

# SYMMETRIC COHOMOLOGY OF GROUPS

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**ABSTRACT.** We investigate the relationship between the symmetric, exterior and classical cohomologies of groups. The first two theories were introduced respectively by Staic and Zarelua. We show in particular, that there is a map from exterior cohomology to symmetric cohomology which is a split monomorphism in general and an isomorphism in many cases, but not always. We introduce two spectral sequences which help to explain the relationship between these cohomology groups. As a sample application we obtain that symmetric and classical cohomologies are isomorphic for torsion free groups.

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## 1. INTRODUCTION

Let  $G$  be a group and  $M$  be a  $G$ -module. In order to better understand 3-algebras arising in lattice field theory [?], Staic defined a variant of group cohomology, which he denoted by  $HS^*(G, M)$  and called *symmetric cohomology of groups* [?]. Some aspects of this theory were later extended by Singh [?] and Todea [?]. There is an obvious natural transformation from the symmetric cohomology to the classical Eilenberg-MacLane cohomology

$$\alpha^n : HS^n(G, M) \rightarrow H^n(G, M), \quad n \geq 0.$$

According to [?],[?],  $\alpha^n$  is an isomorphism if  $n = 0, 1$  and is a monomorphism for  $n = 2$ . By Corollary 2.3 in [?] we know that  $\alpha^2$  is an isomorphism if  $G$  has no elements of order two.

Ten years prior to this, Zarelua had also defined a version of group cohomology, denoted by  $H_\lambda^n(G, M)$  and called *exterior cohomology of groups* [?]. It also comes together with a natural transformation

$$\beta^n : H_\lambda^n(G, M) \rightarrow H^n(G, M),$$

with similar properties. The exterior cohomology has the following striking property: if  $G$  is a finite group of order  $d$ , then  $H_\lambda^i(G, M) = 0$  for all  $i \geq d$ .

The aim of this work is to obtain more information about homomorphisms  $\alpha^*$  and  $\beta^*$ . We construct a natural transformation  $\gamma^n : H_\lambda^n(G, M) \rightarrow HS^n(G, M)$  such that the following diagram commutes:

$$\begin{array}{ccc} H_\lambda^n(G, M) & \xrightarrow{\gamma^n} & HS^n(G, M) \\ & \searrow \beta^n & \swarrow \alpha^n \\ & H^n(G, M) & \end{array}$$

Our results in Section ?? show that the homomorphism  $\gamma^n : H_\lambda^n(G, M) \rightarrow HS^n(G, M)$  is a split monomorphism in general, and an isomorphism in certain cases, namely if  $0 \leq n \leq 4$ , or  $M$  has no elements of order two. In general,  $\gamma^5$  is not an isomorphism.

Our next results are related to the homomorphism  $\beta^n : H_\lambda^n(G, M) \rightarrow H^n(G, M)$ . We construct a spectral sequence for which  $\beta^n$  are edge homomorphisms,  $n \geq 0$ . As any first quadrant spectral sequence, it gives a 5-term exact sequence (see for example [?, Exercise 5.1.3]) which has the following form:

$$0 \rightarrow H_\lambda^2(G, M) \xrightarrow{\beta^2} H^2(G, M) \rightarrow \prod_{C_2 \subset G} H^2(C_2, M) \rightarrow H_\lambda^3(G, M) \xrightarrow{\beta^3} H^3(G, M).$$

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Here the product is taken over all subgroups of order two. The exactness at  $H^2(G, M)$  is an answer to Problem 25 by Singh in [?]. At the very end of Section 4 in [?], Staic wondered about the injectivity of the map  $\alpha^3$  under the assumption that  $G$  has no elements of order 2. A trivial consequence of our spectral sequence says that, if  $G$  has no elements of order two, then one has an exact sequence:

$$\begin{aligned} 0 \rightarrow H_\lambda^3(G, M) \xrightarrow{\beta^3} H^3(G, M) \rightarrow \prod_{C_3 \subset G} H^3(C_3, M) \rightarrow H_\lambda^4(G, M) \xrightarrow{\beta^4} \\ \xrightarrow{\beta^4} H^4(G, M) \rightarrow \prod_{C_3 \subset G} H^4(C_3, M) \rightarrow H_\lambda^5(G, M) \xrightarrow{\beta^5} H^5(G, M). \end{aligned}$$

In particular, if  $G$  has no elements of order two and three, then  $H_\lambda^i(G, M) = H^i(G, M)$ , for  $i = 0, 1, 2, 3, 4$ .

Among other consequences of our spectral sequence, we mention the following: if  $G$  is a torsion free group, then  $\beta^n : H_\lambda^n(G, M) \rightarrow H^n(G, M)$  is an isomorphism for all  $n \geq 0$ .

The paper is organised as follows: In Section 2 we recall the definitions of the symmetric and exterior cohomologies. In the next section we construct the transformation  $\gamma^*$  and prove our first result, which shows that  $\gamma^n$  is quite often an isomorphism, but not always. In the final section we construct a spectral sequence and we prove our main result Theorem ??.

## 2. PRELIMINARIES

**2.1. Classical cohomology.** Let  $G$  be a group and  $M$  be a  $G$ -module. One way to define the cohomology  $H^*(G, M)$  is via cochains, as  $H^*(C^*(G, M))$ . For every  $i \in \mathbb{N}$ , the group of  $i$ -cochains of  $G$  with coefficients in  $M$  is the set of functions from  $G^i$  to  $M$ :

$$C^i(G, M) = \{\phi : G^i \rightarrow M\}.$$

The  $i^{\text{th}}$  differential  $\partial^i : C^i(G, M) \rightarrow C^{i+1}(G, M)$  is the map

$$\begin{aligned} \partial^i(\phi)(g_0, g_1, \dots, g_i) &= g_0 \cdot \phi(g_1, \dots, g_i) \\ &+ \sum_{j=1}^i (-1)^j \phi(g_0, \dots, g_{j-2}, g_{j-1}g_j, g_{j+1}, \dots, g_i) \\ &+ (-1)^{i+1} \phi(g_0, \dots, g_{i-1}), \end{aligned}$$

where  $g_0, \dots, g_i \in G$ . Given a chain complex such as this one, one can define its normalised subcomplex. In each dimension  $i$ , define  $NC^i(G, M)$  to be the group of  $i$ -cochains  $\phi$  which satisfy the normalisation condition. This condition says that  $\phi(g_0, \dots, g_{i-1})$  vanishes whenever some  $g_j = 1$ , for  $0 \leq j \leq i-1$ . The canonical inclusion  $\iota : NC^*(G, M) \rightarrow C^*(G, M)$  is a chain equivalence [?].

Another way to define  $H^*(G, M)$  is via projective resolutions, as  $H^*(K^*(G, M))$ . The standard projective resolution of  $\mathbb{Z}$  by  $G$ -modules is the sequence of  $G$ -module homomorphisms [?]

$$\dots \rightarrow \mathbb{Z}[G^{i+1}] \xrightarrow{\partial_{i-1}} \mathbb{Z}[G^i] \rightarrow \dots \rightarrow \mathbb{Z}[G] \xrightarrow{\epsilon} \mathbb{Z},$$

where

$$\partial_{i-1}(g_0, \dots, g_i) = \sum_{j=0}^i (-1)^j (g_0, \dots, g_{j-1}, g_{j+1}, \dots, g_i),$$

and the mapping  $\epsilon$  sends each generator  $(g)$  to  $1 \in \mathbb{Z}$ . An element of

$$K^i(G, M) = \text{Hom}_G(\mathbb{Z}[G^{i+1}], M)$$

is then a function  $f : G^{i+1} \rightarrow M$  such that

$$f(sg_0, sg_1, \dots, sg_i) = s \cdot f(g_0, g_1, \dots, g_i).$$

The maps

$$K^i(G, M) \xrightarrow{\psi^i} C^i(G, M)$$

defined by

$$\psi^i(f)(g_1, \dots, g_i) = f(1, g_1, g_1 g_2, \dots, g_1 g_2 \cdots g_i)$$

induce an isomorphism of cochain complexes  $K^*(G, M) \rightarrow C^*(G, M)$  [?]. Moreover, one has a commutative diagram

$$\begin{array}{ccc} K^*(G, M) & \xrightarrow{\psi} & C^*(G, M) \\ \uparrow & & \uparrow \\ NK^*(G, M) & \xrightarrow{\psi} & NC^*(G, M) \end{array}$$

where the horizontal maps are isomorphisms and the vertical maps are inclusions and homotopy equivalences. Here  $NK^i(G, M)$  consists of such maps  $f \in K^i(G, M)$  that

$$f(x_0, \dots, x_i) = 0, \text{ if } x_j = x_{j+1}, \text{ for } 0 \leq j < i.$$

Thus

$$H^*(G, M) = H^*(NC^*(G, M)) = H^*(C^*(G, M)) = H^*(K^*(G, M)) = H^*(NK^*(G, M)).$$

**2.2. Symmetric cohomology.** We now discuss a subcomplex of  $C^*(G, M)$  introduced by Staic in [?] and [?]. It is based on an action of  $\Sigma_{n+1}$  on  $C^n(G, M)$  (for every  $n$ ) compatible with the differential. In order to define this action, it is enough to define how the transpositions  $\tau_i = (i, i+1)$ ,  $1 \leq i \leq n$  act. For  $\phi \in C^n(G, M)$  one defines:

$$(\tau_i \phi)(g_1, g_2, g_3, \dots, g_n) = \begin{cases} -g_1 \phi(g_1^{-1}, g_1 g_2, g_3, \dots, g_n), & \text{if } i = 1, \\ -\phi(g_1, \dots, g_{i-2}, g_{i-1} g_i, g_i^{-1}, g_i g_{i+1}, \dots, g_n), & 1 < i < n, \\ -\phi(g_1, g_2, g_3, \dots, g_{n-1} g_n, g_n^{-1}), & \text{if } i = n. \end{cases}$$

Denote by  $CS^n(G, M)$  the subgroup of the invariants of this action. That is,  $CS^n(G, M) = C^n(G, M)^{\Sigma_{n+1}}$ . Staic proved that  $CS^*(G, M)$  is a subcomplex of  $C^*(G, M)$  [?], [?].

**Definition 2.1.** *The homology of this subcomplex is called the symmetric cohomology of  $G$  with coefficients in  $M$  and is denoted by  $HS^n(G, M)$ .*

**Remark 2.2.** *There is a natural map  $\alpha^n : HS^n(G, M) \rightarrow H^n(G, M)$  induced by the inclusion  $CS^*(G, M) \hookrightarrow C^*(G, M)$ .*

**2.3. Exterior powers.** In order to define the chain complex introduced by Zarelua [?] we need to recall some facts about exterior powers.

**Definition 2.3.** *The exterior algebra  $\Lambda^*(A)$  of an abelian group  $A$  is a quotient algebra of the tensor algebra  $T^*(A)$  with respect to the two-sided ideal generated by the elements of the form  $a \otimes a \in T^2(A) = A \otimes A$ .*

A weaker version of this, denoted by  $\tilde{\Lambda}^*(A)$ , can be defined as the quotient algebra of the tensor algebra  $T^*(A)$  with respect to the two-sided ideal generated by the elements of the form  $a \otimes b + b \otimes a \in T^2(A)$ . Since

$$a \otimes b + b \otimes a = (a + b) \otimes (a + b) - a \otimes a - b \otimes b,$$

it is clear that one has the canonical quotient maps

$$\otimes^n(A) \twoheadrightarrow \tilde{\Lambda}^n(A) \twoheadrightarrow \Lambda^n(A).$$

Denote by  $\Delta^n(A)$  the kernel of the projection  $\tilde{\Lambda}^n(A) \twoheadrightarrow \Lambda^n(A)$ . Thus we have a short exact sequence

$$0 \rightarrow \Delta^n(A) \rightarrow \tilde{\Lambda}^n(A) \twoheadrightarrow \Lambda^n(A) \rightarrow 0.$$

Clearly  $\Lambda^1(A) = A = \tilde{\Lambda}^1(A)$ . Hence

$$(2.3.1) \quad \Delta^1(A) = 0.$$

The images of  $a_1 \otimes \cdots \otimes a_n \in \otimes^n(A)$  in  $\tilde{\Lambda}^n(A)$  and  $\Lambda^n(A)$  are denoted by  $a_1 \tilde{\wedge} \cdots \tilde{\wedge} a_n$  and  $a_1 \wedge \cdots \wedge a_n$  respectively. Recall that if  $A = \mathbb{Z}[S]$  is a free abelian group with a set  $S$  as basis, then  $\otimes^n(A)$  is a free abelian group with basis elements  $s_1 \otimes \cdots \otimes s_n$ , where  $s_i \in S$ . It is also well-known that  $\Lambda^n(A)$  is a free abelian group with basis elements  $s_1 \wedge \cdots \wedge s_n$ , where  $s_1 < \cdots < s_n$ . Here  $<$  is a total order on  $S$ .

In  $\tilde{\Lambda}^n(A)$ ,  $A = \mathbb{Z}[S]$ , things are a bit more complicated because of the relation  $2a\tilde{\lambda}a = 0$ , which is a consequence of the relation  $a\tilde{\lambda}b + b\tilde{\lambda}a = 0$ . It implies that  $\Delta^n(A)$  is an  $\mathbb{F}_2$ -vector space. The epimorphism  $\tilde{\Lambda}^n(\mathbb{Z}[S]) \rightarrow \Lambda^n(\mathbb{Z}[S])$  has a splitting given by  $s_1 \wedge \cdots \wedge s_n \mapsto s_1 \tilde{\lambda} \cdots \tilde{\lambda} s_n$ . Here  $s_1, \dots, s_n$  are distinct elements in  $S$ . Thus

$$(2.3.2) \quad \tilde{\Lambda}^n(\mathbb{Z}[S]) \cong \Lambda^n(\mathbb{Z}[S]) \oplus \Delta^n(\mathbb{Z}[S]),$$

Thus expressions of the form  $s_1 \tilde{\lambda} \cdots \tilde{\lambda} s_n$ , where  $s_1 \leq \cdots \leq s_n$ , are canonical generators of  $\tilde{\Lambda}^n(\mathbb{Z}[S])$ . Among these elements, ones with strict inequalities  $s_1 < \cdots < s_n$  form a basis of the summand corresponding to the free abelian group part, while the rest form a basis of the  $\mathbb{F}_2$ -vector space  $\Delta^n(\mathbb{Z}[S])$ .

**2.4. Exterior cohomology of groups.** We now discuss a subcomplex of  $K^*(G, M)$ , denoted by  $K_\lambda^*(G, M)$ , introduced by Zarelua in [?].

According to Lemma 3.1 in [?], there is a differential

$$\partial : \Lambda^{n+1}(\mathbb{Z}[G]) \rightarrow \Lambda^n(\mathbb{Z}[G])$$

in the exterior algebra generated by  $\mathbb{Z}[G]$  given by

$$\partial(g_0 \wedge \cdots \wedge g_n) = \sum_{i=0}^n (-1)^{i+1} g_0 \wedge \cdots \wedge \hat{g}_i \wedge \cdots \wedge g_n,$$

where, as usual, the hat  $\hat{\phantom{x}}$  denotes a missing value. The group  $G$  acts on this chain complex by:

$$g(g_1 \wedge g_2 \wedge \cdots \wedge g_n) = gg_1 \wedge gg_2 \wedge \cdots \wedge gg_n.$$

**Definition 2.4.** *The homology groups of the cochain complex (denoted by  $K_\lambda^*(G, M)$ )*

$$\text{Hom}_G(\Lambda^1(\mathbb{Z}[G]), M) \xrightarrow{\partial} \text{Hom}_G(\Lambda^2(\mathbb{Z}[G]), M) \xrightarrow{\partial} \cdots \xrightarrow{\partial} \text{Hom}_G(\Lambda^n(\mathbb{Z}[G]), M) \xrightarrow{\partial} \cdots$$

are called the exterior cohomology groups of the group  $G$  with coefficients in  $M$  and are denoted by  $H_\lambda^n(G, M)$ .

Therefore,  $K_\lambda^*(G, M)$  is the subcomplex of  $K^*(G, M)$ , which, for a degree  $n$ , consists of all  $G$ -maps  $f \in K^n(G, M)$  such that

$$f(g_0, \dots, g_i, g_i, \dots, g_n) = 0,$$

and

$$f(g_0, \dots, g_i, g_{i+1}, \dots, g_n) = -f(g_0, \dots, g_{i+1}, g_i, \dots, g_n),$$

for all  $0 \leq i < n$ .

**Remark 2.5.** *There is a natural transformation  $\beta^n : H_\lambda^n(G, M) \rightarrow H^n(G, M)$  induced by the inclusion  $K_\lambda^*(G, M) \hookrightarrow K^*(G, M)$ .*

**Remark 2.6.** *Let  $G$  be a finite group of order  $d$ . Since  $\mathbb{Z}[G]$  is a free abelian group of rank  $d$ , we have  $\Lambda^i(\mathbb{Z}[G]) = 0$ , for  $i > d$  and  $H_\lambda^n(G, M) = 0$  for  $n \geq d$ . On the other hand, as we will see later, the groups  $HS^n(C_2, M)$  are nontrivial for infinitely many  $n$ .*

### 3. COMPARISON OF SYMMETRIC AND EXTERIOR COHOMOLOGIES

**3.1. Construction of the map  $\gamma$ .** We need two more complexes:  $C_\lambda^*(G, M)$  and  $KS^*(G, M)$ . They are defined as follows.

**Definition 3.1.** *Let  $KS^*(G, M)$  denote the subcomplex of  $K^*(G, M)$  which consists of all  $G$ -maps  $f \in K^n(G, M)$  such that*

$$(3.1.1) \quad f(g_0, \dots, g_i, g_{i+1}, \dots, g_n) = -f(g_0, \dots, g_{i+1}, g_i, \dots, g_n)$$

for all  $0 \leq i < n$ .

So we have the following subcomplexes:

$$K_\lambda^*(G, M) \hookrightarrow KS^*(G, M) \hookrightarrow K^*(G, M).$$

**Definition 3.2.** Let  $C_\lambda^*(G, M)$  be the complex defined by

$$C_\lambda^*(G, M) = CS^*(G, M) \cap NC^*(G, M)$$

Thus  $\phi \in CS^n(G, M)$  belongs to  $C_\lambda^n(G, M)$  if  $\phi(g_0, \dots, g_{n-1})$  vanishes whenever some  $g_j = 1$ , for  $0 \leq j \leq n-1$ .

This subcomplex has already been considered in [?], where it was shown in Proposition 2.2 that if  $M$  has no elements of order 2, then  $HS^n(G, M)$  equals the  $n^{\text{th}}$  cohomology of  $C_\lambda^n(G, M)$  for all  $n$ . In fact,  $C_\lambda^n(G, M) = CS^n(G, M)$  in this case. We will later prove this fact in a different way.

We have the following subcomplexes:

$$C_\lambda^*(G, M) \hookrightarrow CS^*(G, M) \hookrightarrow C^*(G, M).$$

In order to understand the relationship between all these complexes it is useful to rewrite them in terms of resolutions, which we constructed in Lemma ?? below.

Since  $\mathbb{Z}[G^i] = \mathbb{Z}[G]^{\otimes i}$ , the standard projective resolution can be rewritten as

$$\dots \rightarrow \mathbb{Z}[G]^{\otimes i} \rightarrow \mathbb{Z}[G]^{\otimes i-1} \rightarrow \dots \rightarrow \mathbb{Z}[G]^{\otimes 2} \rightarrow \mathbb{Z}[G].$$

If one replaces the tensor algebra by either version of the exterior algebra, one still obtains a resolution, though in general no longer a projective one. This is the subject of the following lemma.

**Lemma 3.3.** One has a commutative diagram of resolutions of  $\mathbb{Z}$ :

$$\begin{array}{ccccccc} \dots & \longrightarrow & \mathbb{Z}[G]^{\otimes i} & \longrightarrow & \mathbb{Z}[G]^{\otimes i-1} & \longrightarrow & \dots \longrightarrow \mathbb{Z}[G]^{\otimes 2} \longrightarrow \mathbb{Z}[G] \\ & & \downarrow & & \downarrow & & \downarrow & \downarrow Id \\ \dots & \longrightarrow & \tilde{\Lambda}^i(\mathbb{Z}[G]) & \longrightarrow & \tilde{\Lambda}^{i-1}(\mathbb{Z}[G]) & \longrightarrow & \dots \longrightarrow \tilde{\Lambda}^2(\mathbb{Z}[G]) \longrightarrow \mathbb{Z}[G] \\ & & \downarrow & & \downarrow & & \downarrow & \downarrow Id \\ \dots & \longrightarrow & \Lambda^i(\mathbb{Z}[G]) & \longrightarrow & \Lambda^{i-1}(\mathbb{Z}[G]) & \longrightarrow & \dots \longrightarrow \Lambda^2(\mathbb{Z}[G]) \longrightarrow \mathbb{Z}[G] \end{array}$$

One denotes these resolutions by  $(T^{*+1}(\mathbb{Z}[G]), \partial)$ ,  $(\tilde{\Lambda}^{*+1}(\mathbb{Z}[G]), \partial)$  and  $(\Lambda^{*+1}(\mathbb{Z}[G]), \partial)$  respectively.

*Proof.* In this proof, take  $\partial_{-1} = \epsilon : \mathbb{Z}[G] \rightarrow \mathbb{Z}$ . We only present the proof for  $\Lambda^*$ , as the proof for  $\tilde{\Lambda}^*$  is similar. We construct a homomorphism  $h : \Lambda^i(\mathbb{Z}[G]) \rightarrow \Lambda^{i+1}(\mathbb{Z}[G])$  by the formula

$$h(g_0 \wedge \dots \wedge g_i) = 1 \wedge g_0 \wedge \dots \wedge g_i.$$

To show that this is a contracting homotopy, we need to check that  $h \circ \partial + \partial \circ h = Id_{\Lambda^i(\mathbb{Z}[G])}$ . Indeed, we have

$$\begin{aligned} \partial_i \circ h_i(g_0 \wedge \dots \wedge g_i) &= g_0 \wedge \dots \wedge g_i - \sum_{j=0}^i (-1)^j 1 \wedge g_0 \wedge \dots \wedge \hat{g}_j \wedge \dots \wedge g_i \\ &= g_0 \wedge \dots \wedge g_i - h_{i-1} \circ \partial_{i-1}(g_0 \wedge \dots \wedge g_i). \end{aligned}$$

□

**Lemma 3.4.** The differential  $\partial : \tilde{\Lambda}^{n+1}(\mathbb{Z}[G]) \rightarrow \tilde{\Lambda}^n(\mathbb{Z}[G])$  sends  $\Delta^{n+1}(\mathbb{Z}[G])$  to  $\Delta^n(\mathbb{Z}[G])$ . Moreover, it is compatible with the decomposition (??). Hence

$$(\tilde{\Lambda}^{*+1}(\mathbb{Z}[G]), \partial) \cong (\Delta^{*+1}(\mathbb{Z}[G]), \partial) \oplus (\Lambda^{*+1}(\mathbb{Z}[G]), \partial)$$

and  $H_*(\Delta^{*+1}(\mathbb{Z}[G]), \partial) = 0$ .

*Proof.* By Lemma ?? the canonical projection  $\tilde{\Lambda}^{*+1}(\mathbb{Z}[G]) \rightarrow \Lambda^{*+1}(\mathbb{Z}[G])$  is a chain map, inducing an isomorphism in homology, hence  $\Delta^{*+1}(\mathbb{Z}[G])$  is a chain subcomplex with trivial homology. To finish the proof, it suffices to note that the map  $g_1 \wedge \dots \wedge g_n \mapsto g_1 \tilde{\Lambda} \dots \tilde{\Lambda} g_n, g_1, \dots, g_n \in G$ , commutes with differentials and hence defines a splitting of chain complexes.

□

**Lemma 3.5.** *After applying the functor  $\text{Hom}_{\mathbb{Z}[G]}(-, M)$  to the resolutions in Lemma ?? one obtains the following diagram*

$$\begin{array}{ccccc}
\text{Hom}_{\mathbb{Z}[G]}(\Lambda^*(\mathbb{Z}[G]), M) & \longrightarrow & \text{Hom}_{\mathbb{Z}[G]}(\tilde{\Lambda}^*(\mathbb{Z}[G]), M) & \longrightarrow & \text{Hom}_{\mathbb{Z}[G]}(T^*(\mathbb{Z}[G]), M) \\
\downarrow = & & \downarrow = & & \downarrow = \\
K_\lambda^*(G, M) & \longrightarrow & KS^*(G, M) & \longrightarrow & K^*(G, M) \\
\downarrow \psi & & \downarrow \psi & & \downarrow \psi \\
C_\lambda^*(G, M) & \longrightarrow & CS^*(G, M) & \longrightarrow & C^*(G, M)
\end{array}$$

where all horizontal arrows are inclusions and vertical arrows are isomorphisms.

*Proof.* A group homomorphism  $f : T^n(\mathbb{Z}[G]) \rightarrow M$  is uniquely determined by its restriction on the basis elements  $(x_1, \dots, x_n) \in G^n$ . This gives the isomorphism  $\text{Hom}_{\mathbb{Z}[G]}(T^n(\mathbb{Z}[G]), M) \cong K^n(G, M)$ . Such an  $f$  belongs to  $\text{Hom}_{\mathbb{Z}[G]}(\tilde{\Lambda}^n(\mathbb{Z}[G]), M)$  if and only if it satisfies the skew-symmetry property with respect to the elements of  $\mathbb{Z}[G]$ , and this happens if and only if the same skew symmetry conditions holds on the basis elements. This gives the middle isomorphism of the top row, and the left isomorphism follows similarly. For the lower row, a key point is to show that restricting  $\psi$  on  $KS^*(G, M)$  yields an isomorphism between  $KS^*(G, M)$  and  $CS^*(G, M)$ . To this end, take  $\phi \in CS^n(G, M)$  and  $f = (\psi^n)^{-1}(\phi)$ . Then

$$f(g_0, \dots, g_n) = (\psi^n)^{-1}(\phi)(g_0, g_1, \dots, g_n) = g_0 \cdot \phi(g_0^{-1}g_1, g_1^{-1}g_2, \dots, g_{n-1}^{-1}g_n).$$

The equation  $\phi(g_1, g_2, g_3, \dots, g_n) = -g_1\phi(g_1^{-1}, g_1g_2, g_3, \dots, g_n)$  translates to

$$\begin{aligned}
f(g_0, \dots, g_n) &= g_0f(g_0^{-1}g_1, \dots, g_{n-1}^{-1}g_n) \\
&= -g_0 \cdot g_0^{-1}g_1 \cdot f((g_0^{-1}g_1)^{-1}, g_0^{-1}g_1g_1^{-1}g_2, \dots, g_{n-1}^{-1}g_n) \\
&= -g_1 \cdot f((g_1)^{-1}g_0, g_0^{-1}g_2, \dots, g_{n-1}^{-1}g_n) \\
&= -f(g_1, g_0, \dots, g_n).
\end{aligned}$$

In a similar way, the other equations above give the condition ?? for  $n > 0$ . □

If one passes to cohomology, one obtains the homomorphisms

$$\begin{array}{ccccc}
H_\lambda^*(G, M) & \longrightarrow & H^*(KS^*(G, M)) & \longrightarrow & H^*(G, M) \\
& \searrow \gamma & \downarrow \cong & \nearrow \alpha & \\
& & HS^*(G, M) & & 
\end{array}$$

and the commutativity of the diagram in Lemma ?? shows that  $\beta = \alpha\gamma$ .

**Proposition 3.6.** *The homomorphism  $\gamma^n : H_\lambda^n(G, M) \rightarrow HS^n(G, M)$  is a split monomorphism. Moreover, it is an isomorphism provided  $M$  has no elements of order two.*

*Proof.* The first part follows from Lemma ?. Assume  $M$  has no elements of order two. It suffices to show that  $K_\lambda^*(G, M) = KS^*(G, M)$ . Take an element  $f \in KS^n(G, M)$ . Then we have

$$f(x_0, \dots, x_i, x_{i+1}, \dots, x_n) = -f(x_0, \dots, x_{i+1}, x_i, \dots, x_n),$$

for all  $0 \leq i < n$ . If  $x_i = x_{i+1}$ , one obtains  $2f(x_0, \dots, x_i, x_i, \dots, x_n) = 0$  and hence  $f(x_0, \dots, x_i, x_i, \dots, x_n) = 0$ . This implies that  $f \in K_\lambda^n(G, M)$  and the proof is finished. □

3.2.  **$\delta$ -cohomology.** In order to state the relationship between the exterior and symmetric cohomology we need to introduce new groups.

**Definition 3.7.** For a group  $G$  and a  $G$ -module  $M$  one defines the  $\delta$ -homology  $H_\delta^*(G, M)$  by

$$H_\delta^*(G, M) = H^*(\text{Hom}_{\mathbb{Z}[G]}(\Delta^{*+1}(\mathbb{Z}[G]), M)).$$

Since  $\Delta^n(\mathbb{Z}[G])$  is an  $\mathbb{F}_2$ -vector space, it follows that the groups  $H_\delta^n(G, M)$  are also  $\mathbb{F}_2$ -vector spaces,  $n \geq 0$ . The importance of these groups comes from the fact that

$$(3.2.1) \quad HS^n(G, M) \cong H_\lambda^n(G, M) \oplus H_\delta^n(G, M)$$

which is a trivial consequence of Lemma ???. It follows from Proposition ??? that if  $M$  has no elements of order two, then  $H_\delta^*(G, M) = 0$ .

3.3. **Preliminaries on spectral sequences.** To state our main result of this section, let us recall the construction of the hypercohomology spectral sequences. These spectral sequences will also play a prominent role in the next section.

Let  $G$  be a group and  $M$  be a left  $G$ -module. For any chain complex of left  $G$ -modules  $C_* = (C_0 \leftarrow C_1 \leftarrow \dots)$  one defines  $\text{Ext}_{\mathbb{Z}[G]}^*(C_*, M)$  to be the homology of the total complex of the double complex  $\text{Hom}_{\mathbb{Z}[G]}(C_*, I^*)$ , where  $I^*$  is an injective resolution of  $M$ .

There exist two spectral sequences. Both of them abut to the group  $\text{Ext}_{\mathbb{Z}[G]}^*(C_*, M)$ . They are:

$$\begin{aligned} \mathbf{I}_1^{pq} = \text{Ext}_{\mathbb{Z}[G]}^q(C_p, M) &\implies \text{Ext}_{\mathbb{Z}[G]}^{p+q}(C_*, M), \\ \mathbf{II}_2^{pq} = \text{Ext}_{\mathbb{Z}[G]}^p(H_q(C_*), M) &\implies \text{Ext}_{\mathbb{Z}[G]}^{p+q}(C_*, M). \end{aligned}$$

We also need the following easy lemma on spectral sequences.

**Lemma 3.8.** Assume a spectral sequence abuts to zero and  $E_2^{pq} = 0$  if  $q < 0$  or  $p < k$ , where  $k$  is a fixed integer. Then

$$E_2^{k0} = 0 = E_2^{k+10}.$$

3.4. **Vanishing of  $\delta$ -cohomology in low dimensions.** Now we can state the main result of this section:

**Theorem 3.9.** Let  $G$  be a group and  $M$  be a  $G$ -module. Then

$$H_\delta^i(G, M) = 0, \quad \text{for } 0 \leq i \leq 4.$$

Hence  $\gamma^i : H_\lambda^i(G, M) \rightarrow HS^i(G, M)$  is an isomorphism for  $i = 0, 1, 2, 3, 4$ .

*Proof.* In the hypercohomology spectral sequence we take  $C_* = (\Delta^{*+1}(\mathbb{Z}[G]), \partial)$ . Since  $H_*(C_*) = 0$ , the spectral sequence  $\mathbf{II}$  gives  $\text{Ext}_{\mathbb{Z}[G]}^*(C_*, M) = 0$ . Thus, the spectral sequence  $\mathbf{I}$  has the form

$$E_1^{pq} = \text{Ext}_{\mathbb{Z}[G]}^q(\Delta^{p+1}(\mathbb{Z}[G]), M) \implies 0.$$

Since  $E_1^{p0} = \text{Hom}_{\mathbb{Z}[G]}(\Delta^{p+1}(\mathbb{Z}[G]), M)$ , we see that

$$E_2^{*0} = H_\delta^*(G, M).$$

According to (??), we have  $E_1^{pq} = 0$  for  $p < 1$ . It follows from Lemma ??? that  $E_2^{i0} = H_\delta^i(G, M) = 0$  for  $i \leq 2$ . Thus by the same Lemma it suffices to show that  $E_2^{1,q} = 0 = E_2^{2,q}$  if  $q > 0$ .

One checks that the following diagram of  $G$ -modules commutes:

$$\begin{array}{ccccc} \Delta^2(\mathbb{Z}[G]) & \xleftarrow{\partial^1} & \Delta^3(\mathbb{Z}[G]) & \xleftarrow{\partial^2} & \Delta^4(\mathbb{Z}[G]) \\ \uparrow \psi_1 & & \uparrow \psi_2 & & \uparrow \psi_3 \\ \mathbb{F}_2[G] & \xleftarrow{\delta_1} & \mathbb{F}_2[G \times G] & \xleftarrow{\delta_2} & \mathbb{F}_2[G \times G] \end{array}$$

where

$$\psi_1(g) = g\tilde{\wedge}g, \quad \psi_2(g, h) = g\tilde{\wedge}g\tilde{\wedge}h, \quad \psi_3(g, h) = g\tilde{\wedge}g\tilde{\wedge}g\tilde{\wedge}h,$$

$$\delta_1(g, h) = g, \quad \delta_2(g, h) = (g, h) - (g, g).$$

Since the set of elements  $\{s\bar{\lambda}s | s \in G\}$ , (resp.  $\{s\bar{\lambda}s\bar{\lambda}t | s, t \in G\}$ ) forms an  $\mathbb{F}_2$ -basis of  $\Delta^2(\mathbb{Z}[G])$  (resp.  $\Delta^3(\mathbb{Z}[G])$ ), the  $G$ -homomorphism  $\psi_1$  (resp.  $\psi_2$ ) is an isomorphism. In general, the  $G$ -homomorphism  $\psi_3$  is not an isomorphism, but only a split monomorphism. Hence the projective resolutions

$$0 \rightarrow \mathbb{Z}[G] \xrightarrow{2} \mathbb{Z}[G] \rightarrow \mathbb{F}_2[G] \rightarrow 0 \quad \text{and} \quad 0 \rightarrow \mathbb{Z}[G \times G] \xrightarrow{2} \mathbb{Z}[G \times G] \rightarrow \mathbb{F}_2[G \times G] \rightarrow 0$$

can be used to compute  $\text{Ext}_{\mathbb{Z}[G]}^q(\Delta^2(\mathbb{Z}[G]), M)$  and  $\text{Ext}_{\mathbb{Z}[G]}^i(\Delta^3(\mathbb{Z}[G]), M)$ . In both cases

$$\text{Ext}_{\mathbb{Z}[G]}^i(\Delta^2(\mathbb{Z}[G]), M) = 0 = \text{Ext}_{\mathbb{Z}[G]}^i(\Delta^3(\mathbb{Z}[G]), M) \quad \text{if } i > 1.$$

Hence  $E_1^{1,q} = 0 = E_1^{2,q}$  if  $q > 1$ . The first projective resolution gives

$$E_1^{11} = \text{Ext}_{\mathbb{Z}[G]}^1(\Delta^2(\mathbb{Z}[G]), M) = N,$$

where  $N = M/2M$ . Since  $\mathbb{Z}[G \times G] = \bigoplus_{g \in G} \mathbb{Z}[G]$  as a  $G$ -module, the second projective resolution gives

$$E_1^{21} = \text{Ext}_{\mathbb{Z}[G]}^1(\Delta^3(\mathbb{Z}[G]), M) = \text{Maps}(G, N).$$

Moreover, it also shows that the group  $\text{Maps}(G, N)$  is a direct summand of  $\text{Ext}_{\mathbb{Z}[G]}^1(\Delta^4(\mathbb{Z}[G]), M)$ . It follows that there is an isomorphism of chain complexes

$$\begin{array}{ccccccc} E_1^{01} & \xrightarrow{\partial^0} & E_1^{11} & \xrightarrow{\partial^1} & E_1^{21} & \xrightarrow{\partial^2} & E_1^{31} \\ \uparrow & & \uparrow \psi_1^* & & \uparrow \psi_2^* & & \uparrow \psi_3^* \\ 0 & \xrightarrow{\delta^0} & N & \xrightarrow{\delta^1} & \text{Maps}(G, N) & \xrightarrow{\delta^2} & X \oplus \text{Maps}(G, N) \end{array}$$

for some  $X$ . For  $n \in N$ , the map  $\delta^1(n)$  is given by  $(\delta^1(n))(g) = n$ , for  $g \in G$  and  $\delta^2 = \begin{pmatrix} x \\ \delta' \end{pmatrix}$  for some  $x \in X$ , where  $(\delta'(\tau))(g) = \tau(g) - \tau(1)$  for  $\tau \in \text{Maps}(G, N)$ . Since  $\delta^1$  is a monomorphism, it follows that  $E_2^{1,1} = 0$ . And as  $\text{Ker}(\delta') = \text{Im}(\delta^1)$ , we obtain that  $E_2^{2,1} = 0$  and the proof is finished.  $\square$

Now we give an example which shows that  $\gamma^n$ ,  $n \geq 5$  is not an isomorphism in general.

**3.5. The symmetric and exterior cohomologies of  $C_2$ .** Let  $G = C_2 = \{1, t\}$ ,  $t^2 = 1$  be the cyclic group of order two. In this section we compute both symmetric and exterior cohomologies of  $C_2$ . The computation of the exterior cohomology is extremely easy. In fact, for  $G = C_2$ , the resolution  $(\Lambda^*(\mathbb{Z}[G]), \partial)$  has the following form:

$$\begin{array}{ccccccc} \dots & \longrightarrow & 0 & \longrightarrow & \Lambda^3(\mathbb{Z}[C_2]) & \xrightarrow{\partial} & \Lambda^2(\mathbb{Z}[C_2]) & \xrightarrow{\partial} & \mathbb{Z}[C_2] \\ & & & & \downarrow \cong & & \downarrow \cong & & \\ & & & & 0 & & \mathbb{Z}[C_2]/(1+t) & & \end{array}$$

where  $\partial = (1 - t)$ . So,

$$H_\lambda^n(C_2, M) = \begin{cases} H^n(C_2, M), & \text{if } n = 0, 1, \\ 0, & \text{else.} \end{cases}$$

For the symmetric cohomology one has the following result:

**Lemma 3.10.** *For  $G = C_2$  and  $M = \mathbb{F}_2$  with trivial action of  $G$  on  $M$ , one has*

$$HS^i(C_2, \mathbb{F}_2) = \begin{cases} \mathbb{F}_2, & \text{if } i = 0, \text{ or } i \equiv 1 \pmod{4}, \\ 0, & \text{else.} \end{cases}$$

Thus, in general,  $H_\lambda^*(G, M) \neq HS^*(G, M)$ .



*Proof.* Consider the resolution

$$\cdots \rightarrow \tilde{\Lambda}^3(\mathbb{Z}[C_2]) \xrightarrow{\partial_1} \tilde{\Lambda}^2(\mathbb{Z}[C_2]) \xrightarrow{\partial_0} (\mathbb{Z}[C_2]).$$

Fix  $n > 0$  and in  $\tilde{\Lambda}^n(\mathbb{Z}[C_2])$  consider the elements

$$\alpha_i^n = \underbrace{1 \tilde{\wedge} \cdots \tilde{\wedge} 1}_{n-i} \tilde{\wedge} \underbrace{t \tilde{\wedge} \cdots \tilde{\wedge} t}_i, \quad 0 \leq i < \frac{n}{2},$$

$$\beta^n = \underbrace{1 \tilde{\wedge} \cdots \tilde{\wedge} 1}_m \tilde{\wedge} \underbrace{t \tilde{\wedge} \cdots \tilde{\wedge} t}_m, \quad n = 2m.$$

Then  $\alpha_i^n$  and  $\beta^n$  generate  $\tilde{\Lambda}^n(\mathbb{Z}[C_2])$  as a  $C_2$ -module. More accurately,  $\mathbb{Z}[C_2]$  is a free  $\mathbb{Z}[C_2]$ -module with the generator  $\alpha_0^1$ . As a  $\mathbb{Z}[C_2]$ -module,

$$\tilde{\Lambda}^2(\mathbb{Z}[C_2]) = \mathbb{F}_2[C_2] \bigoplus \mathbb{Z}[C_2]/(t+1),$$

with  $\alpha_0^2$  generating  $\mathbb{F}_2[C_2]$  and  $\beta^2$  generating  $\mathbb{Z}[C_2]/(t+1)$ . For odd  $n$ ,  $n = 2m + 1 \geq 3$ ,

$$\tilde{\Lambda}^n(\mathbb{Z}[C_2]) = \mathbb{F}_2[C_2] \bigoplus \cdots \bigoplus \mathbb{F}_2[C_2],$$

with  $\alpha_0^n, \dots, \alpha_m^n$  generating each of the summands. Similarly to  $n = 2$ , for larger  $n = 2m \geq 4$  we have

$$\tilde{\Lambda}^n(\mathbb{Z}[C_2]) = \mathbb{F}_2[C_2] \bigoplus \cdots \bigoplus \mathbb{F}_2[C_2] \bigoplus \mathbb{F}_2[C_2]/(t-1),$$

where the  $\alpha_0^n, \dots, \alpha_m^n$  generate the  $\mathbb{F}_2[C_2]$  summands and  $\beta^n$  generates  $\mathbb{F}_2[C_2]/(t-1)$ .

Beginning from  $\partial_1$ , the coboundary maps are given by the matrices

$$(\partial_{4k+1})_{ij} = \begin{cases} 1, & \text{if } i \text{ is odd and } j = i \text{ or } j = i + 1 \\ 0, & \text{else,} \end{cases}$$

where  $1 \leq i \leq 2k + 2, 1 \leq j \leq 2k + 2$ ,

$$(\partial_{4k+2})_{ij} = \begin{cases} 1, & \text{if } j \text{ is even and } i = j \text{ or } i = j - 1 \\ 0, & \text{else,} \end{cases}$$

where  $1 \leq i \leq 2k + 2, 1 \leq j \leq 2k + 3$ ,

$$(\partial_{4k+3})_{ij} = \begin{cases} 1, & \text{if } i \text{ is odd and } j = i \\ 1, & \text{if } i \text{ is odd and } j = i + 1 \text{ and } i < 2k + 2 \\ 0, & \text{else,} \end{cases}$$

where  $1 \leq i \leq 2k + 3, 1 \leq j \leq 2k + 3$ ,

$$(\partial_{4k})_{ij} = \begin{cases} 1, & \text{if } j \text{ is even and } i = j \text{ or } i = j - 1 \text{ and } j < 2k + 2 \\ t - 1, & \text{if } j = 2k + 2 \text{ and } i = 2k + 1 \\ 0, & \text{else,} \end{cases}$$

where  $1 \leq i \leq 2k + 1, 1 \leq j \leq 2k + 2$ . Based on this the result easily follows.  $\square$

#### 4. RELATIONSHIP BETWEEN EXTERIOR AND CLASSICAL COHOMOLOGY

We start this section with the following easy and probably well-known fact. It will be used in the proof of Theorem ?? below.

**Lemma 4.1.** *Let  $g \in G$  and  $\omega = x_1 \wedge \cdots \wedge x_n \in \Lambda^n(\mathbb{Z}[G])$ , where  $x_1, \dots, x_n$  are distinct elements in  $G$ . If  $g\omega = \pm\omega$ , then the order of  $g$  divides  $n$ .*

*Proof.* If one forgets the sign, it follows from the assumption that the multiplication by  $g$  permutes the  $n$  elements  $x_1, \dots, x_n$ , meaning the cyclic group generated by  $g$  acts on the set  $\{x_1, \dots, x_n\}$ . The action is free, because it is given by the multiplication in  $G$ . Hence all orbits will have the same length equal to the order of  $g$ , dividing  $n$ .  $\square$

Now we can state our main result.

**Theorem 4.2.** *For any group  $G$  and any  $G$ -module  $M$ , there is a first quadrant spectral sequence*

$$E_1^{pq} \implies H^{p+q}(G, M)$$

with properties

- (i)  $E_2^{p0} = H_\lambda^p(G, M)$  and the edge homomorphism  $E_2^{p0} \rightarrow H^p(G, M)$  is precisely  $\beta^p$ ,  $p \geq 0$ .
- (ii) If  $q > 0$ , then  $E_1^{0q} = 0$ .
- (iii) If  $q > 0$ ,  $p > 0$  and the equation  $x^{p+1} = 1$  has only trivial solution in  $G$ , then  $E_1^{pq} = 0$ .
- (iv) If  $\ell$  is a prime number and  $q > 0$ , then

$$E_1^{\ell-1q} = \begin{cases} \prod_{C_\ell \subset G} H^{q+1}(C_\ell, M), & \text{if } \ell = 2, \\ \prod_{C_\ell \subset G} H^q(C_\ell, M), & \text{if } \ell > 2. \end{cases}$$

Here the product is taken over all subgroups of order  $\ell$  and for each such subgroup, the corresponding action of  $C_\ell$  on  $M$  is induced by the inclusion.

**Remark 4.3.** *If  $p+1$  is not prime, then  $E_1^{pq}$ ,  $q > 0$ ,  $p > 0$  can be described as a product (usually of several copies) of the group cohomology of subgroups of order  $k$ , where  $k|p+1$ , but the exact formula is too complex to state here. From this it is easy to deduce that  $E_1^{pq} = E_2^{pq}$  for all  $q > 0$  (compare with the proof of the part i) of Corollary ??).*

*Proof.* In the hypercohomology spectral sequence discussed in Section ??, we take  $R = \mathbb{Z}[G]$  and  $C^* = (\Lambda^{*+1}(\mathbb{Z}[G]), \delta)$ , which we denote simply by  $\Lambda^{*+1}$ . This gives the spectral sequences

$$\begin{aligned} \mathbf{I}_1^{pq} &= \text{Ext}_G^q(\Lambda^{p+1}, M) \implies \text{Ext}_G^{p+q}(\Lambda^{*+1}, M) \\ \mathbf{II}_2^{pq} &= \text{Ext}_G^p(H_q(\Lambda^{*+1}), M) \implies \text{Ext}_G^{p+q}(\Lambda^{*+1}, M). \end{aligned}$$

Let us first consider the second spectral sequence. As  $\Lambda^{*+1}$  is a resolution of  $\mathbb{Z}$ , we have

$$H_q(\Lambda^{*+1}) = \begin{cases} \mathbb{Z}, & \text{for } q = 0, \\ 0, & \text{else.} \end{cases}$$

Therefore, the second spectral sequence degenerates to the isomorphism

$$\text{Ext}_G^p(\Lambda^{*+1}, M) = \text{Ext}_G^p(H_0(\Lambda^{*+1}), M) = \text{Ext}_G^p(\mathbb{Z}, M) = H^p(G, M).$$

Substituting this value into the first spectral sequence, we obtain the spectral sequence

$$E_1^{pq} = \text{Ext}_{\mathbb{Z}[G]}^q(\Lambda^{p+1}(\mathbb{Z}[G]), M) \implies H^{p+q}(G, M).$$

Since the differential  $d_1$  in the first page of the spectral sequence is induced by the boundary map in the resolution  $\Lambda^{*+1}(\mathbb{Z}[G]) \rightarrow \mathbb{Z}$ , it follows that for  $q = 0$ , the chain complex  $(E_1^{p0}, d_1)$  coincides with the Zarelua chain complex and the statement (i) follows.

If  $p = 0$ , then  $E_1^{p0} = \text{Ext}_{\mathbb{Z}[G]}^q(\mathbb{Z}[G], M)$  vanishes for  $q > 0$ . Hence  $E_1^{0q} = 0$  for  $q > 0$ , and the property (ii) holds.

Next, the  $G$ -module  $\Lambda^{p+1}(\mathbb{Z}[G])$  is free as an abelian group with a basis of the form  $x_1 \wedge \dots \wedge x_p$ , where  $x_1 < \dots < x_p$ . Here  $\leq$  is any total order on  $G$ . If one ignores the sign, we see that  $G$  acts on the basis. Thus  $\Lambda^{p+1}(\mathbb{Z}[G])$  decomposes as a direct sum of  $G$ -submodules corresponding to these orbits. In particular, summands corresponding to free orbits are free  $G$ -modules. Now, if the assertion of (iii) holds, all orbits are free thanks to Lemma ?? and hence the Ext-group vanishes and  $E_1^{pq} = 0$  for  $q > 0$ . Thus the property (iii) is proved.

If  $\ell$  is prime and  $C_\ell = \{1, g, \dots, g^{\ell-1}\}$  is a cyclic subgroup of  $G$ , then for the basis element  $\omega = 1 \wedge g \wedge \dots \wedge g^{\ell-1}$  one has  $g\omega = \omega$  for odd  $\ell$ , and  $g\omega = -\omega$  for  $\ell = 2$ . Thus  $\omega$  determines a non free summand of  $\Lambda^\ell(\mathbb{Z}[G])$ . This summand is isomorphic to  $\mathbb{Z}[G]/(g-1)$  for odd  $\ell$  and  $\mathbb{Z}[G]/(g+1)$  for  $\ell = 2$ . This summand has an obvious projective resolution

$$0 \leftarrow \mathbb{Z}[G]/(g-1) \leftarrow \mathbb{Z}[G] \xleftarrow{g-1} \mathbb{Z}[G] \xleftarrow{1+g+\dots+g^{\ell-1}} \dots$$

if  $\ell$  is odd and

$$0 \leftarrow \mathbb{Z}[G]/(g+1) \leftarrow \mathbb{Z}[G] \xleftarrow{g+1} \mathbb{Z}[G] \xleftarrow{g-1} \dots$$

if  $\ell = 2$ . From this it follows that this summand of  $\Lambda^\ell(\mathbb{Z}[G])$  contributes the factor  $H^q(C_\ell, M)$  (resp.  $H^{q+1}(C_\ell, M)$ ) in  $\text{Ext}_{\mathbb{Z}[G]}^q(\Lambda^{p+1}(\mathbb{Z}[G]), M)$  for odd  $\ell$  (resp.  $\ell = 2$ ). By Lemma ?? all non-free summands of  $\Lambda^\ell(\mathbb{Z}[G])$  arise in this way and hence  $E_1^{\ell-1, q}$  has the form described in (iv).  $\square$

The second plane/page of the spectral sequence is:

$q$	0	$\prod_{C_2 \subset G} H^{q+1}(C_2, M)$	$\prod_{C_3 \subset G} H^q(C_3, M)$	$\dots$	$\text{Ext}^q(\Lambda^{p+1}\mathbb{Z}[G], M)$	$\dots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\dots$
2	0	$\prod_{C_2 \subset G} H^3(C_2, M)$	$\prod_{C_3 \subset G} H^2(C_3, M)$	$\dots$	$\text{Ext}^2(\Lambda^{p+1}\mathbb{Z}[G], M)$	$\dots$
1	0	$\prod_{C_2 \subset G} H^2(C_2, M)$	$\prod_{C_3 \subset G} H^1(C_3, M)$	$\dots$	$\text{Ext}^1(\Lambda^{p+1}\mathbb{Z}[G], M)$	$\dots$
0	$H_\lambda^0(G, M)$	$H_\lambda^1(G, M)$	$H_\lambda^2(G, M)$	$\dots$	$H_\lambda^p(G, M)$	$\dots$
	0	1	2	$\dots$	$p$	

As an immediate consequence of Theorem ?? one obtains the following corollary.

**Corollary 4.4.** (i) For any group  $G$  and any  $G$ -module  $M$ , the homomorphism  $\beta^i : H_\lambda^i(G, M) \rightarrow H^i(G, M)$  is an isomorphism for  $i = 0$  and  $i = 1$ , while  $\beta^2$  and  $\beta^3$  can be fit in an exact sequence:

$$0 \rightarrow H_\lambda^2(G, M) \xrightarrow{\beta^2} H^2(G, M) \rightarrow \prod_{C_2 \subset G} H^2(C_2, M) \rightarrow H_\lambda^3(G, M) \xrightarrow{\beta^3} H^3(G, M).$$

(ii) If  $G$  has no elements of order two, then for any  $G$ -module  $M$ , the homomorphism  $\beta^2$  is an isomorphism, while  $\beta^3, \beta^4$  and  $\beta^5$  can be fit in an exact sequence:

$$0 \rightarrow H_\lambda^3(G, M) \xrightarrow{\beta^3} H^3(G, M) \rightarrow \prod_{C_3 \subset G} H^3(C_3, M) \rightarrow H_\lambda^4(G, M) \xrightarrow{\beta^4}$$

$$\xrightarrow{\beta^4} H^4(G, M) \rightarrow \prod_{C_3 \subset G} H^4(C_3, M) \rightarrow H_\lambda^5(G, M) \xrightarrow{\beta^5} H^5(G, M).$$

(iii) If all nontrivial elements of  $G$  are of infinite order, then  $\beta^i : H_\lambda^i(G, M) \rightarrow H^i(G, M)$  is an isomorphism for all  $i \geq 0$ .

*Proof.* (i) We first show that if  $q > 0$ , the differential  $E_1^{1q} \rightarrow E_1^{2q}$  vanishes. In fact, by part (iv) of Theorem ?? the group  $E_1^{1q}$  is annihilated by the multiplication by 2, while the group  $E_1^{2q}$  is annihilated by the multiplication by 3 and hence the corresponding map is zero. This fact implies that  $E_2^{1q} = E_1^{1q}$  for all  $q > 0$ . The rest is a consequence of the 5-term exact sequence, which we have in any first quadrant spectral sequence.

In general, for a first quadrant spectral sequence  $E_2^{pq} \Rightarrow H^{p+q}(A)$ , we have the following exact sequence of low-degree terms:

$$0 \rightarrow E_2^{10} \rightarrow H^1(A) \rightarrow E_2^{01} \rightarrow E_2^{20} \rightarrow H^2(A).$$

In our specific case, the first column vanishes for  $q > 0$ , moving up the indices of the terms of the 5-term exact sequence to

$$0 \rightarrow E_2^{20} \rightarrow H^2(A) \rightarrow E_2^{11} \rightarrow E_2^{30} \rightarrow H^3(A),$$

which calculated, give the desired exact sequence.

- (ii) Assume  $q > 0$ . By part (iii) of Theorem ?? and the fact that  $G$  does not contain an element of order two, we have  $E_1^{pq} = 0$ , if  $q > 0$  and  $p + 1$  is a power of two. It follows that  $E_2^{pq} = E_1^{pq}$ , for  $p = 2$  and hence the result.
- (iii) By part (iii) of Theorem ?? we have  $E_1^{pq} = 0$  for all  $q > 0$ . Hence the spectral sequence degenerates and in particular, the edge homomorphism is an isomorphism. □

**Example.** Let  $\ell$  be a prime number and  $G = C_\ell$  be a cyclic group of order  $\ell$ . Then

$$H_\lambda^i(C_\ell, M) = \begin{cases} H^i(C_\ell, M), & \text{if } i \leq \ell - 1, \\ 0, & \text{if } i \geq \ell. \end{cases}$$

In fact, the case when  $i \geq \ell$  follows from Remark ??, while the case  $i \leq \ell - 1$  follows from part (iv) of Theorem ??.

## 5. COHOMOLOGY LONG EXACT SEQUENCE

Let  $\mathbb{E}$  be the short exact sequence

$$0 \rightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \rightarrow 0$$

of  $G$ -modules. (Note that the homomorphisms  $\alpha$  and  $\beta$  are used in a different context here than in the rest of the paper.) A basic fact in group cohomology claims that there exist connecting homomorphisms

$$\delta^n : H^n(G, C) \rightarrow H^{n+1}(G, A)$$

such that the sequence of abelian groups

$$(5.0.1) \quad \dots \xrightarrow{\delta^{n-1}} H^n(G, A) \xrightarrow{\alpha^n} H^n(G, B) \xrightarrow{\beta^n} H^n(G, C) \xrightarrow{\delta^n} H^{n+1}(G, A) \xrightarrow{\alpha^{n+1}} \dots$$

is exact. A similar statement for  $H_\lambda^*$  or  $HS^*$  is not true in general. Singh constructs such an exact sequence for the symmetric cohomology under quite strong restriction on the exact sequence  $\mathbb{E}$ , see [?, Proposition 5.3]. Namely, he assumes the existence of a map  $\sigma : C \rightarrow A$  with the properties:

$$\beta \circ \sigma = Id_C, \quad \sigma(xc) = x\sigma(c) \quad \text{and} \quad \sigma(-c) = -\sigma(c)$$

for all  $x \in G$  and  $c \in C$ .

In this section we consider an arbitrary extension  $\mathbb{E}$  and we aim to find a relationship between the groups  $H_\lambda^*(G, A)$ ,  $H_\lambda^*(G, B)$  and  $H_\lambda^*(G, C)$ . To state our result, let us observe that the exactness of the sequence (??) is equivalent to the following rather awkward statement.

**Proposition 5.1.** For any short exact sequence  $\mathbb{E}$  there exists a first quadrant spectral sequence which abuts to zero

$$E_1^{pq} \implies 0,$$

and for which

$$E_1^{pq} = \begin{cases} H^q(G, A), & \text{if } p = 0, \\ H^q(G, B), & \text{if } p = 1, \\ H^q(G, C), & \text{if } p = 2, \\ 0, & \text{if } p \geq 3. \end{cases}$$

The differentials in the first page

$$d_1^{0q} : E_1^{0q} \rightarrow E_1^{1q}, \quad d_1^{1q} : E_1^{1q} \rightarrow E_1^{2q}$$

coincide with  $\alpha^q : H^q(G, A) \rightarrow H^q(G, B)$  and  $\beta^q : H^q(G, B) \rightarrow H^q(G, C)$  respectively.

A similar result can be stated for the groups  $H_\lambda^*$ . Namely, the following is true.

**Theorem 5.2.** (a) For any short exact sequence  $\mathbb{E}$  there exists a first quadrant spectral sequence which abuts to zero

$$F_1^{pq} \implies 0,$$

and

$$F_1^{pq} = \begin{cases} H_\lambda^q(G, A), & \text{if } p = 0, \\ H_\lambda^q(G, B), & \text{if } p = 1, \\ H_\lambda^q(G, C), & \text{if } p = 2, \\ \text{Ker} \left( \text{Ext}_{\mathbb{Z}[G]}^1(\Lambda^{q+1}(\mathbb{Z}[G]), A) \rightarrow \text{Ext}_{\mathbb{Z}[G]}^1(\Lambda^{q+1}(\mathbb{Z}[G]), B) \right), & \text{if } p = 3, \\ 0, & \text{if } p \geq 4. \end{cases}$$

The differentials in the first page

$$d_1^{0q} : F_1^{0q} \rightarrow F_1^{1q}, \quad d_1^{1q} : F_1^{1q} \rightarrow F_1^{2q}$$

coincide with  $\alpha_\lambda^q : H_\lambda^q(G, A) \rightarrow H_\lambda^q(G, B)$  and  $\beta_\lambda^q : H_\lambda^q(G, B) \rightarrow H_\lambda^q(G, C)$  respectively.

(b) Moreover, for the groups  $F_1^{3q}$  (which are the obstruction groups for the existence of a long exact sequence for  $H_\lambda^*$ ), the following facts are true:

- (i)  $F_1^{30} = 0$ .
- (ii) If the equation  $x^{q+1} = 1$  has only a trivial solution in  $G$ , then  $F_1^{3q} = 0$ .
- (iii) If  $\ell$  is a prime number, then

$$F_1^{3, \ell-1} = \prod_{C_\ell \subset G} \mathcal{X}(C_\ell, \mathbb{E})$$

where the product is taken over all subgroups of order  $\ell$  and

$$\mathcal{X}(C_\ell, \mathbb{E}) = \frac{\{c \in C | tc = c\}}{\{c = \beta(b) | b \in B \text{ and } tb = b\}}, \quad \text{if } \ell \text{ is odd}$$

and

$$\mathcal{X}(C_\ell, \mathbb{E}) = \frac{\{c \in C | tc + c = 0\}}{\{c = \beta(b) + x - tx | x \in b \in B \text{ and } tb + b = 0\}}, \quad \text{if } \ell = 2.$$

Here  $t$  is a generator of  $C_\ell$ .

*Proof.* Apply the functor  $\text{Hom}_{\mathbb{Z}[G]}(\Lambda^*(\mathbb{Z}[G]), -)$  to the short exact sequence  $\mathbb{E}$  to get the exact sequence

$$0 \rightarrow \text{Hom}_{\mathbb{Z}[G]}(\Lambda^{*+1}, A) \rightarrow \text{Hom}_{\mathbb{Z}[G]}(\Lambda^{*+1}, B) \rightarrow \text{Hom}_{\mathbb{Z}[G]}(\Lambda^{*+1}, C) \rightarrow \text{Ext}_{\mathbb{Z}[G]}^1(\Lambda^{*+1}, A) \rightarrow \dots$$

where we write  $\Lambda^*$  for  $\Lambda^*(\mathbb{Z}[G])$ . We consider this sequence as a double complex, where  $\text{Hom}_{\mathbb{Z}[G]}(\Lambda^{*+1}, A)$  has bidegree  $(0, *)$ . By our construction, the rows of this double complex are exact. As any double complex, it has two spectral sequences abutting to the same groups, namely the homology of the total complex of the double complex. In the first spectral sequence the first page is obtained by first taking the homology in the horizontal direction. Thus, the first page

is zero in all places, thanks to the exactness of rows. It follows that the homology of the total complex is zero in all dimensions.

Now, we turn to the second spectral sequence  $E_r^{pq}$ . We just proved that it abuts to zero. It would be convenient to think of the members of our double complex as the zeroth page of the spectral sequence. Then the first page is obtained by taking the homology in the vertical direction. By the definition of the groups  $H_\lambda^*$  we see that the first three columns (i.e. the terms of bidegree  $(p, *)$ , for  $p = 0, 1, 2$ ) of the first page are respectively  $H_\lambda^*(G, A)$ ,  $H_\lambda^*(G, B)$  and  $H_\lambda^*(G, C)$ . We claim that the terms of bidegree  $(p, *)$ , for  $p \geq 3$ , in the first page are the same as in the zeroth page, which are certain  $\text{Ext}^k$ -groups with  $k > 0$ . This follows from our previous results, namely by the fact that multiplication by  $q$  in the group  $\text{Ext}_{\mathbb{Z}[G]}^k(\Lambda^{q+1}, -)$  is zero. Since  $q$  and  $q + 1$  are coprime, all vertical maps of the double complex vanish in bidegrees  $(p, q)$  for  $p \geq 3$  and  $q \geq 0$ . Since rows are exact in the double complex, it follows that the sequence

$$E_1^{3,*} \xrightarrow{d_1} E_1^{4,*} \xrightarrow{d_1} \dots$$

is exact. This proves the part (a) with

$$F_1^{pq} = \begin{cases} E_1^{pq}, & p \leq 2 \\ \text{Ker}(E_1^{3,*} \xrightarrow{d_1} E_1^{4,*}) & p = 3 \\ 0 & p \geq 4 \end{cases}$$

From this part (b)(i) follows, because  $E_1^{3,*} = \text{Ext}_{\mathbb{Z}[G]}^1(\Lambda^{*+1}, A)$  and  $E_1^{4,*} = \text{Ext}_{\mathbb{Z}[G]}^1(\Lambda^{*+1}, B)$ . Recall that the groups  $\text{Ext}_{\mathbb{Z}[G]}^1(\Lambda^{*+1}, -)$  appears also in Theorem ?? (they are  $E^{*1}$  in the notation of Theorem ??). Thus parts (b)(ii) and (b)(iii) are direct consequences of corresponding parts of Theorem ?? and the well-known formula for computing the cohomology of cyclic groups.  $\square$

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