ANICK SPACES AND KAC-MOODY GROUPS

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Abstract. For primes $p \geq 5$ we prove an approximation to Cohen, Moore and Neisendorfer's conjecture that the loops on an Anick space retracts off the double loops on a mod-p Moore space. The approximation is then used to answer a question posed by Kitchloo regarding the topology of Kac-Moody groups. We show that, for certain rank two Kac-Moody groups K, the based loops on K is p-locally homotopy equivalent to the product of the loops on a 3-sphere and the loops on an Anick space.

1. Introduction

This paper has two purposes. The first is to address an important conjecture in homotopy theory regarding the homotopy type of the double loops on an odd primary Moore space. The second is to establish a connection between rank two Kac-Moody groups and Anick spaces.

Let p be an odd prime and $r \geq 1$. Take homology with mod-p coefficients. For $m \geq 1$ the *Moore space* $P^{m+1}(p^r)$ is the cofibre of the degree p^r map on S^m. Its homotopy theory was investigated in depth by Cohen, Moore and Neisendorfer [CMN1, CMN2, CMN3] and additional properties were proved by Neisendorfer [N2, N3]. In the case of an odd dimensional Moore space $P^{2n+1}(p^r)$, a related space was constructed by Anick [A2] for $p \geq 5$, and reconstructed in a much simpler way in [GT] for $p \geq 3$. For each $n, r \geq 1$ there is a space $T^{2n+1}(p^r)$ which fits in a homotopy fibration

$$S^{2n-1} \longrightarrow T^{2n+1}(p^r) \longrightarrow \Omega S^{2n+1} \tag{1}$$

and has the property that there is a coalgebra isomorphism

$$H_*(T^{2n+1}(p^r)) \cong \Lambda(u_{2n-1}) \otimes \mathbb{Z}/p\mathbb{Z}[v_{2n}]$$

with $\beta^r(v_{2n}) = u_{2n-1}$, where β^r is the r^{th}-Bockstein. It was conjectured by Cohen, Moore and Neisendorfer that $\Omega T^{2n+1}(p^r)$ retracts off $\Omega^2 P^{2n+1}(p^r)$. This was proved by Neisendorfer [N3] for $p \geq 3$ and $r \geq 2$, but the critical case of $r = 1$ remains open.

Our first result is to prove an approximation to this remaining open case, although we state the result for all $r \geq 1$. Define $C^{2n+1}(p^r)$ by the homotopy cofibration

$$P^{4n}(p^r) \xrightarrow{[\nu,\mu]} P^{2n+1}(p^r) \longrightarrow C^{2n+1}(p^r)$$

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where $[\nu,\mu]$ is the mod-p^r Whitehead product of the identity map ν on $P^{2n+1}(p^r)$ and the Bockstein map μ.

Theorem 1.1. *Let $p\geq 5$, $r\geq 1$ and $n>1$. Then $\Omega T^{2n+1}(p^r)$ is a retract of $\Omega^2 C^{2n+1}(p^r)$.*

Philosophically, Theorem 1.1 says that if one gets rid of mod-p^r Whitehead products on $P^{2n+1}(p^r)$ (by coning them out in $C^{2n+1}(p^r)$) then all obstructions to a splitting involving $\Omega T^{2n+1}(p^r)$ vanish. This may or may not be helpful in trying to show that $\Omega T^{2n+1}(p^r)$ retracts off $\Omega^2 P^{2n+1}(p^r)$. However, it is interesting to note that if $T_0^{2n+1}(p^r)$ is the bottom indecomposable factor of $\Omega P^{2n+1}(p^r)$, then Anick [A1] showed that $T_0^{2n+1}(p^r)$ retracts off ΩL, where L is the $4n$-skeleton of $C^{2n+1}(p^r)$. So the obstruction to retracting $\Omega T^{2n+1}(p^r)$ off $\Omega^2 P^{2n+1}(p^r)$ is encoded in the attaching map of the top dimensional cell of $C^{2n+1}(p^r)$.

Theorem 1.1 has practical applications, which leads to the second purpose of the paper. Fix a prime p. Let $k\in\{p,2p\}$ or let k be a divisor of $p-1$ or $p+1$. In [K1, K2], Kitchloo showed that for each such k there is a nonempty set $\mathcal{V}_k$ of positive integers with the property that if $r\in\mathcal{V}_k$ then there is a rank-2 Kac-Moody group K such that

$$H_*(K)\cong\Lambda(z_3,y_{2k-1})\otimes\mathbb{Z}/p\mathbb{Z}[x_{2k}] \tag{2}$$

and $\beta^r(x_{2k})=y_{2k-1}$, where β^r is the r^{th}-Bockstein. Further, K has an S^3 subgroup whose inclusion induces an isomorphism onto the subalgebra $\Lambda(z_3)$ in homology. Taking classifying spaces, this results in a homotopy fibration sequence

$$S^3\longrightarrow K\xrightarrow{\delta}X\longrightarrow BS^3\longrightarrow BK$$

where

$$H_*(X)\cong\Lambda(y_{2k-1})\otimes\mathbb{Z}/p\mathbb{Z}[x_{2k}]$$

and δ_* is the projection. Observe that X has the same homology as the Anick space $T^{2k+1}(p^r)$.

Kitchloo conjectured that there is a p-local homotopy fibration $S^{2k-1}\longrightarrow X\longrightarrow\Omega S^{2k+1}$ that is equivalent to Anick's fibration. A weaker conjecture is that there is a p-local homotopy equivalence $X\simeq T^{2k+1}(p^r)$. We prove that the weaker conjecture holds after looping if $1<k<p-1$. Moreover, the method results in a homotopy decomposition for ΩK.

Theorem 1.2. *Let $p\geq 5$ and let K be a rank two Kac-Moody group satisfying (2). If $1<k<p-1$ and $r\in\mathcal{V}_k$ then there are p-local homotopy equivalences*

$$\Omega X\simeq\Omega T^{2k+1}(p^r)\qquad\textit{and}\qquad\Omega K\simeq\Omega S^3\times\Omega T^{2k+1}(p^r).$$

The approach to proving Theorem 1.2 involves four steps. First, we lift the inclusion of the bottom Moore space $P^{2k}(p^r)\longrightarrow X$ to K. Second, we show that its adjoint $P^{2k+1}(p^r)\longrightarrow BK$ extends to a map $C^{2k+1}(p^r)\longrightarrow BK$. Third, Theorem 1.1 is applied to produce a map $\Omega T^{2k+1}(p^r)\longrightarrow\Omega K$.

Finally, an atomicity style argument is used to show that the composite $\Omega T^{2k+1}(p^r) \longrightarrow \Omega K \longrightarrow \Omega X$ is a p-local homotopy equivalence, from which Theorem 1.2 follows.

The decomposition of ΩK in Theorem 1.2 implies exponent information about K. The *p-primary homotopy exponent* of a space Y is the least power of p that annihilates the p-torsion in the homotopy groups of Y. If this power is r, write $\exp_p(Y) = p^r$. Selick [S] showed that for $p \geq 3$, $\exp_p(S^3) = p$, and Gray [Gr, Corollary 7.28] showed that for $p \geq 5$, $\exp_p(T^{2n+1}(p^r)) = p^r$. Since looping simply shifts homotopy groups down one dimension, Theorem 1.2 immediately implies the following.

Corollary 1.3. *If K is a Kac-Moody group as in Theorem 1.2 then* $\exp_p(K) = p^r$. □

In particular, if $r = 1$ then one obtains the remarkable outcome that $\exp_p(K) = p$.

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2. Background information on the homotopy theory of Moore spaces

In this section we record some of the material from [CMN1] and [N3] that will be needed for later. From here on it will be assumed that all spaces and maps have been localized at an odd prime p.

If A is a co-H-space, let $\underline{p}^r \colon A \longrightarrow A$ be the map of degree p^r and if B is an H-space, let $p^r \colon B \longrightarrow B$ be the p^r-power map. For $m \geq 1$, the Moore space $P^{m+1}(p^r)$ is defined by the homotopy cofibration

$$S^m \xrightarrow{\underline{p}^r} S^m \longrightarrow P^{m+1}(p^r).$$

The sphere S^{2n+1} is an H-space localized at an odd prime p and its p^r-power map is homotopic to the map of degree p^r. Define the space $S^{2n+1}\{p^r\}$ by the homotopy fibration

$$S^{2n+1}\{p^r\} \longrightarrow S^{2n+1} \xrightarrow{p^r} S^{2n+1}.$$

One key result in [CMN1] is that there is a map

$$S^{2n+1}\{p^r\} \longrightarrow \Omega P^{2n+2}(p^r)$$

which has a left homotopy inverse.

It will be necessary to relate Moore spaces of different torsion orders. For $r, s \geq 1$, there is a homotopy pushout diagram

$$\begin{tikzcd}
S^m \arrow[r, "p^r"] \arrow[d, equal] & S^m \arrow[r] \arrow[d, "p^s"] & P^{m+1}(p^r) \arrow[d, "\omega_r^{r+s}"] \\
S^m \arrow[r, "p^{r+s}"] & S^m \arrow[r] \arrow[d] & P^{m+1}(p^{r+s}) \arrow[d, "\rho^s_{r+s}"] \\
& P^{m+1}(p^s) \arrow[r, equal] & P^m(p^s)
\end{tikzcd}$$

that defines the maps ω_r^{r+s} and ρ_{r+s}^s. For fibres of degree maps there is an analogous homotopy pullback diagram

$$\begin{array}{ccccc}
S^{2n+1}\{p^r\} & \xrightarrow{\varpi_r^{r+s}} & S^{2n+1}\{p^{r+s}\} & \xrightarrow{\varrho_{r+s}^s} & S^{2n+1}\{p^s\} \\
\| & & \downarrow & & \downarrow \\
S^{2n+1}\{p^r\} & \longrightarrow & S^{2n+1} & \xrightarrow{p^r} & S^{2n+1} \\
 & & \downarrow{\scriptstyle p^{r+s}} & & \downarrow{\scriptstyle p^s} \\
 & & S^{2n+1} & = & S^{2n+1}
\end{array}$$

that defines the maps ϖ_r^{r+s} and ϱ_{r+s}^s. As in [N3, Diagram 1.10], the map $S^{2n+1}\{p^r\} \longrightarrow \Omega P^{2n+2}(p^r)$ with a left homotopy inverse may be chosen so that it is natural with respect to changes in torsion order. That is, there is a homotopy commutative diagram

$$(3) \qquad \begin{array}{ccccc}
S^{2n+1}\{p^r\} & \xrightarrow{\varpi_r^{r+s}} & S^{2n+1}\{p^{r+s}\} & \xrightarrow{\varrho_{r+s}^s} & S^{2n+1}\{p^s\} \\
\downarrow & & \downarrow & & \downarrow \\
\Omega P^{2n+2}(p^r) & \xrightarrow{\Omega\omega_r^{r+s}} & \Omega P^{2n+2}(p^{r+s}) & \xrightarrow{\Omega\rho_{r+s}^r} & \Omega P^{2n+2}(p^s)
\end{array}$$

For odd dimensional Moore spaces, consider the homotopy fibration sequence

$$\Omega S^{2n+1} \xrightarrow{\partial_r} F^{2n+1}(p^r) \longrightarrow P^{2n+1}(p^r) \xrightarrow{q} S^{2n+1}$$

where q is the pinch map to the top cell, and the fibration sequence defines the space $F^{2n+1}(p^r)$ and the map ∂_r. In [CMN1] it was shown that there is a homotopy equivalence

$$(4) \qquad \kappa\colon S^{2n-1} \times V^{2n+1}(p^{r+1}) \times \Omega R^{2n+1}(p^r) \longrightarrow \Omega F^{2n+1}(p^r).$$

There may be choices of the homotopy equivalence κ, and in Lemma 3.3 a lift of κ will be produced that depends on making a specific choice. To this end, κ will now be described in more detail.

Let $f\colon \Sigma P^s(p^r) \longrightarrow P^m(p^r)$ and $g\colon \Sigma P^t(p^r) \longrightarrow P^m(p^r)$ be maps. Let

$$w(f,g)\colon \Sigma P^s(p^r) \wedge P^t(p^r) \longrightarrow P^m(p^r)$$

be the Whitehead product of f and g. By [N1], as p is odd there is a homotopy equivalence $P^s(p^r) \wedge P^t(p^r) \simeq P^{s+t}(p^r) \vee P^{s+t-1}(p^r)$. The mod-$p^r$ Whitehead product of f and g is the composite

$$[f,g]\colon P^{s+t+1}(p^r) \hookrightarrow P^{s+t+1}(p^r) \vee P^{s+t}(p^r) \simeq \Sigma P^s(p^r) \wedge P^t(p^r) \xrightarrow{w(f,g)} P^m(p^r).$$

Mod-p^r Whitehead products play an important role, and certain ones are distinguished. Let $\nu\colon P^{2n+1}(p^r) \longrightarrow P^{2n+1}(p^r)$ be the identity map and let $\mu\colon P^{2n}(p^r) \longrightarrow P^{2n+1}(p^r)$ be the composite $P^{2n}(p^r) \xrightarrow{q} S^{2n} \longrightarrow P^{2n+1}(p^r)$, where the right map is the inclusion of the bottom cell.

Let $ad^1(\nu)(\mu) = [\nu, \mu]$ and, for $k > 1$, recursively define $ad^k(\nu)(\mu)$ by $ad^k(\nu, \mu) = [\nu, ad^{k-1}(\nu)(\mu)]$. In [CMN1] it was shown that there is an extension

$$\begin{tikzcd} P^{2np^j}(p^r) \arrow[r, "\omega_r^{r+1}"] \arrow[d, "ad^{p^j-1}(\nu)(\mu)"'] & P^{2np^j}(p^{r+1}) \arrow[dl, "e_j"] \\ P^{2n+1}(p^r) & \end{tikzcd} \tag{5}$$

for some map e_j.

We now describe the spaces and maps appearing in (4). First, there is the inclusion $i\colon S^{2n-1} \longrightarrow \Omega F^{2n+1}(p^r)$ of the bottom cell. Second, the space $R^{2n+1}(p^r)$ is a wedge of mod-p^r Moore spaces and there is a map $R^{2n+1}(p^r) \longrightarrow P^{2n+1}(p^r)$ which is a wedge sum of iterated mod-p^r Whitehead products. Each mod-p^r Whitehead product composes trivially with the pinch map $P^{2n+1}(p^r) \xrightarrow{q} S^{2n+1}$ because S^{2n+1} is an H-space and any Whitehead product on an H-space is null homotopic. Thus there is a lift $\psi\colon R^{2n+1}(p^r) \longrightarrow F^{2n+1}(p^r)$. Looping gives a map $\Omega R^{2n+1}(p^r) \xrightarrow{\Omega\psi} \Omega F^{2n+1}(p^r)$. Third, as above, the mod-p^r Whitehead product $ad^{p^j-1}(\nu)(\mu)$ lifts to a map $\ell_j\colon P^{2np^j}(p^r) \longrightarrow F^{2n+1}(p^r)$ and in [CMN1] it is shown that the extension property in (5) occurs at the lifted level as well. That is, there is a homotopy commutative diagram

$$\begin{tikzcd} P^{2np^j}(p^r) \arrow[r, "\omega_r^{r+1}"] \arrow[d, "\ell_j"'] & P^{2np^j}(p^{r+1}) \arrow[dl, "e'_j"] \\ F^{2n+1}(p^r) & \end{tikzcd} \tag{6}$$

for some map e'_j. Thus for each $j \geq 1$ there is a composite $S^{2np^j-1}\{p^{r+1}\} \longrightarrow \Omega P^{2np^j}(p^r) \xrightarrow{\Omega e'_j} \Omega F^{2n+1}(p^r)$. Letting $V^{2n+1}(p^{r+1}) = \prod_{j=1}^{\infty} S^{2np^j-1}\{p^{r+1}\}$ and using the loop space structure on $\Omega F^{2n+1}(p^r)$ to multiply we obtain map $\epsilon\colon V^{2n+1}(p^{r+1}) \longrightarrow \Omega F^{2n+1}(p^r)$. The map κ in (4) is the result of multplying together the maps i, ϵ and $\Omega\psi$.

Remark 2.1. There may have been choices of the lifts ψ and ℓ_j. Any choice of ψ and any choice of ℓ_j that satisfied (6) would do to produce a choice of the homotopy equivalence κ.

Let b_r be the composite

$$b_r\colon \Omega^2 S^{2n+1} \xrightarrow{\Omega\partial_r} \Omega F^{2n+1}(p^r) \xrightarrow{\kappa^{-1}} S^{2n-1} \times V^{2n+1}(p^{r+1}) \times \Omega R^{2n+1}(p^r).$$

Then there is a homotopy fibration sequence

$$\Omega^2 S^{2n+1} \xrightarrow{b_r} S^{2n-1} \times V^{2n+1}(p^{r+1}) \times \Omega R^{2n+1}(p^r) \longrightarrow \Omega P^{2n+1}(p^r) \xrightarrow{\Omega q} \Omega S^{2n+1}.$$

There is a factorization of b_r proved by Neisendorfer. Changes in torsion order will play a role. For any $t \geq 1$, let $V^{2n+1}(p^t) = \prod_{j=1}^{\infty} S^{2np^j-1}\{p^t\}$. Abusing notation, let

$$\varpi_r^{r+s}\colon V^{2n+1}(p^r) \longrightarrow V^{2n+1}(p^{r+s}) \qquad \varrho_{r+s}^s\colon V^{2n+1}(p^{r+s}) \longrightarrow V^{2n+1}(p^s)$$

also denote the product of the maps $S^{2np^j-1}\{p^r\} \xrightarrow{\varpi_r^{r+s}} S^{2np^j-1}\{p^{r+s}\}$ and $S^{2np^j-1}\{p^{r+s}\} \xrightarrow{\varpi_{r+s}^{s}} S^{2np^j-1}\{p^s\}$ respectively. In [N4], Neisendorfer proved the following.

Lemma 2.2. *There is a homotopy commutative diagram*

$$\begin{tikzcd} V^{2n+1}(p^{r+1}) \arrow[r] \arrow[d, "\varrho^r_{r+1}"] & \Omega P^{2n+1}(p^r) \arrow[dl, "\zeta"] \\ V^{2n+1}(p^r) & \end{tikzcd}$$

for some map ζ. □

From the homotopy fibration $S^{2n+1}\{p^r\} \xrightarrow{\varpi_r^{r+s}} S^{2n+1}\{p^{r+s}\} \xrightarrow{\varrho^s_{r+s}} S^{2n+1}\{p^s\}$ we obtain a product homotopy fibration $V^{2n+1}(p^r) \xrightarrow{\varpi_r^{r+s}} V^{2n+1}(p^{r+s}) \xrightarrow{\varrho^s_{r+s}} V^{2n+1}(p^s)$. Therefore, Lemma 2.2 implies that there is a lift

$$\begin{tikzcd} & S^{2n-1} \times V^{2n+1}(p) \times \Omega R^{2n+1}(p^r) \arrow[d, "1\times\varpi_1^{r+1}\times 1"] \\ \Omega^2 S^{2n+1} \arrow[ur] \arrow[r, "b_r"] & S^{2n-1} \times V^{2n+1}(p^{r+1}) \times \Omega R^{2n+1}(p^r). \end{tikzcd}$$

In what follows, we only require a weaker lift. The map ϖ_1^{r+1} factors as the composite $\varpi_r^{r+1} \circ \varpi_1^r$. Therefore we obtain the following.

Lemma 2.3. *There is a homotopy commutative diagram*

$$\begin{tikzcd} & S^{2n-1} \times V^{2n+1}(p^r) \times \Omega R^{2n+1}(p^r) \arrow[d, "1\times\varpi_r^{r+1}\times 1"] \\ \Omega^2 S^{2n+1} \arrow[ur] \arrow[r, "b_r"] & S^{2n-1} \times V^{2n+1}(p^{r+1}) \times \Omega R^{2n+1}(p^r). \end{tikzcd}$$

□

3. A retraction of $\Omega T^{2n+1}(p^r)$ off $\Omega^2 C^{2n+1}(p^r)$

In this section we prove Theorem 1.1 by constructing maps $a\colon \Omega T^{2n+1}(p^r) \longrightarrow \Omega^2 C^{2n+1}(p^r)$ and $b\colon \Omega^2 C^{2n+1}(p^r) \longrightarrow \Omega T^{2n+1}(p^r)$ with the property that $b \circ a$ is a homotopy equivalence. We begin with a description of the properties of Anick spaces that will be needed.

3.1. **Properties of Anick spaces.** As in the Introduction, for each odd prime p and $n, r \geq 1$ there is a homotopy fibration

$$S^{2n-1} \longrightarrow T^{2n+1}(p^r) \longrightarrow \Omega S^{2n+1} \tag{7}$$

and a coalgebra isomorphism

$$H_*(T^{2n+1}(p^r)) \cong \Lambda(u_{2n-1}) \otimes \mathbb{Z}/p\mathbb{Z}[v_{2n}]$$

with $\beta^r(v_{2n}) = u_{2n-1}$, where β^r is the r^{th}-Bockstein.

A simply-connected space X is *atomic* if any self-map $f\colon X \longrightarrow X$ which induces an isomorphism in the least nonvanishing degree in homology is a homotopy equivalence. Atomicity is used to detect indecomposable spaces, those for which no nontrivial product decompositions exist.

Theorem 3.1. *The space $T^{2n+1}(p^r)$ and the homotopy fibration (7) have the following properties:*

(a) *there is a factorization*

$$\begin{tikzcd} \Omega P^{2n+1}(p^r) \arrow[r, "\Omega q"] \arrow[d, "t"] & \Omega S^{2n+1} \arrow[d, equal] \\ T^{2n+1}(p^r) \arrow[r] & \Omega S^{2n+1} \end{tikzcd}$$

for some map t;

(b) *the fibration connecting map for (7) is homotopic to $\Omega^2 S^{2n+1} \xrightarrow{\varphi_r} S^{2n-1}$;*

(c) *if $r \geq 2$ then the map $\Omega^2 P^{2n+1}(p^r) \xrightarrow{\Omega t} \Omega T^{2n+1}(p^r)$ has a right homotopy inverse;*

(d) *if $p \geq 5$ then $T^{2n+1}(p^r)$ is a homotopy associative, homotopy commutative H-space and t is an H-map;*

(e) *$\Omega T^{2n+1}(p^r)$ is atomic.*

Proof. Part (a) is proved in [A2] for $p \geq 5$ and in [GT] for $p \geq 3$, and part (b) - also established in both papers - is a consequence of part (a). Part (c) is proved in [N3], part (d) in [Gr], and part (e) in [Th]. $\square$

3.2. **Constructing a map $\Omega T^{2n+1}(p^r) \longrightarrow \Omega^2 C^{2n+1}(p^r)$.** In general, if X is a path-connected space, let $J_2(\Sigma X)$ be the second stage of the James construction on ΣX. There is a homotopy cofibration

$$\Sigma X \wedge X \xrightarrow{[1,1]} \Sigma X \xrightarrow{j} J_2(\Sigma X)$$

where $[1,1]$ is the Whitehead product of the identity map on ΣX with itself and j can be regarded as the inclusion of $J_1(\Sigma X) = \Sigma X$ into $J_2(\Sigma X)$. In our case take $X = P^{2n}(p^r)$. Let $T\colon X \wedge X \longrightarrow X \wedge X$ be the map that swaps factors. As we are localized at an odd prime, the self-map

$$\frac{1}{2}(1-T)\colon \Sigma P^{2n}(p^r) \wedge P^{2n}(p^r) \longrightarrow \Sigma P^{2n}(p^r) \wedge P^{2n}(p^r)$$

exists, and as in [CW], it is an idempotent because $P^{2n}(p^r)$ is a suspension since $n \geq 1$ Moreover, as in [CW], the Whitehead product $\Sigma P^{2n}(p^r) \wedge P^{2n}(p^r) \xrightarrow{[1,1]} P^{2n+1}(p^r)$ factors through the telescope of $\frac{1}{2}(1-T)$, which is $P^{4n}(p^r)$, giving a factorization of $[1,1]$ as a composite $\Sigma P^{2n}(p^r) \wedge P^{2n}(p^r) \xrightarrow{\mathsf{t}} P^{4n}(p^r) \xrightarrow{[\nu,\mu]} P^{2n+1}(p^r)$ where t is the map to the telescope and has a right homotopy inverse.

Consequently, there is a homotopy pushout diagram

$$\begin{tikzcd}
\Sigma P^{2n}(p^r)\wedge P^{2n}(p^r) \arrow[r, "\mathfrak{t}"] \arrow[d, equal] & P^{4n}(p^r) \arrow[r, "*"] \arrow[d, "{[\nu,\mu]}"] & P^{4n+2}(p^r) \arrow[d] \\
\Sigma P^{2n}(p^r)\wedge P^{2n}(p^r) \arrow[r, "{[1,1]}"] & P^{2n+1}(p^r) \arrow[r, "j"] \arrow[d, "c"] & J_2(P^{2n}(p^r)) \arrow[d, "\varrho"] \\
 & C^{2n+1}(p^r) \arrow[r, equal] & C^{2n+1}(p^r)
\end{tikzcd} \tag{8}$$

that defines the map ϱ.

In general, if Y and Z are simply-connected spaces, let ev_1 and ev_2 be the composites

$$ev_1\colon \Sigma\Omega Y \xrightarrow{ev} Y \xrightarrow{i_1} Y\vee Z$$

$$ev_2\colon \Sigma\Omega Z \xrightarrow{ev} Z \xrightarrow{i_2} Y\vee Z,$$

where i_1 and i_2 are the inclusions of the left and right wedge summands respectively. By [Ga], there is a homotopy fibration

$$\Sigma\Omega Y\wedge\Omega Z \xrightarrow{[ev_1,ev_2]} Y\vee Z \longrightarrow Y\times Z$$

where the right map is the inclusion of the wedge into the product. When $Y=Z$ there is a fold map $\nabla\colon Y\vee Y\longrightarrow Y$. The *universal Whitehead product* on Y is the composite

$$\Psi\colon \Sigma\Omega Y\wedge\Omega Y \xrightarrow{[ev_1,ev_2]} Y\vee Y \xrightarrow{\nabla} Y.$$

It is universal because any Whitehead product on Y factors through Ψ. In [TW] it was shown that if $Y=\Sigma X$ then the composite $\Sigma\Omega\Sigma X\wedge\Omega\Sigma X\xrightarrow{\Psi}\Sigma X\xrightarrow{j}J_2(\Sigma X)$ is null homotopic. In our case, taking $X=P^{2n}(p^r)$, the factorization of c through j in (8) immediately implies the following.

Lemma 3.2. *The composite* $\Sigma\Omega P^{2n+1}(p^r)\wedge\Omega P^{2n+1}(p^r)\xrightarrow{\Psi}P^{2n+1}(p^r)\xrightarrow{c}C^{2n+1}(p^r)$ *is null homotopic.* $\square$

Next, since S^{2n+1} is an H-space when localized at a prime $p\geq 3$, the composite $P^{4n}(p^r)\xrightarrow{[\nu,\mu]}P^{2n+1}(p^r)\xrightarrow{q}S^{2n+1}$ is null homotopic. Thus q extends to a map $q'\colon C^{2n+1}(p^r)\longrightarrow S^{2n+1}$. From this extension we obtain a homotopy fibration diagram

$$\begin{tikzcd}
 & M^{2n+1}(p^r) \arrow[r, equal] \arrow[d] & M^{2n+1}(p^r) \arrow[d] & \\
\Omega S^{2n+1} \arrow[r, "\partial_r"] \arrow[d, equal] & F^{2n+1}(p^r) \arrow[r] \arrow[d, "d"] & P^{2n+1}(p^r) \arrow[r, "q"] \arrow[d, "c"] & S^{2n+1} \arrow[d, equal] \\
\Omega S^{2n+1} \arrow[r, "\overline{\partial}_r"] & D^{2n+1}(p^r) \arrow[r] & C^{2n+1}(p^r) \arrow[r, "q'"] & S^{2n+1}
\end{tikzcd} \tag{9}$$

that defines the spaces $D^{2n+1}(p^r)$ and $M^{2n+1}(p^r)$ and the maps d and $\overline{\partial}_r$.

Lemma 3.3. *There is a choice of the homotopy equivalence* $S^{2n-1}\times V^{2n+1}(p^{r+1})\times \Omega R^{2n+1}(p^r)\xrightarrow{\kappa}\Omega F^{2n+1}(p^r)$ *with the property that there is a homotopy commutative diagram*

$$\begin{tikzcd} V^{2n+1}(p^r)\times \Omega R^{2n+1}(p^r) \arrow[r, "\xi"] \arrow[d, "*\times\varpi_r^{r+1}\times 1"] & \Omega M^{2n+1}(p^r) \arrow[d] \\ S^{2n-1}\times V^{2n+1}(p^{r+1})\times \Omega R^{2n+1}(p^r) \arrow[r, "\kappa"] & \Omega F^{2n+1}(p^r) \end{tikzcd}$$

for some map ξ.

Proof. Start with the homotopy cofibration $P^{4n}(p^r)\xrightarrow{[\nu,\mu]} P^{2n+1}(p^r)\xrightarrow{c} C^{2n+1}(p^r)$. By definition, $[\nu,\mu]=ad^1$ so $c\circ ad^1$ is null homotopic. Since $ad^k=[\nu,ad^{k-1}]$ for $k>1$, the naturality of the mod-p^r Whitehead product implies that $c\circ ad^k$ is null homotopic for all $k\geq 1$. Thus each ad^k lifts to the homotopy fibre $M^{2n+1}(p^r)$ of c. Moreover, any iterated mod-p^r Whitehead product in which $[\nu,\mu]$ appears has the property that it composes trivially with c and so lifts to $M^{2n+1}(p^r)$.

Recall the construction of κ in Section 2. The map $R^{2n+1}(p^r)\longrightarrow P^{2n+1}(p^r)$ was a wedge sum of mod-p^r Whitehead products. Each such Whitehead product factors through the universal Whitehead product on $P^{2n+1}(p^r)$, so Lemma 3.2 implies that the composite $R^{2n+1}(p^r)\longrightarrow P^{2n+1}(p^r)\xrightarrow{c} C^{2n+1}(p^r)$ is null homotopic. Thus the map $R^{2n+1}(p^r)\longrightarrow P^{2n+1}(p^r)$ lifts to $M^{2n+1}(p^r)$, and the lift $R^{2n+1}(p^r)\xrightarrow{\psi} F^{2n+1}(p^r)$ used in forming κ may be chosen to be the composite $R^{2n+1}(p^r)\longrightarrow M^{2n+1}(p^r)\longrightarrow F^{2n+1}(p^r)$. Therefore we obtain a homotopy commutative diagram

$$\begin{tikzcd} & \Omega M^{2n+1}(p^r) \arrow[d] \\ \Omega R^{2n+1}(p^r) \arrow[ur] \arrow[r, "\Omega\psi"] & \Omega F^{2n+1}(p^r). \end{tikzcd} \tag{10}$$

Similarly, each ad^{p^j-1} has $[\nu,\mu]$ appearing in it and so can be chosen to lift to $F^{2n+1}(p^r)$ through $M^{2n+1}(p^r)$. The extension through ω_r^{r+1} may not exist as a map to $M^{2n+1}(p^r)$, but we do not require this. We obtain, for each $j\geq 1$, a homotopy commutative diagram

$$\begin{tikzcd} P^{2np^j}(p^r) \arrow[r] \arrow[d, "\omega_r^{r+1}"] & M^{2n+1}(p^r) \arrow[d] \\ P^{2np^j}(p^{r+1}) \arrow[r, "e_j'"] & F^{2n+1}(p^r). \end{tikzcd}$$

Looping to take products and using (3) we obtain a homotopy commutative diagram

$$\begin{tikzcd} V^{2n+1}(p^r) \arrow[r] \arrow[d, "\varpi_r^{r+1}"] & \prod_{j=1}^{\infty}\Omega P^{2np^j}(p^r) \arrow[r] \arrow[d, "\prod_{j=1}^{\infty}\Omega\omega_r^{r+1}"] & \Omega M^{2n+1}(p^r) \arrow[d] \\ V^{2n+1}(p^{r+1}) \arrow[r] & \prod_{j=1}^{\infty}\Omega P^{2np^j}(p^{r+1}) \arrow[r] & \Omega F^{2n+1}(p^r). \end{tikzcd} \tag{11}$$

By Remark 2.1, we may choose κ to be the product of the inclusion i of the bottom cell S^{2n-1} into $\Omega F^{2n+1}(p^r)$, the map $\Omega\psi$ in (10), and take ϵ to be the bottom row of (11). Then κ is a homotopy equivalence and from (10) and (11) we obtain the homotopy commutative diagram asserted in the statement of the lemma. $\square$

Consider the map $\Omega S^{2n+1} \xrightarrow{\overline{\partial}_r} D^{2n+1}(p^r)$ appearing in (9). We give a factorization of $\Omega\overline{\partial}_r$. Let φ_r be the composite

$$\varphi_r\colon \Omega^2 S^{2n+1} \xrightarrow{b_r} S^{2n-1} \times V^{2n+1}(p^{r+1}) \times \Omega R^{2n+1}(p^r) \xrightarrow{\text{proj}} S^{2n-1}$$

where the right map is the projection.

Proposition 3.4. *There is a homotopy commutative diagram*

$$\begin{tikzcd} \Omega^2 S^{2n+1} \arrow[r, "\varphi_r"] \arrow[d, equal] & S^{2n-1} \arrow[d] \\ \Omega^2 S^{2n+1} \arrow[r, "\Omega\overline{\partial}_r"] & \Omega D^{2n+1}(p^r). \end{tikzcd}$$

Proof. Let κ' be the composite

$$\kappa'\colon S^{2n-1}\times V^{2n+1}(p^r)\times\Omega R^{2n+1}(p^r) \xrightarrow{1\times\varpi_r^{r+1}\times 1} S^{2n-1}\times V^{2n+1}(p^{r+1})\times\Omega R^{2n+1}(p^r) \xrightarrow{\kappa} \Omega F^{2n+1}(p^r).$$

Consider the homotopy fibration $\Omega M^{2n+1}(p^r) \longrightarrow \Omega F^{2n+1}(p^r) \xrightarrow{\Omega d} \Omega D^{2n+1}(p^r)$ from (9). Since Ωd is an H-map and κ is defined by using the loop multiplication on $\Omega F^{2n+1}(p^r)$ to multiply the factors together, the composite $\Omega d\circ\kappa$ is determined by the restriction to each of the factors. By Lemma 3.3, the restriction to $V^{2n+1}(p^r)\times\Omega R^{2n+1}(p^r)$ is null homotopic. Thus there is a homotopy commutative diagram

$$\begin{tikzcd} S^{2n-1} \times V^{2n+1}(p^r) \times \Omega R^{2n+1}(p^r) \arrow[r, "\kappa'"] \arrow[d, "\pi_1"] & \Omega F^{2n+1}(p^r) \arrow[d, "\Omega d"] \\ S^{2n-1} \arrow[r] & \Omega D^{2n+1}(p^r) \end{tikzcd} \tag{12}$$

where π_1 is the projection onto the first factor.

Now consider the diagram

$$\begin{tikzcd} \Omega^2 S^{2n+1} \arrow[r] \arrow[d, equal] & S^{2n-1} \times V^{2n+1}(p^r) \times \Omega R^{2n+1}(p^r) \arrow[d, "\kappa'"] \arrow[dr, "\pi_1"] & \\ \Omega^2 S^{2n+1} \arrow[r, "\Omega\partial_r"] \arrow[d, equal] & \Omega F^{2n+1}(p^r) \arrow[d, "\Omega d"] & S^{2n-1} \arrow[dl, "\iota"] \\ \Omega^2 S^{2n+1} \arrow[r, "\Omega\overline{\partial}_r"] & \Omega D^{2n+1}(p^r) & \end{tikzcd}$$

where ι is the inclusion of the bottom cell. By Lemma 2.3 and the definition of κ', the upper square left homotopy commutes. The lower left square homotopy commutes by (9) and the right triangle homotopy commutes by (12). The diagram as a whole therefore states that $\Omega\overline{\partial}_r$ factors through S^{2n-1}. It remains to identify the map $\varphi'_r\colon \Omega^2 S^{2n+1} \longrightarrow S^{2n-1}$ along the upper direction of the diagram as φ_r. But, by definition, κ' is the identity on the S^{2n-1} factor so φ'_r can be identified as the composite $\Omega^2 S^{2n+1} \xrightarrow{\Omega\overline{\partial}_r} \Omega F^{2n+1}(p^r) \xrightarrow{\text{proj}} S^{2n-1}$, which is the definition of φ_r. $\square$

By Theorem 3.1 (b) there is a homotopy fibration $\Omega T^{2n+1}(p^r) \longrightarrow \Omega^2 S^{2n+1} \xrightarrow{\varphi_r} S^{2n-1}$. Proposition 3.4 therefore implies that the map $\Omega T^{2n+1}(p^r) \longrightarrow \Omega^2 S^{2n+1}$ lifts to the homotopy fibre of $\Omega^2 S^{2n+1} \xrightarrow{\Omega\overline{\partial}_r} \Omega D^{2n+1}(p^r)$, which by (9) is $\Omega^2 C^{2n+1}(p^r)$. Hence we have shown the following, where we explicitly remember that everthing done so far holds for all odd primes.

Corollary 3.5. *If $p \geq 3$ then there is a lift*

$$\begin{tikzcd} & \Omega T^{2n+1}(p^r) \arrow[dl, "\lambda"'] \arrow[d] \\ \Omega^2 C^{2n+1}(p^r) \arrow[r] & \Omega^2 S^{2n+1} \end{tikzcd}$$

for some map λ. $\square$

3.3. **Constructing a map $\Omega^2 C^{2n+1}(p^r) \longrightarrow \Omega T^{2n+1}(p^r)$.** This will be done for $p \geq 5$, and the map will in fact be a loop map. Recall that there is a homotopy cofibration $P^{4n}(p^r) \xrightarrow{[\nu,\mu]} P^{2n+1}(p^r) \xrightarrow{c} C^{2n+1}(p^r)$. As $[\nu,\mu]$ factors through the Whitehead product $\Sigma P^{2n}(p^r) \wedge P^{2n}(p^r) \xrightarrow{[1,1]} P^{2n+1}(p^r)$, there is a homotopy pushout diagram

$$\begin{tikzcd} P^{4n}(p^r) \arrow[r] \arrow[d, equal] & \Sigma P^{2n}(p^r) \wedge P^{2n}(p^r) \arrow[r] \arrow[d, "{[1,1]}"] & P^{4n+1}(p^r) \arrow[d] \\ P^{4n}(p^r) \arrow[r, "{[\nu,\mu]}"] & P^{2n+1}(p^r) \arrow[r, "c"] \arrow[d, "j"] & C^{2n+1}(p^r) \arrow[d, "j'"] \\ & J_2(P^{2n+1}(p^r)) \arrow[r, equal] & J_2(P^{2n+1}(p^r)). \end{tikzcd} \tag{13}$$

that defines the map j'.

By Theorem 3.1 (a), the loops on the pinch map $\Omega P^{2n+1}(p^r) \xrightarrow{\Omega q} \Omega S^{2n+1}$ factors as a composite $\Omega P^{2n+1}(p^r) \xrightarrow{t} T^{2n+1}(p^r) \longrightarrow \Omega S^{2n+1}$ for some map t.

Lemma 3.6. *If $p \geq 5$ then there is a homotopy commutative diagram*

$$\begin{tikzcd} \Omega P^{2n+1}(p^r) \arrow[r, "t"] \arrow[d, "\Omega c"] & T^{2n+1}(p^r) \\ \Omega C^{2n+1}(p^r) \arrow[ur, "\widetilde{t}"'] & \end{tikzcd}$$

for some map $\widetilde{t}$.

Proof. In [TW, Lemma 2.6] it was shown that if Z is a homotopy associative and homotopy commutative H-space and $f\colon \Omega P^{2n+1}(p^r) \longrightarrow Z$ is an H-map then there is a homotopy commutative diagram

$$\begin{tikzcd}
\Omega P^{2n+1}(p^r) \arrow[r, "f"] \arrow[d, "\Omega j"] & Z \\
\Omega J_2(P^{2n+1}(p^r)) \arrow[ur, "\overline{f}"'] &
\end{tikzcd}$$

for some map $\overline{f}$. By (13), the map Ωj factors as the composite $\Omega P^{2n+1}(p^r) \xrightarrow{\Omega c} \Omega C^{2n+1}(p^r) \xrightarrow{\Omega j'} \Omega J_2(P^{2n+1}(p^r))$. Thus if we take $\widetilde{f} = \overline{f} \circ \Omega j'$ then there is a homotopy commutative diagram

$$\begin{tikzcd}
\Omega P^{2n+1}(p^r) \arrow[r, "f"] \arrow[d, "\Omega c"] & Z \\
\Omega C^{2n+1}(p^r). \arrow[ur, "\widetilde{f}"'] &
\end{tikzcd} \tag{14}$$

By Theorem 3.1 (d), if $p \geq 5$ then $T^{2n+1}(p^r)$ is homotopy associative and homotopy commutative, and the map t is an H-map. Thus the assertion of the lemma follows by applying (14) to t. □

3.4. **The proof of Theorem 1.1 and an application.** First, we put the pieces together to obtain a retraction of $\Omega T^{2n+1}(p^r)$ off $\Omega^2 C^{2n+1}(p^r)$, proving Theorem 1.1.

Proof of Theorem 1.1. We will show that if $p \geq 5$ and $n > 1$ then the composite $\Omega T^{2n+1}(p^r) \xrightarrow{\lambda} \Omega^2 C^{2n+1}(p^r) \xrightarrow{\Omega\widetilde{t}} \Omega T^{2n+1}(p^r)$ is a homotopy equivalence. The homotopy commutativity of the diagrams in Corollary 3.5 and Lemma 3.6 imply that both λ and $\Omega\widetilde{t}$ induce an isomorphism in mod-p homology in degree $2n-2$, the least nonvanishing degree. Thus $\Omega\widetilde{t} \circ \lambda$ is a self-map of $\Omega T^{2n+1}(p^r)$ which induces an isomorphism in the least nonvanishing degree in homology. By Theorem 3.1 (e), $\Omega T^{2n+1}(p^r)$ is atomic if $p \geq 5$ and $n > 1$, so $\Omega\widetilde{t} \circ \lambda$ is a homotopy equivalence. □

Next, Theorem 1.1 is applied to produce maps from $\Omega T^{2n+1}(p^r)$ into certain spaces by checking that minimal requirements hold.

Theorem 3.7. *Let $p \geq 5$ and $r \geq 1$. Suppose that there is a map $f\colon P^{2n+1}(p^r) \longrightarrow Z$ for some space Z. If the composite $P^{4n}(p^r) \xrightarrow{[\nu,\mu]} P^{2n+1}(p^r) \xrightarrow{f} Z$ is null homotopic then there is a map $\Omega T^{2n+1}(p^r) \longrightarrow \Omega^2 Z$ whose restriction to the bottom Moore space is the double adjoint of f.*

Proof. The hypothesis that $f \circ [\nu, \mu]$ is null homotopic is equivalent to saying that the map f extends to a map $C^{2n+1}(p^r) \longrightarrow Z$. Theorem 1.1 therefore implies that there is a composite $\Omega T^{2n+1}(p^r) \longrightarrow \Omega^2 C^{2n+1}(p^r) \longrightarrow \Omega^2 Z$ whose restriction to the bottom Moore space is the double adjoint of f. □

By Theorem 3.1 (c), if $r \geq 2$ then $\Omega T^{2n+1}(p^r)$ retracts off $\Omega^2 P^{2n+1}(p^r)$, so in this case, given a map $P^{2n+1}(p^r) \xrightarrow{f} Z$, one automatically obtains a map $\Omega T^{2n+1}(p^r) \longrightarrow \Omega^2 Z$ whose restriction to the bottom Moore space is the double adjoint of f. The additional hypothesis in Theorem 3.7 regarding $f \circ [\nu, \mu]$ being null homotopic is not necessary.

However, the $r = 1$ case is often the vital one, and it is an open conjecture as to whether $\Omega T^{2n+1}(p)$ retracts off $\Omega^2 P^{2n+1}(p)$. So Theorem 3.7 can be thought of as a way of producing a consequence of the conjecture without having to prove it first. Moreover, it is practical in the sense that one can hope to check that a given map $P^{2n+1}(p^r) \xrightarrow{f} Z$ has $f \circ [\nu, \mu]$ null homotopic. In fact, this criterion will be used in the next section in the context of Kac-Moody groups.

4. Kac-Moody groups

As in [K1, K2], Kac-Moody groups of rank two correspond to generalized Cartan matrices of the form

$$A = \begin{pmatrix} 2 & -a \\ -b & 2 \end{pmatrix}.$$

Given such a matrix A, the Kac-Moody group K is the semisimple factor inside the corresponding unitary form. If $ab < 4$ then K is a compact Lie group. We are interested in the case when $ab \geq 4$. Define integers c_i and d_i recursively by:

$$c_0 = d_0 = 1; \qquad c_1 = d_1 = 1; \qquad c_{j+1} = ad_j - c_{j-1}; \qquad d_{j+1} = bc_j - d_{j-1}.$$

Let $g_i = (c_i, d_i)$ be the greatest common divisor of c_i and d_i. Fix an odd prime p and take homology with mod-p coefficients. Let k be the smallest positive integer such that p divides g_k. Then there is an isomorphism of Hopf algebras

$$H_*(K) \cong \Lambda(z_3, y_{2k-1}) \otimes \mathbb{Z}/p\mathbb{Z}[x_{2k}] \tag{15}$$

where the generators are primitive, and if r is the exponent of p in g_k, then $\beta^r(x_{2k}) = y_{2k-1}$. Further, it is known that if k exists then either is it p, $2p$ or a nontrivial divisor of $p+1$ or $p-1$, and in each case there are choices of a and b which produce such a k.

Given such a k, let $\mathcal{V}_k$ be the collection of all possible integers r that arise as the exponent of p in g_k for some choice of integers a and b. In general it is known that $\mathcal{V}_k$ is nonempty but no precise description is known. However, to give some examples, we show that if $p \geq 5$ and $k \in \{2, 3, 4\}$ then $\mathcal{V}_k = \mathbb{N}$. Note that each of 2, 3 and 4 is a proper divisor of either $p+1$ or $p-1$, and so is a valid value of k. Observe that

$$c_2 = a, \quad d_2 = b, \quad c_3 = d_3 = ab - 1, \quad c_4 = a(ab-2), \quad d_4 = b(ab-2).$$

If $k = 2$ then taking $a = b = p^r$ gives $g_2 = (c_2, d_2) = p^r$. If $k = 3$ then taking $a = 1$ and $b = p^r + 1$ gives $g_2 = (c_2, d_2) = 1$ and $g_3 = (c_3, d_3) = p^r$. If $k = 4$ then taking $a = 1$ and $b = p^r + 2$ gives $g_2 = (c_2, d_2) = 1$, $g_3 = (c_3, d_3) = p^r + 1$, and $g_4 = (c_4, d_4) = p^r$. Thus, in all cases, any $r \geq 1$ will do so $\mathcal{V}_k = \mathbb{N}$.

Recall from the introduction that K has an S^3 subgroup whose inclusion induces an isomorphism onto the sub-Hopf-algebra $\Lambda(z_3)$ in homology, resulting in a homotopy fibration sequence

$$S^3 \longrightarrow K \xrightarrow{\delta} X \longrightarrow BS^3 \longrightarrow BK \tag{16}$$

where

$$H_*(X) \cong \Lambda(y_{2k-1}) \otimes \mathbb{Z}/p\mathbb{Z}[x_{2k}]$$

and δ_* is the projection. In particular, this is an isomorphism of coalgebras (as X may not be an H-space).

Now localize all spaces and maps at p. We aim to show that for $p \geq 5$ and $1 < k < p-1$ there is a p-local homotopy equivalence $\Omega X \simeq \Omega T^{2n+1}(p^r)$. Any approach to the problem is limited by the fact that almost nothing is known about the homotopy theory, or homotopy groups, of K. The method is to first produce a map $P^{2k+1}(p^r) \longrightarrow BK$ whose adjoint induces an isomorphism in homology onto the generators x_{2k} and y_{2k-1}, and then to extend it to a map $C^{2k+1}(p^r) \longrightarrow BK$. At both stages certain values of k have to be eliminated in order to ensure that potential obstructions vanish. This will require some information about the homotopy groups of spheres proved by Toda [To].

Remark 4.1. It is worth pointing out beforehand that the only properties of Kac-Moody groups used in showing $\Omega T^{2n+1}(p^r) \simeq \Omega X$ are the existence of the homotopy fibration sequence (16) and the description of $H_*(X)$. The rest of the argument is based on the homotopy theory of spheres, Moore spaces and Anick spaces.

Lemma 4.2. *Let $p \geq 5$. If $3 < m \leq 4p$ then $\pi_m(S^3) \cong 0$ unless $m \in \{2p, 4p-2\}$.* $\square$

Lemma 4.3. *Let $p \geq 5$. If $k \leq p$ and $r \in \mathcal{V}_k$ then there is a map $P^{2k+1}(p^r) \longrightarrow BK$ whose adjoint induces an isomorphism onto the generators x_{2k} and y_{2k-1} in $H_*(K)$.*

Proof. Let $P^{2k}(p^r) \longrightarrow X$ be the inclusion of the bottom Moore space. We aim to show that there is a lift

$$\begin{tikzcd} & P^{2k}(p^r) \arrow[dl, dashed] \arrow[d] & \\ K \arrow[r] & X \arrow[r] & BS^3. \end{tikzcd}$$

The adjoint of this lift is the map asserted in the Lemma. The lift will certainly exist when $[P^{2k}(p^r), BS^3] \cong 0$.

The homotopy cofibration $S^{2k-1} \longrightarrow P^{2k}(p^r) \longrightarrow S^{2k}$ induces an exact sequence $[S^{2k}, BS^3] \longrightarrow [P^{2k}(p^r), BS^3] \longrightarrow [S^{2k-1}, BS^3]$. By Lemma 4.2, the first nontrivial torsion homotopy group of BS^3 occurs in dimension $2p+1$. So if $k \leq p$ then $\pi_{2k}(BS^3)$ and $\pi_{2k-1}(BS^3)$ are trivial groups. Therefore, by exactness, $[P^{2k+1}(p^r), BS^3] \cong 0$, and the asserted lift exists. (Note that when $k = p+1$ the map α_1 generating $\pi_{2k-1}(BS^3)$ is a potential obstruction to a lift, and when $k = 2p$ the map α_2 generating $\pi_{2k-1}(BS^3)$ is a potential obstruction.) $\square$

Lemma 4.4. *Let $p \geq 5$. Suppose that k is a proper divisor of $p-1$ or $p+1$, but $k \neq \frac{p+1}{2}$. If $r \in \mathcal{V}_k$ then the composite $P^{4k}(p^r) \xrightarrow{[\nu,\mu]} P^{2k+1}(p^r) \longrightarrow BK$ is null homotopic.*

Proof. Let $\xi\colon P^{2k+1}(p^r) \longrightarrow BK$ be the map in Lemma 4.3. Consider $\Omega P^{2k+1}(p^r) \xrightarrow{\Omega\xi} K$. Since $P^{2n+1}(p^r)$ is a suspension, the Bott-Samelson theorem implies that $H_*(\Omega P^{2k+1}(p^r)) \cong T(u_{2k-1}, v_{2k})$ where $\beta^r(v_{2k}) = u_{2k-1}$. The homology statement in Lemma 4.3 implies that $(\Omega\xi)_*$ sends u_{2k-1}, v_{2k} to y_{2k-1}, x_{2k} respectively in $H_*(K) \cong \Lambda(z_3, y_{2k-1}) \otimes \mathbb{Z}/p\mathbb{Z}[x_{2k}]$. Since $(\Omega\xi)_*$ is an algebra map, this implies that the image of $(\Omega\xi)_*$ is the subalgebra of $H_*(K)$ generated by y_{2k-1} and x_{2k}. Recall that the map $K \xrightarrow{\delta} X$ in (16) induces a projection in homology onto $H_*(X) \cong \Lambda(y_{2k-1}) \otimes \mathbb{Z}/p\mathbb{Z}[x_{2k}]$. Therefore the composite $\theta\colon \Omega P^{2n+1}(p^r) \xrightarrow{\Omega\xi} K \xrightarrow{\delta} X$ has the property that θ_* is the abelianization of the tensor algebra.

Let N be the homotopy fibre of θ. Since θ_* is the abelianization of the tensor algebra, the Serre exact sequence implies that in degrees $\leq 4k-1$, $H_*(N)$ is the kernel of θ_*. Therefore, in this degree range, $H_*(N)$ consists of the brackets $\langle u_{2n-1}, u_{2n-1}\rangle$ and $\langle v_{2n}, u_{2n-1}\rangle$, which are connected by a Bockstein β^r. Thus there is an inclusion of a bottom Moore space into N which gives a composite

$$\gamma\colon P^{4k-1}(p^r) \longrightarrow N \longrightarrow \Omega P^{2k+1}(p^r). \tag{17}$$

Now we compare γ to known elements in the group $[P^{4k-1}(p^r), \Omega P^{2k+1}(p^r)]$. Consider the homotopy fibration

$$\Omega F^{2k+1}(p^r) \longrightarrow \Omega P^{2k+1}(p^r) \xrightarrow{\Omega q} \Omega S^{2k+1}.$$

Applying the functor $[P^{4k-1}(p^r), \ \ ]$ to this fibration we obtain an exact sequence

$$[P^{4k-1}(p^r), \Omega F^{2k+1}(p^r)] \longrightarrow [P^{4k-1}(p^r), \Omega P^{2k+1}(p^r)] \longrightarrow [P^{4k-1}(p^r), \Omega S^{2k+1}]. \tag{18}$$

By Lemma 4.2, the hypothesis that $k \neq p-1$ implies that $[P^{4k-1}(p^r), \Omega S^{2k+1}] \cong 0$. On the other hand, by (4) there is a homotopy equvalence

$$\Omega F^{2k+1}(p^r) \simeq S^{2k-1} \times \Omega R^{2k+1}(p^r) \times \prod_{j=1}^{\infty} S^{2kp^j-1}\{p^{r+1}\}.$$

Recall that $R^{2k+1}(p^r)$ is a wedge of mod-p^r Moore spaces mapping to $P^{2k+1}(p^r)$ by a wedge sum of mod-p^r Whitehead products. In particular, by [CMN1], the least dimensional Moore space in $R^{2k+1}(p^r)$ is $P^{4k}(p^r)$, which maps to $P^{2k+1}(p^r)$ by $[\nu, \mu]$, and for $p \geq 5$ the second least dimensional Moore space in $R^{2k+1}(p^r)$ is $P^{6k}(p^r)$. This, together with the fact that each factor $S^{2kp^j-1}\{p^{r+1}\}$ is more than $(4k-1)$-connected, implies that

$$[P^{4k-1}(p^r), \Omega F^{2k+1}(p^r)] \cong [P^{4k-1}(p^r), S^{2k-1} \times \Omega P^{4k}(p^r)].$$

Lemma 4.2 and the hypotheses that $k \neq p$ ensures that $[P^{4k-1}(p^r), S^{2k-1}] \cong 0$. Since the suspension map $P^{4k-1}(p^r) \xrightarrow{E} \Omega P^{4k}(p^r)$ is $(8k-6)$-connected, which is greater than the dimension of $P^{4k-1}(p^r)$, it induces an isomorphism $[P^{4k-1}(p^r), P^{4k-1}(p^r)] \cong [P^{4k-1}(p^r), \Omega P^{4k}(p^r)]$. Therefore

$$[P^{4k-1}(p^r), \Omega F^{2k+1}(p^r)] \cong [P^{4k-1}(p^r), P^{4k-1}(p^r)] \cong \mathbb{Z}/p^r\mathbb{Z}$$

where the generator of the group $[P^{4k-1}(p^r), P^{4k-1}(p^r)]$ is the identity map. Thus the exact sequence (18) simplifies to an exact sequence

$$\mathbb{Z}/p^r\mathbb{Z} \cong [P^{4k-1}(p^r), P^{4k-1}(p^r)] \longrightarrow [P^{4k-1}(p^r), \Omega P^{2k+1}(p^r)] \longrightarrow 0. \tag{19}$$

By [CMN2], the factor $\Omega R^{2k+1}(p^r)$ of $\Omega F^{2k+1}(p^r)$ also retracts off $\Omega P^{2k+1}(p^r)$. Therefore there is a retraction of $[P^{4k-1}(p^r), P^{4k-1}(p^r)]$ off $[P^{4k-1}(p^r), \Omega P^{2k+1}(p^r)]$, implying that there is an isomorphism

$$[P^{4k-1}(p^r), \Omega P^{2k+1}(p^r)] \cong [P^{4k-1}(p^r), P^{4k-1}(p^r)] \cong \mathbb{Z}/p^r\mathbb{Z}.$$

Hence if $k \notin \{p-1, p\}$ then $[P^{4k-1}(p^r), \Omega P^{2k+1}(p^r)]$ is isomorphic to $\mathbb{Z}/p^r\mathbb{Z}$, and is generated by the adjoint of the mod-p^r Whitehead product $[\nu, \mu]$.

The adjoint of $[\nu, \mu]$ is equivalently described as the mod-p^r Samelson product $\langle\tilde{\nu}, \tilde{\mu}\rangle$, where $\tilde{\nu}, \tilde{\mu}$ are the adjoints of ν, μ respectively. Thus $\langle\tilde{\nu}, \tilde{\mu}\rangle$ generates $[P^{4k-1}(p^r), \Omega P^{2k+1}(p^r)] \cong \mathbb{Z}/p^r\mathbb{Z}$. In particular, the map γ in (17) must be some multiple of $\langle\tilde{\nu}, \tilde{\mu}\rangle$. To see which multiple, we look at homology. In degree $4k-1$ in mod-p homology, the mod-p^r Samelson product $\langle\tilde{\nu}, \tilde{\mu}\rangle$ has image $\langle v_{2k}, u_{2k-1}\rangle$. Recall that the composite $\gamma\colon P^{4k-1}(p^r) \longrightarrow M \longrightarrow \Omega P^{2k+1}(p^r)$ has the same image in homology. Thus, as γ is a multiple of $\langle\tilde{\nu}, \tilde{\mu}\rangle$, we must have $\gamma \simeq u \cdot \langle\tilde{\nu}, \tilde{\mu}\rangle$ for some unit u in $\mathbb{Z}/p^r\mathbb{Z}$.

Consequently, as γ factors through the homotopy fibre of θ, the map $\langle\tilde{\nu}, \tilde{\mu}\rangle$ also factors through the homotopy fibre of θ. Therefore the composite $P^{4k-1}(p^r) \xrightarrow{\langle\tilde{\nu}, \tilde{\mu}\rangle} \Omega P^{2k+1}(p^r) \xrightarrow{\Omega\xi} K \xrightarrow{\delta} X$ is null homotopic. Hence $\Omega\xi \circ \langle\tilde{\nu}, \tilde{\mu}\rangle$ lifts to the homotopy fibre of δ, which is S^3. By Lemma 4.2, if $k \notin \{\frac{p+1}{2}, p\}$ then $[P^{4k-1}(p^r), S^3] \cong 0$, implying that $\Omega\xi \circ \langle\tilde{\nu}, \tilde{\mu}\rangle$ is null homotopic. Taking adjoints, this is equivalent to saying that $\xi \circ [\nu, \mu]$ is null homotopic. Summarizing, we have shown that if $k \notin \{\frac{p+1}{2}, p-1, p\}$ then $\xi \circ [\nu, \mu]$ is null homotopic, as asserted. $\square$

The case when $k = \frac{p+1}{2}$ can be recovered using a special argument.

Lemma 4.5. *Let $p \geq 5$. If $k = \frac{p+1}{2}$ and $r \in \mathcal{V}_k$ then the composite $P^{4k}(p^r) \xrightarrow{[\nu,\mu]} P^{2k+1}(p^r) \longrightarrow BK$ is null homotopic.*

Proof. The potential obstruction in the case when $k = \frac{p+1}{2}$ in the proof of Lemma 4.4 came about from the composite $P^{2p+1}(p^r) \xrightarrow{\langle\tilde{\nu}, \tilde{\mu}\rangle} \Omega P^{2k+1}(p^r) \xrightarrow{\Omega\xi} K$ lifting to a map $P^{2p+1}(p^r) \longrightarrow S^3$ which could be an extension of the homotopy class $\alpha_1\colon S^{2p} \longrightarrow S^3$ that generates $\pi_{2p}(S^3) \cong \mathbb{Z}/p\mathbb{Z}$. Assume that this occurs. Taking adjoints, we obtain a homotopy commutative diagram

$$\begin{tikzcd} & P^{2p+2}(p^r) \arrow[d, "{[\nu,\mu]}"] \arrow[ddl, "\overline{\alpha}_1"'] \\ & P^{2k+1}(p^r) \arrow[d, "\xi"] \\ BS^3 \arrow[r] & BK \end{tikzcd} \tag{20}$$

where $\overline{\alpha}_1$ is an extension of the adjoint of α_1. Observe that $\Sigma[\nu,\mu]$ is null homotopic since the suspension of any mod-p^r Whitehead product is null homotopic. Therefore if we restrict (20) to S^{2p+1} and suspend we obtain an extension

$$\begin{tikzcd} S^{2p+2} \arrow[r, "\alpha_1"] & S^5 \arrow[r] \arrow[d] & A \arrow[d, dashed, "\zeta"] \\ & \Sigma BS^3 \arrow[r] & \Sigma BK \end{tikzcd}$$

for some map ζ, where A is the homotopy cofibre of α_1. The class α_1 is detected in mod-p cohomology by the Steenrod operation $\mathcal{P}^1$, so the two-cell complex A has its bottom cell attached to its top cell by $\mathcal{P}^1$. The homotopy commutativity of the square in the preceding diagram implies that ζ^* is an isomorphism in degree 5. Therefore, as $\mathcal{P}^1$ is nonzero on $H^5(A)$, it must also be nonzero on $H^5(\Sigma BK)$. By stability, this implies that it is nontrivial in $H^4(BK)$. As the generator of $H^4(BK)$ is the transgression of the generator in $H^3(K)$ in the cohomology Serre spectral sequence for the path-loop fibration $K \longrightarrow * \longrightarrow BK$, and as Steenrod operations commute with the transgression, we obtain that $\mathcal{P}^1$ is nontrivial on $H^3(K)$.

On the other hand, as $k = \frac{p+1}{2}$, we have $H_*(K) \cong \Lambda(z_3, y_p) \otimes \mathbb{Z}/p\mathbb{Z}[x_{p+1}]$, and as this coalgebra is primitively generated, we can dualize to obtain an algebra isomorphism $H^*(K) \cong \Lambda(\bar{z}_3, \bar{y}_p) \otimes \Gamma[\bar{x}_{p+1}]$ where $\bar{z}_3, \bar{y}_p, \bar{x}_{p+1}$ are dual to z_3, y_p, x_{p+1} and $\Gamma[\ \ ]$ is the divided power algebra. The only element in degree $2p+1$ in this algebra is $y_p \cup x_{2p+1}$. The nontriviality of $\mathcal{P}^1$ on $H^3(K)$ therefore implies that $\mathcal{P}^1(z_3) = u \cdot (y_p \cup x_{p+1})$ for some unit $u \in \mathbb{Z}/p\mathbb{Z}$. But $\mathcal{P}^1$ sends primitives to primitives, giving a contradiction. Thus it cannot have been the case that $\Omega\xi \circ \langle \tilde{\nu}, \tilde{\mu} \rangle$ lifted nontrivially to S^3. Thus $\Omega\xi \circ \langle \tilde{\nu}, \tilde{\mu} \rangle$ is null homotopic, as required. □

Let $k < p-1$. Start with the map $f\colon P^{2k+1}(p^r) \longrightarrow BK$ in Lemma 4.3. By Lemmas 4.4 and 4.5, the hypotheses of Proposition 3.7 are satisfied. So there is a map $\Omega T^{2k+1}(p^r) \longrightarrow \Omega K$ whose restriction to the bottom Moore space is the double adjoint of f. The composite

$$g\colon \Omega T^{2k+1}(p^r) \longrightarrow \Omega K \longrightarrow \Omega X$$

therefore induces an isomorphism in the least nonvanishing degree in homology. We claim that this is enough to show that g induces an isomorphism in homology in all degrees, and so is a homotopy equivalence. This is an atomicity style argument, and requires a preliminary lemma.

Lemma 4.6. *There is an abstract isomorphism of vector spaces $H_*(\Omega T^{2k+1}(p^r)) \cong H_*(\Omega X)$.*

Proof. Since $H_*(T^{2k+1}(p^r)) \cong H_*(X)$ as coalgebras, there is an induced isomorphism between cobar constructions. This implies that there is an isomorphism between the outputs of the homology Eilenberg-Moore spectral sequences for the path-loop fibrations $\Omega A \longrightarrow * \longrightarrow A$ and $\Omega B \longrightarrow * \longrightarrow B$ which converge to $H_*(\Omega A)$ and $H_*(\Omega B)$. That is, there are coalgebra isomorphisms between the associated graded modules $E^0(H_*(\Omega T^{2k+1}(p^r)))$ and $E^0(H_*(\Omega X))$. As we are taking homology with coefficients in a field, this implies that there is a vector space isomorphism $H_*(\Omega T^{2k+1}(p^r)) \cong H_*(\Omega X)$. (A coalgebra isomorphism would require resolving potential extension problems, which we do not address.) $\square$

Proposition 4.7. *If $k \geq 2$ then the composite $g\colon \Omega T^{2k+1}(p^r) \longrightarrow \Omega K \longrightarrow \Omega X$ is a homotopy equivalence.*

Proof. In general, suppose that Y is a simply-connected space and there is a self-map $e\colon Y \longrightarrow Y$. In [Th, Lemma 2.2] it is shown that if x is an element of least nontrivial degree the kernel of e_* then x is: (i) primitive, (ii) annihilated by all dual Steenrod operations and higher Bocksteins, and (iii) in the image of the Hurewicz homomorphism or the mod-p Hurewicz homomorphism. Let $HMH(Y)$ be the submodule of $H_*(Y)$ that consists of elements satisfying (i), (ii) and (iii). The argument in [Th, Lemma 2.2] did not really require a self-map of spaces, but only a map $e'\colon Y \longrightarrow Z$ where $H_*(Y)$ is known as a coalgebra over the Steenrod algebra, and $H_*(Z)$ is abstractly isomorphic to $H_*(Y)$ as vector spaces (in order to show at the appropriate moment that an injection $H_m(Y) \longrightarrow H_m(Z)$ is an isomorphism). This fits our case as we have a map $\Omega T^{2k+1}(p^r) \xrightarrow{g} \Omega X$ and, by Lemma 4.6, there is an abstract isomorphism of vector spaces $H_*(\Omega T^{2k+1}(p^r)) \cong H_*(\Omega X)$.

Let $k > 2$. Instead of determining $HMH(\Omega T^{2k+1}(p^r))$ directly, we follow [Th] by making use of the calculation

$$HMH(\Omega^2 T^{2k+1}(p^r)) = \{a_{2k-3}\}$$

for $k > 2$, where a_{2k-3} is a generator of $H_{2k-3}(\Omega^2 T^{2k+1}(p^r)) \cong \mathbb{Z}/p\mathbb{Z}$, the least dimensional nontrivial homology group. Let $x \in HMH(\Omega T^{2k+1}(p^r))$ and suppose x is of degree m. As x is in the image of the Hurewicz homomorphism there is a map $h\colon S^m \longrightarrow \Omega T^{2k+1}(p^r)$ such that $h_*(\iota_m) = x$, where $\iota_m \in H_m(S^m)$ represents a generator. Let $\tilde{h}\colon S^{m-1} \longrightarrow \Omega^2 T^{2k+1}(p^r)$ be the adjoint of h. We claim that $\tilde{h}(\iota_{m-1}) \in HMH(\Omega^2 T^{2k+1}(p^r))$. By definition, $\tilde{h}_*(\iota_{m-1})$ is in the image of the Hurewicz homomorphism (although we have not yet checked if it is nonzero), and as it is a Hurewicz image, it is also primitive and is annihilated by all dual Steenrod operations. It remains to show that $\tilde{h}(\iota_{m-1})$

is nonzero. Consider the composite $S^m \xrightarrow{\Sigma\tilde{h}} \Sigma\Omega^2 T^{2k+1}(p^r) \xrightarrow{ev} \Omega T^{2k+1}(p^r)$, where ev is the canonical evaluation map. This composite is the adjoint of $\tilde{h}$, which is the map h. Therefore, as $h_*(\iota_m)$ is nonzero so is $(\Sigma\tilde{h})_*(\iota_m)$, and hence so is $\tilde{h}_*(\iota_{m-1})$. The calculation $HMH(\Omega^2 T^{2k+1}(p^r)) = \{a_{2k-3}\}$ therefore implies that $HMH(\Omega T^{2k+1}(p^r)) = \{\bar{a}_{2k-2}\}$, where $\bar{a}_{2k-2}$ transgresses to a_{2k-3} in the Serre spectral sequence for the path-loop fibration.

Thus, for the map $\Omega T^{2k+1}(p^r) \xrightarrow{g} \Omega X$, if $x \in \mathrm{Ker}\,(g_*)$ is a nontrivial element of least degree it must be a multiple of $\bar{a}_{2k-2}$. But we have shown that g_* is an isomorphism in degree $2k-2$. Therefore $\mathrm{Ker}\,(g_*) = 0$ and so g_* is an injection. Since $H_*(\Omega X)$ is isomorphic to $H_*(\Omega T^{2k+1}(p^r))$ as vector spaces, they have identical Euler-Poincaré series, so g_* being an injection implies that it is an isomorphism. Hence g is a homotopy equivalence by Whitehead's Theorem.

The $k = 2$ case is different in that $HMH(\Omega^2 T^5(p^r))$ may not be just $\{a_1\}$. This was dealt with separately in [Th, Proof of Theorem 1.1], where it was noted that $HMH(\Omega T^5(p^r)) = \{\bar{a}_2\}$. The rest of the argument in the present proof now goes through as before. $\square$

Finally, we prove the second main statement in the paper.

Proof of Theorem 1.2. The p-local homotopy equivalence for ΩX is given by Proposition 4.7. The p-local homotopy decomposition for ΩK now follows since g factors through $\Omega K \xrightarrow{\Omega\delta} \Omega X$. $\square$

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