UNIVERSITY OF SOUTHAMPTON

VACUUM INSTABILITY IN SCALAR FIELD THEORIES

by

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ABSTRACT

FACULTY OF SCIENCE

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VACUUM INSTABILITY IN SCALAR FIELD THEORIES

by Alan John McKane

Scalar field theories with an interaction of the form $g\phi^N$ have no stable vacuum state for some range of values of their coupling constant, g. This thesis reports calculations of vacuum instability in such theories. Using the idea that the tunnelling out of the vacuum state is described by the instanton solutions of the theory, the imaginary part of the vertex functions is calculated for the massless theory in the one-loop approximation, near the dimension $d_c = 2N/N-2$, where the theory is just renormalisable. The calculation differs from previous treatments in that dimensional regularisation is used to control the ultra-violet divergences of the theory. In this way previous analytic calculations in conformally invariant field theories are extended to the case where the theory is almost conformally invariant, since it is now defined in $d_c - \varepsilon$ dimensions ($\varepsilon > 0$).

To date the main application of the knowledge of the imaginary part of the vertex functions, has been to calculate the asymptotic behaviour of the perturbation expansion in g. The results we obtain in the case N = 4 (d = 4) agree with those of Lipatov and Brézin et al. who used Pauli-Villars regulators in their study of $g\phi^N$ theories (with N even) in d_c dimensions. The results obtained when N is odd indicate that the perturbation series is not Borel summable. The possible appearance of extra singularities in exactly d_c dimensions (renormalons) is discussed.

PREFACE AND ACKNOWLEDGEMENTS.

Some of the work set out in this thesis appears in preprint form and will shortly be published or submitted for publication. Chapter four is based on work carried out jointly with Dr.D.J.Wallace, and is to be published in Journal of Physics A (McKane and Wallace 1978). The contents of chapter five are the subject of a preprint (McKane 1978) which is to be submitted to Nuclear Physics B. The discussion of simple field theoretic models in chapter two is based on lectures given by Dr.D.J.Wallace at Southampton University in the Autumn of 1977. Other publications with myself as author or co-author, deal with topics not discussed in this thesis (McKane et al. 1976, McKane 1977, Elderfield and McKane 1978).

I wish to thank my supervisor Dr.D.J.Wallace for his guidance, advice and encouragement during the past three years. I should also like to thank other members of the theory group at Southampton for their help and friendship, and especially thank David Elderfield for many useful discussions.

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CHAPTER ONE

INTRODUCTION

In quantum field theory, as in most other branches of theoretical physics, there are so few systems which can be treated exactly, that good approximation schemes are vital if quantitative physical predictions are to be made. It might be hoped that the approximation procedures developed in quantum mechanics might be capable of generalisation to field theory Unfortunately, two of the most useful methods in quantum mechanics, the variational method and the WKB method, do not immediately generalise to field theory. Thus one is forced to rely more than ever on perturbation expansions.

This aspect of field theory is well developed. Typically one performs the perturbation expansion about a theory which is exactly soluble (the free theory) in terms of a coupling constant. One then obtains the Green's functions of the theory (for example) as a power series in the coupling g :

$$G(p_i;g) = G_0(p_i) + g\,G_1(p_i) + g^2G_2(p_i) + \dots \qquad (1.1)$$

where the $\{G_K(p_i);\ K = 0,1,\dots\}$ are calculated using the Feynman graph expansion. The question that interests us here is, does the series (1.1) converge for all g such that $|g| < R$, where R is a positive number? If it does not, then the meaning of eqn. (1.1) is in doubt. The answer to the question was first suggested by Dyson (1952) in the case of quantum electrodynamics. He argued that the perturbation series was in fact divergent for all g , and that this was intimately connected with the instability of the vacuum for an unphysical value of the coupling constant.

The divergence of the perturbation expansion

We will now reproduce Dyson's argument. Consider a physical quantity, F, which is calculated in perturbation theory. In quantum electrodynamics the analogous expression to (1.1) is:

$$F(e^2) \;=\; a_0 + a_2 e^2 + a_4 e^4 + \ldots \tag{1.2}$$

where e is the electromagnetic charge. If $F(e^2)$ is analytic for $|e^2| < R$ then the series will converge in this region. Now consider the same theory but with $e^2 \to -e^2$, so that like charges attract instead of repel. In this theory the usual vacuum state is not the state of lowest energy. Thus due to the phenomenon of quantum mechanical tunnelling the vacuum can disintegrate. This leads to the conclusion that $F(-e^2)$ is not analytic for any non-zero e and so the series in eqn. (1.2) has zero radius of convergence. Dyson adds that the series (1.2) will be asymptotic and the divergence will only become noticeable at high orders in perturbation theory. These comments are not special for quantum electrodynamics; in general, perturbation series in field theory are divergent (Jaffe 1965 and references therein).

The next question that might be asked is, how badly does the series diverge? In other words, can we characterise the behaviour of perturbation theory at large order? In order to get some insight into this question, we study two model "field theories".

Firstly, consider a ϕ^4 field theory defined at one space-time point - a "zero dimensional" field theory. The partition function for this theory is given by

$$Z(g) \;=\; \int_{-\infty}^{\infty} d\phi \, \exp - \{\tfrac{1}{2}\phi^2 + \tfrac{g}{4}\phi^4\} \tag{1.3}$$

Expanding the exponential as a power series in g, we obtain

$$Z(g) = \sum_{K=0}^{\infty} Z_K g^K \tag{1.4}$$

where
$$Z_K = \frac{(-1)^K}{K!\ 4^K} \int_{-\infty}^{\infty} \phi^{4K} \exp - \tfrac{1}{2}\phi^2\ d\phi$$

$$\simeq \frac{(-1)^K}{\sqrt{\pi}}\ K!\ 4^K\ K^{-1}\ (1 + 0\ (\tfrac{1}{K})\ ) \tag{1.5}$$

We see in this simple example that the perturbation expansion does indeed have zero radius of convergence and moreover it grows like $(K - 1)!$ for K large.

As a second example we consider the anharmonic one-dimensional oscillator in quantum mechanics. This has the Hamiltonian

$$H = \frac{p^2}{2m} + \tfrac{1}{2}m\omega^2 x^2 + \frac{g}{4}x^4 \tag{1.6}$$

This system can be viewed equivalently as a ϕ^4 field theory in zero space and one time dimension with x playing the role of the field ϕ. Perturbation theory is frequently used for the Hamiltonian (1.6) in order to obtain the energy levels. For example the ground state energy, E_0, can be calculated as a power series in g

$$E_0(g) = \sum_{K=0}^{\infty} E_K\ g^K = \left(\frac{\hbar\omega}{2}\right) . \left[1 + \frac{3g}{8}\left(\frac{\hbar}{m^2\omega^3}\right) + 0(g^2)\right] \tag{1.7}$$

Bender and Wu (1968, 1969, 1973) have calculated the first one hundred and fifty terms in the series (1.7). For large K they obtained the following fit for E_K ($\hbar = m = \omega = 1$)

$$E_K = -\ (-1)^K\ K!\ \left(\frac{3}{4}\right)^K K^{-\frac{1}{2}} \sqrt{\frac{6}{\pi^3}}\ \left(1 + 0\ \left(\frac{1}{K}\right)\right) \tag{1.8}$$

Again one sees the characteristic $K!$ growth at high orders. This system is amenable to a variational calculation and for example it is found that (Graffi and Grecchi 1973)

$$E_0(0.2) = 0.532642754 \ldots \text{ and } E_0(2.0) = 0.696175820 \ldots \quad (1.9)$$

Clearly we would hope to resum the perturbation expansion (1.7), using the information contained in eqn. (1.8), to obtain a result such as (1.9). The divergence of the series stems from the existence of the $K!$ term and so it is natural to use a resummation method which effectively cancels this factor and leaves a series with a finite radius of convergence. Such a method will be discussed in the next section.

The Borel Transformation

Let us begin with a divergent series such as (1.4) or (1.7):

$$A(g) \sim \sum_{K=0}^{\infty} A_K \, g^K \quad (1.10)$$

where $A_K = (-1)^K \, K! \, a^K \, K^b \, c \left[1 + 0\left(\frac{1}{K}\right)\right]$

and a, b and c are real numbers $(a > 0)$. We introduce the representation

$$K! = \int_0^{\infty} t^K e^{-t} \, dt$$

and formally interchanging the summation and integration we obtain

$$A(g) = \int_0^{\infty} e^{-t} \sum_{K=0}^{\infty} (tg)^K (-1)^K a^K K^b c \left(1 + 0\left(\frac{1}{K}\right)\right) dt$$

$$\equiv \int_0^{\infty} e^{-t} B(tg) \, dt \quad (1.11)$$

where $B(t) = \sum_{K=0}^{\infty} \frac{A_K}{K!} t^K \quad (1.12)$

Notice that the series (1.12) has a finite radius of convergence: it is convergent for $|t| < {}^1/_a$. In the next section we will discuss how a knowledge of the singularities of B(t) allows us to obtain improved estimates for A(g) from perturbation theory.

We have been extremely cavalier in obtaining eqns. (1.11) and (1.12). From these results it appears that if we are given the coefficients of the asymptotic series A_K , then we can obtain the function A(g). This contradicts the well known fact (see for example Hardy 1949) that there are infinitely many functions with the same asymptotic series. We clearly need to impose conditions on A(g) if we are to obtain it from its perturbation series. To begin our discussion we give a theorem due to Watson (Watson 1912, Hardy 1949):

<u>Theorem</u> Suppose that

(1) A(g) is analytic in the domain D defined by

$|g| < R$, $|\arg g| < \delta + {}^\pi/_2$ where $\delta > 0$[†]

(2) $$A(g) = \sum_{n=0}^{K} A_n g^n + R_K(g)$$

where $|R_K(g)| < Ca^{K+1}(K+1)!\,|g|^{K+1}$ in D

then the series $\sum_{n=0}^{\infty} A_n g^n$ is Borel summable to

$$A(g) : A(g) = \int_0^\infty e^{-t} B(gt)\, dt$$

$$\text{where } B(t) = \sum_{K=0}^{\infty} \frac{A_K}{K!} t^K \; ; \quad |t| < {}^1/_a \, , \; |\arg t| < \delta$$

Thus if A(g) satisfies conditions (1) and (2) we may go in a unique way from perturbation theory to the full theory. Of course,

[†]Actually we will use the extension of this result which allows a cut from 0 to $-\infty$ (Graffi et al 1970)

these conditions have to be proved outside of the framework of perturbation theory. This is difficult for higher dimensional field theories but proofs for the anharmonic oscillator (Graffi et al 1970) and for the Euclidean Green's functions in $(\phi^4)_2$ (Eckmann et al 1975) have been given (the notation $(\phi^N)d$ means a theory with a ϕ^N interaction in d dimensions). Progress has also been made recently on $(\phi^4)_3$ (Feldman and Osterwalder 1976, Magnen and Sénéor 1976).

If one investigates model field theories with interaction $g\phi^{2N}$ one finds perturbation theory grows like $K[(N-1)]!$ (Bender and Wu 1971). In this case one may use a generalisation of Watson's theorem (Graffi et al 1970): Suppose that

(1) A(g) is analytic in the domain D defined by

$$|g| < R, \ |\arg g| < \delta + \frac{M\pi}{2} \quad \text{where } \delta > 0$$

(2) $$A(g) = \sum_{n=0}^{K} A_n g^n + R_K(g)$$

where $|R_K(g)| < Ca^{K+1} [M(K+1)]! \, |g|^{K+1}$ in D

Then the series is Borel summable (in a generalised sense) to

$$A(g) = \frac{1}{M} \int_0^\infty e^{-t^{1/M}} B(gt)\, t^{1/M-1}\, dt \tag{1.13}$$

$$\text{where } B(t) = \sum_{K=0}^{\infty} \frac{A_K}{(MK)!} t^K; \ |t| < {}^1/a, \ |\arg t| < \delta \tag{1.14}$$

Watson's Theorem indicates that if we know the radius of convergence of the series (1.12), that is, the position of the nearest singularity of B(t) to the origin, then we may determine the coefficient "a" in equation (1.10). We shall now show by a simple example that indeed the singularity of B(t) nearest the origin determines the leading asymptotic behaviour of $A(g)$.

Let the nearest singularity of $B(t)$ to the origin be at $t = a^{-1}$ where a is a real negative number.

Then we may write

$$B(t) = (t - a^{-1})^{-\alpha} C(t)$$

if the singularity is a pole or a branch point and where $C(t)$ is analytic at $t = a^{-1}$. Expanding $C(t)$ about $t = a^{-1}$ we obtain

$$B(t) = C_0(t - a^{-1})^{-\alpha} + C_1(t-a^{-1})^{-\alpha+1} + \dots \qquad (1.15)$$

Using equation (1.11) we obtain from equation (1.15) the coefficient of g^K in the asymptotic series for $A(g)$ to be

$$A_K = CK!\, a^K K^{\alpha-1} \left(1 + 0\left(\frac{1}{K}\right)\right) \qquad (1.16)$$

where C is a constant. Comparing equations (1.10) and (1.16) we see that the number b depends on the type and strength of the singularity. We also note that if there is another singularity further along the negative real axis then its effect on the asymptotic expansion for large K is negligible.

Finally in this section we point out that in one case the high order behaviour of the perturbation expansion tells us that the series is not Borel summable. This is the case where "a" defined in equation (1.10) is a negative number. The nearest singularity of $B(t)$ to the origin then lies on the positive real axis and so for instance the integral

$$A(g) = \int_0^\infty e^{-t} B(gt)\, dt$$

is not defined for $g > 0$ because it has a singularity on the path of integration.

Use of High Order Estimates

When faced with attempting to extract information from a perturbation expansion which is only asymptotic, one method of proceeding is to employ Padé approximants. These try to approximate the perturbation expansion by a ratio of two polynomials. Results obtained are sometimes very good, and in some cases, such as the one dimensional anharmonic oscillator (Loeffel et al 1969), it is known that Padé approximants converge to the correct result.

For theories which have a perturbation expansion which grows like $K!$ at $K^{\underline{th}}$ order ($K\to\infty$), a better strategy might be to Padé approximate the Borel transform of the perturbation expansion, $B(t)$. In practice $B(t) = \sum_K \frac{A_K t^k}{K!}$ will be a polynomial whose degree will depend on how many of the coefficients of the perturbation expansion, A_K, have been calculated. Once $B(t)$ is represented by a Padé approximant the integral (1.11) may be performed for a given value of g. Le Guillou and Zinn-Justin (1977) have calculated the ground state energy of the anharmonic oscillator from the first six terms of the perturbation series using both the Padé method and the Padé-Borel method. For an intermediate value of the coupling they find errors of 10^{-2} and 10^{-3} for the Padé and Padé-Borel methods respectively.

To obtain this increased accuracy, only the fact that the $K^{\underline{th}}$ order of perturbation theory grew like $K!$ for K large was used. If in addition the position of the nearest singularity of $B(t)$ to the origin is known then one can obtain even better accuracy. As an example suppose that $B(t)$ is analytic in the t-plane cut from

$-1/a$ to $-\infty$. Then one can map this cut plane into the interior of the unit circle (Loeffel 1976, Le Guillou and Zinn-Justin 1977, Brézin 1977) by

$$z = \frac{\sqrt{1 + at} - 1}{\sqrt{1 + at} + 1} \tag{1.17}$$

The branch cut is now located on the unit circle. Thus if we work with the variable z we have convergence everywhere inside the unit circle. This method gives improved results, compared to the Padé-Borel method, for the anharmonic oscillator (Le Guillou and Zinn-Justin 1977, Brézin 1977). This method can be refined even further by using information on the strength of the singularity (Le Guillou and Zinn-Justin 1977).

Thus it becomes clear that the more information one has on the asymptotic nature of perturbation theory, the better one can estimate quantities obtained at low orders in perturbation theory. This is the chief motivation for obtaining estimates such as (1.8).

Finally, it should be pointed out that, although in much of the above discussion the example of the anharmonic oscillator was used, (because we have an alternative way of calculating the energy levels — the variational method — which provides a check), we expect these methods to be useful for field theories in higher dimensions too. In particular we expect the K! growth at K$^{\underline{th}}$ order to come from the number of Feynman diagrams (Baker et al. 1976, Bender and Wu 1976). In these cases where the variational method is not applicable and where computer estimates such as (1.8) are not available, it is necessary to find an analytical technique to obtain high order estimates in perturbation theory. A method for doing just this is discussed in the next section.

The Role of Instantons

We shall indicate in this section, using an argument given by 't Hooft (1977a,b), that the singularities of the Borel transforms of Green's functions in field theory, are characterised by finite action solutions of the classical field equations in Euclidean space — the so called instantons.

Firstly, consider a "zero dimensional" field theory with partition function (compare with equation (1.3))

$$Z(g) = \int d\phi \exp - \{\tfrac{1}{2}\phi^2 + \frac{1}{g^2} V(g\phi)\} \tag{1.18}$$

where $V(x) = V_3 x^3 + V_4 x^4 + \ldots$

For convenience we rescale the fields, and the action:

$$H' = g^2 H \ , \ \phi' = g\phi$$

so that $$H'(\phi') = \tfrac{1}{2}(\phi')^2 + V(\phi') \tag{1.19}$$

Thus the integral (1.18) becomes

$$gZ(g) = \int d\phi' \exp - \frac{1}{g^2} H'(\phi') \tag{1.20}$$

Now the Borel transform of $gZ(g)$ as a function of g^2 is $B(t)$ where

$$gZ(g) = \int_0^\infty B(tg^2)\, e^{-t}\, dt = g^{-2}\int_0^\infty B(t)\, e^{-t/g^2}\, dt \tag{1.21}$$

and so comparing equations (1.20) and 1.21) we find

$$B(t) \sim \int d\phi' \, \delta(t - H'(\phi')) \qquad (1.22)$$

In order to perform the integral (1.22) we must find all the solutions of $H'(\phi') = t$, for fixed t. We call these solutions $\phi_1'(t)$, $\phi_2'(t)$, ... The integral (1.22) is then

$$B(t) \sim \sum_i \left| \frac{\partial H'(\phi')}{\partial \phi'} \right|^{-1}_{\phi' = \phi_i'(t)} \qquad (1.23)$$

Thus the singularities of the Borel transform can be found if the solutions of $\partial H'(\phi')/\partial \phi' = 0$ (the classical field equation for this system) are known; if $\{\overline{\phi}_i\}$ are the solutions then the singularities of B(t) occur at $t_i = H'(\overline{\phi}_i')$.

Now consider a multidimensional integral of the same form as (1.18). Going through the same arguments one finds that the Borel transform has singularities at $t = H(\overline{\underset{\sim}{\phi}})$ where $\overline{\underset{\sim}{\phi}}$ is a solution of the equations $\partial H/\partial \underset{\sim}{\phi} = 0$.

Finally, if we go over to multi-infinite dimensional integrals we get singularities at $t = H(\overline{\phi})$ where $\overline{\phi}(x)$ is a solution of the field equations $\delta H/\delta\phi(x) = 0$. The Borel transform of all the functions of the theory have their singularities at the same values of t, although the strengths of the singularities will vary ('t Hooft 1977a,b).

From this and previous discussions, a method of calculating the leading high order behaviour of perturbation theory for scalar theories has emerged. One begins by looking for the instanton solution with least action, since this will give the position of the nearest singularity of the Borel transformed Green's functions to the origin. If we denote this solution by $\phi_c(x)$, then the high order behaviour of the Green's functions in ϕ^4 theories will be (according to equation

(1.16)).

$$G_K \sim K! \; (1/H'(\phi_c'))^K K^b \tag{1.24}$$

for large K, where G_K is the coefficient of g^K in the perturbation expansion for the Green's function and b depends on the Green's function being considered. Of course this argument is quite heuristic, however we shall see that in fact the form of the estimate (1.24) is correct.

In order to get better control of the high orders in perturbation theory a more powerful and direct method to that outlined above is needed. In the next section a step in this direction is made, when the relation between instantons and vacuum tunnelling in field theories is explored.

Tunnelling in Field Theories

We begin this section by a brief historical review of instanton solutions to field theories.

The word "instanton" was first used to describe the pseudoparticle solution to the SU(2) Yang-Mills theory in Euclidean four-space found by Belavin et al (1975). It was 't Hooft (1976a) who suggested that since they are not only localised in three-space, but also localised, or instantaneous, in time they should be called instantons. A physical interpretation for these solutions was then found ('t Hooft 1976a,b, Callan et al. 1976, Jackiw and Rebbi 1976a);they describe vacuum tunnelling between the classically degenerate stable vacua of Yang-Mills theories.

In scalar field theories the instantons also describe tunnelling between classically stable vacua. If, for example, the potential has two relative (non-degenerate) minima, then there will be tunnelling from the state of higher energy density to the state of lower energy density. The use of non-trivial solutions to the field equations to describe the decay of metastable states in this way, was recognized by Langer (1967) when investigating the problem of condensation in statistical mechanics. The quantitative theory of the decay of the "false vacuum" has also been recently discussed by Voloshin et al (1974). The instanton approach to the problem was developed by Coleman (1977) (see also Callan and Coleman (1977)) and Stone (1976,1977); a semiclassical calculation of the lifetime of metastable vacuum states in Minkowski space formalism has been performed by Katz (1978).

From the above review we see that classical solutions of the field equations in imaginary time (ie. in Euclidean space) seem to describe classically forbidden processes. To see how this comes about let us go back to quantum mechanics where we already know how to deal with such problems.

Consider the problem of transmission through the potential barrier $V(x)$ (shown in Figure 1) in one dimensional quantum mechanics. The tunnelling amplitude is easily calculated in the WKB approximation. One finds

$$|T|^2 = \exp - \frac{2}{\hbar} \int_{x_1}^{x_2} dx \sqrt{2m(V(x) - E)}\, (1 + O(\hbar)) \qquad (1.25)$$

where $V(x_1) = V(x_2) = E$. The result (1.25) is exponentially small in $\hbar$ (a typical result found when studying tunnelling phenomena) and thus is never seen in perturbation theory.

Another method of obtaining the result (1.25), using the imaginary time formalism, was first suggested by Freed (1972) and

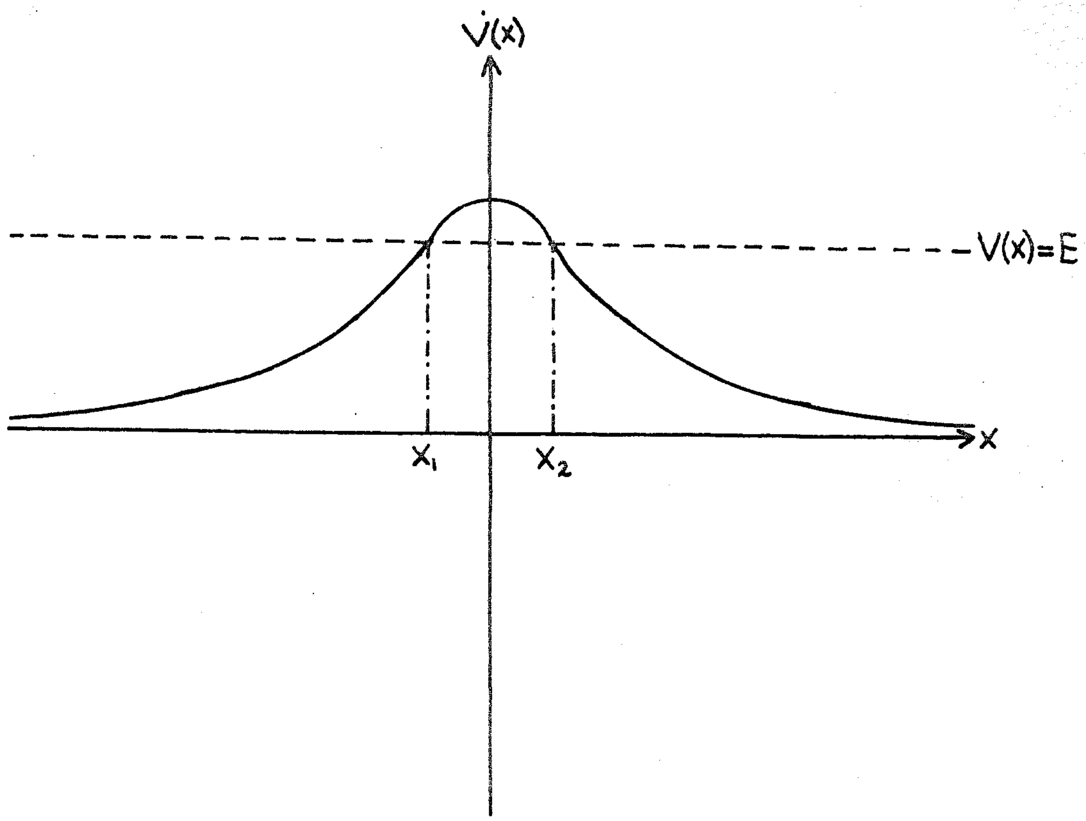

Figure 1. Potential barrier in one dimensional quantum mechanics

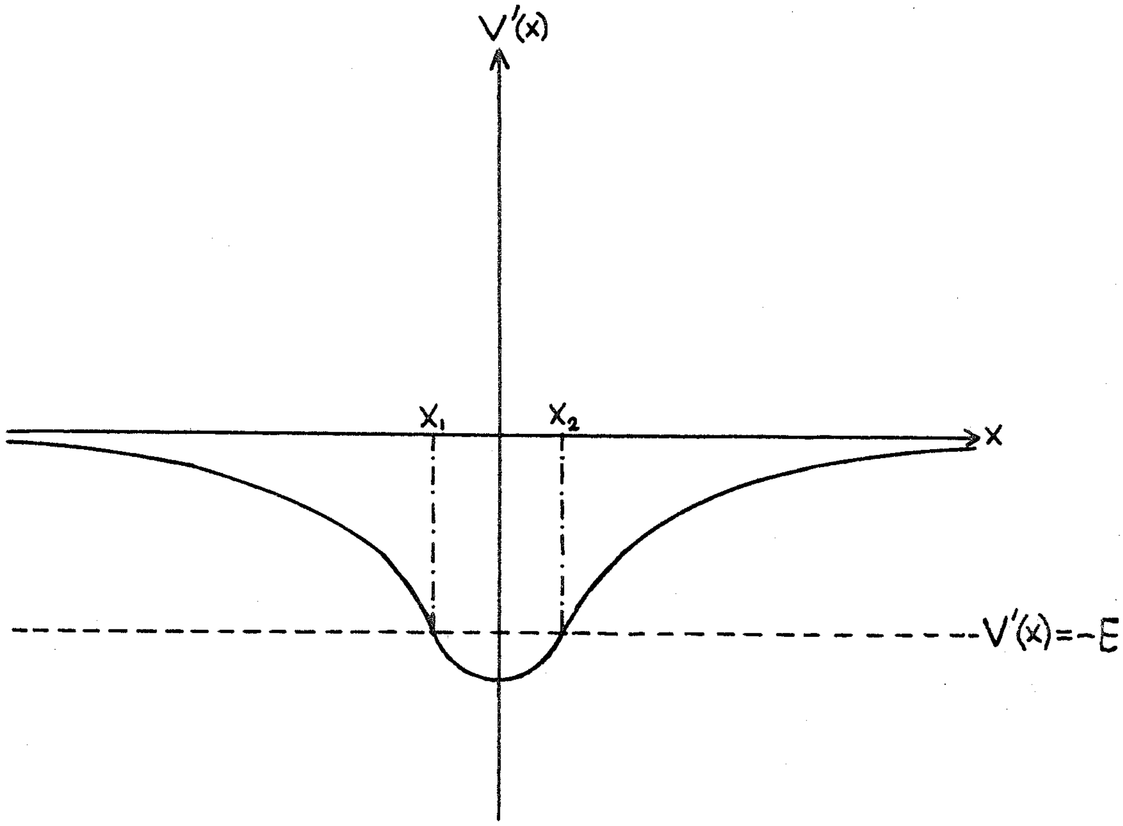

Figure 2. Inversion of the potential barrier shown in figure 1.

McLaughlin (1972). They argued as follows. We know that in Feynman's path integral formulation of quantum mechanics the limit $\hbar \to 0$ picks out paths of stationary action, that is, the functional integral is dominated by the classical paths. However in the classically forbidden region there is no real classical path. We can get around this problem by working in imaginary time because this just inverts the potential and makes classically forbidden regions classical allowed and vice-versa. Thus tunnelling phenomena should be described by considering classical paths in imaginary time inside the classically forbidden region.

Using these ideas we now obtain equation (1.25). We begin from the Hamiltonian

$$H = \frac{m}{2}\left(\frac{dx}{dt}\right)^2 + V(x) \tag{1.26}$$

where V(x) is shown in Figure 1. The equations of motion are then

$$m\frac{d^2x}{dt^2} = -\frac{dV}{dx} \tag{1.27}$$

There is no solution to the equation (1.27) for $x_1 < x < x_2$ when $E < V(x)$.

Now let us consider the theory in imaginary time with t replaced by $-i\tau$. Equation (1.27) now looks like

$$m\frac{d^2x}{d\tau^2} = -\frac{dV'}{dx} \tag{1.28}$$

where $V'(x) = -V(x)$ is shown in Figure 2. A classical solution, x_c, now exists for $x_1 < x < x_2$ and we can integrate equation (1.28) to get

$$\frac{1}{2}\, m\left(\frac{dx_c}{d\tau}\right)^2 = V(x_c) - E \tag{1.29}$$

Now the first approximation to the Feynman path integral

$$\int d\,[x(t)]\, \exp\, {}^{i}/\hbar \int L dt \tag{1.30}$$

is just

$$\exp\ {}^{1}/\hbar \int_{\tau(x_1)}^{\tau(x_2)} \left[-\frac{m}{2}\left(\frac{dxc}{d\tau}\right)^2 - V(x_c) \right] d\tau$$

$$= \exp\ {}^{1}/\hbar \int_{\tau(x_1)}^{\tau(x_2)} \left[- m\left(\frac{dxc}{d\tau}\right)^2 - E \right] d\tau \qquad \text{using equation } (1.29)$$

$$= \exp - \frac{E}{\hbar}\ (\tau(x_2) - \tau(x_1))\ \exp - {}^{1}/\hbar \int_{\tau(x_1)}^{\tau(x_2)} m\left(\frac{dxc}{d\tau}\right)^2 d\tau \tag{1.31}$$

Thus the required amplitude is to leading order

$$\exp - {}^{2}/\hbar \int_{\tau(x_1)}^{\tau(x_2)} m\left(\frac{dxc}{d\tau}\right)^2 d\tau$$

$$= \exp - {}^{2}/\hbar \int_{x_1}^{x_2} m\,\frac{dxc}{d\tau}\ dx_c$$

$$= \exp - {}^{2}/\hbar \int_{x_1}^{x_2} \sqrt{2m(V(x)-E)}\,dx \tag{1.32}$$

using equation (1.29) again. The result (1.32) is in complete agreement with the result (1.25) obtained by the WKB method.

The power of the imaginary time formalism becomes apparent when we generalise it to field theory. This is because the WKB method is not easily generalisable to multidimensional systems and so to obtain analogous results to (1.25) we have to use the imaginary time formalism. The generalisation itself is obvious : we look for finite action solutions of the classical field equations in Euclidean space. Further the most important solutions for the tunnelling amplitude are those with minimum action. Thus the classical solution responsible for tunnelling can be identified with the instanton solutions governing high order behaviour, which were discussed in the last section. A semiclassical description of tunnelling is then obtained by dominating the functional integral by the instanton solutions.

Looking back over the last two sections of this chapter, we see a realisation of Dyson's argument connecting the instability of the vacuum and the divergence of perturbation theory. It is therefore not surprising that the high order behaviour of perturbation theory can be obtained by a semiclassical calculation as outlined above. This was first observed by Lam (1968) and independently by Bender and Wu (1971) in the case of the anharmonic oscillator. The method was rediscovered by Lipatov (1977a) and applied to the problem of obtaining high order estimates in scalar field theories. The work has been extended by Brézin et al. (1977a). The leading high order behavior of perturbation theory has now been characterised for a number of other field theories (Parisi 1977a, Itzykson et al 1977a,b, Balian et al 1978).

In this thesis we will follow Lipatov (1977a,b,) and Brézin et al (1977a) and calculate the high order behaviour of perturbation theory for $g\phi^{2N}$ field theories, in $d = 2N/N-1$ dimensions, using the semiclassical approximation. However, we will not use the Pauli-Villars method in order to regularise the ultra-violet divergences of the theory as these authors did, instead we shall use dimensional regularisation ('t Hooft and Veltman 1972, Bollini and Giambiagi 1972). We shall also investigate $g\phi^{2N-1}$ theories using the same method.

The outline of the thesis is as follows. In Chapter two we will study simple systems which we expect to have the same behaviour (qualitatively) as the field theories discussed later. In the third chapter we examine the existence and form of instanton solutions to ϕ^N field theories with particular emphasis on massless theories in $d = {}^{2N}/_{N-2}$ dimensions. We also discuss how the conformal invariance of these theories allows us to perform calculations analytically. In chapter four the leading high order behaviour of perturbation theory is characterised for ϕ^4 field theory in $4 - \varepsilon$ dimensions, by a one-loop calculation of the imaginary part of the vertex functions generated by tunnelling out of the metastable ground state (for the theory with the wrong sign coupling constant). Using these results the leading asymptotic behaviour of the renormalisation group β function is also obtained in four dimensions. In Chapters five and six we repeat these calculations but for ϕ^3 theory in $6 - \varepsilon$ dimensions and ϕ^N theory $(N > 4)$ in $d_c - \varepsilon$ dimensions ($d_c = {}^{2N}/_{N-2}$) respectively. In the latter case we study the case $N=6$ ($d_c=3$) in slightly more detail. Finally, in chapter seven, we conclude by discussing the validity of these results in view of the extra singularities which appear when the theory is just renormalisable.

CHAPTER TWO

INSTANTON CALCULATIONS IN SIMPLE MODELS

In this chapter we shall explain how to quantify the idea that high order estimates of the perturbation expansion can be obtained by a semiclassical calculation. To do this it is easier to work with simple prototype models of the field theories we will be considering later on. Thus most of this chapter will be concerned with "zero dimensional" field theories, that is, a field theory defined at one space-time point. The partition function for these theories is just a single integral and the semiclassical method is just the method of steepest descent. At the end of the chapter we will apply the knowledge we have gained to find the high order behaviour of a field theory with interaction $g\phi^{2N}$ in one time and no space dimensions. This is just the quantum mechanical anharmonic oscillator and thus we find we reproduce the result of Bender and Wu (1971), but using instanton techniques (Brézin et al (1977a), Brézin (1977),Zinn-Justin (1977), Collins and Soper (1978)).

ϕ^4 field theory in zero dimensions

This theory has already been mentioned in chapter one. The partition function is given by

$$Z(g) = \int_{-\infty}^{\infty} d\phi \, \exp - \{\tfrac{1}{2}\phi^2 + {}^{g}/_{4}\, \phi^4\} \qquad (1.3)$$

If one expands Z(g) as a power series in g, it is found that the power series is divergent; the high order behaviour is given by equation (1.5). This result will now be obtained by a method which can be generalised to actual field theories.

The first point to notice is that the integral (1.3) is convergent, and hence $Z(g)$ is defined, only for $-\pi/2 < \arg g < \pi/2$. However this is only when the contour of integration in the ϕ plane runs along the real axis. If we rotate the contour by a small amount, say δ, then the integral is defined for $-\pi/2 - 4\delta < \arg g < \pi/2 - 4\delta$. We can also continue the function by rotating the contour of integration by $-\delta$, and then the integral is defined for $-\pi/2 + 4\delta < \arg g < \pi/2 + 4\delta$. Let us consider these two cases when $\delta = \pi/4$. The contours, C_1 and C_2, are then as shown in Figure 3. There are now two functions: $Z_1(g)$ defined by C_1 and $Z_2(g)$ defined by C_2, that is,

$$Z_1(g) = \int_{C_1} d\phi \exp - \{\tfrac{1}{2}\phi^2 + g/_4\, \phi^4\} \; ; \; -3\pi/2 < \arg g < -\pi/2 \tag{2.1}$$

and

$$Z_2(g) = \int_{C_2} d\phi \exp - \{\tfrac{1}{2}\phi^2 + g/_4\, \phi^4\} \; ; \quad \pi/2 < \arg g < 3\pi/2$$

In particular, when g is real and negative it is easy to show that both $Z_1(g)$ and $Z_2(g)$ contain an imaginary part and that in fact $[Z_1(g)]^* = Z_2(g)$. Thus we see that the sign of the imaginary part depends on the way that the analytic continuation is performed.

Now we will evaluate the integral (for small g) by the method of steepest descent. This involves finding the saddle points of the function in the argument of the exponential, $H(\phi)$. These are found to be

$$\phi = 0, \quad \phi = \pm\sqrt{-\frac{1}{g}} \tag{2.2}$$

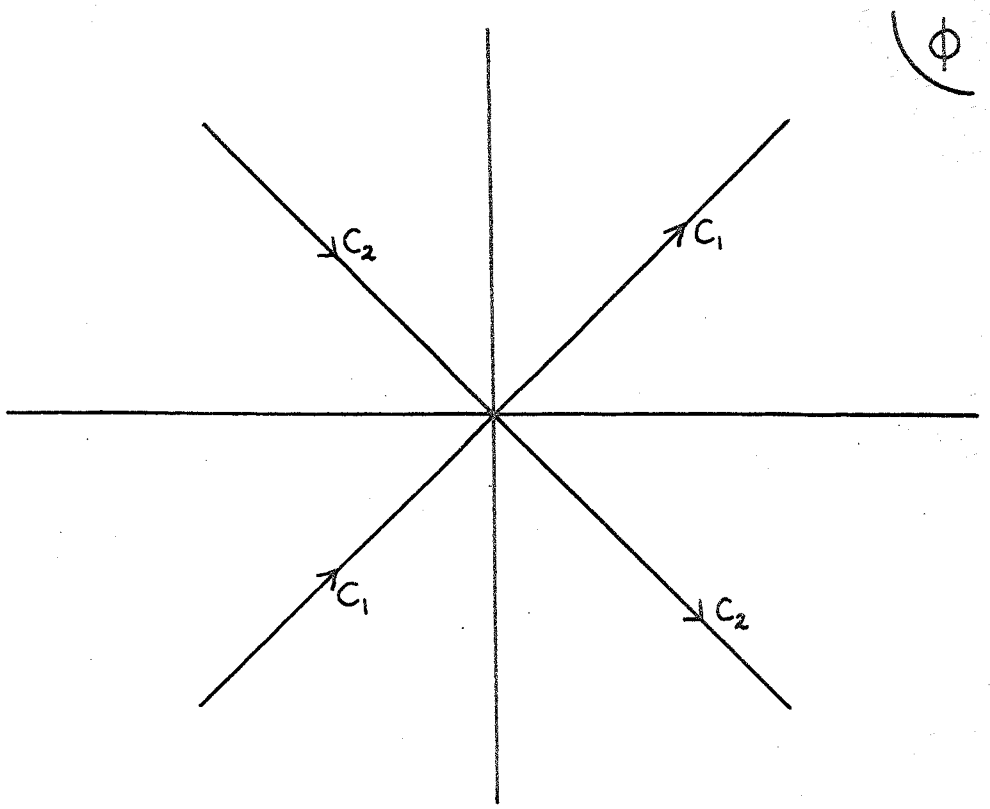

Figure 3. Contours for the integrals given in equation (2.1)

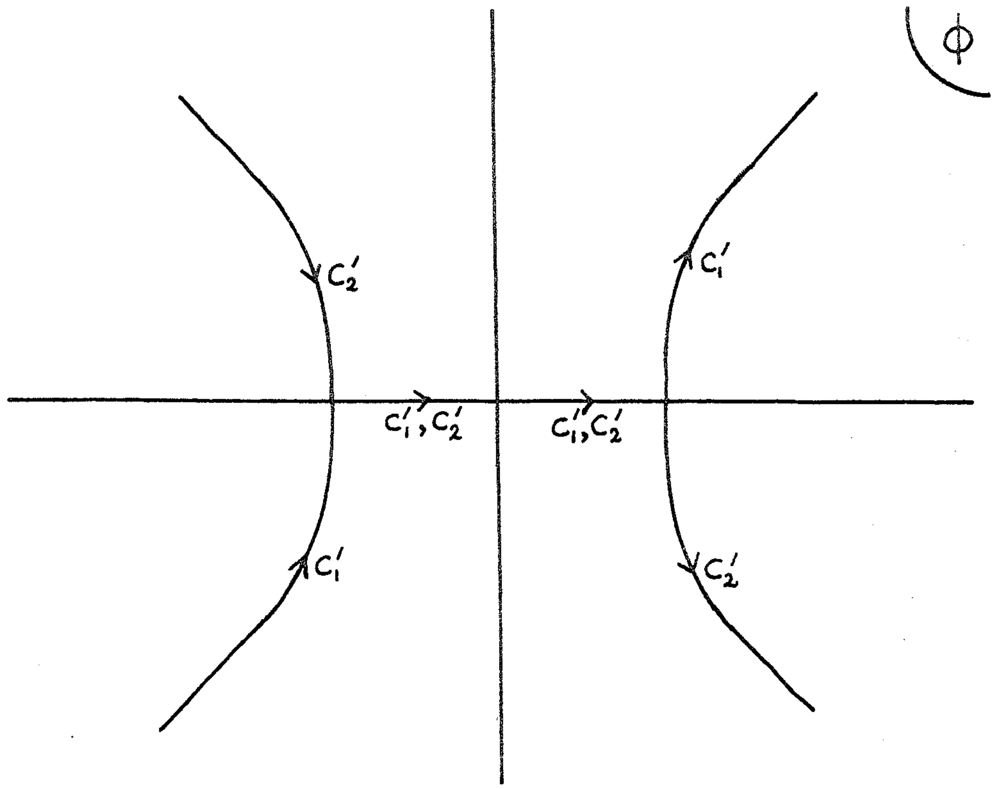

Figure 4. Steepest descent contours for ϕ^4 field theory.

Thus if we study the integral for $g < 0$ we have three real saddle points: the trivial one $\phi = 0$, and two "instanton" saddle points $\phi = \pm \phi_c$. We now have to find lines of zero imaginary part of $H(\phi)$. In order to obtain the best estimate for $Z_1(g)$ and $Z_2(g)$ defined by equation (2.1) the contours C_1 and C_2 have to be distorted in order to follow the paths of steepest descent. We call these contours C_1' and C_2' respectively; they are shown for the case $g < 0$ in Figure 4.

We expect that a good approximation to $Z_1(g)$ and $Z_2(g)$ will be to evaluate the integrand at the saddle points and approximate the corrections to leading order by a Gaussian integral. It turns out that the real part of $Z_1(g)$ and $Z_2(g)$ comes only from that part of the contour lying along the real axis. When the integral is dominated by the trivial saddle point $\phi = 0$ it just gives the usual perturbation expansion for $Z(g)$. The imaginary part comes only from those portions of the contour which go into the complex plane. Thus the imaginary part is dominated by the non-trivial "instanton" saddle-points and we find

$$\operatorname{Im} Z_2(g) = \exp - \{\tfrac{1}{2}\phi_c^2 + {}^{g}/_{4}\,\phi_c^4\} \int_{\infty}^{0} d\tau \exp\left[-\tau^2 + g\,\tau^4\right]$$

$$+ \exp - \{\tfrac{1}{2}\phi_c^2 + {}^{g}/_{4}\,\phi_c^4\} \int_{0}^{-\infty} d\tau \exp\left[-\tau^2 + g\,\tau^4\right] \tag{2.3}$$

where $\phi_c = 1/\sqrt{-g}$ and $g < 0$. Notice that both the "instanton" saddle points contribute only half a Gaussian integral. This observation will be useful when dealing with higher dimensional field theories. Evaluating equation (2.3) we have

$$\text{Im } Z\,(g;\ \arg g = \pi) \;=\; -\exp\,(1/4g)\int_{-\infty}^{\infty} d\tau\,\exp\left[-\tau^2 + g\tau^4\right]$$

$$= -\sqrt{\pi}\,\exp\,(1/4g)\,\left[1 + O\,(g)\right] \qquad (2.4)$$

It is easy to check that if the same analysis is repeated for $Z_1(g)$ then

$$\text{Im } Z(g;\ \arg g = -\pi) \;=\; +\sqrt{\pi}\,\exp\left(1/4g\right)\left[1 + O\,(g)\right] \qquad (2.5)$$

To summarise we may say that although for $g > 0$ $Z(g)$ is real, for $g < 0$ it develops an imaginary part which is exponentially small in g. In field theory this imaginary part develops because for $g < 0$ there is no physical state of lowest energy.

We will now use the result (2.4) to derive the leading asymptotic behaviour of the perturbation expansion for this theory, given already in equation (1.5). As discussed earlier in this section, $Z(g)$ (analytically continued from the integral representation (1.3)) is analytic in the g plane cut from 0 to $-\infty$. We can exploit this analytic structure and write a dispersion relation in g which enables the asymptotic behaviour of $Z(g)$ to be calculated (Bender and Wu 1971). To derive this dispersion relation use is made of Cauchy's integral formula

$$Z(g) \;=\; \frac{1}{2\pi i}\int_C \frac{Z(g')\,dg'}{g'-g}\;,\; g > 0 \qquad (2.6)$$

where C is shown in Figure 5. The contour at infinity gives no contribution since $Z(g) \sim |g|^{-\frac{1}{4}}$ as $|g| \to \infty$. Also the contribution from the small semicircle about the origin vanishes as the radius of the semicircle. Thus from equation (2.6) we deduce that

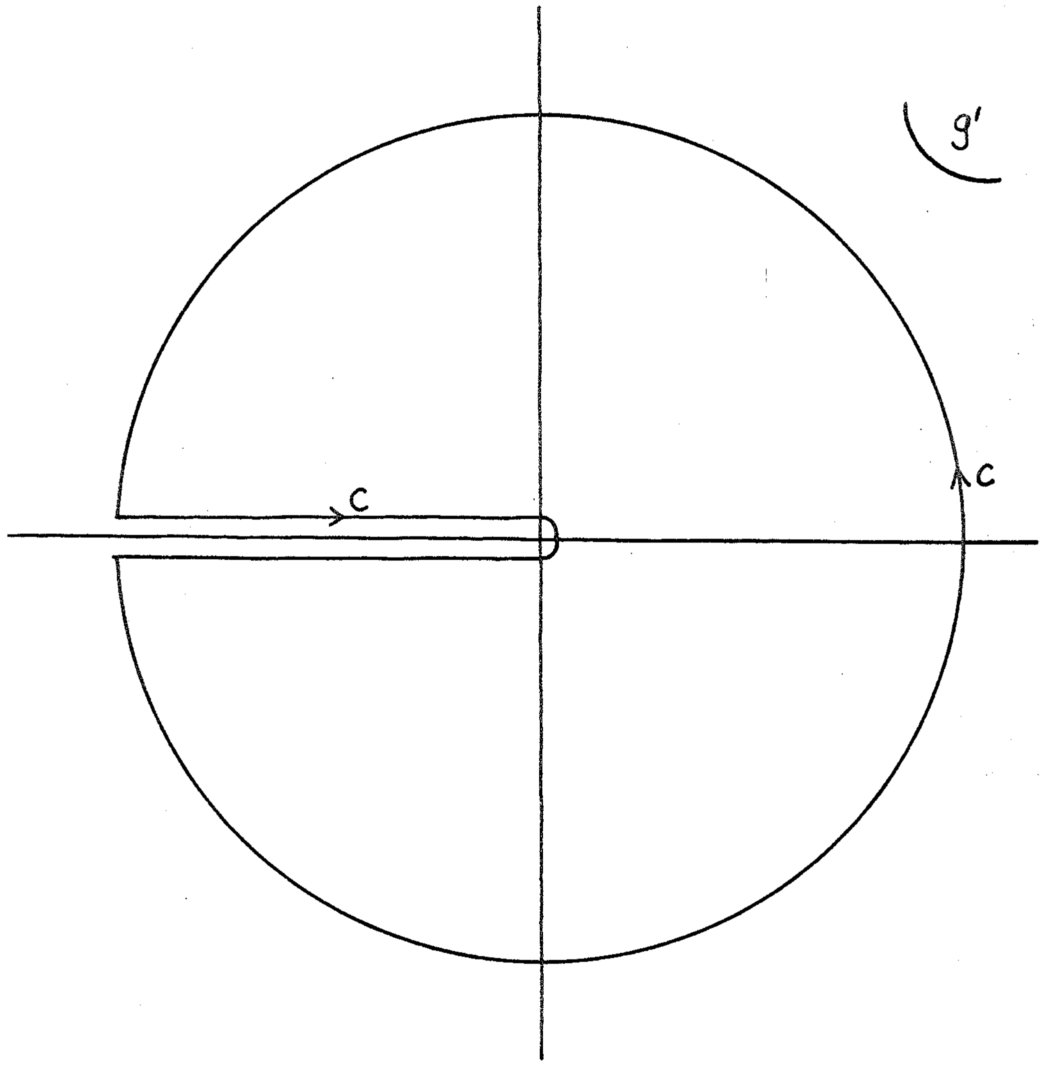

Figure 5. Contour involved in the dispersion relation for ϕ^4 theory.

$$Z(g) = \frac{1}{2\pi i} \int_{-\infty}^{0} \frac{Z(g'; \arg g' = \pi)\, dg'}{g' - g} + \frac{1}{2\pi i} \int_{0}^{-\infty}$$

$$\cdot \frac{Z(g'; \arg g = -\pi)\, dg'}{g' - g} \qquad (2.7)$$

Since $(Z(g; \arg g = \pi))^* = Z(g; \arg g = -\pi)$ equation (2.7) reduces to

$$Z(g) = \frac{1}{\pi} \int_{-\infty}^{0} \frac{\operatorname{Im} Z(g'; \arg g' = \pi)\, dg'}{g' - g} \qquad (2.8)$$

This is the required dispersion relation. If we pick out the coefficient of g^K in equation (2.8) we find

$$Z_K = \frac{1}{\pi} \int_{-\infty}^{0} \frac{\operatorname{Im} Z(g'; \arg g' = \pi)\, dg'}{(g')^{K+1}} \qquad (2.9)$$

Using the result given in equation (2.4) we obtain

$$Z_K = \frac{(-1)^K 4^K \Gamma(K)}{\sqrt{\pi}} \left[1 + O\left(\frac{1}{K}\right) \right] \qquad (2.10)$$

which is exactly the result given by (1.5). To calculate the $O(^1/K)$ corrections we need to know the $O(g)$ corrections to the result (2.4).

We see that this method of obtaining high order estimates involves two steps. Firstly, obtaining the imaginary part of the function under consideration, and secondly, proving a representation such as that in equation (2.8). Both these steps may be non-trivial in field theory.

ϕ^{2N} field theory in zero dimensions

In this section we investigate the asympotic behaviour of

$$Z(g) = \int_{-\infty}^{\infty} d\phi \exp - \{\tfrac{1}{2}\phi^2 + \frac{g}{2N} \phi^{2N}\} , \quad N=2,3, \ldots \tag{2.11}$$

using the methods discussed in the last section. Once again we can analytically continue $Z(g)$ by rotating the contour of integration in the ϕ plane. In particular if we rotate the contour by angles of $\pm\,{}^{\pi}/_{2N}$ then we again get two functions as the analytic continuation of (2.11); they have an imaginary part and are complex conjugates of each other for $g < 0$.

The evaluation of the integral for small negative g is also as before. The saddle points are now

$$\phi = 0, \quad \phi = \pm \left(-\frac{1}{g}\right)^{\frac{1}{2(N-1)}} \tag{2.12}$$

The imaginary part is dominated by the non-trivial saddle points in equation (2.12) and so we find †

$$\mathrm{Im}\, Z(g;\arg g = \pi) = \exp - \{ \frac{(N-1)}{2N} \left(-\frac{1}{g}\right)^{\frac{1}{N-1}} \} \int_{\infty}^{0} d\tau \exp - \tau^2 (N-1) \left[1+0\left(g^{\frac{1}{N-1}}\right)\right]$$

$$+ \exp - \{ \frac{(N-1)}{2N} \left(-\frac{1}{g}\right)^{\frac{1}{N-1}} \} \int_{0}^{-\infty} d\tau \exp - \tau^2 (N-1) \left[1+0\left(g^{\frac{1}{N-1}}\right)\right]$$

$$= - \sqrt{\frac{\pi}{N-1}} \exp - \{ \frac{(N-1)}{2N} \left(-\frac{1}{g}\right)^{\frac{1}{N-1}} \} \left[1+0\left(g^{\frac{1}{N-1}}\right)\right] \tag{2.13}$$

Using equation (2.9) we find that

$$Z_K = \frac{(-1)^K}{\sqrt{\pi}\,(N-1)} \cdot \left[K(N-1)\right]! \left(\frac{2N}{N-1}\right)^{(N-1)K} K^{-1} \left(1+0\left(\frac{1}{K}\right)\right) \tag{2.14}$$

---

† Only the saddle points on the real axis dominate $Z(g;\arg g = \pm\pi)$.

This result may also be obtained by expanding the integral (2.11) as a power series in g. We see that, as pointed out in Chapter one, ϕ^{2N} theories have a perturbation expansion which grows like $[K(N-1)]!$ at $K^{\underline{th}}$ order for K large.

<u>ϕ^3 field theory in zero dimensions</u>

The asymptotic behaviour of theories with an interaction term which involves odd powers of ϕ is very different from those with even powers. To see this consider the simplest theory of this type, namely a ϕ^3 theory defined at one space-time point:

$$Z(g)=\int_{-\infty}^{\infty} d\phi \exp - \{\tfrac{1}{2}\phi^2 + {}^{g}/_{3}\, \phi^3\} \tag{2.15}$$

Expanding the exponential as a power series in g, we find that Z(g) is even in g and

$$Z_{2K} = \frac{K!}{\sqrt{2\pi}} \, 6^K \, K^{-1} \left(1+0\left(\frac{1}{K}\right)\right) \tag{2.16}$$

$$\text{where } Z(g) = \sum_{K=0}^{\infty} Z_{2K}\, g^{2K} \tag{2.17}$$

Thus the coefficient of g^{2K} does not oscillate for real g as $K \to \infty$. In the notation of Chapter one this means that $a > 0$ and so the Borel transform of Z(g) has its nearest singularity to the origin on the positive real axis. Thus the perturbation expansion for these theories is not Borel summable. An indication of this is the existence of real non-trivial saddle points for real values of g; the saddle points being

$$\phi = 0 \;,\; \phi = -\,{}^{1}/_{g} \tag{2.18}$$

The integral (2.15) is not well defined for real positive g. To make it well defined we introduce the Airy integral contours C_I, C_{II} and C_{III} shown in Figure 6. Thus we have three different functions defined by

$$Z_i(g) \quad = \quad \int_{C_i} d\phi \, \exp - \{\tfrac{1}{2}\,\phi^2 + \tfrac{g}{3}\,\phi^3\} \quad i \; = I, II, III \qquad (2.19)$$

As usual we may define $Z_i(g)$ for other values of g by rotation of the contour. Again we find that because of the existence of branch cuts our continuation is not unique. Also it should be noted that although we appear to have three independent functions, one is the difference of the other two, and these are complex conjugates for real g.

As we have already remarked the crucial difference between this theory and those considered in the previous two sections, is that an imaginary part exists for all real values of g. Therefore let us evaluate the integrals (2.19) by the method of steepest descent for g positive. The paths of steepest descent are shown in Figure 7. We see that $Z_I(g)$ is dominated only by the "instanton" saddle point and is therefore pure imaginary, whereas $Z_{II}(g) = [Z_{III}(g)]^*$ has contributions from both saddle points and therefore has real and imaginary parts. The real perturbative part comes from the part of the integral dominated by the trivial $\phi = 0$ saddle point, and gives conventional perturbation theory. The imaginary part is given by

$$\mathrm{Im}\, Z_{II}(g;\, \arg g = 0) \quad = \quad \exp\{-1/6g^2\} \int_0^\infty d\tau \; e^{-\tau^2/2} \left[1 + 0(g^2)\right]$$

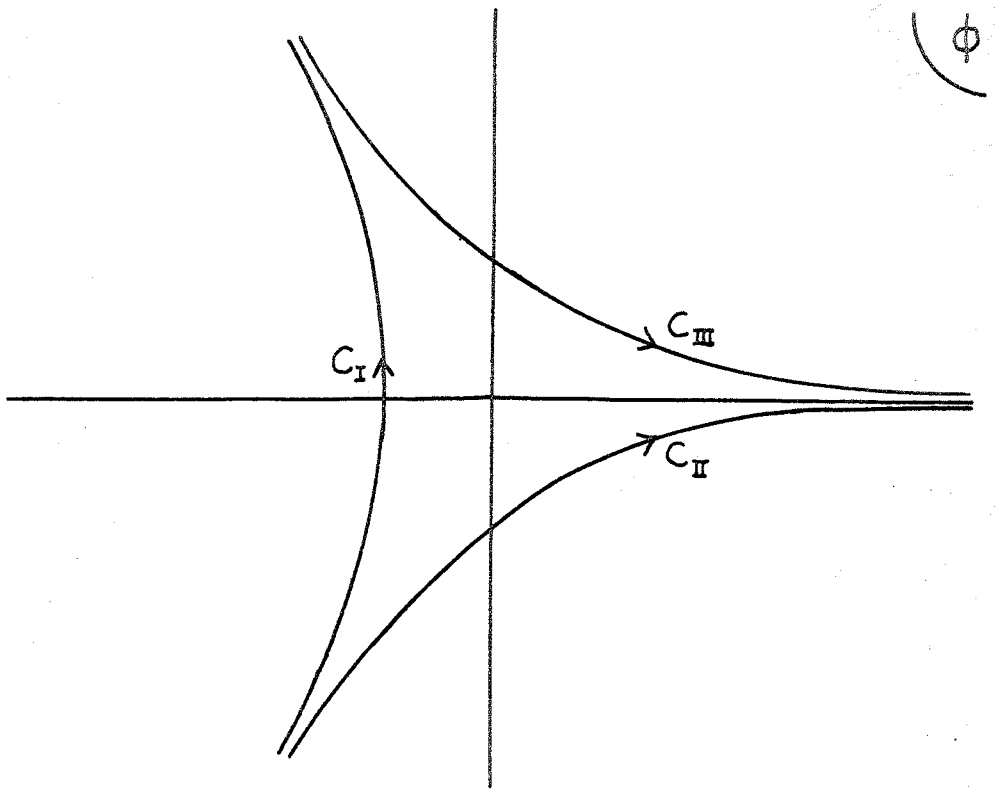

Figure 6. Airy contours used in ϕ^3 theory (equation (2.19))

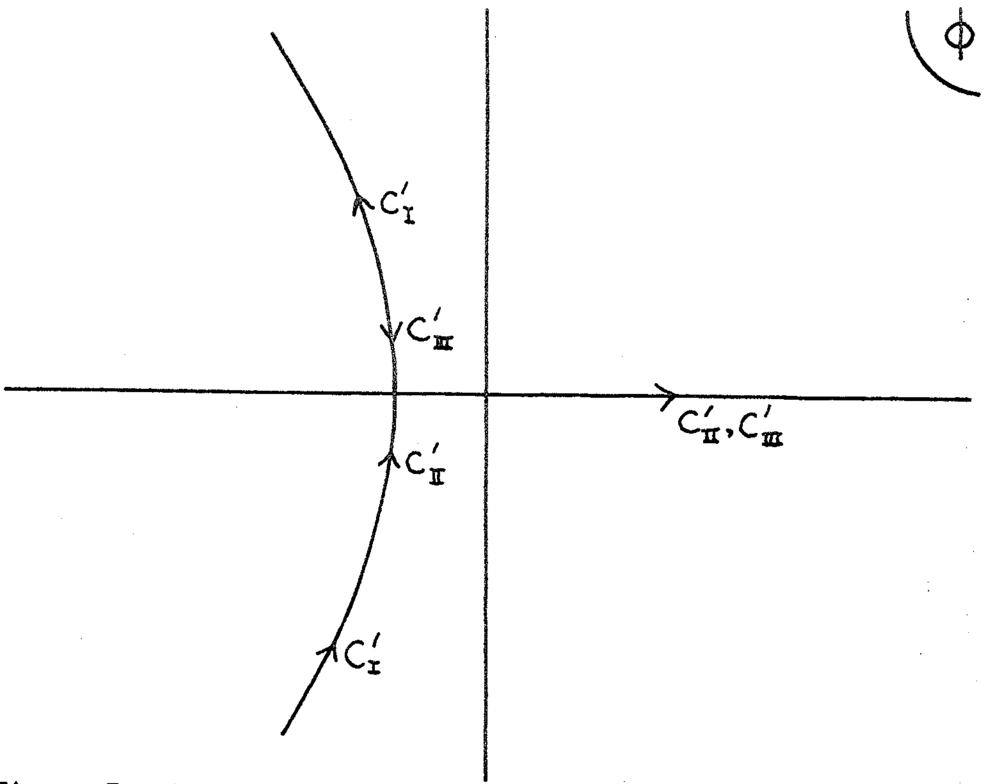

Figure 7. Steepest descent contours for ϕ^3 field theory

$$= \sqrt{\frac{\pi}{2}}\ \exp\{-{}^{1}/{}_{6g^2}\}\ \left[1 + O(g^2)\right] \tag{2.20}$$

A rotation of the ϕ contours by $-{}^{\pi}/{}_{3}$ in the integrals (2.19) defines the integrals for $g < 0$. We may then perform another steepest descent calculation to find that

$$\mathrm{Im}\ Z_{II}(g;\ \arg g = \pi) = \exp\{-1/6g^2\} \int_0^{-\infty} d\tau\ e^{-\tau^2/2} \left[1 + O(g^2)\right]$$

$$= -\sqrt{\frac{\pi}{2}}\ \exp\{-1/6g^2\} \left[1 + O(g^2)\right] \tag{2.21}$$

In order to determine the high order behaviour of Z_{2K} from equations (2.20) and (2.21) we need to know the analyticity properties of $Z(g)$. An examination of the properties of the functions defined by equation (2.19) shows that $Z_{II}(g)$ is entire in the upper half plane. Using this information we obtain an expression for Z_{2K} in terms of $\mathrm{Im}\ Z_{II}(g)$ in two ways.

The first method is to use Cauchy's integral formula with the contour C' defined in Figure 8:

$$Z(g) = \frac{1}{2\pi i} \int_{C'} \frac{Z(g')\ dg'}{g' - g} \quad ;\ g > 0 \tag{2.22}$$

Since $Z(g) \sim |g|^{-1/3}$ as $|g| \to \infty$ we can throw away the contour at infinity and we are left with

$$\frac{1}{2\pi i} \int_{-\infty}^{0} \frac{Z_{II}(g';\ \arg g' = \pi)\ dg'}{g' - g} + \frac{1}{2\pi i} \int_{0}^{\infty} \frac{Z_{II}(g';\ \arg g' = 0)\ dg'}{g' - g}$$

$$+ \frac{1}{2\pi i} \lim_{\delta \to 0} \int_{\pi}^{2\pi} \frac{Z_{II}(g + \delta e^{i\theta})\ i\delta e^{i\theta} d\theta}{\delta e^{i\theta}} = Z_{II}(g) \tag{2.23}$$

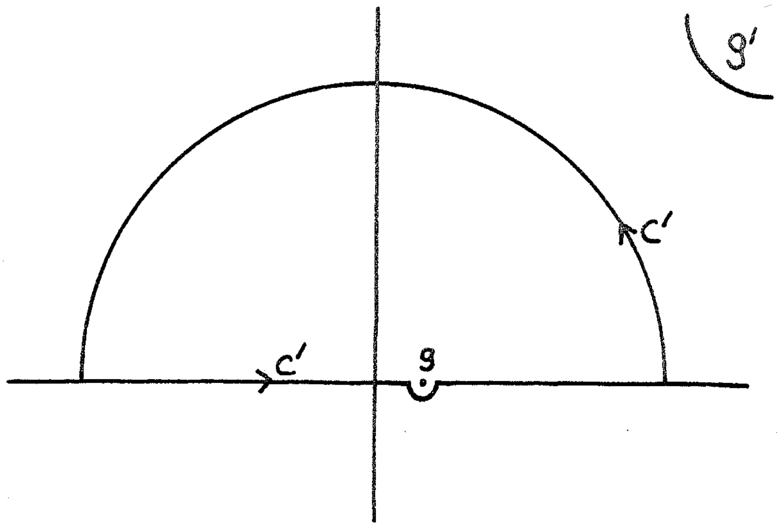

Figure 8. Contour involved in the dispersion relation in g' for ϕ^3 field theory.

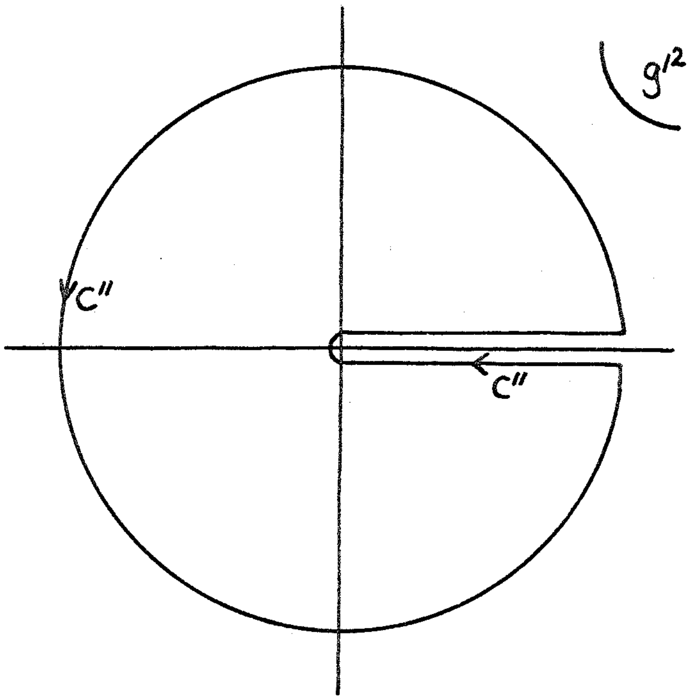

Figure 9. Contour involved in the dispersion relation in g'^2 for ϕ^3 field theory

This result implies that

$$\tfrac{1}{2}\,\mathrm{Re}\,\{Z_{II}(g)\} = \frac{1}{\pi}\int_0^\infty \frac{g'\,\mathrm{Im}\,Z_{II}\,(g';\ \arg g' = 0)\,dg'}{(g')^2 - g^2} \tag{2.24}$$

and therefore

$$Z_{2K} = \frac{2}{\pi}\int_0^\infty \frac{\mathrm{Im}\,Z_{II}\,(g';\ \arg g' = 0)\,dg'}{(g')^{2K+1}} \tag{2.25}$$

This is the required result. The second method of deriving equation (2.25) is to view Z as a function of g^2 and write a dispersion relation in g^2. Using the contour C" defined in Figure 9 we obtain

$$Z(g^2) = \frac{1}{2\pi i}\int_\infty^0 \frac{Z_{II}\,(g'^2;\ \arg g' = \pi)\,d(g'^2)}{(g')^2 - g^2} + \frac{1}{2\pi i}\int_0^\infty \frac{Z_{II}(g'^2;\arg g'=0)d(g'^2)}{(g')^2-g^2}$$

This immediately gives

$$\mathrm{Re}\,\{Z_{II}(g^2)\} = \frac{1}{\pi}\int_0^\infty \frac{\mathrm{Im}\,Z_{II}\,(g'^2;\ \arg g' = 0)\,d(g'^2)}{(g')^2 - g^2}$$

which agrees with equation (2.24).

We can now use equation (2.25) in conjunction with equation (2.20) to recover the result (2.16). Once again we emphasise that the reason that the series is not Borel summable is that "instantons" exist for all real values of g and thus so does an imaginary part of Z(g). In field theory this imaginary part comes about because of tunnelling out of the metastable ground state.

ϕ^{2N-1} field theory in zero dimensions

In this final section on zero dimensional field theories we study the asymptotic behaviour of Z(g) defined by

$$Z(g) \quad = \quad \int_{-\infty}^{\infty} d\phi \; \exp - \{\tfrac{1}{2}\phi^2 + \frac{g}{(2N-1)} \; \phi^{2N-1}\}, \; N=2,3,... \qquad (2.26)$$

This integral as it stands is not well defined for g positive; we require Re $(g\phi^{2N-1}) > 0$ as $|\phi| \to \infty$. This last condition is equivalent to (when arg g = 0)

$$\frac{(2n-\frac{1}{2})\pi}{2N-1} < \arg \phi < \frac{(2n+\frac{1}{2})\pi}{2N-1} \quad , \; |\phi| \to \infty \qquad (2.27)$$

where n is an integer. The bounds on arg ϕ given in equation (2.27) tell us that if the complex ϕ plane is divided into 2(2N-1) equal sectors, then the integral converges if we choose a contour which for large $|\phi|$ lies in one of the 2N-1 sectors defined by equation (2.27). Each of these contours gives a different function and so we have in principle (N-1) (2N-1) functions defined for arg g = 0 (compare with equation (2.19)). Of course, most of these are not independent and in analogy with the last section we choose to work with the two functions

$$Z_i(g) \quad = \quad \int_{C_i'} d\phi \; \exp - \{\tfrac{1}{2} \; \phi^2 + \frac{g}{(2N-1)} \; \phi^{2N-1}\} \; i = II,III \qquad (2.28)$$

where C_{II}' and C_{III}' are defined so that they begin in the sector $\pi + {}^{\pi}/_{2(2N-1)} < \arg \phi < \pi + {}^{3\pi}/_{2(2N-1)}$ and $\pi - {}^{3\pi}/_{2(2N-1)} < \arg \phi < \pi - {}^{\pi}/_{2(2N-1)}$ respectively and end up in the sector $- {}^{\pi}/_{2(2N-1)} < \arg \phi < {}^{\pi}/_{2(2N-1)}$ (see Figure 6 where the contours are drawn for the case N = 2) The two functions defined by equation (2.28) are complex conjugates for real g, and may be defined for other values of g by rotation of the contour. Just as in the ϕ^3 model discussed in the last section, we find that $Z_{II}(g)$ is entire in the upper-half plane, and so performing a dispersion relation in the coupling constant again gives equation (2.25).

The steepest descent calculation is a straightforward generalisation of the method for ϕ^3. The saddle points are for g positive

$$\phi = 0 \quad , \quad \phi = -\left(\frac{1}{g}\right)^{\frac{1}{2N-3}} \tag{2.29}$$

and so †

$$\mathrm{Im}\, Z_{II}\,(g;\ \arg g = 0) = \exp\left\{\frac{-(2N-3)}{2(2N-1)}\left[\frac{1}{g^2}\right]^{\frac{1}{2N-3}}\right\}\int_0^\infty d\tau e^{-\tau^2(2N-3)/2}\left[1+0\left(g^{\frac{2}{2N-3}}\right)\right]$$

$$= \sqrt{\frac{\pi}{2(2N-3)}}\ \exp\left\{\frac{-(2N-3)}{2(2N-1)}\left[\frac{1}{g^2}\right]^{\frac{1}{2N-3}}\right\}\left[1+0\left(g^{\frac{2}{2N-3}}\right)\right] \tag{2.30}$$

Substituting equation (2.30) into equation (2.25) and performing the integration we find

$$Z_{2K} = \frac{[K(2N-3)]\,!}{\sqrt{2\pi(2N-3)}}\left[\frac{2(2N-1)}{2N-3}\right]^{K(2N-3)} K^{-1}\left(1+0\left(\frac{1}{K}\right)\right) \tag{2.31}$$

This ends our discussion of zero dimensional field theories. We will find that the corresponding calculations for field theories, while being technically far more complicated, have the same general

† Again only the saddle points on the real axis dominate Z_{II} (g; arg g = 0)

structure as exhibited here for these simple theories.

ϕ^{2N} field theory in one dimension

In this, the last section of this chapter, we will study the high order behaviour of perturbation theory for the generalised anharmonic oscillator with Hamiltonian

$$H = \frac{p^2}{2m} + \tfrac{1}{2} m\omega^2 x^2 + \frac{g}{2N} x^{2N} \tag{2.32}$$

As is well known, this is equivalent to a one dimensional Euclidean ϕ^{2N}field theory by the change of time variable $t = i\tau$. The work of Bender and Wu on the high order behaviour of the perturbation expansion for the ground state energy in the case N=2, has already been discussed in Chapter one. They have also investigated the more general system given by equation (2.32) (Bender and Wu 1971, 1973) and find that the ground state energy of the system is given by ($\hbar=m=\omega=1$)

$$E_0(g) = \sum_{K=0}^{\infty} E_K g^K$$

where

$$E_K = -(-1)^K \left[K(N-1)\right]! \left[\frac{1}{4N}\left\{\frac{\Gamma(2N/N-1)}{\Gamma^2(N/N-1)}\right\}^{N-1}\right]^K K^{-\frac{1}{2}}$$

$$\left[\frac{(N-1)}{\pi^3}\frac{\Gamma(2N/N-1)}{\Gamma^2\left(\frac{N}{N-1}\right)}\right]^{\frac{1}{2}} \left[1+0\left(\frac{1}{K}\right)\right] \tag{2.33}$$

To obtain this result Bender and Wu performed a WKB calculation to determine the discontinuity of $E_0(g)$ across the branch cut for

$g < 0$. They then used a dispersion relation to obtain E_K for $K \to \infty$. They have also checked the result (2.33) against numerical values for E_K obtained on the computer for low values of N, and find good agreement.

The high order behaviour of E_K given by equation (2.33) can also be obtained by instanton techniques (Brézin et al 1977a). The technical obstacles that have to be overcome are not so severe as those in higher dimensional field theories, nevertheless, the techniques are applicable to the field theory problem.

Before we begin describing the details of the instanton calculation, we point out that it is known rigo rously that $E_0(g)$ is analytic in the complex g plane cut from o to $-\infty$ (Loeffel and Martin 1971, see Simon 1970 for discussion) in the case of the anharmonic oscillator. We would expect this structure to persist in the case of the generalised anharmonic oscillator, that is the theory with an anharmonic term gX^{2N} . Thus we may follow the same procedure as in previous sections to prove that

$$E_K = \frac{1}{\pi} \int_{-\infty}^{0} \frac{\operatorname{Im} E_0(g'; \arg g' = \pi)\, dg'}{(g')^{K+1}} \tag{2.34}$$

Thus in order to obtain the behaviour of E_K for K large we again have only to calculate the imaginary part of $E_0(g)$ generated when $g < 0$.

We begin from the Feynman-Kac formula

$$\lim_{T\to\infty} \exp - T(E_0(g) - E_0) = \lim_{T\to\infty} \left[\frac{\operatorname{Tr} e^{-TH}}{\operatorname{Tr} e^{-TH_0}}\right] \tag{2.35}$$

$$\text{where} \quad \frac{\mathrm{Tr}\, e^{-TH}}{\mathrm{Tr}\, e^{-TH_o}} = \frac{\int d\,[x(\tau)]\, \exp - \int_o^T d\tau \left[\tfrac{1}{2}\dot{x}^2 + \tfrac{1}{2}x^2 + \tfrac{g}{2N}x^{2N} \right]}{\int d\,[x(\tau)]\, \exp - \int_o^T d\tau \left[\tfrac{1}{2}\dot{x}^2 + \tfrac{1}{2}x^2 \right]} \tag{2.36}$$

A few words of explanation are in order here. Firstly, we have chosen for convenience units $\hbar = m = \omega = 1$. Secondly, the functional integral is over closed paths $x(\tau=0) = x(\tau=T)$ (because the trace appears in equation (2.35)), that is, it is a sum over all paths which are periodic in imaginary time with period T. Finally, by $\dot{x}$ we mean $dx/d\tau$.

To calculate the imaginary part of $E_0(g)$, $g<0$, we dominate the functional integral (2.36) by instanton solutions, $x_c(\tau)$, which are periodic in τ : $x_c(\tau) = x_c(\tau+T)$ as $T \to \infty$. They will satisfy the classical equations of motion

$$\ddot{x} = x + gx^{2N-1} \tag{2.37}$$

and will also have finite action, so that $x_c(\tau) \to 0$ as $\tau \to T$. In order to get a more physical picture of the instanton solutions we can employ a "mechanical analogy". This is obtained by remembering that while the potential for the generalised anharmonic oscillator, defined by equation (2.32), is not bounded below for $g < 0$. (see Figure 10), the transition to imaginary time effectively inverts the potential (see discussion at the end of Chapter one) and so the instanton corresponds to classically allowed solutions for a potential

$$V'(x) = -\tfrac{1}{2}x^2 - \tfrac{g}{2N}x^{2N} \quad , g < 0 \tag{2.38}$$

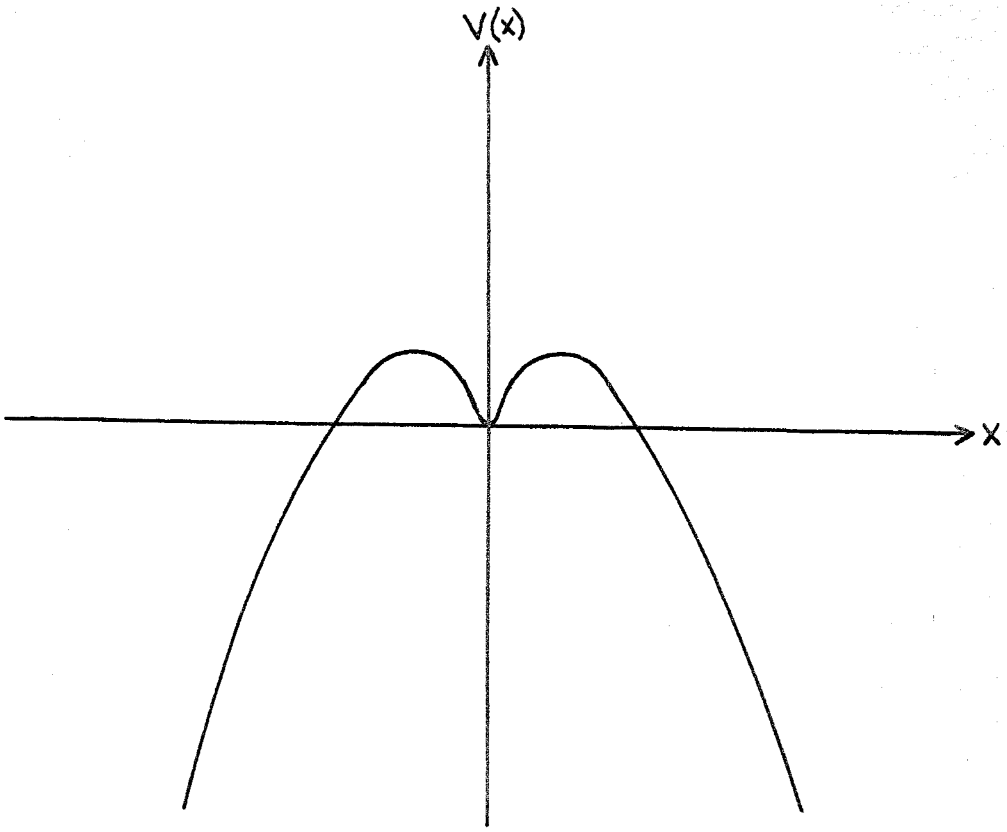

Figure 10. Anharmonic oscillator potential $V(x)=\frac{1}{2}x^2+g/2Nx^{2N}$ $(g<0)$

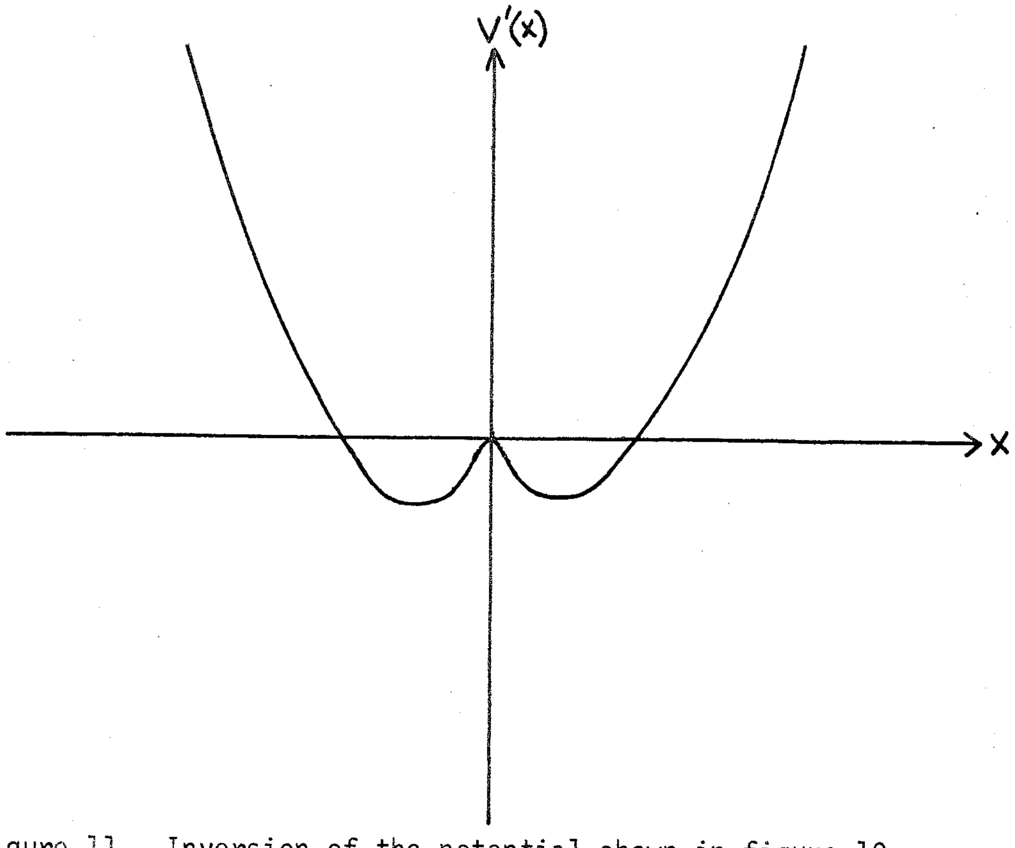

Figure 11. Inversion of the potential shown in figure 10; $V'(x)=-\frac{1}{2}x^2-g/2Nx^{2N}$ $(g<0)$

which is shown in Figure 11. The mechanical analogy then consists of interpreting x as a particle position and τ as time. In this way we can easily recognise a number of solutions to equation (2.37).

The most obvious solution is the trivial one $x_c(\tau) = 0$ corresponding to the particle staying at $x = 0$ for all time. However, if the particle leaves $x = 0$ it will move with an oscillatory motion, coming back to $x = 0$ after a time T. It is these solutions that interest us; in particular, we want the solutions with least action in order to calculate the leading high order behaviour. These solutions correspond to the particle making just one return journey away from $x = 0$. Moreover, since we are interested in the $T\to\infty$ limit, we want the particle to have infinitesimal velocity when leaving $x = 0$ so that it takes an infinitely long time to get back again. Thus in summary we may say that the instantons with least action correspond (in the $T\to\infty$ limit) to particles leaving $x = 0$ at $\tau = -T/2$ with infinitesimally small velocity, reaching $x = \pm \left(\frac{-N}{g}\right)^{\frac{1}{2(N-1)}}$ at time $\tau = 0$, and arriving back at $x = 0$ at $\tau = T/2$. Thus we have to solve equation (2.37) and obtain solutions that satisfy

$$
\begin{aligned}
x(\tau = -T/2) &= x(\tau = T/2) = 0, \quad T\to\infty \\
\dot{x}(\tau = -T/2) &= \dot{x}(\tau = T/2) = 0, \quad T\to\infty \\
x(\tau = 0) &= \pm(-N/g)^{1/2(N-1)}, \quad \dot{x}(\tau = 0) = 0, \quad T\to\infty
\end{aligned}
\tag{2.39}
$$

If we now integrate equation (2.37) we obtain

$$
\tfrac{1}{2}\left(\frac{dx}{d\tau}\right)^2 = \tfrac{1}{2}x^2 + \frac{g}{2N}x^{2N} + E \tag{2.40}
$$

where E is an integration constant which is just the energy of the particle (in our analogy). The conditions (2.39) tell us that as $T \to \infty$ the instantons approach the E = 0 solutions. Thus in the limit $T \to \infty$ the instantons are solutions of the system

$$\frac{1}{2}\left(\frac{dx}{d\tau}\right)^2 = \frac{1}{2}x^2 + \frac{g}{2N}x^{2N}$$

$$X(\tau = \pm\infty) = 0, \; X(\tau = 0) = \pm\left(-\frac{N}{g}\right)^{\frac{1}{2(N-1)}} \tag{2.41}$$

We may obtain an analytic form for the instantons in this case. Writing

$$X(\tau) = \pm\left(-\frac{N}{g}\right)^{\frac{1}{2(N-1)}} \left[f(\tau)\right]^{\frac{1}{N-1}} \tag{2.42}$$

equations (2.41) reduce to

$$\frac{1}{(N-1)^2} \cdot \left(\frac{df}{d\tau}\right)^2 = f^2 - f^4$$

$$f(\tau = \pm\infty) = 0, \; f(\tau = 0) = 1 \tag{2.43}$$

Scaling τ by (N-1) we have

$$\int \frac{df}{\sqrt{f^2 - f^4}} = \int d\tau \tag{2.44}$$

The substitution $f = \tanh\theta$ then gives

$$f(\tau) = \operatorname{sech}(N-1)(\tau - \tau_o) \tag{2.45}$$

Here τ_o is a constant of integration. It is the "time" at which the particle is at $X = \pm\left(-\frac{N}{g}\right)^{\frac{1}{2(N-1)}}$ (taken for convenience to be zero in our earlier discussion). So finally equations (2.42) and (2.45) give the instanton solutions with least action to be

$$\chi_c(\tau;\tau_o) = \pm \left(-\frac{N}{g}\right)^{\frac{1}{2(N-1)}} \left(\operatorname{sech}\,(N-1)\,(\tau-\tau_o)\right)^{\frac{1}{N-1}} \tag{2.46}$$

where their action is given by

$$H(\chi_c) = \lim_{T\to\infty} \int_{-T/2}^{T/2} d\tau \left[\tfrac{1}{2}\left(\frac{d\chi_c}{d\tau}\right)^2 + \tfrac{1}{2}\chi_c^{\,2} + \frac{g}{2N}\,\chi_c^{\,2N} \right] \tag{2.47}$$

$$= \lim_{T\to\infty} \int_{-T/2}^{T/2} d\tau \left[\chi_c^{\,2} + \frac{g}{N}\chi_c^{\,2N} \right] \quad \text{using equation (2.41)}$$

$$= \left[-\frac{4N}{g}\right]^{\frac{1}{N-1}} \frac{\Gamma^2\left(\frac{N}{N-1}\right)}{\Gamma\left(\frac{2N}{N-1}\right)} \tag{2.48}$$

Looking back at the semiclassical calculation for the zero dimensional ϕ^{2N} theory we see that the leading high order behaviour was obtained by approximating the "fluctuations" about the instanton solutions" to be Gaussian. In this case this consists of writing

$$\chi(\tau) = \chi_c(\tau) + \hat{\chi}(\tau) \tag{2.49}$$

then $$H(\chi) = H(\chi_c) + \tfrac{1}{2}\int d\tau_1\, d\tau_2\, \hat{\chi}(\tau_1)\, M(\tau_1,\tau_2)\, \hat{\chi}(\tau_2) + O(\hat{\chi}^3) \tag{2.50}$$

where $$M(\tau_1,\tau_2) = \left.\frac{\delta^2 H}{\delta\chi(\tau_1)\,\delta\chi(\tau_2)}\right|_{\chi=\chi_c} \tag{2.51}$$

In this approximation the functional integral (2.36) becomes

$$\exp - H(X_c) \frac{\int d[\hat{X}(\tau)] \exp - \{\frac{1}{2} \hat{X} M \hat{X} + 0(\hat{X}^3)\}}{\int d[X(\tau)] \exp - \{\frac{1}{2} X M_o X\}} \tag{2.52}$$

where $$M_o(\tau_1,\tau_2) = \frac{\delta^2 H_o}{\delta X(\tau_1) \, \delta X(\tau_2)} \tag{2.53}$$

The functional integrals in equation (2.52) are easily performed and we find

$$\exp - H(X_c) \left(\frac{\det M}{\det M_o}\right)^{-\frac{1}{2}} \cdot \left(1 + 0\left(\frac{1}{gN-1}\right)\right) \tag{2.54}$$

The expression (2.54), is however infinite, due to the fact that the operator M given by equation (2.51) has a zero eigenvalue. To see this we have only to differentiate

$$-\frac{d^2 X_c}{d\tau^2} + X_c + g X_c^{2N-1} = 0 \tag{2.55}$$

with respect to τ_o. We obtain

$$-\frac{d^2}{d\tau^2}\left(\frac{dX_c}{d\tau_o}\right) + \frac{dX_c}{d\tau_o} + (2N-1) g \, X_c^{2N-2} \frac{dX_c}{d\tau_o} = 0 \tag{2.56}$$

but from equations (2.46) and (2.51)

$$M(\tau_1,\tau_2) = \left[-\frac{d^2}{d\tau_1^2} + 1 - \frac{N(2N-1)}{\cosh^2 (N-1)\tau_1}\right] \delta(\tau_1-\tau_2) \tag{2.57}$$

and so equation (2.56) can be written as

$$\int d\tau\ M(\tau',\tau)\ \frac{dx_c}{d\tau_o} = 0 \tag{2.58}$$

and thus $dx_c/d\tau_o$ is an eigenfunction of M with zero eigenvalue. Its existence is due to the fact that the Hamiltonian for the problem is time translation invariant but the solution is not. The explicit form for this eigenfunction is found from equation (2.46) to be

$$\frac{dx_c}{d\tau_o} = \mp \left(-\frac{N}{g}\right)^{\frac{1}{2(N-1)}} \cdot \left(\text{sech}(N-1)\ (\tau-\tau_o)\right)^{\frac{1}{N-1}} \tanh\ (N-1)\ (\tau-\tau_o) \tag{2.59}$$

It can be seen that the eigenfuction in equation (2.59) has one node — when $\tau=\tau_o$. This suggests that there exists one eigenfunction with a smaller eigenvalue, that is, there exists one bound state with a negative eigenvalue.

The existence of zero eigenvalues of M means that the Gaussian approximation used to obtain expression (2.54) from equation (2.52) is invalid. For, in one of the integrals, the quadratic term vanishes and so higher order terms cannot be neglected. We can get around this problem by replacing the troublesome integral by an integral over τ_o, using the method of collective coordinates (Zittartz and Langer 1966, Langer 1967, Christ and Lee 1975, Gervais and Sakita 1975). To make this idea concrete, let us rewrite equation (2.49) as

$$x(\tau) = x_c\ (\tau;\tau_o) + \sum_n a_n\ x_n(\tau;\tau_o) \tag{2.60}$$

where $\{\chi_n(\tau;\tau_0)\}$ are eigenfunctions of the operator M. All we have done is chosen a convenient basis in which to expand $\hat{x}(\tau;\tau_0)$, and so the functional measure $d[\hat{x}(\tau)]$ goes over to $J \prod_n da_n$ where J is the Jacobian of the transformation. If the eigenfunctions are orthonormal then J = 1 (see Appendix I). This is one method of obtaining expression (2.54) from equation (2.52) in the Gaussian approximation. The idea of collective coordinates is to exclude the eigenfunction with zero eigenvalue from the sum in equation (2.60). The space of functions is then specified by coordinates $\{\tilde{a}_n, \tau_0\}$, where the tilde is there to remind us that the coefficient of the eigenfunction with zero eigenvalue is not included. The functional measure now goes over to $J\, d\tau_0 \tilde{\prod_n} da_n$ where the Jacobian, J, is now given by (see Appendix I)

$$J = \left(\int \left(\frac{dx_c}{d\tau_0}\right)^2 d\tau\right)^{\frac{1}{2}} \left(1 + 0\left(g^{\frac{1}{N-T}}\right)\right) \tag{2.61}$$

We can now perform the Gaussian integrals on all non-zero modes in equation (2.52). We obtain

$$\int_{-T/2}^{T/2} d\tau_0 \exp - H(x_c)\ (2\pi)^{-\frac{1}{2}} \left(\frac{\det \tilde{M}}{\det M_0}\right)^{-\frac{1}{2}} J \left(1 + 0\left(g^{\frac{1}{N-T}}\right)\right) \tag{2.62}$$

The tilde again indicates that zero modes have been extracted and the $(2\pi)^{-\frac{1}{2}}$ comes from the fact that there is one more Gaussian integral in the denominator than in the numerator. We have remarked already that M has one negative eigenvalue. Since all the eigenvalues of M_0 are positive, the expression (2.62) must be imaginary. In a proper steepest descent calculation the coordinate corresponding to the bound state would be integrated into the complex plane, and so we would pick up an imaginary part just as in the zero dimensional case discussed earlier. In lieu of such a treatment we take the sign of expression (2.62) from the zero dimensional result (2.13). We also notice from equation (2.13) that while there are two saddle points $x = \pm x_c$, each one only contributes half a Gaussian, and so the bound state effectively contributes one Gaussian integral. Bearing this in mind equations (2.35) and (2.62) give

$$\lim_{T\to\infty} \operatorname{Im} \exp - T\left(E_0(g) - E_0\right) = \lim_{T\to\infty} -T\,(2\pi)^{-\frac{1}{2}}\, J \exp - H(x_c)$$

$$\left|\frac{\det \tilde{M}}{\det M_0}\right|^{-\frac{1}{2}} \left(1 + 0\left(g^{\frac{1}{N-1}}\right)\right)$$

and expanding the left-hand-side gives

$$\operatorname{Im} E_0(g; \arg g = \pi) = (2\pi)^{-\frac{1}{2}}\, J \exp - H(x_c) \left|\frac{\det \tilde{M}}{\det M_0}\right|^{-\frac{1}{2}} \left(1 + 0\left(g^{\frac{1}{N-1}}\right)\right) \quad (2.63)$$

$H(x_c)$ is given by equation (2.48) and the Jacobian can be easily found from equations (2.59) and (2.61) to be

$$J = \left(\frac{-N}{g}\right)^{\frac{1}{2(N-1)}} 2^{\frac{1}{N-1}} \left[\frac{\Gamma^2\left(\frac{N}{N-1}\right)}{\Gamma\left(\frac{2N}{N-1}\right)}\right]^{\frac{1}{2}} \left(1 + 0\left(g^{\frac{1}{N-1}}\right)\right) \quad (2.64)$$

so it only remains to calculate the small oscillations determinant.

In order to compute this ratio, we define the differential operators

$$H = \frac{d^2}{dx^2} - z + \frac{\mathcal{J}(\mathcal{J}+1)}{\cosh^2 x}, \quad H_0 = \frac{d^2}{dx^2} - z \quad (2.65)$$

Then (Brézin et al. 1977a)

$$\frac{\det H}{\det H_0} = \frac{\Gamma(1+\sqrt{z})\,\Gamma(\sqrt{z})}{\Gamma(1+\gamma+\sqrt{z})\,\Gamma(\sqrt{z}-\gamma)} \tag{2.66}$$

The operators M and M_o are of the form (2.65) (after τ has been rescaled) with $\gamma = {}^N/N-1$, and so we find

$$\frac{\det M}{\det M_o} = \lim_{z\to\frac{1}{(N-1)^2}} \frac{\Gamma(1+\sqrt{z})\,\Gamma(\sqrt{z})}{\Gamma(1+{}^N/N-1+\sqrt{z})\,\Gamma(\sqrt{z}-{}^N/N-1)} \tag{2.67}$$

As we expected, we find that det M/ det M_o is formally zero, and that the required ratio is given by

$$\frac{\det \tilde{M}}{\det M_o} = \frac{1}{(N-1)^2} \lim_{z\to\frac{1}{(N-1)^2}} \frac{1}{z-\frac{1}{(N-1)^2}} \frac{\Gamma(1+\sqrt{z})\,\Gamma(\sqrt{z})}{\Gamma(1+\frac{N}{N-1}+\sqrt{z})\,\Gamma(\sqrt{z}-\frac{N}{N-1})} \tag{2.68}$$

so that

$$\frac{\det \tilde{M}}{\det M_o} = -\tfrac{1}{2}\,\frac{\Gamma^2(\frac{N}{N-1})}{\Gamma(\frac{2N}{N-1})} \tag{2.69}$$

Putting results (2.48), (2.63), (2.64), and (2.69) together gives

$$\operatorname{Im} E_o(g;\arg g = \pi) = \frac{1}{\sqrt{\pi}} \left(\frac{-4N}{g}\right)^{\frac{1}{2(N-1)}} \exp - \left\{\left(\frac{-4N}{g}\right)^{\frac{1}{N-1}} \cdot \frac{\Gamma^2(\frac{N}{N-1})}{\Gamma(\frac{2N}{N-1})}\right\} \left(1+0\left(g^{\frac{1}{N-1}}\right)\right) \tag{2.70}$$

This is the required result; when it is used together with equation (2.34) we find that the high order behaviour of E_k is given by the result (2.33) of Bender and Wu. Brézin et al (1977a) were the first to obtain this result using instanton techniques; they also generalised it to the case of an O(n) internal symmetry.

We have discussed this problem in detail because it has all the ingredients found in the analogous calculations in higher dimensional field theories (except for the problem of renormalisation). Moreover, all the calculations can be done analytically and therefore it serves as a good introduction to the field theory problem. Of course, the high order behaviour of the perturbation expansion in quantum mechanical problems of this sort is an interesting problem in itself. Therefore for completeness, we end this chapter with a short account of some of the work that has been done on high order estimates in quantum mechanics.

We have already mentioned the pioneering work of Bender and Wu (1968, 1969, 1971, 1973) on the anharmonic oscillator. In addition Banks et al (1973) and Banks and Bender (1973) have investigated coupled anharmonic oscillators. The methods used by these authors have now been largely superseded by the techniques discussed in this chapter. In addition to the generalised anharmonic oscillator with O(n) symmetry (Brézin et al 1977a), more general quantum mechanical potentials have been investigated using these techniques (Brézin et al 1977b); of especial interest is the case of a potential with degenerate minima (Brézin et al 1977c) where the perturbation series is found to be non-Borel summable.

The problem of finding the asymptotic behaviour of the perturbation theory for the energy levels of a periodic potential has been investigated by Stone and Reeve (1977). They found from a numerical study and a WKB analysis that for the potential $V(x) = \beta^{-2}(1-\cos \beta x)$, the perturbation series was not Borel summable.

Finally, the next to leading term in the high order behaviour has been obtained in the case of the anharmonic oscillator by Bender and Wu (1971) using the WKB approach and by Collins and Soper (1978) who performed a two loop calculation about the instanton solution.

CHAPTER THREE

INSTANTONS AND CONFORMAL INVARIANCE IN ϕ^N FIELD THEORY

The purpose of this chapter is twofold. In the first part we will discuss the existence and form of instanton solutions to $g\phi^N$ field theories in d dimensions. One of the main points to emerge will be the central position which the massless $g\phi^N$ theory in $d = {}^{2N}/_{N-2}$ dimensions occupies; it gives a demarcation line between the existence and non-existence of instanton solutions. This leads on directly into the second part of this chapter, which explores the conformal invariance of these theories. It is the presence of this extra symmetry that allows us to find analytic expressions for the instantons in this case, and even perform the semiclassical calculation analytically.

Instanton solutions to massive ϕ^N field theory

The first theory we will consider in this chapter has Euclidean action

$$H(\phi) = \int d^d x \left[\tfrac{1}{2} (\nabla\phi)^2 + \tfrac{1}{2} m^2\phi^2 + \frac{g}{N} \phi^N\right] \qquad (3.1)$$

where N=3,4, ... The field equations are

$$\nabla^2\phi = m^2\phi + g\phi^{N-1} \qquad (3.2)$$

and so to find instantons we are faced with solving a non-linear partial differential equation. The situation would be hopeless were it not for the fact that to calculate the high order behaviour we are looking for the instanton solutions with least action. Faced with a similar problem, Coleman (1977) conjectured, and later with Glaser and Martin proved (Coleman et al 1978), that for a large class of field theories the instanton solutions with least action are spherically symmetric. The theorem of Coleman, Glaser and Martin applies to the theory defined by equation (3.1) only when $d < {}^{2N}/_{N-2} \equiv d_c$, and so restricting ourselves for the moment to dimensionalities less than d_c we deduce from equation (3.2) that the instantons with least action satisfy the non-linear ordinary differential equation

$$\frac{d^2\phi}{dr^2} + \frac{(d-1)}{r}\frac{d\phi}{dr} = m^2\phi + g\phi^{N-1} \tag{3.3}$$

In order to understand what type of solution equation (3.3) admits, it is useful to think in terms of the "mechanical analogy" again. First of all let us consider the case when N is even. Then an examination of equation (3.3) indicates that instantons only exist for $g<0$. The instanton then corresponds to classically allowed solutions for a potential

$$V[\phi] = -\tfrac{1}{2} m^2\phi^2 - \frac{g}{N}\phi^N \quad , g<0 \tag{3.4}$$

shown in Figure 12(a). The mechanical analogy now consists of interpreting ϕ as a particle position and r as time (Coleman 1977, Brézin 1977). The particle is moving in the potential $V(\phi)$ given by equation(3.4) subject to a viscous damping force $-\frac{(d-1)}{r}\frac{d\phi}{dr}$.

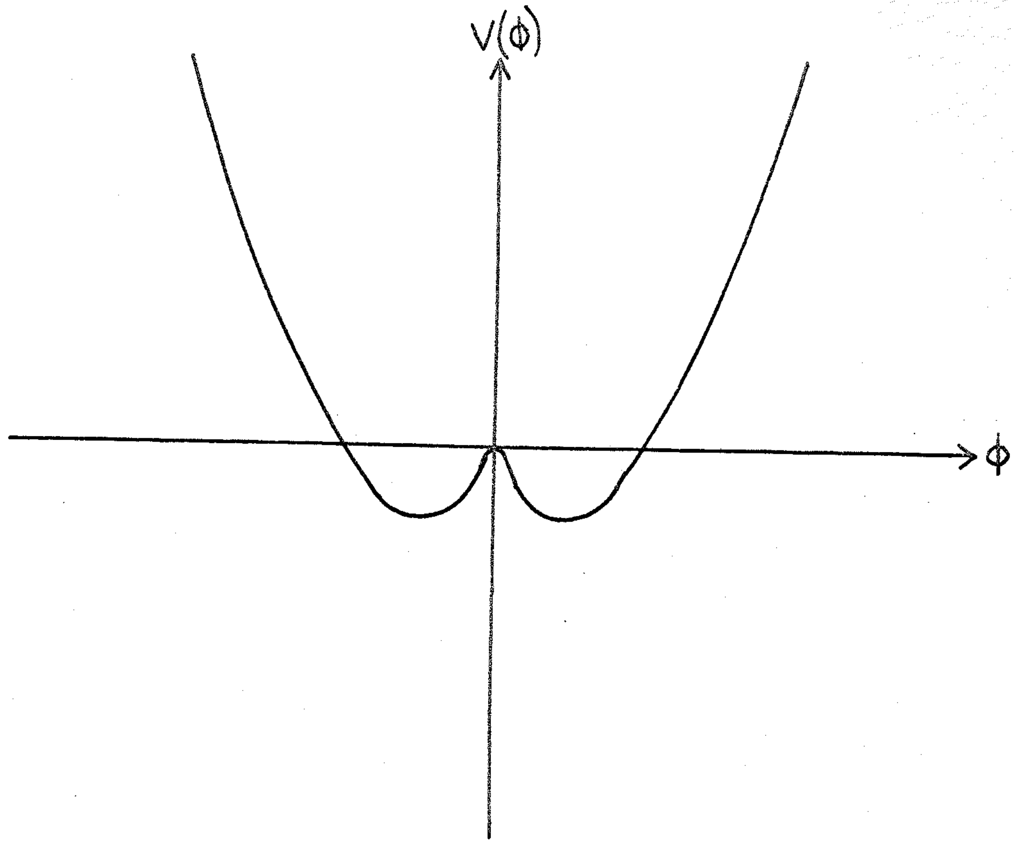

Figure 12(a). The potential $V(\phi) = -\frac{1}{2}m^2\phi^2 - g/N\phi^N$ $(g<0)$ where N is even

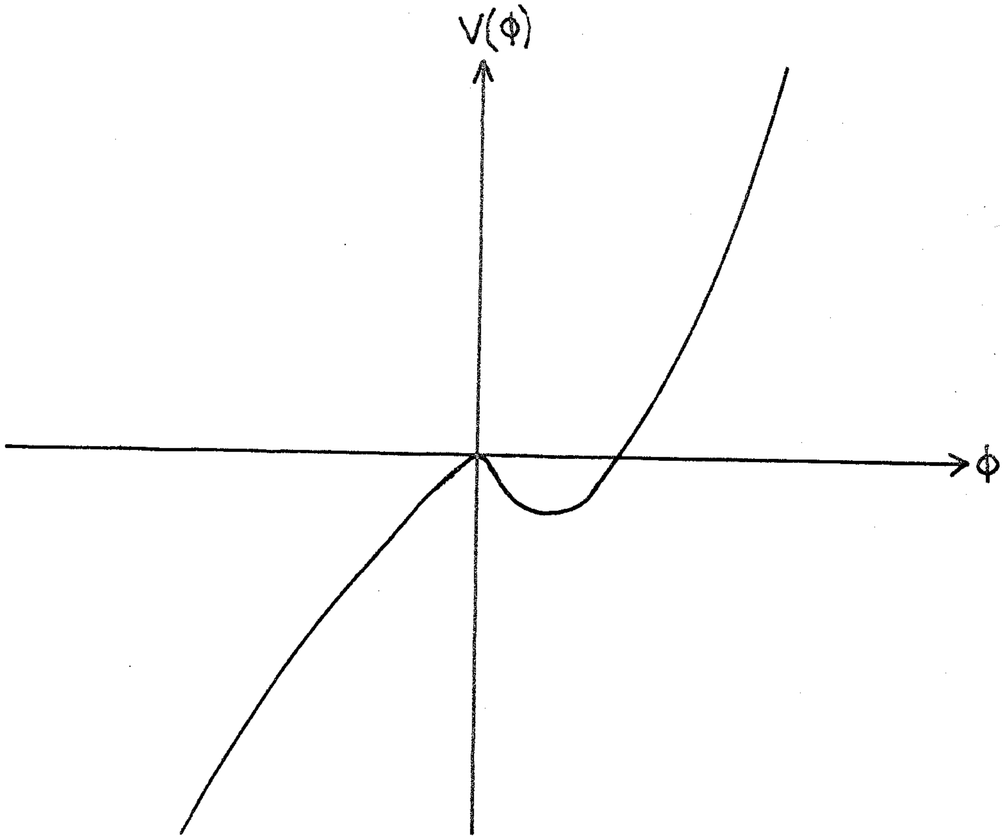

Figure 12(b). The potential $V(\phi) = -\frac{1}{2}m^2\phi^2 - g/N\phi^N$ $(g<0)$ where N is odd

We require solutions of finite action and therefore we have the condition $\phi(\infty) = 0$. What has to be shown is that there exists a finite position $\phi(0)$, where the particle is released, such that it has just enough energy to get to $\phi = 0$ in an infinite time ($\phi(\infty) = 0$) and come to rest there ($d\phi/dr\big|_{r=\infty} = 0$). One can argue using this physical analogy that such solutions do exist for small enough d (Coleman 1977), however, what is not clear is whether or not there exists a critical value of d, d_c, such that the viscous force becomes so large that the particle can never reach $\phi=0$. The theorem of Coleman, Glaser and Martin indicates that d_c must be larger than or equal to $2N/N-2$; in fact we will see in a moment that we can use a simple argument to show that real instanton solutions of equation (3.3) do not exist for $d \geqslant 2N/_{N-2}$ and so $d_c = 2N/N-2$.

Now consider the case when N is odd. If an instanton solution, ϕ_c, exists for $g>0$, then for $g<0$, $-\phi_c$ is a solution. Thus it is sufficient to consider the case $g<0$. The potential given by equation (3.4) is then of the form shown in Figure 12(b). The same arguments as above apply; we will see that there are no finite action solutions to equation (3.3) unless $d<2N/_{N-2}$.

We now prove that for $d \geqslant 2N/_{N-2}$ no real instanton solutions of equation (3.2) exist, by using a method of proof suggested by Derrick (1964) (see also Makhankov (1977)).

Let $\phi(X)$ be an instanton solution of equation (3.2) with action

$$H(\phi) = \int d^d x \left[\tfrac{1}{2} (\nabla\phi)^2 + \tfrac{1}{2} m^2\phi^2 + \tfrac{g}{N}\phi^N\right] \tag{3.5}$$

and define $\phi_\lambda(x) = \lambda^{\frac{2}{N-2}} \phi(\lambda x)$ so that

$$H(\phi_\lambda) = \lambda^{-d+\frac{2N}{N-2}} \int d^d x \left[\tfrac{1}{2}(\nabla\phi)^2 + \tfrac{1}{2}\frac{m^2}{\lambda^2}\phi^2 + \frac{g}{N}\phi^N\right]$$

$$= \lambda^{-d+\frac{2N}{N-2}} H(\phi) + \tfrac{1}{2} m^2 \lambda^{-d+\frac{2N}{N-2}} \left(\frac{1}{\lambda^2} - 1\right) \int \phi^2 d^d x \qquad (3.6)$$

Thus

$$\left.\frac{dH(\phi_\lambda)}{d\lambda}\right|_{\lambda=1} = \left(\frac{2N}{N-2} - d\right) H(\phi) - m^2 \int \phi^2 d^d x \qquad (3.7)$$

Now since $\phi_\lambda(x)$ is a solution of ${}^{\delta H}/_{\delta\phi_\lambda} = 0$ for $\lambda=1$ we have from equation (3.7)

$$\left(\frac{2N}{N-2} - d\right) H(\phi) = m^2 \int \phi^2 d^d x \qquad (3.8)$$

This tells us that real solutions of (3.5) with $H(\phi)>0$ do not exist if $d>{}^{2N}/_{N-2}$ and can only exist when $d = {}^{2N}/_{N-2}$ if $m^2 = 0$. This proves the assertion that massive ϕ^N theory has no real instanton solutions for $d\geqslant{}^{2N}/_{N-2}$.

Summarising the results of this section we may say that the field theory (3.1) has a critical dimensionality $d_c = {}^{2N}/_{N-2}$. For $d\geqslant d_c$ no instantons exist, but for $d<d_c$ they do exist, and moreover those with least action are spherically symmetric and satisfy

$$\frac{d^2\phi}{dr^2} + \frac{(d-1)}{r}\frac{d\phi}{dr} = m^2\phi + g\phi^{N-1}$$

$$\frac{d\phi}{dr}(r=\infty) = 0, \quad \phi(r=\infty) = 0 \tag{3.9}$$

Finally we remark that no analytic solutions of equations (3.9) have been found for d>1, so the equations have to be investigated numerically. The constants m^2 and g may be factored out by writing $\phi(x) = \left(-m^2/g\right)^{\frac{1}{N-2}} \cdot \psi(mx)$ and then equation (3.9) becomes

$$\frac{d^2\psi}{dr^2} + \frac{(d-1)}{r}\frac{d\psi}{dr} = \psi - \psi^{N-1} \tag{3.10}$$

$$\frac{d\psi}{dr}(r=\infty) = 0, \; \psi(r=\infty) = 0$$

For example in the case N=4, d=3 one finds (Brézin et al 1977a, Brézin 1977, Zinn-Justin 1977).

$$H(\phi_c) = -\frac{m\pi}{g}\int_0^\infty r^2\psi_c^4(r)dr = -\frac{m\pi}{g}(6.015182\ \ldots) \tag{3.11}$$

where ϕ_c is the instanton solution with least action.

Instanton solutions to massless ϕ^N field theory

In this section we consider the massless version of the theory we have just been discussing:

$$H(\phi) = \int d^d x \left[\tfrac{1}{2}(\nabla\phi)^2 + \frac{g}{N}\phi^N\right] \quad ;N=3,4,\ \ldots \tag{3.12}$$

where the field equations are now

$$\nabla^2\phi = g\phi^{N-1} \tag{3.13}$$

Our first question is: for what values of d do instanton solutions of equation (3.13) exist? To answer this question we can proceed just like in the massive case and define $\phi_\lambda(x) = \lambda^{\frac{2}{N-2}}\ \phi(\lambda x)$ where $\phi(x)$ is a solution of equation (3.13). Then

$$H(\phi_\lambda) = \lambda^{-d+\frac{2N}{N-2}}\ H(\phi) \tag{3.14}$$

so that

$$0 = \left.\frac{dH(\phi_\lambda)}{d\lambda}\right|_{\lambda=1} = \left(\frac{2N}{N-2} - d\right)\ H(\phi) \tag{3.15}$$

Thus if $H(\phi)$ is non-zero we must have $d = {}^{2N}/_{N-2}$.

This means that there is only a chance of finding instanton solutions to equation (3.13) in $d = {}^{2N}/_{N-2}$ dimensions.

Unfortunately the theorem of Coleman, Glaser and Martin does not apply to the theory defined by equation (3.12). For the moment we will make the ansatz that the instantons of least action will be spherically symmetric and discuss the proof later in the chapter. Thus the required solutions satisfy

$$\frac{d^2\phi}{dr^2} + \frac{(d-1)}{r}\frac{d\phi}{dr} = g\phi^{N-1} \tag{3.16}$$

This equation is well known in the literature; it is called the Emden-Fowler equation (see Wong (1975) for a review and complete list of references). It has a number of interesting properties, one of which is that the action integral

$$I = \int_0^\infty r^{d-1} \phi^N \, dr \tag{3.17}$$

can be integrated exactly for any solution of equation (3.16) (Emden 1907, Milne 1927). To show this it is convenient to eliminate g from equation (3.16) by defining a new function $\psi(r)$ by

$$\phi(r) = \left[- {}^1/_g\right]^{\frac{1}{N-2}} \psi(r) \tag{3.18}$$

so that equation (3.16) becomes

$$\frac{d}{dr} \left(r^{d-1} \frac{d\psi}{dr}\right) = -r^{d-1} \psi^{N-1} \tag{3.19}$$

Integrating by parts and use of equation (3.19) gives the following results:

$$\int_0^\infty r^{d-1} \psi^N \, dr = -\left[r^{d-1}\psi \frac{d\psi}{dr}\right]_0^\infty + \int_0^\infty r^{d-1} \left(\frac{d\psi}{dr}\right)^2 dr \tag{3.20}$$

$$\int_0^\infty \frac{1}{r^{d-1}} \left(r^{d-1} \frac{d\psi}{dr}\right)^2 dr = -\frac{1}{d-2} \left[r^d \left(\frac{d\psi}{dr}\right)^2\right]_0^\infty - \frac{2}{d-2} \int_0^\infty r^d \frac{d\psi}{dr} \psi^{N-1} dr \tag{3.21}$$

$$\int_0^\infty r^d \left(\psi^{N-1} \frac{d\psi}{dr}\right) dr = \frac{1}{N} \left[r^d \psi^N\right]_0^\infty - \frac{d}{N} \int_0^\infty r^{d-1} \psi^N dr \tag{3.22}$$

and combining equations (3.20) - (3.22) gives

$$\int_0^\infty r^{d-1} \psi^N \, dr = \frac{N(d-2)}{2N-d(N-2)} \left[r^{d-1} \psi \frac{d\psi}{dr} + \frac{1}{d-2} r^d \left(\frac{d\psi}{dr}\right)^2 + \frac{2}{N(d-2)} r^d \psi^N\right]_0^\infty \tag{3.23}$$

Thus we need only to know the behaviour of the solutions of the Emden-Fowler equation at the end-points in order to determine their action. Unfortunately we see that equation (3.23) becomes singular when $2N = d(N-2)$, that is $d = d_c$. For $d \neq d_c$ an analysis of the asymptotic behaviour of the solutions of (3.19) will show that there are no solutions with finite action. For $d = d_c$ we see that we may obtain finite action solutions if the bracketed expression on the right-hand side of equation (3.23) is zero. Rather than finding solutions for which this quantity vanishes, we will proceed in a more straightforward fashion and analyse equation (3.19) directly in the case $d = d_c$.

To do this we make the change of variable $x = \ell nr$ in equation (3.19) and define a new function $\bar{\psi} = e^{2x/N-2}\psi$.
The differential equation then becomes

$$\frac{d^2\bar{\psi}}{dx^2} + \frac{d\bar{\psi}}{dx}\left(d - {}^{2N}/_{N-2}\right) + \bar{\psi}\left({}^{4}/_{(N-2)^2} - {}^{2(d-2)}/_{(N-2)}\right) = -\bar{\psi}^{N-1} \tag{3.24}$$

Thus in the particular case $d = d_c = {}^{2N}/_{N-2}$ equation (3.24) reduces to

$$\frac{d^2\bar{\psi}}{dx^2} = \frac{4}{(N-2)^2}\,\bar{\psi} - \bar{\psi}^{N-1} \tag{3.25}$$

Integrating equation (3.25) once gives

$$\left(\frac{d\bar{\psi}}{dx}\right)^2 = \frac{4}{(N-2)^2}\,\bar{\psi}^2 - \frac{2}{N}\,\bar{\psi}^N + C \tag{3.26}$$

Now if the integral (3.17) is to be finite then for large r, $\phi(r) \sim r^{-\alpha}$

where $\alpha > d/_N = {}^2/_{N-2}$ for $d = d_c$. Thus since $\bar{\psi}=r^{2/N-2}\,\psi$ we have $\bar{\psi} \to 0$ as $r\to\infty$ ie. as $x \to \infty$. Similarly $d\bar{\psi}/_{dx} \to 0$ as $x \to \infty$. This determines the integration constant in equation (3.26) to be zero. The differential equation now has exactly the same form as equation (2.41), and it can be solved in the same way. We find that

$$\bar{\psi}(x) = \left[\frac{2N}{(N-2)^2}\right]^{\frac{1}{N-2}} \left[f(x)\right]^{\frac{2}{N-2}} \tag{3.27}$$

where $$\left(\frac{df}{dx}\right)^2 = f^2 - f^4$$

and so $$\bar{\psi}(x) = \left[\frac{2N}{(N-2)^2}\right]^{\frac{1}{N-2}} \left[\text{sech }(x-a)\right]^{\frac{2}{N-2}} \tag{3.28}$$

Writing this in terms of $\psi(r)$ with $\lambda \equiv e^{-a}$ we obtain

$$\psi(r) = \left[\frac{2N}{(N-2)^2}\right]^{\frac{1}{N-2}} \left[\frac{2\lambda}{1+\lambda^2 r^2}\right]^{\frac{2}{N-2}} \tag{3.29}$$

These are finite action solutions of equation (3.19) in which the instanton is centred at the origin. The most general solution will be when the instanton is situated at an arbitary point in space, $x=x_0$. Thus the required solutions are (using equation (3.18))

$$\phi_c(x;x_0,\lambda) = \left[\frac{-\,8N}{g(N-2)^2}\right]^{\frac{1}{N-2}} \left[\frac{\lambda}{1+\lambda^2\,(x-x_0)^2}\right]^{\frac{2}{N-2}} \tag{3.30}$$

We have not seperated the two cases N even and N odd in the above discussion. The only difference is that for N odd the real solution

(3.30) exists for all real g, whereas when N is even it only exists for $g < 0$, but now it has a partner $\phi = -\phi_c$.

The solutions given by equation (3.30), called Emden solutions, contain not (as might be expected) d arbitrary constants, but d+1. The extra degree of freedom corresponds to the dilatational invariance of the theory (3.12). In fact, massless ϕ^N theories in $d = {}^{2N}/_{N-2}$ dimensions are invariant under a still larger group — the conformal group. We will see in the rest of the chapter that if this extra symmetry is made manifest, then the study of instantons in these theories is simplified.

Conformal Invariance

In this section we will study the conformal invariance of the theory (3.12) in $d_c = {}^{2N}/_{N-2}$ dimensions. The conformal group is the group of coordinate transformations that leave invariant the form

$$ds^2 = g_{\mu\nu}\, dx^\mu\, dx^\nu = 0 \tag{3.31}$$

where $\mu,\nu = 0,1, \ldots, d_c-1$ and $g = \mathrm{diag}\,(1,-1,-1, \ldots, -1)$. It consists of the usual Poincaré transformations

$$\begin{aligned} L &: x'_\mu = \Lambda_\mu{}^\nu x_\nu \\ T &: x'_\mu = x_\mu + a_\mu \end{aligned} \tag{3.32}$$

plus dilatations, D, and special conformal transformations, C,

$$\begin{aligned} D &: x'_\mu = \lambda x_\mu \\ C &: x'_\mu = \frac{x_\mu + \alpha_\mu x^2}{1 + 2\alpha_\nu x^\nu + \alpha^2 x^2} \end{aligned} \tag{3.33}$$

These transformations form a group which is a $\frac{1}{2}(d_c+1)(d_c+2)$ parameter

Lie group. The generators of the group form a Lie algebra with commutation relations (Wess 1960, Fubini 1976).

$$
\begin{aligned}
[P\mu, P\nu] &= 0 \\
[P\lambda, M\mu\nu] &= i(g\mu\lambda\ P\nu - g\nu\lambda\ P\mu) \\
[P\mu, D] &= iP\mu \\
[P\mu, K_\nu] &= 2i(g\mu\nu\ D - M\mu\nu) \\
[M\mu\nu, M\sigma\rho] &= i(g\mu\rho M\nu\sigma + g\nu\sigma\ M\mu\rho + g\mu\sigma\ M\rho\nu + g\nu\rho\ M\sigma\mu\ ) \\
[M\mu\nu, D] &= 0 \\
[M\mu\nu, K_\lambda] &= -i(g\mu\lambda\ K\nu - g\nu\lambda K\mu) \\
[D, K\mu] &= iK\mu \\
[K\mu, K\nu] &= 0
\end{aligned}
\tag{3.34}
$$

The scalar fields transform in the following way (Fubini 1976):

$$
\begin{aligned}
[P\mu, \phi(x)] &= -i\ \frac{\partial\phi(x)}{\partial x\mu} \\
[M\mu\nu, \phi(x)] &= -i\ (x^\mu \frac{\partial}{\partial x\nu} - x\nu \frac{\partial}{\partial x\mu})\ \phi(x) \\
[D, \phi(x)] &= -i\ (x_\mu \frac{\partial}{\partial x\mu} + {}^{dc}/_2\ -1)\ \phi(x) \\
[K_\mu, \phi(x)] &= -i(-x^2 \frac{\partial}{\partial x\mu} + 2x\mu\ (x\nu \frac{\partial}{\partial x\nu} + \frac{dc}{2}\ -1))\ \phi(x)
\end{aligned}
\tag{3.35}
$$

We may write the commutation relations in a more compact form if we introduce $L_{AB} = -L_{BA}$; $A,B = 0,1, \ldots, d_c-1, d_c+1, d_c+2$ defined by

$$P_\mu = L_{\mu\, d_c+1} - L_{\mu\, d_c+2}$$

$$K_\mu = L_{\mu\, d_c+1} + L_{\mu\, d_c+2} \qquad \mu=0,1,2,\dots,d_c-1 \tag{3.36}$$

$$M_{\mu\nu} = L_{\mu\nu}$$

$$D = L_{d_c+1\; d_c+2}$$

then equations (3.34) reduce to

$$[L_{AB}, L_{CD}] = i(g_{BC} L_{AD} + g_{AD} L_{BC} - g_{BD} L_{AC} - g_{AC} L_{BD}) \tag{3.37}$$

where $g = \mathrm{diag}\,(1,-1,-1,\dots,-1,-1,1)$. Thus we see that the conformal group is isomorphic to the non-compact group $SO(d_c,2)$.

The fact that the field theory is invariant under this larger group is not immediately obvious. To make the conformal invariance of the theory manifest, Dirac (1936) suggested that the field theory in Minkowski space should be replaced by an equivalent field theory over a six dimensional space (in the case $d_c=4$), η_A,

$$x_\mu = \kappa^{-1}\eta_\mu; \quad \kappa = \eta_5 + \eta_6, \quad \eta^2 = 0 \tag{3.38}$$

with new fields Φ defined by

$$\Phi = \kappa^{-1}\phi \tag{3.39}$$

The group of (pseudo) rotations on η is then isomorphic to the conformal group of transformations on x (Boulware et al 1970). The advantage of this scheme is clear; we can now work with linear representations of the conformal group and not with the nonlinear transformations on x. The analogous expressions to (3.38) and (3.39) for general d_c are

$$\eta_\mu = \mathcal{K} x_\mu, \quad \mu = 0,1,\ldots,d_c-1$$
$$\mathcal{K} = \eta_{d_c+1} + \eta_{d_c+2}, \quad g_{AB}\,\eta^A\eta^B = 0 \tag{3.40}$$

where $A,B = 0,1,\ldots,d_c-1, d_c+1, d_c+2$ and $g = \mathrm{diag}\,(1,-1,-1,\ldots,-1,1)$.

The same structure is found if we begin from the Euclidean version of the theory, the only difference being that the conformal group is now isomorphic to $SO(d_c+1,1)$. In this case let us write (3.40) in the form

$$\eta_\mu = \mathcal{K} x_\mu, \; \eta_{d+1} = \mathcal{K} - \eta_{d+2}, \quad \mu = 1,2,\ldots,d_c$$
$$\text{with} \quad \delta_{ab}\eta^a\eta^b = \eta^2_{d+2} \quad ; \quad a,b = 1,2,\ldots,d_c+1 \tag{3.41}$$

This suggests that we view the d_c+1 coordinates $\{\eta_\mu, \eta_{d_c+1}\}$ as constrained to lie on spheres of arbitrary radius. In this picture elements of the conformal group map d_c+1 - dimensional spheres into d_c+1 - dimensional spheres†.

Thus one way of making the conformal invariance of theories in d_c dimensions manifest is to work with the theory defined on the d_c+1 - dimensional hypersphere. In the next section we will therefore study massless ϕ^N field theories in spherical space.

Field Theory on the d+1 - dimensional Hypersphere

The advantages of studying conformally invariant field theories in spherical space were first exploited by Adler(1972,1973). He studied massless quantum electrodynamics on the five-dimensional

---

† By a "d_c+1 - dimensional sphere" we mean a sphere with a d_c - dimensional surface imbedded in a d_c+1 - dimensional Euclidean space.

hypersphere. The case of scalar field theories has been investigated by Drummond (1975) and also by Fubini (1976) who studied the conformal properties of the instanton solution given by equation (3.30). Jackiw and Rebbi (1976b) have studied SU(2) Yang-Mills theory using this "O(d+1) formalism" in order to study the conformal properties of the instanton solution of Belavin et al (1975).

In order to arrive at the spherical formulation of the theory, d - dimensional flat space is sterographically projected onto the d+1 - dimensional hypersphere. Flat space is thought of as the equatorial plane of the sphere (see Figure 13), and the projection associates a point P of the plane to a unique point P' (other than the north pole, N) of the sphere and conversely.

Let $x_1,\ldots,x_d$ be coordinates of P in flat space and $\eta_1\ldots,\eta_{d+1}$ be coordinates of P' on the sphere $\eta^2 = R^2$. Then from Figure 13. we see that by similar triangles

$$\frac{OM}{OP} = \frac{\eta_1}{x_1} = \frac{\eta_2}{x_2} = \cdots = \frac{\eta_d}{x_d} \tag{3.42}$$

and also

$$\frac{OP}{KP'} = \frac{ON}{KN} = \frac{R}{R-\eta_{d+1}} \tag{3.43}$$

and since KP' = OM equations (3.42) and (3.43) give

$$\frac{R-\eta_{d+1}}{R} = \frac{\eta_1}{x_1} = \cdots = \frac{\eta_d}{x_d} \tag{3.44}$$

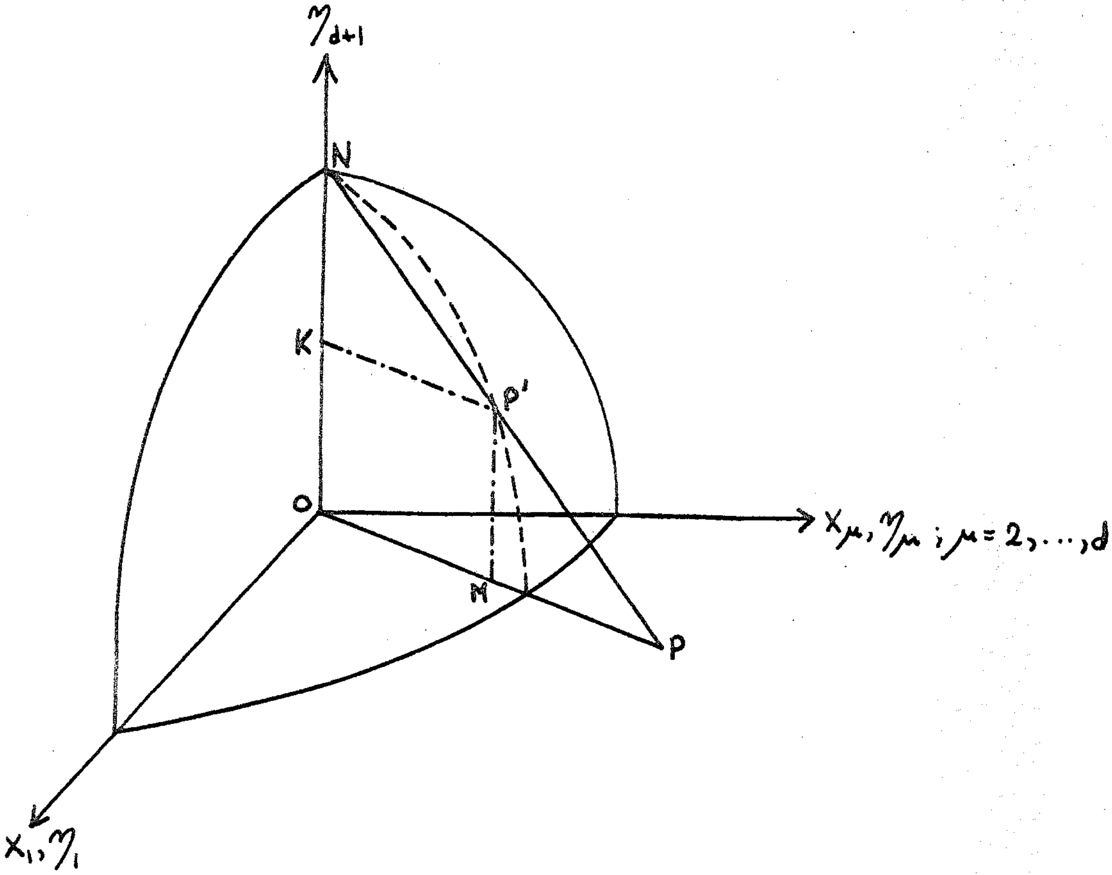

Figure 13. Sterographic projection of flat space onto spherical space.

and thus

$$n_\mu = \frac{2x_\mu}{1+x^2/R^2} \quad , \quad n_{d+1} = \frac{R(x^2/R^2-1)}{x^2/R^2+1} \quad ; \mu=1,2,\ldots,d \tag{3.45}$$

Equation (3.45) is a concrete realisation of equation (3.41).

We shall now state a number of results we need in order to study the spherical form of the field theory (Adler 1972, 1973, Drummond 1975).

(1) If $d\Omega$ is the surface element of the d+1 dimensional hypersphere then

$$d^d x = R^d \kappa^{-d} d\Omega \tag{3.46}$$

where

$$\kappa = \frac{2}{1+x^2/R^2} \tag{3.47}$$

(2) The spherical operator corresponding to ∇^2 is V_0 where

$$V_0 \Phi = \kappa^{-1-d/2} \nabla^2 \phi \tag{3.48}$$

and

$$\Phi = \kappa^{1-d/2} \phi \tag{3.49}$$

(3) V_0 may be written down in terms of the square of the angular momentum operator in d+1 dimensions:

$$V_0 = -L^2 - {}^d/_2 \left(d/2 - 1 \right) /R^2 \tag{3.50}$$

where

$$L_{ab} = -i\left(n_a \frac{\partial}{\partial n_b} - n_b \frac{\partial}{\partial n_a} \right) \quad ; a,b = 1,2,\ldots,d+1 \tag{3.51}$$

These results enable us to investigate instantons in massless ϕ^N theory on the d_c+1 - dimensional hypersphere.

The spherical form of the action (3.12) is

$$H(\Phi) = R^{d_c} \int d\Omega \left[\tfrac{1}{2}\,\Phi\,(-V_0)\,\Phi + \frac{g}{N}\,\Phi^N\right] \;;\; N=3,4,\ldots \tag{3.52}$$

and thus the field equations are

$$V_0\,\Phi = g\Phi^{N-1} \tag{3.53}$$

or using equation (3.50)

$$L^2\Phi + {}^{d}/_{2}\,({}^{d}/_{2}-1)\Phi = -gR^2\Phi^{N-1} \tag{3.54}$$

The simplest ansatz we can make in looking for solutions of equation (3.54) is to look for $O(d+1)$ invariant solutions: $L^2\Phi=0$. We will now prove that such solutions are the solutions with least action.

We shall need the result

$$\int d\Omega\,\left[\Phi^* L^2\Phi\right] \geqslant 0 \tag{3.55}$$

with equality only when Φ = constant. This follows from the fact that L^2 is positive definite. For an explicit demonstration we may use the completeness of the spherical harmonics to write

$$\Phi = \sum_{L,\underset{\sim}{\alpha}} C_{L\underset{\sim}{\alpha}}\, Y_{L\underset{\sim}{\alpha}}(\theta_1,\ldots,\theta_{d-1},\phi) \tag{3.56}$$

and so

$$\int d\Omega\left[\Phi^* L^2\Phi\right] = \sum_{L,\underset{\sim}{\alpha}} |C_{L\underset{\sim}{\alpha}}|^2\, L(L+d-1) \tag{3.57}$$

(using Result 1 Appendix II) which proves (3.55).

Using the equation of motion (3.53) the action of an instanton, Φ_c, is given by

$$H(\Phi_c) = \frac{(N-2)}{2N}\, R^{d_c} \int d\Omega\;\Phi_c(-V_0)\Phi_c \tag{3.58}$$

and so using equation (3.50)

$$H(\Phi_c) = R^{d_c-2} \frac{(N-2)}{2N} \int d\Omega \, \{ \Phi_c L^2 \Phi_c + \frac{d_c}{2}(\frac{d_c}{2} -1) \Phi_c^2 \} \quad (3.59)$$

The R dependence of Φ_c is easily found from equation (3.54) and then we may write $\Phi_c(n)$ in the form

$$\Phi_c(n) = \frac{A}{R^{2/(N-2)}} f(\Theta_1, \Theta_2, \ldots, \Theta_{d-1}, \phi) \quad (3.60)$$

where we will leave A to be an arbitrary constant but choose f to be normalised by

$$\int d\Omega \quad f^2(\Theta_1, \ldots, \Theta_{d-1}, \phi) = 1 \quad (3.61)$$

Then by equation (3.57)

$$H(\Phi_c) \geqslant R^{d_c-2} \frac{(N-2)}{2N} \frac{d_c}{2} (\frac{d_c}{2} -1) \int d\Omega \, \Phi_c^2 (n)$$

$$= A^2 \frac{(N-2)}{2N} \frac{d_c}{2} (\frac{d_c}{2} -1) \quad (3.62)$$

by equations (3.60) and (3.61). The right-hand-side of equation (3.62) is exactly the action for the constant solution $\Phi_c = AR^{-2/(N-2)}$ and moreover equality is only achieved in equation (3.62) for the constant solution. Thus, for fixed R, the solution with minimum action is the constant solution.

If we now look for real solutions of equation (3.54) of the form $\Phi = AR^{-2/(N-2)}$, where A is a constant, we find

$$\Phi_c = 0, \quad \Phi_c = (\pm)^{N+1} \left[\frac{-2N}{g(N-2)^2 R^2} \right]^{\frac{1}{N-2}} \quad (3.63)$$

Notice that the R factor in the non-trivial solution means that it depends on the radius of the spherical space and so is not conformally invariant.

Using equations (3.47) and (3.49) we find that the solutions corresponding to (3.63) in flat space are

$$\phi_c = 0, \quad \phi_c = (\pm)^{N+1} \left[\frac{-8N}{g(N-2)^2}\right]^{\frac{1}{N-2}} \left[\frac{1/R}{1+x^2/R^2}\right]^{\frac{2}{N-2}} \tag{3.64}$$

Therefore the flat space instanton solutions previously obtained go over to constant solutions in the spherical case. Since $H(\phi) = H(\Phi)$ we have proved that indeed the instanton solutions (3.30) are those with minimum action.

It is amusing to note that we can prove this directly in Euclidean space if we assume that the instanton solution is seperable:

$$\phi_c(x) = f(r)g\left(\Theta_1, \ldots, \Theta_{d-2}, \phi\right) \tag{3.65}$$

where $g(\underset{\sim}{\Theta}, \phi)$ is chosen so that

$$\int d\Omega \; g^2(\underset{\sim}{\Theta} \; \phi) = \int d\Omega \tag{3.66}$$

Then from equations (3.12) and (3.13) we obtain

$$H(\phi_c) = \frac{N-2}{2N} \int d^dx \left[\phi_c(-\nabla^2)\phi_c\right]$$

$$= \frac{N-2}{2N} \int d^dx \; \{g^2 f \left(- \frac{d^2}{dr^2} - \frac{(d-1)}{r} \frac{d}{dr} \right) f + \frac{f^2}{r^2} gL^2 g\} \tag{3.67}$$

where we have used equation (A2.8). If we now use equation (3.55) we have

$$H(\phi_c) \geqslant H(f) \tag{3.68}$$

with equality only when g is a constant (provided that the integral $\int_0^\infty r^{d-3} f^2(r)dr$ is finite).

The fact that the instanton solutions with minimum action are constant in spherical space causes great simplifications when calculating fluctuations about the instanton. As an example of this simplification we will find an explicit form for the small oscillations determinant using the spherical formalism (Lipatov 1977b).

In Chapter 2 we saw that Gaussian fluctuations about instanton solutions produce a factor

$$\left(\frac{\det M}{\det M_0}\right)^{-\frac{1}{2}} \tag{3.69}$$

where
$$M(x,x') = \left.\frac{\delta^2 H}{\delta\phi(x)\ \delta\phi(x')}\right|_{\phi=\phi_c} \tag{3.70}$$

and
$$M_0(x,x') = \left.\frac{\delta^2 H_0}{\delta\phi(x)\ \delta\phi(x')}\right|_{\phi=0} \tag{3.71}$$

In the case of the theory (3.12) with instanton solutions given by equation (3.30) these operators have the form

$$M(x,x') = \left[-\nabla^2 - \frac{4\gamma(\gamma+1)\lambda^2}{(1+\lambda^2(x-x_0)^2)^2}\right]\delta^d(x-x') \tag{3.72}$$

and

$$M_0(x,x') = -\nabla^2\ \delta^d(x-x') \tag{3.73}$$

where $\gamma = {}^{d_c}/_2 = {}^N/_{N-2}$†. In order to calculate the factor (3.69)

---

† We will study M and M_0 in d dimensions since later we will want to work in $d=d_c-\varepsilon$ dimensions in order to regulate the theory.

an attempt might be made to calculate the eigenvalues and phase shifts of the operators (3.72) and (3.73). However, the general eigenfunctions of M are not known analytically and so expression (3.69) cannot be found by this direct method. This problem is avoided by exploiting the O(d+1) invariance of the quadratic form. After translating x by x_0 and rescaling by λ we have that

$$M = M_0 - \frac{4\gamma(\gamma+1)}{(1+x^2)^2} \tag{3.74}$$

and so defining

$$V = V_0 + \gamma(\gamma+1) \tag{3.75}$$

equations (3.48), (3.49) and (3.74) give

$$\begin{aligned} V_0\,\Phi &= -k^{-1-d/2} M_0\,\phi \\ V\Phi &= -k^{-1-d/2} M\,\phi \end{aligned} \tag{3.76}$$

where $\Phi = k^{1-d/2}\phi$

and thus

$$\int d^dx\,\phi\, M\,\phi = \int d\Omega\Phi\, V\,\Phi$$

and (3.77)

$$\int d^dx\,\phi\, M_0\,\phi = \int d\Omega\Phi\, V_0\,\Phi$$

The invariance of the quadratic form indicated in equations (3.77) means that

$$\frac{\det M}{\det M_0} = \frac{\det V}{\det V_0} \tag{3.78}$$

This solves the problem, because all the eigenfunctions and eigenvalues of V and V_0 are known analytically. To construct them we only need to recall that (equations (3.50), (3.75))

$$V_0 = -L^2 - \frac{d}{2}\left(\frac{d}{2} - 1\right) \tag{3.79}$$

and
$$V = -L^2 - \frac{d}{2}\left(\frac{d}{2} - 1\right) + \frac{d_c}{2}\left(\frac{d_c}{2} + 1\right)$$

$$= -L^2 - \left(\frac{d}{2} + \frac{d_c}{2}\right)\left(\frac{d}{2} - \frac{d_c}{2} - 1\right) \tag{3.80}$$

Thus the eigenfunctions of V and V_0 are the spherical harmonics in d+1 dimensions. The properties of these functions are derived in Appendix II. We find that the eigenvalues of L^2 in d+1 dimensions are $L(L+d-1)$ $L = 0,1,2,\ldots$, with degeneracy

$$\nu_L(d+1) = \frac{\Gamma(L+d-1)\,(2L+d-1)}{\Gamma(d)\,\Gamma(L+1)} \tag{3.81}$$

Therefore from equations (3.78) - (3.81) we find

$$\left(\frac{\det M}{\det M_0}\right)^{-\frac{1}{2}} = \exp -\tfrac{1}{2}\left\{\sum_{L=0}^{\infty} \nu_L(d+1)\, \ln\left[\frac{(L+\frac{1}{2}d+\frac{1}{2}d_c)\;(L+\frac{1}{2}d-\frac{1}{2}d_c-1)}{(L+\frac{1}{2}d)\;(L+\frac{1}{2}d-1)}\right]\right. \tag{3.82}$$

This is the required result. Let us suppose for the moment that we did not use the spherical formalism to obtain this result but used other methods ('t Hooft 1976b, Brézin et al 1977a). Then we have a sum over n and ℓ (ℓ is the d dimensional analogue of L and $n = 0,1,2,\ldots$; the principal quantum number, $n_p = n+\ell$, goes over to L). Now for a given function f we have

$$\sum_{L=0}^{\infty} \nu_L(d+1)\ f(L)$$

$$= \sum_{\ell+n=0}^{\infty} \ \sum_{m=0}^{\ell+n} \nu_m(d)\ f(\ell+n) \quad \dagger$$

$$= \sum_{\ell=0}^{\infty} \nu_\ell(d) \ \sum_{n=0}^{\infty} f(n+\ell)$$

and so equation (3.82) can be written in the equivalent form

$$\left(\frac{\det M}{\det M_o}\right)^{-\frac{1}{2}} = \exp -\tfrac{1}{2}\ \{ \sum_{\ell,n=0}^{\infty} \nu_\ell(d)\ \ln \left[\frac{(n+\ell+\frac{1}{2}d+\frac{1}{2}d_c)\ (n+\ell+\frac{1}{2}d-\frac{1}{2}d_c-1)}{(n+\ell+\frac{1}{2}d)\ (n+\ell+\frac{1}{2}d-1)}\right] \} \tag{3.83}$$

The sum over n can now be performed and the result

$$\left(\frac{\det M}{\det M_o}\right)^{-\frac{1}{2}} = \exp-\tfrac{1}{2}\ \{ \sum_{\ell=0}^{\infty} \nu_\ell(d)\ \ln \left[\frac{\Gamma(\ell+\frac{1}{2}d)\ \Gamma(\ell+\frac{1}{2}d-1)}{\Gamma(\ell+\frac{1}{2}d+\frac{1}{2}d_c)\ \Gamma(\ell+\frac{1}{2}d-\frac{1}{2}d_c-1)}\right] \} \tag{3.84}$$

is the form obtained by Brézin et al (1977a).

It might now appear that the problem of determining the high order behaviour perturbation theory for ϕ^N field theory in $d={}^{2N}/_{N-2}$ dimensions is straightfoward; the instantons are known (equation (3.30)), and so is the small oscillations determinant (equation (3.82)). However, this ratio of determinants is just the one loop correction to the classical result and thus contains the usual ultraviolet divergences we find in field theories. In order to give the calculation any meaning we have to work with a suitably regularised theory. Both Lipatov (1977a,b) and Brézin et al (1977a) used Pauli-Villars

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† We have used $\nu_L(d+1) = \sum_{m=0}^{L} \nu_m(d)$ (see Appendix II).

regulators whereas we will use dimensional regularisation ('t Hooft and Veltman 1972, Bollini and Giambiagi 1972). The method will be discussed in some detail for ϕ^4 theory in $4-\varepsilon$ dimensions in the next chapter and then also applied to ϕ^3 in $6-\varepsilon$ in chapter five and ϕ^N ($N>4$) in $d_c-\varepsilon$ ($d_c={}^{2N}/N-2$) in chapter six.

CHAPTER FOUR

ASYMPTOTIC BEHAVIOUR OF THE PERTURBATION EXPANSION FOR ϕ^4 FIELD THEORIES IN $4-\varepsilon$ DIMENSIONS

The investigation of instanton effects in field theories gets increasingly more complicated for higher dimensional systems. Firstly, one has to find solutions to field equations which are non-linear partial differential equations and this cannot usually be done analytically. As we have already remarked, massless ϕ^N field theory in $d = {}^{2N}/_{N-2}$ dimensions is an exception. Secondly, rigo rous results concerning, say, the analytic properties of Green's functions in the coupling constant are either totally absent or rather weak. Thirdly, one has the added problem of renormalisation. For instance, there are indications that the semiclassical approximation fails to pick out extra singularities in the Borel transform when the theory is just renormalisable. Since the dimension in which ϕ^N theories are just renormalisable are the ones in which they are conformally invariant, we find ourselves in the unfortunate position that the semiclassical approximation may be inadequate in the only dimension (greater than one) in which instanton calculations can be done analytically.

In this chapter we will temporarily ignore these difficulties (they are discussed in Chapter seven) and calculate the high order behaviour of perturbation theory for ϕ^4 near four dimensions using the semiclassical approximation. We begin by justifying the use of the four dimensional instanton in $d = 4-\varepsilon$ dimensions.

Instantons and Dimensional Regularisation

We are interested in calculating the imaginary part of the

Euclidean Green's functions

$$G^{(2M)}(x_1,\ldots,x_{2M}) = \frac{\int D\phi\ \phi(x_1)\ldots\ \phi(x_{2M})\ \exp - H(\phi)}{\int D\phi\ \exp - H(\phi)} \tag{4.1}$$

which develops due to tunnelling out of the metastable ground state. The Euclidean action, $H(\phi)$, is given by

$$H(\phi) = \int d^d x \left[\tfrac{1}{2}(\nabla\phi)^2 + {}^g/_4\ \phi^4\right] \tag{4.2}$$

where $d=4-\varepsilon$, $\varepsilon>0$. The imaginary part is determined by instanton solutions to the classical field equations found from equation (4.2). In Chapter three we saw such solutions exist when d=4 and in particular we obtained the analytic form (equation (3.30))

$$\phi = \pm\phi_c \tag{4.3}$$

where

$$\phi_c(x;x_0,\lambda) = \sqrt{\frac{8}{-g}}\ \frac{\lambda}{1+\lambda^2(x-x_0)^2} \tag{4.4}$$

The parameters x_0 and λ characterise the position and scale size of the instanton ϕ_c, and are associated with the translation and dilatation invariance of H.

In order to regularise the ultra-violet divergences of the theory we work in $4-\varepsilon$ dimensions, but we will still expand $H(\phi)$ about $\phi = \phi_c$ where ϕ_c is given by equation (4.4).

Thus writing $\phi = \phi_c + \hat{\phi}$ we have

$$H(\phi) = H(\phi_c) + \tfrac{1}{2}\int d^d x\hat{\phi}M\hat{\phi} - \frac{4\sqrt{2}}{\sqrt{-g}}\,\varepsilon\int d^d x\ \frac{\lambda^3}{(1+\lambda^2(x-x_0)^2)^2}\ \hat{\phi}$$

$$- \sqrt{-8g}\int d^d x\ \frac{\lambda}{1+\lambda^2(x-x_0)^2}\ \hat{\phi}^3 + \frac{g}{4}\int d^d x\ \hat{\phi}^4 \tag{4.5}$$

where $M(x,x') = \left.\dfrac{\delta^2 H}{\delta\phi(x)\delta\phi(x')}\right|_{\phi=\phi_c}$

$$= \left[-\nabla^2 - \frac{24\lambda^2}{(1+\lambda^2(x-x_0)^2)^2}\right] \delta^d(x-x') \tag{4.6}$$

(see also equation (3.72)). We notice that there is a term linear in $\hat{\phi}$ because ϕ_c is not a solution to the field equations in $4-\varepsilon$ dimensions ($\varepsilon \neq 0$). Normally if one has a term linear in the field in the Hamiltonian, the functional integral must be brought into a proper Gaussian form by translating the field to eliminate the linear term. This is equivalent in our case to solving the non-linear equation (3.13) in the case N=4 exactly. However in our particular case, the linear term in equation (4.5) can be handled perturbatively.

We will look into the consequences of the existence of a linear term later; we will now say a few words concerning the handling of collective coordinates in such a situation.

In exactly four dimensions we would expect the operator M, given by equation (4.6), to have five eigenfunctions with zero eigenvalues. These zero modes come about because the instanton solution (4.4) breaks the translation and dilatation invariance of the theory. The eigenfunctions are given by

$$\frac{\partial\phi_c}{\partial\lambda} \equiv \phi_\lambda = \sqrt{\frac{8}{-g}} \; \frac{1-\lambda^2(x-x_0)^2}{(1+\lambda^2(x-x_0)^2)^2} \tag{4.7}$$

and

$$\frac{\partial\phi_c}{\partial x_0{}^\mu} \equiv \phi_\mu = -\sqrt{\frac{8}{-g}} \; \frac{2\lambda^3(x-x_0)_\mu}{(1+\lambda^2(x-x_0)^2)^2} \tag{4.8}$$

In $4-\varepsilon$ dimensions these are no longer eigenfunctions of M with zero eigenvalue. We will refer to them as "zero modes" - the inverted commas signifying that they are only true zero modes in exactly four dimensions.

To deal with the problems posed by these "zero modes" we introduce collective coordinates as described in Chapter two. We keep dilatation as a collective coordinate in d dimensions as well as translations, thus we have d+1 collective coordinates. In this way we ensure that M has no eigenvalues of order ε, and so the propagator in the presence of an instanton has no divergences which are generated by the "zero modes". As we will see later this is vital if we are to handle the linear term in equation (4.5) perturbatively.

We now follow an analogous procedure to that outlined in chapter two in order to introduce the collective coordinates. We write

$$\phi(x) = \phi_c(x;x_o,\lambda) + \tilde{\sum_n} a_n \phi_n(x;x_o,\lambda) \tag{4.9}$$

where the tilde indicates that the sum is not taken over the "zero modes". We regard $\{\tilde{a}_n, x_o, \lambda\}$ as new variables replacing $\phi(x)$ and $\{\phi_n\}$ to be normalised eigenfunctions of M corresponding to the non-zero modes. The functional measure $D\phi$ is then replaced by $J^M\, d\lambda d^d x_o \tilde{\prod_n} da_n$, where J^M is the Jacobian of the transformation (the superscript M refers to its association with the differential operator M).

If we now dominate the functional integral in equation (4.1) by the instanton ϕ_c we obtain to lowest order

$$G^{(2M)}(x_1,\ldots,x_{2M}) = \int d\lambda\, d^dx_0\, J^M\, \phi_c(x_1)\ldots\,\phi_c(x_{2M})\,\exp\text{-}H(\phi_c)$$

$$\frac{\tilde{\prod_n} \int da_n \exp - \{\tfrac{1}{2} a_n^2 E_n^M - 4\sqrt{2\varepsilon}\, a_n /\sqrt{-g} \int d^dx\, \lambda^3\phi_n/(1+\lambda^2(x-x_0)^2)^2 + O(a_n^3)\}}{\prod_n \int da_n \exp - \{\tfrac{1}{2} a_n^2 E_n^{M_0} + O(a_n^3)\}} \tag{4.10}$$

where $\{E_n^M\}$ and $\{E_n^{M_0}\}$ are the eigenvalues of M and M_0 (see equation (3.73)) respectively. We have expanded only the numerator of equation (4.1) about $\phi=\phi_c$, because the imaginary part from the denominator turns out to be higher order in g and can be neglected at this order.

Performing the Gaussian integrals leads to

$$G^{(2M)}(x_1,\ldots,x_{2M}) = \int d\lambda\, d^dx_0\, J^M\, \phi_c(x_1)\ \ldots\ \phi_c(x_{2M})\exp - H(\phi_c)$$

$$\times\, C_1\, (2\pi)^{-\frac{d+1}{2}} \left(\frac{\det \tilde{M}}{\det M_0}\right)^{-\frac{1}{2}} (1+O(\varepsilon;g)) \tag{4.11}$$

where the tilde again indicates that the zero modes have been extracted and the factor $(2\pi)^{-\frac{d+1}{2}}$ comes from the fact that there are d+1 more Gaussian integrals in the denominator than in the numerator. The factor C_1 represents the effect of the linear term and will be investigated later in this chapter.

Just as for the case of the generalised anharmonic oscillator discussed in Chapter two, the determinant factor is pure imaginary (a glance at equation (3.82) will show this), so once again we have to fall back on the intuition we obtained from studying the zero dimensional field theory to obtain the sign of Im $G^{(2M)}$. We may then write equation (4.11) as

$$\mathrm{Im}\, G^{(2M)}(x_1,\ldots,x_{2M};g) = -\int d\lambda\, d^d x_0\, J^M\, \phi_c(x_1) \ldots \phi_c(x_{2M})$$

$$(\arg g = \pi)$$

$$\times \exp - H(\phi_c)\, C_1\, (2\pi)^{-\frac{d+1}{2}} \left|\frac{\det \tilde{M}}{\det M_0}\right|^{-\frac{1}{2}} (1+O(\varepsilon,g)) \quad (4.12)$$

We again note that the two saddle points $\phi = \pm\phi_c$ contribute half a Gaussian integral and so the bound state mode effectively contributes a full Gaussian integral.

In order to obtain the imaginary part of the vertex function $\Gamma^{(2M)}$, we take the Fourier transform of equation (4.12) and amputate the external legs. We then have (removing a factor $(2\pi)^d \delta(\sum q_i)$)

$$\underset{(\arg g=\pi)}{\mathrm{Im}\,\Gamma^{(2M)}(q_i)} = -\int d\lambda\, J^M \exp - H(\phi_c)\, C_1\, (2\pi)^{-\frac{d+1}{2}} \left|\frac{\det \tilde{M}}{\det M_0}\right|^{-\frac{1}{2}}$$

$$\times \prod_{i=1}^{2M} q_i^2 \int d^d x_i\, e^{iq_i \cdot x_i}\, \phi_c(x_i) \left[1+O(\varepsilon,g)\right] \quad (4.13)$$

This is the main result of this section. It gives the imaginary part of the vertex function, generated by tunnelling out of the metastable ground state $\phi=0$ (with $g<0$), in the one loop approximation.

Use of the Spherical Formalism†

At the end of the last chapter we gave an example of the use of the spherical or "$O(d+1)$" formalism. This consisted in showing that

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† The $O(5)$ formalism has been used extensively in instanton calculations in Yang-Mills theories. See for instance Belavin and Polyakov (1977), Chadda et al (1977) and Ore (1977a,b,c).

the ratio $^{\det M}/_{\det M_0}$ was equal to $^{\det V}/_{\det V_0}$ where V and V_0 had eigenvalues which could be easily calculated. In this way $^{\det M}/_{\det M_0}$ was found to be given by equation (3.82). However, we see from equation (4.13) that it is $\det \tilde{M}/\det M_0$ that we require, and this will not in general, be equal to $^{\det V}/_{\det V_0}$. In fact.

$$\left|\frac{\det \hat{M}}{\det M_0}\right|^{-\frac{1}{2}} = \left(\frac{E_\lambda^M}{E_\lambda^V}\right)^{\frac{1}{2}} \left(\frac{E_\mu^M}{E_\mu^V}\right)^{d/2} \left|\frac{\det \tilde{V}}{\det V_0}\right|^{-\frac{1}{2}} \tag{4.14}$$

where E_λ^M, E_μ^M (μ=1,...,d) and E_λ^V, E_μ^V (μ=1,...,d) are the regularised eigenvalues of M and V respectively corresponding to the dilatation and translation eigenfunctions.

From the discussion at the end of the last chapter we see that we can calculate everything on the right-hand-side of equation (4.14) except E_λ^M and E_μ^M. These can, however, be calculated to lowest order in ε in straightforward perturbation theory:

$$E_\lambda^M = \frac{\int d^dx\ \phi_\lambda(x)\ \left[M_d - M_4\right]\ \phi_\lambda(x)}{d^dx\ \phi_\lambda\ (x)\ \phi_\lambda\ (x)} \tag{4.15}$$

and

$$E_\mu^M = \frac{\int d^dx\ \phi_\mu(x)\ \left[M_d - M_4\right]\ \phi_\mu(x)}{\int d^dx\ \ \phi_\mu(x)\ \ \phi_\mu(x)} \tag{4.16}$$

and thus the left-hand-side of equation (4.14) can be found to lowest in ε.

We will now indicate a more instructive way of dealing with the above problem. Suppose that the true eigenfunctions of M and V in d dimensions are denoted by $\{\psi_n\}$ and $\{\Psi_n\}$ respectively. Then

$$M\psi_n = E_n^M \psi_n \tag{4.17}$$

and

$$V\Psi_n = E_n^V \Psi_n \tag{4.18}$$

and in particular E_λ^M, E_μ^M, E_λ^V and E_μ^V are of order ε. We can now proceed as in chapter three and write $V\Psi_n = \kappa^{-1-d/2} M\psi'_n$ where $\Psi_n = \kappa^{1-d/2}\psi'_n$. It then follows that

$$M\psi'_n = \frac{4\lambda^2 E_n^V}{(1+\lambda^2(x-x_o)^2)^2} \psi'_n \tag{4.19}$$

From equations (4.17) and (4.19) we deduce that

$$\frac{E_n^M}{E_n^V} = \frac{\int d^d x \; \psi_n(x) \; M \; \psi_n(x)}{\int d^d x \; \psi_n(x) \; \psi_n(x)} \cdot \frac{\int d^d x \; \psi'_n(x) \frac{4\lambda^2}{(1+\lambda^2(x-x_o)^2)^2} \psi'_n(x)}{\int d^d x \; \psi'_n(x) \; M \; \psi'_n(x)} \tag{4.20}$$

The eigenfunctions ψ_n are not known (if they were we could have calculated $^{\det M}/_{\det M_o}$ directly) and so equation (4.20) does not appear to be very useful. However let us now imagine doing perturbation theory in ε:

$$M_d = M_4 + \varepsilon M_{int.}$$

$$\psi_n = \psi_n^{(o)} + \varepsilon\psi_n^{(1)} + O(\varepsilon^2) \tag{4.21}$$

$$\psi'_n = \psi_n^{(o)\prime} + \varepsilon\psi_n^{(1)\prime} + O(\varepsilon^2)$$

From equations (4.17) - (4.19) we see that for the "zero modes",

$\psi = \psi'$ in d=4 and thus $\psi_\lambda^{(o)} = \psi_\lambda^{(o)'}$, $\psi_\mu^{(o)} = \psi_\mu^{(o)'}$ $(\mu=1,\ldots,d)$.

Thus

$$\frac{\int d^d x\ \psi_\lambda\ (x)\ M\ \psi_\lambda\ (x)}{\int d^d x\ \psi'_\lambda\ (x)\ M\ \psi'_\lambda\ (x)} = 1+0(\varepsilon) \tag{4.22}$$

and similarly in the case of the translation eigenfunctions. So to lowest order we have from equation (4.20)

$$\frac{E_\lambda^M}{E_\lambda^V} = \frac{\int d^d x\ \phi_\lambda\ (x) \frac{4\lambda^2}{(1+\lambda^2\ \ (x-x_0)^2)^2}\ \phi_\lambda(x)}{\int d^d x\ \phi_\lambda\ (x)\ \ \phi_\lambda\ (x)} \tag{4.23}$$

and

$$\frac{E_\mu^M}{E_\mu^V} = \frac{\int d^d x\ \phi_\mu\ (x) \frac{4\lambda^2}{(1+\lambda^2\ (x-x_0)^2)^2}\ \phi_\mu(x)}{\int d^d x\ \phi_\mu\ (x)\ \ \phi_\mu\ (x)} \tag{4.24}$$

since $\psi_\lambda = \psi'_\lambda = \phi_\lambda$, $\psi_\mu = \psi'_\mu = \phi_\mu$ in d = 4.

If we calculate the norm of the dilatation eigenfunction appearing in the denominator of equation (4.24) we find that it is logarithmically divergent in four dimensions. This indicates the presence of another factor which will cancel off this spurious infrared divergence. The factor comes from the Jacobian, J^M, which is given by (for a derivation see Appendix I)

$$J^M = \{\int d^d x\ \phi_\lambda\ (x)\ \phi_\lambda\ (x) \left[\frac{1}{d}\int d^d x\ \phi_\mu\ (x)\ \phi_\mu\ (x)\right]^d\}^{\frac{1}{2}}\ (1+0(\varepsilon,g)). \tag{4.25}$$

Thus from equations (4.23) -(4.25) we have

$$J^M \left(\frac{E_\lambda^M}{E_\lambda^V}\right)^{\frac{1}{2}} \left(\frac{E_\mu^M}{E_\mu^V}\right)^{d/2} = J^V \tag{4.26}$$

where

$$J^V = \{ \int d^d x\, \phi_\lambda\, (x) \frac{4\lambda^2}{(1+\lambda^2\ (x-x_0)^2)^2}\ \phi_\lambda(x) \left[\frac{1}{d}\int d^d x\ \phi_\mu\ (x)\right.$$

$$\left.\frac{4\lambda^2}{(1+\lambda^2(x-x_0)^2)^2}\ \phi_\mu\ (x)\right]^d \}^{\frac{1}{2}}\ (1+0(\varepsilon,g)) \tag{4.27}$$

We use the superscript V because if we define fields on the d+1 dimensional hypersphere by

$$\Phi_\lambda = K^{1-d/2}\,\phi_\lambda\ ,\ \Phi_\mu = K^{1-d/2}\phi_\mu \quad (\mu=1,\dots,d) \tag{4.28}$$

where $K=2(1+\lambda^2x^2)^{-2}$ then by equation (3.46)

$$J^V = \{ \int d\Omega\Phi_\lambda\ (n)\ \Phi_\lambda\ (n) \left[\frac{1}{d}\int d\Omega\Phi_\mu\ (n)\ \Phi_\mu(n)\right]^d \}^{\frac{1}{2}}\ (1+0(\varepsilon,g)) \tag{4.29}$$

Thus J^V is the spherical analogue of J^M given by equation (4.25).

In summary we may say that the combination of the Jacobian and the determinant factors without the zero modes remains invariant (to the lowest order) under the transformation from flat space to spherical space, that is,

$$J^M \left(\frac{\det \hat{M}}{\det M_o}\right)^{-\frac{1}{2}} = J^V \left(\frac{\det \hat{V}}{\det V_o}\right)^{-\frac{1}{2}}\ (1+0(\varepsilon,g)) \tag{4.30}$$

We may now write equation (4.13) in a form which is better suited for calculational purposes:

$$\underset{(\arg g=\pi)}{\mathrm{Im}\,\Gamma^{(2M)}(q_i)} = -\int d\lambda J^V \exp - H(\phi_c)\, C_1 (2\pi)^{-\frac{d+1}{2}} \left|\frac{\det \hat{V}}{\det V_0}\right|^{-\frac{1}{2}}$$

$$\times \prod_{i=1}^{2M} q_i^2 \int d^d x_i e^{iq_i \cdot x_i}\, \phi_c(x_i) \left[1+O(\varepsilon,g)\right] \qquad (4.31)$$

Finally we comment that C_1 can also be calculated in the spherical formalism. We need the spherical analogue of equation (4.5) for this purpose. This is obtained (after translating x by x_0 and rescaling by λ) by defining new fields $\hat{\Phi}$ in the usual way (see chapter three). We find

$$H(\phi) = H(\phi_c) + \tfrac{1}{2}\int d\Omega\hat{\Phi}(\hat{V})\hat{\Phi} - \sqrt{\frac{-2}{g}}\,\varepsilon \int d\Omega \mathcal{K}^{-1+\varepsilon/2}\,\hat{\Phi}$$

$$- \sqrt{-2g}\int d\Omega \mathcal{K}^{-\varepsilon/2}\,\hat{\Phi}^3 + \frac{g}{4}\int d\Omega \mathcal{K}^{-\varepsilon}\,\hat{\Phi}^4 \qquad (4.32)$$

The Contribution from the Linear Term

In the expression (4.31) for one loop approximation to $\mathrm{Im}\Gamma^{(2M)}$, all the quantities appearing are well defined except C_1. We now turn our attention to the specification of C_1 which arises from the presence of a linear term in the Hamiltonian (4.5) or equivalently (4.32). This term is effectively a "source term" of order ε which we will treat perturbatively. Since our ultimate aim is to calculate the leading high order behaviour of vertex functions we may neglect O(g) contributions generated by the source term, since these give $O(K^{-1})$ corrections to the leading $K^{\underline{th}}$ order asymptotic behaviour (see chapter two). Thus, since an L loop graph gives L-1 powers of g, it is sufficient to restrict ourselves to tree diagrams and one loop graphs. Further, tree diagrams and one loop graphs with M "source terms" give contributions $O(\varepsilon^{M})$ and $O(\varepsilon^{M-1})$ respectively (with the "zero modes" extracted). Thus to this order we need only calculate the divergent part of one loop diagrams with one source term. There is only one such diagram - it is shown in Figure 14.

Since lines in this diagram represent the propagator in the presence of an instanton, it is much simpler to use the Hamiltonian on the sphere, expression (4.32), where the propagator can be written explicitly in terms of the spherical harmonics $Y_L^{\alpha}(n)$ in d+1 dimensions

$$\tilde{V}^{-1}(n,n') = - \tilde{\sum_{L,\tilde{\alpha}}} \frac{Y_L^{\alpha}(n)\, Y_L^{\alpha *}(n')}{(L+\frac{1}{2}d+2)\,(L+\frac{1}{2}d-3)} \tag{4.33}$$

where the tilde means L=l is excluded. The diagram in Figure 14 gives a contribution

$$3\left(-\sqrt{\frac{-2}{g}}\,\varepsilon\right)\left(-\sqrt{-2g}\right)\int d\Omega_1\, d\Omega_2\, \kappa_1^{-1+\varepsilon/2}\, \tilde{V}^{-1}(n_1,n_2)\, \kappa_2^{-\varepsilon/2}\, \tilde{V}^{-1}(n_2,n_2)$$

$$= 6\varepsilon \int d\Omega_1\, d\Omega_2\, \kappa_1^{-1}\, \tilde{V}^{-1}(n_1,n_2)\, \tilde{V}^{-1}(n_2,n_2)\Big(1+O(\varepsilon)\Big) \tag{4.34}$$

The factor C_1 is then just the exponential of expression (4.34).

In the last section we discussed how the combination of Jacobian and determinant factor is independent of the method of performing the calculation, that is, independent of whether we work in flat space or on the sphere. Therefore we expect that the same value for this graph should be obtainable directly from the Hamiltonian (4.5) in flat space rather than equation (4.32) on the sphere. The expression corresponding to (4.34) is (to

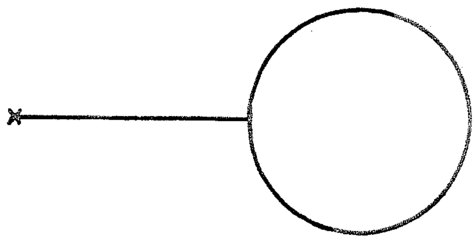

Figure 14. The only graph involving the linear term which contributes at this order in ϕ^3 and ϕ^4 theories.

simplify we translate by x_o and rescale by λ)

$$48\varepsilon \int \frac{d^dx\, d^dy}{(1+x^2)^2}\, \tilde{M}^{-1}(x,y)\, \frac{1}{(1+y^2)}\, \tilde{M}^{-1}(y,y) \tag{4.35}$$

We will discuss the evaluation of the integrals (4.34) and (4.35) in the next section.

Calculation of the Imaginary Part of the Vertex Functions

In this section we will calculate the various factors appearing in equation (4.31), and so find $\mathrm{Im}\Gamma^{(2M)}$ at the one-loop level.

The classical action $H(\phi_c)$ and the Jacobian factor J^V are easily calculated. The classical action is found to be

$$H(\phi_c) = -\frac{\lambda^\varepsilon}{g}\frac{8\pi^2}{3}\left[1-\tfrac{1}{2}\varepsilon\left(2+\ln\pi+\gamma\right)+O(\varepsilon^2)\right] \tag{4.36}$$

where γ=0.577 215 ... is Euler's constant. We have to evaluate $H(\phi_c)$ to the first order in ε because after a one-loop renormalisation $g^{-1} \to g^{-1}_R + O(\varepsilon^{-1})$(see the next section). The Jacobian defined by equation (4.27) is

$$J^V = \lambda^{d-1}\left(\frac{-16\pi^2\lambda^\varepsilon}{15g}\right)^{\frac{d+1}{2}}(1+O(\varepsilon,g)) \tag{4.37}$$

where equations (4.7) and (4.8) have been used.

To evaluate the determinant factor we use equations (3.78) and (3.82)

$$\left(\frac{\det \tilde{V}}{\det V_o}\right)^{-\frac{1}{2}} = \exp -\tfrac{1}{2}\tilde{\sum_{L=0}^{\infty}} \frac{\Gamma(L+d-1)\,(2L+d-1)}{\Gamma(L+1)\Gamma(d)} \ln\left\{\frac{(L+\frac{1}{2}d-3)\,(L+\frac{1}{2}d+2)}{(L+\frac{1}{2}d-1)\,(L+\frac{1}{2}d)}\right\}$$

(4.38)

Evaluating the contribution from the L=0 modes and the L=1 modes of V_0 explicitly, we find

$$\left|\frac{\det \tilde{V}}{\det V_0}\right|^{-\frac{1}{2}} = 2^2 3^{5/2} \exp -\tfrac{1}{2} \sum_{L=2}^{\infty} \frac{\Gamma(L+3-\varepsilon)\ (2L+3-\varepsilon)}{\Gamma(4-\varepsilon)\,\Gamma(L+1)}$$

$$\ln\left\{\frac{(L-1-\varepsilon/2)\ (L+4-\varepsilon/2)}{(L+1-\varepsilon/2)\ (L+2-\varepsilon/2)}\right\} \quad \times \quad (1+0(\varepsilon)) \tag{4.39}$$

The factor (4.39) is, in the Feynman graph language, represented by the diagram shown in Figure 15. We therefore expect to find the usual one-loop ultra-violet divergences in equation (4.39). To isolate these we need only look at the sum in (4.39) for large L. Using

$$\frac{\Gamma(L+3-\varepsilon)\ (2L+3-\varepsilon)}{\Gamma(4-\varepsilon)\ \Gamma(L+1)} = \frac{2L^{3-\varepsilon}}{\Gamma(4-\varepsilon)}\left(1 + 0\left(\frac{1}{L}\right)\right) \tag{4.40}$$

and $$\ln\left\{\frac{(L-1-\varepsilon/2)\ (L+4-\varepsilon/2)}{(L+1-\varepsilon/2)\ (L+2-\varepsilon/2)}\right\} = \frac{-6}{L^2}\left(1 + 0\left(\frac{1}{L}\right)\right) \tag{4.41}$$

for L large, we can write the sum as

$$\frac{-12}{\Gamma(4-\varepsilon)} \sum_{L=2}^{\infty} \left\{ \frac{1}{L^{\varepsilon-1}} + \frac{A_0(\varepsilon)}{L^{\varepsilon}} + \frac{A_1(\varepsilon)}{L^{1+\varepsilon}} + \frac{A_2(\varepsilon)}{L^{2+\varepsilon}} + \ldots \right\} \tag{4.42}$$

where the $A_i(\varepsilon)$ i=0,1,2,..., are just polynomials in ε. Each of the sums in expression (4.42) can be expressed in terms of the Riemann zeta function, $\zeta(s)$. Thus we may write (4.42) as

$$\frac{-12}{\Gamma(4-\varepsilon)} \left\{ (\zeta(\varepsilon-1)-1) + A_0(\varepsilon)\ (\zeta(\varepsilon)-1) + A_1(\varepsilon)\ (\zeta(1+\varepsilon)-1) + \ldots \right\} \tag{4.43}$$

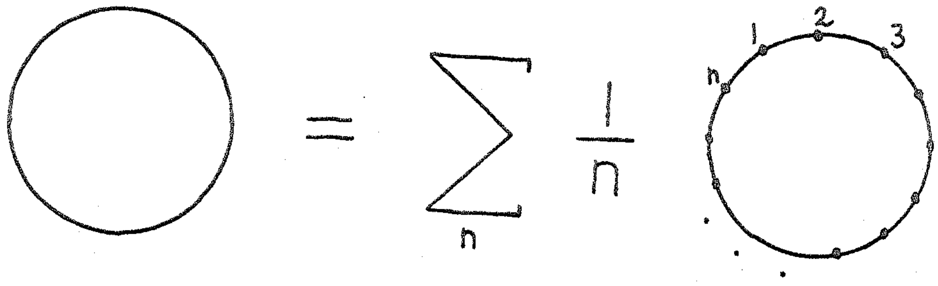

Figure 15. Diagrammatic representation of the small oscillations determinant. The dots represent insertions of $-(N-1)g\phi_c^{N-2}$ for a theory with interaction $g/N\phi^N$.

Now $\zeta(s)$ is one-valued and regular everywhere with the exception of s=1 where $\zeta(s)$ has a simple pole, with residue 1 (Erdélyi 1955). Thus expression (4.43) gives

$$\frac{-2A_1(0)}{\varepsilon} + O(1) \qquad (4.44)$$

and so from equation (4.39)

$$\left|\frac{\det \tilde{V}}{\det V_0}\right|^{-\frac{1}{2}} = 2^2 3^{5/2} \exp\left\{\frac{A_1(0)}{\varepsilon} + O(1)\right\} \qquad (4.45)$$

Thus to calculate the divergent part of the sum we need only look at the asymptotic behaviour of expressions (4.40) and (4.41) when $\varepsilon=0$. We find

$$\frac{\Gamma(L+3)\ (2L+3)}{\Gamma(4)\,\Gamma(L+1)} = \frac{2L^3}{\Gamma(4)}\left[1 + \frac{9}{2L} + \frac{13}{2L^2} + 0\left(\frac{1}{L^3}\right)\right] \qquad (4.46)$$

and

$$\ln\left\{\frac{(L-1)\ (L+4)}{(L+1)(L+2)}\right\} = \frac{-6}{L^2}\left[1 - \frac{3}{L} + \frac{10}{L^2} + 0\left(\frac{1}{L^3}\right)\right] \qquad (4.47)$$

therefore $A_1(0)=3$, which means

$$\left|\frac{\det \tilde{V}}{\det V_0}\right|^{-\frac{1}{2}} = 2^2 3^{5/2} \exp\left\{\frac{3}{\varepsilon} + 0(1)\right\} \qquad (4.48)$$

To find the finite part of the above expression by this method is more difficult - we have essentially to calculate $\{A_i(0)\}$ and then evaluate the double sum (4.42). A more effective way of proceeding is to expand the summand in (4.39) in ε not in $^1/_L$.

To do this we shift the summation variable independently for each of the four logarithmic factors so that the sums may be written in the form

$$\sum_{L=2}^{\infty} L^{\rho} \frac{\Gamma(L-\varepsilon)}{\Gamma(L)} \ln(L-\varepsilon/2) \tag{4.49}$$

where ρ is an integer. Using the asymptotic expansion

$$\begin{aligned}\frac{\Gamma(L-\varepsilon)}{\Gamma(L)} = L^{-\varepsilon} \{ 1 + \varepsilon \left(\frac{1}{2L} + \frac{1}{12L^2} + 0\left(\frac{1}{L^4}\right)\right) \\ + \varepsilon^2 \left(\frac{1}{2L} + \frac{3}{8L^2} + 0\left(\frac{1}{L^3}\right)\right) + 0(\varepsilon^3) \}\end{aligned} \tag{4.50}$$

we find that the sums reduce to $\sum_{L=2}^{\infty} L^{1-\varepsilon} \ln L$ or $\sum_{L=2}^{\infty} L^{-1-\varepsilon} \ln L$. These are respectively $-\zeta'(-1+\varepsilon)$ and $-\zeta'(1+\varepsilon)$, where $\zeta'(s)$ is the derivative of the Riemann ζ function. We may write $\zeta'(-1+\varepsilon)$ in terms of $\zeta'(2-\varepsilon)$ using the functional relation (Erdélyi 1955)

$$\zeta(s) = 2^s \pi^{s-1} \sin\left(\tfrac{1}{2}\pi s\right) \Gamma(1-s)\, \zeta(1-s) \quad . \tag{4.51}$$

A tedious but straightforward calculation yields

$$\begin{aligned}\left| \frac{\det \tilde{V}}{\det V_0} \right|^{-\frac{1}{2}} = (2\pi)^{-\frac{1}{2}}\, 5^{5/2} \exp \{ \frac{3}{\epsilon} + \frac{3}{4} - \frac{7}{2}\gamma + \\ \frac{3}{\pi^2}\zeta'(2) + 0(\varepsilon) \}\end{aligned} \tag{4.52}$$

where $\zeta'(2) = -0.937\,548 \ldots$; the divergent term is seen to agree with the earlier result (4.48).

In the case when the critical dimensionality of theory, d_c, is an even integer, we can also compute $|\det \tilde{V}/\det V_0|^{-\frac{1}{2}}$ from equation (3.84) in a straightforward manner. In these cases the arguments of the gamma functions differ by integers, and so the argument of the logarithm is a ratio of products of factors linear in ℓ. For example, in this particular case, $d_c=4$, we have

$$\left(\frac{\det \hat{V}}{\det V_0}\right)^{-\frac{1}{2}} = \exp -\tfrac{1}{2} \left\{ \sum_{\ell=0}^{\tilde{\infty}} \frac{\Gamma(\ell+2-\varepsilon)\ (2\ell+2-\varepsilon)}{\Gamma(3-\varepsilon)\Gamma(\ell+1)} \ln\left[\frac{(\ell-\varepsilon/2)\ (\ell-1-\varepsilon/2)}{(\ell+3-\varepsilon/2)\ (\ell+2-\varepsilon/2)}\right] \right\} \tag{4.53}$$

This is exactly of the same form as equation (4.38) and so we can repeat the above calculation to obtain the result (4.52).

We now go on to evaluate the Fourier transform of $\phi_c(x)$ which appears in equation (4.31). We will find it convenient to define

$$\tilde{\phi}(q) = \frac{\sqrt{3}}{\pi} \int \frac{d^d x\ e^{iq\cdot x}}{(1+x^2)} \tag{4.54}$$

so that

$$\int d^d x\ e^{iq\cdot x}\ \phi_c\ (x;x_0,\lambda) = \left(\frac{-\lambda^{\varepsilon} 8\pi^2}{3g}\right)^{\frac{1}{2}} \lambda^{-1-d/2}\ \tilde{\phi}(q/\lambda) \tag{4.55}$$

We can evaluate the integral in equation (4.54) in terms of a modified Bessel function of the second kind:

$$\tilde{\phi}(q) = 2^{d/2}\ \pi^{(d-2)/2}\ 3^{\frac{1}{2}}\ |q|^{1-d/2}\ K_{d/2-1}\ (|q|) \tag{4.56}$$

and so obtain an explicit form for the Fourier transform (4.55).

It now only remains to calculate C_1. Using equations (4.33) and (4.34) we find the expression for the graph in Figure 14 to be (to lowest order)

$$\tilde{\sum_{L,L',\underset{\sim}{\alpha},\underset{\sim}{\alpha}'}} 6\varepsilon \int \frac{d\Omega_1\, d\Omega_2\, (1+\cos\theta_1)^{-1}\, Y_L^{\underset{\sim}{\alpha}}(n_1)\, Y_L^{\underset{\sim}{\alpha}}(n_2)^*\, Y_{L'}^{\underset{\sim}{\alpha}'}(n_2)\, Y_{L'}^{\underset{\sim}{\alpha}'}(n_2)^*}{(L+\frac{1}{2}d+2)\,(L+\frac{1}{2}d-3)\,(L'+\frac{1}{2}d+2)\,(L'+\frac{1}{2}d-3)} \tag{4.57}$$

where θ_1 is the principal polar angle of n_1. Using Result 3 in Appendix II we have

$$\sum_{\underset{\sim}{\alpha}'} Y_{L'}^{\underset{\sim}{\alpha}'}(n_2)\, Y_{L'}^{\underset{\sim}{\alpha}'}(n_2)^* = \frac{(2L'+d-1)\Gamma(d/2-\frac{1}{2})}{4\pi^{\frac{d+1}{2}}}\, C_{L'}^{d/2-\frac{1}{2}}(1)$$

$$= \frac{(2L'+d-1)\,\Gamma(d/2-\frac{1}{2})\,\Gamma(d-1+L')}{4\pi^{\frac{d+1}{2}}\,\Gamma(L'+1)\,\Gamma(d-1)} \tag{4.58}$$

and so expression (4.57) reduces to

$$\tilde{\sum_{L,L',\underset{\sim}{\alpha}}} \frac{3\varepsilon\,(2L'+d-1)\,\Gamma(d/2-\frac{1}{2})\,\Gamma(d-1+L')}{2\pi^{\frac{d+1}{2}}\,\Gamma(L'+1)\,\Gamma(d-1)}$$

$$\times \frac{\int d\Omega_1\,(1+\cos\theta_1)^{-1}\, Y_L^{\underset{\sim}{\alpha}}(n_1) \int d\Omega_2\, Y_L^{\underset{\sim}{\alpha}}(n_2)}{(L+\frac{1}{2}d+2)\,(L+\frac{1}{2}d-3)\,(L'+\frac{1}{2}d+2)\,(L'+\frac{1}{2}d-3)} \tag{4.59}$$

The n_2 integral is now trivial - we just get a contribution when $L=0$, $\underset{\sim}{\alpha}=0$, that is, only the bound state contributes. Therefore we are left with only one sum, and , since the overall expression has an ε factor multiplying it we require only the $1/\varepsilon$ pole from the ultra-violet divergence of this sum. The explicit form for expression (4.59) is

$$-\frac{3\varepsilon}{8} \tilde{\sum_{L'}} \frac{(2L'+3-\varepsilon)\,\Gamma(L'+3-\varepsilon)}{\Gamma(L'+1)\,(L'+4-\varepsilon/2)\,(L'-1-\varepsilon/2)} \cdot (1+O(\varepsilon))$$

$$= - \frac{3\varepsilon}{8} \tilde{\sum_{L'}} \frac{(L' + 3)^{-\varepsilon}(2L' + 3)\ (L' + 2)\ (L' + 1)}{(L' + 4)\ (L' - 1)} . \quad (1 + O\ (\varepsilon))$$

$$= - \frac{9\varepsilon}{2} \zeta (1+\varepsilon) + O(\varepsilon)$$

$$= - \frac{9}{2} + O(\varepsilon) \qquad (4.60)$$

Thus to this order we find

$$C_1 = \exp\left(- \frac{9}{2} \right) \qquad (4.61)$$

As we pointed out in the previous section we should also be able to evaluate C_1 from expression (4.35). To do this we will assume again that we require only the $^1/_\varepsilon$ pole from the ultra-violet divergent loop integral. It is useful in this context to view $M^{-1}(y,y)$ as just the sum of free propagators with zero, one, two, ... insertions of $-3g\ \phi_c^2(x) = 24\ (1+x^2)^{-2}$. The only ultraviolet divergent contribution in dimensional regularisation is from the single insertion and thus the finite part of equation (4.35) is contained in

$$48\varepsilon \int \frac{d^dx\ d^dy}{(1+x^2)^2}\ \tilde{M}^{-1}\ (x,y)\ \frac{1}{(1+y^2)} \int \frac{d^dp}{(2\pi)^d}\ \frac{d^dk}{(2\pi)^d}\ \frac{e^{ip.y}\ f(p)}{(p+k)^2\ k^2} \qquad (4.62)$$

where $f(p)$ is defined by

$$f(p) = \int d^dx\ e^{-ip.x}\ \frac{24}{(1+x^2)^2} \qquad (4.63)$$

The k integral in equation (4.62) is straightforward

$$\int \frac{d^dk}{(2\pi)^d}\ \frac{1}{k^2(p+k)^2} = p^{-\varepsilon} \left[\frac{1}{8\pi^2\varepsilon} + O(1) \right] \qquad (4.64)$$

and so the finite part of equation (4.62) is

$$\frac{6}{\pi^2} \int \frac{d^dx\, d^dy}{(1+x^2)^2} \tilde{M}^{-1}(x,y) \frac{1}{1+y^2} \int \frac{d^dp}{(2\pi)^d} e^{ip.y} f(p)$$

$$= \frac{144}{\pi^2} \int \frac{d^dx\, d^dy}{(1+x^2)^2} \tilde{M}^{-1}(x,y) \frac{1}{(1+y^2)^3} \tag{4.65}$$

from equation (4.63). We now make use of the identity

$$\int d^dz\, M(y,z) \frac{1}{(1+z^2)^2} = \frac{4(d-6)}{(1+y^2)^3} \tag{4.66}$$

to write expression (4.65) as

$$\frac{144}{\pi^2} \frac{1}{4(d-6)} \int \frac{d^dx\, d^dy\, d^dz}{(1+x^2)^2} \tilde{M}^{-1}(x,y)\, M(y,z) \frac{1}{(1+z^2)^2} \tag{4.67}$$

Now by the completeness of the eigenfunctions of M

$$\int d^dy\, \tilde{M}^{-1}(x,y)\, M(y,z) = \delta^d(x-z) - \frac{\phi_\lambda(x)\phi_\lambda(z)}{\int d^dy \phi_\lambda(y)\, \phi_\lambda(y)} - \frac{\phi_\mu(x)\, \phi_\mu(z)}{\int d^dy\, \phi_\nu(y)\phi_\nu(y)} \tag{4.68}$$

(where ϕ_λ and ϕ_μ are defined by equations (4.7) and (4.8)) we find that (4.67) is equal to

$$-\frac{18}{\pi^2} \int \frac{d^dx}{(1+x^2)^4} (1+O(\varepsilon)) + \frac{18}{\pi^2} \left[\int d^dx \frac{\phi_\lambda(x)}{(1+x^2)^2}\right]^2$$

$$\left[\int d^dx\, [\phi_\lambda(x)]^2\right]^{-1} (1+O(\varepsilon)) \tag{4.69}$$

If we examine the second term in the above expression we see that although the two integrals in the numerator are finite as $\varepsilon \to 0$, the one in the denominator (the norm of the dilatation eigenfunction) is logarithmically divergent, as we in fact pointed out earlier in this chapter. Thus the second term is of order ε and therefore we may write (4.69) as

$$- \frac{18}{\pi^2} \int \frac{d^4x}{(1+x^2)^2} + O(\varepsilon)$$

$$= -3 + O(\varepsilon) \tag{4.70}$$

We have not been able to elucidate the discrepancy between this calculation and the one leading to the result (4.60). The former result $-\frac{9}{2}$ is to be believed since the calculation is completely controlled; presumably some assumption in the second calculation is false, for example, the zero modes may not have been correctly excluded.

Collecting all the factors from equations (4.36), (4.37), (4.52), (4.55) and (4.61), and substituting them into the one-loop result (4.31), we have

$$\underset{(\arg g=\pi)}{\mathrm{Im}\Gamma^{(2M)}(q_i)} = - C_b \int_0^\infty \frac{d\lambda}{\lambda} \lambda^{d-M(d-2)} \left(- \frac{\lambda^\varepsilon 8\pi^2}{3g} \right)^{\frac{d+1+2M}{2}}$$

$$\times \exp \frac{\lambda^\varepsilon}{g} \cdot \frac{8\pi^2}{3} \left(1-{}^\varepsilon/_2(\ln\pi+\gamma+2) + O(\varepsilon^2)\right) \prod_{i=1}^{2M} \left[{}^{q_i^2}/_{\lambda^2} \, \tilde{\phi} \, ({}^{q_i}/_\lambda) \right] (1+O(\varepsilon,g)) \tag{4.71}$$

where

$$C_b = 2^{-\frac{1}{2}} \pi^{-3} \exp\left(\frac{3}{\varepsilon} + \frac{3}{\pi^2}\zeta'(2) - \frac{7}{2}\gamma - \frac{15}{4}\right) \tag{4.72}$$

Equations (4.71) and (4.72) contain the final result for the bare theory. In order to render the result finite we have to remove the divergence by renormalisation. We discuss this in the next section.

Renormalisation

In the massless theory which we are considering, the divergences in perturbation theory are removed by a coupling constant and a wave-function renormalisation. In the latter case we write

$$\Gamma_R^{(2M)} = Z^M \Gamma^{(2M)} \tag{4.73}$$

and so, formally at least,

$$\mathrm{Im}\Gamma_R^{(2M)} = Z^M \,\mathrm{Im}\Gamma_R^{(2M)} \tag{4.74}$$

if Z is real. Since $Z = 1 + O(g^2)$ we see that wave-function renormalisation only contributes to $\mathrm{Im}\Gamma$ at the three-loop level. Thus for the one-loop result in expressions (4.71) and (4.72) we only require a one-loop coupling constant renormalisation.

We will use the minimal subtraction method of 't Hooft (1973) in order to carry this out. If we define $g_R(\mu_0)$, to be the renormalised coupling at the momentum scale μ_0, then

$$\mu_0^{-\varepsilon} g = g_R(\mu_0) + \frac{9}{8\pi^2\varepsilon} g_R^2(\mu_0) + O(g_R^3) \tag{4.75}$$

In the expression (4.71) we see that the combination λ^ε/g naturally appears in the exponential, thus the effective coupling for instantons of scale-size $1/\lambda$ is the renormalised coupling at the momentum scale λ. Writing†

$$\frac{\lambda^\varepsilon}{g} = \frac{1}{g_r(\lambda)} - \frac{9}{8\pi^2\varepsilon} \tag{4.76}$$

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† The subscript r means "one-loop renormalised".

we obtain the renormalised imaginary part

$$\underset{(\arg g=\pi)}{\mathrm{Im}\Gamma_r^{(2M)}(q_i)} = -C_r \int_0^\infty \frac{d\lambda}{\lambda} \lambda^{d-M(d-2)} \left(\frac{-8\pi^2}{3g_r(\lambda)}\right)^{\frac{d+1+2M}{2}}$$

$$\exp \frac{8\pi^2}{3g_r(\lambda)} \quad \times \prod_{i=1}^{2M} \left[q_i^2/_{\lambda^2} \tilde{\phi}\,(q_i/_\lambda) \right] (1+O(\varepsilon,g_r)) \tag{4.77}$$

where

$$C_r = C_b \exp\left(-\frac{3}{\varepsilon} + \frac{3}{2}(\ln\pi+\gamma+2) + O(\varepsilon)\right)$$

$$= 2^{-\frac{1}{2}}\pi^{-3/2} \exp\left(\frac{3}{\pi^2} \zeta'(2) - 2\gamma - 3/4\right) \tag{4.78}$$

In order to simplify the λ integration in equation (4.77) we make λ dimensionless by writing $\lambda \to \lambda\mu_o$ where μ_o is some fixed momentum scale. From equation (4.76) we have

$$\frac{1}{g_r(\mu_o\lambda)} = \frac{\lambda^\varepsilon}{g_r(\mu_o)} - \frac{9}{8\pi^2\varepsilon}(\lambda^\varepsilon - 1) \tag{4.79}$$

so that

$$\underset{(\arg g=\pi)}{\mathrm{Im}\Gamma_r^{(2M)}(q_i)} = -C_r \int_0^\infty \frac{d\lambda}{\lambda} (\lambda\mu_o)^{d-M(d-2)} \left(-\frac{8\pi^2\lambda^\varepsilon}{3g_r(\mu_o)} + 3\ln\lambda\right)^{\frac{d+1+2M}{2}}$$

$$\exp\left\{\frac{8\pi^2\lambda^\varepsilon}{3g_r(\mu_o)} - 3\ln\lambda\right\} \prod_{i=1}^{2M} \left[\frac{q_i^2}{\mu_o^2\lambda^2}\tilde{\phi}\,(q_i/_{\mu_o\lambda})\right] (1+O(\varepsilon,g_r)) . \tag{4.80}$$

We will not carry out the λ integration explicitly here, but merely note that the exponential decrease of $\tilde{\phi}(x)$ for large x (see equation (4.56)) ensures that the integral converges for small λ. On the other hand for large λ the exponential factor gives convergence when

$\varepsilon>0$. When $\varepsilon=0$, the convergence for $M\geqslant 1$ is due to the presence of the factor $\exp(-3\ln\lambda) = \lambda^{-3}$. This is a direct consequence of the asymptotic freedom of the theory in $d=4$ (for $g<0$). Equations (4.77) and (4.78) contain the final result for the renormalised imaginary part.

High-Order Estimates

The existence of an imaginary part of the vertex functions for $g<0$, implies that they have a cut in the g plane which can be placed along the negative g axis. In the absence of stronger singularities (from other sources) at the origin in the g plane, one can obtain the leading behaviour of the late terms in the perturbation expansion by means of a dispersion relation in g, just as we did in chapter two :

$$\Gamma_r^{(2M)}(q_i;g) = \frac{1}{\pi}\int_{-\infty}^{0} \frac{dg'}{g'-g} \, \mathrm{Im}\Gamma_r^{(2M)}(q_i;g',\arg g'=\pi) \tag{4.81}$$

Thus if we write

$$\Gamma_r^{(2M)}(q_i;g) = \sum_K \Gamma_{r,K}^{(2M)} g^K \tag{4.82}$$

we have from equation (4.81)

$$\Gamma_{r,K}^{(2M)}(q_i) = \frac{1}{\pi}\int_{-\infty}^{0} \frac{\mathrm{Im}\Gamma_r^{(2M)}(q_i;g', \arg g' = \pi)}{(g')^{K+1}} \tag{4.83}$$

The results (4.80) and (4.83) enable us in principle to obtain the high order behaviour of $\Gamma_{r,K}^{(2M)}(q_i)$. To be specific, we write a dispersion relation in the one-loop renormalised (minimally subtracted) coupling $g \equiv g_r(\mu_o)$. Using equations (4.80) and (4.83) we find the coefficient of $[g_r(\mu_o)]^K$ in $\Gamma_r^{(2M)}(q_i;g_r(\mu_o))$ to be

$$\Gamma^{(2M)}_{r,K}(q_i) = \frac{C_r}{\pi}\left(\frac{-3}{8\pi^2}\right)^K \Gamma\left(\frac{d+1+2M}{2}+K\right)\mu_o^{d-M(d-2)}$$

$$\times \int_o^\infty \frac{d\lambda}{\lambda}\ \lambda^{(d-M(d-2)-\varepsilon K-3)} \prod_{i=1}^{2M}\left[\frac{q_i^2}{\mu_o^2\lambda^2}\,\Phi(q_i/\mu_o\lambda)\right] \left(1+0\left(\varepsilon, \frac{1}{K}\right)\right) \quad (4.84)$$

If we tentatively assume that as $\varepsilon\to 0$, εK also tends to zero†, we may write the above result in four dimensions as

$$\Gamma^{(2M)}_{r,K}(q_i) = \frac{C_r}{\pi}\left(-\frac{3}{8\pi^2}\right)^K \Gamma(K+M+5/2)\ \mu_o^{4-2M}$$

$$\times \int_o^\infty \frac{d\lambda}{\lambda^{2M}} \prod_{i=1}^{2M}\left[\frac{q_i^2}{\mu_o^2\lambda^2}\Phi(q_i/\mu_o\lambda)\right] \left(1+0\left(\frac{1}{K}\right)\right)$$

$$= \frac{C_r}{\pi}\left(-\frac{3}{8\pi^2}\right)^K K!\,K^{M+3/2}\,\mu_o^3\int_o^\infty \frac{d\lambda}{\lambda^{2M}} \prod_{i=1}^{2M}\left[\frac{q_i^2}{\lambda^2}\Phi\left(\frac{q_i}{\lambda}\right)\right] \left(1+0\left(\frac{1}{K}\right)\right)$$

(4.85)

This gives the high order behaviour of $\Gamma^{(2M)}_{r,K}(q_i)$ defined by equation (4.82). However it is the high order behaviour of $\Gamma^{(2M)}_{R,K}(q_i)$, the coefficient of the $K^{\underline{th}}$ power of the fully renormalised coupling constant g_R in the fully renormalised vertex function $\Gamma^{(2M)}_R(q_i)$, that is required:

$$\Gamma^{(2M)}_R(q_i) = \sum_K \Gamma^{(2M)}_{R,K}\ \left[g_R(\mu)\right]^K \quad (4.86)$$

Fortunately the K! growth of $\Gamma^{(2M)}_{r,K}$ means that to leading order the transformation between $\Gamma^{(2M)}_{r,K}$ and $\Gamma^{(2M)}_{R,K}$ can be found.

To see this let us first find the behaviour of the coefficient of g^K in the wave-function renormalisation constant for K large.

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† That is, K is large but finite.

The bare theory is related to the fully renormalised theory in the case M=1 by

$$\Gamma_R^{(2)}(q;g,\varepsilon) = Z(g,\varepsilon)\Gamma^{(2)}(q;g,\varepsilon) \tag{4.87}$$

with the renormalisation condition

$$\frac{\partial}{\partial q^2}\Gamma_R^{(2)}(q;g,\varepsilon)\bigg|_{q^2=\mu^2} = 1 \tag{4.88}$$

and thus

$$Z^{-1}(\mu,g,\varepsilon) = \frac{\partial}{\partial q^2}\Gamma^{(2)}(q,g,\varepsilon)\bigg|_{q^2=\mu^2} \tag{4.89}$$

Now by dimensional analysis $\Gamma_K^{(2)}(q,\varepsilon) = q^{2-\varepsilon K}\Gamma_K^{(2)}(\varepsilon)$ and thus

$$\left[Z^{-1}(\mu,g,\varepsilon)\right]_{(K)} = (1-\varepsilon K/2)\,\mu^{-\varepsilon K}\Gamma_K^{(2)}(\varepsilon) \tag{4.90}$$

where $\left[Z^{-1}(g)\right]_{(K)}$ means the coefficient of g^K in $Z^{-1}(g)$.

From (4.90) we easily find that

$$-Z_K(\varepsilon) = (1-\varepsilon K/_2)\,\Gamma_K^{(2)}(\varepsilon)\,\left(1+0\left(\frac{1}{K}\right)\right) \tag{4.91}$$

and thus for large K, $Z_K \sim K!\, K^{5/2}$.

Using this result we may check that the coefficient of g^K in $\left[Z(g)\right]^M$ is just $MZ_K(1+0(\frac{1}{K}))$ and thus the leading behaviour of the coefficient of g^K in $Z^M(g)\Gamma^{(2M)}(q_i;g)$ is $\Gamma_K^{(2M)} + MZ_K\Gamma_0^{(2M)} + MZ_{K-1}\Gamma_1^{(2M)} + \ldots$. For M>1 this is $\Gamma_K^{(2M)}\left[1+0\left(\frac{1}{K}\right)\right]$ and therefore we have the useful result that to obtain the relation between

$\Gamma^{(2M)}_{R,K}(q_i)$ and $\Gamma^{(2M)}_{r,K}(q_i)$ we may ignore wave-function renormalisation to leading order in K for M>1.

Let us now write

$$g_r(\mu_o) = g_R(\mu) - \delta_2 . g_R^2(\mu) - \delta_3 . g_R^3(\mu) \cdots - \delta_K . g_R^K(\mu) \cdots \qquad (4.92)$$

and thus from equation (4.82) we have the coefficient of g_R^K in $\Gamma^{(2M)}_R$ to be (for M>1)

$$\Gamma^{(2M)}_{r,1}(-\delta_K) + \Gamma^{(2M)}_{r,2}(+2\delta_{K-1} + 2\delta_2\,\delta_{K-2} \cdots)$$

$$+ \quad \Gamma^{(2M)}_{r,3}(-3\delta_{k-2} + \ldots) + \ldots + \Gamma^{(2M)}_{r,K-3}\left(-\frac{(K-3)\ (K-4)\ (K-5)}{3!}\ \delta_2^3 + \ldots\right)$$

$$+ \quad \Gamma^{(2M)}_{r,K-2}\left(\frac{(K-2)\ (K-3)}{2!}\ \delta_2^2 + \ldots\right) + \Gamma^{(2M)}_{r,K-1}\left((K-1)\ \delta_2\right) + \Gamma^{(2M)}_{r,K} \qquad (4.93)$$

Using equation (4.85) we may sum up the last terms and therefore obtain to leading order

$$\Gamma^{(2M)}_{R,K} = -\Gamma^{(2M)}_{r,1}(\delta_K) - 2\Gamma^{(2M)}_{r,2}(\delta_{K-1} - \delta_2\delta_{K-2} + \ldots) - 3\Gamma^{(2M)}_{r,3}(\delta_{K-2} \cdots)$$

$$+ \ \ldots + \Gamma^{(2M)}_{r,K}\ e^{-\delta_2/a}\ \left(1 + 0\left(\frac{1}{K}\right)\right) \quad ;\ M>1 \qquad (4.94)$$

where $a = {}^3/8\pi^2$. In the case M=2 we have by definition

$$-\frac{1}{6}\Gamma^{(4)}_R(q_i;g)\Big|_{SP(\mu)} = g_R(\mu) \qquad (4.95)$$

and so from (4.94) we have

$$\delta_K = -\frac{1}{6}\Gamma^{(4)}_{r,K}(q_i)\Big|_{SP(\mu)} e^{-\delta_2/a}\ \left(1 + 0\left(\frac{1}{K}\right)\right) . \qquad (4.96)$$

We may obtain this result another way. If we write

$$g_R(\mu) \;=\; g_r(\mu_o) \;+\; \delta_2' \; g_r^2(\mu_o) + \delta_3' \; g_r^3(\mu_o) \;+\; \ldots \;+\; \delta_K' g_r^K(\mu_o) \;+\; \ldots \tag{4.97}$$

then

$$- \frac{1}{6} \, \Gamma^{(4)}_{r,K}(q_i) \Big|_{SP(\mu)} \;=\; \delta_K' \; (1+0 \; (\frac{1}{K})) \tag{4.98}$$

If we invert (4.97) and compare it with (4.92) we find

$$\delta_K \;=\; \delta_K' \;\; e^{-\delta_2/a} \;\; (1+0 \; (\frac{1}{K})) \tag{4.99}$$

and so equations (4.98) and (4.99) reproduce equation (4.96). From equations (4.85)and (4.96) we see that $\delta_K \sim K! \; K^{7/2}$ for K large.

For M>2 we may use equation (4.85) and the estimate of the growth of δ_K, to deduce from equation (4.94) that

$$\Gamma^{(2M)}_{R,K}(q_i) \;=\; \Gamma^{(2M)}_{r,K}(q_i) \; e^{-\alpha/a} \; (1+0 \; (\frac{1}{K})) \;\; ; \; M>2 \tag{4.100}$$

where $\alpha=\delta_2$ is given by

$$g_r(\mu_o) \;=\; g_R(\mu) - \alpha g_R^2(\mu) + 0(g_R^3) \tag{4.101}$$

The coefficient α comes from straightforward perturbation theory. Evaluating the usual one loop diagram at the symmetry point μ gives (see Appendix III)

$$\mu^{\varepsilon} g_R(\mu) \;=\; g - \frac{9}{8\pi^2\varepsilon} \, g^2 \mu^{-\varepsilon} \left[1+\varepsilon \; (\tfrac{1}{2}\ln\pi + \tfrac{1}{2}\ln 3 - \tfrac{1}{2}\gamma + 1) + 0(\varepsilon^2) \right] \; + 0(g^3) \tag{4.102}$$

where the symmetry point is defined by $q_i . q_j = \mu^2/3(4\delta_{ij}-1)$ and g means the bare coupling constant. Eliminating g between equations (4.76) and (4.102) gives in four dimensions

$$\alpha \;=\; \frac{9}{8\pi^2} \; (\tfrac{1}{2}\ln\pi + \tfrac{1}{2}\ln 3 - \tfrac{1}{2}\gamma + 1 - \ln \mu/\mu_o) \tag{4.103}$$

and thus using equations (4.85) and (4.100) we obtain

$$\Gamma^{(2M)}_{R,K}(q_i) = K! \left(-\frac{3}{8\pi^2}\right)^K K^{M+3/2} \mu^3 \frac{C_r}{\pi} \exp\{-\frac{3}{2}\ln\pi - \frac{3}{2}\ln 3 + \frac{3}{2}\gamma - 3\}$$

$$\times \int_0^\infty \frac{d\lambda}{\lambda^{2M}} \prod_{i=1}^{2M} \left[\frac{q_i^2}{\lambda^2} \tilde{\phi}\left(q_i/\lambda\right)\right] (1+0(\frac{1}{K})) \; ; \; M>2 \qquad (4.104)$$

If we now use the explicit forms (4.56) for $\tilde{\phi}$ and (4.78) for C_r, we obtain the required result

$$\Gamma^{(2M)}_{R,K}(q_i) = K! \left(-\frac{3}{8\pi^2}\right)^K K^{M+3/2} \frac{(48\pi^2)^M}{(54)^{\frac{1}{2}} \pi^4} \exp\left(\frac{3}{\pi^2}\zeta'(2) - \tfrac{1}{2}\gamma - \frac{15}{4}\right)$$

$$\times \mu^3 \int_0^\infty \frac{d\lambda}{\lambda^{2M}} \prod_{i=1}^{2M} \left[\frac{q_i}{\lambda} K_1(q_i/\lambda)\right] (1+0(\frac{1}{K})) \; ; \; M>2 \qquad (4.105)$$

We can also obtain high order estimates for the renormalisation group β function in four dimensions. This is defined by

$$\beta(g_R) = \mu \frac{d}{d\mu}\bigg|_{\text{fixed bare theory}} g_R(\mu) = \mu \frac{d}{d\mu}\bigg|_{\mu_0, g_r(\mu_0) \equiv g} g_R(\mu) \qquad (4.106)$$

and so using equation (4.95)

$$\beta(g_R) = -\frac{1}{6} \mu \frac{d}{d\mu}\bigg|_{\mu_0, g} \Gamma^{(4)}_R(q_i; g)\bigg|_{SP(\mu)} . \qquad (4.107)$$

If we write

$$\beta(g_R) = \sum_K \beta_K g^K \qquad (4.108)$$

then we obtain

$$\beta_K = -\frac{1}{6}\mu \left.\frac{d}{d\mu}\right|_{\mu_0} \left.\Gamma^{(4)}_{r,K}(q_i)\right|_{SP(\mu)} \tag{4.109}$$

We may now use the result (4.85) to deduce that

$$\beta_K = \frac{-C_r}{6\pi}\left(-\frac{3}{8\pi^2}\right)^K K!\, K^{7/2}\,\mu_0{}^3 \int\frac{d\lambda}{\lambda^4}\mu\frac{d}{d\mu}\left[\frac{\mu^2}{\lambda^2}\tilde{\phi}\,({}^{\mu}/\lambda)\right]^4 \quad (1+0\,(\frac{1}{K})) \tag{4.110}$$

Simplifying the integral and then integrating by parts gives

$$\beta_K = \frac{C_r}{2\pi}\left(-\frac{3}{8\pi^2}\right)^K K!K^{7/2}\left(\frac{\mu_0}{\mu}\right)^3 \int_0^\infty dx\; x^{10}\left[\tilde{\phi}\,(x)\right]^4(1+0(\frac{1}{K})) \tag{4.111}$$

Recall that this is the coefficient of g^K, that is $\left[g_r(\mu_0)\right]^K$, in the β function. In order to obtain the β function in terms of the full renormalised coupling $g_R(\mu)$, we must use † (compare with equation (4.100))

$$\tilde{\beta}_K = \beta_K\, e^{-\alpha/a}\left[1+0\,(\frac{1}{K})\right] \tag{4.112}$$

(The expression corresponding to (4.94) is $\tilde{\beta}_K = -\delta_K\beta_1 + \beta_K\, e^{-\alpha/a}$, to leading order, and since $\beta_1=0$ equation (4.112) follows). Using the value of α given in (4.103) and $a={}^3/_{8\pi^2}$, together with the explicit forms (4.56) for $\tilde{\phi}$ and (4.78) for C_r, we get

$$\tilde{\beta}_K = K!\left(-\frac{3}{8\pi^2}\right)^K K^{7/2}\, 2^{13/2}\, 3^{\frac{1}{2}}\exp\left(\frac{3}{\pi^2}\zeta'(2)-\tfrac{1}{2}\gamma-\frac{15}{4}\right)\int_0^\infty dx\; x^6\,(K_1(x))^4$$
$$\left[1+0\,(\frac{1}{K})\right] \tag{4.113}$$

Up to the convention of the normalisation of coupling constant (${}^g/_{4!}$ or ${}^g/_4$), this result is identical to that obtained by Lipatov (1977a,b) and Brézin et al (1977a)‡. To verify this one can evaluate

---

† $\tilde{\beta}_K$ is defined by $\beta(g_R) = \sum_K \tilde{\beta}_K g_R^K$ ‡ See also Bogomolny (1977).

the sum

$$\Sigma = \exp -\frac{1}{12}\sum_{\ell=2}^{\infty}(\ell+1)(\ell+2)(2\ell+3)\left\{\ell n\left(1-\frac{6}{(\ell+1)(\ell+2)}\right)+\frac{6}{(\ell+1)(\ell+2)}+\frac{18}{(\ell+1)^2(\ell+2)^2}\right\} \tag{4.114}$$

(see, for example, the results in Section (B)(ii) of Brézin et al (1977a) for n=1). This can be done following the method of 't Hooft (1976b) by placing an integer cut off Λ on the sum on ℓ, and evaluating each of the three terms in the sum independently for large Λ. For the first term, the trick is again to change the summation variable independently for each of the four logarithmic factors, so that each is of the form $\sum P(\ell)\ln\ell$ where $P(\ell)$ is a polynomial in ℓ. Apart from "end terms " from changing the summation variable one ends up with

$$\sum_{\ell=2}^{\Lambda}\ell\ln\ell = \tfrac{1}{2}\Lambda^2\ln\Lambda + \tfrac{1}{2}\Lambda\ln\Lambda + \frac{1}{12}\ln\Lambda - \frac{1}{4}\Lambda^2 + \frac{\ln 2\pi+\gamma}{12} - \frac{\zeta'(2)}{2\pi^2} \tag{4.115}$$

Combining all three terms, the Λ-dependence cancels as it must and the final result is

$$\sum = 2^{-5/2}\,3^{-5/2}\,5^{5/2}\,\pi^{-\frac{1}{2}}\exp\left\{\frac{3}{\pi^2}\zeta'(2) - \frac{7}{2}\gamma + \frac{17}{4}\right\} \tag{4.116}$$

Substituting this expression into the result of Brézin et al, reproduces equation (4.113) up to a factor 6 for the different choice of coupling normalisations.

<u>ϕ^4 theory with O(n) internal symmetry near four dimensions</u>

We now discuss the generalisation of the calculations of the previous sections to the case of $(\underset{\sim}{\phi}^2)^2$ interactions with O(n) internal symmetry (Brézin et al 1977a).

The Euclidean action, $H(\phi)$, is given by

$$H(\phi) = \int d^dx\left[\tfrac{1}{2}(\nabla\phi_i)(\nabla\phi_i) + \frac{g}{4}(\phi_i\phi_i)^2\right] \tag{4.117}$$

where $d=4-\varepsilon$ and $i=1,2,\ldots,n$. The instanton therefore satisfies

$$\nabla^2\phi_i = g(\phi_j\phi_j)\phi_i \tag{4.118}$$

We are thus faced with solving n coupled non-linear differential equations. Fortunately, as we will now prove, the instanton solution of least action can be written as

$$\phi_i(x) = u_i \phi_c(x) \qquad i=1,\ldots,n \tag{4.119}$$

where u_i is independent of x.

To prove (4.119) let us write the instanton action as

$$H(\phi) = \frac{1}{4}\int d^d x \, (\nabla\phi_i)\,(\nabla\phi_i) \tag{4.120}$$

where we have used equations (4.117) and (4.118). If we now express the instanton in the form $\phi_i(x) = u_i(x)\phi_c(x)$ we find

$$H(\phi) = \frac{1}{4}\int d^d x \,\{\phi_c^2 \, (\nabla u_i)\,(\nabla u_i) + 2\phi_c\,(\nabla\phi_c)\,u_i\,(\nabla u_i)$$

$$+ \; u_i u_i\,(\nabla\phi_c)^2\} \tag{4.121}$$

Now without loss of generality we may choose $u_i(x)$ so that $u_i(x)\, u_i(x) = 1$ and therefore

$$H(\phi) = \frac{1}{4}\int d^d x \;\{\phi_c^2\,(\nabla u_i)\,(\nabla u_i) + (\nabla\phi_c)^2\} \tag{4.122}$$

$$\geqslant \frac{1}{4}\int d^d x \;\{\,(\nabla\phi_c)^2\} \tag{4.123}$$

Equality is achieved when $\phi_c=0$ or $\nabla u_i=0$ $i=1,\ldots,n$, that is, u_i is independent of x. Using

$$H(\phi) \geqslant H(\phi)\Big|_{\underline{u}=\text{constant}} \tag{4.124}$$

, that is, the instanton with minimum action is of the form (4.119), we find from equations (4.118) that ϕ_c satisfies

$$\nabla^2\phi_c = g\phi_c{}^3 \tag{4.125}$$

which is the same equation as the n=1 case, and thus

$$\phi_c(x;x_0,\lambda) = \sqrt{\frac{8}{-g}}\,\frac{\lambda}{1+\lambda^2(x-x_0)^2} \tag{4.126}$$

Note that the Hamiltonian is O(n) invariant, and so the expression (4.36) for $H(\phi_c)$ is unchanged. The differential operator

$$M_{ij}(x,x') = \left.\frac{\delta^2 H}{\delta\phi_i(x)\delta\phi_j(x')}\right|_{\phi=\phi_c} \tag{4.127}$$

has the form

$$M_{ij} = \left(-\nabla^2 - \frac{8\lambda^2}{(1+\lambda^2(x-x_o)^2)^2}\right)\delta_{ij} - \frac{16u_iu_j\,\lambda^2}{(1+\lambda^2(x-x_o)^2)^2} \tag{4.128}$$

This operator can be decomposed into longitudinal and transverse components

$$M_{ij} = M_L u_i u_j + M_T(\delta_{ij} - u_i u_j) \tag{4.129}$$

Translating by x_o and scaling by λ we have

$$\begin{aligned} M_L &= -\nabla^2 - \frac{24}{(1+x^2)^2} \\ M_T &= -\nabla^2 - \frac{8}{(1+x^2)^2} \end{aligned} \tag{4.130}$$

The determinant of M_L is as before (compare with equation (4.6)). The determinant of the (n-1) - fold degenerate modes in M_T is obtained by the same procedure as outlined at the end of Chapter three (note that it is of the form (3.74) with $\zeta=1$). The equation analogous to (4.38) is

$$\left(\frac{\det V_T}{\det V_o}\right)^{-\frac{1}{2}} = \exp -\tfrac{1}{2}\sum_{L=0}^{\infty} \nu_L(d+1)\,\ln\left\{\frac{(L+\frac{1}{2}d-2)\,(L+\frac{1}{2}d+1)}{(L+\frac{1}{2}d-1)\,(L+\frac{1}{2}d)}\right\} \tag{4.131}$$

In the numerator on the right-hand-side we see the L=0 "zero modes" with eigenvalue

$$E^V = -\frac{3\varepsilon}{2}(1+O(\varepsilon)) \tag{4.132}$$

corresponding to the spontaneous breaking of the O(n) symmetry. Extracting this factor from the expression (4.131), and following the same procedure as that outlined in the one component case we find

$$\left(\frac{\det \tilde{V}_T}{\det V_o}\right)^{\frac{(n-1)}{2}} = 2^{\frac{n-1}{2}} \exp - \frac{(n-1)}{2} \sum_{L=1}^{\infty} \nu_L(d+1) \ln\left\{ \frac{(L+\frac{1}{2}d-2)\ (L+\frac{1}{2}d+1)}{(L+\frac{1}{2}d-1)\ (L+\frac{1}{2}d)} \right\}$$

$$= (2^{-1/6}\ \pi^{-1/6}\ 3^{\frac{1}{2}})^{n-1} \exp (n-1) \left\{ \frac{1}{3\varepsilon} + \frac{\zeta'(2)}{\pi^2} -\tfrac{1}{2}\gamma + \frac{1}{4} \right\} (1+O(\varepsilon)) \tag{4.133}$$

The (n-1)-fold "zero modes" with eigenvalue given by (4.132) are replaced by collective coordinate integrations over the unit vectors $\{\hat{u}_i\}$. Following the arguments leading up to equation (4.27) the appropriate Jacobian factor is

$$J^{V_T} = \left[\int d^dx\ \phi_c(x)\ \frac{4\lambda^2}{(1+\lambda^2(x-x_o)^2)^2}\ \phi_c(x) \right]^{\frac{(n-1)}{2}} (1+O(\varepsilon,g))$$

$$= \left[-\frac{16\pi^2\lambda^\varepsilon}{3g} \right]^{\frac{n-1}{2}} (1+O(\varepsilon,g)) \tag{4.134}$$

The final correction factor that is required for the bare theory nec essitates the evaluation of Figure 14 for general n. The statement corresponding to (4.34) is

$$2\varepsilon u_i u_j \int d\Omega_1\ d\Omega_2\ K_1^{-1}\ \tilde{V}_{ij}^{-1}\ (\eta_1,\eta_2)\ \tilde{V}_{kk}^{-1}\ (\eta_2,\eta_2)\ \left[1+O(\varepsilon)\right]$$

$$+\ 4\varepsilon u_i u_j \int d\Omega_1\ d\Omega_2\ K^{-1}\ \tilde{V}_{ik}^{-1}\ (\eta_1,\eta_2)\ \tilde{V}_{kj}^{-1}\ (\eta_2,\eta_2)\ \left[1+O(\varepsilon)\right] \tag{4.135}$$

Now if $V_{ij} = V_L\ u_i u_j + V_T\ (\delta_{ij} - u_i u_j)$ then $V_{ij}^{-1} = V_T^{-1}\ \delta_{ij} + (V_L^{-1} - V_T^{-1})\ u_i u_j$ and so we may write expression (4.135) as

$$6\varepsilon \int d\Omega_1\, d\Omega_2\, K_1^{-1}\, \tilde{V}_L^{-1}(n_1,n_2)\, \tilde{V}_L^{-1}(n_2,n_2)\, \left[1+O(\varepsilon)\right]$$

$$+\ 2\varepsilon(n-1) \int d\Omega_1\, d\Omega_2\, K_1^{-1}\, \tilde{V}_L^{-1}\, (n_1,n_2)\, \tilde{V}_T^{-1}(n_2,n_2)\, \left[1+O(\varepsilon)\right] \qquad (4.136)$$

Thus we have the additional factor given by the second term in (4.136). This is easily calculated following the methods of equations (4.57) - (4.60) and we find it to be equal to

$$2\varepsilon(n-1)\left[-\frac{1}{4\varepsilon} + O(1)\right]$$

$$= -\frac{(n-1)}{2} + O(\varepsilon) \qquad (4.137)$$

Thus for $n\neq 1$ the graph in Figure 14 contributes a factor

$$\exp -\tfrac{1}{2}(n-1) \qquad (4.138)$$

in addition to the factor $C_1 = \exp - {}^9/_2$ in equation (4.61).

Combining expressions (4.133), (4.134) and (4.138) and remembering the factor $(2\pi)^{-\frac{(n-1)}{2}}$ for the Gaussians replaced by collective coordinates, one obtains the additional factor

$$\left(-\frac{8\pi^2\lambda^\varepsilon}{3g}\right)^{\frac{(n-1)}{2}} \left(3^{\frac{1}{2}}\pi^{-2/3}\, 2^{-1/6}\right)^{(n-1)} \exp\,(n-1)\,\left\{\frac{1}{3\varepsilon} + \frac{1}{\pi^2}\zeta'(2) - \tfrac{1}{2}\gamma - \tfrac{1}{4}\right\}$$

$$\times \tfrac{1}{2}\int d\vec{u}\ u_{i_1} u_{i_2} \ldots u_{i_{2M}} \qquad (4.139)$$

for $n\neq 1$, for the imaginary part of the 2M-point vertex function. Note that we have put in a factor $\frac{1}{2}$ since, according to the discussion in Chapter two, there is half a Gaussian integral for the bound state at each saddle point. Expression (4.139) can be simplified using the result

$$\tfrac{1}{2}\int d\vec{u}\ u_{i_1}\ u_{i_2} \ldots u_{i_{2M}} = \frac{\pi^{n/2}}{\Gamma(n/2)}\ \frac{(\delta_{i_1 i_2}\,\delta_{i_3 i_4} \cdots \delta_{i_{2M-1} i_{2M}} + \text{perms.})}{n(n+2)\ \ldots\ (n+2M-2)}$$

$$(4.140)$$

It is seen explicitly that (4.139) is equal to 1 when n=1.

Expression (4.139) contains a divergent term which is again removed by coupling constant renormalisation; for $n\neq 1$, equation (4.76) becomes

$$\frac{\lambda^{\varepsilon}}{g} = \frac{1}{g_r(\lambda)} - \frac{n+8}{8\pi^2\varepsilon} \tag{4.141}$$

Making the substitution for g yields

$$\underset{(\arg g=\pi)}{\operatorname{Im}\Gamma_r^{(2M)}(q_i)} = -C_r \int_0^\infty \frac{d\lambda}{\lambda}\, \lambda^{d-M(d-2)} \left[\frac{-8\pi^2}{3g_r(\lambda)}\right]^{\frac{d+n+2M}{2}} \prod_{i=1}^{2M} \frac{q_i^{\,2}}{\lambda^2}$$

$$\times\ \tilde{\phi}\,(q_i/_\lambda) \quad \times \quad \exp \frac{8\pi^2}{3g_r(\lambda)} \ (1-{}^{\varepsilon}/_2\ (\ln\pi+\gamma+2)).$$

$$\times \frac{(\delta_{i_1i_2}\ \delta_{i_3i_4}\cdots\ \delta_{i_{2M-1}i_{2M}} \ + \text{perms.})}{1.3\ \ldots\ (2M-1)} \quad (1+0(\varepsilon,g_r)) \tag{4.142}$$

$$\text{where } C_r = 2^{-\frac{1}{2}}\ \pi^{-3/2} \exp\left(\frac{3}{\pi^2}\zeta'(2) - 2\gamma - \frac{3}{4}\right).\left(3^{\frac{1}{2}}\ 2^{-1/6}\right)^{(n-1)}$$

$$\times \frac{\Gamma(M+\frac{1}{2})}{\Gamma(M+{}^n/_2)} \quad \times \quad \exp\,(n-1)\left(\frac{1}{\pi^2}\zeta'(2) - \frac{1}{3}\gamma + \frac{1}{12}\right) \tag{4.143}$$

This generalises the results (4.77) and (4.78) for $n\neq 1$.

The calculations to obtain the asymptotic behaviour of the coefficient of g_R^K for K large follow the previous section. We shall content ourselves with outlining the calculation in the case of the β function.

The expression generalising (4.111) is

$$\beta_K = \frac{C_r}{2\pi}\left(-\frac{3}{8\pi^2}\right)^K K!\ K^{3+\,n/2} \left(\frac{\mu_o}{\mu}\right)^{\frac{n+8}{3}} \frac{(n+8)}{9} \cdot \int_0^\infty dx\ x^{\frac{n+8}{3}+7}$$

$$\times\ [\tilde{\phi}(x)]^4\ (1+0\ (\frac{1}{K})) \tag{4.144}$$

where C_r is given in (4.143). The multiplicity of the one-loop graph modifies the coefficient α in expression (4.103):

$$\alpha = \frac{n+8}{8\pi^2} \left(\tfrac{1}{2}\ln\pi + \tfrac{1}{2}\ln 3 - \tfrac{1}{2}\gamma + 1 - \ln \mu/\mu_0\right) \tag{4.145}$$

The final result for $\tilde{\beta}_K$ becomes

$$\tilde{\beta}_K = K! \left(-\frac{3}{8\pi^2}\right)^K K^{3+n/2}\, 2^{13/2}\, 3^{\frac{1}{2}} \exp\left(\frac{3}{\pi^2}\zeta'(2) - \tfrac{1}{2}\gamma - \frac{15}{4}\right).$$

$$\frac{n+8}{9} \cdot \frac{\Gamma(5/2)}{\Gamma(2+n/2)} \times \left(\frac{9}{2\pi}\right)^{\frac{n-1}{6}} \exp (n-1) \left(\frac{1}{\pi^2}\zeta'(2) - \frac{1}{6}\gamma - \frac{1}{4}\right)$$

$$\int_0^\infty dx\, x^{\frac{n+8}{3}+3} (K_1(x))^4 \quad \left(1+0\left(\frac{1}{K}\right)\right) \tag{4.146}$$

Using the expression

$$\sum_{\ell=2}^{\infty} (\ell+1)(\ell+2)(2\ell+3) \left\{ \ln\left(1 - \frac{2}{(\ell+1)(\ell+2)}\right) + \frac{2}{(\ell+1)(\ell+2)} + \frac{2}{(\ell+1)^2(\ell+2)^2}\right\}$$

$$= -\frac{12}{\pi^2}\zeta'(2) + 6\gamma - 24\ln 2 + 24\ln 3 + 2\ln 2\pi - \frac{55}{3} \tag{4.147}$$

(evaluated in the same way as the sum in expression (4.114)) one may verify that the expression (4.136) agrees with the result of Brézin et al (1977a).

To summarise this chapter we may say that, given the assumptions discussed in the introduction to the chapter, we may calculate the high order behaviour of perturbation theory for ϕ^4 field theory in $4-\varepsilon$ dimensions analytically. In particular we can reproduce the results of Lipatov and Brézin et al. in four dimensions. We now have to ask whether or not the assumptions are justified. Discussion of this point is deferred until Chapter seven, where we will discuss the appearance of new singularities, the renormalons, in four dimensions. However, we will merely point out here that they are believed to be absent in $d<4$ and since we have calculated the imaginary part using dimensional regularisation, we are in an ideal position to obtain high order estimates for quantities which are believed to be free of renormalon singularities.

CHAPTER FIVE

VACUUM INSTABILITY IN ϕ^3 FIELD THEORIES NEAR SIX DIMENSIONS

In the preceding chapter, we saw that for ϕ^4 field theory the perturbation expansion was oscillatory at large orders providing $g>0$, and this therefore indicated that the perturbation series was Borel summable. This is directly related to the fact that for $g>0$ the theory possesses a stable ground state with perturbation theory about this ground state giving an oscillating series. If, however, the theory is studied with $g<0$ for which no stable ground state exists, perturbation theory does not tell the whole story - there are terms exponentially small in the coupling constant due to tunnelling, which are never seen in perturbation theory. In this case perturbation theory gives a series which is not Borel summable (Brézin et al 1977b).

For some field theories, real instanton solutions (and therefore tunnelling) exist for all real values of the coupling constant. An important example of a theory of this kind is that of non-abelian gauge theories when the gauge group contains SU(2) as a subgroup. In this case instantons are known to exist in four dimensions (Belavin et al 1975) and so presumably the perturbation series is not Borel summable; however, the situation is very complicated ('t Hooft 1977a,b). Another, much simpler, class of field theories in which tunnelling occurs for all real values of the coupling constant are ϕ^N field theories where N is odd. In this chapter we will be investigating ϕ^3 field theories near their critical dimension, $d_c=6$. Although it was recognized many years ago that these theories have no stable ground state (Baym 1960, Jaffe 1965 and references therein), it has only been recently, with the advent of instanton techniques, that we have been able to explore the consequences of this instability quantitatively.

Structure of the Imaginary Part of the Vertex Functions

In this section we will obtain the one-loop approximation to the imaginary part of the vertex functions, generated by tunnelling out of the metastable ground state $\phi=0$. We will work in $6-\varepsilon$ dimensions in order to regulate the divergences of the theory. The approach will be very similar to that discussed for ϕ^4 in $4-\varepsilon$ dimensions in the previous chapter, and so we will not go into many of the details of the derivation.

The Euclidean action for this theory is given by

$$H(\phi) = \int d^d x \left[\tfrac{1}{2}(\nabla\phi)^2 + \frac{g}{3}\phi^3\right] \tag{5.1}$$

where $d = 6-\varepsilon$, $\varepsilon>0$. From the study of this equation in chapter three we know that real instanton solutions exist in six dimensions, and that the instanton solution with least action in this case is given by

$$\phi_c(x;x_0,\lambda) = \frac{-24\lambda^2}{g(1+\lambda^2(x-x_0)^2)^2} \tag{5.2}$$

The crucial difference between this theory and the one considered in the previous chapter is that now ϕ_c is a real solution of the field equations for all real values of g. Thus the Euclidean Green's functions

$$G^{(M)}(x_1,\ldots,x_M) = \frac{\int D\phi\ \phi(x_1)\ldots\phi(x_M)\exp-H(\phi)}{\int D\phi\exp-H(\phi)} \tag{5.3}$$

have an imaginary part for all real g due to tunnelling out of the metastable ground state. The "zero dimensional ϕ^3 field theory" discussed in chapter two brings out this point very well.

As before we will expand $H(\phi)$ in $6-\varepsilon$ dimensions about $\phi=\phi_c$ where ϕ_c (given by equation (5.2)) is an extrema of $H(\phi)$ only when $\varepsilon=0$. Therefore writing $\phi = \phi_c+\hat\phi$ we have

$$H(\phi) = H(\phi_c) + \tfrac{1}{2}\int d^d x\ \hat\phi M\hat\phi + \frac{96\varepsilon}{g}\int d^d x\ \frac{\lambda^4}{(1+\lambda^2(x-x_o)^2)^3}\ \hat\phi$$

$$+ \frac{g}{3}\int d^d x\ \hat\phi^3 \tag{5.4}$$

where $$M(x,x') = \left[-\nabla^2 - \frac{48\lambda^2}{(1+\lambda^2(x-x_o)^2)^2}\right]\delta^d(x-x') \tag{5.5}$$

and where once again there is a term linear in $\hat\phi$.

After introducing collective coordinates to deal with the "zero modes" we find the analogous expression to (4.11) is

$$G^{(M)}(x_1,\ldots,x_M) = \tfrac{1}{2}\int d\lambda d^d x_o J^M \phi_c(x_1)\ldots\phi_c(x_M)\exp-H(\phi_c)$$
$$C_1(2\pi)^{\frac{-d+1}{2}}\left(\frac{\det \hat M}{\det M_o}\right)^{-\frac{1}{2}}(1 + O(\varepsilon,g^2)) \tag{5.6}$$

where the components J^M, M_o, C_1 have the same meaning as before. The factor of $\frac{1}{2}$ is present because the saddle point $\phi=\phi_c$ contributes only half a Gaussian integral (see equation (2.20)). The right-hand-side of equation (5.6) is pure imaginary and thus we have obtained the magnitude of $\mathrm{Im}G^{(M)}$ generated by vacuum decay. Its sign will depend on the method of analytic continuation from the well defined theory, and so using the discussion of chapter two as our guide we may write equation (5.6) as

$$\underset{(\arg g=0)}{\mathrm{Im}G^{(M)}(x_1,\ldots,x_M)} = \tfrac{1}{2}\int d\lambda d^d x_o J^M \phi_c(x_1)\ldots\phi_c(x_M)\exp-H(\phi_c)$$
$$C_1(2\pi)^{\frac{-d+1}{2}}\left|\frac{\det \hat M}{\det M_o}\right|^{-\frac{1}{2}}(1 + O(\varepsilon,g^2)) \tag{5.7}$$

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The imaginary part of the vertex function $\Gamma^{(M)}(q_i;g)$ is then

$$\underset{(\arg g=0)}{\mathrm{Im}\Gamma^{(M)}(q_i)} = \tfrac{1}{2}\int d\lambda J^M \exp -H(\phi_c)C_1(2\pi)^{\frac{-(d+1)}{2}} \left|\frac{\det \tilde{M}}{\det M_0}\right|^{-\frac{1}{2}}$$

$$\times \prod_{i=1}^{M} q_i^2 \int d^d x_i e^{iq_i.x_i} \phi_c(x_i) \Big(1 + O(\varepsilon, g^2)\Big) \quad (5.8)$$

We may again check that the combination of the Jacobian and the determinant factors without the zero modes remains invariant (to lowest order in ε) under the transformation from flat space to spherical space (see equation (4.30)). Thus equation (5.8) may be written in the more useful form

$$\underset{(\arg g=0)}{\mathrm{Im}\Gamma^{(M)}(q_i)} = \tfrac{1}{2}\int d\lambda J^V \exp -H(\phi_c)C_1(2\pi)^{\frac{-(d+1)}{2}} \left|\frac{\det \tilde{V}}{\det V_0}\right|^{-\frac{1}{2}}$$

$$\times \prod_{i=1}^{M} q_i^2 \int d^d x_i e^{iq_i.x_i} \phi_c(x_i) \Big(1 + O(\varepsilon, g^2)\Big) \quad (5.9)$$

where J^V, V and V_0 are defined in the same way as in previous chapters - they are the spherical analogues of J^M, M and M_0.

It remains to study what graphs, if any, contribute to C_1. This factor arises because of the term linear in ϕ in equation (5.4), and is treated perturbatively as in the corresponding problem in the previous chapter. Thus we may argue in the same way that we may neglect $O(g^2)$ contributions generated by this term, since these give $O(K^{-1})$ corrections to the leading asymptotic behaviour, which is all we are attempting to calculate. Since an L loop graph gives 2(L-1) powers of g, it is sufficient to restrict ourselves to tree diagrams and one-loop graphs. In the same way as before, we then find that to lowest order in ε, only the graph shown in figure 14 contributes to C_1.

In order to calculate C_1 it is easier to use the Hamiltonian on the sphere which is obtained from equation (5.4) (after translating x by x_0 and rescaling by λ) by defining new fields $\hat{\Phi}$, and introducing the operators V and V_0, in the usual way (see chapter three). We find

$$H(\phi) = H(\phi_c) + \tfrac{1}{2}\int d\Omega\hat{\Phi}(-V)\hat{\Phi} + \frac{12\varepsilon}{g}\int d\Omega K^{-1+\varepsilon/2}\hat{\Phi} + \frac{g}{3}\int d\Omega K^{-\varepsilon/2}\hat{\Phi}^3 \tag{5.10}$$

The propagator in the presence of an instanton can be written in terms of the spherical harmonics in d+1 dimensions in a form analogous to (4.33):

$$\tilde{V}^{-1}(n,n') = -\sum_{L,\tilde{\alpha}} \frac{Y_L^{\alpha}(n)Y_L^{\alpha}(n')^*}{(L+\frac{1}{2}d+3)(L+\frac{1}{2}d-4)} \tag{5.11}$$

where the tilde means L=1 is excluded. The contribution from the graph in figure 14 in this notation is

$$12\varepsilon\int d\Omega_1 d\Omega_2 K_1^{-1+\varepsilon/2}\tilde{V}^{-1}(n_1,n_2)K_2^{-\varepsilon/2}\tilde{V}^{-1}(n_2,n_2) \tag{5.12}$$

The corresponding expression in flat space is found from equation (5.4) to be (after translating x by x_0 and rescaling by λ)

$$96\varepsilon\int\frac{d^dx\,d^dy}{(1+x^2)^3}\,\tilde{M}^{-1}(x,y)\,\tilde{M}^{-1}(y,y) \tag{5.13}$$

In the next section we will go on to evaluate the various components appearing in $\mathrm{Im}\Gamma^{(M)}$, expression (5.9).

<u>One-loop Calculation About the Instanton</u>

The imaginary part of the vertex functions at "zeroth-order", that is, ignoring fluctuations about the instanton, is easily calculated - we only have to find $H(\phi_c)$:

$$H(\phi_c) = \frac{192\pi^3\lambda^{\varepsilon}}{5g^2}\left(1-\varepsilon/2(\ln\pi+\gamma+1) + O(\varepsilon^2)\right) \tag{5.14}$$

Of course, if we were not going on to find the one-loop corrections to the "classical result" for $\mathrm{Im}\mathcal{T}^{(M)}$ we would not need to continue in d, (we could set ε=0) since we would encounter no ultra-violet divergences. As it is, we will find that the $O(\varepsilon)$ term in $H(\phi_c)$ will contribute to the one-loop result after renormalisation: $g^{-2} \to g_R^{-2} + O(\varepsilon^{-1})$.

The Jacobian J^V is given by equation (4.27) to leading order but with

$$\phi_\lambda \equiv \frac{\partial \phi_c}{\partial \lambda} = \frac{-48\lambda(1-\lambda^2(x-x_0)^2)}{g(1+\lambda^2(x-x_0)^2)^3} \tag{5.15}$$

and

$$\phi_\mu \equiv \frac{\partial \phi_c}{\partial x_0{}^\mu} = \frac{96\lambda^4(x-x_0)\mu}{g(1+\lambda^2(x-x_0)^2)^3} \tag{5.16}$$

where $\phi_c(x;x_0,\lambda)$ is given by equation (5.2). Substituting (5.15) and (5.16) into (4.27) gives

$$J^V = \lambda^{d-1} \left(\frac{768\pi^3\lambda^\varepsilon}{35g^2}\right)^{\frac{d+1}{2}} (1 + O(\varepsilon,g^2)) \tag{5.17}$$

We now come to the evaluation of the small oscillations determinant. From equations (3.78) and (3.82) we find

$$\left(\frac{\det \tilde{V}}{\det V_0}\right)^{-\frac{1}{2}} = \exp -\tfrac{1}{2}\tilde{\sum_{L=0}^{\infty}} \frac{\Gamma(L+d-1)(2L+d-1)}{\Gamma(L+1)\Gamma(d)} \ln\left\{ \frac{(L+\frac{1}{2}d-4)(L+\frac{1}{2}d+3)}{(L+\frac{1}{2}d-1)(L+\frac{1}{2}d)} \right\} \tag{5.18}$$

where the tilde means that the "zero modes", that is, the L=1 modes of V, are excluded. Evaluating the contribution from the L=0 modes and the L=1 modes of V_0 explicitly, we find

$$\left| \frac{\det \tilde{V}}{\det V_0} \right|^{-\frac{1}{2}} = 2^7 3^{7/2} \exp -\tfrac{1}{2}\sum_{L=2}^{\infty} \frac{\Gamma(L+5-\varepsilon)(2L+5-\varepsilon)}{\Gamma(L+1)\Gamma(6-\varepsilon)} \ln\left\{ \frac{(L-1-\varepsilon/2)(L+6-\varepsilon/2)}{(L+2-\varepsilon/2)(L+3-\varepsilon/2)} \right\}$$
$$\times (1 + O(\varepsilon)) \tag{5.19}$$

We saw how to extract the divergent part and the finite part from a similar sum in the previous chapter. Using the method described there, we find the sum in (5.19) is equal to

$$\frac{1}{\Gamma(6-\varepsilon)}\ \{ -480\zeta'(-3+\varepsilon) - 2640\zeta'(-1+\varepsilon) + 864\varepsilon\zeta'(1+\varepsilon)$$

$$-\frac{9319}{5}\varepsilon^2\zeta'(1+\varepsilon) + O(\varepsilon)\ \} + 7(\ln 3 + 2\ln 2 - \ln 7) \qquad (5.20)$$

Using the functional relation (4.51) together with the result $\zeta'(1+\varepsilon) = -1/\varepsilon^2 + O(1)$, we find

$$\left|\frac{\det \tilde{V}}{\det V_0}\right|^{-\frac{1}{2}} = 7^{7/2}(2\pi)^{-9/10}\exp\{\frac{18}{5\varepsilon} + \frac{193}{144} - \frac{9}{2}\gamma + \frac{11\zeta'(2)}{2\pi^2} - \frac{3\zeta'(4)}{2\pi^4}$$

$$+ O(\varepsilon)\ \} \qquad (5.21)$$

where $\zeta'(4) = -0.068911\ldots$.

In the calculation of the small oscillations determinant for ϕ^4 in $d = 4-\varepsilon$, we commented that if d_c is an even integer we may also use equation (3.84) to evaluate $|\det \tilde{V}/\det V_0|^{-\frac{1}{2}}$ in a straightforward manner. Now d_c is an even integer only for ϕ^3 and ϕ^4 theories, and so this is the only other theory for which we may use equation (3.84) as a (relatively) easy check for the result (5.21). One finds

$$\left|\frac{\det \tilde{V}}{\det V_0}\right|^{-\frac{1}{2}} = 2^7 3^{7/2} 5^{7/2} 7^{7/2}\exp-\tfrac{1}{2}\sum_{\ell=2}^{\infty}\frac{\Gamma(\ell+4-\varepsilon)(2\ell+4-\varepsilon)}{\Gamma(\ell+1)\Gamma(5-\varepsilon)}$$

$$\times \ln\{\frac{(\ell+1-\varepsilon/2)(\ell-\varepsilon/2)(\ell-1-\varepsilon/2)}{(\ell+5-\varepsilon/2)(\ell+4-\varepsilon/2)(\ell+3-\varepsilon/2)}\}\ (1 + O(\varepsilon))$$

$$(5.22)$$

The sum in equation (5.22) is equal to

$$\frac{1}{\Gamma(5-\varepsilon)}\{ -96\psi'(-3+\varepsilon) - 528\psi'(-1+\varepsilon) + \frac{864}{5}\varepsilon\psi'(1+\varepsilon)$$

$$\frac{-1691}{120}\varepsilon^2\psi'(1+\varepsilon) + O(\varepsilon)\} + 7(\ell n3+2\ell n2+\ell n5) \tag{5.23}$$

Equations (5.22) and (5.23) used together enable us to recover the result (5.21).

Turning now to the product over the Fourier transforms of $\phi_c(x_i)$, we will find it convenient to define $\tilde{\phi}(q)$ by

$$\lambda^{-1-d/2}\left(\frac{192\pi^3\lambda^\varepsilon}{5g^2}\right)^{\frac{1}{2}}\tilde{\phi}\left(\frac{q}{\lambda}\right) = \int d^d x\, e^{iq.x}\phi_c(x) \tag{5.24}$$

so that using the result

$$\int\frac{d^d x\, e^{iq.x}}{(1+x^2)^\nu} = \frac{2^{1+d/2-\nu}\pi^{d/2}}{\Gamma(\nu)}(|q|)^{\nu-d/2}K_{d/2-\nu}(|q|) \tag{5.25}$$

(K is a modified Bessel function of the second kind) we have

$$\tilde{\phi}(q) = -\left(\frac{15}{\pi^3}\right)^{1/2}. 2^{d/2-1}\pi^{d/2}(|q|)^{2-d/2}K_{d/2-2}(|q|) \tag{5.26}$$

We will use this explicit form for $\tilde{\phi}(q)$ to show that the dilatation integral in equation (5.9) converges.

Finally we come to the calculation of C_1. Using equations (5.11) and (5.12) we find the expression for the graph in figure 14 to be (to lowest order)

$$\tilde{\sum_{L,L',\underset{\sim}{\alpha},\underset{\sim}{\alpha}'}} 12\varepsilon\frac{\int d\Omega_1 d\Omega_2(1+\cos\theta_1)^{-1}Y_L^{\underset{\sim}{\alpha}}(n_1)Y_L^{\underset{\sim}{\alpha}}(n_2)^*Y_{L'}^{\underset{\sim}{\alpha}'}(n_2)Y_{L'}^{\underset{\sim}{\alpha}'}(n_2)^*}{(L+\frac{1}{2}d+3)(L+\frac{1}{2}d-4)(L'+\frac{1}{2}d+3)(L'+\frac{1}{2}d-4)} \tag{5.27}$$

where θ_1 is the principal polar angle of n_1 (see Appendix II). The evaluation of (5.27) follows the ϕ^4 case exactly. We eventually find that expression (5.27) is equal to

$$\frac{-\varepsilon}{48} \sum_{L'}^{\infty} \frac{(2L'+5-\varepsilon)\Gamma(L'+5-\varepsilon)}{\Gamma(L'+1)(L'+6-\varepsilon/2)(L'-1-\varepsilon/2)} \cdot (1 + O(\varepsilon)) \tag{5.28}$$

The ultra-violet divergence in (5.28) comes from large L' contributions and therefore evaluating the sum for large L' we find that expression (5.28) equals

$$\frac{-\varepsilon}{48}\left(240\zeta(1+\varepsilon) + O(1)\right)$$

$$= -5 + O(\varepsilon) \tag{5.29}$$

Thus to this order we find

$$C_1 = \exp(-5) \tag{5.30}$$

We will now calculate the graph in figure 14 using the Hamiltonian in flat space, equation (5.4). In the analogous calculation for ϕ^4 the result obtained differed from the result of the calculation on the sphere by a factor $^3/2$. We find a similar discrepancy in this case. Let us expand $\tilde{M}^{-1}(y,y)$ as the sum of free propagators with zero, one, two, ... insertions of $-2g\phi_c(x) = 48(1+x^2)^{-2}$, then we find that the only ultra-violet divergences come from the one and two insertion contributions. Thus the finite part of expression (5.13) is contained in

$$96\varepsilon \int \frac{d^d x\, d^d y}{(1+x^2)^3} \tilde{M}^{-1}(x,y) \int \frac{d^d p}{(2\pi)^d} \frac{d^d k}{(2\pi)^d} \frac{e^{ip.y} f(p)}{(p+k)^2 k^2}$$

$$+ 96\varepsilon \int \frac{d^d x\, d^d y}{(1+x^2)^3} \tilde{M}^{-1}(x,y) \int \frac{d^d p}{(2\pi)^d} \frac{d^d k}{(2\pi)^d} \frac{d^d \ell}{(2\pi)^d} \frac{e^{i(p+k).y} f(p) f(k)}{\ell^2 (p+\ell)^2 (k-\ell)^2} \tag{5.31}$$

where $f(p)$ is defined by

$$f(p) = \int d^d x\, e^{-ip.x} \frac{48}{(1+x^2)^2} \tag{5.32}$$

The k integral in the first term gives (see Appendix III)

$$p^{2-\varepsilon}\left[-\frac{1}{192\pi^3\varepsilon}+O(1)\right] \tag{5.33}$$

and the ℓ integral in the second term gives (see Appendix III)

$$\frac{(p+k)^{-\varepsilon}}{64\pi^3\varepsilon}+O(1) \tag{5.34}$$

and so (5.31) reduces to

$$-\frac{1}{2\pi^3}\int\frac{d^d x\, d^d y}{(1+x^2)^3}\tilde{M}^{-1}(x,y)\int\frac{d^d p}{(2\pi)^d}e^{ip.y}p^{2-\varepsilon}f(p)$$

$$+\frac{3}{2\pi^3}\int\frac{d^d x\, d^d y}{(1+x^2)^3}\tilde{M}^{-1}(x,y)\left[\int\frac{d^d p}{(2\pi)^d}e^{ip.y}f(p)\right]^2+O(\varepsilon) \tag{5.35}$$

Now the Fourier transform of $f(p)$ is $48(1+x^2)^{-2}$ by definition, and from equation (5.25) (using $K_\nu(x) = K_{-\nu}(x)$) we find that the Fourier transform of $p^{2-\varepsilon}f(p)$ is $48.2^{2-\varepsilon}\Gamma(4-\varepsilon)\,.\,(1+x^2)^{-4+\varepsilon}$. Therefore (5.35) reduces to

$$\frac{2^6.3^2.5}{\pi^3}\int\frac{d^d x\, d^d y}{(1+x^2)^3}M^{-1}(x,y)\frac{1}{(1+y^2)^4}+O(\varepsilon) \tag{5.36}$$

To evaluate the integrals in (5.36) we use the identity

$$\int d^d z M(y,z)\frac{1}{(1+z^2)^3} = -\frac{6(8-d)}{(1+y^2)^4} \tag{5.37}$$

to reduce expression (5.36) to

$$-\frac{240}{\pi^3}\int\frac{d^d x}{(1+x^2)^6}+\frac{240}{\pi^3}\left[\int d^d x\,\frac{\phi_\lambda(x)}{(1+x^2)^3}\right]^2\left[\int d^d x\,[\phi_\lambda(x)]^2\right]^{-1}+O(\varepsilon) \tag{5.38}$$

Unlike the ϕ^4 case the second term in (5.38) is finite as $\varepsilon\to 0$, since the dilatation eigenfunction, $\phi_\lambda(x)$, in ϕ^3 theory has a finite norm in d=6. Using the explicit form (5.15) for $\phi_\lambda(x)$ we find that (5.38) equals $-15/4 + O(\varepsilon)$, a factor of $4/3$ different from the result (5.29) obtained by a calculation on the sphere. As we have already remarked we cannot find the origin of the discrepancy, however, it is amusing to note that the factor is just

$$\frac{\int d\Omega^{d+1}(1+\cos\theta)^{-1}}{\int d\Omega^{d+1}} = \begin{cases} 3/2 \text{ when } d=4 \\ 4/3 \text{ when } d=6 \end{cases} \tag{5.39}$$

As for the ϕ^4 case we will take the result obtained by the calculation on the sphere to be correct, since far fewer assumptions were involved in the derivation of this result.

Substituting the results (5.14), (5.17), (5.21), (5.24) and (5.30) into equation (5.9), we obtain

$$\underset{(\arg g=0)}{\mathrm{Im}\,\Gamma^{(M)}(q_i)} = C_b \int_0^\infty \frac{d\lambda}{\lambda}\, \lambda^{d-M(d-2)/2} \left(\frac{192\pi^3}{5g^2}\lambda^\varepsilon\right)^{\frac{d+1+M}{2}} \prod_{i=1}^{M}\left[q_i{}^2/\lambda^2\, \tilde{\phi}\left(q_i/\lambda\right)\right]$$

$$\times \exp -\frac{\lambda^\varepsilon}{g^2}\cdot\frac{192\pi^3}{5}\left\{ 1 - \varepsilon/2(\ell n\pi + \gamma + 1) + O(\varepsilon^2) \right\}$$

$$\times\,(1 + O(\varepsilon, g^2)) \tag{5.40}$$

where $$C_b = 2^{5/2}\pi^{-7/2}(2\pi)^{-9/10} \exp\left\{ \frac{18}{5\varepsilon} - \frac{527}{144} - \frac{9}{2}\gamma + \frac{11\zeta'(2)}{2\pi^2} - \frac{3\zeta'(4)}{2\pi^4} \right\} \tag{5.41}$$

This is the final result for the bare theory. To get a finite result we have to perform a coupling constant renormalisation (no wavefunction renormalisation is required at this order apart from that implicit in the coupling constant renormalisation). Since λ is the natural momentum scale in equation (5.40), we define the renormalised coupling constant at momentum scale λ by

$$\frac{\lambda^{\varepsilon}}{g^2} = \frac{1}{g_r^{\,2}(\lambda)} + \frac{3}{32\pi^3\varepsilon} \tag{5.42}$$

This result gives

$$\underset{(\arg g=0)}{\mathrm{Im}\Gamma_r^{(M)}(q_i)} = C_r \int_0^{\infty} \frac{d\lambda}{\lambda}\, \lambda^{d-M(d-2)/2} \left(\frac{192\pi^3}{5g_r^{\,2}(\lambda)}\right)^{\frac{d+1+M}{2}}$$

$$\times \exp - \frac{192\pi^3}{5g_r^{\,2}(\lambda)} \prod_{i=1}^{M} \left[\frac{q_i^{\,2}}{\lambda^2}\,\tilde{\phi}\left(q_i/\lambda\right)\right] \left(1 + O(\varepsilon, g_r^{\,2})\right) \tag{5.43}$$

where $C_r = C_b \exp\left(-\frac{18}{5\varepsilon} + \frac{9}{5}\left(\ell n\pi + \gamma + 1\right)\right)$

$$= 2^{8/5}\pi^{-13/5} \exp\left(\frac{11\zeta'(2)}{2\pi^2} - \frac{3\zeta'(4)}{2\pi^4} - \frac{27}{10}\gamma - \frac{1339}{720}\right) \tag{5.44}$$

In order to simplify the λ integration in equation (5.43) we introduce a fixed momentum scale, μ_0, and rewrite (5.43) in terms of $g_r(\mu_0)$, which is related to $g_r(\lambda\mu_0)$ by

$$\frac{1}{g_r^{\,2}(\mu_0\lambda)} = \frac{\lambda^{\varepsilon}}{g_r^{\,2}(\mu_0)} + \frac{3}{32\pi^3\varepsilon}\left(\lambda^{\varepsilon}-1\right) \tag{5.45}$$

We may now write

$$\underset{(\arg g=0)}{\mathrm{Im}\Gamma_r^{(M)}(q_i)} = C_r \int_0^{\infty} \frac{d\lambda}{\lambda}\, (\lambda\mu_0)^{d-M(d-2)/2} \left(\frac{192\pi^3\lambda^{\varepsilon}}{5g_r^{\,2}(\mu_0)} + \frac{18}{5}\,\ell n\lambda\right)^{\frac{d+1+M}{2}}$$

$$\times \exp - \left\{\frac{192\pi^3\lambda^{\varepsilon}}{5g_r^{\,2}(\mu_0)} + \frac{18}{5}\,\ell n\lambda\right\} \prod_{i=1}^{M} \left(\frac{q_i^{\,2}}{\mu_0^{\,2}\lambda^2}\right) \tilde{\phi}\left(q_i/\mu_0\lambda\right)$$

$$\times \left[1 + O(\varepsilon, g_r^{\,2})\right] \tag{5.46}$$

That the λ integral converges, follows from similar observations to those made in the ϕ^4 case (equation (4.80) et seq). In particular, ϕ^3 theory in six dimensions is asymptotically free for all real values of the coupling constant, and it is this that gives rise to the convergence of (5.46) for large λ when $\varepsilon=0$ (through the factor $\lambda^{-18/5}$).

Equation (5.46) together with equation (5.44) gives the renormalised imaginary part of the vertex functions in the one-loop approximation.

High-order Estimates

In this chapter we will again assume that the semiclassical approximation is able to pick out the nearest singularity to the origin in the Borel-transformed vertex functions and so enable us to obtain the high-order behaviour of the perturbation expansion for the vertex functions. To be specific we shall assume that the result (2.25) still holds near six dimensions, that is, the analytic properties of $\Gamma^{(M)}(q_i;g)$ in the complex g plane are such that we may write a dispersion relation which allows us to extract the leading behaviour of the coefficient of g^K for large K. The generalisations of (2.25) for the vertex functions are

$$\Gamma^{(M)}_{r,2K} \sim \frac{2}{\pi}\int_0^\infty \frac{\mathrm{Im}\Gamma_r^{(M)}(g';\arg g'=0)dg'}{(g')^{2K+1}} \quad , \quad M \text{ even}$$

and (5.47)

$$\Gamma^{(M)}_{r,2K+1} \sim \frac{2}{\pi}\int_0^\infty \frac{\mathrm{Im}\Gamma_r^{(M)}(g';\arg g'=0)dg'}{(g')^{2K+2}} \quad , \quad M \text{ odd}$$

The integration variable in equations (5.47), g', is taken to be the one-loop renormalised (minimally subtracted) coupling $g' = g_r(\mu_0)'$. Thus equation (5.46), when used in conjunction with equation (5.47) gives the coefficient of $\left[g_r(\mu_0)\right]^K$ in $\Gamma_r^{(M)}(q_i;g_r(\mu_0))$ to be

$$\Gamma^{(M)}_{r,L}(q_i) = \frac{C_r}{\pi}\left(\frac{5}{192\pi^3}\right)^{L/2}\Gamma\left(\frac{d+1+M}{2}+\frac{L}{2}\right)\mu_o^{d-M(d-2)/2}$$

$$\times\int_o^\infty \frac{d\lambda}{\lambda}\lambda^{(d-M(d-2)/2-\frac{\varepsilon L}{2}-18/5)}\prod_{i=1}^{M}\left[\frac{q_i^2}{\mu_o^2\lambda^2}\tilde{\phi}\left(q_i/\mu_o\lambda\right)\right]$$

$$\times\ (1+0(\varepsilon,\tfrac{1}{K})) \tag{5.48}$$

where L=2K if M is even and L=2K+1 if M is odd. Taking the limit $\varepsilon\to 0$ (and therefore for fixed K, $\varepsilon K\to 0$) we obtain the above result in six dimensions:

$$\Gamma^{(M)}_{r,L}(q_i) = \frac{C_r}{\pi}\left(\frac{5}{192\pi^3}\right)^{L/2}\Gamma\left(\frac{M+7}{2}+\frac{L}{2}\right)\mu_o^{6-2M}$$

$$\times\int_o^\infty \frac{d\lambda}{\lambda}\lambda^{6-2M-18/5}\prod_{i=1}^{M}\left[\frac{q_i^2}{\mu_o^2\lambda^2}\tilde{\phi}\left(q_i/\mu_o\lambda\right)\right]\left(1+0\left(\frac{1}{K}\right)\right) \tag{5.49}$$

We now wish to find the large L behaviour of $\Gamma^{(M)}_{R,L}(q_i)$, the coefficient of the Lth power of the fully renormalised coupling constant, g_R, in the fully renormalised vertex function $\Gamma^{(M)}_R(q_i)$. To do this we can proceed as in equation (4.86) et seq, and prove first of all that since the wavefunction renormalisation constant satisfies an equation of the type given by (4.89), then it has the same growth at high orders as the two point vertex function. That is (see equation (5.49)) $Z_{2K}\sim K!\,K^{7/2}$ where $Z(g) = \sum_K Z_{2K}g^{2K}$.

Next, we find that using this result, the coefficient of $g^{2K}(g^{2K+1})$ for M even (odd) in $Z^{M/2}(g)\Gamma^{(M)}(q_i;g)$ is

$$\begin{aligned}&\Gamma^{(M)}_{2K}+\frac{M}{2}Z_{2K}\Gamma^{(M)}_o+\frac{M}{2}Z_{2(K-1)}\Gamma^{(M)}_2+\ldots\ (\text{M even})\\ &\Gamma^{(M)}_{2K+1}+\frac{M}{2}Z_{2K}\Gamma^{(M)}_1+\frac{M}{2}Z_{2(K-1)}\Gamma^{(M)}_3+\ldots\ (\text{M odd})\end{aligned} \tag{5.50}$$

to leading order, and so for M>2 we may again ignore the effects of wavefunction renormalisation when working to leading order in large K.

To investigate the effects of coupling constant renormalisation, we define $\{\delta_{2K+1} : K = 1, 2, \ldots\}$, in analogy with equation (4.92), by

$$g_r(\mu_0) = g_R(\mu) - \delta_3 g_R{}^3(\mu) - \delta_5 g_R{}^5(\mu) - \ldots - \delta_{2K+1} g_R{}^{2K+1}(\mu) - \ldots \tag{5.51}$$

The equations corresponding to (4.94) are now

$$\Gamma^{(M)}_{R,2K} = -2\Gamma^{(M)}_{r,2}(\delta_{2K-1} - \delta_3\delta_{2K-3} + \ldots) - 4\Gamma^{(M)}_{r,4}(\delta_{2K-3} - \ldots)$$
$$+ \ldots + \Gamma^{(M)}_{r,2K}\, e^{-2\delta_3/a}\left(1 + O\left(\frac{1}{K}\right)\right) \;;\; \text{M even and M>2} \tag{5.52}$$

and for M odd

$$\Gamma^{(M)}_{R,2K+1} = -\Gamma^{(M)}_{r,1}(\delta_{2K+1}) - 3\Gamma^{(M)}_{r,3}(\delta_{2K-1} - \delta_3\,\delta_{2K-3} + \ldots)$$
$$- 5\Gamma^{(M)}_{r,5}(\delta_{2K-3} - \ldots) + \ldots + \Gamma^{(M)}_{r,2K+1}\, e^{-2\delta_3/a}\left(1 + O\left(\frac{1}{K}\right)\right) \tag{5.53}$$

where $a = 5/192\pi^3$. In the case M=3, we have by definition

$$\tfrac{1}{2}\Gamma^{(3)}_{R}(q_i;g)\Big|_{SP(\mu)} = g_R(\mu) \tag{5.54}$$

where the factor $\frac{1}{2}$ comes from the normalisation of g and $SP(\mu)$ means that the vertex function has been evaluated at momenta $q_i . q_j = \frac{1}{2}(3\delta_{ij} - 1)\mu^2$ where i,j = 1, 2, 3. Using equations (5.53) and (5.54) we find that

$$\delta_{2K+1} = \tfrac{1}{2}\Gamma^{(3)}_{r,2K+1}\Big|_{SP(\mu)} e^{-2\delta_3/a}\left(1 + O\left(\frac{1}{K}\right)\right) \tag{5.55}$$

and thus $\delta_{2K+1} \sim K!\, K^{9/2}$ for K large. Using this estimate we find from equations (5.52) and (5.53) that

$$\Gamma^{(M)}_{R,L} = \Gamma^{(M)}_{r,L}\, e^{-2\alpha/a}\left(1 + O\left(\frac{1}{K}\right)\right) \quad ; \quad M>3 \tag{5.56}$$

where as usual L=2K if M is even or L=2K+1 if M is odd, and where $\alpha=\delta_3$ is given by

$$g_r(\mu_o) = g_R(\mu) - \alpha g_R^3(\mu) + O(g_R^5) \tag{5.57}$$

To determine α we simply need to perform a one-loop calculation of $\Gamma^{(2)}$ and $\Gamma^{(3)}$. It is found that (see Appendix III)

$$\mu^{\varepsilon/2} g_R = g - \frac{3\mu^{-\varepsilon} g^3}{64\pi^3}\left(\frac{1}{\varepsilon} + \frac{31}{18} - \tfrac{1}{2}\gamma + \ln 2 + \tfrac{1}{2}\ln\pi + \frac{16\pi^2}{81} - \frac{8\psi'\left(\frac{1}{3}\right)}{27} + O(\varepsilon)\right) + O(g^5) \tag{5.58}$$

where here g means the bare coupling constant, and $\psi'\left(\frac{1}{3}\right) = \sum_{n=0}^{\infty} 9(3n+1)^{-2} =$ 10.0954 ... is the trigamma function with argument $^1/_3$. Elimination of g between equations (5.42) and (5.58) gives in six dimensions

$$\alpha = \frac{1}{128\pi^3}\left(\frac{31}{3} - 3\gamma + 6\ln 2 + 3\ln\pi - \frac{16}{9}\psi'\left(\frac{1}{3}\right) + \frac{32\pi^2}{27}\right) + \frac{3}{64\pi^3}\ln\left({}^{\mu_o}/_{\mu}\right) \tag{5.59}$$

and thus

$$\Gamma^{(M)}_{R,L} = \Gamma^{(M)}_{r,L}\left(\frac{\mu}{\mu_o}\right)^{18/5} \exp\left\{-\frac{31}{5} + \frac{9}{5}\gamma - \frac{18}{5}\ln 2 - \frac{9}{5}\ln\pi + \frac{16}{15}\psi'\left(\frac{1}{3}\right) - \frac{32\pi^2}{45}\right\}\left(1 + O\left(\frac{1}{K}\right)\right) \quad ; \quad M>3 \tag{5.60}$$

and together with equations (5.26) and (5.44) this finally gives

$$\Gamma^{(M)}_{R,L}(q_i) = \left(\frac{5}{192\pi^3}\right)^{L/2} \Gamma\left(\frac{M+7}{2} + \frac{L}{2}\right) \frac{\left(-2^2 3^{1/2} 5^{1/2} \pi^{3/2}\right)^M}{4\pi^{27/5}}$$

$$\times \exp\left\{ \frac{11\psi'(2)}{2\pi^2} - \frac{3\psi'(4)}{2\pi^4} - \frac{9}{10}\gamma - \frac{5803}{720} + \frac{16}{15}\psi'\left(\frac{1}{3}\right) - \frac{32}{45}\pi^2 \right\}$$

$$\times \mu^{6-2M} \int_0^\infty \frac{d\lambda}{\lambda} \lambda^{6-2M-18/5} \prod_{i=1}^{M} \left[\frac{q_i}{\lambda\mu} K_1\left({q_i}/{\lambda\mu}\right)\right]\left(1 + O\left(\frac{1}{K}\right)\right)$$

(5.61)

where it is understood that $M>3$.

We note from equation (5.61) that, as expected, the perturbation series does not oscillate at high orders and so it is not Borel summable. Cases in which ϕ^3 theories do give oscillating series at high order are discussed in the next section.

Finally in this section, we will obtain high order estimates for the renormalisation group β function in six dimensions. If we write

$$\beta(g_R) = \sum_K \beta_{2K+1} \left[g_r(\mu_o)\right]^{2K+1} \tag{5.62}$$

then from equations (4.106) and (5.54) we obtain

$$\beta_{2K+1} = \tfrac{1}{2}\mu \frac{d}{d\mu}\Bigg|_{\mu_o} \Gamma^{(3)}_{r,2K+1}(q_i)\Bigg|_{SP(\mu)} . \tag{5.63}$$

Using equation (5.49) this gives (after integration by parts)

$$\beta_{2K+1} = -\frac{9C_r}{5\pi}\left(\frac{5}{192\pi^3}\right)^{K+\frac{1}{2}} \Gamma(K+{11}/{2})\left(\frac{\mu_o}{\mu}\right)^{18/5}$$

$$\cdot \int_0^\infty x^{43/5}\left[\phi(x)\right]^3 dx\left(1 + O\left(\frac{1}{K}\right)\right) \tag{5.64}$$

To find the coefficient of $\left[g_R(\mu)\right]^{2K+1}$ in $\beta(g_R)$ we use the fact that the required coefficient, $\tilde{\beta}_{2K+1}$, satisfies (compare with equation (5.53)) $\tilde{\beta}_{2K+1} = \left(-\delta_{2K+1}\,\beta_1 + \beta_{2K+1}\,e^{-2\alpha/a}\right)\left(1 + O\left(\frac{1}{K}\right)\right)$, and since $\beta_1=0$ we may use equations (5.26), (5.44) and (5.59) to obtain

$$\tilde{\beta}_{2K+1} = K!\left(\frac{5}{192\pi^3}\right)^K K^{9/2}\,270\pi^{-12/5}\int_0^\infty x^{28/5}\,K_1{}^3(x)\,dx$$

$$\times \exp\left\{\frac{11\zeta'(2)}{2\pi^2} - \frac{3\zeta'(4)}{2\pi^4} - \frac{9}{10}\gamma - \frac{5803}{720} + \frac{16}{15}\psi'\left(\frac{1}{3}\right) - \frac{32}{45}\pi^2\right\}\left(1 + O\left(\frac{1}{K}\right)\right) \qquad (5.65)$$

The integral in equation (5.65) appears in the literature (Ragab 1956), but it does not seem to be expressible in terms of simple functions.

We will discuss the validity of the assumptions which lead to the results (5.61) and (5.65) in chapter seven.

Other ϕ^3 Interactions

In this section we will discuss a generalisation of the field theory defined by (5.1):

$$H(\phi) = \int d^dx\left[\tfrac{1}{2}(\nabla\phi_i)(\nabla\phi_i) + \frac{g}{3}\,d_{ijk}\phi_i\phi_j\phi_k\right] \qquad (5.66)$$

where $i,j,k = 1, 2, \ldots, n$ and summation over repeated indices is understood. Here d_{ijk} is a symmetric third rank invariant tensor of some symmetry group G. To find the instanton solutions for this theory we have to analyse the field equations obtained from equation (5.66), which are given by

$$\nabla^2\phi_i = g d_{ijk}\phi_j\phi_k \quad ; \quad i=1, 2, \ldots, n \qquad (5.67)$$

We may now appeal to the result proved in the final section of the previous chapter, namely, the instanton solution of least action has the form $\phi_i(x) = u_i\phi_c(x)$ where u_i is independent of x. Thus $\phi_c(x)$ is given by (5.2) where the vector u_i satisfies $u_i = d_{ijk}u_ju_k$.

We may now repeat the analysis of this chapter to find that the high order behaviour of the vertex functions is given by

$$\Gamma^{(M)}_{2K} \sim K! \left(\frac{5}{192\pi^3 u.u}\right)^K K^{\frac{M+r+5}{2}} C\left(1 + 0\left(\frac{1}{K}\right)\right) \quad , \quad M \text{ even}$$

and (5.68)

$$\Gamma^{(M)}_{2K+1} \sim K! \left(\frac{5}{192\pi^3 u.u}\right)^K K^{\frac{M+r+6}{2}} C'\left(1 + 0\left(\frac{1}{K}\right)\right) \quad , \quad M \text{ odd}$$

where r is the number of collective coordinates that have to be introduced due to the breaking of the internal symmetry by the instanton and c,c′ are numbers depending on the theory under study. In the case where G is discrete no collective coordinates for the internal symmetry are required and so r=0. An example of a theory of this kind is the n component Potts model (Potts 1952) where $G = S_{n+1}$ (Zia and Wallace 1975). If the internal symmetry is continuous (for example G = SU(n) (McKane et al 1976)), then r will be a positive integer depending on u_i.

The result (5.68) again shows the malignant growth found in the one component case (equation (5.61)). Oscillatory growth at high orders only seems possible if uu<0. Clearly this condition cannot be satisfied for real vectors, however such statements are common in theories where the limit $n \to 0$ has been taken. This suggests that ϕ^3 theories which involve taking the limit $n \to 0$ may have benign growth at high order. Two of the most studied theories in which this limit is taken are the percolation problem (which is the $n \to 0$ limit of the n component Potts model (Fortuin and Kasteleyn 1972)) and the Edwards-Anderson model of a spin glass (Edwards and Anderson 1975). For both these theories $u.u \sim n$ as $n \to 0$ and so the result (5.68) becomes meaningless as $n \to 0$ (Houghton et al 1978, Elder field and McKane (unpublished)). The problem for the percolation model has been overcome (Houghton et al 1978) by reformulating the whole theory so as to avoid the $n \to 0$ limit. The growth is found to be oscillatory at high orders.

Finally, we make two general comments concerning the calculation of the high order behaviour of ϕ^3 theories with internal symmetry. These theories are usually encountered in the theory of phase transitions, where one looks for infrared stable fixed points of the renormalisation group. If such fixed points exist in $6-\varepsilon$ dimensions then the theory is not asymptotically free in six dimensions. Thus firstly, the dilatation integral is not guaranteed to converge for large λ in six dimensions (compare with equation (5.46)). Secondly, in $6-\varepsilon$ dimensions all ϕ^3 theories are asymptotically free, thus, if in addition an infrared stable fixed point of order ε exists in $6-\varepsilon$ dimensions the integral (5.46) converges at both end points for $d<6$, even in the case $M=0$. Therefore the vacuum decay rate in such models is a finite calculable quantity.

Apart from these $n \to 0$ field theories, the perturbation expansion for field theories studied in this chapter is not Borel summable for real values of the coupling constant. However, if the coupling constant is pure imaginary, then the perturbation expansion is real (or pure imaginary) but has oscillating behaviour at large orders. Thus for example, the perturbation expansions in the reggeon field theory are Borel summable (Cardy 1977) and improved predictions for the critical exponents can be obtained. A more conventional ϕ^3 model with a pure imaginary coupling constant has recently been discussed by Fisher (1978) in connection with the Yang-Lee edge singularity in ferromagnets. In this case too, the high order estimates will enable critical exponents to be calculated more accurately.

CHAPTER SIX

ASYMPTOTIC BEHAVIOUR OF THE PERTURBATION EXPANSION AND VACUUM INSTABILITY IN ϕ^N FIELD THEORIES ($N > 4$)

In this chapter we shall generalise the work of the previous two chapters and study ϕ^N field theories ($N > 4$) near the dimension in which they are conformally invariant, that is, near $d_c = 2N/N-2$ dimensions. We shall find, as we would expect, that theories with N an even integer have a perturbation series which oscillates at high order, but theories with N an odd integer have a perturbation series which is not Borel summable. The case where N is even has been investigated by Lipatov (1977a,b) and Brézin et al(1977a) in $d = d_c$ dimensions, the theory being regularised by the Pauli-Villars method. We shall again employ dimensional regularisation by working in $d = d_c - \varepsilon$ dimensions. One of the major advantages is that mass renormalisations are effectively avoided and so one finds that the theories we are interested in here do not require renormalisation at the one-loop level. Since we will only perform a one-loop calculation of the imaginary part of the vertex functions, we will encounter no ultraviolet divergences at all, and so the calculation in these cases is somewhat simplified.

Calculation of the Imaginary Part of the Vertex Functions.

We begin, as usual, with the Euclidean action:

$$H(\phi) = \int d^d x \left[\tfrac{1}{2}(\nabla\phi)^2 + g/N\ \phi^N\right] \quad ; N > 4 \tag{6.1}$$

where $d = d_c - \varepsilon$, $\varepsilon > 0$. The instanton solutions with least action for this theory are given by eqn.(3.30):

$$\phi_c(x;x_0,\lambda) = \left[\frac{-8N}{g(N-2)^2}\right]^{1/N-2}\left[\frac{\lambda}{1+\lambda^2(x-x_0)^2}\right]^{2/N-2} \tag{6.2}$$

From our discussion of the saddle points for the corresponding "zero dimensional field theories" in Chapter two, we know that for the particular analytic continuation we choose to make, we have only to consider real saddle points. Thus it is understood by eqn.(6.2) that for N even there are two saddle points $\phi = \pm\phi_c$ given by (6.2) with $g < 0$, and for N odd there is only one saddle point which is positive for $g < 0$ and negative for $g > 0$.

Expanding $H(\phi)$ about ϕ_c we find

$$H(\phi_c+\phi) = H(\phi_c)+\tfrac{1}{2}\int d^dx\phi M\phi - \frac{4\varepsilon}{N-2}\left[\frac{-8N}{g(N-2)^2}\right]^{1/N-2}$$
$$\cdot\int d^dx\ \lambda\left[\frac{\lambda}{1+\lambda^2(x-x_0)^2}\right]^{N/N-2}\phi + \frac{g(N-1)(N-2)}{3!}\int d^dx\phi_c^{N-3}\phi^3$$
$$+\frac{g(N-1)(N-2)(N-3)}{4!}\int d^dx\phi_c^{N-4}\phi^4 + \ldots + g/N\int d^dx\phi^N \qquad (6.3)$$

where M is given by eqn.(3.72). We will see later that in the one-loop approximation the linear term does not contribute for $N > 4$. The form of the imaginary part of the vertex functions is found in exactly the same way as for ϕ^3 and ϕ^4. The generalisation of eqn.(4.31) is

$$\underset{(\arg g=\pi)}{\mathrm{Im}\Gamma^{(M)}(q_i)} = -\int d\lambda J^V \exp - H(\phi_c)C_1(2\pi)^{-(d+1)/2}\left|\frac{\det\tilde{V}}{\det Vo}\right|^{-\frac{1}{2}}$$
$$\prod_{i=1}^{M} q_i^2\int d^dx_i e^{iq_i\cdot x_i}\phi_c(x_i)\left[1+O(\varepsilon,g^{2/N-2})\right];\ \text{N even} \qquad (6.4)$$

and of eqn.(5.9) is

$$\underset{(\arg g=0)}{\mathrm{Im}\Gamma^{(M)}(q_i)} = \tfrac{1}{2}\int d\lambda J^V\exp-H(\phi_c)C_1(2\pi)^{-(d+1)/2}\left|\frac{\det\tilde{V}}{\det Vo}\right|^{\frac{1}{2}}$$
$$\prod_{i=1}^{M} q_i^2\int d^dx_i e^{iq_i\cdot x_i}\phi_c(x_i)\left[1+O(\varepsilon,g^{2/N-2})\right];\ \text{N odd} \qquad (6.5)$$

The signs of (6.4) and (6.5) together with the factor $\frac{1}{2}$ in eqn.(6.5) are, as usual, taken from the discussion of the corresponding "zero dimensional" models in chapter two. The factor C_1 represents the effect of the linear term.

As we have already mentioned we expect the theories for which $N > 4$ to be free of $1/\varepsilon$ poles at the one-loop level. In particular we expect

$$c_2 = \left|\frac{\det \tilde{V}}{\det V_o}\right|^{-\frac{1}{2}} \tag{6.6}$$

to be finite as $\varepsilon \to 0$. Thus we do not have to calculate the $O(\varepsilon)$ corrections to $H(\phi_c)$ in $d = d_c$ dimensions at this order. One easily finds

$$H(\phi_c) = \lambda^{\varepsilon} S_{d_c}\left[\frac{-8N}{g(N-2)^2}\right]^{2/N-2} \frac{2}{N-2} \frac{\Gamma^2(dc/2)}{\Gamma(d_c)}(1+O(\varepsilon)) \tag{6.7}$$

where S_{d_c} is the surface area of a d_c-dimensional sphere of unit radius.

The Jacobian J^V takes its usual form (see eqn.(4.27)) but with

$$\phi_\lambda \equiv \frac{\partial\phi c}{\partial\lambda} = 2\left[\frac{-8N}{g(N-2)^2}\right]^{1/N-2} \frac{\lambda^{(2/N-2)-1}}{N-2}\left\{\frac{1-\lambda^2(x-X_o)^2}{[1+\lambda^2(x-X_o)^2]^{N/N-2}}\right\} \tag{6.8}$$

and

$$\phi_\mu \equiv \frac{\partial\phi c}{\partial x_o^\mu} = -4\left[\frac{-8N}{g(N-2)^2}\right]^{1/N-2} \frac{\lambda^{(2/N-2)+1}}{N-2}\left\{\frac{\lambda(x-X_o)\mu}{[1+\lambda^2(x-X_o)^2]^{N/N-2}}\right\} \tag{6.9}$$

and therefore we find

$$J^V = \lambda^{d-1}\left\{\frac{8\lambda^{\varepsilon}}{(N-2)(3N-2)}\left[\frac{-8N}{g(N-2)^2}\right]^{2/N-2} S_{d_c}\frac{\Gamma^2(dc/2)}{\Gamma(dc)}\right\}^{d+1/2}(1+O(\varepsilon,g^{2/N-2})) \tag{6.10}$$

We notice that $H(\phi_c)$ and J^V have many factors in common and therefore for convenience we will introduce the quantity $\rho(N)$ defined by

$$\rho(N) = S_{d_c}\left[\frac{8N}{(N-2)^2}\right]^{2/N-2} \frac{2}{N-2} \cdot \frac{\Gamma^2(dc/2)}{\Gamma(dc)} \tag{6.11}$$

which implies

$$H(\phi_c) = \lambda^{\varepsilon}\left[-1/g\right]^{2/N-2}\rho(N)(1+O(\varepsilon)) \tag{6.12}$$

and

$$J^V = \lambda^{d-1}\left\{\frac{4\lambda^{\varepsilon}}{(3N-2)}\left[-1/g\right]^{2/N-2}\rho(N)\right\}^{d+1/2}(1+O(\varepsilon,g^{2/N-2})) \tag{6.13}$$

We define $\tilde{\Phi}(q)$ in an analogous way to (4.54) and (5.24):

$$\lambda^{-1-d/2}\left\{\rho(N)\left[-1/g\right]^{2/N-2}\lambda^{\varepsilon}\right\}^{\frac{1}{2}}\tilde{\Phi}(q/\lambda) \doteq \int d^d x e^{iq.x}\phi_c(x) \tag{6.14}$$

so that

$$\tilde{\Phi}(q) = \left\{\frac{2S_{d_c}\Gamma^2(dc/2)}{N-2\ \Gamma(dc)}\right\}^{-\frac{1}{2}}\int\frac{d^d x e^{iq.x}}{(1+x^2)^{2/N-2}} \tag{6.15}$$

and therefore using eqn.(5.25) we find

$$\tilde{\Phi}(q) = (-1)^N 4 \left\{ \frac{\Gamma(d_c)\pi^{d_c/2}}{2\Gamma^2(d_c/2)\Gamma(d_c/2-1)} \right\}^{\frac{1}{2}} (2\pi)^{-\varepsilon/2} K_{1-\varepsilon/2}(|q|)\,|q|^{-1+\frac{1}{2}\varepsilon} \tag{6.16}$$

where we have used $S_{d_c} = 2\pi^{d_c/2}/\Gamma(d_c/2)$ and $2/(N-2) = d_c/2 - 1$.

We can now use eqns.(6.6),(6.12),(6.13) and (6.14) to write eqns.(6.4) and (6.5) as

$$\begin{aligned} \underset{(\arg g=\pi)}{\mathrm{Im}\,\Gamma^{(M)}(q_i)} = {} & -C_1 C_2 \left[\frac{2}{\pi(3N-2)}\right]^{d_c+1/2} \int \frac{d\lambda}{\lambda}\, \lambda^{d-\frac{M}{2}(d-2)} \\ & \left[\lambda^{\varepsilon}\rho(N)(-1/g)^{2/N-2}\right]^{(d+1+M)/2} \exp-\left[\lambda^{\varepsilon}\rho(N)(-1/g)^{2/N-2}\right] \\ & \prod_{i=1}^{M} \frac{q_i^2}{\lambda^2}\, \tilde{\Phi}(q_i/\lambda)\left[1+0(\varepsilon, g^{2/N-2})\right]; \text{ N even} \end{aligned} \tag{6.17}$$

and

$$\begin{aligned} \underset{(\arg g=0)}{\mathrm{Im}\,\Gamma^{(M)}(q_i)} = {} & \frac{C_1 C_2}{2} \left[\frac{2}{\pi(3N-2)}\right]^{d+1/2} \int \frac{d\lambda}{\lambda}\, \lambda^{d-\frac{M}{2}(d-2)} \\ & \left[\lambda^{\varepsilon}\rho(N)(1/g)^{2/N-2}\right]^{(d+1+M)/2} \exp-\left[\lambda^{\varepsilon}\rho(N)(1/g)^{2/N-2}\right] \\ & \prod_{i=1}^{M} \frac{q_i^2}{\lambda^2}\, \tilde{\Phi}(q_i/\lambda)\left[1+0(\varepsilon, g^{2/N-2})\right]; \text{ N odd} \end{aligned} \tag{6.18}$$

where $\rho(N)$ is given by eqn.(6.11) and $\tilde{\Phi}(q)$ by eqn.(6.16). Thus it only remains to calculate C_1 and C_2.

We begin by considering C_1. The loop parameter now is $g^{2/N-2}$, and so using similar arguments to those presented in chapters four and five we deduce that to this order we need again only calculate the ultraviolet divergent part of the graph shown in Fig.14 in order to evaluate C_1, The graph gives a contribution

$$\frac{16N(N-1)\varepsilon}{(N-2)^2} \int \frac{d^d x\, d^d y}{(1+x^2)^{N/N-2}}\, \tilde{M}^{-1}(x,y) \frac{1}{(1+y^2)^{2(N-3)/N-2}}\, \tilde{M}^{-1}(y,y) \tag{6.19}$$

in flat space and

$$\frac{2N(N-1)\varepsilon}{(N-2)^2} \int d\Omega_1 d\Omega_2 K_1^{-1+\varepsilon/2} \tilde{V}^{-1}(n_1,n_2) K_2^{-\varepsilon/2}\, \tilde{V}^{-1}(n_2,n_2) \tag{6.20}$$

in spherical space. Let us consider the structure of (6.19) first. As usual we write $M^{-1}(y, y)$ as the sum of propagators with zero, one, two,... insertions of $4d_c(d_c + 1)(1 + x^2)^{-2}$. The loop integral corresponding to the case with zero insertions is zero in dimensional regularisation. However, the loop integral corresponding to one insertion is ultra-violet finite for $d_c < 4$ (see eqn.(4.64)) and therefore so are all the other integrals corresponding to more than one insertion. Thus the integrals in expression (6.19) give no $1/\varepsilon$ ultra-violet pole for $N > 4$ and so the whole expression is order ε which can be neglected at this order. We can also check this from eqn.(6.20). If we evaluate the integrals as in chapters four and five we are left with

$$\frac{-(N-1)\varepsilon}{4\Gamma(N+2)(N-2)} \sum_{L'\neq 1} \frac{(2L'+d-1)\Gamma(L'+d-1)}{\Gamma(L'+1)(L'+d/2+d_c/2)(L'+d/2-d_c/2-1)} \cdot (1+O(\varepsilon)) \tag{6.21}$$

For large L' we may expand the summand as follows:

$$\frac{(2L'+d-1)\Gamma(L'+d-1)}{\Gamma(L'+1)(L'+d/2+d_c/2)(L'+d/2-d_c/2-1)} = 2(L')^{d-3}\left[1+O(1/L)\right] \tag{6.22}$$

(compare with eqn.(4.40)) ; thus for L' large the sum may be written as

$$2\sum_{L'}\left\{\frac{1}{L'^{\varepsilon+3-d_c}} + \frac{Bo(\varepsilon)}{L'^{\varepsilon+4-d_c}} + \frac{B_1(\varepsilon)}{L'^{\varepsilon+5-d_c}} + \cdots\right\} \tag{6.23}$$

where $Bo(\varepsilon)$, $B_1(\varepsilon)$,... are polynomials in ε. Now for theories with $N > 4$, d_c is only an integer in one instance and that is when $N=6$, $d_c=3$. Since the Riemann zeta function, $\zeta(s)$, has a simple pole only when $s=1$, we see that it is only in this case that we can possibly pick up an ultra-violet divergence. We then find that the sum in (6.23) gives

$$\frac{2Bo(0)}{\varepsilon} + O(1) \qquad (N=6,\ d_c=3) \tag{6.24}$$

$Bo(0)$ is easily found to be zero from calculating the $O(L^{-1})$ correction in eqn.(6.22) for $d=d_c=3$. Therefore the sum in expression (6.21) gives no ultra-violet $1/\varepsilon$ pole for any $N > 4$ and thus the whole expression is of order ε. These observations mean that to this order the constant C_1 in eqns.(6.17) and (6.18) is equal to one.

We now come to the investigation of the term C_2 which arises from the small oscillations determinant. We expect it to be finite since massless theories with $N > 4$ need no renormalisation at the one-loop level. To see this we notice that one loop diagrams exist only for the vertex functions $\Gamma^{M(N-2)}$; $M = 1, 2, \ldots$. The one loop contribution to $\Gamma^{(N-2)}$ (Fig.16) has the form $\int d^d p/p^2$ which is zero in dimensional regularisation. The one loop contributions to $\Gamma^{M(N-2)}$; $M=2,3,\ldots$ are convergent for $d_c < 4$ $(N > 4)$ and so all vertex functions are finite at the one loop level.

The precise form of the small oscillations term can be read off from eqns. (3.78) and (3.82):

$$\left(\frac{\det \tilde{V}}{\det V_0}\right)^{-\frac{1}{2}} = \exp -\tfrac{1}{2} \sum_{L=0}^{\infty} \frac{\Gamma(L+d-1)(2L+d-1)}{\Gamma(L+1)\Gamma(d)} \ln\left\{\frac{(L+\frac{1}{2}d+\frac{1}{2}d_c)(L+\frac{1}{2}d-\frac{1}{2}d_c-1)}{(L+\frac{1}{2}d)(L+\frac{1}{2}d-1)}\right\} \tag{6.25}$$

or evaluating the L=0 mode and L = 1 modes of Vo explicitly

$$C_2 = \left(\frac{d_c-2}{4}\right)^{\frac{1}{2}}\left(\frac{d_c(d_c+2)}{4}\right)^{\frac{1}{2}(d_c+1)} \exp -\tfrac{1}{2} \sum_{L=2}^{\infty} \frac{\Gamma(L+d_c-1-\varepsilon)(2L+d_c-1-\varepsilon)}{\Gamma(L+1)\Gamma(d_c-\varepsilon)} \ln\left\{\frac{(L+d_c-\varepsilon/2)(L-1-\varepsilon/2)}{(L+\frac{1}{2}d_c-\varepsilon/2)(L+\frac{1}{2}d_c-1-\varepsilon/2)}\right\}\left(1+O(\varepsilon)\right) \tag{6.26}$$

We now have to check that for $N > 4$ the sum in (6.26) gives no $1/\varepsilon$ poles. Expanding the summand for large L we may rewrite the sum as (see eqn.(4.40) et seq)

$$\frac{-d_c(d_c+2)}{2\Gamma(d_c-\varepsilon)} \sum_{L=2}^{\infty} \left\{\frac{1}{L^{3-d_c+\varepsilon}} + \frac{C_0(\varepsilon)}{L^{4-d_c+\varepsilon}} + \frac{C_1(\varepsilon)}{L^{5-d_c+\varepsilon}} + \ldots \right\} \tag{6.27}$$

where as usual $C_0(\varepsilon)$, $C_1(\varepsilon), \ldots$ are polynomials in ε. Applying the same arguments as for the sum (6.23) we see that we only have to show that $C_0(0)=0$ for $d=d_c=3$. Using $\ln\{(L+3)(L-1)/(L+3/2)(L+\frac{1}{2})\} = -15/4L^2 + 15/2L^3 + O(1/L^4)$ this is easily verified. Thus we have to calculate only a pure number for each d_c from the sum in eqn.(6.26). This is easier said then done, since d_c is not now an even integer, and so we are unable to use the usual method and shift the summation variable independently for each of the logarithmic factors.

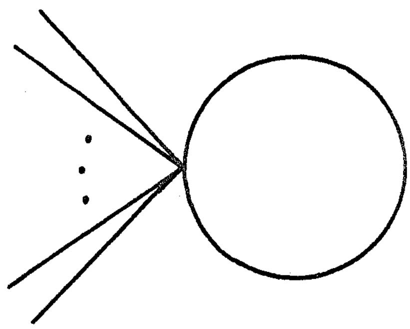

Figure 16. Graph contributing to $\Gamma^{(2N-2)}$ which vanishes in the massless theory in dimensional regularisation.

One could try to write an integral representation for either the logarithm or the degeneracy factor; however one is then left with integrals which seem equally intractable. We shall content ourselves with evaluating it for the case N=6 when d_c is an integer and some simplifications occur (see final section of this chapter).

High Order Estimates

As we have seen many times in this thesis the perturbation expansion for $g\phi^N$ theories behaves very differently at high orders in the two cases N even and N odd. Therefore in this section we will look at these two cases separately. We begin with N an even integer.

As usual we shall assume that the analyticity properties of the vertex functions are such that we can generalise eqn.(2.8) to eqn.(4.83). We shall write the dispersion relation in the dimensionless bare coupling constant $\bar{g}=g\mu^{-(N-2)\varepsilon/2}$. Using eqns.(4.83) and (6.17) we then find the coefficient of $(\bar{g})^K$ in $\Gamma^{(M)}(q_i;\bar{g})$ to be

$$\bar{\Gamma}_K^{(M)}(q_i) = \frac{C_2(N-2)}{2\pi}\left[\frac{2}{\pi(3N-2)}\right]^{(d_c+1)/2}(1/\rho(N))^{K(N-2)/2}\Gamma((d+1+M)/2+K(N-2)/2)$$

$$(-1)^K\int_0^\infty\frac{d\lambda}{\lambda}\lambda^{d-\frac{M}{2}(d-2)}(\mu/\lambda)^{\varepsilon K(N-2)/2}\prod_{i=1}^{M}\frac{q_i^2}{\lambda^2}\tilde{\phi}(q_i/\lambda)(1+O(\varepsilon,1/K)) \qquad (6.28)$$

As usual the integral converges for small λ, due to the exponential decrease of ϕ. For large λ the integral converges in d_c dimensions only if M > N, since for N > 4 we have no anomalous λ factors from the conventional one-loop coupling constant renormalisation. When M=N we find that the integral diverges logarithmically:

$$\int_0^\infty\frac{d\lambda}{\lambda}\lambda^{\varepsilon(N-2)/2}(\mu/\lambda)^{\varepsilon K(N-2)/2}\prod_{i=1}^{N}\frac{q_i^2}{\lambda^2}\tilde{\phi}(q_i/\lambda)(1+O(\varepsilon,1/K))$$

$$=\left\{\frac{8\Gamma(d_c)\pi^{d_c/2}}{\Gamma^2(d_c/2)\Gamma(d_c/2-1)}\right\}^{N/2}\mu^{\varepsilon(N-2)/2}\left[\frac{2}{\varepsilon(N-2)K}+\text{finite}\right](1+O(\varepsilon,1/K)) \qquad (6.29)$$

using eqn.(6.16). Thus we find that

$$\bar{\Gamma}_K^{(N)}(q_i) = \frac{C_2}{\pi K}\left[\frac{2}{\pi(3N-2)}\right]^{(d_c+1)/2}(1/\rho(N))^{K(N-2)/2}\Gamma\left(\frac{d+1+N}{2}+\frac{K(N-2)}{2}\right)$$

$$\times(-1)^K\left[\frac{8\Gamma(d_c)\pi^{d_c/2}}{\Gamma^2(d_c/2)\Gamma(d_c/2-1)}\right]^{N/2}\mu^{\varepsilon(N-2)/2}\,[1/\varepsilon+0(1)](1+0(1/K)) \tag{6.30}$$

The pole in ε comes from the divergence of the leading diagrams at $K^{\underline{th}}$ order. As M decreases from N these divergences get worse until in the case M=2 we find a quadratic divergence which vanishes in dimensional regularisation. The divergences are removed by performing $K^{\underline{th}}$ loop renormalisations;wavefunction and coupling constant renormalisation in the cases M=2 and M=N and for $2 < M < N$ we choose the fully renormalised vertex functions to be identically zero.

We can obtain the high order behaviour of the β function quite simply since we can check that wave-function renormalisation can again be ignored to leading order in K. Thus

$$\beta_K = -\frac{1}{(N-1)!}\,\mu\frac{d}{d\mu}\Gamma_K^{(N)}(q_i)\Big|_{SP(\mu)}$$

$$= -\frac{1}{(N-1)!}\,\mu\frac{d}{d\mu}\left[\bar{\Gamma}_K^{(N)}(q_i)\Big|_{SP(\mu)}\,\mu^{-K\varepsilon(N-2)/2}\right] \tag{6.31}$$

where β_K is the coefficient of g^K in $\beta(g_R)$. Therefore from eqn.(6.30). we obtain

$$\beta_K = \frac{C2(N-2)}{2\pi\Gamma(N)}\left[\frac{2}{\pi(3N-2)}\right]^{d_c+1/2}\left[\frac{8\Gamma(d_c)\pi^{d_c/2}}{\Gamma^2(d_c/2)\Gamma(d_c/2-1)}\right]^{N/2}\mu^{\varepsilon(N-2)(1-K)/2}$$

$$\times(-1)^K\left(\frac{1}{\rho(N)}\right)^{K(N-2)/2}\Gamma((d+1+N)/2+K(N-2)/2)(1+0(\varepsilon,1/K)) \tag{6.32}$$

The coefficient of $(\bar{g})^K$ in the β function, which we denote by $\bar{\beta}_K$, is of course related to β_K in a trivial way:

$$\bar{\beta}_K = \beta_K\,\mu^{\varepsilon(N-2)K/2} \tag{6.33}$$

It is straightforward to check that the coefficient of $(g_R)^K$ in $\beta(g_R)$, $\widetilde{\beta}_K$, is related to $\bar{\beta}_K$ by $\widetilde{\beta}_K = \bar{\beta}_K(1+0(1/K))$ in $d = d_c$ dimensions if $N > 4$. Thus we may write

$$\tilde{\beta}_K = \frac{C_2(N-2)}{2\pi\Gamma(N)}\left[\frac{2}{\pi(3N-2)}\right]^{(d_c+1)/2}\left[\frac{8\Gamma(d_c)\pi^{d_c/2}}{\Gamma^2(d_c/2)\Gamma(d_c/2-1)}\right]^{N/2}$$

$$\left\{\left[\frac{-8N}{(N-2)^2}\right]^K\left[\frac{(N-2)\Gamma(d_c)}{4\pi^{d_c/2}\Gamma(d_c/2)}\right]^{K(N-2)/2}\right\}\Gamma\left(\frac{(d_c+1+N)}{2}+\frac{K(N-2)}{2}\right)(1+O(1/K)) \quad (6.34)$$

for $N > 4$. This equation gives the required high order behaviour of the β function. We will evaluate it explicitly in the case N=6 in the final section of this chapter.

The case where N is an odd integer can be similarly investigated. This time we use eqns.(5.47) and (6.18) in order to calculate the asymptotic behaviour of the coefficient of $(\bar{g})^L$ in $\Gamma^{(M)}(q_i;\bar{g})$. We find

$$\bar{\Gamma}_L^{(M)}(q_i) = \frac{-C_2(N-2)}{2\pi}\left[\frac{2}{\pi(3N-2)}\right]^{d_c+1/2}\left(\frac{1}{\rho(N)}\right)^{L(N-2)/2}\Gamma\left(\frac{(d+1+M)}{2}+\frac{L(N-2)}{2}\right)$$

$$\int_0^\infty \frac{d\lambda}{\lambda}\,\lambda^{d-M(d-2)/2}\,(\mu/\lambda)^{\varepsilon L(N-2)/2}\prod_{i=1}^{M}\frac{q_i^{\,2}}{\lambda^2}\phi(q_i/\lambda)(1+O(\varepsilon,1/K)) \quad (6.35)$$

where L=2K if M is even and L=2K+1 if M is odd. Notice that this has almost exactly the same form as the case where N is even (eqn.(6.28)); the only difference being the $(-1)^K$ factor that appears in theories with an even number of interactions. It follows that the asymptotic structure of the β function has the form (6.34), again with the appropriate minus signs absent.

High Order Estimate for the β function in ϕ^6 theory.

In this final section of this chapter we will evaluate the constant C_2 defined by eqn.(6.26) in the special case N=6. This will enable us to obtain an explicit form for $\tilde{\beta}_K$ from eqn.(6.34).

For ϕ^6 theory $d_c=3$ and so eqn.(6.26) becomes

$$C_2 = \tfrac{1}{2}\left(\frac{15}{4}\right)^2\exp-\tfrac{1}{2}\sum_{L=2}^{\infty}\frac{\Gamma(L+2-\varepsilon)(2L+2-\varepsilon)}{\Gamma(L+1)\Gamma(3-\varepsilon)}$$

$$\ln\left\{\frac{(L+3-\varepsilon/2)(L-1-\varepsilon/2)}{(L+3/2-\varepsilon/2)(L+\frac{1}{2}-\varepsilon/2)}\right\}(1+O(\varepsilon)) \quad (6.36)$$

The difficulty with evaluating C_2 when $2d_c$ is not an integer is that we cannot shift the summation variable for all the sums. In this case we can perform the shift for two of the sums and so we do these first. Defining

$$S_1 = \sum_{L=2}^{\infty} \frac{\Gamma(L+2-\varepsilon)(2L+2-\varepsilon)}{\Gamma(3-\varepsilon)\Gamma(L+1)} \ln\{L+3-\varepsilon/2)(L-1-\varepsilon/2)\} \tag{6.37}$$

we find using the method described in chapters four and five

$$S_1 = \frac{1}{\Gamma(3-\varepsilon)}\{-4\zeta'(\varepsilon-2)-16\zeta'(\varepsilon)-8\varepsilon\zeta(1+\varepsilon)+\frac{149}{6}\varepsilon^2\ \zeta'(1+\varepsilon)+0(\varepsilon)\}-\ln 3-4\ln 4$$

$$= -2\zeta'(-2)-8\zeta'(0)-\frac{197}{12}-\ln 3-4\ln 4+0(\varepsilon) \tag{6.38}$$

We now investigate

$$S_2 = -\sum_{L=2}^{\infty} \frac{\Gamma(L+2-\varepsilon)(2L+2-\varepsilon)}{\Gamma(3-\varepsilon)\Gamma(L+1)} \ln\{(L+3/2-\varepsilon/2)(L+\tfrac{1}{2}-\varepsilon/2)\} \tag{6.39}$$

Expanding the logarithm in powers of L+1 we find

$$S_2 = -2\sum_{L=2}^{\infty} \frac{\Gamma(L+2-\varepsilon)(2L+2-\varepsilon)}{\Gamma(3-\varepsilon)\Gamma(L+1)} \ln(L+1)$$

$$+\sum_{L=2}^{\infty} \frac{\Gamma(L+2-\varepsilon)(2L+2-\varepsilon)}{\Gamma(3-\varepsilon)\Gamma(L+1)}\left\{\frac{\varepsilon}{L+1}+\frac{\varepsilon}{4(L+1)^3}+\varepsilon 0(1/L^5)\right\}$$

$$+2\sum_{L=2}^{\infty} \frac{\Gamma(L+2-\varepsilon)(2L+2-\varepsilon)}{\Gamma(3-\varepsilon)\Gamma(L+1)}\left\{\frac{(\frac{1}{2})^2}{2(L+1)^2}+\frac{(\frac{1}{2})^4}{4(L+1)^4}+\ldots\right\}+0(\varepsilon) \tag{6.40}$$

To evaluate the first two sums and the first term of the third sum we use

$$\frac{\Gamma(L+2-\varepsilon)(2L+2-\varepsilon)}{\Gamma(L+1)} = (2(L+1)^2-3\varepsilon(L+1)+\varepsilon^2)\frac{\Gamma(L+1-\varepsilon)}{\Gamma(L+1)} \tag{6.41}$$

and so we find

$$S_2 = 2\zeta'(-2) + 8\ln 2+\tfrac{1}{4}\zeta(0) - \frac{7}{12}$$

$$+2\sum_{L=2}^{\infty}(L+1)^2\sum_{n=2}^{\infty}\frac{(\frac{1}{2})^{2n}}{2n(L+1)^{2n}} + 0(\varepsilon) \tag{6.42}$$

Thus from eqns.(6.38) and (6.42) we have

$$S_1 + S_2 = -8\zeta'(0) + \tfrac{1}{4}\zeta(0)-\frac{33}{2}+4\ln 3-18\ln 2+4\ln 5$$

$$+2\sum_{m=1}^{\infty}\sum_{n=2}^{\infty}\frac{(\frac{1}{2})^{2n}}{2n\ m^{2n-2}} + 0(\varepsilon) \tag{6.43}$$

The sum in eqn.(6.43) is

$$\sum_{n=2}^{\infty} \frac{(\frac{1}{2})^{2n}}{2n} \sum_{m=1}^{\infty} \frac{1}{m^{2n-2}} = \sum_{n=2}^{\infty} \frac{(\frac{1}{2})^{2n}\zeta(2n-2)}{2n} \tag{6.44}$$

We can write this sum in terms of an integral as follows. Define F(a) by

$$F(a) = \sum_{n=2}^{\infty} \frac{\zeta(2n-2)a^{2n}}{2n} = \frac{1}{2}\left(\sum_{m=4}^{\infty} \frac{\zeta(m-2)(-a)^m}{m} + \sum_{m=4}^{\infty} \frac{\zeta(m-2)a^m}{m}\right) \tag{6.45}$$

with $|a|<1$. Then using (Erdélyi 1955)

$$\zeta(m-2) = \frac{1}{(m-3)!} \int_0^{\infty} \frac{x^{m-3}dx}{e^x-1} \tag{6.46}$$

we have

$$F'(a) = \frac{a^2}{2} \int_0^{\infty} \frac{e^{ax}-e^{-ax}}{e^x-1}\, dx = \frac{a}{2} - \frac{\pi a^2}{2} \cot(\pi a) \tag{6.47}$$

and thus we find

$$F(\tfrac{1}{2}) = \sum_{n=2}^{\infty} \frac{\zeta(2n-2)(\frac{1}{2})^{2n}}{2n} = \frac{1}{16} - \frac{1}{2\pi^2} \int_0^{\pi/2} y^2 \cot y\, dy \tag{6.48}$$

So finally eqns.(6.36), (6.37), (6.39), (6.43) and (6.48) give

$$C_2 = \frac{4}{\pi^2} \exp\left\{ \frac{17}{2} + \frac{1}{2\pi^2} \int_0^{\pi/2} y^2 \cot y\, dy \right\}(1+O(\varepsilon)) \tag{6.49}$$

where we have used $\zeta(0) = -\frac{1}{2}$, $\zeta'(0) = -\frac{1}{2}\ln(2\pi)$ (Erdélyi 1955).

Let us now return to eqn.(6.34) in the case N=6:

$$\tilde{\beta}_K = (2K)! \left(\frac{-16}{3\pi^4}\right)^K K^4 \frac{C2}{2^8.3.5.\pi^2} (1+O(1/K)) \tag{6.50}$$

and so using eqn.(6.49) we find

$$\tilde{\beta}_K = (2K)! \left(\frac{-16}{3\pi^4}\right)^K K^4 \frac{1}{960\pi^4} \exp\left\{ \frac{17}{2} + \frac{1}{2\pi^2} \int_0^{\pi/2} y^2 \cot y\, dy\right\}(1+O(1/K)) \tag{6.51}$$

Since no new singularities appear in ϕ^6 theory in three dimensions (see chapter seven) we expect the result (6.51) to be a correct estimate of the asymptotic behaviour of $\tilde{\beta}_K$, in contrast with the corresponding results for ϕ^3 in d = 6 and ϕ^4 in d = 4 dimensions.

CHAPTER SEVEN

SUMMARY AND DISCUSSION

In this final chapter we will briefly summarise the methods and results of the previous three chapters paying particular attention to the assumptions we have made. We make the point that all the calculations exhibited in this thesis, and other calculations of a similar nature, require two main assumptions:

1. The imaginary part of the partition function, Green's function, etc. generated by vacuum decay, is characterised by non-trivial classical structures (the instantons), and this enables the imaginary part to be calculated by a semiclassical approximation.
2. The analytic properties of the partition function, Green's function, etc. in the coupling constant, are such that we may write a contour integral that relates the imaginary part of the function for unphysical values of the coupling constant to the real perturbative part of the function for physical values of the coupling constant. Examples are eqns.(2.8),(2.24), (4.81) and (5.47).

As we have remarked previously, such assumptions which lie outside the realm of perturbation theory, are difficult to investigate for higher dimensional field theories, nevertheless we will attempt to present some current views on the validity of assumption 1 in the final section of this chapter.

Summary of the Instanton Calculations in ϕ^N theory.

In this thesis we have calculated the imaginary part of vertex functions associated with tunnelling in ϕ^N field theories at or near $d_c = 2N/N-2$ dimensions, in the semiclassical approximation. As discussed in chapter three we chose to work in d_c dimensions since in these cases analytic forms may be obtained for the instanton solutions, and indeed the whole of the calculation of the imaginary part may be performed analytically. This calculation of the imaginary part together with suitable analyticity assumptions (see assumption 2 above) enabled us to estimate the asymptotic behaviour of vertex functions etc. as functions of the coupling constant.

In the case where N is even, these results are not new - they were previously obtained by Lipatov (1977a,b) and Brézin et al (1977a). The new feature of our calculation was the use of dimensional regularisation to control the ultra-violet divergences of d_c dimensions. A major aspect was that even in the regularised theory ($d \neq d_c$) it was adequate to use the instanton solution of d_c dimensions. Since this field configuration was not a solution in $d(\neq d_c)$ dimensions, one was able to obtain additional insight into the Jacobian factor when collective coordinates were introduced.

In ϕ^3 and ϕ^4 theories it was necessary to perform a one-loop coupling constant renormalisation. Another advantage of our regularisation scheme was in evidence here: the direct appearance of the dimensionally regularised, renormalised, running coupling constant; instantons of scale size $1/\lambda$ having an effective coupling $g_R(\lambda)$. The asymptotic freedom of the theories in $d = d_c$ implied the convergence of the dilatation integral over instanton scale sizes $\lambda \to \infty$, and for fixed external momenta there was exponential convergence for $\lambda \to 0$.

Finally, we mention a perhaps unimportant feature, that is, the evaluation of the contribution of the one extra diagram (Fig.14) in ϕ^3 and ϕ^4 theories. The calculation can be completely controlled by working on the sphere, and it is this result which ensures agreement with the calculations of Lipatov and Brézin et al in the ϕ^4 case. We feel the graph should also be directly calculable in flat space, however we have not been able to elucidate the discrepancy discussed in chapters four and five.

We will now go on to give a very brief discussion of the results we obtained. The form of the asymptotic estimates is familiar from the work of Lipatov and Brézin et al. We merely re-emphasise the difference between ϕ^N theories with N even and those with N odd. The former have a perturbation series which oscillates at high order (for positive coupling constant) and we say it has a benign growth, whereas the latter have a perturbation series in which all the terms have the same sign at high order (for real coupling constant) and we say it has a malignant growth. As discussed at the end of chapter five, not all ϕ^3 models are diseased in this way, in particular those models with an imaginary coupling constant have a perturbation series with benign growth at high order.

The Appearance of Renormalon Singularities in d_c dimensions.

In this section we will give some arguments (Lautrup 1977, 't Hooft 1977b, Parisi 1977c) which indicate that assumption 1 may break down in d_c dimensions, that is, new singularities in the Borel-transformed Green's functions appear which are not associated with the instanton solutions of the theory.

We begin by recalling the results of Bender and Wu (1976)(see also Parisi (1977b)), who, from a statistical treatment of Feynman diagrams, found that the $[K(N-1)]!$ growth of the K^{th} term in the perturbation series in ϕ^{2N} theories came from the number of diagrams at this order, while individual diagrams grew no faster than C^K where C is a constant. Thus the instanton singularity comes from the number, and not the individual magnitude, of Feynman diagrams. However, it has been pointed out (Lautrup 1977, 't Hooft 1977b) that in d_c dimensions a new type of singularity appears which comes from the individual magnitude, and not the number, of Feynman diagrams. These new singularities cannot be analysed by semi-classical methods; as we shall see they are a short distance phenomena which appear when the theory is just renormalisable. For this reason 't Hooft (1976b) has named them "renormalons".

To investigate the nature of the renormalon singularities it is again convenient to study ϕ^N theories with N even and N odd separately. Let us first of all consider the case when N is even, where an example of a diagram contributing to $\Gamma_K^{(M)}(q_i)$ which has an integral proportional to K! can be constructed by examining the high momentum behaviour of the diagram shown in Fig.17. With large external momentum p the diagram with L loops grows like $(C\ln p^2)^{L(N-1)}$, where C is a constant. Thus if we (for example) insert this diagram into the relevant diagram shown in Fig.18 we obtain integrals of the form

$$I \sim \int \frac{d^{d_c}p}{(p^2 + m^2)^n} (\ln p^2)^K \qquad \text{for } K \to \infty,\ p^2 \to \infty \tag{7.1}$$

where n is an integer. We have neglected external momenta since we are interested in the large internal momentum region of the integral. For large p

$$I \sim \int \frac{dx}{x} x^{\frac{N}{N-2} - n} (\ln x)^K \sim \int dy\, y^K e^{-(n-N/N-2)y}$$

and thus

$$I \sim K!a^K \text{ for large K} \tag{7.2}$$

Notice that in $d = d_c - \varepsilon$ dimensions where the theory is super-renormalisable this K! growth is no longer found:

$$I \sim \int \frac{d^{d_c-\varepsilon}p}{(p^2 + m^2)^n} p^{-\varepsilon K(N-2)/2} \quad \text{for } K \to \infty,\ p^2 \to \infty \tag{7.3}$$

$$\sim \int dx\, x^{\frac{N}{N-2} - n - 1 - \varepsilon K(N-2)/4}$$

$$\sim \frac{1}{\frac{N}{N-2} - n - \varepsilon K(N-2)/4} \tag{7.4}$$

In summary, we may say that when the theory becomes just renormalisable new short distance (high momenta) effects appear which give rise to a K! growth for $\Gamma_K^{(M)}$ not previously encountered. Now in a super-renormalisable theory we know that at high momentum we approach a free field theory and so we expect no anomalous high momentum behaviour. However, if the theory

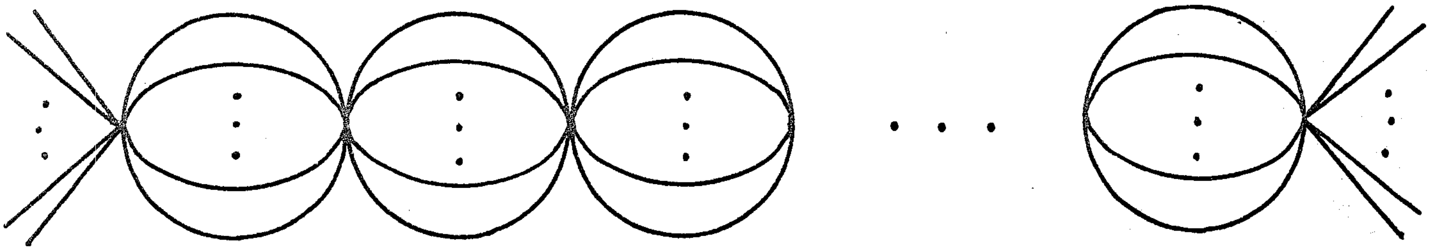

Figure 17. The "chain of bubbles" in ϕ^N field theory (N even). Each "bubble" contains N/2 - 1 loops.

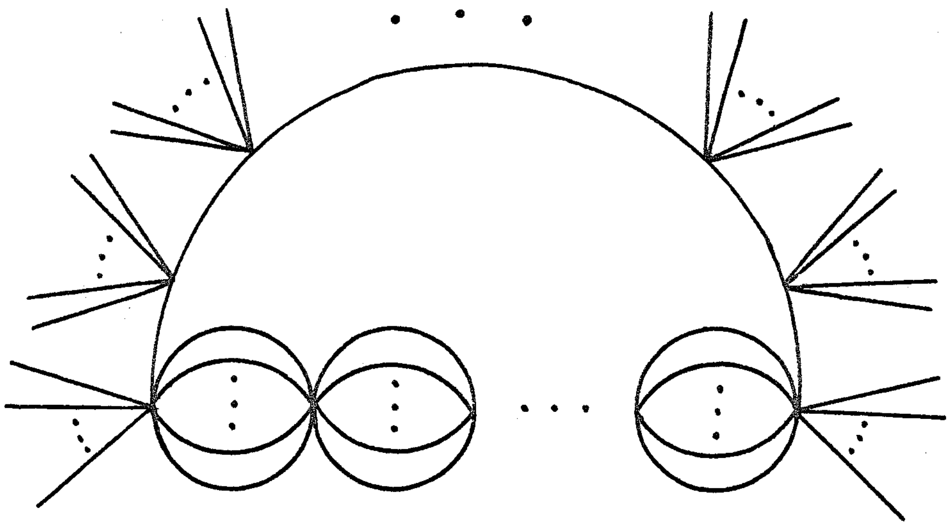

Figure 18. Example of a graph in ϕ^N theory (N even) which contributes to $\Gamma_K^{(M)}$ and has K! growth.

is just renormalisable, then there is no guarantee that the theory at high momentum and at low momentum are the same. In fact it is well known (Landau 1955) that in the large momentum region of these theories, one finds a pole in the N point vertex function, commonly referred to as the Landau ghost. One can easily check that in the leading logarithm approximation, in the $1/n$ expansion etc., the pole occurs at momentum p given by

$$p^2 = \mu^2 e^{2/\beta_2 g} \tag{7.5}$$

Here μ is a momentum scale and β_2 is the coefficient of g^2 in the β function.

We know that the renormalon, or Landau ghost, gives the asymptotic growth shown in (7.2), however, the sign and magnitude of a is crucial if we are to know whether or not renormalons dominate the high order estimates obtained from the semiclassical calculations. The value of a is easily obtained if we follow Parisi (1978) and solve the Borel-transformed renormalisation group equation to lowest order:

$$[-p\frac{\partial}{\partial p} + \beta_2 b + d_M]\hat{\Gamma}^{(M)}(p,\mu;b) = 0 \tag{7.6}$$

where $\hat{\Gamma}^{(M)}$ is the Borel transform of $\Gamma^{(M)}$ (which has naive dimension d_M). Solving eqn.(7.6) for large p gives

$$\hat{\Gamma}^{(M)}(p,\mu;b) = f^{(M)}(b)p^{d_M}(p/\mu)^{\beta_2 b} \tag{7.7}$$

which clearly shows the anomalous $(P/\mu)^{\beta_2 b}$ term which appears in the large p region. To find the form of singularities produced, consider (for example) the contribution to $\Gamma_K^{(n(N-2))}$ shown in Fig.19. In the large momentum region we get a contribution

$$\int \frac{d^d p}{(p^2+m^2)^n}[\Gamma^{(N)}(p_i;g)]^n \tag{7.8}$$

We may take the Borel transform of eqn.(7.8) by using the convolution theorem (Parisi 1978)

$$B[F_1 \dots F_n] = \int_0^b \prod_{i=1}^n db_i f_i(b_i)\ \delta(\sum_{i=1}^n b_i - b) \tag{7.9}$$

where the Borel transform of F_i is $B[F_i] = f_i$. Equations (7.7) and (7.9) give the Borel transform of expression (7.8) as

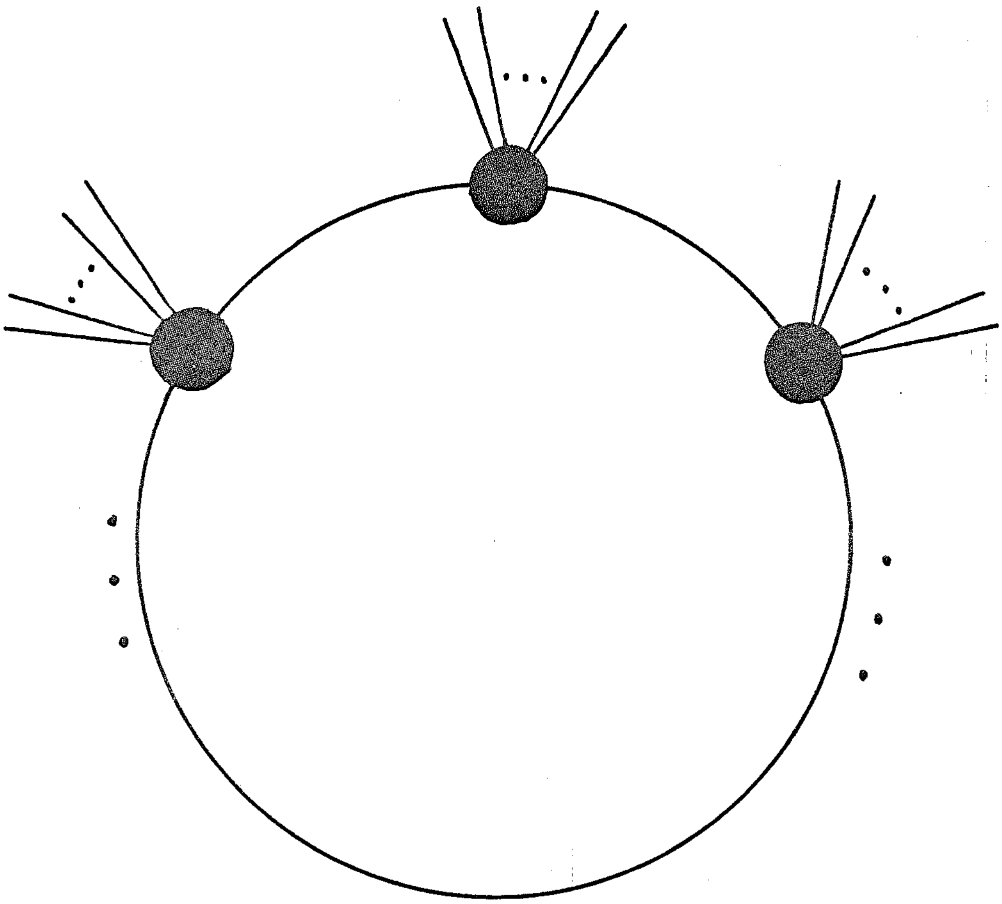

Figure 19. Graph giving a contribution to $\Gamma_K^{(n(N-2))}$ in ϕ^N field theory (N even). The shaded areas represent vertex insertions.

$$f(b)\int \frac{d^{d_c}p}{(p^2+m^2)^n}\,(p/\mu)^{\beta_2 b} \tag{7.10}$$

We see that the integral in (7.10) is divergent for $\beta_2 b + d_c - 2n \geqslant 0$, that is, the Borel transform of $\Gamma^{(M)}$ has singularities for

$$\beta_2 b \geqslant \frac{2(M-N)}{N-2} \qquad ; \; M > N \tag{7.11}$$

For all ϕ^N theories (N even) $\beta_2 > 0$ and thus the theories are not asymptotically free. This means that the renormalon singularities given by eqn.(7.11) are on the positive real axis of the b plane thereby casting doubt on the Borel summability of the theory. For ϕ^4 theory (N = 4) they occur at $b = 2/\beta_2, 4/\beta_2, 6/\beta_2, \ldots..$ and thus the constant a in eqn.(7.2) is given by $\beta_2/(M-4)$ in $\Gamma^{(M)}$. Since for one component ϕ^4 theory $\beta_2 = 9/8\pi^2$ (with our normalisation) we have, in for instance $\Gamma_K^{(6)}$, the asymptotic behaviour $K!(9/16\pi^2)^K$ from the renormalons compared with $K!(-3/8\pi^2)^K$ from the instantons. Clearly if these extra singularities are present the earlier estimates have to be discarded. However the "derivation" we have just outlined is far from satisfactory and it is possible that some of the diagrams producing the singularity are cancelled by other diagrams in some way not yet totally clear.

An interesting aspect of these new singularities is that we do not get a $[(N-2/2)K]!$ growth in ϕ^N. Instead we find only a K! growth. Recalling the definition of the generalised Borel transform relevant to these theories (see chapter 1), we would not expect renormalons to produce extra singularities. One can easily check that this is so and therefore we may (effectively) say that ϕ^6, ϕ^8, ... theories are free from renormalon singularities.

We now go on to consider renormalons in ϕ^N theories where N is odd. The structure of the diagrams we have to study is quite different. For example, the chain of bubbles shown in Fig.17does not exist in these theories. Instead we consider the dressed propagator shown in Fig.20. Its high

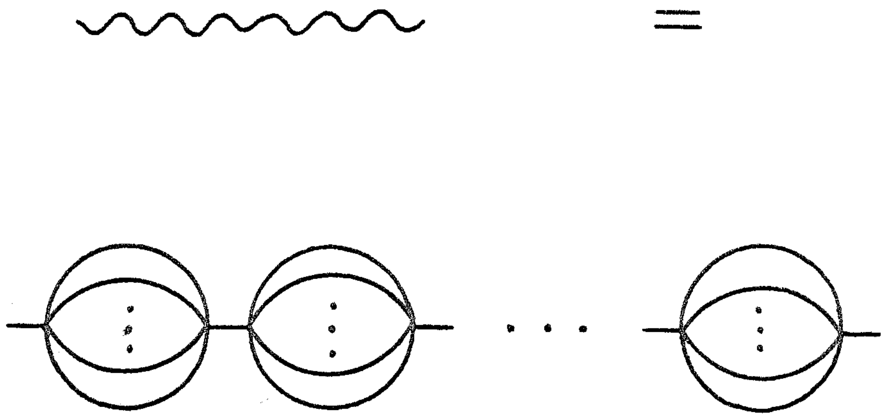

Figure 20. Chain of self-energy "bubbles" in ϕ^N field theory (N odd). Each "bubble" contains N-2 loops.

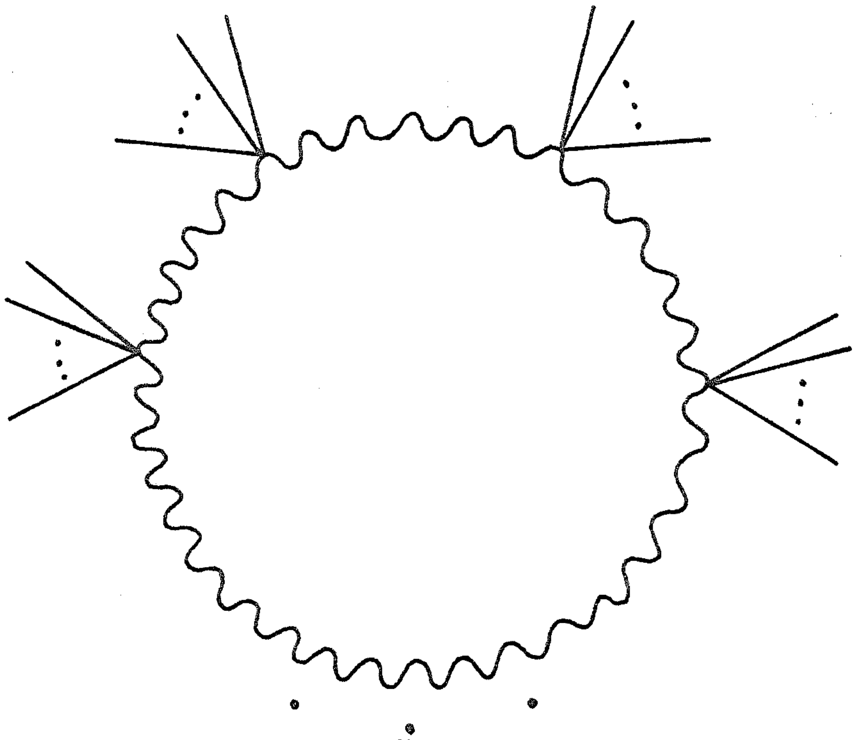

Figure 21. Example of a graph in ϕ^N theory (N odd) which contributes to $\Gamma_{2K}^{(M)}$ or $\Gamma_{2K+1}^{(M)}$ and has K! growth.

momentum behaviour is given by $p^{-2}(C\ln p^2)^{L/N-2}$, where C is a constant and L is the number of loops. Therefore if we use this propagator as for example shown in Fig.21 we get contributions of the form

$$\int \frac{d^{d_c}p}{(p^2)^{M/(N-2)}} (\ln p^2)^K \quad \text{for } K \to \infty,\ p^2 \to \infty \tag{7.12}$$

to $\Gamma_{2K}^{(M)}$ or $\Gamma_{2K+1}^{(M)}$. The large momentum region of the integral in (7.12) gives a K! growth just as before.

The expression corresponding to (7.5) is

$$p^2 = \mu^2 e^{1/\beta_3 g^2} \tag{7.13}$$

and a similar analysis of the Borel transform of $\Gamma^{(M)}$ to that carried out for theories with an even number of interactions, reveals an anomalous term of the form $(p/\mu)^{2\beta_3 b}$. This leads to new singularities in the Borel-transformed vertex functions which in ϕ^3 theory occur at $b = 1/\beta_3, 2/\beta_3, \ldots$ For one component ϕ^3 theory $\beta_3 < 0$, that is, the theory is asymptotically free in six dimensions, and so these are left-hand-side singularities. In ϕ^N theories (N odd, N > 3) the K! growth coming from the renormalons is totally insignificant compared with the $[(N-2)K]!$ growth from the instantons, and so no new singularities appear in d_c dimensions.

We may summarise the results of this section in the following way. In ϕ^4 theories in four dimensions or ϕ^3 theories in six dimensions the high order estimates found in chapters four and five respectively may need modification due to the appearance of the renormalon singularities. If the theory is asymptotically free these are left-hand-side singularities, and so relatively harmless. If the theory is not asymptotically free these are right-hand-side singularities, and these may adversely affect the Borel summability of the theory. What is worse is that in most theories the instanton and renormalon singularities are on different sides of the Borel plane and so the theories are never Borel summable. In ϕ^4 theories there are a few exceptions to this, in cases where certain (mathematically dubious) limits are taken. For instance, in theories with $O(n)$ internal symmetry, both types of singularities are on the left-hand-side of the Borel plane for $n < -8$ (Parisi 1978).

Paradoxically, it is in ϕ^3 theories that one finds more theories which are potentially Borel summable. If one considers theories with an interaction $g/3\ d_{ijk}\phi_i\phi_j\phi_k$ then it is easily found that

$$\beta_3 = \frac{\alpha - 4\beta}{64\pi^3} \tag{7.14}$$

where

$$\alpha\delta_{ij} = d_{iab}\, d_{jab}, \quad \beta d_{ijk} = d_{iab} d_{jbc} d_{kca} \tag{7.15}$$

Thus if we are studying one component ϕ^3 theory $\alpha = \beta = 1$ and $\beta_3 < 0$ showing that the theory is asymptotically free. However, there exist in the literature examples of interactions where $\alpha - 4\beta > 0$ (Harris et al 1975, Priest and Lubensky 1976, McKane et al 1976, McKane 1977). In these theories both the instanton and renormalon singularities are on the right-hand-side for real coupling, but go over to being left-hand-side singularities when the coupling constant is pure imaginary. Thus these theories may be well defined even in six dimensions if they have an imaginary coupling constant.

APPENDIX I

THE JACOBIAN OF THE TRANSFORMATION TO COLLECTIVE COORDINATES

We begin by considering the problem for the quantum mechanical anharmonic oscillator. This is relatively simple - there is only one collective coordinate since we are working in one dimension.

Consider first the Jacobian due to a change of basis from a field $\hat{x}(\tau)$ to a complete set of orthogonal functions $x_n(\tau)$:

$$\hat{x}(\tau) = \sum_n a_n x_n(\tau) \tag{A1.1}$$

so that

$$d[\hat{x}(\tau)] = J\prod_n d\alpha_n \tag{A1.2}$$

where J is the Jacobian of the transformation. The matrix relating the integration variables $d[\hat{x}(\tau)]$ to $d\alpha_n$ is given by

$$A = \begin{bmatrix} \frac{\partial\hat{x}(\tau_1)}{\partial a_1} & \frac{\partial\hat{x}(\tau_1)}{\partial a_2} & \cdots & \frac{\partial\hat{x}(\tau_1)}{\partial a_\ell} \\ \frac{\partial\hat{x}(\tau_2)}{\partial a_1} & \frac{\partial\hat{x}(\tau_2)}{\partial a_2} & & \frac{\partial\hat{x}(\tau_2)}{\partial a_\ell} \\ \vdots & \vdots & & \vdots \\ \frac{\partial\hat{x}(\tau K)}{\partial a_1} & \frac{\partial\hat{x}(\tau K)}{\partial a_2} & \cdots & \frac{\partial\hat{x}(\tau K)}{\partial a_\ell} \end{bmatrix} \tag{A1.3}$$

where $n=1,2,\ldots,\ell$ and where we have divided the real line into $(K-1)$ sections and it is understood that $K\to\infty$. Taking the Hermitian conjugate of A and multiplying it into A we obtain

$$(A^{\dagger}A)_{nn'} = \int d\tau \left(\frac{\partial \hat{x}(\tau)}{\partial a_n}\right)^{*} \left(\frac{\partial \hat{x}(\tau)}{\partial a_{n'}}\right)$$

$$= \int d\tau\ x^{*}_{n}(\tau)\ x_{n'}(\tau) \quad \text{by equation (A1.1)}$$

$$= \delta_{nn'}$$

if the eigenfunctions are orthonormal. Thus $A^{\dagger}A = I$ and so $J = |\det A| = 1$.

Now let us look at how the above discussion is modified when a collective coordinate is introduced. The analogous statements to (A1.1) and (A1.2) are (see equation (2.60))

$$x(\tau) = x_c(\tau;\tau_0) + \tilde{\sum_n} a_n x_n(\tau;\tau_0) \tag{A1.4}$$

and

$$d[\hat{x}(\tau)] = J\, d\tau_0 \tilde{\prod_n} da_n \tag{A1.5}$$

where, as usual, the tilde means that the zero mode is excluded. The matrix relating the integration variables $d[\hat{x}(\tau)]$ to $\{da_n, d\tau_0\}$ is given by

$$A = \left[\begin{array}{c|cccc} \frac{\partial x(\tau_1)}{\partial \tau_0} & \frac{\partial x(\tau_1)}{\partial a_1} & \cdots & \frac{\partial x(\tau_1)}{\partial a_\ell} \\ \frac{\partial x(\tau_2)}{\partial \tau_0} & \frac{\partial x(\tau_2)}{\partial a_1} & \cdots & \frac{\partial x(\tau_2)}{\partial a_\ell} \\ \vdots & \vdots & & \vdots \\ \frac{\partial x(\tau K)}{\partial \tau_0} & \frac{\partial x(\tau K)}{\partial a_1} & \cdots & \frac{\partial x(\tau K)}{\partial a \ell} \end{array}\right] \tag{A1.6}$$

and so

$$(A^{\dagger}A)_{11} = \int d\tau \left(\frac{\partial x}{\partial \tau_o}\right)^* \left(\frac{\partial x}{\partial \tau_o}\right)$$

$$(A^{\dagger}A)_{n1} = (A^{\dagger}A)^*_{1n} = \int d\tau \left(\frac{\partial x}{\partial a_n}\right)^* \left(\frac{\partial x}{\partial \tau_o}\right) \tag{A1.7}$$

$$(A^{\dagger}A)_{nn'} = \int d\tau \left(\frac{\partial x}{\partial a_n}\right)^* \left(\frac{\partial x}{\partial a_{n'}}\right)$$

$$= \delta_{nn'} \text{ if the } x_n \text{ are orthonormal}$$

Using equation (A1.7) we find

$$\det (A^{\dagger}A) = \int d\tau \left(\frac{\partial x}{\partial \tau_o}\right)^* \left(\frac{\partial x}{\partial \tau_o}\right) - \tilde{\sum_n} \left| \int d\tau \left(\frac{\partial x}{\partial a_n}\right)^* \left(\frac{\partial x}{\partial \tau_o}\right)\right|^2 \tag{A1.8}$$

From equation (A1.4) we have

$$\frac{dx}{d\tau_o} = \frac{dx_c}{d\tau_o} + \tilde{\sum_n} a_n \frac{dx_n}{d\tau_o} \tag{A1.9}$$

and so from (A1.8) we obtain the Jacobion to lowest order:

$$J = \sqrt{\int d\tau \left|{dx_c}/{d\tau_o}\right|^2} \tag{A1.10}$$

The corrections to this result are just those of the loop expansion, that is, for a theory with anharmonic term $g\, x^N$ we have

$$J = ||\, \dot{x}_c ||^{\frac{1}{2}} \left(1+0 \left(g^{\frac{2}{N-2}} \right)\right) \tag{A1.11}$$

We will now consider the transition to the d-dimensional case. For the moment let $d<d_c$ so that we do not have to worry about the collective coordinate for dilatations. We then have d collective coordinates : $x_o^1, x_o^2, \ldots, x_o^d$ and so writing

$$\phi(x) = \phi_c(x;x_o) + \sum_n^{\sim} a_n \phi_n(x;x_o) \tag{A1.12}$$

the equations (A1.7) generalise to

$$\begin{aligned}
(A^{\dagger}A)_{\mu\nu} &= \int d^dx \left(\frac{\partial\phi}{\partial x_o^{\mu}}\right)^* \left(\frac{\partial\phi}{\partial x_o^{\nu}}\right) \\
(A^{\dagger}A)_{n\mu} = (A^{\dagger}A)^*_{\mu n} &= \int d^dx \left(\frac{\partial\phi}{\partial a_n}\right)^* \left(\frac{\partial\phi}{\partial x_o^{\mu}}\right) \quad ; \ \mu,\nu = 1,2,\ldots,d \\
(A^{\dagger}A)_{nn'} &= \delta_{nn'}
\end{aligned} \tag{A1.13}$$

To leading order we may re place $\partial\phi/\partial x_o^{\mu}$ by $\partial\phi_c/\partial x_o^{\mu}$ so that equations (A1.13) become

$$\begin{aligned}
(A^{\dagger}A)_{\mu\nu} &= \int d^dx \left(\frac{\partial\phi_c}{\partial x_o^{\mu}}\right)^* \left(\frac{\partial\phi_c}{\partial x_o^{\nu}}\right) = \frac{\delta\mu\nu}{d} \int d^dx \left(\frac{\partial\phi_c}{\partial x_o^{k}}\right)^* \left(\frac{\partial\phi_c}{\partial x_o^{k}}\right) \\
(A^{\dagger}A)_{n\mu} = (A^{\dagger}A)^*_{\mu n} &= \int d^dx \ \phi_n^* \frac{\partial\phi_c}{\partial x_o^{\mu}} = 0 \\
(A^{\dagger}A)_{nn'} &= \delta_{nn'}
\end{aligned} \tag{A1.14}$$

where we have used the fact that ϕ_c is spherically symmetric about x_o and also that ϕ_n^* and $\partial\phi_c/\partial x_o^{\mu}$ are orthogonal. From equations (A1.14) it is easy to see that

$$(\det A)^2 = \left(\frac{1}{d}\int d^dx \ \phi_{\mu}^*(x) \ \ \phi_{\mu}(x)\right)^d \tag{A1.15}$$

where $\phi_{\mu}(x) = \partial\phi_c/\partial x_o^{\mu}$, and therefore

$$J = \left(\frac{1}{d}\int d^dx \ \ \phi_{\mu}^*(x) \ \phi_{\mu}(x)\right)^{d/2} (1+0 \ (g^{\frac{2}{N-2}})) \tag{A1.16}$$

If $d=d_c$, then we will also have to introduce a collective coordinate due to the dilatation invariance of the theory. The collective coordinates are now $\{x_0^\mu, \lambda\}$ and so writing

$$\phi(x) = \phi_c(x;x_0,\lambda) + \tilde{\sum_n} a_n \phi_n(x;x_0,\lambda) \tag{A1.17}$$

the matrix A is now given by

$$\begin{aligned}
(A^\dagger A)_{\lambda\lambda} &= \int d^d x \left(\frac{\partial\phi}{\partial\lambda}\right)^* \left(\frac{\partial\phi}{\partial\lambda}\right) \\
(A^\dagger A)_{\mu\lambda} = (A^\dagger A)^*_{\lambda\mu} &= \int d^d x \left(\frac{\partial\phi}{\partial x_o^\mu}\right)^* \left(\frac{\partial\phi}{\partial\lambda}\right) \quad \mu = 1,\ldots,d_c \\
(A^\dagger A)_{n\lambda} = (A^\dagger A)^*_{\lambda n} &= \int d^d x \left(\frac{\partial\phi}{\partial a_n}\right)^* \left(\frac{\partial\phi}{\partial\lambda}\right)
\end{aligned} \tag{A1.18}$$

together with the relations (A1.13). To leading order these reduce to

$$\begin{aligned}
(A^\dagger A)_{\lambda\lambda} &= \int d^d x \ \phi_\lambda^*(x)\, \phi_\lambda(x) \\
(A^\dagger A)_{\mu\nu} &= \frac{\delta_{\mu\nu}}{d_c} \int d^d x \ \phi_\kappa^*(x)\, \phi_\kappa(x) \\
(A^\dagger A)_{nn'} &= \delta_{nn'}
\end{aligned} \tag{A1.19}$$

all other entries being zero. Therefore we find

$$J = \left(\int d^{d_c} x \ \phi_\lambda^*(x)\, \phi_\lambda(x) \left[\frac{1}{d_c} \int d^{d_c} x \ \phi_\mu^*(x) \ \phi_\mu(x) \right]^{d_c} \right)^{\frac{1}{2}}$$

$$(1 + 0\,(g^{\frac{2}{N-2}})) \tag{A1.20}$$

Finally, if we work in $d=d_c-\varepsilon$ (keeping dilatations as a collective coordinate as discussed in Chapter four) the eigenfunctions corresponding to the "zero modes" are again just $\phi_\lambda(x)$ and $\phi_\mu(x)$, $\mu=1,\ldots,d$, to leading order. Thus in this case

$$J = \left(\int d^d x\ \phi^*_\lambda(x)\ \phi_\lambda(x) \left[\frac{1}{d} \int d^d x\ \phi^*_\mu(x)\ \phi_\mu(x) \right]^d \right)^{\frac{1}{2}} \left(1+O\left(\varepsilon, g^{\frac{2}{N-2}}\right)\right) \tag{A1.21}$$

APPENDIX II

SPHERICAL HARMONICS IN d DIMENSIONS

Consider a polyno mial, f, in $\{x_1,\ldots,x_d\}$ which is homogenous of degree ℓ, that is,

$$f(\underset{\sim}{\lambda x}) = \lambda^{\ell} f(\underset{\sim}{x}) \quad , \quad \underset{\sim}{x} = (x_1,\ldots,x_d) \tag{A2.1}$$

Then if it satisfies Laplace's equation

$$\nabla^2 f(\underset{\sim}{x}) = 0 \tag{A2.2}$$

it is known as a harmonic polynomial of degree ℓ. Condition (A2.1) means that we can write

$$f(\underset{\sim}{x}) = u_{i_1 i_2 \cdots i_\ell} \, x_{i_1} x_{i_2} \cdots x_{i_\ell} \tag{A2.3}$$

where summation over repeated indices is understood and each index takes on the values 1,2,...,d. The condition (A2.2) implies that u is traceless, that is,

$$u_{iii_3 i_4 \ldots i_\ell} = 0 \tag{A2.4}$$

Suppose we now introduce spherical polar coordinates $\{r,\theta_1,\ldots,\theta_{d-2},\phi\}$ defined by (Appell and Kampé de Fériet 1926, Erdélyi 1953)

$$x_1 = r\cos\theta_1$$

$$x_2 = r\sin\theta_1\cos\theta_2$$

$$x_3 = r\sin\theta_1\sin\theta_2\cos\theta_3$$

$$\vdots$$

$$x_{d-2} = r \sin\theta_1 \sin\theta_2 \ldots \sin\theta_{d-3} \cos\theta_{d-2} \tag{A2.5}$$

$$x_{d-1} = r \sin\theta_1 \sin\theta_2 \ldots \sin\theta_{d-2} \cos\phi$$

$$x_d = r \sin\theta_1 \sin\theta_2 \ldots \sin\theta_{d-2} \sin\phi$$

where $r = |\underset{\sim}{x}| = \sqrt{x_1^2 + x_2^2 + \ldots + x_d^2} \geqslant 0$ and $0 \leqslant \theta_i \leqslant \pi$ for $i=1,2,\ldots,d-2$, $0 \leqslant \phi \leqslant 2\pi$. We can now write the harmonic polynomial (A2.3) as

$$f(\underset{\sim}{x}) = r^\ell \, Y^u_\ell \, (\theta_1,\ldots,\theta_{d-2},\phi) \tag{A2.6}$$

where $\{Y^u_\ell\}$ are called spherical harmonics.

Result 1

$$L^2 \, Y^u_\ell \, (\{\theta_i\},\phi) = \ell(\ell+d-2) \, Y^u_\ell \, (\{\theta_i\},\phi) \tag{A2.7}$$

where

$$L^2 = \tfrac{1}{2} L_{ij}L_{ij}, \quad L_{ij} = -i \left(x_i \frac{\partial}{\partial x_j} - x_j \frac{\partial}{\partial x_i}\right) \tag{A2.8}$$

Proof

$$L_{ij}L_{ij} = -2x_i x_i \frac{\partial^2}{\partial x_j \partial x_j} \quad -2(1-d) \, x_i \frac{\partial}{\partial x_i} \quad + 2x_i x_j \frac{\partial^2}{\partial x_i \partial x_j}$$

from the definition of L_{ij}. Using the results

$$x_i \frac{\partial}{\partial x_i} = r \frac{\partial}{\partial r} \qquad \text{and} \qquad x_i x_j \frac{\partial^2}{\partial x_i \partial x_j} = r^2 \frac{\partial^2}{\partial r^2}$$

we then find

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{(d-1)}{r} - \frac{L^2}{r^2} \tag{A2.9}$$

If we now write the Laplacian in terms of the spherical polar coordinates defined by (A2.5) we find

$$\nabla^2 f = \frac{\partial^2 f}{\partial r^2} + \frac{(d-1)}{r}\frac{\partial f}{\partial r} + \frac{1}{r^2 \sin^2\theta_1 \ldots \sin^2\theta_{d-2}} \frac{\partial^2 f}{\partial \phi^2}$$

$$+ \frac{1}{r^2} \sum_{i=1}^{d-2} \frac{1}{\sin^2\theta_1 \ldots \sin^2\theta_{i-1}} \frac{1}{\sin^{d-i-1}\theta_i} \frac{\partial}{\partial\theta_i} \left(\sin^{d-i-1}\theta_i \frac{\partial f}{\partial\theta_i}\right) \tag{A2.10}$$

and so comparing (A2.9) and (A2.10) we see that L^2 consists only of angular factors and angular derivatives.

From equations (A2.2) and (A2.6) we have

$$\nabla^2[r^\ell Y_\ell^u(\theta_1,\ldots,\theta_{d-2},\phi)] = 0$$

and so using equation (A2.9) we find

$$[\ell(\ell-1) + \ell(d-1)]\, r^{\ell-2} Y_\ell^u(\{\theta_i\},\phi) = r^{\ell-2} L^2 Y_\ell^u(\{\theta_i\},\phi)$$

which proves the result.

Using equations (A2.7), (A2.9), and (A2.10) we may write down a differential equation for the spherical harmonics:

$$\sum_{i=1}^{d-2} \frac{1}{\sin^2\theta_1 \ldots \sin^2\theta_{i-1}} \frac{1}{\sin^{d-i-1}\theta_i} \frac{\partial}{\partial\theta_i}\left(\sin^{d-i-1} \frac{\partial Y_\ell}{\partial\theta_i}\right)$$

$$+ \frac{1}{\sin^2\theta_1 \ldots \sin^2\theta_{d-2}} \frac{\partial^2 Y_\ell}{\partial\phi^2} + \ell(\ell+d-2)\, Y_\ell = 0 \tag{A2.11}$$

The polynomial solutions of equation (A2.11) can be given explicitly in terms of Gegenbauer polynomials (Appell and Kampé de Fériet 1926, Erdélyi 1953):

$$Y_{\ell\alpha_1\cdots\alpha_{d-2}}(\{\theta_i\},\phi) = A \sin^{\alpha_1}\theta_1\, C_{\ell-\alpha_1}^{\alpha_1+d/2-1}(\cos\theta_1) \sin^{\alpha_2}\theta_2$$

$$\times\, C_{\alpha_1-\alpha_2}^{\alpha_2+d/2-3/2}(\cos\theta_2) \ldots \sin^{\alpha_{d-2}}\theta_{d-2}\, C_{\alpha_{d-3}-\alpha_{d-2}}^{\alpha_{d-2}+\frac{1}{2}}(\cos\theta_{d-2})$$

$$\times \exp \pm (i\, \alpha_{d-2}\, \phi) \tag{A2.12}$$

where $\{\ell, \alpha_1, \ldots\ldots, \alpha_{d-2}\}$ are integers such that

$$\ell \geqslant \alpha_1 \geqslant \ldots\ldots \geqslant \alpha_{d-2} \geqslant 0 \qquad \text{(A2.13)}$$

and A is an arbitrary constant. We note that the spherical harmonics in d dimensions are related to those in d-1 dimensions by

$$Y_{\ell\alpha_1\ldots\ldots\alpha_{d-2}}(\theta_1,\theta_2,\ldots,\phi) = B_d \sin^{\alpha_1}\theta_1 C_{\ell-\alpha_1}^{\alpha_1+d/2-1}(\cos\theta_1) Y_{\alpha_1\ldots\alpha_{d-2}}(\theta_2,\ldots,\phi) \qquad \text{(A2.14)}$$

where B_d is a constant which we will now fix by demanding that the spherical harmonics are normalised in a particular way.

<u>Result 2</u>

$$\int Y_{\ell\underset{\sim}{\alpha}}(\underset{\sim}{\theta},\phi) Y^*_{\ell'\underset{\sim}{\alpha}'}(\underset{\sim}{\theta},\phi) d\Omega^d = \delta_{\ell\ell'}\delta_{\underset{\sim}{\alpha}\underset{\sim}{\alpha}'} \qquad \text{(A2.15)}$$

if $$|B_d| = 2^{\alpha_1}\Gamma(\alpha_1+d/2-1)\left\{\frac{\Gamma(\ell-\alpha_1+1)(\ell+d/2-1)2^d}{8\pi\Gamma(\alpha_1+d+\ell-2)}\right\}^{\frac{1}{2}} \qquad \text{(A2.16)}$$

Here $d\Omega^d$ is the surface element in d dimensions:

$$d\Omega^d = (\sin\theta_1)^{d-2}(\sin\theta_2)^{d-3}\ldots\sin\theta_{d-2}d\theta_1\ldots d\theta_{d-2}d\phi \qquad \text{(A2.17)}$$

<u>Proof</u>

(i) When d = 3 we have

$$\int_0^{2\pi}\int_0^{\pi} Y_{\ell m}(\theta,\phi)Y^*_{\ell' m'}(\theta,\phi)\sin\theta d\theta d\phi$$

$$= |B|^2\int_0^{\pi} d\theta \sin^{m+m'+1}\theta C_{\ell-m}^{m+\frac{1}{2}}(\cos\theta)\, C_{\ell'-m'}^{m'+\frac{1}{2}}(\cos\theta)\int_0^{2\pi} e^{\pm i(m-m')\phi}d\phi$$

The last integral vanishes when $m \neq m'$ and also (Erdélyi 1954)

$$\int_0^{\pi}\sin^{2\nu}\theta C_n^{\nu}(\cos\theta)C_m^{\nu}(\cos\theta)d\theta = \frac{\pi 2^{1-2\nu}\Gamma(2\nu+n)}{n!(n+\nu)[\Gamma(\nu)]^2}\delta_{nm} \qquad \text{(A2.18)}$$

and thus we only get a contribution when $m = m', \ell = \ell'$.

(ii) Suppose that

$$\int Y_{\underset{\sim}{\alpha}}(\theta_2,\ldots\ldots,\phi)Y^*_{\underset{\sim}{\alpha}'}(\theta_2,\ldots\ldots,\phi)d\Omega^{d-1} = \delta_{\underset{\sim}{\alpha}\underset{\sim}{\alpha}'}$$

then from eqns.(A2.14) and (A2.17) we have that

$$\int Y_{\ell\underset{\sim}{\alpha}}(\theta_1,\theta_2,\ldots\ldots,\phi)Y^*_{\ell'\underset{\sim}{\alpha}'}(\theta_1,\theta_2,\ldots,\phi)d\Omega^d$$

$$= |B_d|^2\int_0^{\pi}\sin^{d-2+2\alpha_1}\theta_1 C_{\ell-\alpha_1}^{\alpha_1+d/2-1}(\cos\theta_1)C_{\ell'-\alpha_1}^{\alpha_1+d/2-1}(\cos\theta_1)d\theta_1\delta_{\underset{\sim}{\alpha}\underset{\sim}{\alpha}'}$$

$$= |Bd|^2 \frac{\pi 2^{-2\alpha_1-d+3}\Gamma(\ell+\alpha_1+d-2)}{(\ell-\alpha_1)!(\ell+d/2-1)[\Gamma(\alpha_1+d/2-1)]^2} \delta_{\ell\ell}\delta_{\underset{\sim}{\alpha}\underset{\sim}{\alpha}'}, \text{ using eqn.(A2.18)}$$

Thus by induction the spherical harmonics are orthogonal and moreover if they are chosen to be orthonormal, according to eqn.(A2.15), then $|Bd|$ is indeed given by expression (A2.16).

We will now prove a very useful result, namely

Result 3

$$\sum_{\underset{\sim}{\alpha}} Y_{\ell\underset{\sim}{\alpha}}(\underset{\sim}{\eta}')Y^*_{\ell\underset{\sim}{\alpha}}(\underset{\sim}{\eta}) = \frac{(2\ell+d-2)\Gamma(d/2-1)}{4\pi^{d/2}} C_\ell^{d/2\,-1}(\underset{\sim}{\eta}\cdot\underset{\sim}{\eta}'/R^2) \tag{A2.19}$$

where $\underset{\sim}{\eta}$ and $\underset{\sim}{\eta}'$ are coordinates on the sphere: $\underset{\sim}{\eta}^2 = \underset{\sim}{\eta}'^2 = R^2$.

Proof

(i) We will firstly prove eqn.(A2.19) in the case $d = 3$. Instead of expressing $Y_{\ell m}(\theta,\phi)$ as Gegenbauer polynomials we will write them in a form often used in elementary quantum mechanics (see for example Schiff 1968)

$$Y_{\ell m}(\theta,\phi) = \left[\frac{2\ell+1}{4\pi}\,\frac{(\ell-|m|)!}{(\ell+|m|)!}\right]^{\frac{1}{2}} P_\ell^m(\cos\theta)e^{im\phi} \tag{A2.20}$$

(up to a phase) where $P_\ell^m(\cos\theta)$ are the associated Legendre polynomials. If we follow Schiff and use $P_\ell^{-m}(\cos\theta)=P_\ell^m(\cos\theta)$ we find

$$\sum_{m=-\ell}^{\ell} Y_{\ell m}(\theta,\phi)Y^*_{\ell m}(\theta',\phi') = \frac{2\ell+1}{4\pi} P_\ell(\cos\theta)P_\ell(\cos\theta')$$
$$+2\sum_{m=1}^{\ell}\frac{2\ell+1}{4\pi}\,\frac{(\ell-m)!}{(\ell+m)!}\; P_\ell^m(\cos\theta)P_\ell^m(\cos\theta')\cos m(\phi-\theta') \tag{A2.21}$$

If we now make use of the addition theorem for Legendre polynomials (Erdélyi 1953)

$$P_\ell(\cos\theta\cos\theta'+\sin\theta\sin\theta'\cos(\phi-\theta')) = P_\ell(\cos\theta)P_\ell(\cos\theta')$$
$$+2\sum_{m=1}^{\ell}\frac{(\ell-m)!}{(\ell+m)!}\; P_\ell^m(\cos\theta)P_\ell^m(\cos\theta')\cos m(\phi-\phi') \tag{A2.22}$$

we obtain

$$\sum_m Y_{\ell m}(\theta,\phi)Y^*_{\ell m}(\theta',\phi') = \frac{2\ell+1}{4\pi} P_\ell(\cos\theta\cos\theta'+\sin\theta\sin\theta'\cos(\phi-\phi')) \tag{A2.23}$$

If $\eta = R(\sin\theta\cos\phi,\ \sin\theta\sin\phi,\ \cos\theta)$ and η' is defined similarly, then $\eta\cdot\eta' = R^2(\cos\theta\cos\theta'+\sin\theta\sin\theta'\cos(\phi-\phi'))$. Also Legendre polynomials and Gegenbauer polynomials are related by $P_\ell(x) = C_\ell^{\frac{1}{2}}(x)$. Thus we may write eqn.(A2.23) as

$$\sum_m Y_{\ell m}(\underset{\sim}{\eta})Y^*_{\ell m}(\underset{\sim}{\eta}') = \frac{2\ell+1}{4\pi}\, C_\ell^{\frac{1}{2}}(\underset{\sim}{\eta}\cdot\underset{\sim}{\eta}'/R^2)$$

which is the required result.

(ii) We will now suppose that the result (A2.19) holds in d-1 dimensions and prove that that it is then true in d dimensions. So suppose

$$\sum_{\alpha_2,\ldots\alpha_{d-2}} Y_{\underset{\sim}{\alpha}}(\theta_2,\ldots,\phi)Y^*_{\underset{\sim}{\alpha}}(\theta_2',\ldots,\phi') = \frac{(2\alpha_1+d-3)\Gamma(d/2-3/2)}{4\pi^{(d-1)/2}}C_{\alpha_1}^{d/2-3/2}(\cos\psi) \tag{A2.24}$$

where ψ is the angle between the vectors specified by $\{\theta_2, \ldots\ldots,\phi\}$ and $\{\theta_2',\ldots,\phi'\}$. Using eqns.(A2.14) and (A2.16) we have

$$\begin{aligned}\sum_{\underset{\sim}{\alpha}} Y_{\ell\underset{\sim}{\alpha}}(\theta_1,\ldots,\phi)Y^*_{\ell\underset{\sim}{\alpha}}(\theta_1',\ldots,\phi') &= \frac{(\ell+d/2-1)2^d\Gamma(d/2-3/2)}{32\pi^{(d+1)/2}}\\ &\times\sum_{\alpha_1=0}^{\ell}\frac{2^{2\alpha_1}\Gamma^2(\alpha_1+d/2-1)\Gamma(\ell-\alpha_1+1)(2\alpha_1+d-3)\sin^{\alpha_1}\theta_1 C_{\ell-\alpha_1}^{\alpha_1+d/2-1}(\cos\theta_1)}{\Gamma(\alpha_1+\ell+d-2)}\\ &\quad \sin^{\alpha_1}\theta_1' C_{\ell-\alpha_1}^{\alpha_1+d/2-1}(\cos\theta_1')C_{\alpha_1}^{d/2-3/2}(\cos\psi)\end{aligned} \tag{A2.25}$$

To perform the sum we use the addition theorem for Gegenbauer polynomials (Erdélyi 1953, Vilenkin 1968)

$$\begin{aligned}C_n^\nu(\cos\theta\cos\theta'+\sin\theta\sin\theta'\cos\psi) &= \sum_{m=0}^{n}\frac{2^{2m}(2\nu+2m-1)(n-m)!\,[(\nu)_m]^2}{(2\nu-1)_{n+m+1}}\\ &\quad\sin^m\theta\, C_{n-m}^{\nu+m}(\cos\theta)\sin^m\theta'\, C_{n-m}^{\nu+m}(\cos\theta')C_m^{\nu-\frac{1}{2}}(\cos\psi)\end{aligned} \tag{A2.26}$$

where $(a)_n \equiv \Gamma(n+a)/\Gamma(a)$. We obtain for eqn.(A2.25)

$$\sum_{\underset{\sim}{\alpha}} Y_{\ell\underset{\sim}{\alpha}}(\theta_1,\ldots,\phi)Y^*_{\ell\underset{\sim}{\alpha}}(\theta_1',\ldots,\phi') = \frac{(2\ell+d-2)\Gamma(d/2-1)}{4\pi^{d/2}}C_\ell^{d/2-1}(\cos\psi') \tag{A2.27}$$

where $\cos\psi' = \cos\theta\cos\theta' + \sin\theta\sin\theta'\cos\psi$. This is the required result for d dimensions. Thus by induction we have proved eqn.(A2.19).

Result 4

The range of values $\underset{\sim}{\alpha}$ takes on is given by

$$\nu_\ell(d) = \Gamma(\ell+d-2)(2\ell+d-2)/\Gamma(\ell+1)\Gamma(d-1) \tag{A2.28}$$

Proof

This result follows very easily from Result 3. We set $\underset{\sim}{\eta}=\underset{\sim}{\eta}'$ in eqn.(A2.19) and integrate over $\underset{\sim}{\eta}$. Then since the spherical harmonics are orthonormal we obtain;

$$\nu_\ell(d) = \frac{(2\ell+d-2)\Gamma(d/2-1)}{4\pi^{d/2}} S_d C_\ell^{d/2-1}(1) \tag{A2.29}$$

S_d is the surface area of a d-dimensional unit sphere and is easily found to be $2\pi^{d/2}/\Gamma(d/2)$. $C_\ell^{d/2-1}(1)$ is merely a binomial coefficient: $C_\ell^{d/2-1}(1) = \Gamma(\ell+d-2)/\Gamma(\ell+1)\Gamma(d-2)$. After some algebraic manipulations we can now find the required result from eqn.(A2.29).

Looking back at Result 1 we see that the eigenvalues of L^2 are degenerate - they only depend on ℓ. Their degeneracy is precisely $\nu_\ell(d)$ given by eqn.(A2.28).

Result 5

$$\nu_L(d+1) = \sum_{\ell=0}^{L} \nu_\ell(d) \tag{A2.30}$$

Proof

Suppose

$$\nu_N(d+1) = \sum_{\ell=0}^{N} \nu_\ell(d),$$

then if we can prove that

$$\nu_{N+1}(d+1) - \nu_N(d+1) = \nu_{N+1}(d) \tag{A2.31}$$

the result (A2.30) follows by induction on N.

Using eqn.(A2.28) the right-hand-side of expression (A2.31) is

$$\frac{\Gamma(N+d-1)}{\Gamma(N+2)\Gamma(d)}\left[(2N+d+1)(N+d-1)-(2N+d-1)(N+1)\right]$$

$$= \frac{\Gamma(N+d-1)(d-1)(2N+d)}{\Gamma(N+2)\Gamma(d)} \qquad = \nu_{N+1}(d)$$

and thus we have proved (A2.31).

APPENDIX III

ONE-LOOP PERTURBATIVE CALCULATIONS

IN ϕ^3 AND ϕ^4 FIELD THEORIES

In order to calculate the high order behaviour of perturbation theory in terms of the fully renormalised coupling constant for ϕ^3 and ϕ^4 field theories, it is necessary to perform a one-loop perturbative calculation. This enables us to relate the renormalised coupling constant defined at a symmetry point to the bare coupling constant and so determine the coefficient α (see eqns. (4.88) and (5.51)).

The calculation in the ϕ^4 case is straightforward. We need to calculate the one-loop contributions to $\Gamma^{(2)}$ and $\Gamma^{(4)}$. The only one-loop diagram contributing is shown in Fig.22 (the one-loop diagram in $\Gamma^{(2)}$ is of the type shown in Fig.16 and is zero in dimensional regularisation) and is equal to

$$54g^2 \int \frac{d^dp}{(2\pi)^d} \frac{1}{p^2(p+q)^2} \tag{A3.1}$$

where the coupling constant is normalised according to (4.2) and the external momenta are related to q by $q_1 + q_2 = q = q_3 + q_4$. When $d = 4 - \varepsilon$ expression (A3.1) is equal to

$$54g^2 . \frac{S_d}{2(2\pi)^d} q^{-\varepsilon} \left[\frac{\Gamma(\varepsilon/2)\Gamma(2-\varepsilon/2)\Gamma(1-\varepsilon/2)\Gamma(1-\varepsilon/2)}{\Gamma(2-\varepsilon)}\right] \tag{A3.2}$$

where as usual S_d is the surface area of the unit sphere in d dimensions.

Using the renormalisation conditions

$$\frac{\partial}{\partial q^2}\Gamma_R^{(2)}(q;g)\Big|_{q^2=\mu^2} = 1$$

and (A3.3)

$$-\frac{1}{6}\Gamma_R^{(4)}(q_i;g)\Big|_{SP(\mu)} = g_R\mu^{\varepsilon}$$

where the symmetry point S.P. is defined as $q_i . q_j = \frac{\mu^2}{3}(4\delta_{ij} - 1)$, we easily find using eqn.(4.73) that $Z = 1 + O(g^2)$ and so

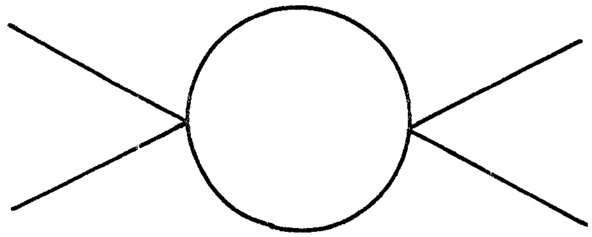

Figure 22. The one-loop contribution to $\Gamma^{(4)}$ in ϕ^4 field theory.

$$\mu^{\varepsilon} g_R = g - 9g^2 \frac{S_d}{(2\pi)^d}\left(\frac{4}{3}\mu^2\right)^{-\varepsilon/2}\left[1/\varepsilon + \tfrac{1}{2} + O(\varepsilon)\right] + O(g^3)$$

$$= g - \frac{9}{8\pi^2\varepsilon} g^2 \mu^{-\varepsilon}\left[1 + \varepsilon(\tfrac{1}{2}\ln\pi + \tfrac{1}{2}\ln 3 - \tfrac{1}{2}\gamma + 1) + O(\varepsilon^2)\right] + O(g^3) \qquad \text{(A3.4)}$$

If we use the minimal subtraction method ('t Hooft 1973) then we have correspondingly

$$g\mu^{-\varepsilon} = g_R + \frac{9}{8\pi^2\varepsilon} g_R^2 + O(g_R^3) \qquad \text{(A3.5)}$$

We now turn to the analogous calculation for ϕ^3 field theory. There are now two graphs (shown in Figs.23 and 24) to consider. The self energy graph in Fig.23 gives a contribution

$$2g^2 \int \frac{d^dp}{(2\pi)^d} \frac{1}{p^2(p-q)^2}$$

$$= - g^2 q^{2-\varepsilon} \frac{S_d}{(2\pi)^d}\left[\frac{2}{3\varepsilon} + \frac{7}{18} + O(\varepsilon)\right] \qquad \text{(A3.6)}$$

where q is the external momentum and $\varepsilon = 6 - d$. The triangle graph in Fig.24 is trickier; it gives a contribution

$$8g^3 \int \frac{d^dp}{(2\pi)^d} \frac{1}{p^2(p-q_1)^2(p+q_2)^2} \qquad \text{(A3.7)}$$

where the external momenta are q_1, q_2 and $q_3 = -q_1 - q_2$. If we evaluate the integral in (A3.7) using Feynman parameters we find that it is equal to (at the symmetry point) (Amit 1976)

$$\mu^{-\varepsilon} \frac{S_d}{(2\pi)^d}\left[\frac{1}{\varepsilon} - \frac{3}{4} - L + O(\varepsilon)\right] \qquad \text{(A3.8)}$$

where $L = \int_0^1 dx \int_0^1 dy\ \theta(1-x-y)\ln\left[x(1-x)+y(1-y)-xy\right]$ (A3.9)

and SP(μ) is given by $q_i \cdot q_j = \frac{\mu^2}{2}(3\delta_{ij} - 1)$. Making the change of variable $x + y = \lambda$, $x = \mu\lambda$ it is straightforward to show that

$$L = -\frac{3}{2} - \frac{1}{2}\int_0^1 \frac{d\mu}{(\mu^2-\mu+1)} - \int_0^1 \frac{\ln\mu\, d\mu}{(\mu^2-\mu+1)^2} \qquad \text{(A3.10)}$$

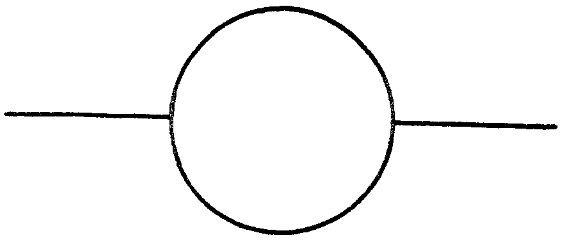

Figure 23. The one-loop contribution to $\Gamma^{(2)}$ in ϕ^3 field theory.

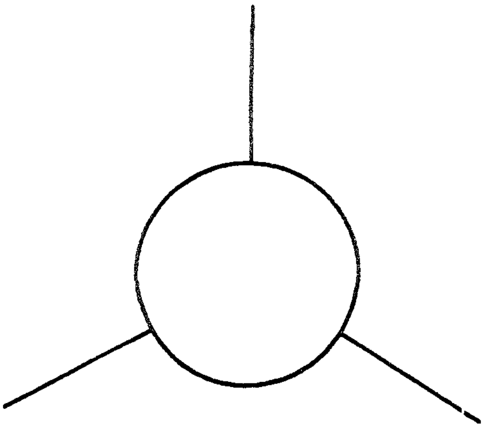

Figure 24. The one-loop contribution to $\Gamma^{(3)}$ in ϕ^3 field theory.

We have also checked that results (A3.8) and (A3.10) are obtained if we evaluate the integral in (A3.7) using Schwinger parameters.

The first integral in eqn.(A3.10) is trivial to evaluate; it equals $2\sqrt{3}\pi/9$. The second is more difficult, but if we write it as

$$I = \int_0^1 \frac{\ln\mu d\mu}{(\mu^2-\mu+1)^2} = \int_0^1 \frac{\mu^2 \ln\mu d\mu}{[\mu^3+1]^2} + 2\int_0^1 \frac{\mu\ln\mu d\mu}{[\mu^3+1]^2} + \int_0^1\frac{\ln\mu d\mu}{[\mu^3+1]^2} \quad \text{(A3.11)}$$

then we can use the result

$$\int_0^1 \frac{x^a \ln x\, dx}{(1+x)^2} = \tfrac{1}{2}\left[\psi((a+1)/2)-\psi(a/2)\right] + a/4\left[\psi'((a+1)/2)-\psi'(a/2)\right] \quad \text{(A3.12)}$$

(where ψ and ψ' are the digamma and trigamma functions respectively) to evaluate it. We find, after using properties of these polygamma functions, that

$$I = -\frac{\sqrt{3}\pi}{9} - \frac{2}{9}\psi'(1/3)+\frac{4\pi^2}{27} \quad \text{(A3.13)}$$

and therefore using eqn.(A3.8) and (A3.10) we find the expression (A3.7) equals

$$\frac{8g^2\mu^{-\varepsilon}S_d}{(2\pi)^d}\left[\frac{1}{\varepsilon} + \frac{3}{4} + \frac{4\pi^2}{27} - \frac{2}{9}\psi'(1/3) + O(\varepsilon)\right] \quad \text{(A3.14)}$$

Thus using (A3.6) and (A3.14) one finds

$$\Gamma^{(2)}(q) = q^2\left[1+\frac{q^{-\varepsilon}g^2}{96\pi^3}\left(1/\varepsilon+4/3-\tfrac{1}{2}\gamma+\ln 2+\tfrac{1}{2}\ln\pi+O(\varepsilon)\right)+O(g^4)\right] \quad \text{(A3.15)}$$

and

$$\Gamma^{(3)}(q_i)\Big|_{SP(\mu)} = 2g + \frac{\mu^{-\varepsilon}g^3}{8\pi^3}\left(1/\varepsilon+3/2-\tfrac{1}{2}\gamma+\ln 2+\tfrac{1}{2}\ln\pi+\frac{4\pi^2}{27}-\frac{2}{9}\psi'(1/3)+O(\varepsilon)\right)+O(g^5) \quad \text{(A3.16)}$$

and therefore using

$$\Gamma_R^{(M)} = Z^{M/2}\Gamma^{(M)}$$

$$\frac{\partial^2\Gamma_R^{(2)}(q)}{\partial q^2}\Big|_{q^2=\mu^2} = 1 \quad \text{(A3.17)}$$

and

$$\tfrac{1}{2}\Gamma_R^{(3)}(q_i)\Big|_{SP(\mu)} = g_R\mu^{\varepsilon/2}$$

we find

$$\mu^{\varepsilon/2} g_R = g + \frac{3g^3\mu^{-\varepsilon}}{64\pi^3}\left(1/\varepsilon+31/18-\tfrac{1}{2}\gamma+\ell n2+\tfrac{1}{2}\ell n\pi+16\pi^2/81-\frac{8\psi'(1/3)}{27}+O(\varepsilon)\right)+O(g^5) \qquad \text{(A3.18)}$$

The corresponding result in the minimal subtraction scheme is

$$g\mu^{-\varepsilon/2} = g_R - \frac{3g_R^3}{64\pi^3\varepsilon} + O(g_R^5) \qquad \text{(A3.19)}$$

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