parallel & & \downarrow \phi & & \downarrow \psi & & \\ 1 & \longrightarrow & K & \xrightarrow{\iota} & G & \xrightarrow{\epsilon} & Q & \longrightarrow & 1 \end{array}$$

commutes. In addition, the following three statements hold.

- (i) $\text{solv}(G^*) = \text{solv}(G) \times_Q \text{solv}(Q^*)$.
- (ii) $\text{Fitt}(G^*) = \text{Fitt}(G) \times_Q \text{Fitt}(Q^*)$.
- (iii) *If N is a nilpotent subgroup of G of class c such that $\psi^{-1}(\epsilon(N))$ is nilpotent of class $\leq c$, then $\phi^{-1}(N)$ is nilpotent of class c .* \square

We now briefly touch on coverings of finite groups.

Lemma 3.18. *Let G be a finite group with a normal nilpotent subgroup N such that $\text{nil } N = c$ and $\text{der}(N) = d$. Then there is a torsion-free, virtually polycyclic group G^* and an epimorphism $\phi : G^* \rightarrow G$ such that*

$$\text{nil } \phi^{-1}(N) = \max\{c, 1\} \text{ and } \text{der}(\phi^{-1}(N)) = \max\{d, 1\}.$$

Proof. Put $c' = \max\{c, 1\}$ and $d' = \max\{d, 1\}$. Let F be a finitely generated free group admitting an epimorphism $\theta : F \rightarrow G$. Define $K = \phi^{-1}(N)$ and $L = K^{(d')} \gamma_{c'+1} K$. Notice $L \leq \text{Ker } \theta$. Moreover, according to [26, Theorem 3], K/L is torsion-free. Also, F/K' is torsion-free by virtue of [9, Theorem 2]. It follows, then, that F/L is torsion-free. Hence F/L is a cover of G that fulfills our requirements. \square

We finish our preliminary discussion of coverings by proving a lemma describing a well-known way to cover a finite $\mathbb{Z}G$ -module, followed by a corollary stating that every $\mathbb{Z}G$ -module in \mathfrak{M}_1^π has a $\mathbb{Z}G$ -module cover that is torsion-free and π -minimax.

Lemma 3.19. *Let G be a group and A a finite $\mathbb{Z}G$ -module. Then there is a $\mathbb{Z}G$ -module A^* and an $\mathbb{Z}G$ -module epimorphism $\phi : A^* \rightarrow A$ such that A^* is torsion-free and polycyclic as an abelian group.*

Proof. Let A^* be the free abelian group on the set of elements of A , and let $\phi^* : A^* \rightarrow A$ be the group epimorphism that maps each of these generators to itself. There is, then, a unique $\mathbb{Z}G$ -module structure on A^* that makes ϕ into an $\mathbb{Z}G$ -module epimorphism. \square

Corollary 3.20. *Let G be a group and A a $\mathbb{Z}G$ -module whose underlying additive group is virtually torsion-free and π -minimax. Then there is a $\mathbb{Z}G$ -module A^* and a $\mathbb{Z}G$ -module epimorphism $\phi : A^* \rightarrow A$ such that A^* is torsion-free and π -minimax as an abelian group.*

Proof. Let A_0 be a $\mathbb{Z}G$ -submodule of A such that A/A_0 is finite and A_0 is torsion-free *qua* abelian group. According to Lemma 3.19, A/A_0 can be covered by a $\mathbb{Z}G$ -module B whose additive group is torsion-free and polycyclic. Hence, letting $A^* = A \times_{A/A_0} B$ establishes the conclusion. \square

3.4 π -Integral actions

We conclude §3 by investigating π -integral actions (see Definition 1.10), focusing on their importance for identifying π -minimax groups (Propositions 3.25 and 3.28). The most elementary aspect of the connection between π -integrality and the π -minimax property is captured in our first lemma.

Lemma 3.21. *Let π be a set of primes and G a group. If A is a $\mathbb{Z}G$ -module that is virtually torsion-free and π -minimax as an abelian group, then G acts π -integrally on A .*

Proof. In view of Corollary 3.20, we can suppose that A is torsion-free as an abelian group. Take $\alpha : A \rightarrow A$ to be an automorphism of A *qua* abelian group arising from the action of an element of G . This automorphism induces an automorphism α_π of $A \otimes \mathbb{Z}[\pi^{-1}]$. Moreover, α_π is a root of a monic polynomial with coefficients in $\mathbb{Z}[\pi^{-1}]$. Multiplying by a large enough integer yields a polynomial $f(t) \in \mathbb{Z}[t]$ such that $f(\alpha_\pi) = 0$ and the leading coefficient of $f(t)$ is a π -number. It follows, then, that $f(\alpha) = 0$. Therefore G acts π -integrally on A . \square

Lemma 3.21 allows us to establish Proposition 1.11.

Proof of Proposition 1.11. Statement (b) is plainly true since every quotient of a π -minimax group is π -minimax. To show (a) and (c), let $\phi : G^* \rightarrow G$ be an epimorphism where G^* is an \mathfrak{M}_1^π -group. Put $N^* = \text{Fitt}(G^*)$ and $c = \text{nil } N^*$. By Proposition 1.3(ii), G^*/N^* is finitely generated. Since $\phi(N^*) \leq N$, it follows that statement (a) holds.

Now we establish assertion (c). That N^* is virtually torsion-free implies that $Z_i(N^*)/Z_{i-1}(N^*)$ is virtually torsion-free for $i \geq 1$. According to Lemma 3.21, this means that, for $i = 1, \dots, c$, G acts π -integrally on $Z_i(N^*)/Z_{i-1}(N^*)$. Taking the images of the subgroups $Z_i(N^*)$ under the composition $N^* \xrightarrow{\phi} N \rightarrow N_{\text{ab}}$ yields a chain $0 = A_0 \subseteq A_1 \subseteq \dots \subseteq A_c$ of $\mathbb{Z}G$ -submodules of N_{ab} . Because Q acts π -integrally on each factor in this chain, the action of Q on A_c must be π -integral. Moreover, N_{ab}/A_c is polycyclic since G^*/N^* is polycyclic. Hence G acts \emptyset -integrally on N_{ab}/A_c and therefore π -integrally on N_{ab} . \square

Next we prove that the π -integrality property is inherited by tensor products.

Lemma 3.22. *Let π be a set of primes and G a group. If G acts π -integrally on the $\mathbb{Z}G$ -modules A and B , then G acts π -integrally on $A \otimes B$, viewed as a $\mathbb{Z}G$ -module via the diagonal action.*

The above lemma follows readily from

Lemma 3.23. *Let A and B be abelian groups and π a set of primes. Let $\phi \in \text{End}(A)$ and $\psi \in \text{End}(B)$. Suppose that there are polynomials $f(t), g(t) \in \mathbb{Z}[t]$ such that $f(\phi) = 0$, $g(\psi) = 0$, and the leading coefficients of $f(t)$ and $g(t)$ are π -numbers. Then there is a polynomial $F(t) \in \mathbb{Z}[t]$ such that the leading coefficient of $F(t)$ is a π -number and $F(\phi \otimes \psi) = 0$.*

Proof. Let θ be the ring homomorphism from the polynomial ring $\mathbb{Z}[s, t]$ to the ring $\text{End}(A \otimes B)$ such that $\theta(s) = \phi \otimes 1_B$ and $\theta(t) = 1_A \otimes \psi$. Define I to be the ideal of $\mathbb{Z}[s, t]$ generated by $f(s)$ and $g(t)$. Then $\theta(I) = 0$. Put $R = \mathbb{Z}[\pi^{-1}]$, and take J to be the ideal in the polynomial ring $R[s, t]$ generated by $f(s)$ and $g(t)$. The images of s and t in $R[s, t]/J$ are both integral over R . Hence the same is true for the image of st in $R[s, t]/J$. In other words, there is a monic polynomial F' over R such that $F'(st) = p(s, t)f(s) + q(s, t)g(t)$, where $p(s, t)$ and $q(s, t)$ are polynomials in $R[s, t]$. Multiplying by a large enough π -number, we acquire a polynomial F over \mathbb{Z} whose leading coefficient is a π -number and such that $F(st) \in I$. It follows, then, that $F(\phi \otimes \psi) = 0$. \square

Lemma 3.22 gives rise to the properties below.

Lemma 3.24. *Let π be a set of primes and G a group. Let N be a G -group such that G acts π -integrally on N_{ab} . Then the following two statements are true.*

- (i) *The group G acts π -integrally on $\gamma_i N / \gamma_{i+1} N$ for $i \geq 1$.*
- (ii) *If N is nilpotent, then, for every G -subgroup M of N , G acts π -integrally on M_{ab} .*

Proof. Assertion (i) follows from Lemma 3.22 and the fact that the iterated commutator map induces a $\mathbb{Z}G$ -module epimorphism from the i th tensor power of N_{ab} to $\gamma_i N / \gamma_{i+1} N$.

To prove (ii), set $c = \text{nil } N$. Then the $\mathbb{Z}G$ -module M_{ab} has a series $0 = M_{c+1} \subseteq M_c \subseteq \dots \subseteq M_1 = M_{\text{ab}}$ of submodules such that, for $1 \leq i \leq c$, M_i / M_{i+1} is isomorphic to a $\mathbb{Z}G$ -module section of $\gamma_i N / \gamma_{i+1} N$. It follows, then, from (i) that G acts π -integrally on each factor M_i / M_{i+1} . Therefore G acts π -integrally on M_{ab} . \square

In the next proposition, we examine a group-theoretic application of π -integrality.

Proposition 3.25. *Let π be a set of primes and G a group with a normal subgroup N such that the following conditions are satisfied.*

- (i) *N is nilpotent and virtually torsion-free.*
- (ii) *G/N is virtually polycyclic.*
- (iii) *G acts π -integrally on N_{ab} .*
- (iv) *N is generated as a G -group by a π -minimax subgroup.*

Then G must be π -minimax.

The first step in proving Proposition 3.25 is the following lemma.

Lemma 3.26. *Let π be a set of primes and G a polycyclic group. Let A be a $\mathbb{Z}G$ -module that is generated as a $\mathbb{Z}G$ -module by a π -minimax additive subgroup. Assume further that the action of G on A is π -integral. Then there is a finite subset π_0 of π such that the underlying additive group of A is an extension of a π_0 -torsion group by a π -minimax one.* \square

Proof. Our strategy is to induct on the length r of a series $1 = G_0 \triangleleft G_1 \triangleleft \cdots \triangleleft G_r = G$ in which each factor G_i/G_{i-1} is cyclic. If $r = 0$, then A is plainly π -minimax. Suppose $r > 0$, and let B be a π -minimax subgroup of A that generates A as a $\mathbb{Z}G$ -module. Also, take $g \in G$ such that gG_{r-1} is a generator of G/G_{r-1} . In addition, let C be the $\mathbb{Z}G_{r-1}$ -submodule of A generated by B . Notice that, in view of the inductive hypothesis, the underlying abelian group of C is $(\pi_1\text{-torsion})\text{-by-}(\pi\text{-minimax})$ for some finite subset π_1 of π .

Let $\alpha_0, \alpha_1, \dots, \alpha_m \in \mathbb{Z}$ such that α_0 and α_m are π -numbers and

$$(\alpha_0 + \alpha_1 g + \cdots + \alpha_m g^m) \cdot a = 0 \quad (3.2)$$

for all $a \in A$. Furthermore, let π_0 be the union of the following three sets: $\text{spec}(C)$, π_1 , and the set of prime divisors of $\alpha_0 \alpha_m$. Then π_0 is a finite subset of π . Next set $U = A \otimes \mathbb{Z}[\pi_0^{-1}]$ and $V = C \otimes \mathbb{Z}[\pi_0^{-1}]$. Observe that the kernel of the canonical homomorphism $A \rightarrow U$ is a

π_0 -torsion group, and that V is π -minimax. Since $U = \sum_{i=-\infty}^{\infty} g^i \cdot V$, equation (3.2) implies $U = \sum_{i=0}^{m-1} g^i \cdot V$. Therefore U is π -minimax, yielding the conclusion of the lemma. \square

Proof of Proposition 3.25. Let G_0 be a subgroup of finite index in G such that $N \leq G_0$ and G_0/N is polycyclic. Then N is generated as a G_0 -group by finitely many π -minimax subgroups. Therefore N_{ab} is generated as a $\mathbb{Z}G_0$ -module by a single π -minimax subgroup. Hence Lemma 3.26 implies that there is a finite subset π_0 of π such that N_{ab} is an extension of a π_0 -torsion group by one that is π -minimax. As a result, any tensor power of N_{ab} is also $(\pi_0\text{-torsion})\text{-by-}(\pi\text{-minimax})$. The canonical epimorphism from the i th tensor power of N_{ab} to $\gamma_i N / \gamma_{i+1} N$, then, yields that $\gamma_i N / \gamma_{i+1} N$ is an extension of the same form. Therefore, appealing to Lemma 3.27 below, we can argue by induction on j that $\gamma_{c-j} N$ is π -minimax for all $j \geq 0$, where $c = \text{nil } N$. In particular, N is π -minimax, and so G is π -minimax. \square

It remains to prove the elementary lemma below, which will also find later use.

Lemma 3.27. *Let π be a finite set of primes. Let N be a virtually torsion-free nilpotent group. Suppose further that N contains a normal π -minimax subgroup M such that N/M is π -torsion. Then N is π -minimax.*

Proof. We induct on $\text{nil } N$. Assume $\text{nil } N = 1$. Then $N \otimes \mathbb{Z}[\pi^{-1}] \cong M \otimes \mathbb{Z}[\pi^{-1}]$, so that $N \otimes \mathbb{Z}[\pi^{-1}]$ is π -minimax. Hence N is π -minimax. Suppose $\text{nil } N > 1$, and let $Z = Z(N)$. Then N/Z and Z are π -minimax by virtue of the inductive hypothesis. Thus N is π -minimax. \square

The following consequence of Proposition 3.25 will be invoked in the proof of Theorem 1.13.

Proposition 3.28. *Let π be a set of primes and G a group with a normal subgroup N such that the following conditions are satisfied.*

- (i) N is nilpotent and virtually torsion-free.
- (ii) G/N is virtually polycyclic.
- (iii) G acts π -integrally on N_{ab} .

If H is a π -minimax subgroup of N and V is a finitely generated subgroup of G , then $\langle H, V \rangle$ is π -minimax.

For the proof of Proposition 3.28, we need the following simple observation.

Lemma 3.29. *Let π be a set of primes and N a nilpotent group. If H and K are π -minimax subgroups of N , then $\langle H, K \rangle$ is π -minimax.*

Proof. Set $L = \langle H, K \rangle$. Then L_{ab} is generated by the images of H and K . Thus L_{ab} is π -minimax, which implies that L is π -minimax. \square

Proof of Proposition 3.28. Put $U = \langle H, V \rangle$. We will describe U in a form that permits us to apply Proposition 3.25. First we observe that, as a virtually polycyclic group, the quotient $V/(N \cap V)$ is finitely presented. Hence $N \cap V$ is generated as a V -group by a finite set S . Take L to be the subgroup of N generated by the set $H \cup S$. Then Lemma 3.29 implies that L is π -minimax. Next define K to be the V -subgroup of N generated, as a V -group, by L . Then $U = KV$. Hence, because $N \cap V \leq K$, the quotient U/K is virtually polycyclic. In addition, Lemma 3.24(ii) yields that U acts π -integrally on K_{ab} . Consequently, U is seen to be π -minimax by invoking Proposition 3.25. \square

4 Preliminary results about topological groups

As explained in §2.1, the proof of Theorem 1.13 will utilize a dense embedding of a nilpotent \mathfrak{M} -group in a topological group belonging to the class \mathfrak{N}_p , defined below. In the present section, we establish the properties of groups in this class required for the proof of Theorem 1.13. The culmination of this discussion are two factorization results for topological groups with a normal \mathfrak{N}_p -subgroup (Propositions 4.15 and 4.20).

4.1 Introducing the class \mathfrak{N}_p

Definition 4.1. For any prime p , we define \mathfrak{N}_p to be the class of all nilpotent topological groups N that have a series

$$1 = N_0 \trianglelefteq N_1 \trianglelefteq \cdots \trianglelefteq N_r = N \quad (4.1)$$

such that, for every $i \geq 1$, N_i/N_{i-1} is of one of the following two types with respect to the quotient topology:

- (i) a cyclic pro- p group; or
- (ii) a discrete quasicyclic p -group.

Notice that the compact groups in \mathfrak{N}_p are exactly the polycyclic pro- p groups. Also, every discrete \mathfrak{N}_p -group is a Černikov p -group (see Proposition 1.3(iii)). Some more properties of \mathfrak{N}_p -groups are described in Lemma 4.2.

Lemma 4.2. *Let p be a prime and N a topological group in \mathfrak{N}_p . Then the following eight statements are true.*

- (i) N is locally compact, totally disconnected, σ -compact, and locally elliptic.

- (ii) Every open subgroup of N belongs to \mathfrak{N}_p .
- (iii) If U is an open normal subgroup of N , then N/U belongs to \mathfrak{N}_p .
- (iv) Every compact subgroup of N belongs to \mathfrak{N}_p .
- (v) Every closed subgroup of N belongs to \mathfrak{N}_p .
- (vi) If M is a closed normal subgroup of N , then N/M belongs to \mathfrak{N}_p .
- (vii) If G is a totally disconnected, locally compact group and $\phi : N \rightarrow G$ is a continuous homomorphism, then $\text{Im } \phi$ is closed in G .
- (viii) If H and K are closed subgroups of N with $K \trianglelefteq N$, then HK is also closed.

Proof. To show (i), observe that cyclic pro- p groups and discrete quasicyclic p -groups enjoy these four properties. Moreover, these properties are preserved by extensions of locally compact groups; for the first three, see [28, Theorem 6.15], and for the last, see [22, Theorem 2] or [4, Proposition 4.D.6(2)]. It follows, then, that (i) is true.

Statements (ii), (iii), and (iv) plainly hold if N is a cyclic pro- p group or a discrete quasicyclic p -group. The general cases can then be proved by inducting on the length of the series (4.1) and employing the fact that images under quotient maps of open (compact) sets are open (compact).

To prove (v) and (vi), we fix a compact open subgroup C of N , and, exploiting the subnormality of C , choose a subgroup series

$$C = M_0 \trianglelefteq M_1 \trianglelefteq \cdots \trianglelefteq M_s = N.$$

Because C is open, so are the other subgroups in this series. Thus, by (ii), each M_i is a member of \mathfrak{N}_p . Also, (iii) implies that M_i/M_{i-1} is a Černikov p -group for $0 \leq i \leq s$. Taking H to be a closed subgroup of N , consider the series

$$H \cap C = (H \cap M_0) \trianglelefteq (H \cap M_1) \trianglelefteq \cdots \trianglelefteq (H \cap M_s) = H,$$

in which every subgroup is open in H and each quotient $(H \cap M_i)/(H \cap M_{i-1})$ is a Černikov p -group. Since $H \cap C$ is compact, it is a polycyclic pro- p group. It follows, then, that H belongs to \mathfrak{N}_p . This proves (v).

For (vi), let $\epsilon : N \rightarrow N/M$ be the quotient map. The class of polycyclic pro- p groups is closed under forming continuous Hausdorff homomorphic images. Thus $\epsilon(C)$ is a polycyclic pro- p group. Furthermore, $\epsilon(M_i)$ is an open subgroup of N/M for $0 \leq i \leq s$. In addition, each quotient $\epsilon(M_i)/\epsilon(M_{i-1})$ is a Černikov p -group. Therefore N/M is a member of \mathfrak{N}_p .

To prove (vii), we induct on the length of the series (4.1). Since G is Hausdorff, the statement is true for $r = 0$. Suppose $r > 0$, and let $L = N_{r-1}$. Then $\phi(L)$ is closed in G . Let $A = \overline{\phi(N)}/\phi(L)$, and let $\psi : N/L \rightarrow A$ be the homomorphism induced by ϕ . Our goal is to prove that $\text{Im } \psi$ is closed in A , which will imply the desired conclusion. If N/L is compact, this is immediate. Assume, then, that N/L is not compact, which means that it is quasicyclic. We will argue now that A must be discrete. Suppose that this is not true. Then A has an infinite compact open subgroup B . Since $\text{Im } \psi$ is dense in A , the subgroup $B \cap \text{Im } \psi$ must be infinite. Consequently, $\psi^{-1}(B)$ is infinite and hence equal to N/L . But this is impossible because B is profinite and there are no nontrivial homomorphisms from a quasicyclic group to a profinite group. Therefore A must be discrete, so that $\text{Im } \psi$ is closed in A .

Lastly, we establish assertion (viii). According to (v), H belongs to \mathfrak{N}_p . Hence (vii) implies that HK/K is closed in N/K . Therefore HK is closed in N . \square

Acknowledgement. Property (vii) in Lemma 4.2 and its proof were provided to the authors by an anonymous referee.

Lemma 4.2(viii) leads to a version of the Schreier refinement theorem for \mathfrak{N}_p -groups.

Lemma 4.3. *Let p be a prime. In an \mathfrak{N}_p -group, any two series of closed subgroups have isomorphic refinements that also consist of closed subgroups.*

Proof. We can prove the lemma with the same reasoning employed in [24] to establish Schreier's famous result. The argument there relies on the "Zassenhaus lemma" to construct the refinements. For our purposes, all that we need to verify is that the subgroups obtained in this manner are closed. This property, however, is ensured by Lemma 4.2(viii). \square

Lemma 4.3 has the following consequence.

Corollary 4.4. *Let p be a prime, and let N be a topological group in the class \mathfrak{N}_p . In any series in N whose factors are all cyclic pro- p groups or quasicyclic p -groups, the number of infinite pro- p factors and the number of infinite factors are both invariants.*

Proof. Forming a refinement of such a series with closed subgroups fails to affect the number of factors that are isomorphic to \mathbb{Z}_p , as well as the number of infinite factors. Hence the conclusion follows immediately from the above lemma. \square

Corollary 4.4 permits us to make the following definition.

Definition 4.5. Let N be a group in \mathfrak{N}_p . The p -Hirsch length of N , denoted $h_p(N)$, is the number of factors isomorphic to \mathbb{Z}_p in any series of finite length in which each factor is either a cyclic pro- p group or a quasicyclic group.

In our next lemma, we examine subgroups of finite index in \mathfrak{N}_p -groups.

Lemma 4.6. *Let p be a prime and N an \mathfrak{N}_p -group. Then the following four statements hold.*

- (i) *Every subgroup of finite index in N is open.*
- (ii) *For each natural number m , N possesses only finitely many subgroups of index m .*
- (iii) *$R(N)$ is closed and $N/R(N)$ compact.*
- (iv) *$R(N)$ is radicable.*

Proof. It is straightforward to see that the property that every subgroup of finite index is open is preserved by extensions of locally compact groups. Thus, since this property holds for any cyclic pro- p group and any quasicyclic p -group, assertion (i) must be true.

Next we jump ahead to prove assertion (iii). First we observe that (i) implies that the profinite completion map $c^N : N \rightarrow \hat{N}$ is continuous. Hence (iii) will follow if we show that c^N is surjective. Since the profinite completion functor is right-exact, the property of having a surjective profinite completion map is preserved by extensions. Moreover, this property holds for both cyclic pro- p groups and quasicyclic p -groups. Therefore c^N is surjective.

Now we prove (ii). Since $c^N : N \rightarrow \hat{N}$ is surjective and continuous, \hat{N} must be a polycyclic pro- p group and thus finitely generated as a topological group. As a consequence, the set $\text{Hom}(N, F)$ is finite for any finite group F . Statement (ii), then, follows.

We establish assertion (iv) by demonstrating that $R := R(N)$ fails to have any proper subgroups of finite index. This will then imply that R_{ab} must be a divisible abelian group, which will allow us to conclude from Lemma 3.9(i) that R is radicable. Let H be a subgroup of R with finite index. According to statement (ii), H has only finitely many conjugates in N . If we form the intersection of these conjugates, we obtain a normal subgroup K of N such that $K \leq H$ and $[R : K] < \infty$. This means that N/K is a polycyclic pro- p group. Hence $R \leq K$, so that $R = H$. Therefore R has no proper subgroups of finite index. \square

4.2 Abelian \mathfrak{N}_p -groups

We now focus our attention on the abelian groups in \mathfrak{N}_p . First we prove that they are topological \mathbb{Z}_p -modules.

Lemma 4.7. *Let p be a prime and A an abelian group in \mathfrak{N}_p .*

- (i) *There is a unique topological \mathbb{Z}_p -module structure on A that extends its \mathbb{Z} -module structure.*
- (ii) *A subgroup B of A is closed if and only if it is a \mathbb{Z}_p -submodule.*
- (iii) *If B is an abelian \mathfrak{N}_p -group, then a map $\phi : A \rightarrow B$ is a continuous homomorphism if and only if it is a \mathbb{Z}_p -module homomorphism.*

Proof. Being σ -compact and locally elliptic, A is isomorphic as a topological group to the inductive limit of its compact open subgroups. If C is a compact open subgroup of A , then C is a pro- p group. Hence there is a unique continuous map $\mathbb{Z}_p \times C \rightarrow C$ that renders C a \mathbb{Z}_p -module and restricts to the map $(m, c) \mapsto mc$ from $\mathbb{Z} \times C$ to C . As a result, there is a unique continuous map $\mathbb{Z}_p \times A \rightarrow A$ that imparts a \mathbb{Z}_p -module structure to A and restricts to the integer-multiplication map $\mathbb{Z} \times A \rightarrow A$. This proves statement (i). Viewing A as an inductive limit in this fashion also permits us to deduce assertions (ii) and (iii) from the compact cases. \square

Lemmas 4.6 and 4.7 allow us to completely describe the structure of abelian \mathfrak{N}_p -groups.

Proposition 4.8. *For any prime p , a topological abelian group A belongs to the class \mathfrak{N}_p if and only if A is a direct sum of finitely many groups of the following four types:*

- (i) *a finite cyclic p -group;*
- (ii) *a quasicyclic p -group;*
- (iii) *a copy of \mathbb{Z}_p ;*
- (iv) *a copy of \mathbb{Q}_p .*

Proof. Let $D = R(A)$ and $C = A/D$. Since D is an injective \mathbb{Z}_p -module, we have $A \cong D \oplus C$ as \mathbb{Z}_p -modules. Invoking the classification of injective modules over a principal ideal domain, we observe that D can be expressed as a direct sum of copies of \mathbb{Q}_p and $\mathbb{Z}(p^\infty)$. Moreover, since D belongs to \mathfrak{N}_p , we can ascertain from Corollary 4.4 that this direct sum can only involve finitely many summands. Also, according to Lemma 4.6(iii), C is a finitely generated \mathbb{Z}_p -module, which means that it can be expressed as a direct sum of finitely many modules of types (i) and (iii). This completes the proof of the proposition. \square

Remark 4.9. Proposition 4.8 also follows from [3, Lemma 7.11].

Our understanding of abelian \mathfrak{N}_p -groups allows us to examine the lower central series of an \mathfrak{N}_p -group.

Lemma 4.10. *Let p be a prime. If N is a member of the class \mathfrak{N}_p , then the two properties below hold.*

- (i) $\gamma_i N$ is a closed subgroup of N for every $i \geq 1$.
- (ii) The iterated commutator map induces a \mathbb{Z}_p -module epimorphism

$$\theta_i : \underbrace{N_{\text{ab}} \otimes_{\mathbb{Z}_p} \cdots \otimes_{\mathbb{Z}_p} N_{\text{ab}}}_i \rightarrow \gamma_i N / \gamma_{i+1} N$$

for each $i \geq 1$.

Proof. We will dispose of (i) and (ii) with a single argument. If $i \geq 0$, forming iterated commutators of weight i defines a continuous function

$$f_i : \underbrace{N/\overline{N'} \times \cdots \times N/\overline{N'}}_i \rightarrow \overline{\gamma_i N} / \overline{\gamma_{i+1} N}$$

that is \mathbb{Z} -linear in each component. As f_i is thus also \mathbb{Z}_p -linear in each component, it gives rise to a \mathbb{Z}_p -module homomorphism

$$\theta_i : \underbrace{N/\overline{N'} \otimes_{\mathbb{Z}_p} \cdots \otimes_{\mathbb{Z}_p} N/\overline{N'}}_i \rightarrow \overline{\gamma_i N} / \overline{\gamma_{i+1} N}.$$

By Lemma 4.7(ii), $\text{Im } \theta_i$ is closed in $\overline{\gamma_i N} / \overline{\gamma_{i+1} N}$. Therefore, since $\text{Im } \theta_i = \gamma_i N / \gamma_{i+1} N$, the subgroup $\gamma_i N$ must be closed in N . \square

4.3 Extensions of finite p -groups by \mathfrak{N}_p -groups

For the proof of Theorem 1.13, we will require the closure property of \mathfrak{N}_p described in our next lemma.

Lemma 4.11. *Let p be a prime and N a topological group in \mathfrak{N}_p . Also, let F be a finite p -group. If $1 \rightarrow F \rightarrow E \xrightarrow{\epsilon} N \rightarrow 1$ is an abstract group extension with E nilpotent, then there is a unique topology on E that makes it into a member of \mathfrak{N}_p and $\epsilon : E \rightarrow N$ a quotient map.*

Proof. In view of Lemma 4.2(i), there is a family $\{H_i : i \in \mathbb{N}\}$ of compact open subgroups of N whose union is N such that, for any $i, j \in \mathbb{N}$, there is a $k \in \mathbb{N}$ with $H_i \cup H_j \subseteq H_k$. For each i , we endow $\epsilon^{-1}(H_i)$ with its pro- p topology. According to Lemma 4.6(i), every subgroup of finite index in H_i is open. Thus the map $\epsilon^{-1}(H_i) \rightarrow H_i$ induced by ϵ is a quotient map. Now we construct a topology on E by defining $U \subseteq E$ to be open if and only if $U \cap \epsilon^{-1}(H_i)$ is open in $\epsilon^{-1}(H_i)$ for all $i \in \mathbb{N}$. If $i, j \in \mathbb{N}$, then $H_i \cap H_j$ is open in H_j and thus has finite index in H_j . Thus $\epsilon^{-1}(H_i) \cap \epsilon^{-1}(H_j)$ has finite index in $\epsilon^{-1}(H_j)$, and so $\epsilon^{-1}(H_i) \cap \epsilon^{-1}(H_j)$ is open in $\epsilon^{-1}(H_j)$. As this holds for every $j \in \mathbb{N}$, $\epsilon^{-1}(H_i)$ is open in E for all $i \in \mathbb{N}$. Since E is the union of the $\epsilon^{-1}(H_i)$, it follows from Lemma 4.12 below that E is a topological group with respect to the topology we have defined. In addition, ϵ is a quotient map, implying that F is closed.

According to Lemma 4.13 below, $\epsilon^{-1}(H_i)$ is residually finite; that is, it is Hausdorff with respect to its pro- p topology. Thus E must be Hausdorff, which means that E induces the discrete topology on F . As a result, a series like (4.1) in N extends to a series of the same type in E , and so E belongs to \mathfrak{N}_p . This, then, establishes the existence of a topology on E with the desired properties.

To verify the uniqueness of the topology described above, suppose that \mathcal{T} is a topology on E making E into a Hausdorff topological group and $\epsilon : E \rightarrow N$ a quotient map. Then, for any $i \in \mathbb{N}$, \mathcal{T} induces a topology on $\epsilon^{-1}(H_i)$ that renders it a polycyclic pro- p group. Thus the topology induced on $\epsilon^{-1}(H_i)$ by \mathcal{T} will coincide with its full pro- p topology. This means that \mathcal{T} is identical with the topology defined above. \square

To complete the above argument, it remains to prove the following two lemmas, the first of which will be invoked again in the next section.

Lemma 4.12. *Let G be an abstract group that is also a topological space. Assume that G is a union of a family of subgroups \mathcal{H} that are all open in the topology on G and such that, for any pair $H, K \in \mathcal{H}$, there is a subgroup $L \in \mathcal{H}$ with $H, K \leq L$. Suppose further that each subgroup in \mathcal{H} is a topological group with respect to this topology. Then G is a topological group.*

Proof. To show that the multiplication map $G \times G \rightarrow G$ is continuous, let $x \in G$ and U_x be an open neighborhood of x . Also, take $g, h \in G$ such that $gh = x$. Then we can find a subgroup $H \in \mathcal{H}$ such that $g, h, x \in H$. Since H is a topological group, there are open subsets U_g and U_h of H such that $g \in U_g$, $h \in U_h$, and $U_g U_h \subseteq U_x$. Also, because H is open in G , so are U_g and U_h . This establishes the continuity of the multiplication map $G \times G \rightarrow G$. Moreover, the continuity of the inversion map $G \rightarrow G$ may be deduced by a similar argument. It follows, then, that G is a topological group. \square

Lemma 4.13. *Let $1 \rightarrow F \rightarrow G \rightarrow Q \rightarrow 1$ be an abstract group extension such that F is finite and Q is a polycyclic pro- p group. Then G is residually finite.*

Proof. In this argument, we employ the notation $H_{\text{Gal}}^*(\Gamma, A)$ for the Galois cohomology of a profinite group Γ with coefficients in a discrete Γ -module A . Taking A to be an arbitrary finite $\mathbb{Z}Q$ -module, consider the canonical maps

$$H_{\text{Gal}}^*(\hat{Q}, A) \longrightarrow H^*(\hat{Q}, A) \longrightarrow H^*(Q, A), \quad (4.2)$$

where the second and third groups are ordinary (discrete) cohomology groups. Since every subgroup of finite index in Q is open, the map $c^Q : Q \rightarrow \hat{Q}$ is an isomorphism. Hence the second map in (4.2) is an isomorphism. However, by [6, Theorem 2.10], the first map in (4.2) is also an isomorphism. As a consequence, the composition is an isomorphism; in other words, Q is cohomologically “good” in the sense of J-P. Serre [25, Exercises 1&2, Chapter I.§2]. According to these exercises, this means that $1 \rightarrow F \rightarrow \hat{G} \rightarrow \hat{Q} \rightarrow 1$ is an exact sequence of profinite groups. Therefore the residual finiteness of Q implies that G must be residually finite. \square

Our interest is primarily in the following extension of Lemma 4.11.

Corollary 4.14. *Let p be a prime and G a topological group with an open normal \mathfrak{N}_p -subgroup N . Let F be a finite p -group. If $1 \rightarrow F \rightarrow E \xrightarrow{\epsilon} G \rightarrow 1$ is an abstract group*

extension with $\epsilon^{-1}(N)$ nilpotent, then there is a unique topology on E that makes it into a topological group such that $\epsilon : E \rightarrow G$ is a quotient map and $\epsilon^{-1}(N)$ is an open \mathfrak{N}_p -subgroup.

Proof. Lemma 4.11 enables us to equip $M := \epsilon^{-1}(N)$ with a unique topology that makes it a member of \mathfrak{N}_p and the map $M \rightarrow N$ induced by ϵ a quotient map. Our plan is to show that the automorphisms of M induced by conjugation in E are continuous. We will then be able to endow E with a topology enjoying the properties sought. Moreover, the uniqueness of this topology will follow from the uniqueness of the topology on M .

Let α be an automorphism of M arising from conjugation by an element of E . To verify that α is continuous, let H be an open subgroup of M and $\bar{\alpha}$ the automorphism of M/F induced by α . Since $\epsilon(H)$ is open in N , $F\alpha^{-1}(H) = \epsilon^{-1}(\bar{\alpha}^{-1}(\epsilon(H)))$ is open in M . Moreover, $\alpha^{-1}(H)$ has finite index in $F\alpha^{-1}(H)$, which means that $\alpha^{-1}(H)$ is open in $F\alpha^{-1}(H)$ by Lemma 4.6(i). Thus $\alpha^{-1}(H)$ is open in M . Therefore α is continuous, which completes the proof. \square

4.4 Factoring certain topological groups

The main result of this section is Proposition 4.15, which, next to Proposition 3.12, is the most important of the preliminary propositions required for the proof of Theorem 1.13.

Proposition 4.15. *Let p be a prime and G a topological group containing an open normal \mathfrak{N}_p -subgroup N such that G/N is finitely generated and virtually abelian. Then G possesses a subgroup X such that $G = R(N)X$ and $N \cap X$ is compact.*

The proof of Proposition 4.15 relies on the cohomological classification of group extensions, as well as the classification of extensions of modules via the functor $\text{Ext}_R^1(\ , \)$. We will begin by establishing the proposition in the case that $R(N)$ is torsion; for this, it is only necessary to assume that G/N is virtually polycyclic. Our reasoning is based on the following two elementary lemmas, whose proofs are left to the reader. (See [16, 10.1.15] for a similar result.)

Lemma 4.16. *Let R be a ring, and let A and B be R -modules such that the underlying abelian group of B is divisible. Let $0 \rightarrow B \rightarrow E \rightarrow A \rightarrow 0$ be an R -module extension and ξ the element of $\text{Ext}_R^1(A, B)$ corresponding to this extension. If $m \cdot \xi = 0$ for some $m \in \mathbb{Z}$, then E has a submodule X such that $E = B + X$ and $B \cap X = \{b \in B \mid m \cdot b = 0\}$. \square*

Lemma 4.17. *Let G be a group and A a $\mathbb{Z}G$ -module whose underlying abelian group is divisible. Let $0 \rightarrow A \rightarrow E \rightarrow G \rightarrow 1$ be a group extension giving rise to the given $\mathbb{Z}G$ -module structure on A , and let ξ be the corresponding element of $H^2(G, A)$. If $m \cdot \xi = 0$ for some $m \in \mathbb{Z}$, then E has a subgroup X such that $E = AX$ and $A \cap X = \{a \in A \mid m \cdot a = 0\}$. \square*

Equipped with these two lemmas, we dispose of the special case of Proposition 4.15.

Lemma 4.18. *Let p be a prime, and let G be a topological group with an open normal \mathfrak{N}_p -subgroup N such that G/N is virtually polycyclic. Suppose further that $R(N)$ is torsion. Then G contains a subgroup X such that $G = R(N)X$ and $N \cap X$ is compact.*

Proof. Set $R = R(N)$ and $Q = G/N$. First we treat the case where R is infinite and every proper G -subgroup of R is finite. Since N is not compact, Lemma 4.10 implies that the same holds for N_{ab} . As a result, $R \cap N'$ is a proper subgroup of R , and so $R \cap N'$ is finite. From Lemmas 4.2(vii) and 4.6(iii), we deduce that N' is compact. Hence it suffices

to consider the case $N' = 1$. Under this assumption, N can be viewed as a $\mathbb{Z}_p Q$ -module. Because Q is virtually polycyclic, $\mathbb{Z}_p Q$ is a Noetherian ring. Thus, as a finitely generated $\mathbb{Z}_p Q$ -module, N/R must have type FP_∞ . This means that $\text{Ext}_{\mathbb{Z}_p Q}^n(N/R, R)$ is p -torsion for $n \geq 0$. Invoking Lemma 4.16, we acquire a compact $\mathbb{Z}_p Q$ -submodule V of N such that $N = R + V$ and $R \cap V$ is finite.

We now examine the group extension $0 \rightarrow A \rightarrow G/V \rightarrow Q \rightarrow 1$, where $A = N/V \cong R/(R \cap V)$. Being a virtually polycyclic group, Q is of type FP_∞ . As a result, $H^n(Q, A)$ is p -torsion for $n \geq 0$. Consequently, by Lemma 4.17, G/V has a subgroup X^\dagger such that $G/V = AX^\dagger$ and $A \cap X^\dagger$ is finite. Thus, if we let X be the preimage of X^\dagger in G , then X has the desired properties. This concludes the argument for the case where R is infinite and every proper G -subgroup of R is finite.

Finally, we handle the general case by inducting on $m_p(R)$. The case $m_p(R) = 0$ being trivial, suppose $m_p(R) \geq 1$. Choose K to be a radicable G -subgroup of R such that $m_p(K)$ is as large as possible while still remaining less than $m_p(R)$. Then every proper G -subgroup of R/K is finite. Hence, by the case established above, we can find a subgroup Y such that $K \leq Y$, $G = RY$, and $(N \cap Y)/K$ is compact. Notice further that K is the finite residual of $N \cap Y$. The inductive hypothesis furnishes, then, a subgroup $X \leq Y$ such that $N \cap X$ is compact and $Y = KX$. Since $G = RX$, this completes the proof. \square

The proof of the general case of Proposition 4.15 will depend heavily upon the following result.

Proposition 4.19. (Kropholler, Lorensen, and Robinson [18, Proposition 2.1]) *Let G be an abelian group and R a principal ideal domain such that R/Ra is finite for every nonzero element a of R . Let A and B be RG -modules that are R -torsion-free and have finite R -rank. Suppose further that A fails to contain a nonzero RG -submodule that is isomorphic to a submodule of B . Then there is a positive integer m such that $m \cdot \text{Ext}_{RG}^n(A, B) = 0$ for all $n \geq 0$.* \square

Proof of Proposition 4.15. Let $R = R(N)$ and $Q = G/N$. To begin with, we treat the case where R is torsion-free and abelian, as well as simple when viewed as a $\mathbb{Q}_p G$ -module. Suppose first that R has a nontrivial compact G -subgroup C . Then R/C is torsion, so that Lemma 4.18 provides a subgroup X of G containing C such that $G = RX$ and $(N \cap X)/C$ is compact. It follows that $N \cap X$ is compact, yielding the conclusion sought. Assume next that R has no nontrivial compact G -subgroups. Since N is not compact, Lemma 4.10 implies that the same holds for N_{ab} . As a result, $R \cap N'$ is a proper subgroup of R . Hence the finite residual R_0 of $R \cap N'$ is a proper radicable G -subgroup of R . Thus $R_0 = 1$; that is, $R \cap N'$ is compact, and so $R \cap N' = 1$. Furthermore, N' is compact, which means that there is no real loss of generality in supposing $N' = 1$.

Consider now the $\mathbb{Z}_p Q$ -module extension $0 \rightarrow R \rightarrow N \rightarrow N/R \rightarrow 0$. Take Q_0 to be a normal abelian subgroup of Q with finite index. Observe that any compact $\mathbb{Z}_p Q_0$ -submodule of R must be contained in a compact $\mathbb{Z}_p Q$ -submodule. Hence R cannot possess any nonzero compact $\mathbb{Z}_p Q_0$ -submodules. It follows, then, from Proposition 4.19 that $\text{Ext}_{\mathbb{Z}_p Q_0}^1(N/R, R)$ is torsion. Being a vector space over \mathbb{Q} , this Ext-group must therefore be trivial. Thus N splits as a $\mathbb{Z}_p Q_0$ -module over R . Let V_0 be a $\mathbb{Z}_p Q_0$ -module complement to R in N . Then V_0 is contained in a compact $\mathbb{Z}_p Q$ -submodule V of N . Hence $R \cap V = 0$ and $N = R + V$.

Having obtained V , we shift our attention to the group extension $0 \rightarrow R \rightarrow G/V \rightarrow Q \rightarrow 1$. We maintain that this extension, too, splits. To see this, notice $H^n(Q_0, R) \cong \text{Ext}_{\mathbb{Z}_p Q_0}^n(\mathbb{Z}_p, R)$ for $n \geq 0$. Appealing again to Proposition 4.19, we conclude that $H^n(Q_0, R)$

is torsion for all $n \geq 0$. Hence $H^n(Q_0, R) = 0$ for every $n \geq 0$, which implies $H^n(Q, R) = 0$ for $n \geq 0$. This means that the desired splitting occurs; in other words, G has a subgroup X such that $G = NX$ and $N \cap X = V$. This subgroup, then, fulfills our requirements, completing the argument for the case where R is torsion-free abelian and simple as a $\mathbb{Q}_p G$ -module.

Now we tackle the case where R is torsion-free, but not necessarily abelian. We proceed by induction on $h_p(R)$, the case $h_p(R) = 0$ being trivial. Suppose $h_p(R) \geq 1$. Select K to be a closed radicable G -subgroup of R with R/K abelian such that $h_p(K)$ is as large as possible while still remaining less than $h_p(R)$. Then R/K is simple when regarded as a $\mathbb{Q}_p G$ -module. Consequently, by the case established above, there is a subgroup Y such that $K \leq Y$, $(N \cap Y)/K$ is compact, and $G = RY$. Because K is the finite residual of $N \cap Y$, the inductive hypothesis provides a subgroup $X \leq Y$ such that $N \cap X$ is compact and $Y = KX$. It follows, then, that $G = RX$, thus completing the argument for the case where R is torsion-free.

Finally, we deal with the general case. Letting T be the torsion subgroup of R , we apply the torsion-free case to R/T , thereby obtaining a subgroup Y containing T such that $G = RY$ and $(N \cap Y)/T$ is compact. Since T is the finite residual of $N \cap Y$, we can apply Lemma 4.18 to Y , acquiring a subgroup $X \leq Y$ such that $Y = TX$ and $N \cap X$ is compact. Then $G = RX$, so that X enjoys the desired properties. \square

The proof of Theorem 1.13 will make use of a further factorization result for topological groups.

Proposition 4.20. *Let p be a prime and π a finite set of primes. Let N be a nilpotent Hausdorff topological group with a closed normal \mathfrak{N}_p -subgroup K such that N/K is $(\pi$ -torsion)-by- $(\pi$ -minimax). Then N contains a $(\pi$ -torsion)-by- $(\pi$ -minimax) subgroup X such that $N = KX$.*

Before proving Proposition 4.20, we establish two lemmas.

Lemma 4.21. *Let π be a set of primes and N be a nilpotent group that is $(\pi$ -torsion)-by- $(\pi$ -minimax). Then N contains a finitely generated subgroup H such that, for every $n \in N$, there is a π -number m such that $n^m \in H$.*

Proof. We argue by induction on $\text{nil } N$. Suppose that N is abelian. Let T be a π -torsion subgroup of N such that N/T is π -minimax, and take $\epsilon : N \rightarrow N/T$ to be the quotient map. The group N/T contains a finitely generated subgroup \bar{H} such that, for every $n \in N$, there is a π -number m such that $\epsilon(n^m) \in \bar{H}$. Thus, if we let H be a finitely generated subgroup of N such that $\epsilon(H) = \bar{H}$, then H can serve as the subgroup sought.

Now we treat the case where $\text{nil } N > 1$. Let $Z = Z(N)$. By the abelian case of the lemma, Z contains a finitely generated subgroup A such that, for every $z \in Z$, there is a π -number m such that $z^m \in A$. Moreover, we can deduce from the inductive hypothesis that N possesses a finitely generated subgroup H_0 such that, for every $n \in N$, there is a π -number m such that $n^m \in ZH_0$. Therefore, if we set $H = AH_0$, then H fulfills our requirements. \square

Lemma 4.22. *Let π be a finite set of primes and N a nilpotent group with a normal subgroup K such that K is π -radicable and N/K is $(\pi$ -torsion)-by- $(\pi$ -minimax). Then N contains a subgroup X such that $N = KX$ and X is $(\pi$ -torsion)-by- $(\pi$ -minimax).*

Proof. We induct on $\text{nil}_N K$. First suppose that N centralizes K . Writing $Q = N/K$ and invoking Lemma 4.21, we choose a finitely generated subgroup $\bar{H} \leq Q$ such that, for every $q \in Q$, there is a π -number m such that $q^m \in \bar{H}$. Now take H to be a finitely generated subgroup of N whose image in Q is \bar{H} . Define X to be the π -isolator of H in N . Then Lemma 3.27 implies that X is an extension of a π -torsion group by one that is π -minimax.

We claim further that the π -radicability of K ensures $N = KX$. To verify this, take n to be an arbitrary element of N . Then the image of n^m is in \bar{H} for some π -number m . It follows that there exists $k \in K$ such that $kn^m \in H$. Selecting $l \in K$ such that $l^m = k$, we have $(ln)^m \in H$. Thus $ln \in X$, and so $n \in KX$.

Finally, we treat the case where $\text{nil}_N K > 1$. Set $Z = Z^N(K)$. We can suppose that K is π -torsion-free. Under this assumption, Z must be π -radicable. By the inductive hypothesis, N contains a subgroup Y such that $Z \leq Y$, Y/Z is $(\pi\text{-torsion})\text{-by-}(\pi\text{-minimax})$, and $N = KY$. The base case yields, then, a $(\pi\text{-torsion})\text{-by-}(\pi\text{-minimax})$ subgroup X of Y such that $Y = ZX$. Hence $N = KX$, completing the proof. \square

Lemma 4.22 is the basis for the proof of Proposition 4.20.

Proof of Proposition 4.20. We argue by induction on $\text{nil}_N K$. First we dispose of the case where N acts trivially on K . Since \mathbb{Z}_p/\mathbb{Z} is divisible, it follows from Proposition 4.8 that every abelian \mathfrak{N}_p -group contains an abstractly finitely generated subgroup giving rise to a divisible quotient. Select L to be such a subgroup in K . Applying Lemma 4.22 to N/L , we acquire a subgroup X of N such that $L \leq X$, X/L is $(\pi\text{-torsion})\text{-by-}(\pi\text{-minimax})$, and $N = KX$. Lemma 3.27, then, implies that X must be $(\pi\text{-torsion})\text{-by-}(\pi\text{-minimax})$.

Next we handle the case where $\text{nil}_N K > 1$. Write $Z = Z^N(K)$. That N is Hausdorff implies that Z is closed; thus both Z and K/Z are members of \mathfrak{N}_p . Applying the inductive hypothesis to N/Z , we can find a subgroup Y of N such that $Z \leq Y$, Y/Z is $(\pi\text{-torsion})\text{-by-}(\pi\text{-minimax})$, and $N = KY$. We can now invoke the case of a trivial action to obtain a $(\pi\text{-torsion})\text{-by-}(\pi\text{-minimax})$ subgroup X of Y such that $Y = ZX$. Therefore $N = KX$, which finishes the argument. \square

5 Tensor p -completion

5.1 p -Completing nilpotent groups

Let p be a prime. Throughout this section, \mathfrak{U}_p will represent the class of topological groups whose topologically finitely generated closed subgroups are all pro- p groups. Note that \mathfrak{U}_p contains the class \mathfrak{N}_p defined in Section 4 as a proper subclass. The embedding of a nilpotent \mathfrak{M} -group in an \mathfrak{N}_p -group employed in the proof of Theorem 1.13 will arise from a functor $N \mapsto N_p$ from the class of nilpotent groups of finite torsion-free rank to \mathfrak{U}_p such that N_p belongs to \mathfrak{N}_p if N is minimax. The present section focuses on the construction and investigation of this functor, which will coincide with classical notions in three cases:

- If N is finitely generated, then N_p is naturally isomorphic to the pro- p completion of N .
- If N is abelian, then N_p is naturally isomorphic to $N \otimes \mathbb{Z}_p$.
- If N is torsion-free, then N_p is naturally isomorphic to the closure of N in its p -adic Mal'cev completion.

Our discussion in this section will rely on the following two well-known properties of pro- p completions of finitely generated nilpotent groups.

Lemma 5.1. *Let N be a finitely generated nilpotent group and p a prime.*

- (i) *If $1 \rightarrow M \xrightarrow{\iota} N \xrightarrow{\epsilon} Q \rightarrow 1$ is an extension of abstract groups, then $1 \rightarrow \hat{M}_p \xrightarrow{\hat{\iota}_p} \hat{N}_p \xrightarrow{\hat{\epsilon}_p} \hat{Q}_p \rightarrow 1$ is an extension of pro- p groups.*
- (ii) *If $H \leq N$ and $\phi : H \rightarrow N$ is the inclusion monomorphism, then $\hat{\phi}_p : \hat{H}_p \rightarrow \hat{N}_p$ is injective for every prime p . \square*

Proof. That the sequence $\hat{M}_p \xrightarrow{\hat{\iota}_p} \hat{N}_p \xrightarrow{\hat{\epsilon}_p} \hat{Q}_p \rightarrow 1$ is exact holds even if N is taken to be an arbitrary group. The injectivity of $\hat{\iota}_p$ when N is finitely generated and nilpotent follows from the cohomological property of finitely generated nilpotent groups contained in [17, Corollary 1.1]. Statement (ii) can be readily deduced from (i) by using the fact that every subgroup of a nilpotent group is subnormal. \square

Let p be a prime and N a nilpotent group of finite torsion-free rank. Let \mathcal{H} be the set of all finitely generated subgroups of G . For any pair $H, K \in \mathcal{H}$ with $H \leq K$, define $\iota^{HK} : H \rightarrow K$ to be the inclusion map. Notice that, according to Lemma 5.1(ii), the map $\hat{\iota}_p^{HK} : \hat{H}_p \rightarrow \hat{K}_p$ must be injective. The *tensor p -completion* of N , denoted N_p , is defined to be the inductive limit of the groups \hat{H}_p for $H \in \mathcal{H}$. Then N_p is a nilpotent group with $\text{nil } N_p \leq \text{nil } N$.

For every $H \in \mathcal{H}$, we identify \hat{H}_p with its image in N_p . Moreover, we endow N_p with a topology by making $U \subseteq N_p$ open if and only if $U \cap \hat{H}_p$ is open in \hat{H}_p for every $H \in \mathcal{H}$. In the proposition below, we show that N_p is a topological group with respect to this topology. An alternative approach to constructing N_p is mentioned in Remark 5.10.

Lemma 5.2. *Let p be a prime and N a nilpotent group of finite torsion-free rank.*

- (i) *If H is a finitely generated subgroup of N such that $h(H) = h(N)$, then \hat{H}_p is open in N_p .*
- (ii) *N_p is a topological group that is the union of a family \mathcal{V} of polycyclic pro- p open subgroups of p -Hirsch length $h(N)$ such that, for every pair $H, K \in \mathcal{V}$, there is a subgroup $L \in \mathcal{V}$ with $H \cup K \subseteq L$.*

Proof. To prove (i), we let K be an arbitrary finitely generated subgroup of N . Then $[K : H \cap K]$ is finite. It can thus be deduced from Lemma 5.1 that $[\hat{K}_p : \hat{H}_p \cap \hat{K}_p] < \infty$. Hence $\hat{H}_p \cap \hat{K}_p$ is open in \hat{K}_p . Therefore \hat{H}_p is open in N_p .

Next we dispose of (ii). Notice that N_p is the union of all the \hat{H}_p such that H is a finitely generated subgroup of N with $h(H) = h(N)$. Hence (ii) follows from (i) and Lemma 4.12. \square

For any prime p and nilpotent group N of finite torsion-free rank, we let $t_p^N : N \rightarrow N_p$ be the homomorphism induced by the pro- p completion maps $c_p^H : H \rightarrow \hat{H}_p$ for all the finitely generated subgroups H of N . The map $t_p^N : N \rightarrow N_p$ enjoys the universal property described in Proposition 5.3 below.

Proposition 5.3. *Let N be a nilpotent group with finite torsion-free rank and p a prime. Suppose that $\phi : N \rightarrow G$ is a group homomorphism such that G is a \mathfrak{U}_p -group. Then there exists a unique continuous homomorphism $\psi : N_p \rightarrow G$ such that the diagram*

$$\begin{array}{ccc} N & \xrightarrow{\phi} & G \\ t_p^N \downarrow & \nearrow \psi & \\ N_p & & \end{array} \quad (5.1)$$

commutes.

Proof. For each finitely generated subgroup H of N , $\overline{\phi(H)}$ is a pro- p group. Hence there is a unique continuous homomorphism $\psi_H : \hat{H}_p \rightarrow \overline{\phi(H)}$ such that the diagram

$$\begin{array}{ccc} H & \xrightarrow{\phi} & \overline{\phi(H)} \\ c_p^H \downarrow & \nearrow \psi_H & \\ \hat{H}_p & & \end{array}$$

commutes. The universal properties of direct limits for abstract groups and topological spaces yield, then, a unique continuous homomorphism $\psi : N_p \rightarrow G$ that makes diagram (5.1) commute. \square

As a consequence of the preceding result, any homomorphism $\phi : N \rightarrow M$ between nilpotent groups of finite torsion-free rank induces a unique continuous homomorphism $\phi_p : N_p \rightarrow M_p$ that renders the diagram

$$\begin{array}{ccc} N & \xrightarrow{\phi} & M \\ t_p^N \downarrow & & \downarrow t_p^M \\ N_p & \xrightarrow{\phi_p} & M_p \end{array}$$

commutative. In this way, tensor p -completion defines a functor from the category of nilpotent groups with finite torsion-free rank to the category of \mathfrak{U}_p -groups. This functor enjoys several convenient properties, listed below.

Lemma 5.4. *Let N be a nilpotent group with finite torsion-free rank and p a prime. Then the following hold.*

- (i) *If N is abelian, then there is a unique topological \mathbb{Z}_p -module structure on N_p that extends the topological \mathbb{Z}_p -module structures possessed by its pro- p open subgroups. Moreover, the canonical \mathbb{Z}_p -module homomorphism $N \otimes \mathbb{Z}_p \rightarrow N_p$ is an isomorphism.*
- (ii) *The kernel of $t_p^N : N \rightarrow N_p$ is the p' -torsion subgroup of N .*
- (iii) *If $1 \rightarrow M \xrightarrow{\iota} N \xrightarrow{\epsilon} Q \rightarrow 1$ is an extension of abstract groups, then $1 \rightarrow M_p \xrightarrow{\iota_p} N_p \xrightarrow{\epsilon_p} Q_p \rightarrow 1$ is an extension of topological groups.*
- (iv) *If N is minimax, then N_p lies in the class \mathfrak{N}_p .*

Proof. The first sentence of (i) follows immediately from the definition of N_p . The second is a consequence of the well-known fact that the canonical homomorphism $H \otimes \mathbb{Z}_p \rightarrow \hat{H}_p$ is an isomorphism for any finitely generated subgroup H of N .

The case of (ii) where N is finitely generated follows from the properties of the pro- p completion of a nilpotent group. The general case is then an immediate consequence of this.

Next we prove (iii). By Lemma 5.1(i), if N is finitely generated, then $1 \rightarrow M_p \xrightarrow{\iota_p} N_p \xrightarrow{\epsilon_p} Q_p \rightarrow 1$ is an extension of pro- p groups. But inductive limits commute with exact sequences of abstract groups. As a result, $1 \rightarrow M_p \xrightarrow{\iota_p} N_p \xrightarrow{\epsilon_p} Q_p \rightarrow 1$ is also exact if N is infinitely generated. Moreover, since the groups involved are locally compact, σ -compact Hausdorff groups, it follows that $1 \rightarrow M_p \xrightarrow{\iota_p} N_p \xrightarrow{\epsilon_p} Q_p \rightarrow 1$ is an extension of topological groups.

Statement (iv) is proved by taking a series of finite length whose factors are cyclic or quasicyclic and then applying (i) and (iii). \square

In deriving further properties of tensor p -completions, it will be helpful to have a notion embracing all the homomorphisms that obey the same universal property as t_p^N .

Definition 5.5. Let G be an abstract group and \bar{G} a topological group. A homomorphism $\tau : G \rightarrow \bar{G}$ *p-completes* G if \bar{G} belongs to \mathfrak{U}_p and, for any homomorphism ϕ from G to a \mathfrak{U}_p -group H , there is a unique continuous homomorphism $\psi : \bar{G} \rightarrow H$ such that the diagram

$$\begin{array}{ccc} G & \xrightarrow{\phi} & H \\ \tau \downarrow & \nearrow \psi & \\ \bar{G} & & \end{array}$$

commutes.

The next lemma is an immediate consequence of the above definition; the proof is left to the reader.

Lemma 5.6. Let N be a nilpotent group of finite torsion-free rank and \bar{N} a topological group in \mathfrak{U}_p . Let $\tau : N \rightarrow \bar{N}$ be a homomorphism. If $\phi : N_p \rightarrow \bar{N}$ is the unique continuous homomorphism such that $\phi t_p^N = \tau$, then τ *p-completes* N if and only if ϕ is an isomorphism of topological groups. \square

Lemma 5.4(iii) gives rise to the following two lemmas about maps that p -complete.

Lemma 5.7. Let p be a prime and $1 \rightarrow M \rightarrow N \rightarrow Q \rightarrow 1$ be a group extension such that N is nilpotent and minimax. Let $1 \rightarrow \bar{M} \rightarrow \bar{N} \rightarrow \bar{Q} \rightarrow 1$ be a topological group extension such that \bar{N} is nilpotent. Suppose further that these two extensions fit into a commutative diagram

$$\begin{array}{ccccccc} 1 & \longrightarrow & M & \longrightarrow & N & \longrightarrow & Q \longrightarrow 1 \\ & & \downarrow \phi & & \downarrow \theta & & \downarrow \psi \\ 1 & \longrightarrow & \bar{M} & \longrightarrow & \bar{N} & \longrightarrow & \bar{Q} \longrightarrow 1. \end{array}$$

If any two of the homomorphisms ϕ , θ , and ψ *p-complete*, then so does the third.

Proof. First we point out that the hypothesis implies that \bar{M} , \bar{N} , and \bar{Q} are all members of \mathfrak{N}_p . Appealing to Lemma 5.4(iii), we form the commutative diagram

$$\begin{array}{ccccccc} 1 & \longrightarrow & M_p & \longrightarrow & N_p & \longrightarrow & Q_p \longrightarrow 1 \\ & & \downarrow \phi' & & \downarrow \theta' & & \downarrow \psi' \\ 1 & \longrightarrow & \bar{M} & \longrightarrow & \bar{N} & \longrightarrow & \bar{Q} \longrightarrow 1, \end{array}$$

where ϕ' , θ' , and ψ' are induced by ϕ , θ , and ψ , respectively. Since two of the maps ϕ' , θ' , and ψ' are isomorphisms, the same holds for the third. This, then, proves the lemma. \square

Lemma 5.8. *Let N be a nilpotent minmax group and p a prime. For any $i \geq 1$, the homomorphism $\gamma_i N \rightarrow \gamma_i N_p$ induced by $t_p^N : N \rightarrow N_p$ p -completes.*

Proof. Invoking Lemma 5.4(iii), we consider the topological group extension $1 \rightarrow (\gamma_i(N))_p \rightarrow N_p \rightarrow (N/\gamma_i(N))_p \rightarrow 1$. Let M be the image of $(\gamma_i(N))_p$ in N_p . Then M is the closure of the image of $\gamma_i(N)$ in N_p . But $\gamma_i(N_p) \leq M$, and, by Lemma 4.10(i), $\gamma_i(N_p)$ is closed. Therefore $\gamma_i N_p = M$, which yields the conclusion. \square

5.2 Partial tensor p -completion

We conclude this section by discussing how the tensor p -completion functor can be applied to a normal nilpotent subgroup of finite torsion-free rank, thereby embedding the ambient group densely in a topological group with an open normal, locally pro- p subgroup. This process is inspired by Hilton's notion of a *relative localization* from [11]. Essential to the construction is the following result of his concerning group extensions.

Proposition 5.9. (Hilton [11]) *Let $1 \rightarrow K \rightarrow G \rightarrow Q \rightarrow 1$ be a group extension and $\phi : K \rightarrow L$ a group homomorphism. Suppose further that there is an action of G on L that satisfies the following two properties:*

- (i) $k \cdot l = \phi(k)l\phi(k^{-1})$ for all $l \in L$ and $k \in K$;
- (ii) $g \cdot \phi(k) = \phi(gkg^{-1})$ for all $g \in G$ and $k \in K$.

In addition, let $K^\dagger = \{(\phi(k), k^{-1}) \mid k \in K\} \subset (L \rtimes G)$. Then $K^\dagger \triangleleft (L \rtimes G)$. Furthermore, if $G^ = (L \rtimes G)/K^\dagger$, the diagram*

$$\begin{array}{ccccccc} 1 & \longrightarrow & K & \longrightarrow & G & \longrightarrow & Q \longrightarrow 1 \\ & & \downarrow \phi & & \downarrow \psi & & \parallel \\ 1 & \longrightarrow & L & \longrightarrow & G^* & \longrightarrow & Q \longrightarrow 1 \end{array}$$

commutes, where $\psi : G \rightarrow G^$ is the map $g \mapsto (1, g)K^\dagger$.* \square

Now assume that G is a discrete group with a nilpotent normal subgroup N with finite torsion-free rank, and suppose that p is a prime. The functorial property of N_p supplies a unique continuous action of G on N_p that makes $t_p^N : N \rightarrow N_p$ a G -group homomorphism. Furthermore, this action fulfills the condition $n \cdot x = t_p^N(n)xt_p^N(n^{-1})$ for all $x \in N_p$ and $n \in N$. Hence, by Hilton's proposition, $N^\dagger := \{(t_p(n), n^{-1}) \mid n \in N\}$ is a normal subgroup of $N_p \rtimes G$. We define the *partial tensor p -completion of G with respect to N* , denoted $G_{(N,p)}$, by $G_{(N,p)} = (N_p \rtimes G)/N^\dagger$. Then $G_{(N,p)}$ fits into a commutative diagram

$$\begin{array}{ccccccc}
1 & \longrightarrow & N & \longrightarrow & G & \longrightarrow & Q \longrightarrow 1 \\
& & \downarrow t_p^N & & \downarrow & & \parallel \\
1 & \longrightarrow & N_p & \longrightarrow & G_{(N,p)} & \longrightarrow & Q \longrightarrow 1,
\end{array}$$

and $G_{(N,p)}$ is a topological group containing an isomorphic copy of N_p as an open normal subgroup.

Remark 5.10. It was pointed out to the authors by an anonymous referee that both the tensor p -completion and the partial tensor p -completion can be subsumed under a single construction, using the notion of a *relative profinite completion* from [27, §3]. (See also the additional references provided there.) To see this, take N and G to be as described above and form the relative profinite completion \hat{G}_H of G with respect to a finitely generated subgroup H of N with maximal Hirsch length. Next, divide out by the closed subgroup of \hat{G}_H generated by all the Sylow q -subgroups, for $q \neq p$, of the closure of the image of N in \hat{G}_H . Then the resulting quotient can be shown to be naturally isomorphic to $G_{(N,p)}$.

6 Proofs of Theorem 1.13 and its corollaries

6.1 Theorem 1.13

As related in §2.1, the key step in our argument for Theorem 1.13 involves constructing a virtually torsion-free, locally compact cover for $G_{(N,p)}$. For the sake of readability, we extract this step from the proof of the theorem, making it a separate proposition.

Proposition 6.1. *Let p be a prime, and let G be a topological group with an open normal \mathfrak{N}_p -subgroup N such that G/N is finitely generated and virtually abelian. Set $S = \text{solv}(G)$. Then there is a virtually torsion-free topological group Γ and a continuous epimorphism $\theta : \Gamma \rightarrow G$ such that the following statements hold, where $\Lambda := \theta^{-1}(N)$.*

- (i) Λ belongs to \mathfrak{N}_p .
- (ii) $\text{der}(\theta^{-1}(S)) = \text{der}(S)$, $\text{nil } \Lambda = \text{nil } N$, and $\text{der}(\Lambda) = \text{der}(N)$.
- (iii) If G acts π -integrally on N_{ab} for some set of primes π , then Γ acts π -integrally on Λ_{ab} .

Proof. Put $R_0 = R(N)$. Also, let P be the torsion subgroup of R_0 , making P a direct product of finitely many quasicyclic p -groups. Appealing to Proposition 4.15, we obtain a subgroup X of G such that $N \cap X$ is compact and $G = R_0 X$. Write $Y = X \cap N$, so that we also have the factorization $N = R_0 Y$. In addition, let $X_0 = \text{solv}(X)$.

For the convenience of the reader, we divide the proof into three steps.

Step 1: Covering R_0 with a radicable nilpotent torsion-free X -group Ω .

Applying Proposition 3.12 *ad infinitum* allows us to construct a sequence

$$\cdots \xrightarrow{\psi_{i+1}} R_i \xrightarrow{\psi_i} R_{i-1} \xrightarrow{\psi_{i-1}} \cdots \xrightarrow{\psi_2} R_1 \xrightarrow{\psi_1} R_0 \quad (6.1)$$

of X -group epimorphisms such that the following two statements are true.

- (1) R_i is nilpotent and radicable for each $i \geq 0$.

- (2) For each $i \geq 1$, there is an X -group isomorphism $\nu_i : P \rightarrow \psi_i^{-1}\psi_{i-1}^{-1}\dots\psi_1^{-1}(P)$ such that $\psi_i\nu_i(u) = \nu_{i-1}(u^p)$ for all $u \in P$ and $i \geq 2$.

The three assertions below about (6.1) also hold. Statement (3) is implied by the last sentence in Proposition 3.12, and the other two follow from Lemma 3.11.

- (3) $\text{Ann}_{\mathbb{Z}X}((R_i)_{\text{ab}}) = \text{Ann}_{\mathbb{Z}X}((R_0)_{\text{ab}})$ if $i \geq 0$.
(4) For every $i \geq 0$, the subgroup Y of X acts nilpotently on R_i with $\text{nil}_Y R_i = \text{nil}_Y R_0$ and $\text{der}_Y(R_i) = \text{der}_Y(R_0)$.
(5) For every $i \geq 0$, $\text{der}_{X_0}(R_i) = \text{der}_{X_0}(R_0)$.

Next we define the X -group Ω to be the inverse limit of the system (6.1) and ψ to be the canonical map $\Omega \rightarrow R_0$. Furthermore, for each $i \geq 0$, set $\Gamma_i = R_i \rtimes X$ and $N_i = R_i \rtimes Y$. Corollary 4.14 implies that there is a sequence of topologies on the groups Γ_i making each N_i an open normal \mathfrak{N}_p -subgroup of Γ_i and each homomorphism $\Gamma_i \rightarrow \Gamma_{i-1}$ induced by ψ_i a quotient map. Using the topologies imparted thereby to the R_i , we can make Ω into a topological group by endowing it with the topology induced by the product topology. Since X acts continuously on each R_i , the action of X on Ω must also be continuous. Moreover, the subgroup $\Psi := \psi^{-1}(P)$ is closed and isomorphic, as a topological group, to the inverse limit of the system $\dots \rightarrow P \rightarrow P \rightarrow P$, in which every homomorphism $P \rightarrow P$ is the map $u \mapsto u^p$. As a result, Ψ is isomorphic to the direct sum of finitely many copies of \mathbb{Q}_p . Since $\Omega/\Psi \cong R_0/P$ as topological groups, we conclude that Ω is a member of the class \mathfrak{N}_p , and that Ω is torsion-free and radicable.

Step 2: Defining the cover Γ and establishing (i) and (ii).

We now set $\Gamma = \Omega \rtimes X$ and take θ to be the epimorphism from Γ to G such that $\theta(r, x) = \psi(r)x$ for all $r \in \Omega$ and $x \in X$. Since X acts continuously on Ω , Γ is a topological group and θ is continuous.

Writing $\Lambda = \theta^{-1}(N)$, we have $\Lambda = \Omega \rtimes Y$. According to Proposition 3.8(i), $\text{nil } N = \max\{\text{nil}_Y R_0, \text{nil } Y\}$. But property (4) in Step 1 implies $\text{nil}_Y \Omega = \text{nil}_Y R_0$. Thus, invoking Proposition 3.8(i) again, we conclude $\text{nil } \Lambda = \text{nil } N$. Moreover, a similar argument, this time using Proposition 3.8(ii), can be adduced to show $\text{der}(\Lambda) = \text{der}(N)$. Observe further that, since Ω and Y both belong to \mathfrak{N}_p , Λ must be a member of \mathfrak{N}_p .

We can also employ property (5) in Step 1 and Proposition 3.8(ii) to show $\text{der}(\Sigma) = \text{der}(S)$, where $\Sigma = \text{solv}(\Gamma) = \theta^{-1}(S)$. To accomplish this, notice $S = R_0 X_0$ and $\Sigma = \Omega \rtimes X_0$. Therefore

$$\text{der}(S) = \max\{\text{der}_{X_0}(R_0), \text{der}(X_0)\} = \max\{\text{der}_{X_0}(\Omega), \text{der}(X_0)\} = \text{der}(\Sigma).$$

Step 3: Showing statement (iii).

By Lemma 3.24(ii), X acts π -integrally on both $(R_0)_{\text{ab}}$ and Y_{ab} . It can therefore be deduced from property (3) in Step 1 that X acts π -integrally on $\varprojlim (R_i)_{\text{ab}}$. Now we examine the action of X on Ω_{ab} . Observe first that the canonical maps $\Omega_{\text{ab}} \rightarrow (R_i)_{\text{ab}}$ give rise to a continuous $\mathbb{Z}X$ -module homomorphism $\eta : \Omega_{\text{ab}} \rightarrow \varprojlim (R_i)_{\text{ab}}$. Also, if $r \in \Omega$ such that the image of r in R_i lies in R'_i for each i , then the fact that Ω' is closed in Ω ensures $r \in \Omega'$. In other words, η is injective. As a result, X must act π -integrally on Ω_{ab} . Since there is

a $\mathbb{Z}X$ -module epimorphism $\Omega_{\text{ab}} \oplus Y_{\text{ab}} \rightarrow \Lambda_{\text{ab}}$, we conclude that the action of X on Λ_{ab} is π -integral. Thus Γ acts π -integrally on Λ_{ab} . \square

We proceed now with the proof of our main result.

Proof of Theorem 1.13. The implication (I) \implies (II) was already established in Proposition 1.11. Our plan is to first show (II) \implies (I), in the process demonstrating that the \mathfrak{M}_1^π -cover we obtain can be made to satisfy conditions (ii)-(iv) in Theorem 1.5. Secondly, we prove the equivalence of (II) and (III).

For the first part of the proof, we establish Assertion A below. This statement will imply (II) \implies (I), together with conditions (ii)-(iv) in Theorem 1.5.

Assertion A. *Let G be an \mathfrak{M}^π -group with $S = \text{solv}(G)$. For any nilpotent normal subgroup N of G with G/N finitely generated and virtually abelian and the action of G on N_{ab} π -integral, there is an \mathfrak{M}_1^π -group G^* and an epimorphism $\phi : G^* \rightarrow G$ such that the following two statements hold.*

- $\phi^{-1}(N)$ is nilpotent of the same class and derived length as N .
- $\text{der}(S^*) = \text{der}(S)$, where $S^* = \text{solv}(G^*)$.

We will prove Assertion A by induction on the number of primes for which G contains a quasicyclic subgroup. If this number is zero, then we simply make $G^* = G$ and $\phi : G^* \rightarrow G$ the identity map. Suppose that this number is positive. Let p be a prime for which G has a quasicyclic p -subgroup, and let P be the p -torsion part of $R(G)$. By the inductive hypothesis, G/P can be covered by an \mathfrak{M}_1^π -group with a solvable radical of derived length $\text{der}(S/P)$ such that N/P lifts to a nilpotent subgroup of the same class and derived length. Thus, in view of Lemma 3.17, there is no real loss of generality in assuming that G/P is virtually torsion-free. Lemma 3.17 also allows us to reduce to the case where the p' -torsion subgroup of N is trivial. With this assumption, the tensor p -completion map $t_p^N : N \rightarrow N_p$ becomes an injection.

Form the partial tensor p -completion $G_{(N,p)}$, and, for convenience, identify G with its image in $G_{(N,p)}$. Put $S^\dagger = \text{solv}(G_{(N,p)})$. Since N and S are dense in N_p and S^\dagger , respectively, we have $\text{nil } N_p = \text{nil } N$ and $\text{der}(S^\dagger) = \text{der}(S)$. Write $Q = G/N$. According to Lemmas 5.8 and 5.7, the $\mathbb{Z}Q$ -module homomorphism $N_{\text{ab}} \rightarrow (N_p)_{\text{ab}}$ induced by $t_p^N : N \rightarrow N_p$ must be p -complete. Thus $\text{Ann}_{\mathbb{Z}Q}(N) \subseteq \text{Ann}_{\mathbb{Z}Q}((N_p)_{\text{ab}})$. As a result, Q must act π -integrally on $(N_p)_{\text{ab}}$, which means that $G_{(N,p)}$ acts π -integrally on $(N_p)_{\text{ab}}$.

Invoking Proposition 6.1, we obtain a virtually torsion-free topological group Γ and a continuous epimorphism $\theta : \Gamma \rightarrow G_{(N,p)}$ such that the following statements hold.

- (1) $\Lambda := \theta^{-1}(N_p)$ belongs to \mathfrak{N}_p .
- (2) $\text{der}(\theta^{-1}(S^\dagger)) = \text{der}(S)$, $\text{nil } \Lambda = \text{nil } N$, and $\text{der}(\Lambda) = \text{der}(N)$.
- (3) Γ acts π -integrally on Λ_{ab} .

We now show that Γ contains an \mathfrak{M}_1^π -subgroup G^* that covers G . We begin by picking elements g_1, \dots, g_k of G whose images generate Q . For each $j = 1, \dots, k$, choose $g_j^* \in \Gamma$ such that $\theta(g_j^*) = g_j$. Let V be the subgroup of Γ generated as an abstract group by g_1^*, \dots, g_k^* . Applying Proposition 4.20 to the extension

$$1 \rightarrow \text{Ker } \theta \rightarrow \theta^{-1}(N) \xrightarrow{\theta} N \rightarrow 1,$$

we acquire a π -minimax subgroup H of $\theta^{-1}(N)$ such that $\theta(H) = N$. We then define G^* to be the subgroup of Γ generated by H and V as an abstract group. Notice that Proposition 3.28 implies that G^* is π -minimax. Moreover, our selection of H and g_1^*, \dots, g_k^* guarantees $\theta(G^*) = G$. In addition, assertion (2) above implies that the preimage of N in G^* has the same nilpotency class and derived length as N , and that $\text{der}(S^*) = \text{der}(S)$, where $S^* = \text{solv}(G^*)$.

This now completes the proof of Assertion A. Hence (II) \implies (I), and the \mathfrak{M}_1^π -covering satisfies statements (ii)-(iv) from Theorem 1.5.

Our final task is to prove the equivalence of (II) and (III). That (II) implies (III) is plain. Suppose that (III) holds, and write $R = R(G)$. For each $i \geq 0$, let Z_i/R be the i th term in the upper central series of N/R . Since Z_i/Z_{i-1} is a $\mathbb{Z}Q$ -module that is virtually torsion-free and π -minimax *qua* abelian group, Q acts π -integrally on Z_i/Z_{i-1} for all $i \geq 1$. Now take \bar{Z}_i to be the image of Z_i in N_{ab} for each $i \geq 0$. Then, for $i \geq 1$, the action of Q on \bar{Z}_i/\bar{Z}_{i-1} is π -integral. In addition, the hypothesis says that Q acts π -integrally on \bar{Z}_0 . Therefore (II) is true. We have thus shown (III) \implies (II). \square

Remark 6.2. We point out that it seems unlikely that the kernel of the covering constructed in the proof of Theorem 1.13 can be made to be abelian in general. This is because it is not always possible to choose the subgroup X in the proof of Proposition 6.1 so that the kernel of the epimorphism $R_0 \rtimes X \rightarrow G$ is abelian: that would require that $R_0 \cap X$ be abelian. Of course, this still leaves open the possibility that the cover can be modified to obtain an abelian kernel. We conjecture, however, that this cannot always be accomplished, which would mean that Open Question 1.6 has a negative answer.

6.2 Torsion-free covers

Example 1.4 suggests that it might be feasible to strengthen Theorem 1.13 to furnish a cover that is entirely torsion-free, rather than merely virtually torsion-free. This suspicion is buttressed by Lemma 3.18, which states that any finite group is a homomorphic image of a torsion-free polycyclic group. In Corollary 6.3 below, we succeed in establishing such an extension of Theorem 1.13. For our discussion of torsion-free covers, we use \mathfrak{M}_0^π to represent the subclass of \mathfrak{M}_1^π consisting of all the \mathfrak{M}_1^π -groups that are torsion-free.

Corollary 6.3. *Let G be an \mathfrak{M} -group satisfying the three equivalent statements in Theorem 1.13 for a set of primes π . Write $N = \text{Fitt}(G)$ and $S = \text{solv}(G)$. Then there is an \mathfrak{M}_0^π -group G^* and an epimorphism $\phi : G^* \rightarrow G$ possessing the following three properties, where $S^* = \text{solv}(G^*)$ and $N^* = \text{Fitt}(G^*)$.*

- (i) $S^* = \phi^{-1}(S)$.
- (ii) $\text{der}(S^*) = \max\{\text{der}(S), 1\}$.
- (iii) *If $S \neq 1$, then N^* contains a normal subgroup N_0^* of finite index such that $\text{nil } N_0^* \leq \text{nil } N$ and $\text{der}(N_0^*) \leq \text{der}(N)$.*

It will not have escaped the reader that, in advancing to a torsion-free cover, we have weakened the properties regarding the Fitting subgroups from Theorem 1.13. Our next

corollary allows us to recover these properties, but at the expense of diluting the assertion about the derived lengths of the solvable radicals.

Corollary 6.4. *Let G be an \mathfrak{M} -group satisfying the three equivalent statements in Theorem 1.13 for a set of primes π . Set $N = \text{Fitt}(G)$ and $S = \text{solv}(G)$. Then there is an \mathfrak{M}_π^π -group G^* and an epimorphism $\phi : G^* \rightarrow G$ with the following three properties, where $N^* = \text{Fitt}(G^*)$ and $S^* = \text{solv}(G^*)$.*

- (i) $N^* = \phi^{-1}(N)$; hence $S^* = \phi^{-1}(S)$.
- (ii) If $S \neq 1$, then S^* contains a normal subgroup S_0^* of finite index such that $N^* \leq S_0^*$ and $\text{der}(S_0^*) = \text{der}(S)$.
- (iii) $\text{nil } N^* = \max\{\text{nil } N, 1\}$ and $\text{der}(N^*) = \max\{\text{der}(N), 1\}$.

The second open question of the article addresses whether it might be possible to completely preserve both the nilpotency class of the Fitting subgroup and the derived length of the group when constructing a cover entirely bereft of torsion.

Open Question 6.5. *Let G be an \mathfrak{M} -group satisfying the three equivalent statements in Theorem 1.13 for a set of primes π . Is there an \mathfrak{M}_π^π -group G^* and an epimorphism $\phi : G^* \rightarrow G$ satisfying conditions (ii)-(iv) in Theorem 1.5?*

The proofs of Corollaries 6.2 and 6.3, provided below, reveal that the above question may be reduced to the case where G is a finite group.

We find it convenient to first prove the second corollary.

Proof of Corollary 6.4. First we observe that the case $S = 1$ is a consequence of Lemma 3.18. Assume $S \neq 1$. Let G^\dagger be an \mathfrak{M}_1^π -cover of G satisfying properties (ii)-(iv) in Theorem 1.5. Pick a torsion-free solvable normal subgroup L in G^\dagger such that G^\dagger/L is finite. Writing $N^\dagger = \text{Fitt}(G^\dagger)$, we apply Lemma 3.18 to cover G^\dagger/L with an \mathfrak{M}_0^\emptyset -group H such that N^\dagger/L lifts to a nilpotent subgroup of the same nilpotency class and derived length. Now put $G^* = G^\dagger \times_{G^\dagger/L} H$. Then G^* is an \mathfrak{M}_π^π -group that covers G . Moreover, according to parts (ii) and (iii) of Lemma 3.17, G^* satisfies properties (i) and (iii) of Corollary 6.4.

It remains to establish statement (ii) of the corollary. For this, we let $\epsilon : H \rightarrow G^\dagger/L$ be the covering defined above and put $S^\dagger = \text{solv}(G^\dagger)$. Also, set $U = L \times_Q (\text{Ker } \epsilon)$. We have $\text{der}(L) \leq \text{der}(S)$ and

$$\text{der}(\text{Ker } \epsilon) \leq \text{der}(N^\dagger/L) \leq \text{der}(S^\dagger) = \text{der}(S).$$

As a result, $\text{der}(U) \leq \text{der}(S)$. Write $S^* = \text{solv}(G^*)$. Since $\text{der}(S^*) \geq \text{der}(S)$, it is straightforward to see that S^* must therefore contain a subgroup S_0^* such that $U \leq S_0^*$ and $\text{der}(S_0^*) = \text{der}(S)$. Because G^*/U is finite, this, then, proves assertion (iii). \square

Proof of Corollary 6.3. The case $S = 1$ is again a result of Lemma 3.18. Suppose $S \neq 1$. Let G^\dagger and L be as described in the proof of Corollary 6.4. As before, we invoke Lemma 3.18 to obtain an epimorphism $\epsilon : H \rightarrow G^\dagger/L$ such that H belongs to \mathfrak{M}_0^\emptyset ; this time, however, we ensure that $\epsilon^{-1}((S^\dagger/L)^{(d-1)})$ is abelian, where $S^\dagger = \text{solv}(G^\dagger)$ and $d = \text{der}(S)$. Since $\text{solv}(H) = \epsilon^{-1}(S^\dagger/L)$, $\text{der}(\text{solv}(H)) \leq d$. As in the previous proof, set $G^* = G^\dagger \times_{G^\dagger/L} H$, making G^* an \mathfrak{M}_π^π -cover of G . Writing $S^* = \text{solv}(G^*)$, we have $S^* = S^\dagger \times_{G^\dagger/L} \text{solv}(H)$. Hence assertions (i) and (ii) of the corollary are true.

Finally, we verify that G^* satisfies property (iii). To show this, we set $N_0^* = (N^\dagger \cap L) \times (\text{Ker } \epsilon)$. Then N_0^* is a normal subgroup of finite index in $N^* := \text{Fitt}(G^*)$, and $\text{nil } N_0^* = \text{nil}(N^\dagger \cap L) \leq \text{nil } N$. Similarly, we have $\text{der}(N_0^*) \leq \text{der}(N)$. \square

We conclude this section by stating an important special case of Corollary 6.4.

Corollary 6.6. *Let π be a set of primes and N a nilpotent π -minimax group. Then N can be covered by a nilpotent torsion-free π -minimax group N^* such that $\text{nil } N^* = \text{nil } N$ and $\text{der}(N^*) = \text{der}(N)$.* \square

7 Examples of finitely generated \mathfrak{M}_∞ -groups

In this section, we collect some examples of finitely generated \mathfrak{M}_∞ -groups. The principal purpose of these examples is to show that there are \mathfrak{M}_∞ -groups that fail to admit an \mathfrak{M}_1 -covering whose kernel is polycyclic. This is demonstrated by Examples 7.1 and 7.6, the first of which has two primes in its spectrum and the second just a single prime.

Example 7.1. For any pair of distinct primes p and q , we define a finitely generated solvable minimax group $G_1(p, q)$ with the following properties.

- The spectrum of $G_1(p, q)$ is $\{p, q\}$.
- The finite residual $R(G_1(p, q))$ is isomorphic to $\mathbb{Z}(p^\infty)$, and its centralizer has infinite index.
- The kernel of every \mathfrak{M}_1 -covering of $G_1(p, q)$ has q in its spectrum.

Let V be the group of upper triangular 3×3 matrices (a_{ij}) such that $a_{12} \in \mathbb{Z}[1/pq]$, $a_{13} \in \mathbb{Z}[1/pq]$, $a_{23} \in \mathbb{Z}[1/p]$, $a_{33} = 1$, $a_{11} = q^s$, and $a_{22} = p^r$, where $r, s \in \mathbb{Z}$. Now let A be the normal abelian subgroup of V consisting of those matrices in V that differ from the identity matrix at most in the a_{13} entry and where $a_{13} \in \mathbb{Z}[1/q]$. Set $G_1(p, q) = V/A$. Then $G_1(p, q)$ is a finitely generated solvable minimax group, and the first two properties listed above hold. The third assertion is proved in Lemma 7.2 below.

Acknowledgement. Example 7.1 was suggested to the authors by an anonymous referee.

Lemma 7.2. *Let p and q be distinct primes. If K is the kernel of an \mathfrak{M}_1 -covering of $G_1(p, q)$, then $q \in \text{spec}(K)$.*

Proof. Let $\phi : G^* \rightarrow G_1(p, q)$ be an \mathfrak{M}_1 -covering of $G_1(p, q)$ such that $\text{Ker } \phi = K$. Write $P = R(G_1(p, q))$, $N^* = \text{Fitt}(G^*)$, and $P^* = N^* \cap \phi^{-1}(P)$. Since G^*/N^* is virtually polycyclic, we have $\phi(P^*) = P$. Choose i to be the largest integer such that $\phi(Z_i(P^*)) \neq P$, and take $g \in G^*$ to be a preimage under ϕ of the image in $G_1(p, q)$ of the 3×3 matrix that differs from the identity in only the $(1, 1)$ position, which is occupied by q . Viewing P as a $\mathbb{Z}G^*$ -module, we have $g \cdot a = qa$ for all $a \in P$; in other words, $g - q \in \text{Ann}_{\mathbb{Z}G^*}(P)$.

Let $\pi = \text{spec}(K) \cup \{p\}$. Invoking Lemma 3.21, we obtain a polynomial $f(t) \in \mathbb{Z}[t]$ such that the constant term of $f(t)$ is a π -number and $f(g) \cdot a = 0$ for all $a \in Z_{i+1}(P^*)/Z_i(P^*)$. It follows that $f(g)$ annihilates a nontrivial $\mathbb{Z}G^*$ -module quotient \bar{P} of P . Moreover, we can find $r \in \mathbb{Z}$ and a polynomial $g(t) \in \mathbb{Z}[t]$ such that $f(t) = (t - q)g(t) + r$. This means $ra = 0$ for all $a \in \bar{P}$, and so $r = 0$. As a result, q divides the constant term in $f(t)$, which implies $q \in \text{spec}(K)$. \square

For our next family of examples, we employ the following general construction.

Construction 7.3. Let Q be a group and A a $\mathbb{Z}Q$ -module. Denote the exterior square $A \wedge A = A \wedge_{\mathbb{Z}} A$ by B , and let $f : A \times A \rightarrow B$ be the alternating bilinear map $(a, a') \mapsto a \wedge a'$. Let N denote the nilpotent group with underlying set $B \times A$ and multiplication

$$(b, a)(b', a') := (b + b' + a \wedge a', a + a').$$

We extend the action of Q on A diagonally to an action of Q on N and set $G = N \rtimes Q$. Then G is finitely generated if and only if Q is finitely generated and A is finitely generated as a $\mathbb{Z}Q$ -module.

To study our examples, we require Lemmas 7.4 and 7.5 below. In 7.4 and Example 7.6, when A is an abelian group and R a commutative ring, we write A_R for the R -module $R \otimes_{\mathbb{Z}} A$. Furthermore, in statement (ii) of Lemma 7.4, we adhere to the convention $\binom{n}{k} := 0$ if $n < k$.

Lemma 7.4. *Let A be a torsion-free abelian minimax group and let p be a prime. Then the following formulae hold.*

- (i) $h(A) = m_p(A) + \dim_{\mathbb{F}_p}(A_{\mathbb{F}_p})$.
- (ii) $m_p(A \wedge A) = \binom{h(A)}{2} - \binom{h(A) - m_p(A)}{2}$.

Proof. Assertion (i) can be easily checked for the case $h(A) = 1$. The general case may then be deduced by inducting on $h(A)$.

Statement (ii) follows readily from (i):

$$\begin{aligned} m_p(A \wedge A) &= h(A \wedge A) - \dim_{\mathbb{F}_p}(A \wedge A)_{\mathbb{F}_p} \\ &= \dim_{\mathbb{Q}}(A_{\mathbb{Q}} \wedge_{\mathbb{Q}} A_{\mathbb{Q}}) - \dim_{\mathbb{F}_p}(A_{\mathbb{F}_p} \wedge_{\mathbb{F}_p} A_{\mathbb{F}_p}) \\ &= \binom{h(A)}{2} - \binom{h(A) - m_p(A)}{2}. \end{aligned} \quad \square$$

Lemma 7.5. *Let $a_0, \dots, a_m \in \mathbb{Z}$ such that $a_0 a_m \neq 0$ and $\gcd(a_0, \dots, a_m) = 1$. Regarding $f(t) := a_m t^m + a_{m-1} t^{m-1} + \dots + a_1 t + a_0$ as an element of the Laurent polynomial ring $\mathbb{Z}[t, t^{-1}]$, let J be the principal ideal of $\mathbb{Z}[t, t^{-1}]$ generated by $f(t)$. Furthermore, let π denote the set of primes dividing $a_0 a_m$. Then the underlying additive group of the ring $\mathbb{Z}[t, t^{-1}]/J$ is torsion-free and π -minimax and has Hirsch length m .*

Proof. Set $A = \mathbb{Z}[t, t^{-1}]/J$. The hypothesis $\gcd(a_0, \dots, a_m) = 1$ renders A torsion-free as an abelian group. As a result, we can embed A in the ring $A' := \mathbb{Z}[\frac{1}{a_0 a_m}, t, t^{-1}]/(f(t))$. It is easy to see that A' is generated by the images of $1, t, \dots, t^{m-1}$ as a $\mathbb{Z}[\pi^{-1}]$ -module, and that these images are linearly independent over $\mathbb{Z}[\pi^{-1}]$. Therefore the additive group of A is π -minimax with Hirsch length m . \square

The examples below appear in the first author's doctoral thesis [13], although our observation about their coverings in Lemma 7.7 is new.

Example 7.6. For each prime p , we define a finitely generated solvable minimax group $G_2(p)$ which will turn out to have the following properties.

- The spectrum of $G_2(p)$ is $\{p\}$.
- The finite residual $R(G_2(p))$ is isomorphic to $\mathbb{Z}(p^\infty)$, and its centralizer has infinite index.
- The kernel of every \mathfrak{M}_1 -covering of $G_2(p)$ has p in its spectrum.

Let $f(t)$ denote the polynomial $pt^3 + t^2 - t + p$ in the Laurent polynomial ring $\mathbb{Z}[t, t^{-1}]$. According to Lemma 7.5, $A := \mathbb{Z}[t, t^{-1}]/(f(t))$ is a torsion-free p -minimax group of Hirsch length 3. Moreover, $\dim_{\mathbb{F}_p}(A_{\mathbb{F}_p}) = 1$, so that Lemma 7.4(i) implies $m_p(A) = 2$. Setting $B = A \wedge A$, we conclude from Lemma 7.4(ii) that $h(B) = m_p(B) = 3$. Therefore $B \cong \mathbb{Z}[1/p]^3$.

Since $f(t)$ has no rational root, it is irreducible over \mathbb{Q} . Computing its discriminant, we ascertain that $f(t)$ must have three distinct roots in \mathbb{C} . Taking α, β , and γ to be these roots, we have

$$\begin{aligned}\alpha + \beta + \gamma &= -1/p, \\ \alpha\beta + \beta\gamma + \gamma\alpha &= -1/p, \\ \alpha\beta\gamma &= -1.\end{aligned}$$

Now let $\phi : A_{\mathbb{C}} \rightarrow A_{\mathbb{C}}$ and $\psi : B_{\mathbb{C}} \rightarrow B_{\mathbb{C}}$ be the linear transformations induced by the left-action of t . Note that for $A_{\mathbb{C}}$ this action arises from left-multiplication, and for $B_{\mathbb{C}}$ it is the resulting diagonal action. Since α, β , and γ are the eigenvalues of ϕ , the complex numbers $\hat{\alpha} := \beta\gamma$, $\hat{\beta} := \gamma\alpha$, and $\hat{\gamma} := \alpha\beta$ are the eigenvalues of ψ . Moreover, we find

$$\begin{aligned}\hat{\alpha} + \hat{\beta} + \hat{\gamma} &= -1/p, \\ \hat{\alpha}\hat{\beta} + \hat{\beta}\hat{\gamma} + \hat{\gamma}\hat{\alpha} &= 1/p, \\ \hat{\alpha}\hat{\beta}\hat{\gamma} &= 1.\end{aligned}$$

Thus the characteristic polynomial of ψ is $g(t) := pt^3 + t^2 + t - p$. Hence $g(t)$ annihilates B . Notice further that $g(t)$, too, is irreducible over \mathbb{Q} . As a consequence, $\mathbb{Q}[t, t^{-1}]/(g(t))$ is a simple $\mathbb{Q}[t, t^{-1}]$ -module, and every nonzero cyclic $\mathbb{Z}[t, t^{-1}]$ -submodule of B is isomorphic to $\mathbb{Z}[t, t^{-1}]/(g(t))$. Take B_0 to be one such submodule. By the same analysis as for A , we determine that B_0 is torsion-free p -minimax of Hirsch length 3 with $m_p(B_0) = 2$. Therefore $B/B_0 \cong \mathbb{Z}(p^\infty)$, and, since B_0 is a rationally irreducible $\mathbb{Z}[t, t^{-1}]$ -module, so is B . Moreover, B_0 is normal in the group $G := N \rtimes Q$ described in Construction 7.3, where $Q := \langle t \rangle$ and the underlying set of N is $B \times A$. We define $G_2(p)$ to be the quotient G/B_0 . Then $G_2(p)$ is a finitely generated solvable p -minimax group and $R(G_2(p)) = B/B_0$. Furthermore, the image of t in $G_2(p)$ acts on $R(G_2(p)) \cong \mathbb{Z}(p^\infty)$ in the same fashion as some element λ of \mathbb{Z}_p^* . Since $g(\lambda) = 0$, we have $\lambda \equiv -1 \pmod{p}$. As $\lambda \neq -1$, this means that λ has infinite order. Consequently, the centralizer of $R(G_2(p))$ has infinite index in $G_2(p)$.

Lemma 7.7. *Let p be a prime. If K is the kernel of an \mathfrak{M}_1 -covering of $G_2(p)$, then $p \in \text{spec}(K)$.*

To prove Lemma 7.7 we employ a theorem of B. Wehrfritz.

Theorem 7.8. [30, Theorem 1] *Let π be a set of primes and G an \mathfrak{M}_1^π -group. Then the holomorph $G \rtimes \text{Aut}(G)$ is isomorphic to a subgroup of $\text{GL}_m(\mathbb{Z}[\pi^{-1}])$ for some natural number m .* \square

We also require the following special case of [1, Theorem 6.3].

Theorem 7.9. (R. Baer) *If π is a set of primes and N is a nilpotent π -minimax group, then every solvable subgroup of $\text{Aut}(N)$ is π -minimax.* \square

Baer's result has the corollary below.

Corollary 7.10. *Let π be a finite set of primes. Then every virtually solvable subgroup of $\text{GL}_n(\mathbb{Z}[\pi^{-1}])$ is π -minimax.* \square

Proof of Lemma 7.7. Pick an arbitrary \mathfrak{M}_1 -cover G^* of $G_2(p)$, and let K be the kernel of the corresponding covering. In view of Lemmas 3.17 and 3.18, we will not lose any real generality if we assume that G^* is torsion-free. Set $N = \text{Fitt}(G_2(p))$, $N^* = \text{Fitt}(G^*)$, $C = C_{G^*}(K)$, $N^\dagger = N^* \cap C$, and $K^\dagger = K \cap C$. By Theorem 7.8 and Corollary 7.10, G^*/C is π -minimax, where $\pi = \text{spec}(K)$. Hence G^*/N^\dagger is also π -minimax. Let P^\dagger be the intersection of N^\dagger with the preimage of the quasicyclic subgroup P of G . Since $K^\dagger \leq Z(N^\dagger)$ and P^\dagger/K^\dagger is torsion, we have $P^\dagger \leq Z(N^\dagger)$.

Suppose $p \notin \pi$. This implies that the image of P^\dagger in G is P . Also, since N/P is a rationally irreducible $\mathbb{Z}G$ -module, the covering induces a $\mathbb{Z}G^*$ -module isomorphism from N^\dagger/P^\dagger to a submodule of finite index in N/P . Put $B^\dagger = (N^\dagger/P^\dagger) \wedge (N^\dagger/P^\dagger)$ and $B = N/P \wedge N/P$. In the discussion of Example 7.6, it is shown that B is rationally irreducible as a $\mathbb{Z}G^*$ -module. Hence the same holds for B^\dagger . Furthermore, by Lemma 7.4(ii), we have $m_p(B^\dagger) = 3$.

The commutator maps $N \times N \rightarrow P$ and $N^\dagger \times N^\dagger \rightarrow P^\dagger$ induce $\mathbb{Z}G^*$ -module homomorphisms $\eta : B \rightarrow P$ and $\theta : B^\dagger \rightarrow P^\dagger$, respectively, with η being surjective. Consider the commutative square

$$\begin{array}{ccc} B^\dagger & \xrightarrow{\theta} & P^\dagger \\ \downarrow & & \downarrow \\ B & \xrightarrow{\eta} & P \end{array}$$

of $\mathbb{Z}G^*$ -module homomorphisms, in which both vertical maps are induced by the covering. Since the image of B^\dagger in B has finite index, the composition $B^\dagger \rightarrow B \xrightarrow{\eta} P$ must be epic, which means $\theta \neq 0$. The rational irreducibility of B^\dagger implies, then, that $\text{Ker } \theta = 0$. Thus $m_p(P^\dagger) \geq 3$, and so $m_p(K^\dagger) \geq 2$, contradicting the assumption $p \notin \pi$. Therefore $p \in \pi$. \square

In the finitely generated \mathfrak{M}_∞ -groups described above, the finite residual lies in the center of the Fitting subgroup. However, it is important to recognize that this is not always the case. We illustrate this with Example 7.11, which also appears in [15], though the discussion there focuses on another aspect. Our description of the example is based on Construction 7.3.

Example 7.11. For each prime p and positive integer m , we define a finitely generated solvable minimax group $G_3(p, m)$ such that $R(G_3(p, m))$ is not contained in the center of the Fitting subgroup when $m \geq 2$. Apply Construction 7.3 with $A = \mathbb{Z}[1/p]^{m+1}$ and Q free abelian of rank two generated by the automorphisms ϕ and ψ of A with the following definitions:

$$\phi(a_0, a_1, \dots, a_m) = (a_0, a_1, a_2 + a_1, a_3 + a_2, \dots, a_m + a_{m-1});$$

$$\psi(a_0, a_1, \dots, a_m) = (p^{-1}a_0, pa_1, \dots, pa_m).$$

If N and G are as defined as in 7.3, then G is a solvable p -minimax group. Furthermore, since the cyclic group $\langle \phi \rangle$ acts nilpotently on A with class m , we have $\text{Fitt}(G) = \langle N, \phi \rangle$. Also, G is finitely generated since A is finitely generated as a $\mathbb{Z}Q$ -module.

Letting e_0, \dots, e_m be the standard basis of A , take B_0 to be the subgroup of B generated by $e_0 \wedge e_1, \dots, e_0 \wedge e_m$. Then B_0 is centralized by ψ and so is normal in G . We define $G_3(p, m) = G/B_0$. Observe that the action of $\langle \phi \rangle$ on $R(G_3(p, m))$ is nilpotent of class m , and that the subgroup of $G_3(p, m)$ generated by $R(G_3(p, m))$ and the image of ϕ is contained in $\text{Fitt}(G_3(p, m))$. As a result, if $m \geq 2$, $R(G_3(p, m))$ is not contained in the center of $\text{Fitt}(G_3(p, m))$.

As observed in the introduction to the paper, part of the significance of Theorem 1.5 derives from the fact that there are uncountably many isomorphism classes of finitely generated \mathfrak{M}_∞ -groups. We conclude the article by using Example 1.4 to provide a simple proof of this well-known proposition. Another approach to showing this is mentioned in [16, p. 104].

Proposition 7.12. *There are uncountably many isomorphism classes of finitely generated \mathfrak{M}_∞ -groups.*

The proof of the proposition is based on the following lemma.

Lemma 7.13. [8, Lemma III.C.42] *Let G be a finitely generated group. Then G has uncountably many nonisomorphic quotients if and only if it has uncountably many normal subgroups.* \square

Proof of Proposition 7.12. In fact, we show that, for any prime p , there are uncountably many nonisomorphic, finitely generated p -minimax groups. Fix p and let Γ be the direct product of two copies of the group G from Example 1.4. The center $Z(\Gamma)$ is isomorphic to $\mathbb{Z}(p^\infty) \oplus \mathbb{Z}(p^\infty)$, which has uncountably many subgroups. Therefore, by Lemma 7.13, Γ has uncountably many nonisomorphic quotients. \square

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