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Developing the *3DMath* Dynamic Geometry Software: Theoretical Perspectives on Design

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Designing successful learning environments entails drawing on theoretical perspectives on learning while, at the same time, being cognisant of the affordances and constraints of the technology. This paper reports on the development of a software environment called 3DMath, *a dynamic three-dimensional geometry microworld aimed at enabling learners to construct, observe and manipulate geometrical figures in a 3D-like space. During the development of* 3DMath, *the key elements of visualisation, including theoretical ideas of mental images, external representations, and the processes and abilities of visualisation, were taken into consideration. The aim of this paper is to illustrate how the design of this particular software was informed by these elements of visualisation, as well as by theories related to the philosophical basis of mathematical knowledge and by semiotics. The paper illustrates how the features of software can be designed to take account of relevant theoretical notions and to satisfy the characteristics of instructional techniques that are appropriate to theoretical perspectives on learning.*

1 INTRODUCTION

During recent years the design of educational software to support pupils in developing mathematics concepts has been prolific. Many of the pedagogical advantages and limitations of devising such materials have been described (see, for example, Lagrange, 2005; Luengo, 2005; National Council for Education Technology (NCET), 2000; Nicaud, Bouhineau and Chaachoua, 2004).

Yet, in terms of the design of successful educational software, much remains to be done and a number of studies and national reports set out a radical agenda specifying the need for the research community not only to bridge the gap between technology and pedagogy, but also to develop solid theoretical frameworks (International Federation for Information Processing, 2005, NCET, 2000). For the most part, this is because the fields of computer-supported learning, of pedagogy, and of software and technology design, have frequently been treated separately; with pedagogy often being based on what the technology appears to permit, rather than being fully integrated as a basis for technological design. This means that the possibility of relying on solid theoretical frameworks is, according to the International Federation for Information Processing (IFIP, 2005), "one of the key factors that can enable conception of the many positive experiences already taking place in order to reach the definition of reliable innovative reference models".

This paper is a modest contribution to meeting the challenges to research set out in these recent research reports. It does this by reporting on the theoretical perspectives underpinning the design of a 3D geometry software environment called *3DMath*. With the emergence of dynamic geometry software (DGS) in the late 1980s, the teaching of geometry in general and geometric theorems in particular, has aroused renewed interest (de Villiers, 1996; Hanna, 2000). There are many DGS programs, including *Cabri*, *Cinderella*, *GeoGebra*, *Geometer's SketchPad*, *Geometry Inventor*, *Geometric Supposer*, *Geometry Assistant*, *Euklid*, *CaR*, *GeoNET*, *GeoLOG*, *Jeometry*, *GRACE*, *DrGEO*, *Wingeom*, etc. Many schools in different countries across the world use DGS to enhance both curricula and student learning in geometry. DGS is used successfully in the teaching and learning of geometry because of its interactive style of direct manipulation of geometrical objects (Jones, 2001). However, at present, and for the most part, such use remains restricted to the 2D drawing canvas on the computer screen and to Euclidean geometry, with development related to three dimensions being in its infancy (Accascina and Rogora, 2006; McClintock, Jiang and July, 2002; Yeh and Nason, 2004).

While an emphasis on 2D geometry within the school curriculum is well established in practice, it is not necessarily the best way of developing learners' visualization and spatial thinking abilities. This is because the construction of a sense of 3D geometry (stereometry) *from* 2D geometry is neither necessarily easy nor a natural way to achieve 3D spatial sense (Gutiérrez, 1996). What is more, even though commonly available forms of DGS can be used to construct 3D figures, the software may lack the features necessary to enable students to develop the ability to visualize 3D objects. For example, someone experienced in 3D geometry, when viewing such constructions, may easily recognise that the flat shape on the computer screen is meant to be a 3D object. Yet for learners, such a representation of a 3D spatial figure may not have the sort of spatial depth that supports learning and this may inhibit the learning of stereometry.

The purpose of the *3DMath* software, reported in this paper, is to enhance the teaching of stereometry in the middle school years (when pupils are aged 8-12) in a way that permits students to explore the interrelationships within a single figure or amongst figures. The design of the *3DMath* software aims to provide opportunities for students to experiment with 3D objects and shapes as total entities rather than solely as relationships amongst the component parts of these objects and shapes. Hence, the purpose of this paper is to present the theoretical principles informing the design of the *3DMath* dynamic geometry software environment, and to review its design, capabilities, and likely teaching potential. In what follows, the theories upon which the design of the software is based are set out, and, in later sections, the design methodology and the capabilities of the software for the teaching of spatial geometry are analysed in terms of these theoretical notions.

2 DESIGN CONTEXT

In many classrooms, if three-dimensional geometry is taught at all, it is done so using static pictures of geometric solids presented in textbooks. Yet students are known to have difficulty reasoning from two-dimensional representations of three-dimensional objects (Raquel, 2002; Parzysz, 1988). Of course, physical 3D manipulatives (in the form of models of 3D shapes) are available to enhance teaching, and it is known that students who use manipulatives in their mathematics classes usually outperform those who do not, although the benefits may be slight (Clements, 1999). The primary reason for this is that the physical nature of a manipulative does not necessarily carry to the learner the meaning of a mathematical idea. Thus the reason that, in mathematics education, there is significant interest in computer-based representations is because such representations are built on the relevant mathematical ideas (Balacheff and Kaput, 1996).

In terms of computer-based representations of geometrical objects, Olkun (2003), for example, compared the experiences of a group of learners using computer-based representations with another group using concrete manipulatives and found that while both computer and concrete groups improved significantly, the computer group improved slightly more, with older pupils (fifth grade) benefiting more from the computer-based manipulatives. Nevertheless, Dixon (1995) showed that while a DGS might be an effective tool in improving students' two dimensional visualisation, with that particular software no such improvement occurred with students' three dimensional visualisation. Such evidence illustrates the need for software tools for 3D geometry to be based on a suitable theoretical framework.

3 THEORETICAL FRAMEWORK

To overcome the known difficulties in teaching 3D geometry, this paper reports on the design of a 3D software named *3DMath*. The main objective of the *3DMath* project has been to develop a dynamic three-dimensional geometry microworld that enables students to construct, observe and manipulate geometrical figures in space, and to focus on modelling geometric situations, and supports teachers in helping their students to construct a suitable understanding of stereometry. To meet these purposes, the design of the proposed software followed three major fields of educational theory: first, the constructivist perspective on knowledge which argues that knowledge is personally constructed and is achieved by designing and making artifacts (both metal and physical) that are personally meaningful (Kafai and Resnick, 1996); secondly, the semiotic perspective about mathematics as a meaning-making endeavour which argues that any single sign (e.g. icon, diagram, symbol) is an incomplete representation of the object or concept, and thus multiple representations of knowledge should be encouraged during learning (Yeh and Nason, 2004); and thirdly, the fallibilist nature of mathematics which argues that mathematical knowledge is subject to revision and is a construction of humans (Ernest, 1994).

Further to these theoretical perspectives, the aim of developing the *3DMath* software was to enhance the abilities and processes in students that are closely associated with the idea of visual imagery as a mental scheme depicting spatial information (Presmeg, 1986). It is generally accepted that learning 3D geometry is strongly associated with spatial and visual ability (Dreyfus, 1991). From this perspective, Tso and Liang (2002) suggest that spatial abilities are important cognitive factors in learning geometry and incorporating spatial visualization and manipulation into learning activity could improve geometric learning. As a result, mathematics curricula in a range of countries (see, as an example, the National Council of Teachers of Mathematics, 2000) emphasise the importance of spatial abilities in mathematics education and recommend that 2D and 3D spatial visualisation and reasoning are core skills that all students should develop.

Spatial ability has had many definitions in the literature. Tartre (1990) defines spatial ability as the mental skills concerned with understanding, manipulating, reorganising, and interpreting relationships visually, while Linn and Petersen (1985) define it as the process of representing, transforming, generating, and recalling symbolic, non-linguistic information. Lohman (1988) proposes a three-factor model for spatial ability, covering 'spatial visualisation', 'spatial orientation', and 'spatial relations'. Accordingly, 'spatial visualisation' is the ability to comprehend imaginary movements in a three-dimensional space or the ability to manipulate objects in imagination, 'spatial orientation' is defined as a measure of one's ability to remain unconfused by the changes in the orientation of visual stimuli that require only a mental rotation of configuration, and 'spatial relation' is defined by the speed in manipulating simple visual patterns such as mental rotations and describes the ability to rotate mentally a spatial object quickly and correctly. Yakimanskaya (1991) argues that the creation of images is possible because of the accumulation of representations that serve as the starting point. The richer and more diverse the store of spatial representations, the easier it is to use images in solving problems.

Students learning 3D geometry thus need to acquire, and improve, a set of 'abilities' of visualisation to perform the necessary processes with specific mental images for a given 3D problem. Depending on the characteristics of the mathematics problems to be solved, and the images created, students need to be able to choose amongst several visual abilities which may have quite different foundations.

Hence, and following Gutiérrez (1996), the core visual abilities that should be taken into account in developing 3D dynamic geometry software are:

(a) 'Perceptual constancy', the ability to recognise that some properties of an object are independent of size, colour, texture, or position, and to remain unconfused when an object or picture is perceived in different orientations,
(b) 'Mental rotation', the ability to produce dynamic mental images and to visualise a configuration in movement,
(c) 'Perception of spatial positions', the ability to relate an object, picture, or mental image to oneself,
(d) 'Perception of spatial relationships', the ability to relate several objects, pictures, and/or mental images to each other, or simultaneously to oneself and
(e) 'Visual discrimination', the ability to compare several objects, pictures, and/or mental images to identify similarities and differences among them. 3D dynamic geometry software should aim to provide the learner with a variety and richness of spatial images.

Based on the above literature, and given that there is general agreement that visualisation is a basic component in learning and teaching 3D geometry (Gutiérrez, 1996), a rich concept of visualisation guided the design and the construction of the software. Visualisation, according to Gutiérrez and Jaime (1998) and Presmeg (1986), is an integration of four main elements: mental images, external representations, visualization processes, and visualization abilities. These four elements of visualisation are used in the account of the *3DMath* software that follows.

4 DESIGN PRINCIPLES FOR *3DMATH*

4.1 General Characteristics of *3DMath*

As with other DGS software, one of the distinguishing features implemented in *3DMath* is the ability to construct geometrical objects and specify relationships between them. Thus, within the computer environment, the geometrical objects created on the screen can be manipulated, moved and reshaped interactively by means of the mouse. The tools, definitions, exploration techniques, and visual representations associated with dynamic geometry contribute to a learning environment in a way that is quite different to its ruler-and-compass counterpart (Laborde, 1998). The realisation of the *3DMath* software allows students to see a geometric solid represented in several possible ways on the screen and to transform it, the intention here being to help students to acquire and develop abilities of visualisation in the context of 3D geometry. As Gutiérrez (1996) argues, when a person handles a real 3-dimensional solid and rotates it, the rotations made with the hands are so fast and unconscious that the person can hardly reflect on such actions. The design of *3DMath* facilitates the informed manipulation of directions of rotation, with the intention that this supports the student in devising strategies for moving objects and anticipating the result of a given rotation.

The design of *3DMath* aims to provide an integrated exploratory environment for creating, analysing, and investigating 3-D figures. This is in accordance with the fallibilist approach to mathematics since it allows teaching to focus on open-ended investigations in mathematics. Furthermore, *3DMath* is designed to enable teachers to integrate geometry with other areas of mathematics and other subjects (such as art, physics, etc.), and afford new conceptual and visual metaphors for building dynamic mathematical models. An aspect of the project that it is not possible to report on here (for reasons of space) is the design of accompanying teaching scenarios that are aimed at supporting teachers in providing opportunities for students for situated, authentic problem solving and modelling. The design of such teaching scenarios will be the subject of future papers.

In terms of the interface of the software (as illustrated in Figure 1), this is designed to be intuitive and to provide an open and generative environment that enables students to learn through making and designing personally meaningful artefacts. The interface also employs rich semiotic resources that enable multiple perspectives and representations for mathematical meaning-making (for example, students can represent a solid in 3D or its correspondence in 2D). Additionally, *3DMath* aims to be adaptable to meet the needs of teachers and students. It allows the teacher, for example, to decide which primitives and operations are made available to the students.

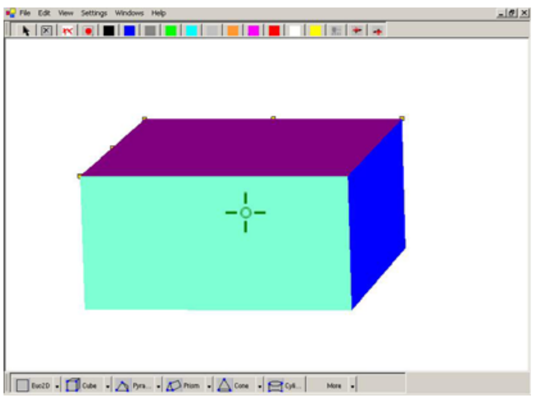

Figure 1 The Interface of *3DMath*

4.2 Visualisation Elements in *3DMath*

The usefulness of visualisation and multiple representations in the teaching of mathematics is recognised by mathematics educators and teachers of mathematics (Booth and Thomas, 2000). Based on previous research (see Gutiérrez, 1996), the design of *3DMath* aims to provide the means for students to focus on the mental images they create, and the processes and abilities of visualisation they use to solve problems, and, attention was paid to the four main elements: mental images, external representations, visualization processes, and visualisation abilities. These are considered in turn with a view to demonstrating how each was taken into account in designing the *3DMath* software.

4.3 Mental Images

A mental image is any kind of cognitive representation of a mathematical concept by means of spatial elements. Thus the design of *3DMath* aims, for example, to make it straightforward for students to construct different solids and perceive them in a concrete or pictorial form. The repetition of this process is known to help students to formulate a "picture in their mind's eyes" (Presmeg, 1986) so *3DMath* was designed to enable students to see solids in many positions on the screen and consequently gain a rich experience that allows them to form richer mental images than from textbooks or other static resources.

4.4 External representations

In cognitive theory, a distinction is often made between internal and external representation in that, while an internal representation is a hypothesised mental construct, an external representation is a material notation of some kind, such as a graph, an equation, or a geometrical figure (Ainley, Barton, Jones, Pfannkuch and Thomas, 2002). In this context, the *3DMath* environment is designed to be a rich environment for manipulating and transforming representations of solids. Given that most middle school students are known to find it difficult to understand that Figure 2, for example, could be the representation of either a pyramid or an octahedron, *3DMath* is designed so that students can rotate such a representation of a pyramid and see that Figure 2 is a special position of it.

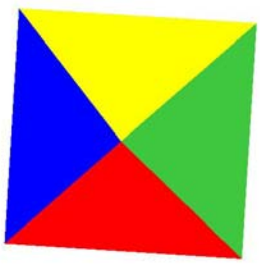

Figure 2 A Special Position of a Pyramid or Octahedron

4.5 Visualisation Processes

According to Presmeg (1986), a visualisation process is a mental or physical action where mental images are involved. Bishop (1980) identified two relevant processes of visualisation: firstly, interpreting figural information and the visual processing of abstract information; and secondly, the translation of abstract relationships and non-figural data into visual terms, the manipulation and extrapolation of visual imagery, and the transformation of one visual image into another. The design of *3DMath* incorporates Bishop's ideas by focusing on the processes of observation, construction and exploration in the ways described below.

Observing: observation allows students to see and understand the third dimension by changing the spatial system of reference (axes), choosing perspectives and displaying visual feedback on objects. The *3DMath* software was designed so that students can rotate a geometric object with reference to the three axes and thus gain a holistic view of the object. Features designed into the software include easy variation of the speed and the direction of the rotation of any object, directly controlled by the user of the software. What is more, the software is designed such that the drawing style of any object can be in a 'solid colour' view or in a 'transparent line' view, as illustrated in Figure 3, and students can select, label and colour the edges and faces of the objects.

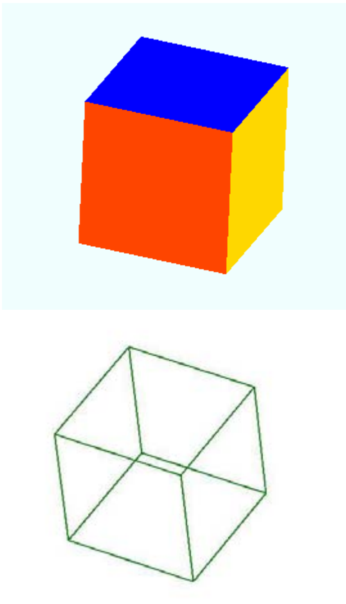

Figure 3 A Solid and Transparent View of a Cube

Constructing: construction allows a dynamic construction of geometrical figures from elementary objects (points, lines, planes) and construction primitives (intersection, parallel, etc.). The *3DMath* software was designed so that students can use such elementary objects to create 3D shapes, including being able to select the appropriate 2D figures and then forming the solids by dynamic animations, as illustrated in Figure 4.

Exploring: exploration allows students to explore and discover the geometrical properties of a figure. This is the main procedure adopted in most of the teaching scenarios that are available to accompany the *3DMath* software.

4.6 Visualisation Abilities

Informed by relevant research (see Gutiérrez, 1996, for a review), *3DMath* is designed in such a way as to accommodate the development of the following visualisation abilities: (a) the figureground perception, (b) perceptual constancy, (c) mental rotation, (d) perception of spatial positions, (e) perception of spatial relationships, and (f) visual discrimination.

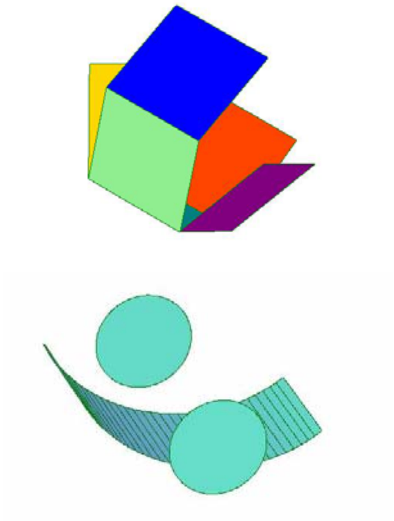

Figure 4 The Construction of a Cube and a Cylinder

In order to contribute to the development of these abilities, the design of *3DMath* encompasses the following features:

'*Dragging*': the dragging capability of the software enables students to rotate, move and resize 3D objects in much the same way as the commonly available 2D dynamic geometry software environments. The design approach for *3DMath* focused on enabling rotation to be executed in all directions through the provision of an on-screen rotation cursor that could also be used to determine the speed of the rotation. In addition, the design was made such that students are able to resize, proportionally, all the dimensions of the object or resize it only in one dimension, according to the requirements of the problem. For example, students can resize a cuboid, as illustrated in Figure 5, by shrinking or enlarging the side *AB* or *AC* or *AD*, and a cylinder by shrinking or enlarging the diameter *KL* of its base or by enlarging the height *LM*.

'*Tracing*': tracing is a particular instance of the design of the *3DMath* interface where only parts of the figure are displayed. The purpose of this feature is to provide learners with a way of performing a visual filtering of the main construction represented on the screen; that is, to allow them to extract and observe parts of the construction in an independent view.

'*Measuring*': the *3DMath* software was designed, as with commonly available 2D dynamic geometry software, so that students can measure the length of edges and the area of faces. In *3DMath*, students can also obtain the measure the volume of a solid. In providing these features, the *3DMath* software was designed such that all measurements are dynamic as the solids are being resized.

'*History*': in the design of *3DMath*, "history" is a textual feature that represents the declarative description of the figure. In designing this feature, the aim was to provide learners with a textual and chronological list of all the geometrical objects involved in the construction of the figure. Additionally, the *History* file can be used as input to the system. For example, a *History* file created by a student (or group of students) in one country can be used by another student in another country to reconstruct (or re-use) the same model. Using this feature, it would be possible to construct not only *Interactive* models, but also *Declarative* models (by importing *History* files) and *Interactive Programming* models (in which new parts of a construction are developed while existing parts are in use). Thus, while a *History* file is a textual feature, its use can support the development of visualisation abilities.

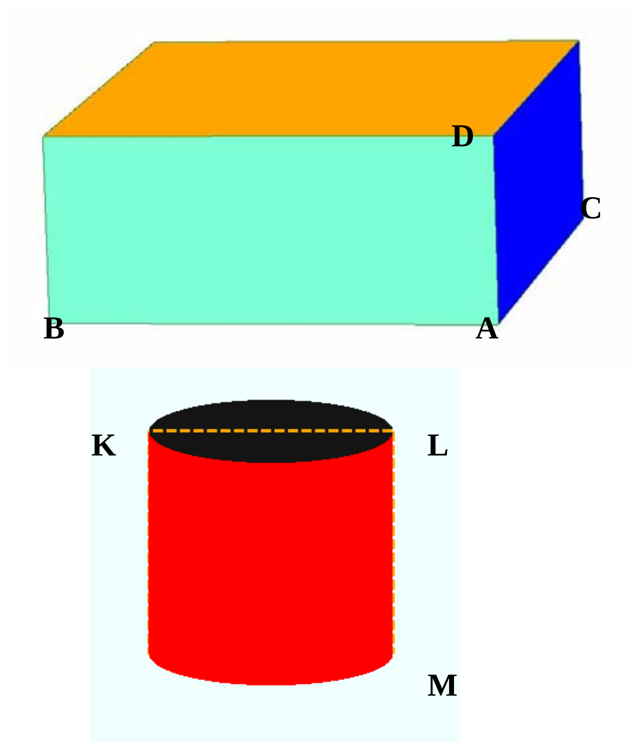

Figure 5 Resizing a Cuboid and a Cylinder

A further feature of the design of *3DMath* is the facility to export constructions as images (BMP, JPEG, etc), or in other rendering format (PS, XML,). The aim of this feature is to help learners record their work and thereby illustrate their developing visualisation ability. Teachers, of course, can also use the feature to create supporting educational materials.

5 CONCLUDING COMMENTS

As illustrated by this paper, the design of *3DMath* has been informed by theories based on the philosophy of mathematical knowledge, such as constructivism, and by semiotics. The main purpose of *3DMath* is to enhance students' understanding of 3D geometry with an emphasis on visualisation. Thus, during the developmental process of producing the software, the key elements of visualization, as defined by Presmeg (1986), Bishop (1980), Clements (1982) and Gutiérrez (1996) (mental images, external representations, processes, and abilities of visualisation), were carefully taken into consideration.

The aim is to continue developing the 3*DMath* software in such a way that it constitutes a powerful tool in the teaching of geometry. The output of the *3DMath* project is available, as open source, for improving practice, decision-making, and resources development in mathematics education.

The expectation is that *3DMath* might be used to enhance students' dynamic visualisation ability and enable them to gain a greater understanding of 3D mathematical concepts. The interactive, representationally-rich, environment of *3DMath* is intended to promote investigation and experimentation with 3D mathematical objects, and may contribute to the integration of mathematics with other subject areas, such as art and physics.

In developing the 3*DMath* software, the idea has been to seek to bridge the gap between technology and pedagogy, and develop solid theoretical frameworks that inform software design (NCET, 2000; IFIP, 2005). This is so that the pedagogy is fully integrated as a basis for technological design, rather than, as is often the case, the pedagogy being based on what the technology appears to permit. Finally, it is important to stress that the expected outcomes, both for teachers and for students, depend intimately on them, since the software cannot guarantee that users (teachers and students) take advantage of the possible potentials and affordances of the software.

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Constantinos Christou *is Professor of mathematics education at the University of Cyprus. His research focuses on the cognitive development of mathematical concepts. He has published more than 80 books, book chapters, and journal articles. He has participated in four European research projects and was the co-ordinator of four other research projects funded by the University of Cyprus and the Cyprus Research Foundation.*

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